# MEMS1045 Automatic control

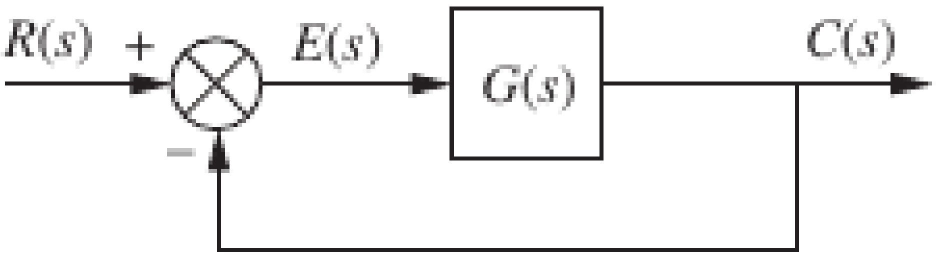
Lecture 8

Steady state error

# Objectives

- Describe relationships between static error constants and steady state errors
- Determine the steady state errors to step, ramp and parabola inputs based on system types
- Determine the steady state errors for disturbances and non-unity feedback systems
- Analyze the sensitivity of a function to changes in parameter

# Steady state error




In the previous lecture :

- ❖ For step input $R(s) = k/s$, $e_{step}(\infty) = \frac{k}{1+\lim_{s\to 0} G(s)}$
- ❖ For ramp input $R(s) = k/s^2$, $e_{ramp}(\infty) = \frac{k}{\lim_{s\to 0} sG(s)}$
- ❖ For parabola input $R(s) = k/s^3$, $e_{parabola}(\infty) = \frac{k}{\lim_{s\to 0} s^2 G(s)}$

# Static error constants

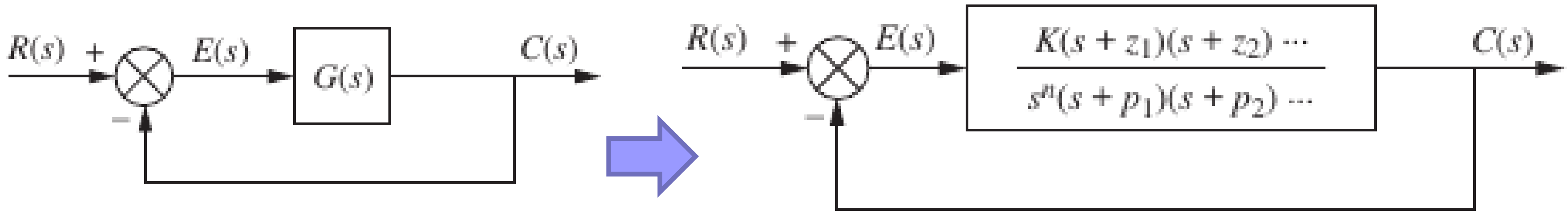
Defined the following static error constants:

- ❖ Position constant $K_p$ where $K_p = \lim_{s \to 0} G(s)$
- ❖ Velocity constant $K_v$ where $K_v = \lim_{s \to 0} sG(s)$
- ❖ Acceleration constant $K_a$ where $K_a = \lim_{s \to 0} s^2 G(s)$

Then the steady state error can be specified as

- ❖ For step input $R(s) = k/s$, $e_{step}(\infty) = \frac{k}{1+\lim_{s \to 0} G(s)} = \frac{k}{1+K_p}$
- ❖ For ramp input $R(s) = k/s^2$, $e_{ramp}(\infty) = \frac{k}{\lim_{s \to 0} sG(s)} = \frac{k}{K_v}$
- ❖ For parabola input $R(s) = k/s^3$, $e_{parabola}(\infty) = \frac{k}{\lim_{s \to 0} s^2 G(s)} = \frac{k}{K_a}$

# System types




Forward transfer function expressed in poles-zero format:

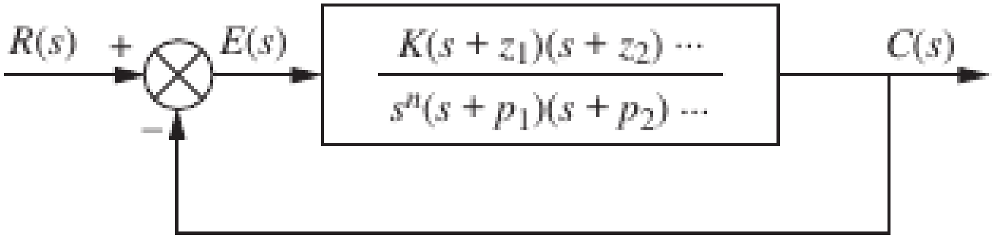
- $S^n$ indicates the number of poles at the origin for the open-loop transfer function, where $n$ = system type
- Important:

a) Do not mix up the poles and zeros of the open-loop forward transfer function with that of the closed-loop transfer function. They are NOT the same

b) System stability depends on the location of the closed-loop poles and NOT on the locations of the open-loop poles

c) System step responses are determined by the locations of the closed-loop poles and zeros and NOT on the locations of the open-loop poles

