MEMS1045 Automatic control

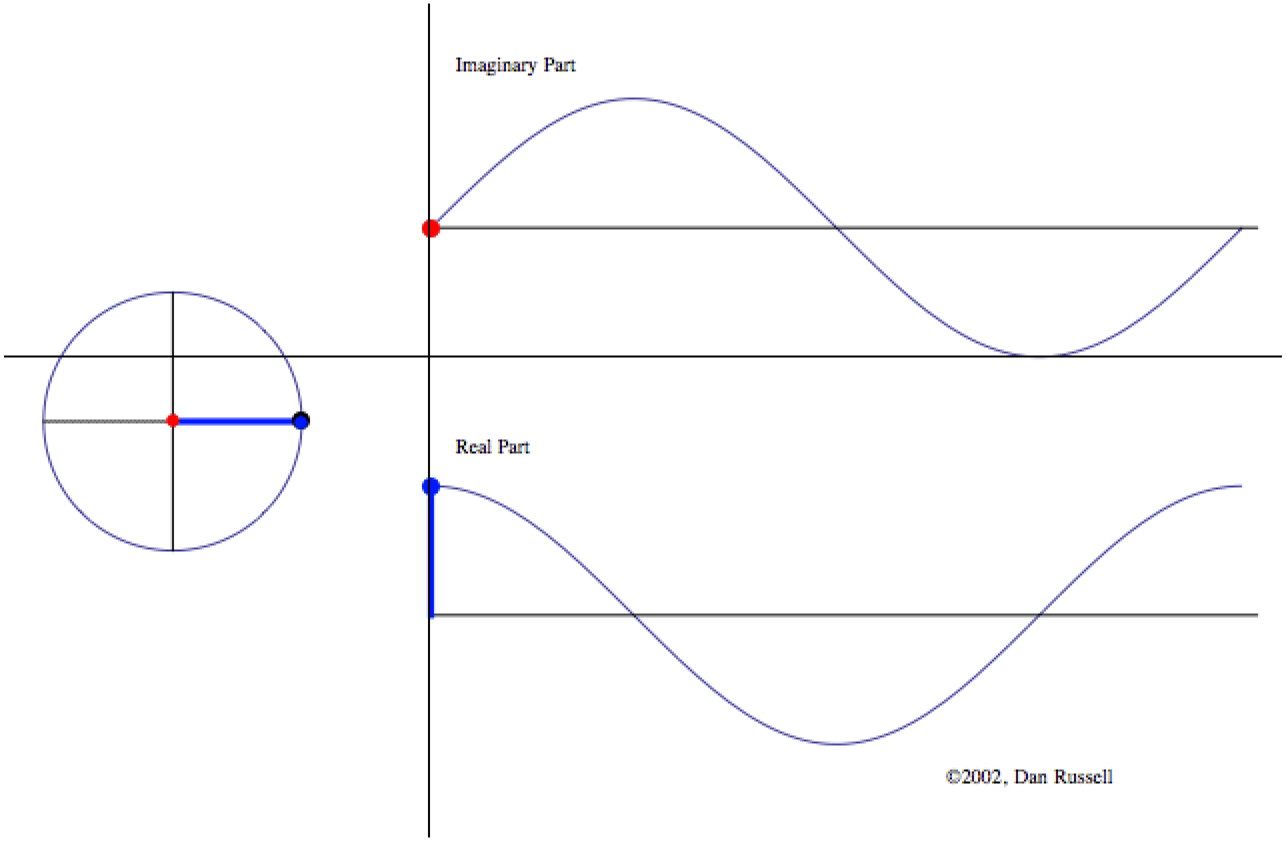
Lecture 11

Frequency response 1

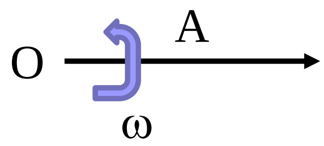
Objectives

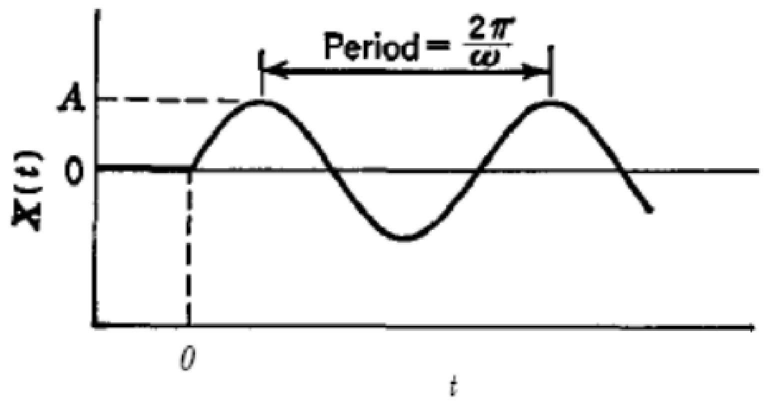
- Explain the concept of frequency response
- Determine frequency response from transfer functions
- Represent the frequency response graphically using Bode diagrams
- Sketch the Bode diagrams from transfer functions
- Represent the frequency response graphically using Nichols chart and Nyquist diagram

Revision - trigonometry



Sinusoidal signal



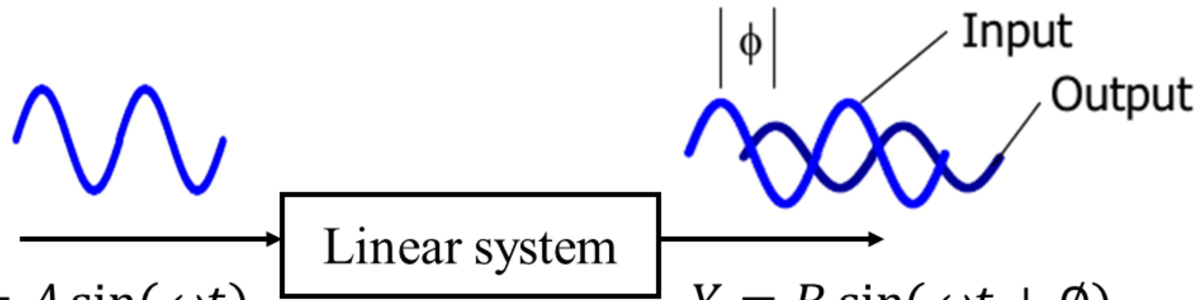


Consider the sinusoidal input:

- At $t < 0$: $x(t) = 0$;
- For $t \geq 0$: $x(t) = A\sin(\omega t)$
- Input frequency ω (rad/s) = $2\pi f$; where f = frequency in cycles/sec (Hz)
- Period $T = \frac{1}{f} = \left(\frac{2\pi}{\omega}\right)$ sec.
- Sinusoids can be represented as rotating phasors using complex numbers

$$Ae^{j\omega t} = |A|\measuredangle$$

Frequency response



$U(t) = A\sin(\omega t)$ $\quad$ $Y = B\sin(\omega t + \emptyset)$

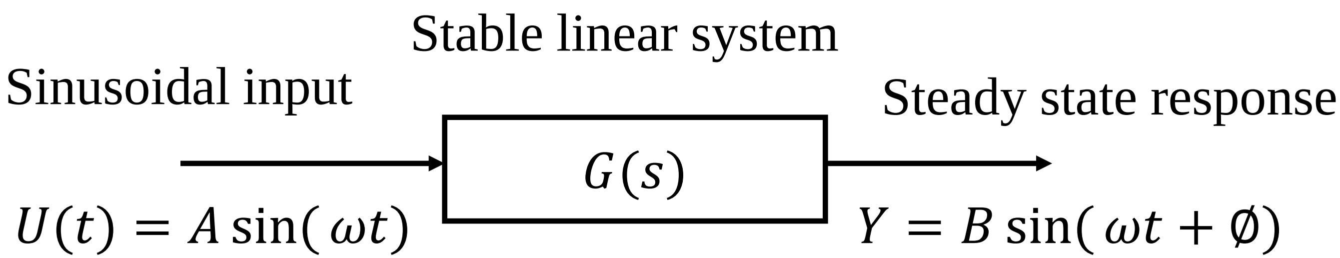
In the steady state, a sine input will generate a sine output

