MEMS1045 Automatic control

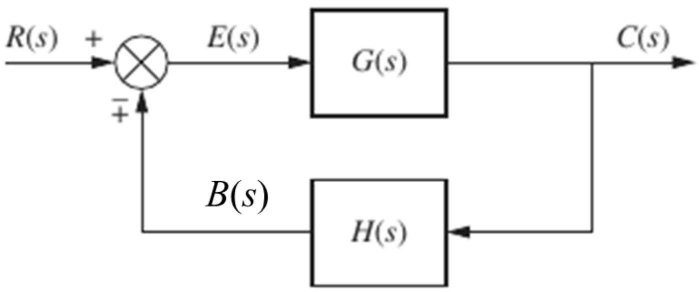
Lecture 12

Frequency response 2

Objectives

- Determine the system stability, gain and phase margins from the open-loop frequency response
- Determine the static error constants from Bode diagram
- Describe the relationships between open loop frequency response parameters with the closed-loop characteristics

Nyquist stability criterion



The closed-loop transfer function of the feedback system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristics equation is:

$$1 + G(s)H(s)$$

Let $G(s) = \frac{n_G}{d_G}$ and $H(s) = \frac{n_H}{d_H}$ or $G(s)H(s) = \frac{n_G n_H}{d_G d_H}$

Open-loop poles express by $d_G d_H$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{n_G}{d_G}\right)}{1 + \left(\frac{n_G}{d_G}\right)\left(\frac{n_H}{d_H}\right)} = \frac{n_G d_H}{d_G d_H + n_G n_H}$$

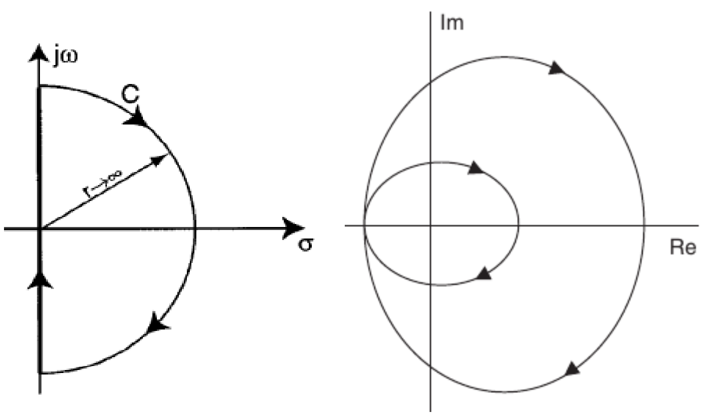
Closed-loop poles express by $d_G d_H + n_G n_H$

Nyquist stability criterion

$$1 + G(s)H(s) = 1 + \left(\frac{n_G}{d_G}\right)\left(\frac{n_H}{d_H}\right) = \frac{d_G d_H + n_G n_H}{d_G d_H}$$

Denominator is expression for open-loop poles $d_G d_H$
Numerator is expression for closed-loop poles $d_G d_H + n_G n_H$
Use this expression to relate open-loop poles to closed loop poles
Use contour in s-plane to search for poles and zeros in the right half plane

Nyquist contour in s-plane



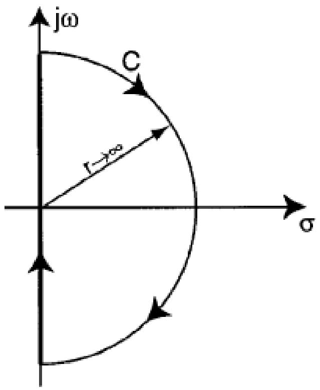
If $1 + G(s)H(s)$ has Z zeros and P poles in the RHP, then the Nyquist plot of $1 + G(s)H(s)$ will encircle the origin in a CW direction N times where N=Z-P

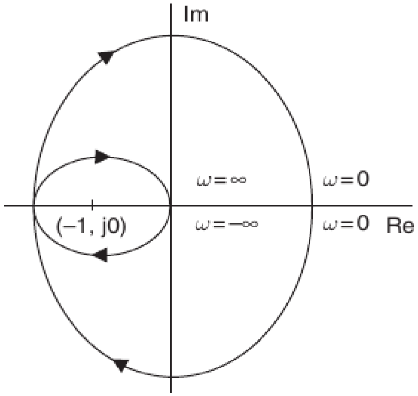
Note: no encirclement can also mean $Z = P$!

Nyquist stability criterion

Let $1 + G(s)H(s) = \Delta(s)$ and the characteristics equation can be reformulated to $G(s)H(s) = \Delta(s) - 1$

Nyquist contour in s-plane





Nyquist plot of $G(s)H(s)$ will now encircle the point -1 in a CW direction N times
Note that $G(s)H(s)$ is the open-loop transfer function

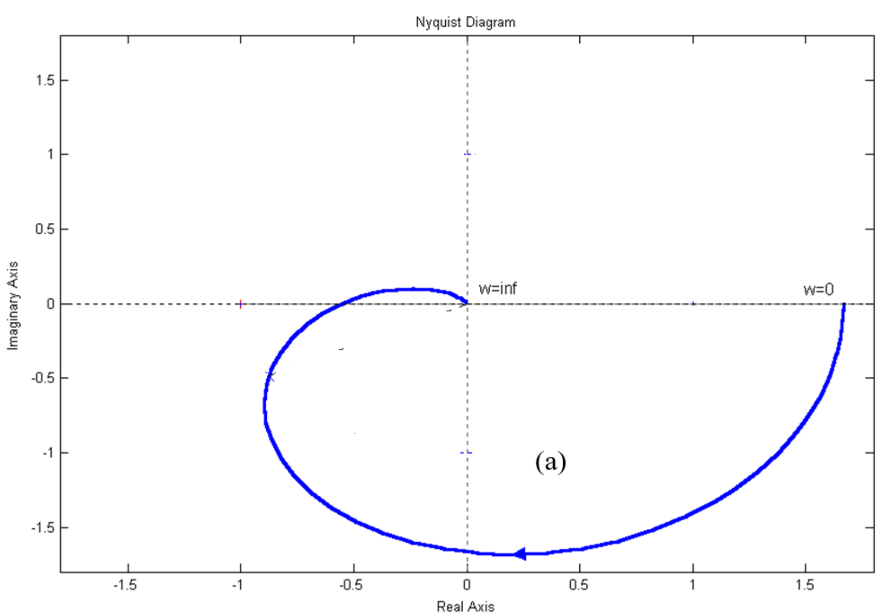
Stability criterion: if the open loop transfer function is stable, then the closed loop system will be stable if Nyquist diagram of the OLTF is not encircling -1+j0 point, as 0<ω<∞
The criterion uses the open-loop transfer function to determine the closed loop stability

