ME1020
Mechanical vibrations

Lecture 12

Multi DOF system vibration 4 (Damped vibration absorber)

Objectives

- Analyze the steady-state vibration of multi-DOF systems with damping to harmonic excitation
- Design of vibration absorber with damping

Forced vibration (2-DOF) damped

The equations of motion of a general 2-DOF damped system under harmonic external forces can be written in matrix form as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \sin \omega t$$

The steady-state solutions will have the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(\omega t) + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(\omega t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \omega \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \cos(\omega t) - \omega \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \sin(\omega t)$$

$$\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(\omega t) - \omega^2 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(\omega t)$$

- ❖ Substituting these back into the equations of motion
- ❖ Generate 4 equations by equating the coefficients of the sine and cosine terms for U_1, U_2, V_1, and V_2
- ❖ Solve the 4 equations for U_1, U_2, V_1, and V_2

Example 1

Determine the steady-state responses of the following system

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \sin 5t$$

The steady-state solutions will have the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(5t) + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(5t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = 5 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \cos(5t) - 5 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \sin(5t)$$

$$\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = -25 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(5t) - 25 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(5t)$$

- Substituting these back into the equations of motion

Example 1

$$\begin{bmatrix} -50 & 0 \\ 0 & -25 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(5t) + \begin{bmatrix} -50 & 0 \\ 0 & -25 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(5t)$$
$$+ \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \cos(5t) - \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \sin(5t)$$
$$+ \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sin(5t) + \begin{bmatrix} 300 & -200 \\ -200 & 400 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cos(5t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \sin 5t$$

- Collect all the sine and cosine terms for U_1, U_2, V_1, and V_2 in the 2 equations:

$$-50\sin(5t)U_1 - 50\cos(5t)V_1 + 150\cos(5t)U_1 - 100\cos(5t)U_2$$
$$-150\sin(5t)V_1 + 100\sin(5t)V_2 + 300\sin(5t)U_1 - 200\sin(5t)U_2$$
$$+300\cos(5t)V_1 - 200\cos(5t)V_2 = 2\sin 5t$$
$$-25\sin(5t)U_2 - 25\cos(5t)V_2 - 100\cos(5t)U_1 + 100\cos(5t)U_2$$
$$+100\sin(5t)V_1 - 100\sin(5t)V_2 - 200\sin(5t)U_1 + 400\sin(5t)U_2$$
$$-200\cos(5t)V_1 + 400\cos(5t)V_2 = 3\sin 5t$$

Example 1

- Simplify the 2 equations

$$250\sin(5t)U_1 - 200\sin(5t)U_2 - 150\sin(5t)V_1 + 100\sin(5t)V_2$$
$$+ 150\cos(5t)U_1 - 100\cos(5t)U_2 + 250\cos(5t)V_1 - 200\cos(5t)V_2$$
$$= 2\sin 5t$$
$$-200\sin(5t)U_1 + 375\sin(5t)U_2 + 100\sin(5t)V_1 - 100\sin(5t)V_2$$
$$- 100\cos(5t)U_1 + 100\cos(5t)U_2 - 200\cos(5t)V_1 + 375\cos(5t)V_2$$
$$= 3\sin 5t$$

- Collect all the sine and cosine terms for U_1, U_2, V_1, and V_2 into matrix and solve

$$\begin{bmatrix} 250 & -200 & -150 & 100 \\ 150 & -100 & 250 & -200 \\ -200 & 375 & 100 & -100 \\ -100 & 100 & -200 & 375 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Example 1

- The solution will give

$U_1 = 0.0212$, $U_2 = 0.0203$, $V_1 = -0.0077$, and $V_2 = -0.0039$

- The steady-state equation is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.0212 \\ 0.0203 \end{bmatrix} \sin(5t) + \begin{bmatrix} -0.0077 \\ -0.0039 \end{bmatrix} \cos(5t)$$

- Note that the sine and cosine terms can be combined to give

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.0225 \sin(5t + 0.348) \\ 0.0207 \sin(5t + 0.188) \end{bmatrix}$$