- ❖ Output signal will oscillate at the same frequency as input signal
- ❖ The magnitude of the output signal is different from the input signal
- ❖ The magnitude (or amplitude) ratio is defined ed as $M = \frac{B}{A}$
- ❖ There is a phase shift (or difference) ϕ between the input and output signals

Note:

- ➢ Only 2 parameters are needed to describe the frequency response: amplitude ratio M and phase difference ϕ
- ➢ Linear system can be represented using transfer functions

Frequency response



Given the transfer function, to determine the amplitude ratio and phase shift:

❖ Substitute $s = j\omega$ into the transfer function $G(s)$

Note: $s = j\omega$ is a complex variable with ω = input signal frequency

❖ The amplitude ratio is the magnitude of the complex function $G(j\omega)$, i.e.

$$M = |G(j\omega)|$$

❖ The phase shift is the phase angle of the complex function $G(j\omega)$, i.e.

$$\emptyset = \measuredangle G(j\omega)$$

Note: for complex numbers $z_1 = a_1 + b_1 j = M_1 e^{j\phi_1}, z_1 = a_2 + b_2 j = M_2 e^{j\phi_2}, \ldots$

$$z = \frac{z_1 z_2}{z_3 z_4} = \frac{M_1 M_2}{M_3 M_4} e^{j(\phi_1 + \phi_2 - \phi_3 - \phi_4)} = |z| e^{j\phi}$$

Example 1

Find the steady state response for $y(t)$ for input $f(t) = 5\sin(0{,}2t)$ given the transfer function

$$G(s) = \frac{Y(s)}{F(s)} = \frac{20}{(10s+1)(4s+1)}$$

Note: input signal frequency is $\omega = 0.2$ and system is stable;

❖ Substitute $s = j\omega = 0.2j$ into the transfer function $G(s)$

$$G(j\omega) = \frac{Y(j\omega)}{F(j\omega)} = \frac{20}{(2j+1)(0.8j+1)}$$

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|F(j\omega)|} = \frac{20}{\left(\sqrt{4+1}\right)\left(\sqrt{0.64+1}\right)} = 7$$

$\measuredangle G(j\omega) = 0 - \tan^{-1}(2/1) - \tan^{-1}(0.8) = -102.1°$ or -1.78 rad.

$$|Y(j\omega)| = 7|F(j\omega)| = 7(5) = 35$$

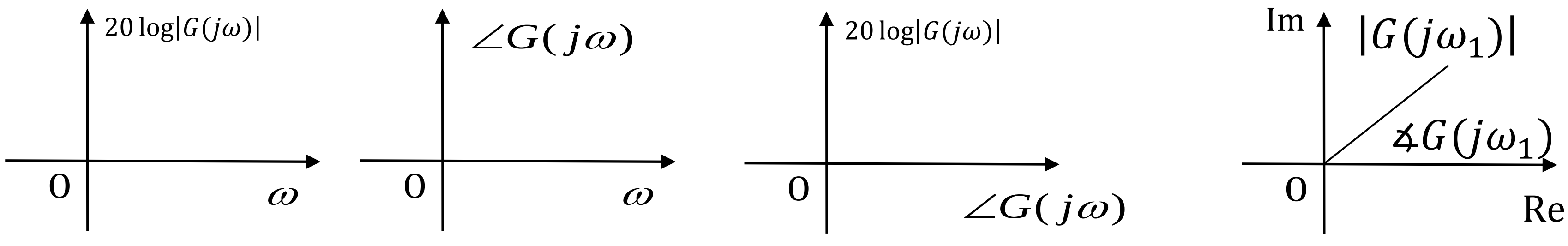
Output signal is $|Y(j\omega)|\sin(\omega t + \emptyset) = 35\sin(0.2t - 1.78)$

Graphical representation

Consider a system given by transfer function $G(s)$

- The frequency response is described by amplitude ratio $M = |G(j\omega)|$ and phase shift $\emptyset = \measuredangle G(j\omega)$
- There are three graphical methods of representing the system frequency response:

1) Bode diagram – separately plot $20\log|G(j\omega)|$ vs. ω and plot ϕ vs. ω
2) Nichols chart – plot $20\log|G(j\omega)|$ vs. $\measuredangle G(j\omega)$
3) Nyquist (or polar) diagram – plot (M, ϕ) as ω changes



Bode diagram

Nichols chart

Nyquist diagram

Bode diagram

Sketch the Bode diagram for the following system:

$$G(s) = \frac{1}{s+2}$$

$$G(j\omega) = \frac{1}{2 + j\omega}$$

$$G(j\omega) = \frac{1\angle 0^0}{\sqrt{4+\omega^2}\angle \tan^{-1}(\omega/2)}$$

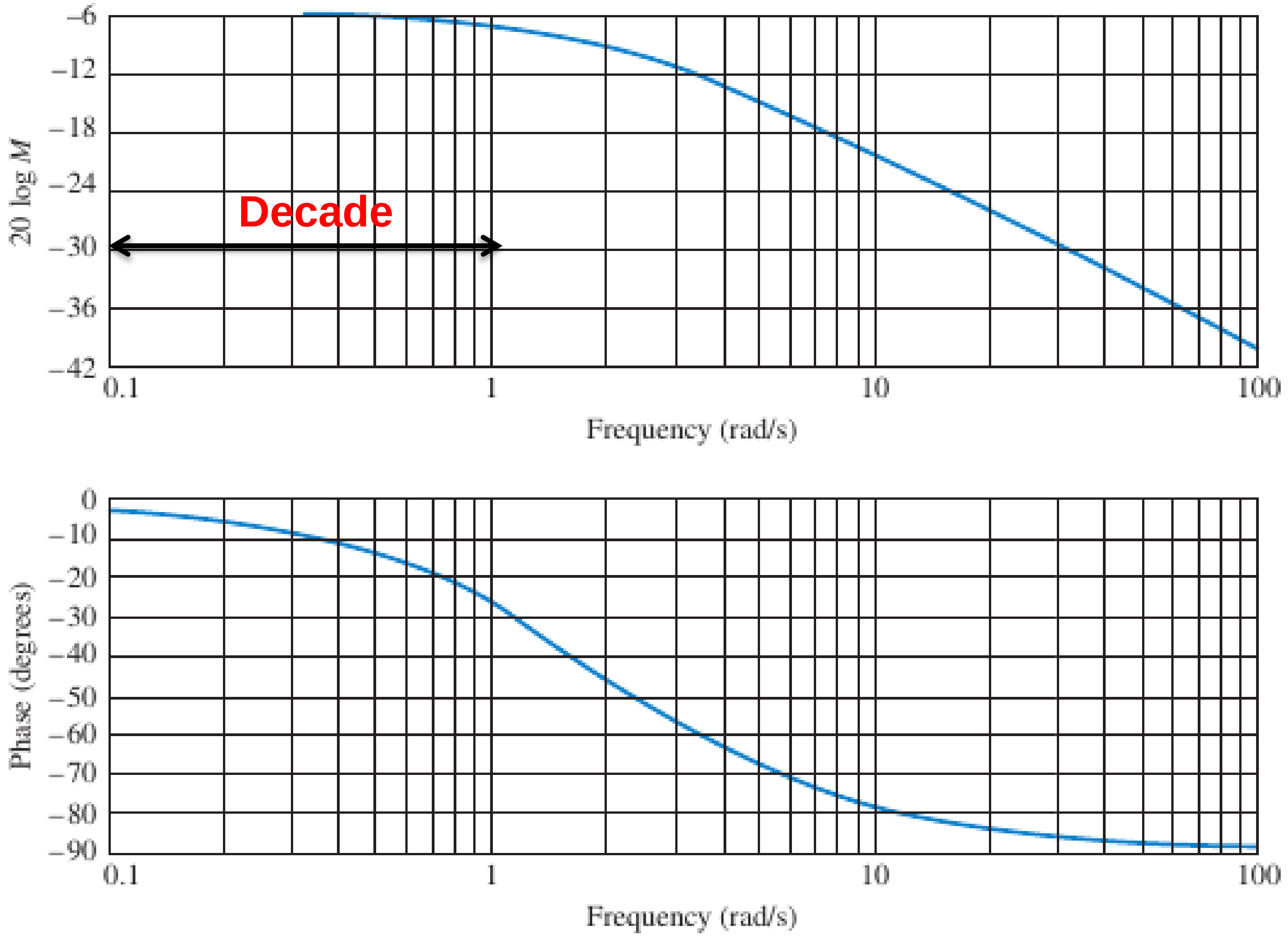
$$|G(j\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$

$$20\log|G(j\omega)| = 20\log\frac{1}{\sqrt{4+\omega^2}}$$

$$\angle G(j\omega) = -\tan^{-1}(\omega/2)$$

ω	$20\log\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	-6.02	0
1	-6.99	-26.57^0
5	-14.6	-68.2^0
10	-20.17	-78.69^0
50	-33.99	-87.71^0
∞	0	-90^0