Example 1

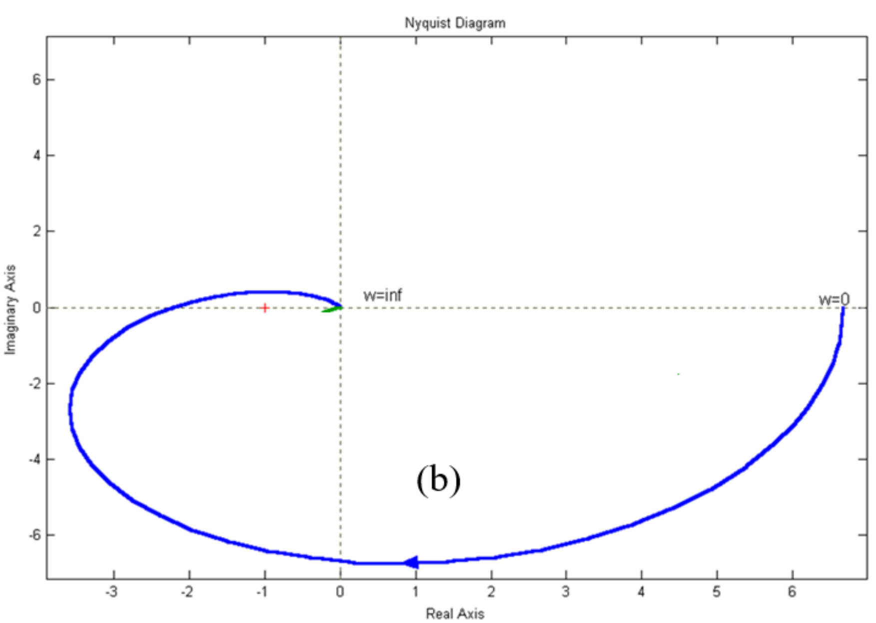
Using the given Nyquist diagrams of the open-loop systems, determine stability of the closed-loop unity feedback systems for:

a) $G_1(s) = \dfrac{5}{s^3+3s^2+4s+3}$

b) $G_2(s) = \dfrac{20}{s^3+3s^2+4s+3}$

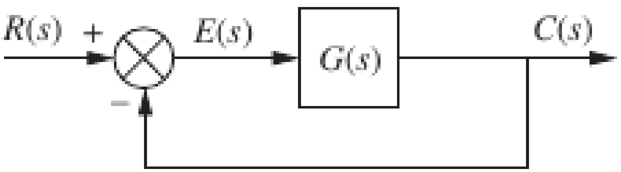


(a)



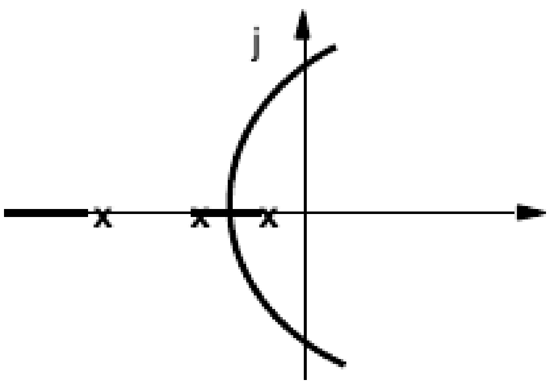
(b)

Review – system stability

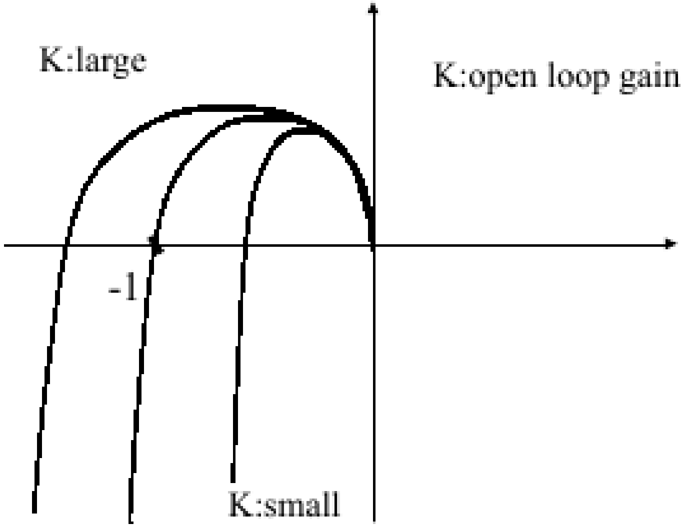


Consider the open-loop transfer function is: $G(s) = \frac{K}{(s+1)(s+2)(s+3)}$

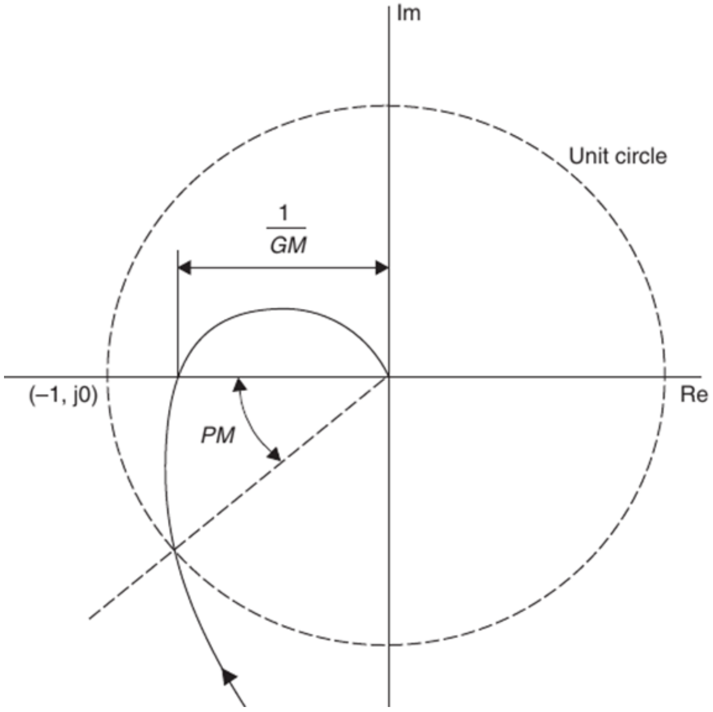
Root locus of G(s)



Nyquist diagrams around -1 as gain changes



Margins of stability



$$GM = \frac{1}{|G(j\omega_{GM})H(j\omega)|}$$

$$PM = 180° - G(j\omega_{PM})H(j\omega_{PM})$$

- The closer the open-loop frequency response locus is to the $(-1, j0)$ point, the nearer the closed-loop system is to instability
- Margin of Stability refers to the gain and phase margins
- Gain Margin (GM): the gain increase for system to become unstable, i.e. at phase cross over frequency ω_{GM} when $\measuredangle G(j\omega)H(j\omega) = 180°$
- Phase margin (PM): change in phase for system to become unstable, i.e. at gain cross over frequency ω_{PM} when $|G(j\omega)H(j\omega)| = 1$

