MEMS1045 Automatic control

Lecture 9

Root Locus 1

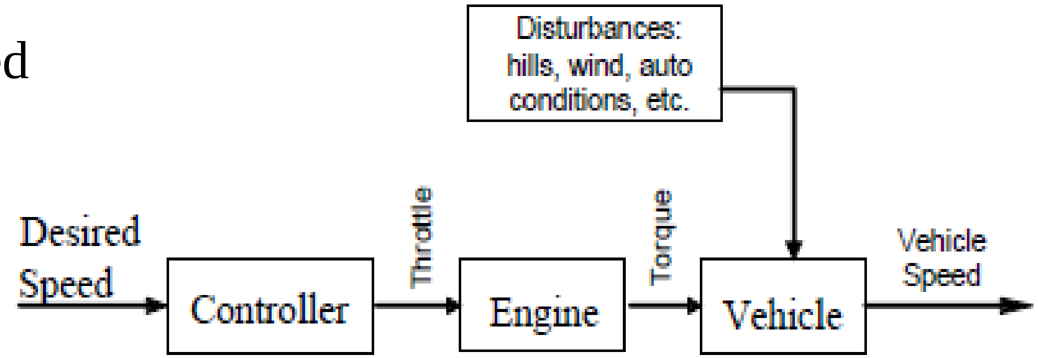
Objectives

- Explain the concept of feedback control and describe how the proportional gain can regulate the time response of the controlled variable
- Construct root locus from the open-loop transfer function
- Refine the root locus sketch based on the magnitude and angle criteria