Bode diagram



Bode approximation

The four different factors that may occur in a transfer function are:

1) Constant gain, i.e. K
2) Poles (or zeros) at the origin, i.e. s
3) Poles (or zeros) on the real axis; i.e. $(Ts+1)$
4) Complex conjugate poles (or zeros), i.e. $(s^2+2\zeta\omega_n s+\omega_n^2)$

Given $G(s)=\frac{G_1(s)G_2(s)}{G_3(s)G_4(s)}$

- Amplitude ratio $(M,\text{db})=20\log|G(s)|=20\log\left\{\frac{|G_1(s)||G_2(s)|}{|G_3(s)||G_4(s)|}\right\}$

  $(M,\text{db})=20\log|G_1(s)|+20\log|G_2(s)|-20\log|G_3(s)|-20\log|G_4(s)|$
- Phase shift $\emptyset=\measuredangle G(j\omega)=\phi_1+\phi_2-\phi_3-\phi_4$
- We will determine the logarithmic magnitude plot and phase angle for the four factors and then use them to approximate the Bode diagram for any general form of a transfer function

Bode approximation

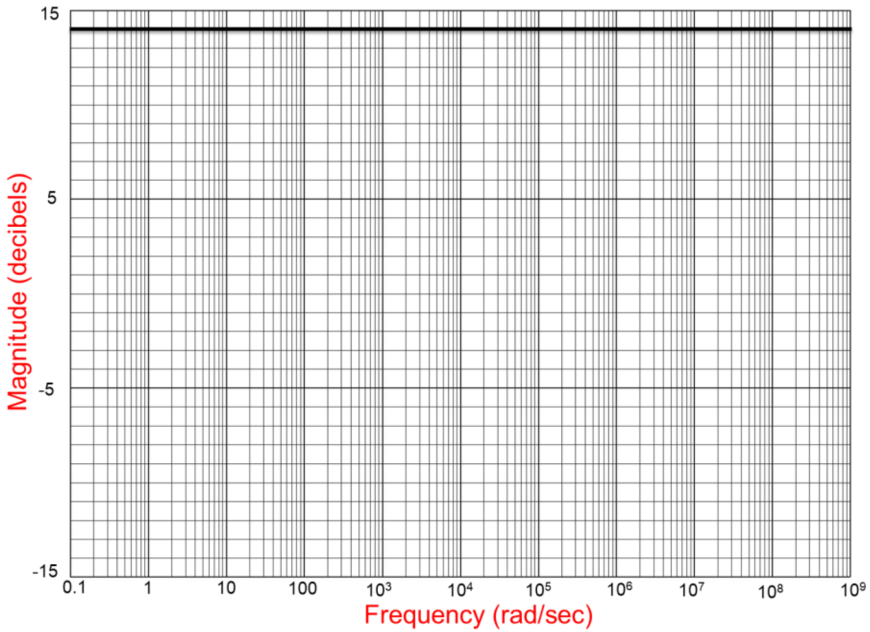
1) Constant gain $G(s) = K$, or $G(j\omega) = K\measuredangle 0$

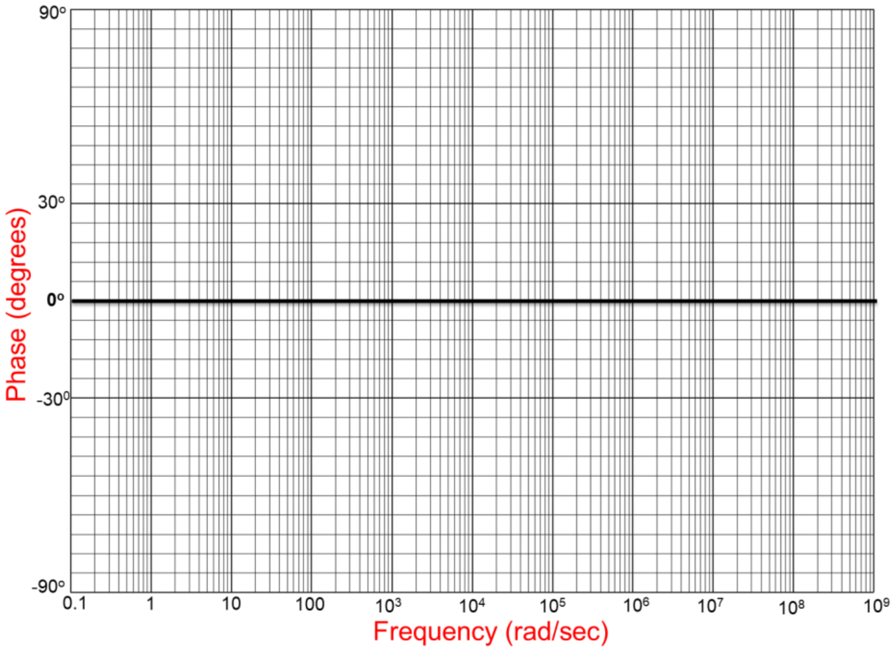
- ❖ $(M, \text{db}) = 20\log|K|$ =constant
- ❖ $\phi = 0$

- The log-magnitude curve for a constant gain *K* is a horizontal straight line at the magnitude of 20 log(*K*) decibels.
- The phase angle of the gain *K* is zero.
- The effect of varying the gain *K* in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

Bode approximation

If $K = 5$, then $20\log(5) = 14\text{db}$ and $\phi = 0$





Bode approximation

2) Poles (or zeros) at the origin:

❖ For pole $G(s) = \frac{1}{s}$, or $G(j\omega) = \frac{1}{j\omega} = \frac{1\measuredangle 0}{\omega\measuredangle 90°}$

$$(M, \mathrm{db}) = 20\log|1| - 20\log|\omega| = -20\log(\omega)$$

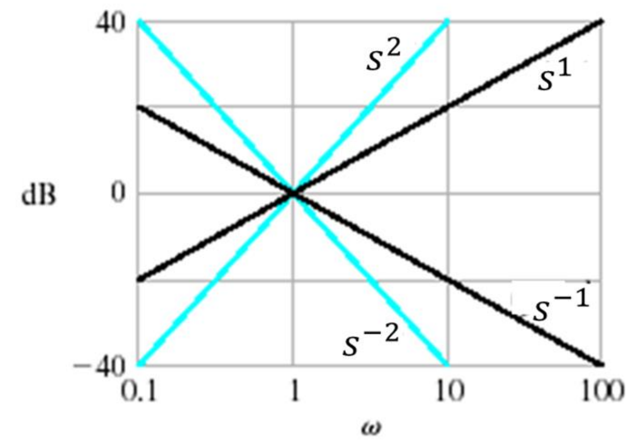
$$\phi = -90°$$

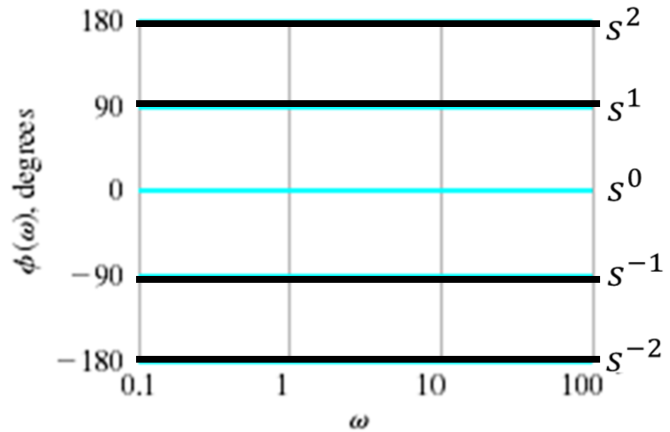
❖ For zero $G(s) = s$, or $G(j\omega) = j\omega = \omega\measuredangle 90°$