Example 2

Calculate the gain and phase margins for

$$G_1(s) = \frac{5}{s^3 + 3s^2 + 4s + 3}$$

$$G_1(j\omega) = \frac{5}{(j\omega)^3 + 3(j\omega)^2 + 4(j\omega) + 3} = \frac{5}{j(4\omega - \omega^3) + (3 - 3\omega^2)}$$

$$G_1(j\omega) = \frac{5}{j(4\omega - \omega^3) + (3 - 3\omega^2)} \times \frac{-j(4\omega - \omega^3) + (3 - 3\omega^2)}{-j(4\omega - \omega^3) + (3 - 3\omega^2)}$$

$$G_1(j\omega) = \frac{5\{(3 - 3\omega^2) - j(4\omega - \omega^3)\}}{(4\omega - \omega^3)^2 + (3 - 3\omega^2)^2}$$

$$|G_1(j\omega)| = \frac{5}{(4\omega - \omega^3)^2 + (3 - 3\omega^2)^2}\sqrt{(3 - 3\omega^2)^2 + (4\omega - \omega^3)^2}$$

$$\measuredangle G_1(j\omega) = \tan^{-1}\left(\frac{-4\omega + \omega^3}{3 - 3\omega^2}\right)$$

Example 2

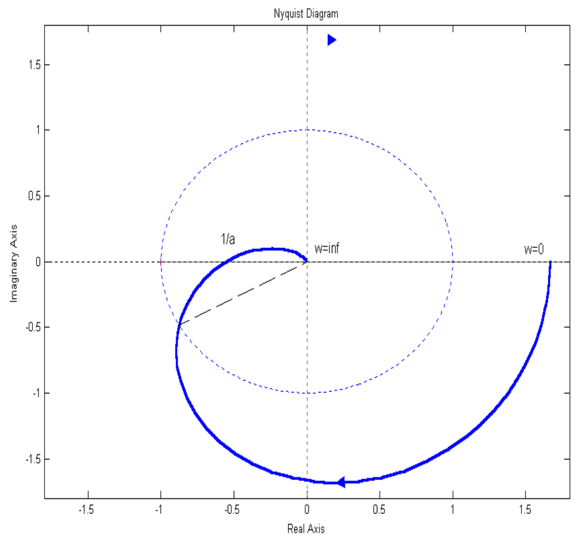
For gain margin, first find $\omega = \omega_{GM}$ at $\measuredangle G_1(j\omega) = 180°$

$$\measuredangle G_1(j\omega) = \tan^{-1}\left(\frac{-4\omega + \omega^3}{3 - 3\omega^2}\right)$$

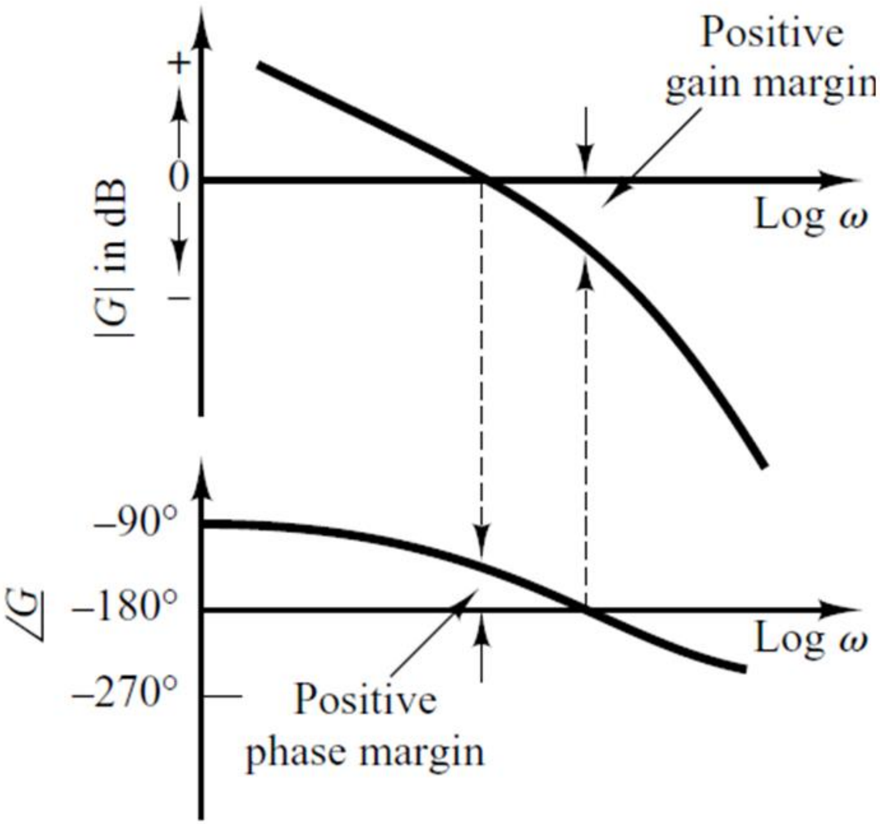
$-4\omega + \omega^3 = 0$ or $\omega_{GM} = \sqrt{2} = 1.414$; substitute this for ω into GM:

$$GM = \frac{1}{|G_1(j\omega)|} = \frac{1}{5} \times \frac{\left(4\omega - \omega^3\right)^2 + (3 - 3\omega^2)^2}{\sqrt{(3-3\omega^2)^2 + (4\omega - \omega^3)^2}} = 0.5 \text{ or } 20\log 0.5 = -5.9\text{db}$$

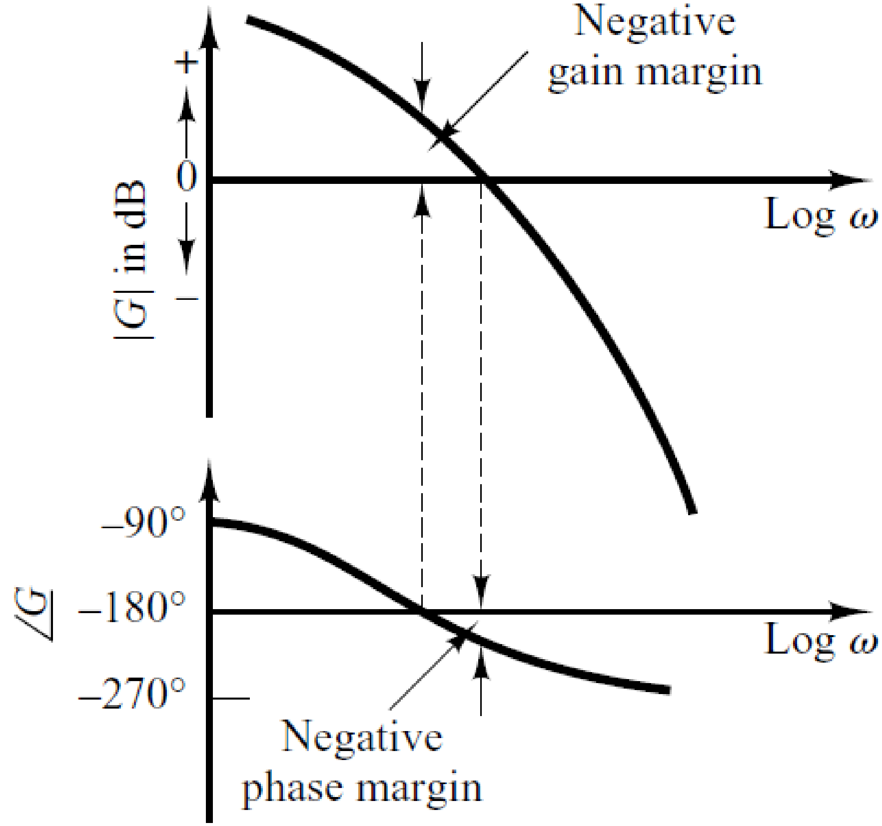
For phase margin, draw a unit circle with centre at origin and measured the required angle