Forced vibration analysis

We can use complex algebra to represent the harmonic external forces as:

$F_i(t) = F_{i0}e^{j\omega t}$ for $i = 1, 2$ and ω is the forcing frequency

- For the 2-DOF damped system:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} e^{j\omega t}$$

- The steady state response will have the form $x_i(t) = X_i e^{j\omega t}$
- Note that $\dot{x}_i(t) = j\omega X_i e^{j\omega t}$ and $\ddot{x}_i(t) = -\omega^2 X_i e^{j\omega t}$
- Substituting this back into the equations of motion:

$$-\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} + j\omega \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} e^{j\omega t}$$

$$\begin{bmatrix} -\omega^2 m_{11} + j\omega c_{11} + k_{11} & -\omega^2 m_{12} + j\omega c_{12} + k_{12} \\ -\omega^2 m_{21} + j\omega c_{21} + k_{21} & -\omega^2 m_{22} + j\omega c_{22} + k_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Forced vibration analysis

The equation can be rewritten as:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \Leftrightarrow [Z(j\omega)] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

- Where the mechanical impedance is defined as (for $r, s = 1,2$)

$$Z_{rs}(j\omega) = -\omega^2 m_{rs} + j\omega c_{rs} + k_{rs}$$

- The inverse of the impedance matrix is given as

$$[Z(j\omega)]^{-1} = \frac{1}{Z_{11}Z_{22} - Z_{21}Z_{12}} \begin{bmatrix} Z_{22} & -Z_{21} \\ -Z_{12} & Z_{11} \end{bmatrix}$$

- We can solve for

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(j\omega)]^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Forced vibration analysis

Note that the frequency response function is defined as:

$$[H(j\omega)] = [Z(j\omega)]^{-1}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(j\omega)]^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} = [H(j\omega)] \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

❖ The four frequency response functions are

$$\frac{X_1}{F_{10}} = |H_{11}| = \left| \frac{k_{22} - \omega^2 m_{22} + j\omega c_{22}}{\det([Z(\omega)]} \right|$$

$$\frac{X_1}{F_{20}} = |H_{12}| = \left| \frac{-k_{12} + \omega^2 m_{12} - j\omega c_{12}}{\det([Z(\omega)]} \right|$$

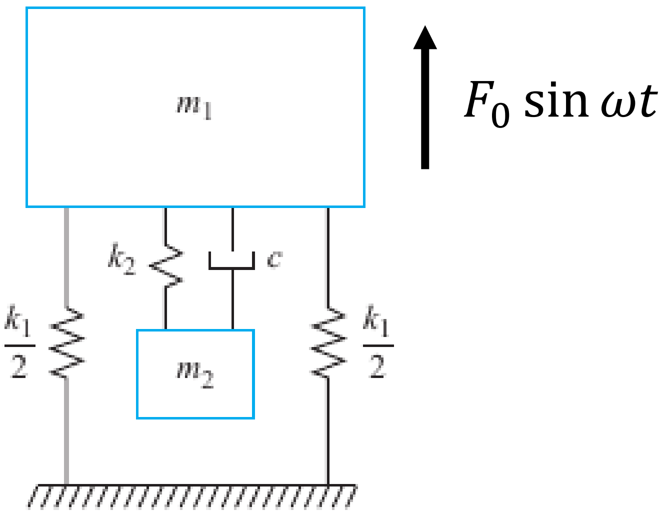
$$\frac{X_2}{F_{10}} = |H_{21}| = \left| \frac{-k_{21} + \omega^2 m_{21} - j\omega c_{21}}{\det([Z(\omega)]} \right|$$

$$\frac{X_2}{F_{20}} = |H_{22}| = \left| \frac{k_{11} - \omega^2 m_{11} + j\omega c_{11}}{\det([Z(\omega)]} \right|$$

Example 2

Determine the amplitudes of the steady-state response of mass m_1 of the following system