$$(M, \mathrm{db}) = 20\log|\omega|$$

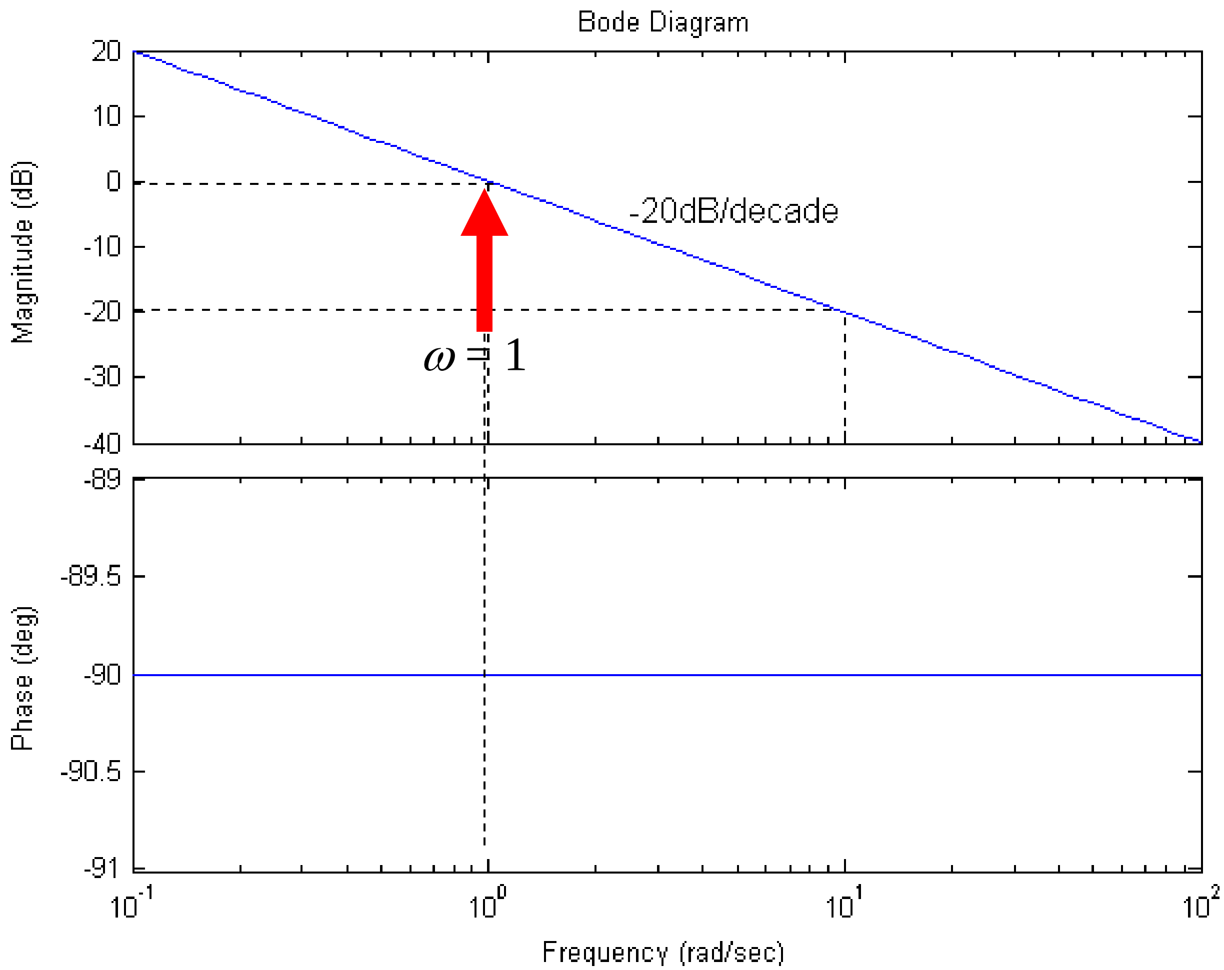
$$\phi = 90°$$

- The factor $s^{\pm k}$ has a magnitude diagram which is a straight line with slope equal to $\pm 20k$ [dB/decade] and constant phase, equal to $\pm k\pi/2$. This line crosses the horizontal axis (0[dB]) at $\omega = 1$





Bode approximation



$$G(s) = \frac{1}{s}$$

Bode approximation

3) Poles (or zeros) on the real axis; i.e. $(Ts + 1)$:

- For pole $G(s) = \frac{1}{Ts+1}$, or $G(j\omega) = \frac{1}{j\omega T+1} = \frac{1\measuredangle 0}{\sqrt{1+\omega^2T^2}\measuredangle \tan^{-1}(\omega T)}$

$$(M, \text{db}) = 20\log|1| - 20\log\left|\sqrt{1+\omega^2T^2}\right| = -20\log(\sqrt{1+\omega^2T^2})$$

$$\phi = -\tan^{-1}(\omega T)$$

As $\omega \to 0, (M, \text{db}) \longrightarrow 0$ and $\phi \to 0$;

At $\omega = \left(\frac{1}{T}\right)$, $\phi = -45°$ with slope ≈ -45deg/decade;

As $\omega \to \infty, \phi \to -90°$; $(M, \text{db}) \approx -20\log(\omega T)$;

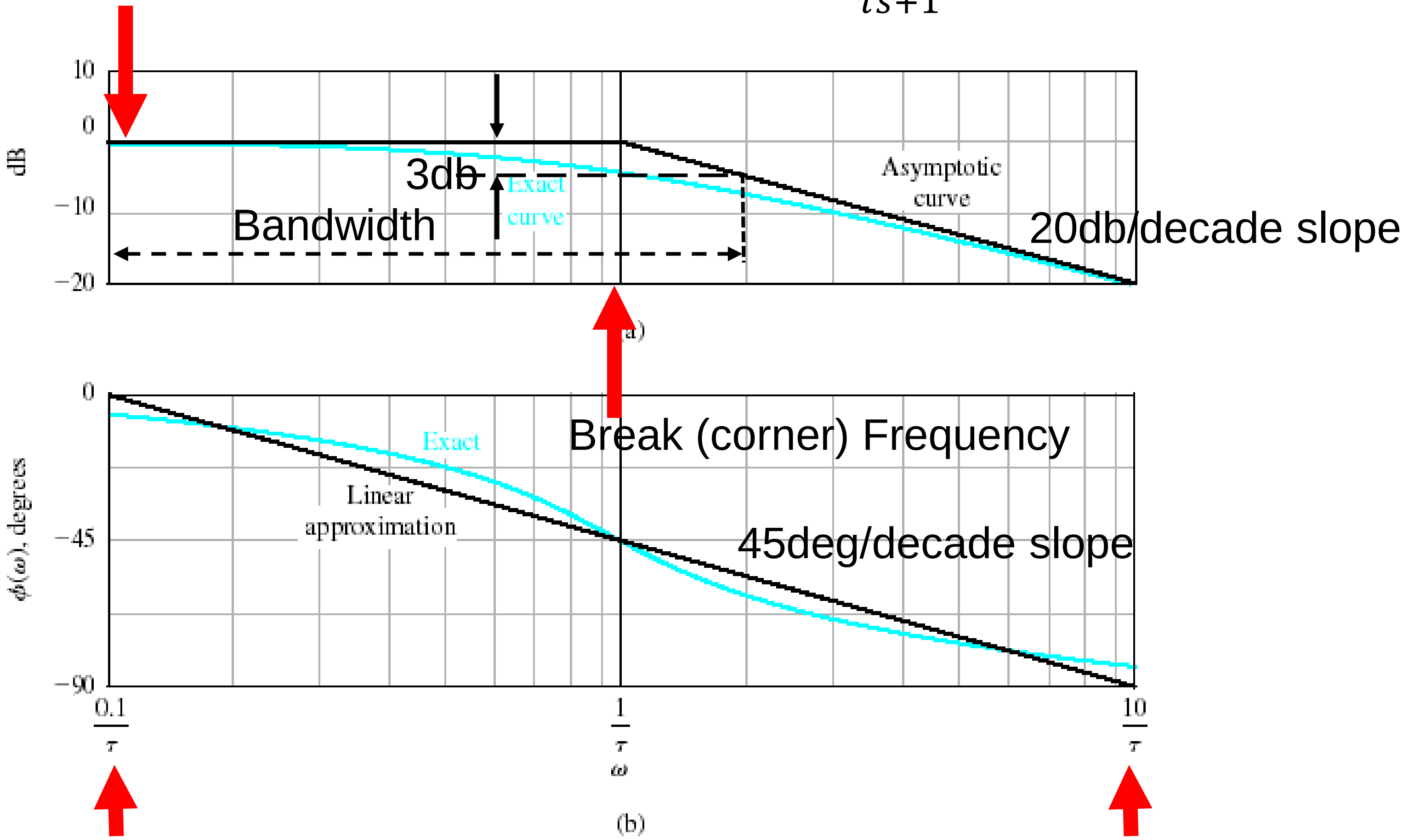
We can approximate the amplitude ratio as

$(M, \text{db}) \approx -20\log\omega - 20\log T$; after $\omega = \frac{1}{T}$ it has -20db/decade slope.