# System type 0




❖ For type 0 or $s^0 = 1$;

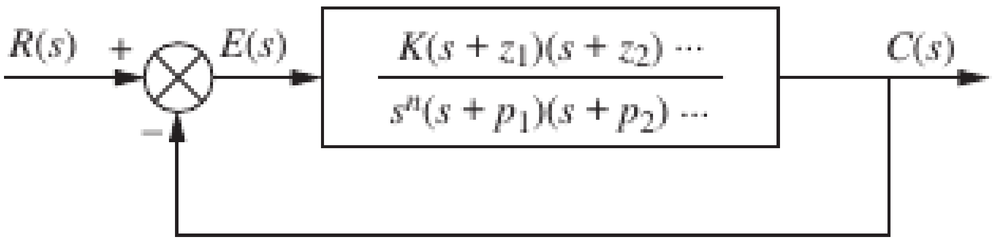
$K_p = \lim_{s\to 0} G(s)$ =constant; $K_v = \lim_{s\to 0} sG(s) = 0$; and $K_a = \lim_{s\to 0} s^2 G(s) = 0$; which implies that

a) For step input $R(s) = k/s$, $e_{step}(\infty) = \frac{k}{1+K_p}$

b) For ramp input $R(s) = k/s^2$, $e_{ramp}(\infty) = \frac{k}{K_v} = \infty$

c) For parabola input $R(s) = k/s^3$, $e_{parabola}(\infty) = \frac{k}{K_a} = \infty$

# System type 1




❖ For type 1 or $s^1 = s$;

$K_p = \lim_{s\to 0} G(s) = \infty$; $K_v = \lim_{s\to 0} sG(s) =$constant; and $K_a = \lim_{s\to 0} s^2 G(s) = 0$;

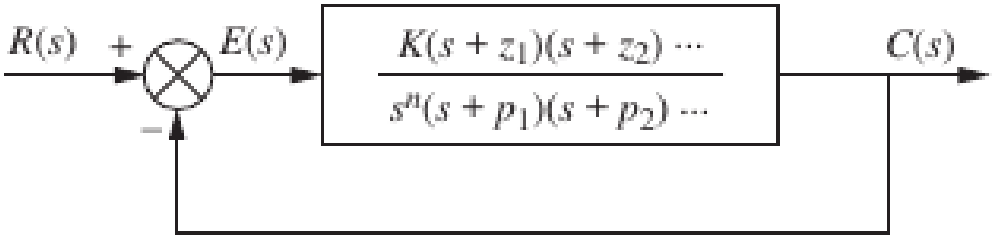
which implies that

a) For step input $R(s) = k/s$, $\quad e_{step}(\infty) = \frac{k}{1+K_p} = 0$

b) For ramp input $R(s) = k/s^2$, $\quad e_{ramp}(\infty) = \frac{k}{K_v}$

c) For parabola input $R(s) = k/s^3$, $\quad e_{parabola}(\infty) = \frac{k}{K_a} = \infty$

# System type 2




❖ For type 2 or $s^2 = \mathrm{s}^2$;

$K_p = \lim_{s\to 0} G(s) = \infty$; $K_v = \lim_{s\to 0} sG(s) = \infty$; and $K_a = \lim_{s\to 0} s^2 G(s) =$constant;

which implies that

a) For step input $R(s) = k/s$, $\quad e_{step}(\infty) = \frac{k}{1+K_p} = 0$

b) For ramp input $R(s) = k/s^2$, $\quad e_{ramp}(\infty) = \frac{k}{K_v} = 0$

c) For parabola input $R(s) = k/s^3$, $\quad e_{parabola}(\infty) = \frac{k}{K_a}$

# Mini summary

| Input | Steady-state error formula | Type 0 Static error constant | Type 0 Error | Type 1 Static error constant | Type 1 Error | Type 2 Static error constant | Type 2 Error |
|---|---|---|---|---|---|---|---|
| Step, $u(t)$ | $\frac{1}{1+K_p}$ | $K_p =$ Constant | $\frac{1}{1+K_p}$ | $K_p = \infty$ | 0 | $K_p = \infty$ | 0 |
| Ramp, $tu(t)$ | $\frac{1}{K_v}$ | $K_v = 0$ | $\infty$ | $K_v =$ Constant | $\frac{1}{K_v}$ | $K_v = \infty$ | 0 |
| Parabola, $\frac{1}{2}t^2u(t)$ | $\frac{1}{K_a}$ | $K_a = 0$ | $\infty$ | $K_a = 0$ | $\infty$ | $K_a =$ Constant | $\frac{1}{K_a}$ |