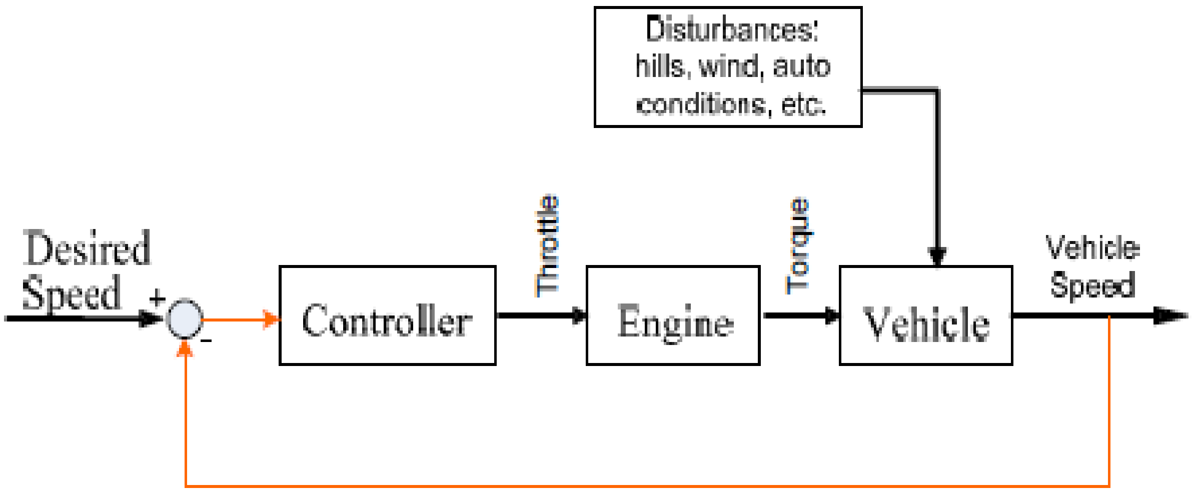
Concept of feedback control

Control of vehicle speed

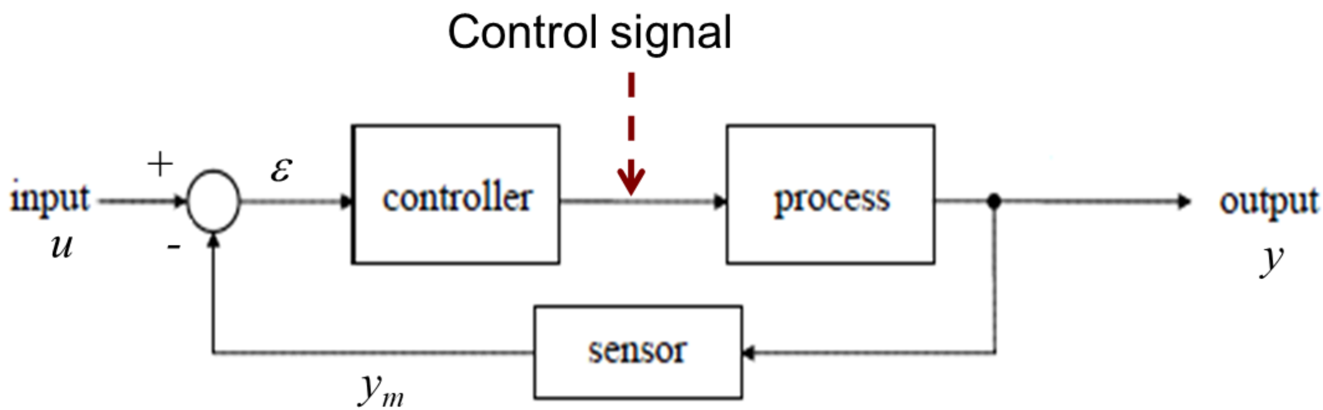
Open Loop
Solution

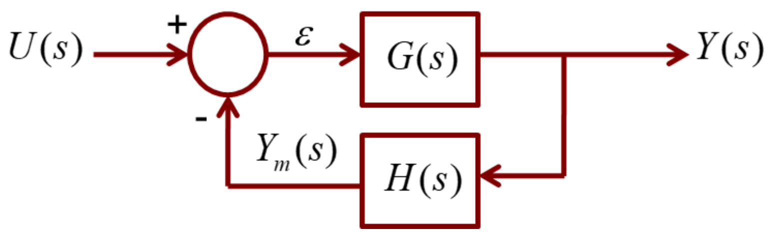


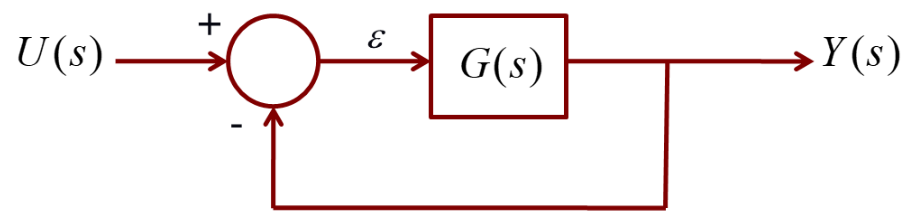
Closed Loop
Solution



Concept of feedback control







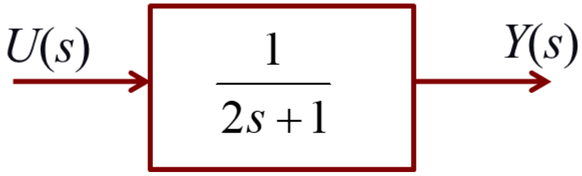
CLTF: $\dfrac{Y(s)}{U(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

CLTF: $\dfrac{Y(s)}{U(s)} = \dfrac{G(s)}{1+G(s)}$

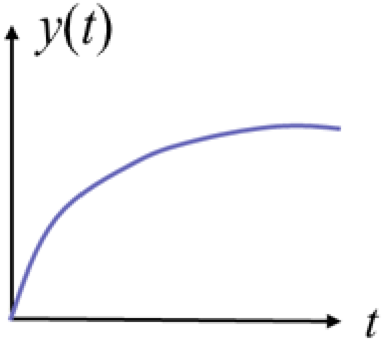
Desired time response for $Y(s)$ include stability, settling time, steady state error, etc

Example 1

Assume a toy car modelled by the transfer function shown with $U(s)$ = input voltage and $Y(s)$ = speed. Sketch the time response for a unit step input.

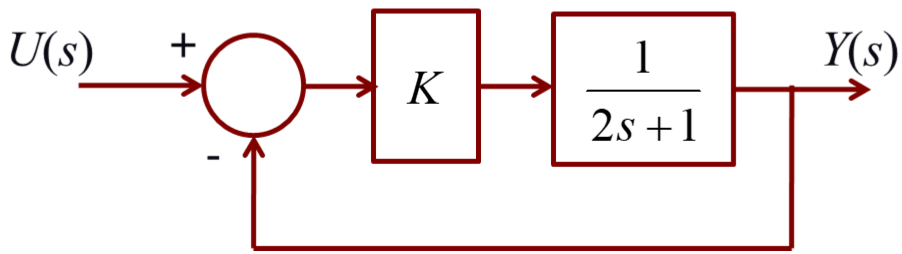


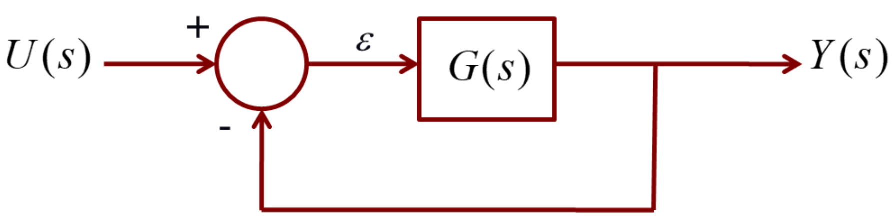
Pole at $s = -0.5$
Settling time $T_s = 8$ sec.
Unit step response steady state is 1 rad/s (no oscillation)



Example 1

A proportional controller is design to regulate the voltage based on the measured speed as shown. Determine the pole locations as gain K takes the value of 0, 1, 2, 5, and 10. Sketch the time responses at each value of K.