Bode approximation

Gain at low freq.

Bode diagram for $G(s) = \frac{1}{\tau s+1}$



Bode approximation

3) Poles (or zeros) on the real axis; i.e. $(Ts+1)$:

❖ For zero $G(s) = Ts + 1$, or $G(j\omega) = j\omega T + 1 = \sqrt{1+\omega^2 T^2} \measuredangle \tan^{-1}(\omega T)$

$$(M, \mathrm{db}) = 20 \log \left| \sqrt{1+\omega^2 T^2} \right|$$

$$\phi = \tan^{-1}(\omega T)$$

As $\omega \to 0, (M, \mathrm{db}) \longrightarrow 0$ and $\phi \to 0$;

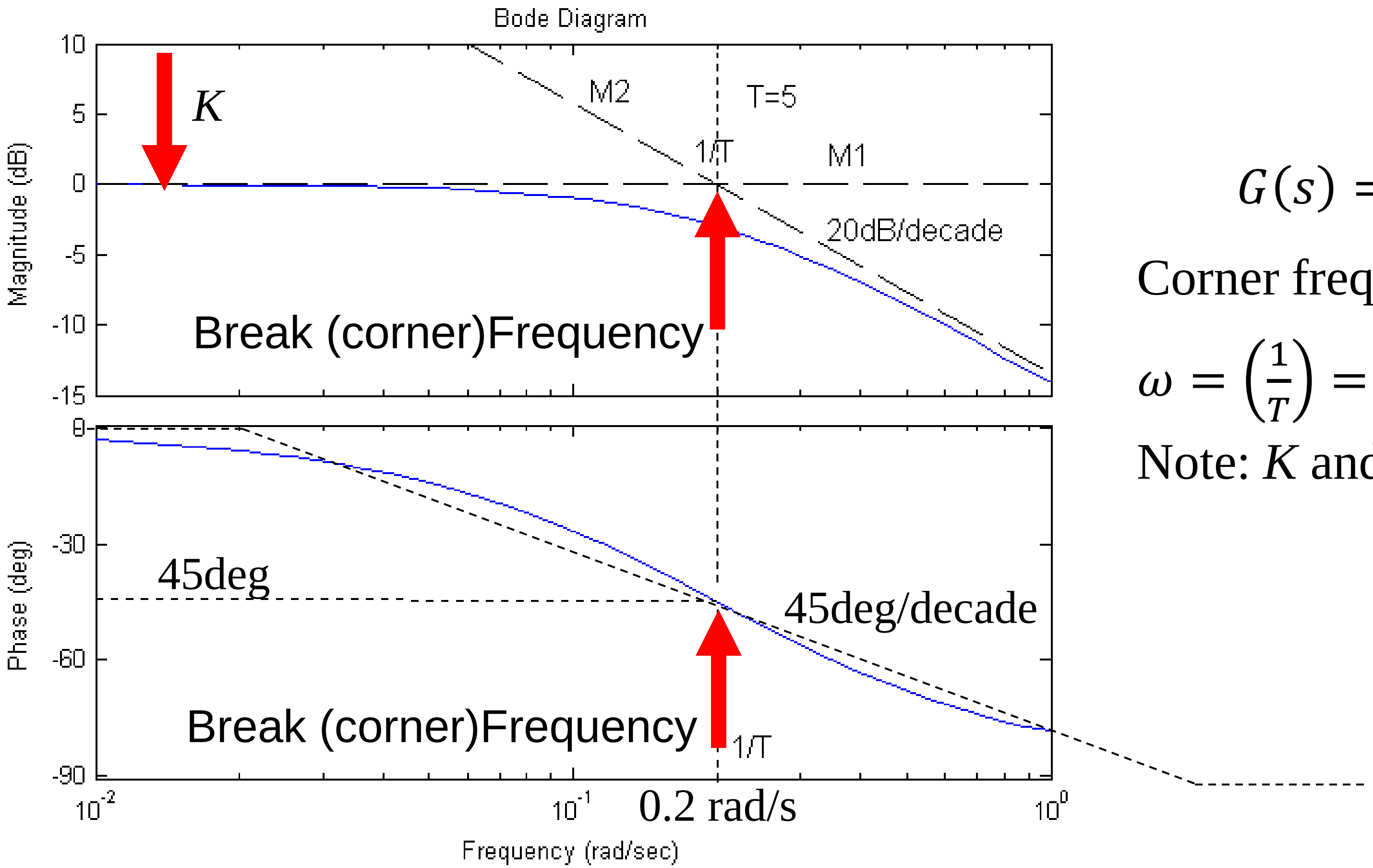
At $\omega = \left(\frac{1}{T}\right)$, $\phi = 45°$ with slope $\approx$ 45deg/decade;

As $\omega \to \infty, \phi \to 90°$ and $(M, \mathrm{db}) \approx 20 \log(\omega T)$;

We can approximate the amplitude ratio as

$(M, \mathrm{db}) \approx 20 \log \omega + 20 \log T$; after $\omega = \frac{1}{T}$ it has 20db/decade slope

Bode approximation



$$G(s) = \frac{1}{5s+1}$$

Corner frequency at

$$\omega = \left(\frac{1}{T}\right) = \frac{1}{5} = 0.2\text{rad/s}$$

Note: K and corner freq.

Bode approximation

4) Complex conjugate poles (or zeros), i.e. $(s^2 + 2\zeta\omega_n s + \omega_n^2)$:

- For pole $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, or $G(j\omega) = \frac{\omega_n^2}{\omega_n^2-\omega^2+2\zeta\omega_n\omega j} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+2\zeta\left(\frac{\omega}{\omega_n}\right)j}$

$$(M,\text{db}) = -20\log\left|\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2}\right|$$

$$\phi = -\tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\}$$

As $\omega \to 0, (M,\text{db}) \longrightarrow 0$ and $\phi \to 0$;

At $\omega = \omega_n$, $(M,\text{db}) = -20\log(2\zeta)$ and $\phi = -90°$; slope ≈ -90deg/decade

As $\omega \to \infty, (M,\text{db}) \approx -20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\omega + 40\log\omega_n$ and $\phi \to -180°$;

Note after $\omega = \omega_n$ it has -40db/decade slope

Bode approximation

4) Complex conjugate poles (or zeros), i.e. $(s^2 + 2\zeta\omega_n s + \omega_n^2)$:

- For zero $G(s) = \frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$, or $G(j\omega) = 1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{\omega}{\omega_n}\right)j$

$$(M, \text{db}) = 20\log\left|\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2}\right|$$

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\}$$

As $\omega \to 0, (M, \text{db}) \longrightarrow 0$ and $\phi \to 0$;

At $\omega = \omega_n, (M, \text{db}) = 20\log(2\zeta)$ and $\phi = 90°$; slope ≈ 90deg/decade