# Example 1

A unity feedback system has the following forward transfer function

$$G(s) = \frac{1000(s+8)}{(s+7)(s+9)}$$

a) Evaluate system type, $K_p$, $K_v$, and $K_a$
b) Use your answers find the steady-state errors for the standard step, ramp, and parabolic inputs
c) Determine the system closed-loop transfer function
d) Determine the system stability
e) Describe the unit step response of the system

❖ Using the forward transfer function, the system is type 0:

$K_p = \lim_{s\to 0} G(s) =$constant;

$K_v = \lim_{s\to 0} sG(s) = 0;$ and

$K_a = \lim_{s\to 0} s^2 G(s) = 0;$

# Example 1

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{1000(s+8)}{(s+7)(s+9)} = \frac{1000(8)}{(7)(9)}$$

$K_p = 127;$

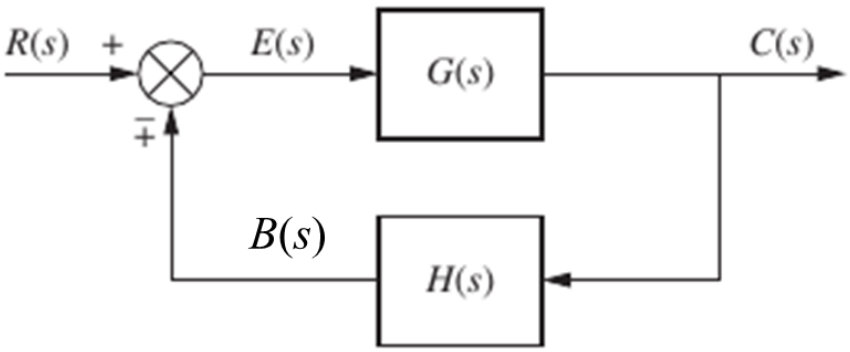
$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\frac{1000(s+8)}{(s+7)(s+9)} = 0$$

$$K_a = \lim_{s\to 0} s^2G(s) = \lim_{s\to 0} s^2\frac{1000(s+8)}{(s+7)(s+9)} = 0$$

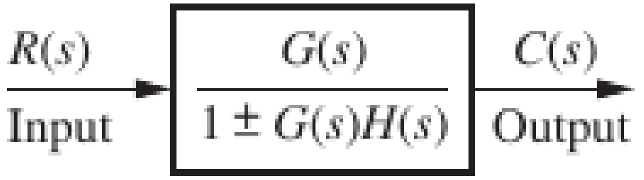
b) For step input $u(t)$=1, $R(s) = 1/s$, $e_{step}(\infty) = \frac{1}{1+K_p} = 0.008$

For ramp input $e_{ramp}(\infty) = \infty$ and for parabola input, $e_{parabola}(\infty) = \infty$

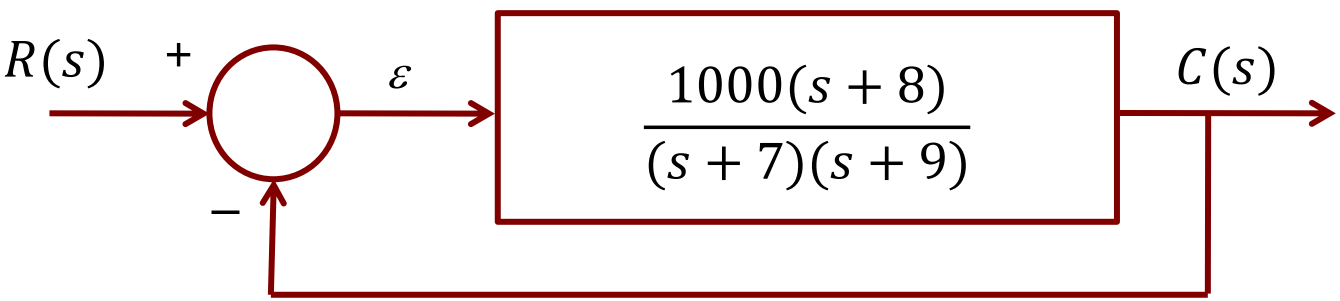
# Example 1



R(s)
+
E(s)
G(s)
C(s)
∓
B(s)
H(s)