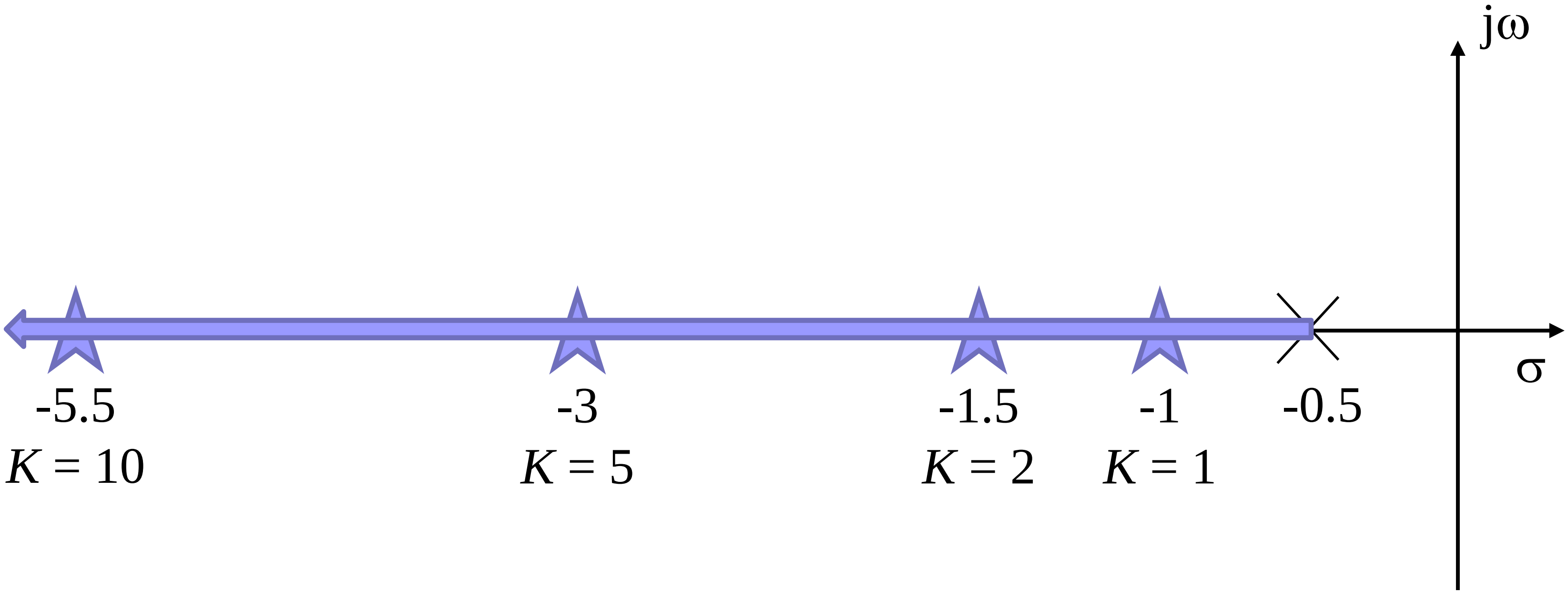
CLTF:

$$\frac{Y(s)}{U(s)} = \frac{K}{2s + 1 + K}$$

Characteristics equation

$$2s + 1 + K = 0$$

Example 1



CLTF: $\frac{Y(s)}{U(s)} = \frac{K}{2s+1+K}$

$$2s + 1 + K = 0$$

K	0	1	2	5	10
Root	-0.5	-1.0	-1.5	-3	-5.5
T_s	8s	4s	2.67s	1.33s	0.72s
e_{step}	0	0.5	0.3	0.167	0.9

Introduction to root locus

- The movement of the pole locations when a parameter (such as K) is varied can be shown by drawing the path traced out in the s-plane as the parameter is increased from very small to very large values
- The path is called a root locus
- The pole locations will determine the system stability
- The pole locations at each value of K will also dictate the speed and shape of the response
- One way to determine the location of the poles is to solve the characteristics equation. You will now learn how to plot the root locus without solving the characteristics equation

Sketching rules for root locus

1. A root locus is symmetrical about the real axis
2. A root locus comprises n separate branch loci, where n is the order of the system's CE (or n=number of OLTF poles)
3. A system's root locus branches start at each of its OLTF poles for K=0
4. As K increases from 0, one branch of the locus departs from each of the OLTF poles
5. As K approaches ∞, the branches of the locus approaches the OLTF zeros
6. Let m=number of OLTF zeros. If $m \neq n$, the remaining n-m branches will approach infinity along asymptotes
7. A root locus occupies those parts of the real axis which have an odd number of real poles and zeroes to the right
8. When the real axis between two adjacent OLTF poles is occupied by the locus, breakaway will occur
9. When the real axis between two adjacent OLTF zeros is occupied by the locus, break-in will occur

Procedure

Step 1: Locate the open-loop poles and zeros in s-plane

- ❖ It has n separate branches (where n = number of open loop poles). Each branch starts from $K = 0$ at an open-loop pole and approaches an open-loop zero as $K \rightarrow \infty$ (Let m=number of open loop zeros; remaining $(n - m)$ branches will approach asymptotes – see step 3)

Step 2: Determine the root loci occupying the real axis

- ❖ It occupies those parts of the real axis which have an odd number of real poles and zeroes to the right

Step 3: Determine the asymptotes of the root loci (if any)

- ❖ Add all the open-loop pole values $= \sum P_O$
- ❖ Add all the open-loop zero values $= \sum Z_O$
- ❖ Asymptotes are located at $\bar{x} = \frac{\sum P_O - \sum Z_O}{n-m}$
- ❖ Asymptotes are oriented at angles $\theta = \frac{(2b+1)180^0}{n-m}$ for $b = 0, \pm 1, \pm 2, \ldots$