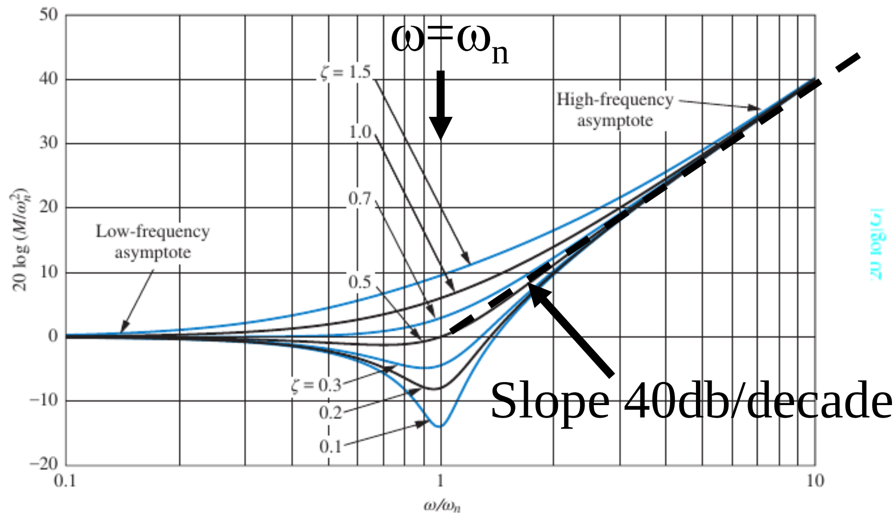
As $\omega \to \infty, (M, \text{db}) \approx 20\log\left(\frac{\omega}{\omega_n}\right)^2 = 40\log\omega - 40\log\omega_n$ and $\phi \to -180°$;

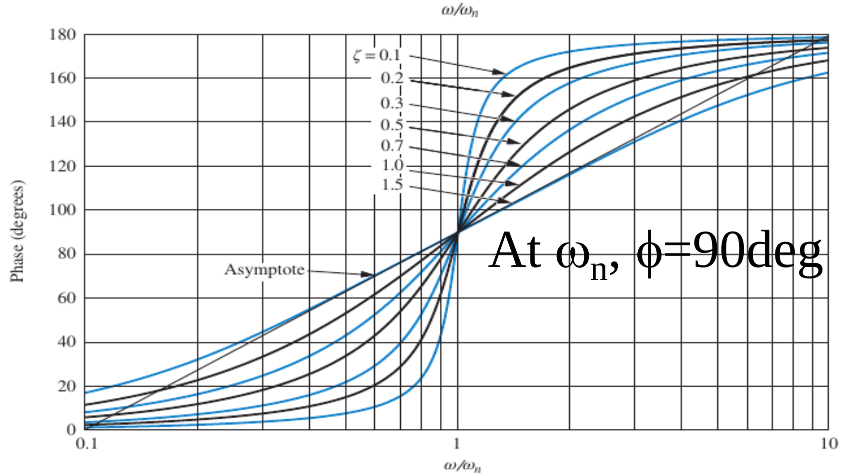
Note after $\omega = \omega_n$ it has 40db/decade slope

Resonance frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

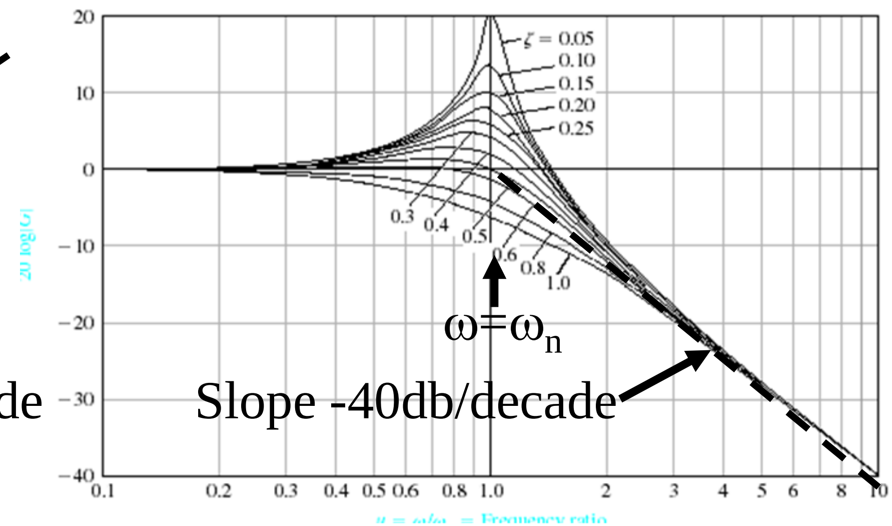
At resonance $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

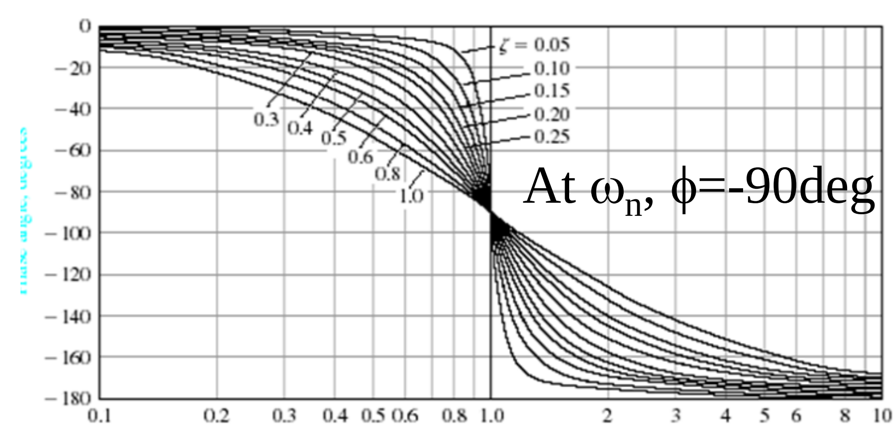
Complex Zero





Complex Pole





Example 2

Sketch the Bode diagram for the transfer function

$$G(s) = \frac{2500(s+10)}{s(s+2)(s^2+30s+2500)}$$

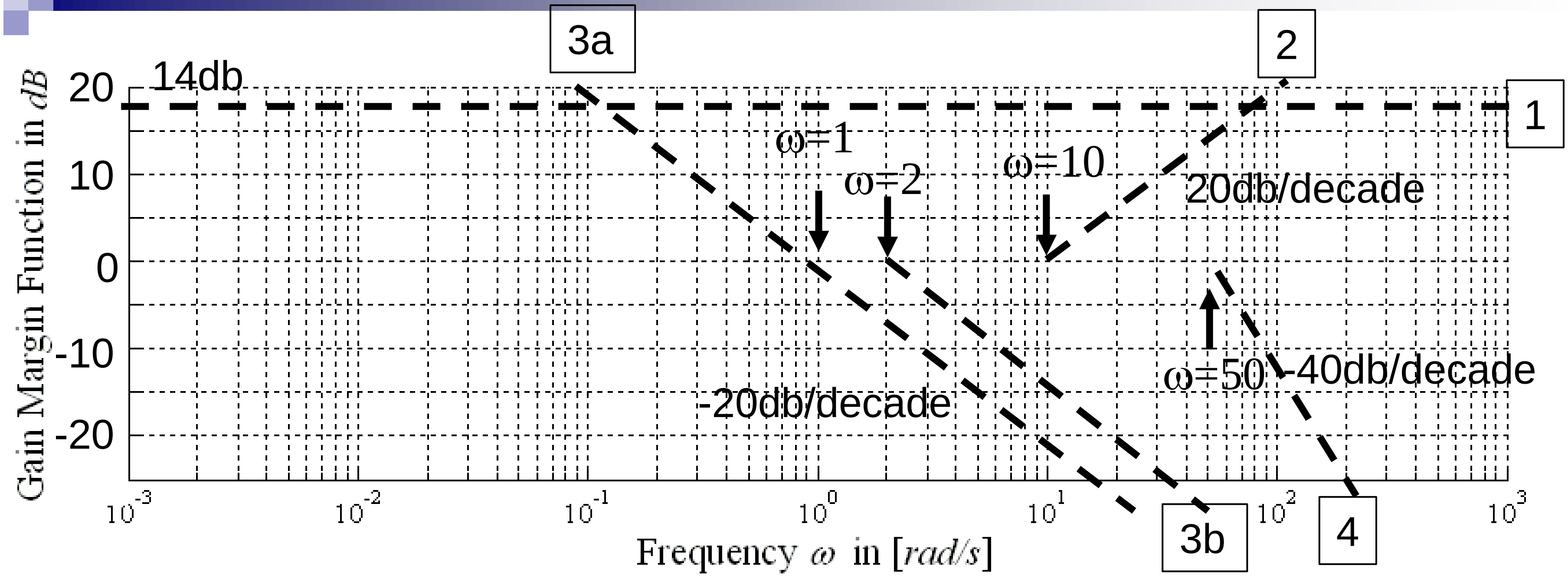
Reformat the transfer function:

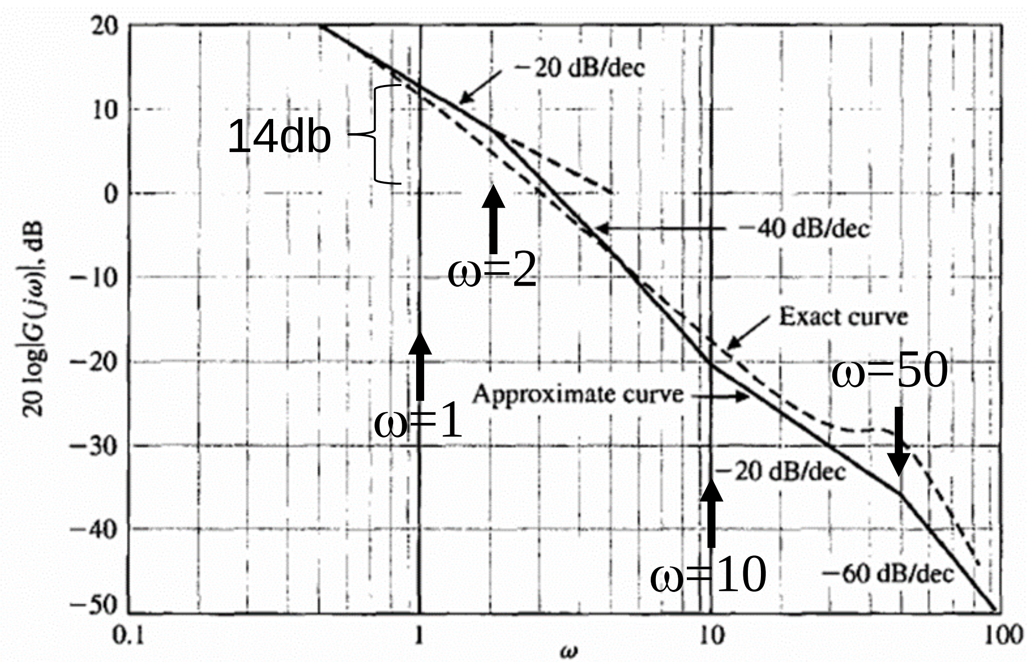
$$G(s) = \frac{5(0.1s+1)}{s(0.5s+1)\left(\frac{1}{50^2}s^2+\frac{0.6}{50}s+1\right)}$$

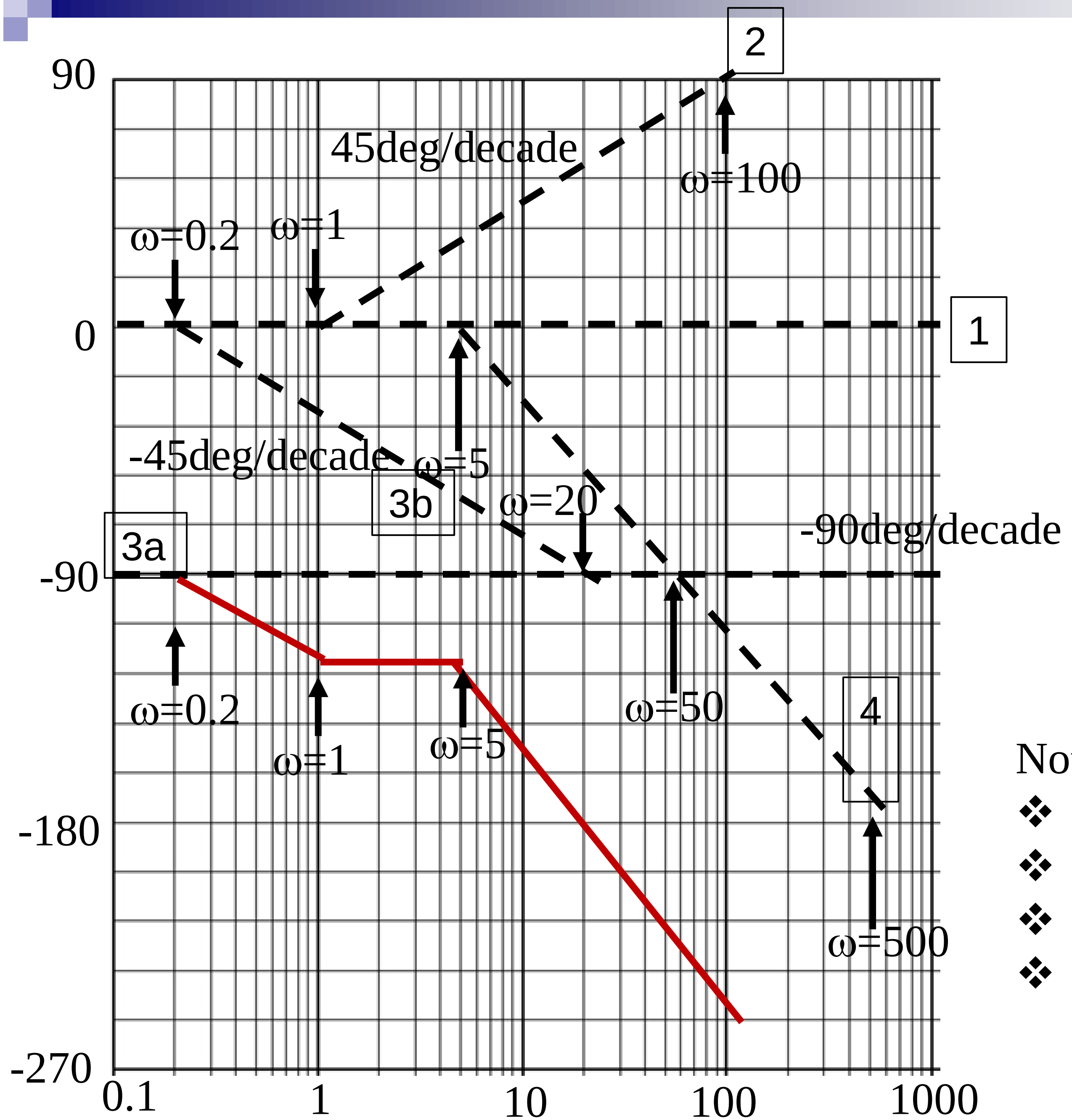
$$G(j\omega) = \frac{5(0.1\omega j+1)}{j\omega(0.5j\omega+1)\left(0.6\left(\frac{\omega}{50}\right)j+1-\left(\frac{\omega}{50}\right)^2\right)}$$

Factors:

- Gain $K = 5$ or $20\log 5 = 14\text{db}$
- Zero at $\omega = 10$
- Pole at origin and pole at $\omega = 2$
- A pair of complex pole at $\omega_n = 50$


3a
2
1
14db
20
10
0
-10
-20
ω=1
ω=2
ω=10
20db/decade
ω=50
-40db/decade
-20db/decade
3b
4
Gain Margin Function in dB
Frequency ω in [rad/s]


−20 dB/dec
14db
ω=2
−40 dB/dec
Exact curve
ω=50
ω=1
Approximate curve
−20 dB/dec
ω=10
−60 dB/dec
20 log|G(jω)|, dB
ω