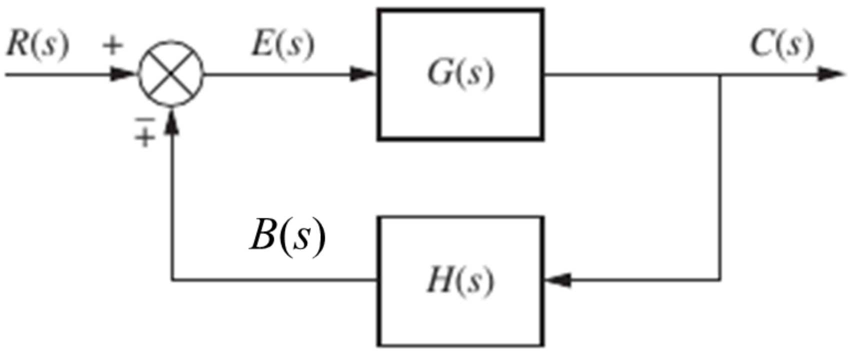



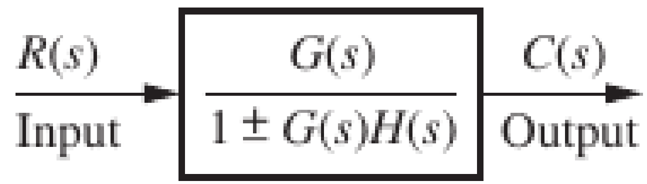
R(s)
Input
$\frac{G(s)}{1 \pm G(s)H(s)}$
C(s)
Output

R(s)
+
ε
$\frac{1000(s+8)}{(s+7)(s+9)}$
C(s)
−

# Example 1




c) Closed-loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{1000(s+8)}{(s+7)(s+9)}}{1+\frac{1000(s+8)}{(s+7)(s+9)}} = \frac{1000(s+8)}{(s+7)(s+9)+1000(s+8)}$$

$$T(s) = \frac{1000(s+8)}{s^2+1016s+8063}$$

# Example 1

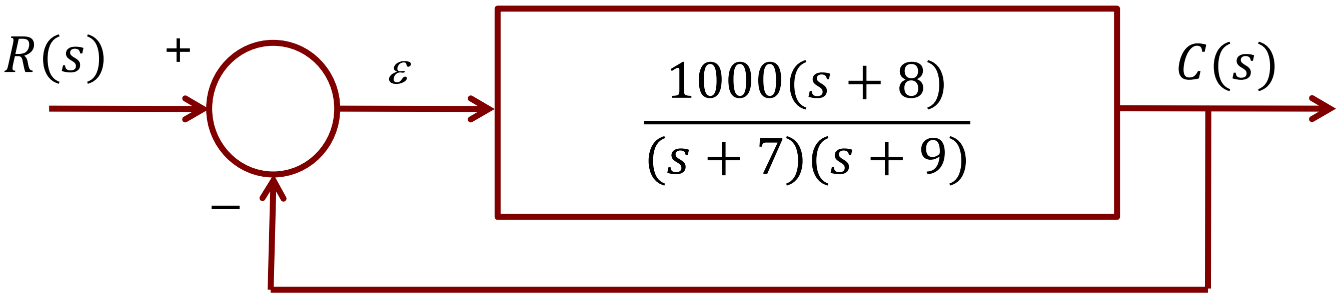
Closed loop transfer function

$$T(s) = \frac{1000(s+8)}{s^2 + 1016s + 8063}$$

For 2nd order system, no need to use Routh's criterion as roots are:

$$s_{1,2} = \frac{-1016 \pm \sqrt{(1016)^2 - 4(1016)(8063)}}{2} = -7.999 \text{ and } -1008$$

Hence, system is stable



# Example 1

Forward transfer function:

$$\frac{1000(s+8)}{(s+7)(s+9)}$$

Closed-loop transfer function

$$T(s)=\frac{C(s)}{R(s)}=\frac{1000(s+8)}{(s+7.999)(s+1008)}$$

The system poles are at s=-7.999 and s=-1008 (and NOT at s=-7 and s=-9)