Procedure

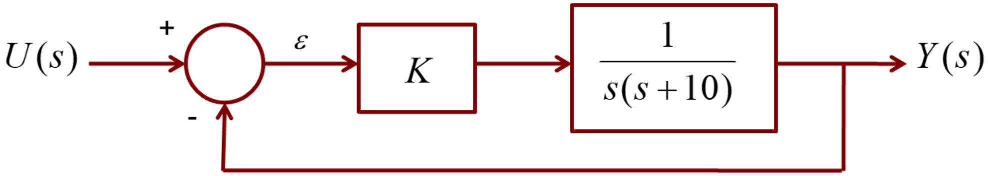
Step 4: Locate the breakaway or break-in points (if any)

- A root locus is symmetrical about the real axis
- When the real axis between two adjacent OLTF poles is occupied by the locus, breakaway will occur
- When the real axis between two adjacent OLTF zeros is occupied by the locus, break-in will occur
- We can locate the breakaway or break-in points by expressing the characteristics equation as $K=f(s)$ and then solve for s when:

$$\frac{\partial K}{\partial s} = \frac{\partial}{\partial s} f(s) = 0$$

Example 2

Sketch the root locus of the given system:



Step 1: Locate the open-loop poles and zeros in s-plane

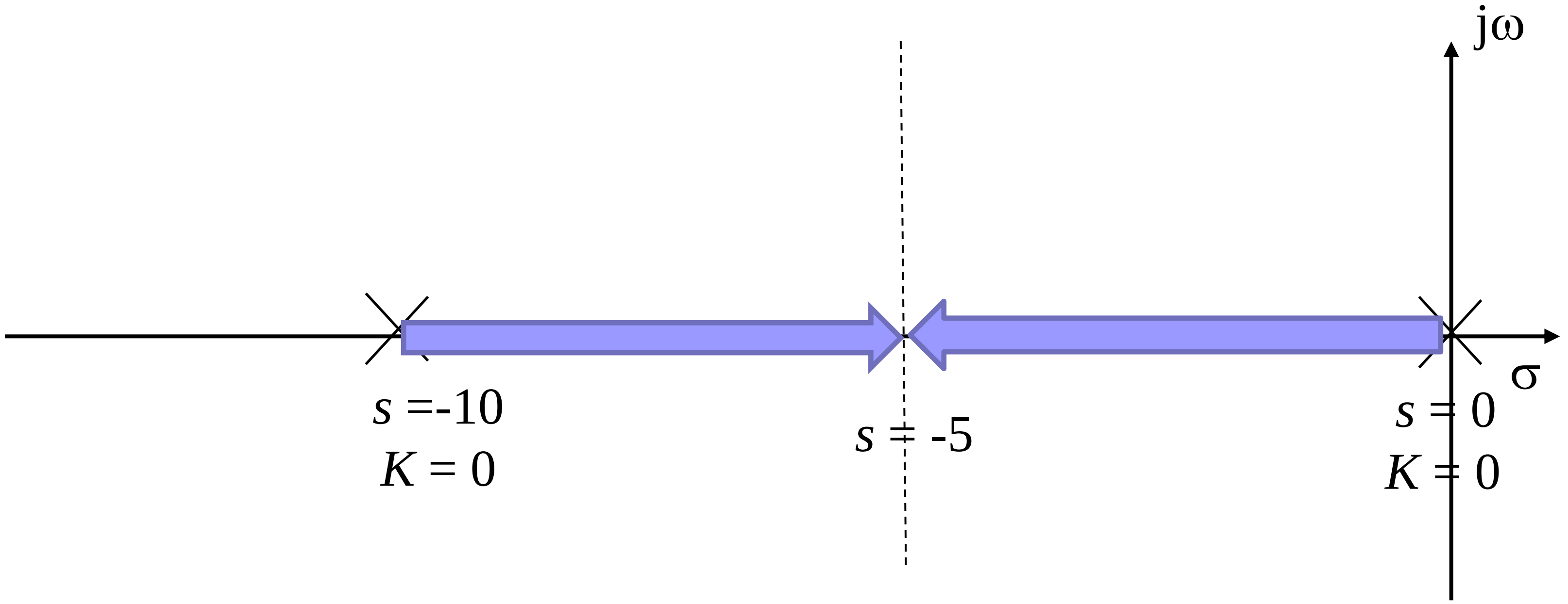
- Note: n = number of open loop poles = 2 (i.e. 2 branches)
- Two poles at s = 0 and s = -10 (2 branches start from K = 0 at these points
- Note: m=number of open loop zeros = 0; (there are $(n - m = 2)$ asymptotes as $K \to \infty$)

Step 2: Determine the root loci occupying the real axis

- It occupies those parts of the real axis which have an odd number of real poles and zeroes to the right (i.e. it will occupy the real axis from 0 to -10)