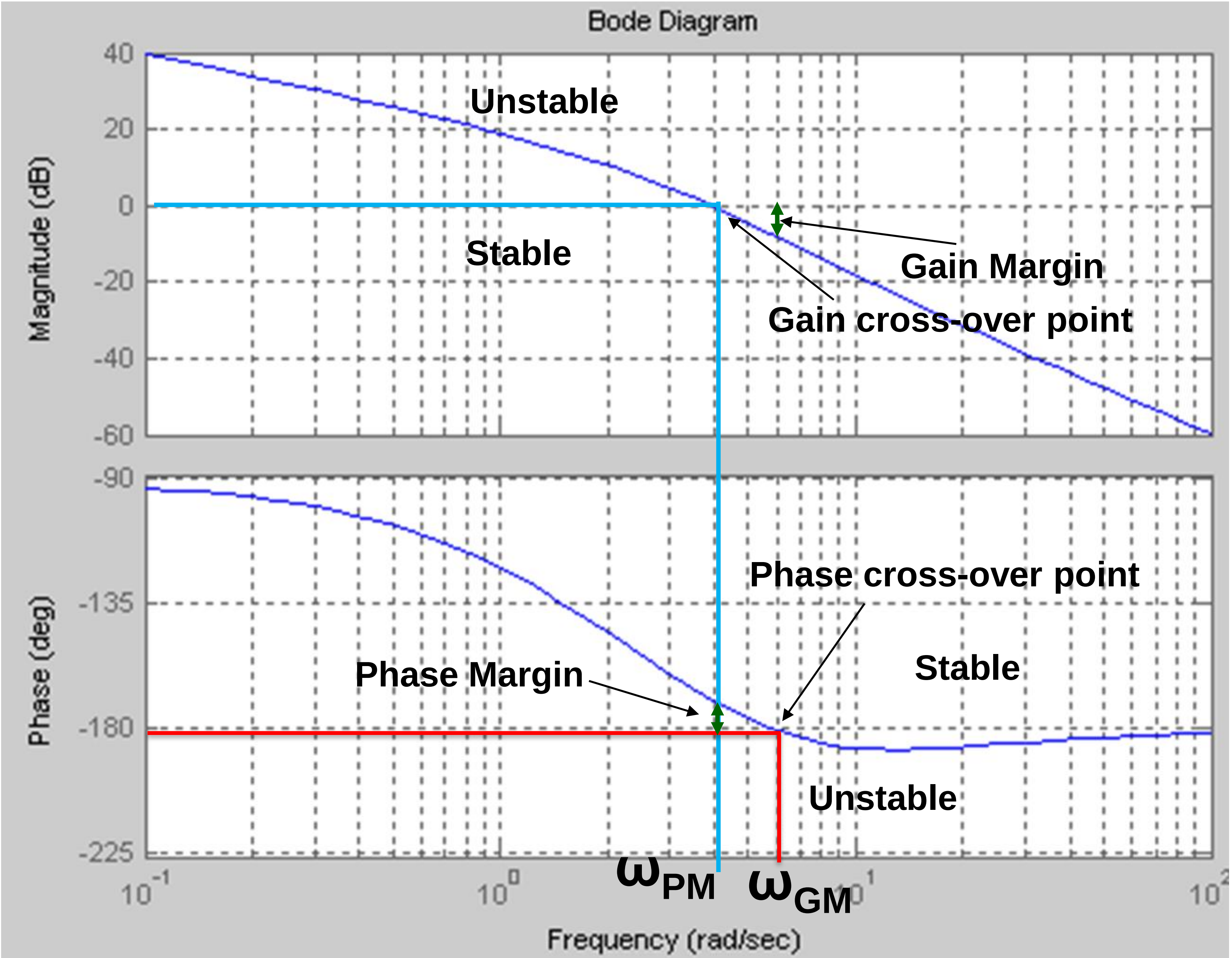
Stability via Bode plots



Stable system



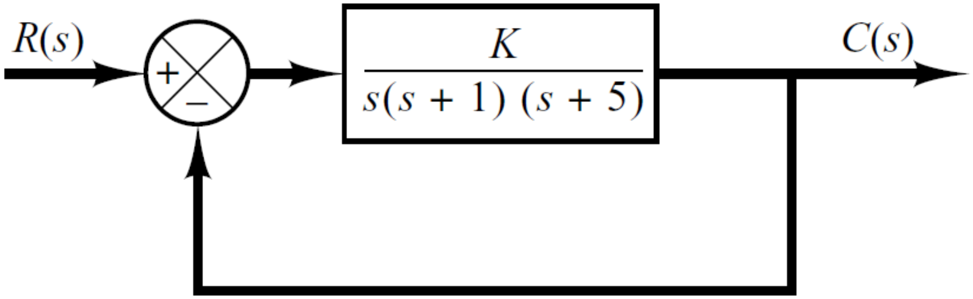
Unstable system

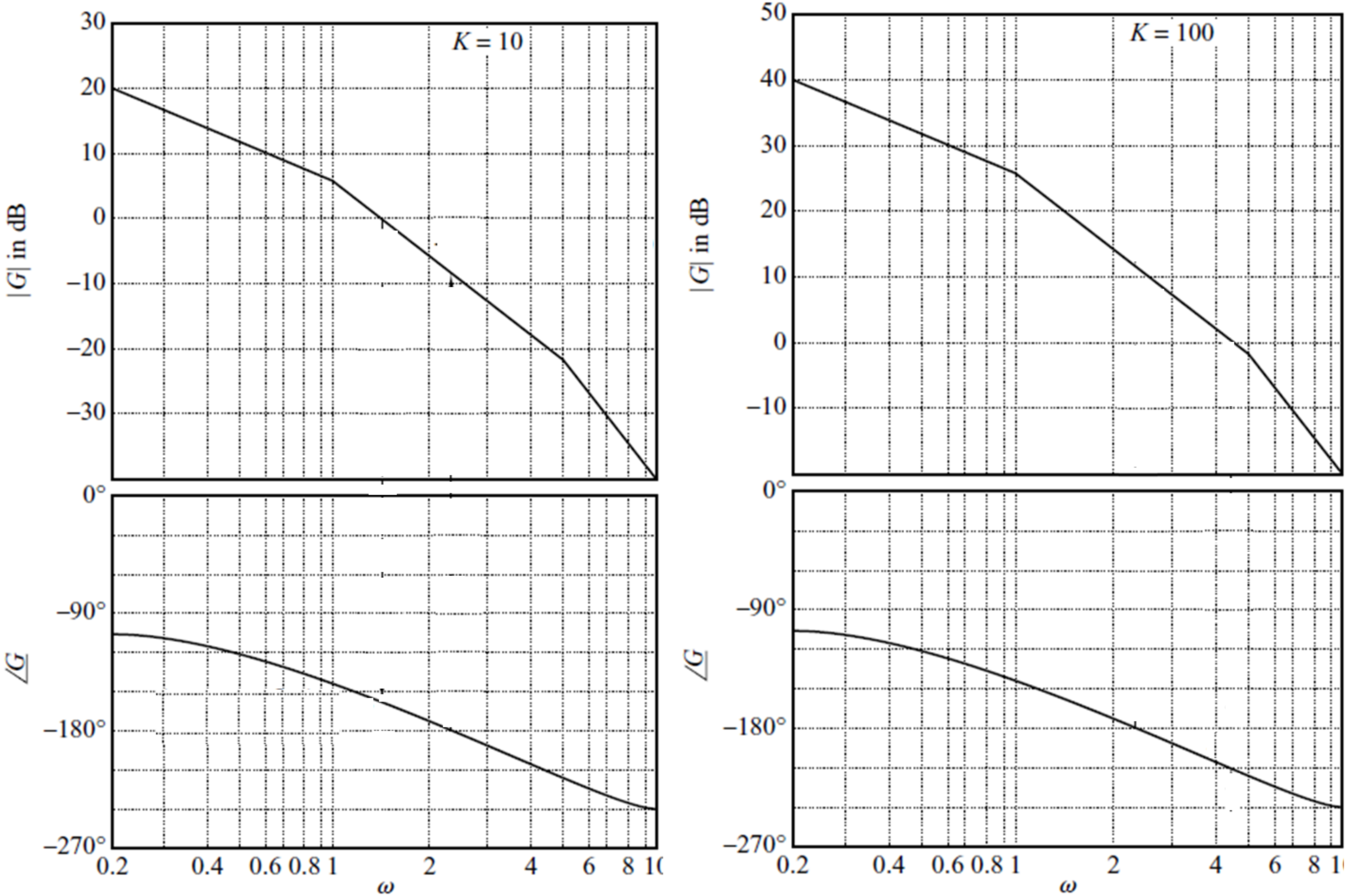


Bode Diagram
Unstable
Stable
Gain Margin
Gain cross-over point
Phase cross-over point
Stable
Phase Margin
Unstable
Magnitude (dB)
40
20
0
-20
-40
-60
Phase (deg)
-90
-135
-180
-225
10^{-1}
10^{0}
10^{1}
10^{2}
ω_{PM}
ω_{GM}
Frequency (rad/sec)

Example 3

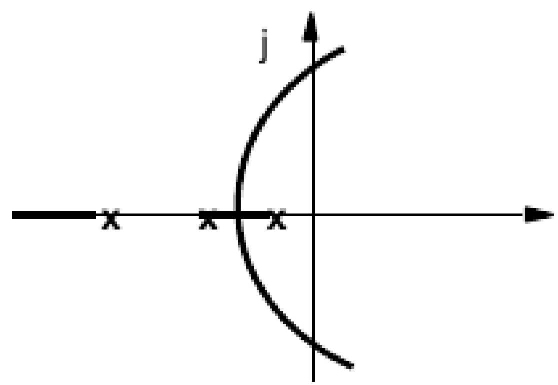