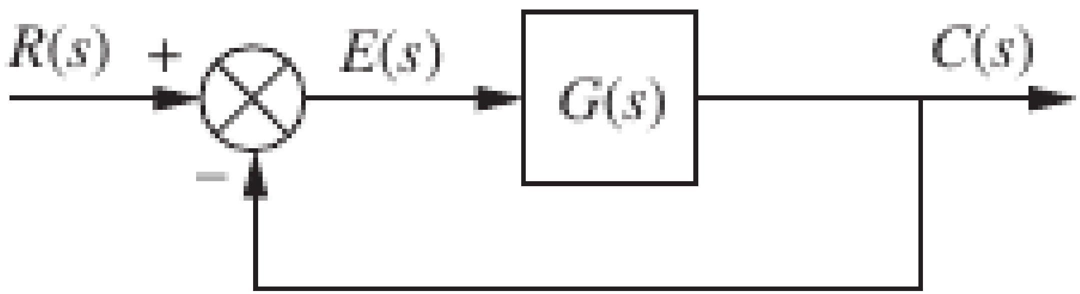
e) This is an overdamped 2$^{nd}$ order system. The dominant pole is at s=-7.999; There could be a pole-zero cancellation effect and the zero at s=-8 will affect the initial magnitude of the response. The worst case settling time is about 0.5 sec. Steady state value to a unit step response is 0.992 (hence a steady state error of 0.008 to a unit step input).

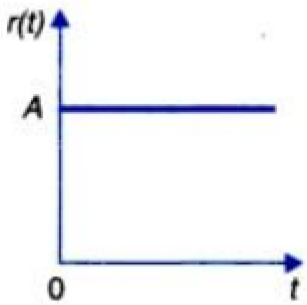
# Lab2

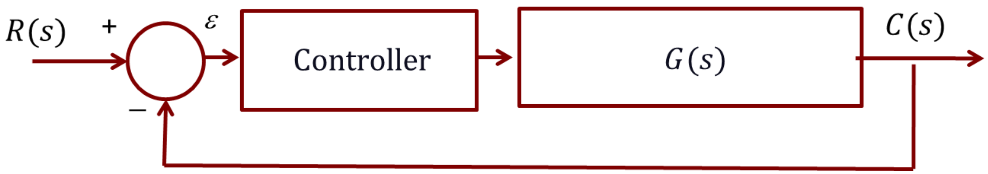
- Lab 2 is available on BB
- This is on application of Matlab control toolbox
- You will do this lab at your convenience but must submit the lab2 exercise by Sunday 15 Nov. through BB (lab/project folder)
- To do the lab, activate Matlab and go through commands given in lab 2  note
- The root locus and frequency response parts will be covered in the next few classes

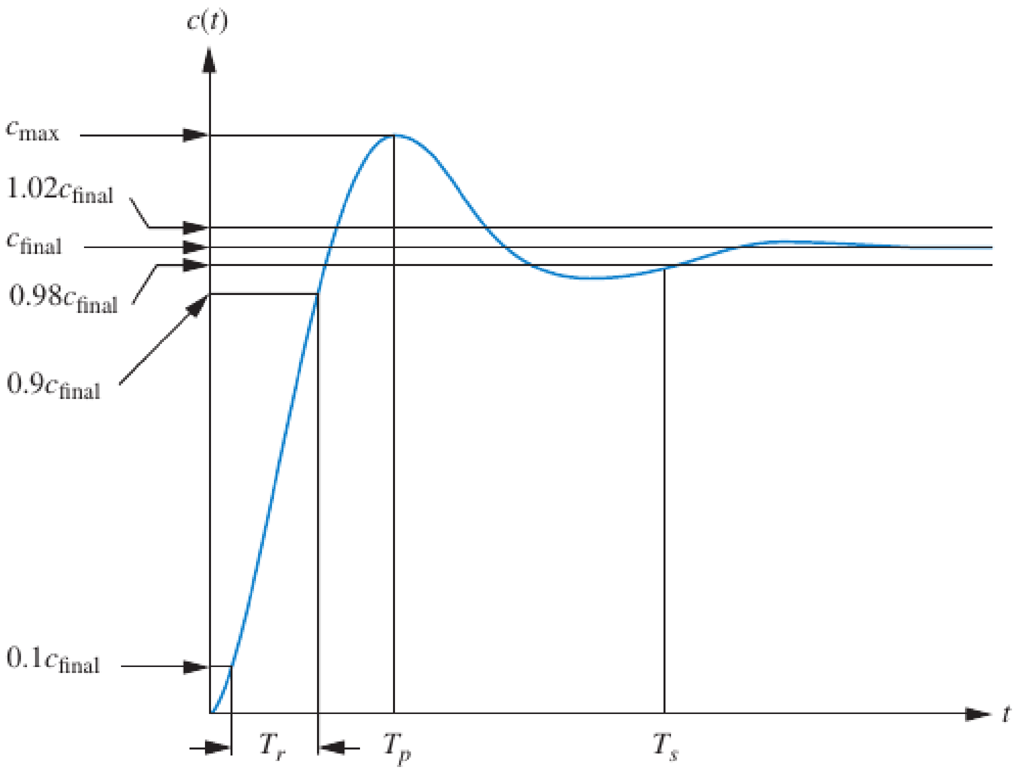
# Control response specifications

- The aim of feedback control is to regulate the output response $c(t)$ to follow the reference input $r(t)$ with desired response characteristics
- Desired transient response for the controlled output can be specified using damping ratio, settling time, peak time, and percent overshoot
- Desired steady-state characteristics of the controlled output can be specified using static error constants