Note: characteristics equation is $s^2 + 10s + K = 0$ or $K = -s^2 - 10s$

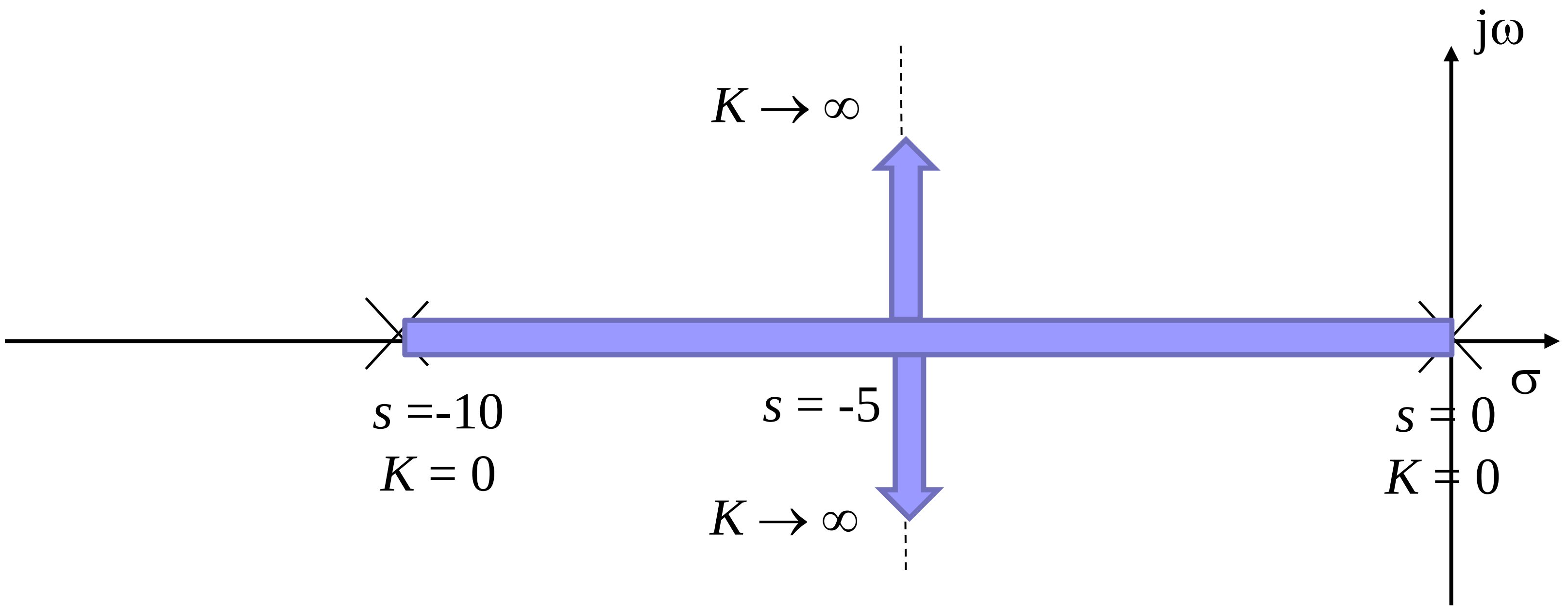
Example 2



Step 3: Determine the asymptotes of the root loci (if any)

- ❖ Add all the open-loop pole values $= \sum P_O = -10$
- ❖ Add all the open-loop zero values $= \sum Z_O = 0$
- ❖ Asymptotes are located at $\bar{x} = \frac{\sum P_O - \sum Z_O}{n-m} = -5$
- ❖ Asymptotes are oriented at angles $\theta = \frac{(2b+1)180^0}{n-m} = \pm 90^0$ for $b = 0$

Example 2

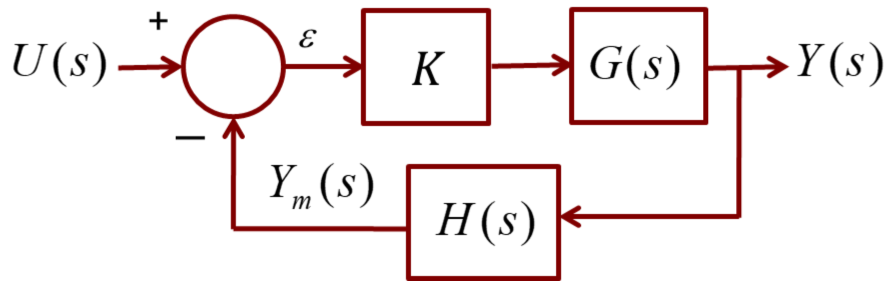


Step 4: Locate the breakaway or break-in points (if any)

- It is symmetrical about the real axis; when the real axis between two adjacent OLTF poles is occupied by the locus, breakaway will occur
- Get characteristic equation, differentiate, equate to zero to solve for *s*:

$$\frac{\partial K}{\partial s} = \frac{\partial}{\partial s} f(s) = \frac{\partial}{\partial s}(-s^2 - 10s) = -2s - 10 = 0 \text{ or } s = -5$$

Magnitude & angle criteria



The closed-loop transfer function of the feedback system is

$$\frac{Y(s)}{U(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

The characteristics equation is $1 + KG(s)H(s)$ or $KG(s)H(s) = -1$

Note: $s = \sigma + j\omega$ is a complex number and $G(s)H(s)$ is also a complex number

- ❖ Therefore $|KG(s)H(s)| = 1$ or $K = 1/|G(s)H(s)|$ (magnitude condition)
- ❖ $\angle KG(s)H(s) = \angle -1 = \pm 180^0(k)$ for $k = 1, 3, \ldots$ (angle condition)