The open-loop Bode plots for the two cases where K=10 and K=100 are shown (in the next slide). Which system is unstable? Explain how the stability changes with the gain K using the root locus plot of the open-loop transfer function

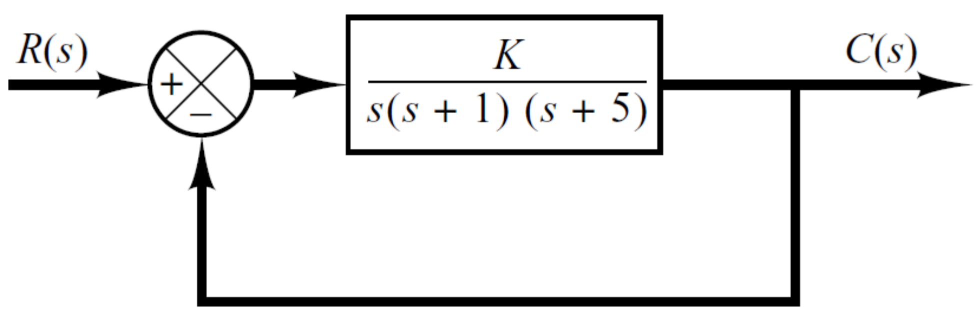




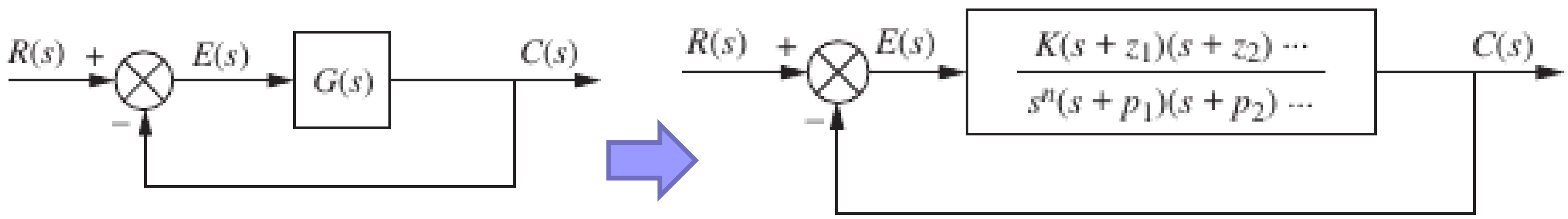
K = 10
K = 100
30
20
10
0
−10
−20
−30
50
40
|G| in dB
0°
−90°
−180°
−270°
∠G
0.2
0.4
0.6
0.8
1
2
4
6
8
10
ω

Root locus of G(s)