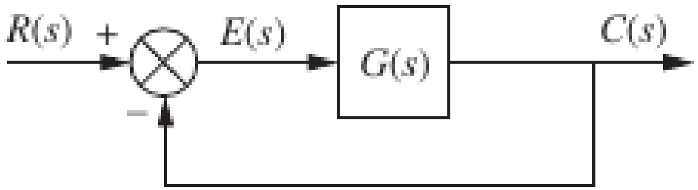


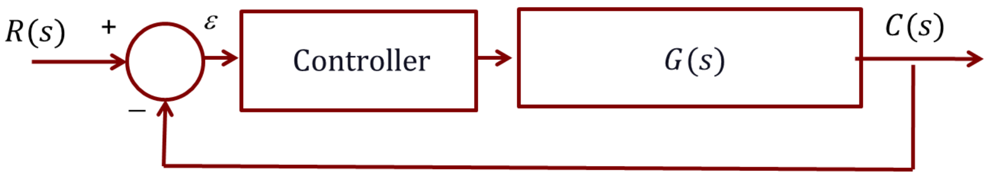

r(t)
A
0
t


R(s)
+
−
ε
Controller
G(s)
C(s)


c(t)
c_max
1.02c_final
c_final
0.98c_final
0.9c_final
0.1c_final
t
T_r
T_p
T_s

# Steady state error specifications




If a control system has the specification $K_v = 1000$, the controller to be designed must ensure the following:

- System is stable (Why?)
- The system is type 1 (why?)
- The test signal used for the input $R(s)$ is a ramp function
- The steady state error to the ramp input $R(s) = k/s^2$ is

$$e_{ramp}(\infty) = \frac{k}{K_v}$$

# Example 2

A unity feedback system has the following forward transfer function:

$$G(s) = \frac{K(s+12)}{(s+14)(s+18)}$$

Find the value of $K$ to yield a 10% error in the steady state to step input

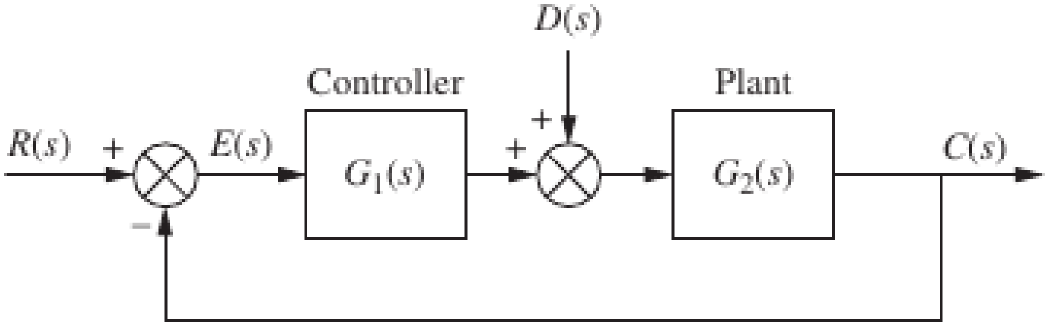
Using the forward transfer function, the system is type 0:

- ❖ The steady state error applied to the step input
- ❖ $K_p = \lim_{s \to 0} G(s) = \frac{12K}{14(18)} = 0.0476K;$
- ❖ $e_{step}(\infty) = 0.1 = \frac{1}{1+K_p}$ (i.e. 10% error)

Hence $\frac{1}{1+0.0476K} = 0.1$; solve to get $K = 189$

# Disturbances

Disturbances can enter the system as shown:




Note that the output $C(s)$ is affected by 2 inputs:

$$C(s) = G_1(s)G_2(s)E(s) + G_2(s)D(s)$$

Where $C(s) = R(s) - E(s)$

$$G_1(s)G_2(s)E(s) + G_2(s)D(s) = R(s) - E(s)$$

$$[1 + G_1(s)G_2(s)]E(s) = R(s) - G_2(s)D(s)$$

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

# Disturbances

Apply the final value theorem:

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)} R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$e_{ss}(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s) - \lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$e_{ss}(\infty) = e_R(\infty) + e_D(\infty)$$

where

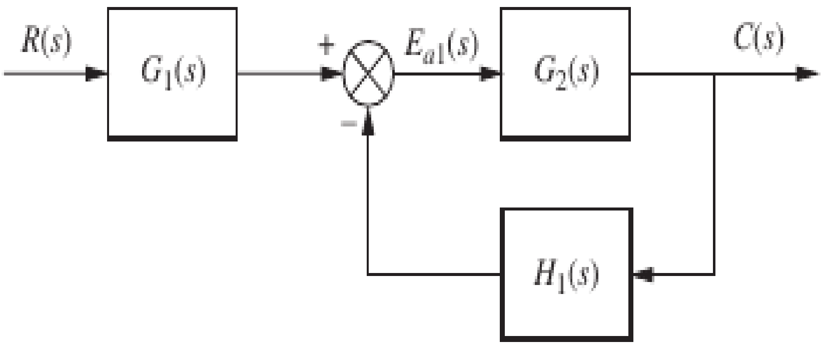
- ❖ $e_R(\infty)$ is the steady state error to the input $r(t)$:

$$e_R(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

- ❖ $e_D(\infty)$ is the steady state error to the input $d(t)$

$$e_D(\infty) = -\lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

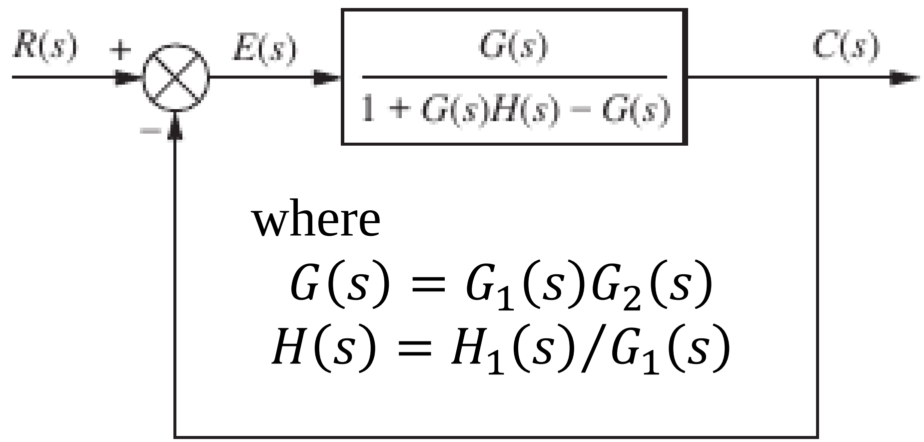
# Non-unity feedback system




Input transducer = $G_1$(s),
Controller and plant = $G_2$(s),
Feedback = $H_1$(s)

$$e_{a1}(\infty) = \lim_{s \to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

Where $e_{a1}$ = actuating signal



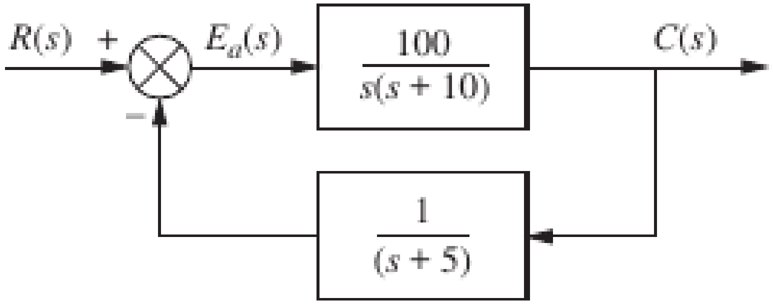

where
$G(s) = G_1(s)G_2(s)$
$H(s) = H_1(s)/G_1(s)$

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

Determine the system type and proceed to find $K_p = \lim_{s \to 0} G_e(s)$, $K_v = \lim_{s \to 0} sG_e(s)$; $K_a = \lim_{s \to 0} s^2 G_e(s)$; and use these to find the steady state error for step, ramp & parabola inputs

# Example 3

a) Find the steady-state error for a unit step input assume input and output units are the same
b) Find the steady-state actuating signal for the system for a unit step input