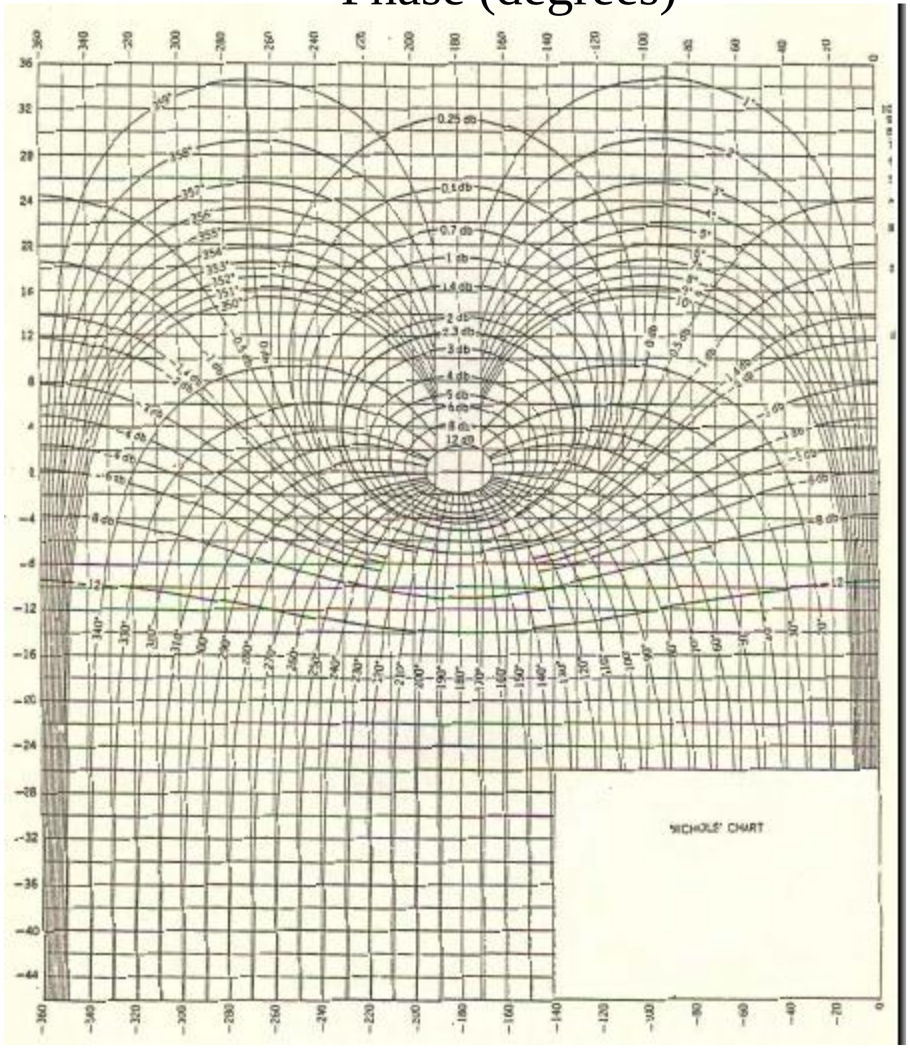
Note critical frequencies at

- $\omega = 2$
- $\omega = 10$
- $\omega = 50$
- Add the contributions

Nichols chart

Phase (degrees)

Magnitude (db)



Closed-loop frequency response for unity feedback can be obtained (read section 10.9 if interested in the construction of these circles)

Nichols chart

Sketch the Nichols chart for the following system:

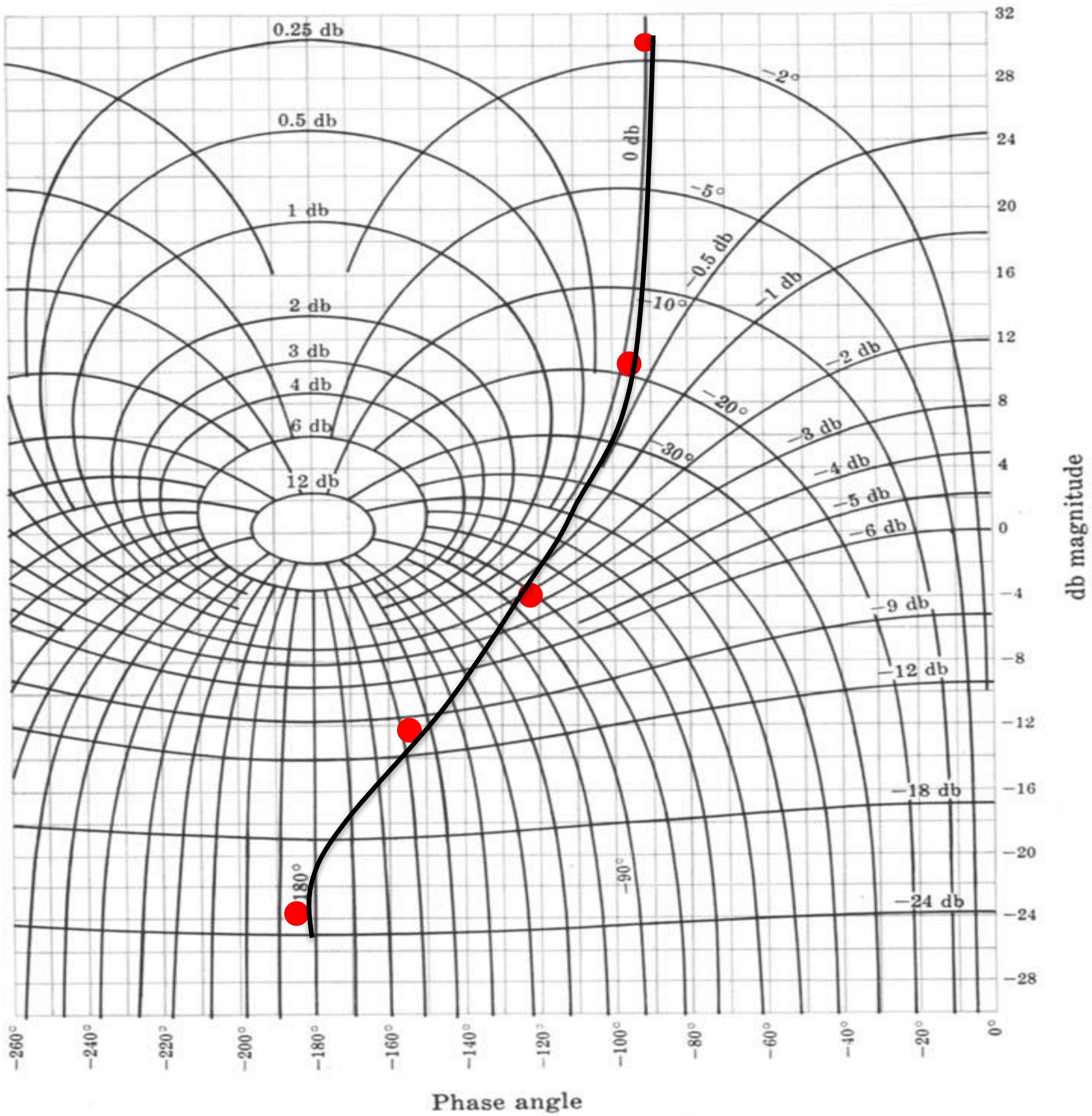
$$G(s) = \frac{1}{s(s+1)(s+3)}$$

ω	$\lvert G(j\omega)\rvert_{db}$	$\angle G(j\omega)$
0.01	30.4	-90.7°
0.1	10.4	-97.6°
0.5	-4.6	-126°
1	-13	-153°
2	-24.1	-187°

$$G(j\omega) = \frac{1}{(j\omega)(j\omega+1)(j\omega+3)}$$

$$G(j\omega) = \frac{1\angle 0^0}{\omega\left(\sqrt{1+\omega^2}\right)\left(\sqrt{9+\omega^2}\right)\angle 90^0 + \tan^{-1}\omega + \angle\tan^{-1}(\omega/3)}$$

ω	$\|G(j\omega)\|_{db}$	$\angle G(j\omega)$
0.01	30.4	-90.7°
0.1	10.4	-97.6°
0.5	-4.6	-126°
1	-13	-153°
2	-24.1	-187°



Nyquist diagram

Sketch the Nyquist diagram for the following system:

$$G(s) = \frac{1}{0.2s + 1}$$

$$G(j\omega) = \frac{1}{1 + j\omega 0.2}$$

$$G(j\omega) = \frac{1\angle 0^0}{\sqrt{1 + \omega^2 0.04}\angle \tan^{-1} \omega\, 0.2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 0.04}}$$

$$\angle G(j\omega) = -\tan^{-1}(\omega 0.2)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0
1	0.96	-11.31^0
5	0.5	-45^0
10	0.2	-63.43^0
30	0.02	-80.54^0
∞	0	-90^0

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0
1	0.96	-11.31^0
5	0.5	-45^0
10	0.2	-63.43^0
30	0.02	-80.54^0
∞	0	-90^0