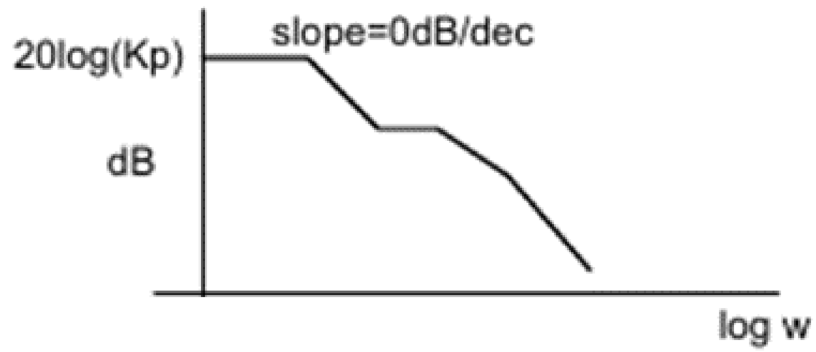


Steady-state error

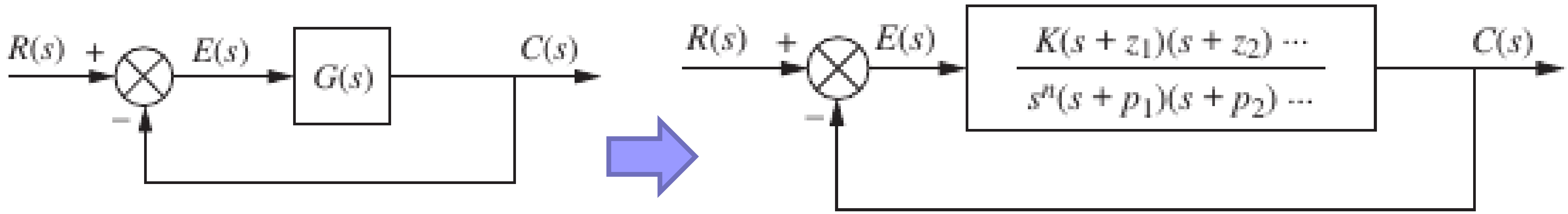


For type 0, static error constant $K_p = \lim_{s\to 0} G(s)$;

❖ The sketch Bode diagram of the open-loop transfer function $G(j\omega)$ will have a horizontal line at low frequency with value $20\log(K_p)$

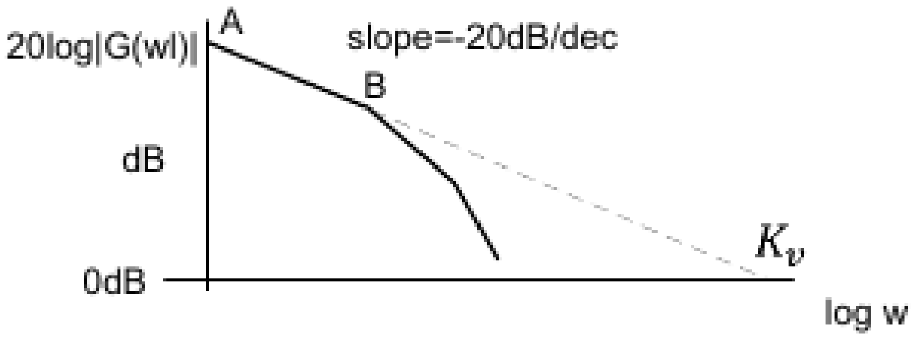


Steady-state error

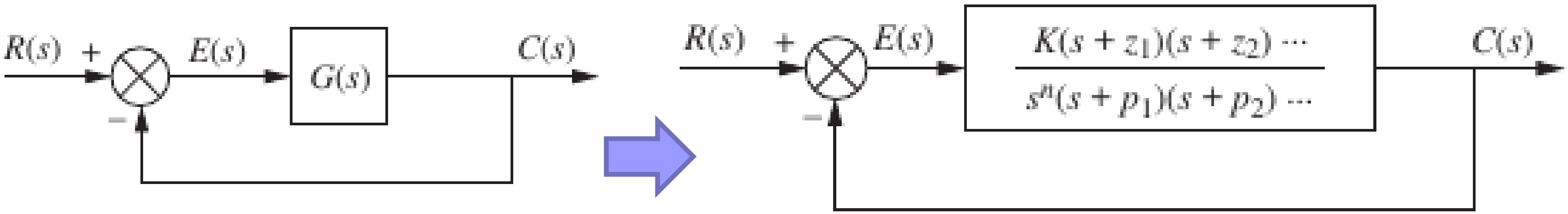


For type 1, static error constant $K_v = \lim_{s\to 0} sG(s)$;

- ❖ The sketch Bode diagram of the open-loop transfer function $G(j\omega)$ will have a line sloping at -20db/decade at low frequency; Extend the sloping line to meet the 0db line to get $\omega = K_v$