Note: $G_1(s) = 1$
$G(s) = G_1(s)G_2(s) = G_2(s)$

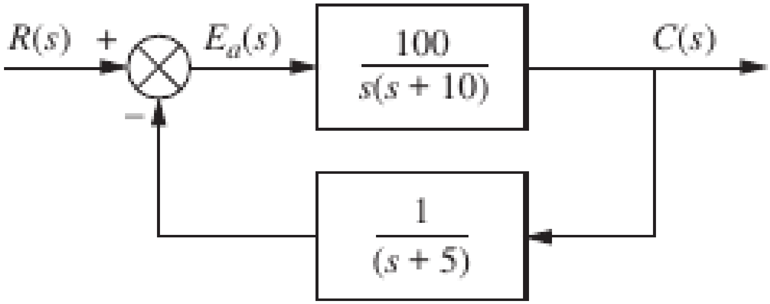
$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{\frac{100}{s(s+10)}}{1 + \frac{100}{s(s+10)(s+5)} - \frac{100}{s(s+10)}}$$

$$G_e(s) = \frac{100(s+5)}{s(s+10)(s+5) + 100 - 100(s+5)}$$

Type 0 system: $K_p = \lim_{s\to 0} G_e(s) = -1.25$, and $e_{step}(\infty) = \frac{1}{1+K_p} = -4$

# Example 3

a) Find the steady-state error for a unit step input assume input and output units are the same
b) Find the steady-state actuating signal for the system for a unit step input




Note: $G_1(s) = 1$

$$e_{a1}(\infty) = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

$$e_{a1}(\infty) = \lim_{s\to 0} \frac{1}{1 + \dfrac{100}{s(s+10)(s+5)}} = \frac{s(s+10)(s+5)}{s(s+10)(s+5) + 100} = 0$$

# Sensitivity

Given $F$ = function which contains the parameter "$a$"
The sensitivity of the function to changes in the parameter "$a$" is defined as:

$$S_{F,a} = \frac{a}{F}\left(\frac{\delta F}{\delta a}\right)$$

For example:

Given transfer function $F(s) = \frac{K}{s^2+as+K}$

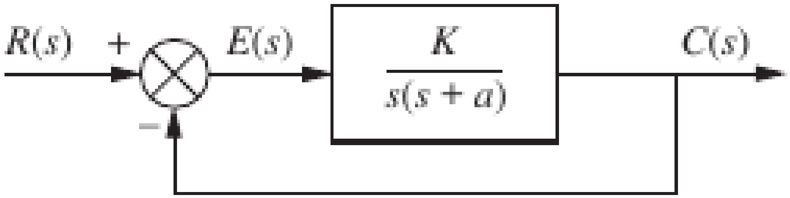
The sensitivity of the transfer function $F(s)$ to changes in parameter "$a$" is:

$$S_{F,a} = \frac{a}{F}\left(\frac{\delta F}{\delta a}\right) = \frac{a}{\left\{\frac{K}{s^2+as+K}\right\}}\left(\frac{\delta}{\delta a}\left\{\frac{K}{s^2+as+K}\right\}\right)$$

$$S_{F,a} = \frac{a}{F}\left(\frac{\delta F}{\delta a}\right) = \frac{a}{\left\{\frac{K}{s^2+as+K}\right\}}\left\{\frac{-Ks}{(s^2+as+K)^2}\right\} = \frac{-as}{s^2+as+K}$$

# Example 4

For the given system, determine the sensitivity of the steady-state error to changes in parameter $K$ and parameter "$a$" with ramp inputs




Type 1 system: $K_v = \lim_{s \to 0} sG(s) = K/a$, and $e_{ramp}(\infty) = \frac{1}{K_v} = \frac{a}{K}$

$$S_{e,K} = \frac{K}{e_{ramp}(\infty)}\left(\frac{\delta e_{ramp}(\infty)}{\delta K}\right) = \frac{K^2}{a}\left(-\frac{a}{K^2}\right) = -1$$

$$S_{e,a} = \frac{a}{e_{ramp}(\infty)}\left(\frac{\delta e_{ramp}(\infty)}{\delta a}\right) = K\left(\frac{1}{K}\right) = 1$$

# Mini summary

**TABLE 7.2** Relationships between input, system type, static error constants, and steady-state errors

| | | Type 0 | | Type 1 | | Type 2 | |
|---|---|---|---|---|---|---|---|
| Input | Steady-state error formula | Static error constant | Error | Static error constant | Error | Static error constant | Error |
| Step, $u(t)$ | $\frac{1}{1+K_p}$ | $K_p =$ Constant | $\frac{1}{1+K_p}$ | $K_p = \infty$ | 0 | $K_p = \infty$ | 0 |
| Ramp, $tu(t)$ | $\frac{1}{K_v}$ | $K_v = 0$ | $\infty$ | $K_v =$ Constant | $\frac{1}{K_v}$ | $K_v = \infty$ | 0 |
| Parabola, $\frac{1}{2}t^2u(t)$ | $\frac{1}{K_a}$ | $K_a = 0$ | $\infty$ | $K_a = 0$ | $\infty$ | $K_a =$ Constant | $\frac{1}{K_a}$ |