Any point that lies on the root locus must satisfy these conditions

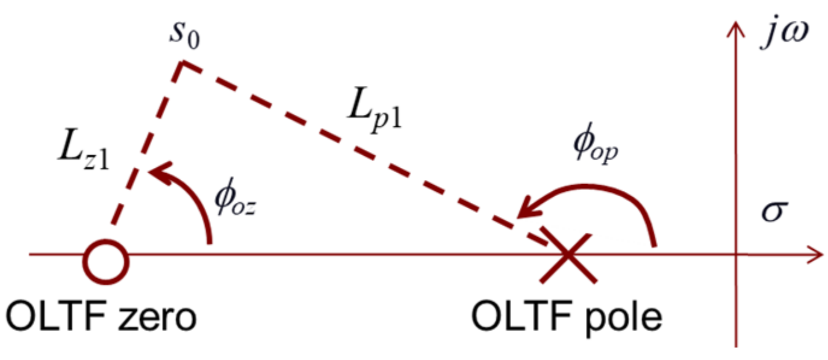
Magnitude & angle criteria

1) Magnitude condition can be used to determine the value of K for a point s_0 on the root locus

❖ $K = \frac{1}{|G(s)||H(s)|} = \frac{\prod(\text{lengths from } s_0 \text{ to OLTF poles})}{\prod(\text{lengths from } s_0 \text{ to OLTF zeros})} = \frac{L_{p1}L_{p2}\ldots L_{pn}}{L_{z1}L_{z2}\ldots L z_m}$

2) The angle condition is used to determine if a point s_0 lies on the root locus

❖ $\angle KG(s)H(s) = 180^0(2k+1) = \sum \emptyset_{oz} - \sum \emptyset_{op} = (\emptyset_{oz1} + \cdots + \emptyset_{ozm}) - (\emptyset_{op1} \ldots + \emptyset_{opn})$ where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$



- If there is no OLTF zero
  $K = \prod(\text{lengths from } s_0 \text{ to poles})$
- Angles are measured from the OLTF poles or zeros to s_0 with reference to the positive x-axis (real axis)

Refining the sketch

- The angle of departure of a branch from a complex open-loop pole s_0 can be found using:

$$\psi_{depart} = \sum \phi_{oz} - \sum \phi_{op} - 180^0$$

- The angle of arrival of a branch to a complex open-loop zero s_0 can be found using:

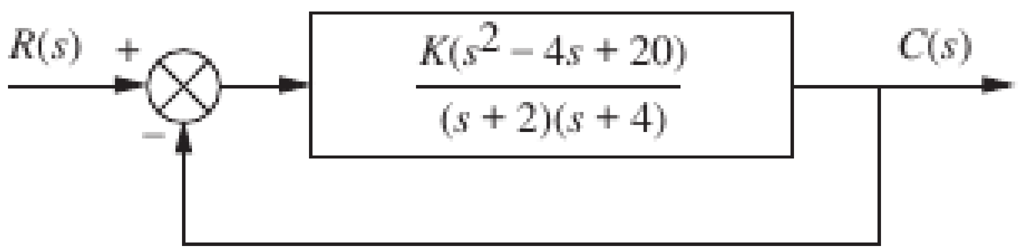
$$\psi_{arrive} = \sum \phi_{op} - \sum \phi_{oz} + 180^0$$

- Points "*w*" at which branches cross the imaginary axis are found by putting "j*w*" for "*s*" in the characteristics equation and solving for "*K*", and "*w*" through separating the complex function into its real and imaginary parts

Example 3

Sketch the root locus for the system shown and find the following:

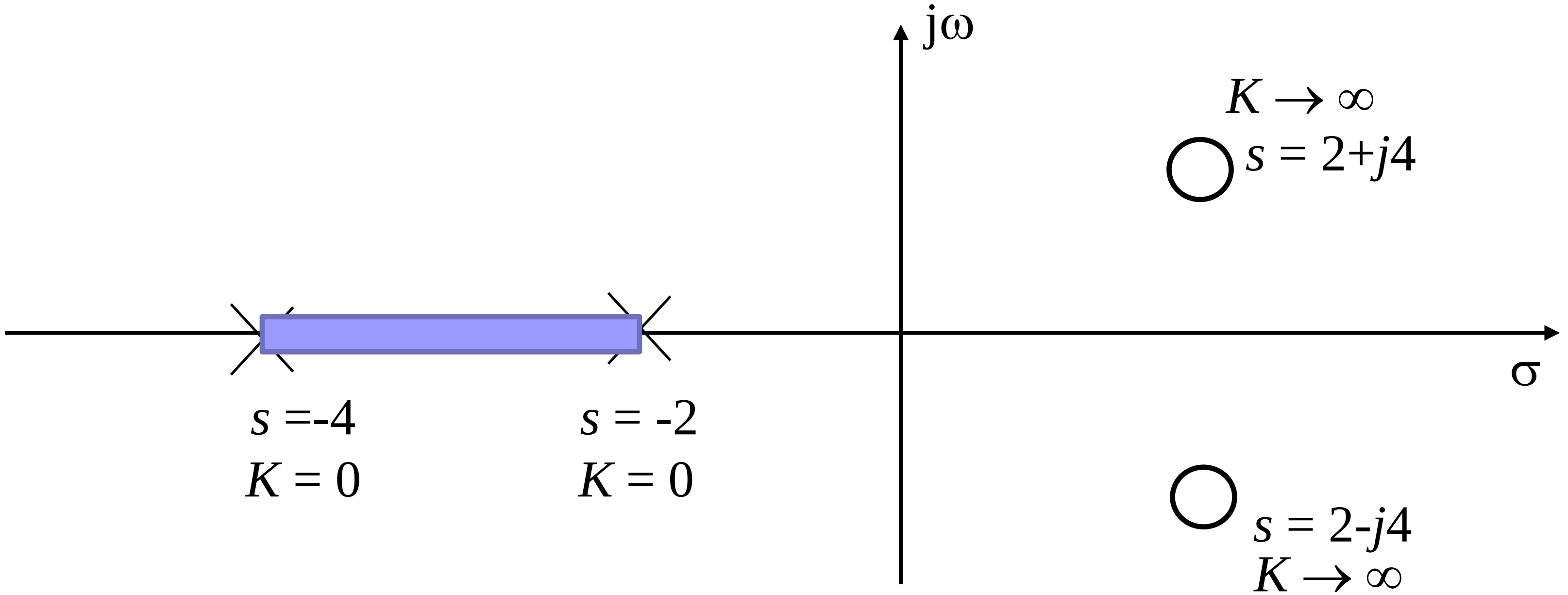
a) The exact point and gain where the locus crosses the jω-axis
b) The breakaway point on the real axis
c) The range of K within which the system is stable
d) The angle of arrival to the complex zeros