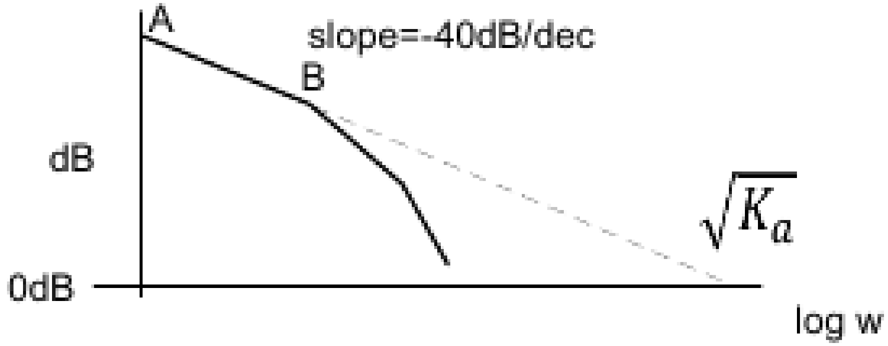


Steady-state error



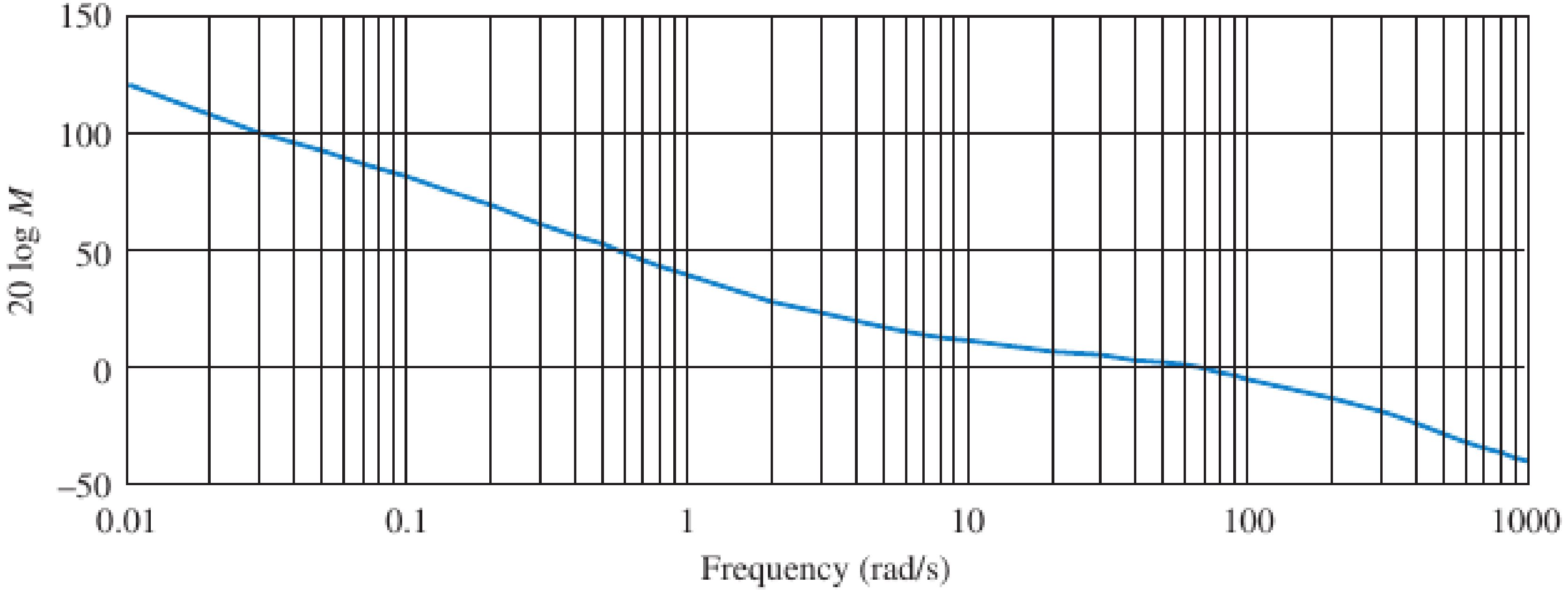
For type 2, static error constant $K_a = \lim_{s\to 0} s^2 G(s)$;

- ❖ The sketch Bode diagram of the open-loop transfer function $G(j\omega)$ will have a line sloping at -40db/decade at low frequency; Extend the sloping line to meet the 0db line to get $\omega = \sqrt{K_a}$



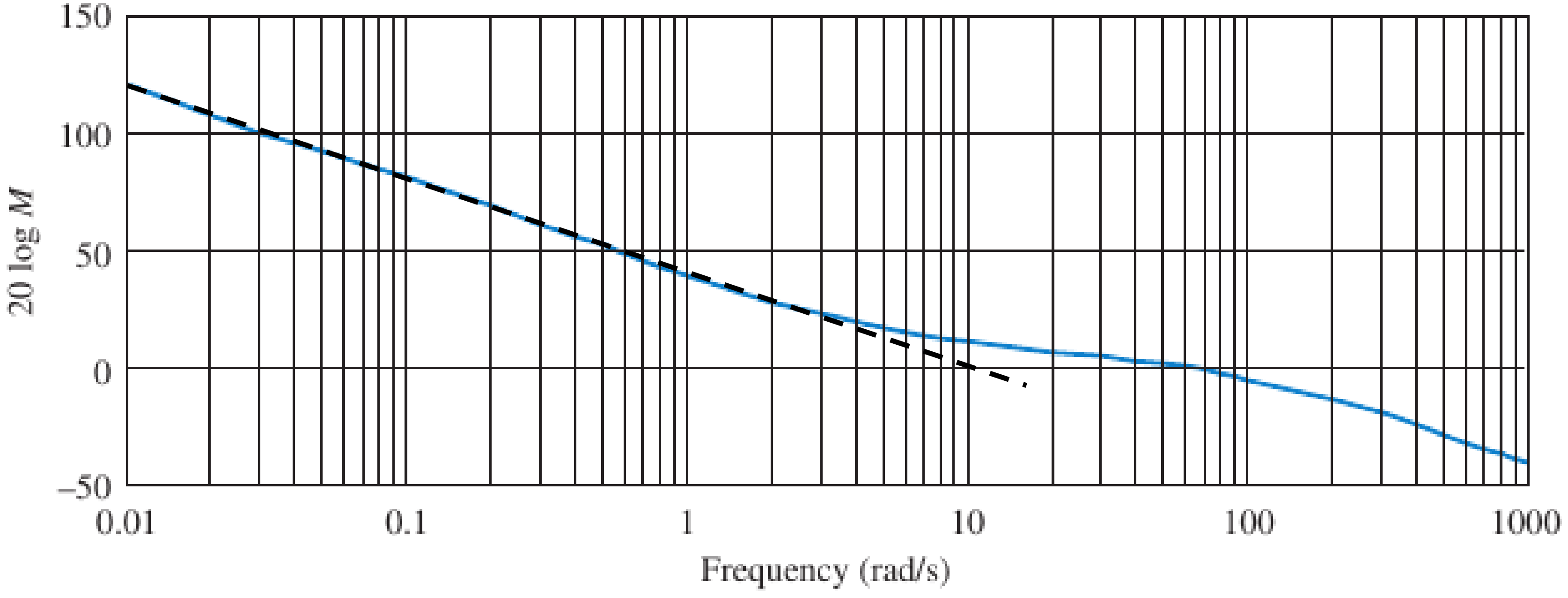
Example 4

Find the static error constants for a stable unity feedback system whose open-loop transfer function has the Bode magnitude plot shown

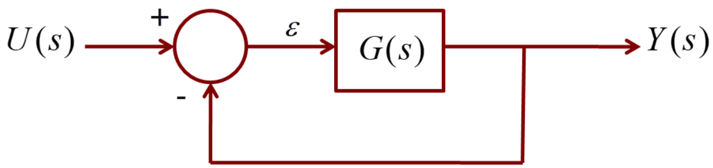


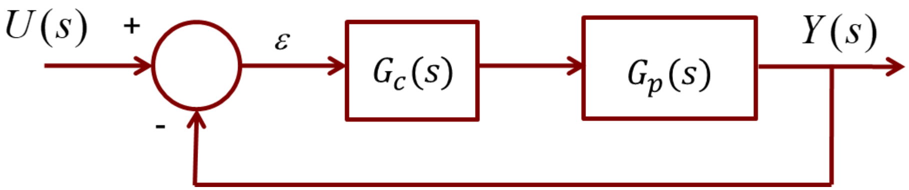
Example 4

Initial slope = -40db/decade (type 2). Intersect 0db line at $\omega = \sqrt{K_a} = 10$ or $K_a \approx 100$, $K_p = K_v = \infty$



Open-loop frequency response





- In the time domain, we have used the root locus of the open-loop transfer function $G(s)$ to design the feedback controller so that the closed-loop time response meets desired specifications
- In the frequency domain, we will now use the Bode diagram of the open loop transfer function $G(s)$ to design the feedback controller so that the closed-loop frequency response meets desired specifications

Open-loop frequency response

The desired specifications in the frequency domain include:

- ❖ System stability
- ❖ Gain and phase margins (transient response)
- ❖ Steady state error