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t$$



Example 2

- The matrix equation can be rewritten as

$$[Z(j\omega)]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

- The impedance matrix is (for $r, s = 1{,}2$) $Z_{rs}(j\omega) = -\omega^2 m_{rs} + j\omega c_{rs} + k_{rs}$

$$[Z(j\omega)] = \begin{bmatrix} -\omega^2 m_1 + j\omega c + k_1 + k_2 & -j\omega c - k_2 \\ -j\omega c - k_2 & -\omega^2 m_2 + j\omega c + k_2 \end{bmatrix}$$

$$[Z(j\omega)] = \begin{bmatrix} k_1 + k_2 - \omega^2 m_1 + j\omega c & -k_2 - j\omega c \\ -k_2 - j\omega c & k_2 - \omega^2 m_2 + j\omega c \end{bmatrix}$$

$$\det[Z(j\omega)] = Z_{11}Z_{22} - Z_{21}Z_{12}$$

$$\det[Z(j\omega)] = \{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2\} + j\omega c\{k_1 - m_1\omega^2 - m_2\omega^2\}$$

Example 2

$$\begin{bmatrix} Z_{22} & -Z_{21} \\ -Z_{12} & Z_{11} \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \begin{bmatrix} (k_2 - \omega^2 m_2 + j\omega c)F_0 \\ k_2 + j\omega c \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(j\omega)]^{-1} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{1}{Z_{11}Z_{22} - Z_{21}Z_{12}} \begin{bmatrix} Z_{22} & -Z_{21} \\ -Z_{12} & Z_{11} \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

- Solving for amplitude X_1

$$X_1 = \frac{(k_2 - \omega^2 m_2 + j\omega c)F_0}{\{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2\} + j\omega c\{k_1 - m_1\omega^2 - m_2\omega^2\}}$$

- The analysis is equivalent to solving the frequency response function

$$\frac{X_1}{F_{10}} = |H_{11}| = \left| \frac{k_{22} - \omega^2 m_{22} + j\omega c_{22}}{\det([Z(\omega)]} \right|$$

Example 2

$$\frac{X_1}{F_0} = \frac{(k_2 - \omega^2 m_2 + j\omega c)}{\{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2\} + j\omega c\{k_1 - m_1\omega^2 - m_2\omega^2\}}$$

- Note that this is a complex expression of the form

$$\frac{X_1}{F_0} = \frac{A + jB}{C + jD} = \frac{A + jB}{C + jD}\left\{\frac{C - jD}{C - jD}\right\} = \frac{(AC + BD) + j(BC - AD)}{C^2 + D^2}$$

$$\left|\frac{X_1}{F_0}\right| = \sqrt{\left(\frac{AC + BD}{C^2 + D^2}\right)^2 + \left(\frac{BC - AD}{C^2 + D^2}\right)^2} = \sqrt{\frac{A^2C^2 + B^2D^2 + B^2C^2 + A^2D^2}{(C^2 + D^2)^2}}$$

$$\left|\frac{X_1}{F_0}\right| = \sqrt{\frac{(A^2+B^2)(C^2 + D^2)}{(C^2 + D^2)^2}} = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}}$$

Example 2

- Hence for

$$\frac{X_1}{F_0} = \frac{(k_2 - \omega^2 m_2 + j\omega c)}{\{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2\} + j\omega c\{k_1 - m_1\omega^2 - m_2\omega^2\}}$$

- Amplitude ratio

$$\left|\frac{X_1}{F_0}\right| = \sqrt{\frac{(k_2 - \omega^2 m_2)^2 + \omega^2 c^2}{\{(-m_1\omega^2 + k_1)(-m_2\omega^2 + k_2) - m_2\omega^2 k_2\}^2 + \omega^2 c^2\{k_1 - m_1\omega^2 - m_2\omega^2\}^2}}$$

- The amplitude of the steady-state response of mass m_1 to $F_0 \sin \omega t$ is X_1