Step 1: Locate the open-loop poles and zeros in s-plane

- Note: n = number of open loop poles = 2 (i.e. 2 branches)
- Two poles at s = -2 and s = -4 (2 branches start from K = 0 at these points
- Note: m=number of open loop zeros = 2; (there are $(n - m = 0)$ asymptotes as $K \to \infty$); 2 complex zeros at $s = 2 \pm j4$

Example 3



Step 2: Determine the root loci occupying the real axis

- It occupies those parts of the real axis which have an odd number of real poles and zeroes to the right (i.e. it will occupy the real axis from -2 to -4)

Note: characteristics equation is $(s+2)(s+4)+K(s^2-4s+20)=0$ or

$$K = \frac{-s^2-6s-8}{s^2-4s+20}$$

Example 3

Step 3: Determine the asymptotes of the root loci (if any) – no asymptotes

Step 4: Locate the breakaway or break-in points (if any)

- ❖ It is symmetrical about the real axis; when the real axis between two adjacent OLTF poles is occupied by the locus, breakaway will occur
- ❖ Get characteristic equation, differentiate, equate to zero to solve for *s*:

$$\frac{\partial K}{\partial s} = \frac{\partial}{\partial s} f(s) = \frac{\partial}{\partial s}\left(\frac{-s^2-6s-8}{s^2-4s+20}\right) = \frac{(s^2-4s+20)(-2s-6)-(-s^2-6s-8)(2s-4)}{(s^2-4s+20)^2} = 0$$

Or $10s^2 - 24s - 152 = 0$

Solve to get $s = -2.88$ or $s = 5.28$ (Note: breakaway must be between -2 and -4)

Therefore breakaway at $s = -2.88$

- ❖ To determine where the branches cross the jω-axis, substitute $s = jw$ into the characteristics equation: $(s + 2)(s + 4) + K(s^2 - 4s + 20) = 0$

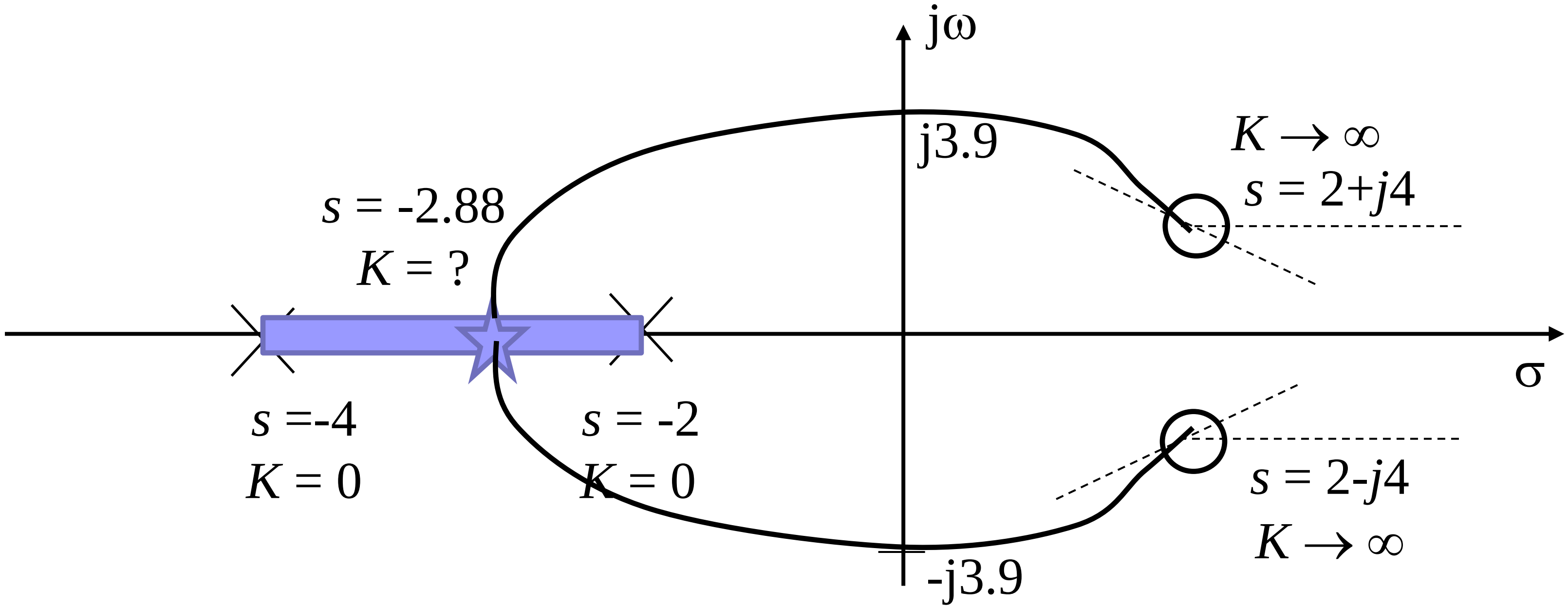
$$(jw + 2)(jw + 4) + K\big((jw)^2 - 4jw + 20\big) = 0$$