- The steady state error is affected by the system type and the proportional gain
- The changes of the proportional gain will affect the closed loop pole location and will affect the transient response and stability (hence it will affect the gain and phase margins)
- Adjustment of the gain may led to a dominant real pole, which will affect the settling time and the bandwidth (Note: for first order system: $BW = 1/\tau$ where τ = time constant)

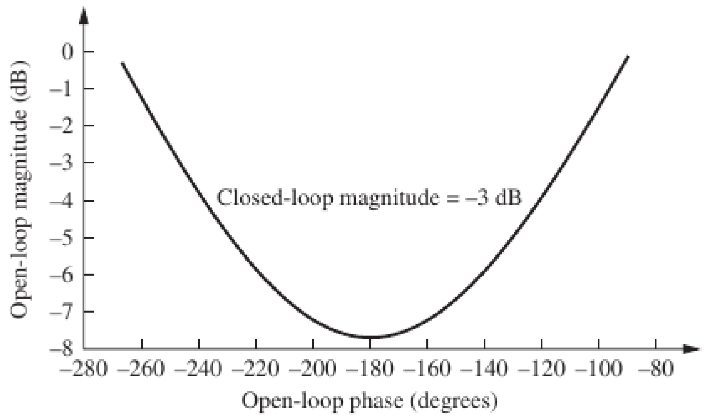
Open-loop frequency response

Adjustment of the gain may led to a pair of dominant complex poles, which will also affect the settling time and the bandwidth BW. For a pair of dominant complex conjugate pole:

- ❖ open-loop phase margin and the closed-loop damping ratio approximated by

$$PM = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$

- ❖ Closed-loop bandwidth, ω_{BW} (frequency at which closed-loop magnitude is 3dB), is the frequency at which the open-loop magnitude response is between -6 and -7:5 dB if the open-loop phase response is between 135° and 225°



Open-loop frequency response

The closed-loop settling time and peak time are related to the closed-loop bandwidth ω_{BW} and closed-loop damping ratio ζ:

- Closed-loop settling time:

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- Closed-loop peak time:

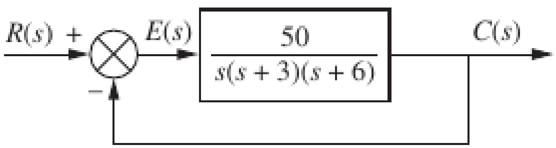
$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- Note that

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

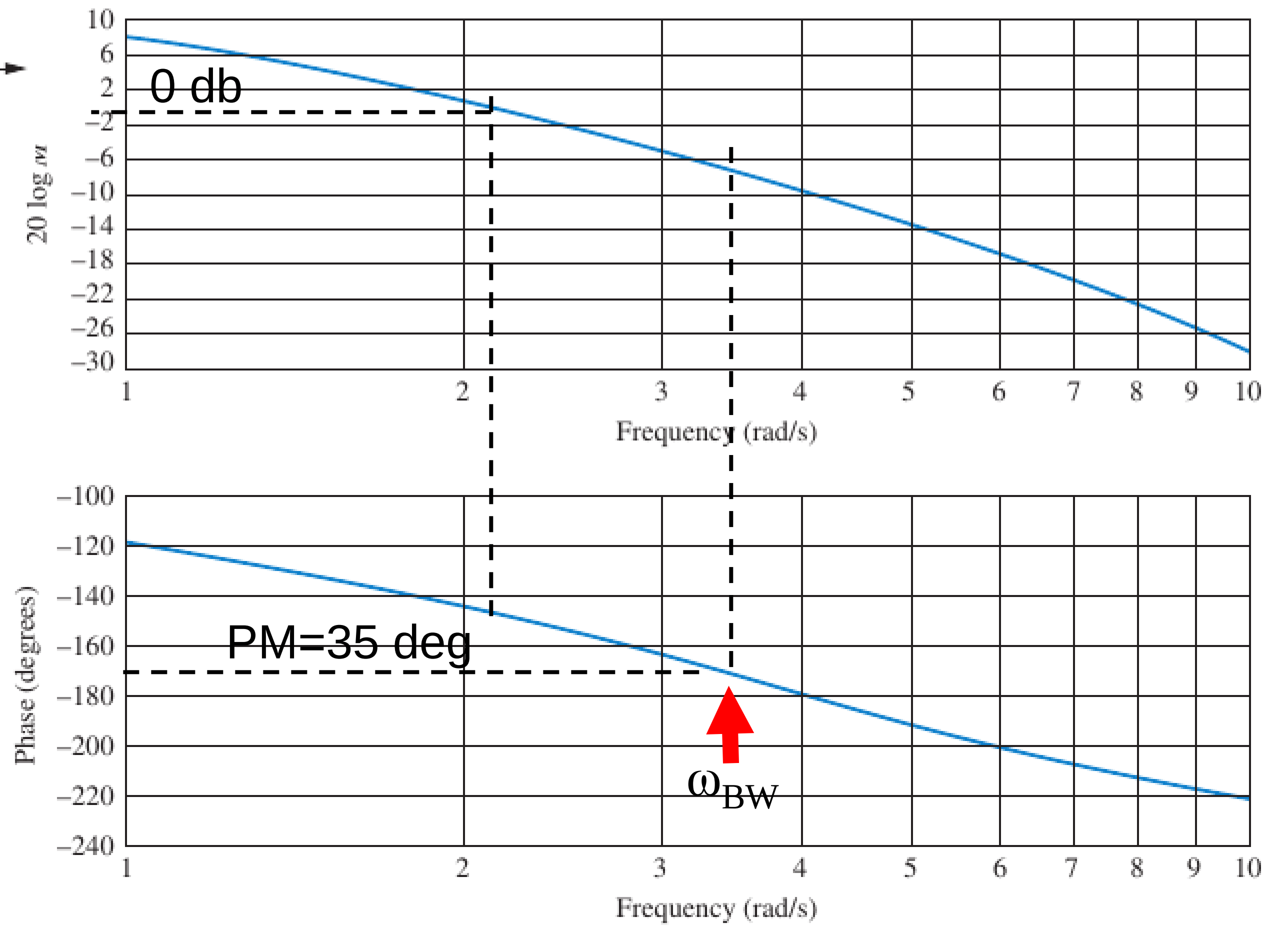
Example 5

The open-loop transfer function and Bode diagrams are given. Estimate the closed-loop settling time and peak time



Check if open-loop $-6 \le |G(j\omega)| \le -7.5$ is between 135° and 225° :- yes; then closed-loop $BW = \omega_{BW} \approx 3.75$ rad/s

Open loop $PM = 35°$



Example 5

Estimate the closed loop damping ratio from the open-loop phase margin:

$PM = 35° = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$ or closed-loop damping ratio $\zeta = 0.36$

Using closed-loop BW = $\omega_{BW} \approx 3.75$ rad/s and closed-loop damping ratio $\zeta = 0.36$:

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Closed-loop settling time $T_s = 4.5$sec.

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Closed-loop peak time $T_p = 1.47$sec.

We can also find the percent overshoot using damping ratio if needed