$$-w^2 - Kw^2 + 8 + 20K + j(6w - 4Kw) = 0$$

Imaginary part = 0: $K = 1.5$; Real part = 0: $2.5w^2 = 38$ or $w = \pm 3.9$

Note: system is stable for $0 \le K \le 1.5$

Example 3

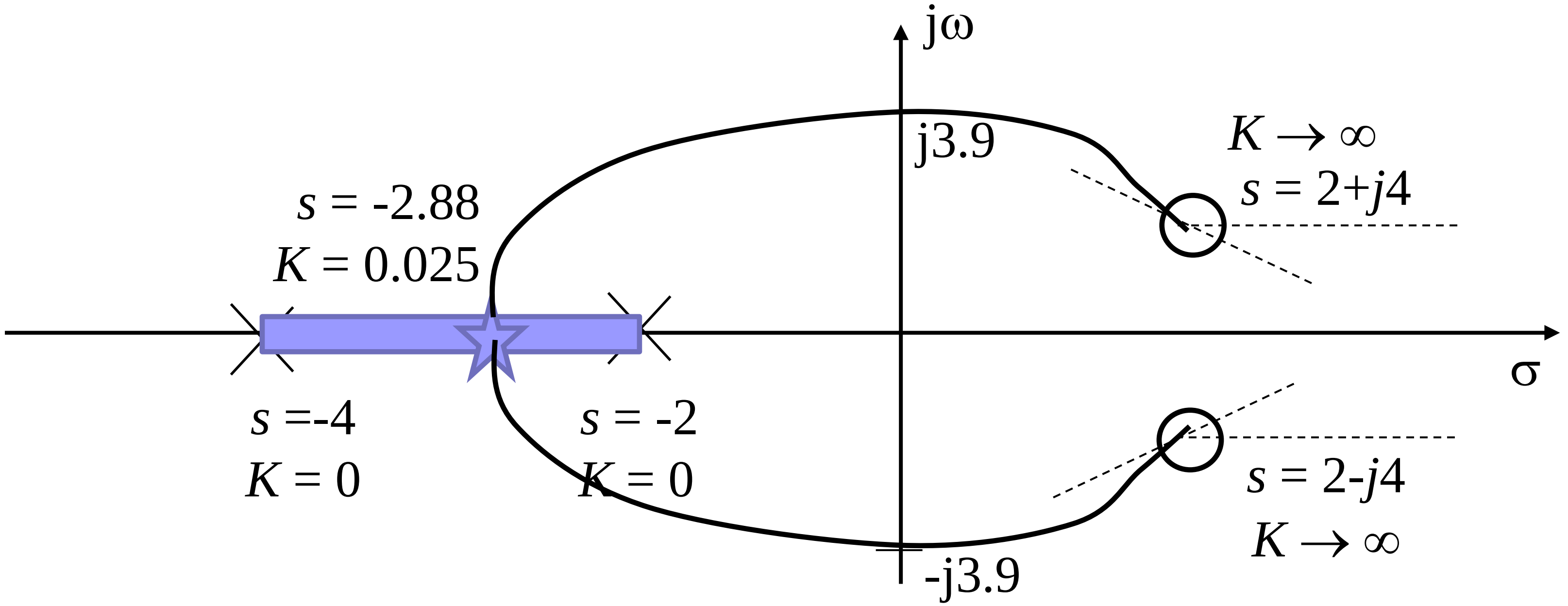


❖ The angle of arrival to the complex zeros can be found from

$$\psi_{arrive} = \sum \phi_{op} - \sum \phi_{oz} + 180^0 = \tan^{-1}(4/4) + \tan^{-1}(4/8) - 90^0 + 180^0$$

$$\psi_{arrive} = 161.6^0$$

Example 3



❖ To find the gain at the breakaway point $s = -2.88$, we can use

$$K = \frac{\prod(\text{lengths from } s_0 \text{ to OLTF poles})}{\prod(\text{lengths from } s_0 \text{ to OLTF zeros})} = \frac{(0.88)(1.12)}{(\sqrt{4^2+4.88^2})(\sqrt{4^2+4.88^2})} = 0.025$$