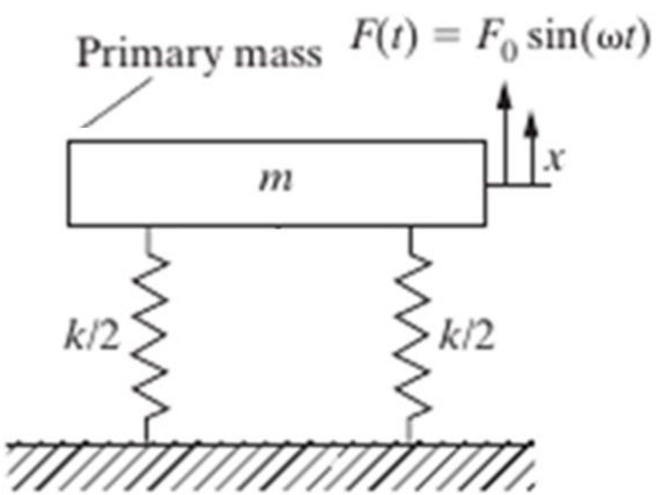
Vibration absorber (damped)

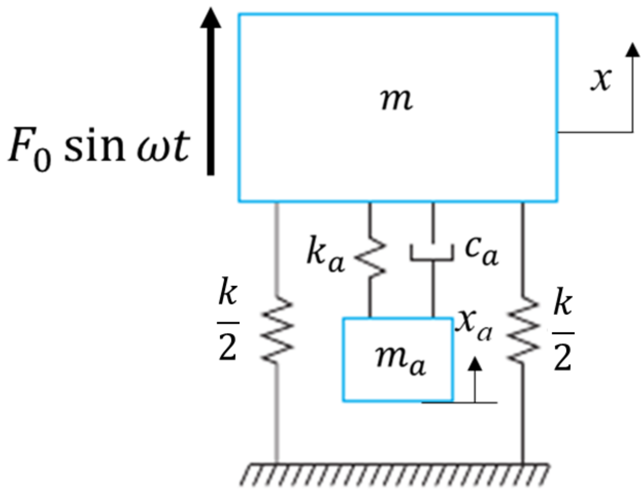
- ❖ Two problems exist when a undamped vibration absorber is used

a) The lowest natural frequency of the two degree-of-freedom system must be passed through in order to build up to the operating speed

b) If the absorber is slightly mistuned, the vibration amplitude of the primary system can be large

- ❖ Consider a primary undamped system before and after adding an absorber with damping as shown:

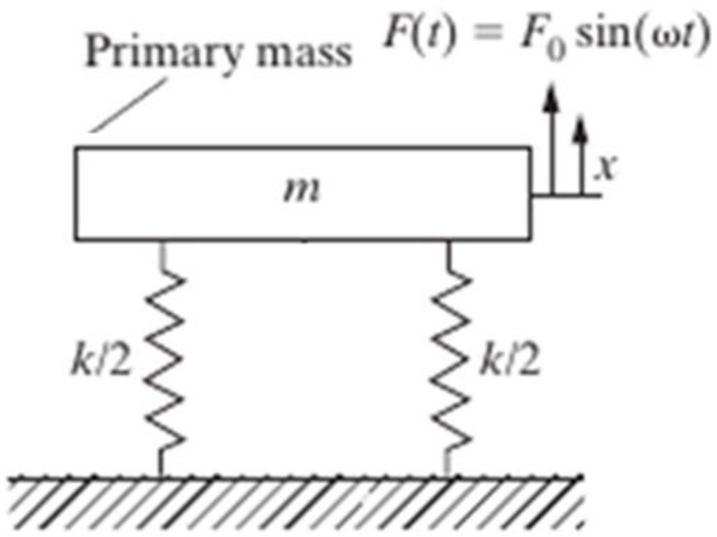




Vibration absorber (damped)

- The basic principle is to attached a second mass-spring-damper to the primary spring-mass system which is subjected to harmonic excitation
- The secondary mass-spring-damper system is designed to absorb the vibration of the primary system at the specified frequency
- The critical frequency of the primary system is usually its resonance frequency. In this case, the vibration of the primary system can be reduced to zero at its resonance frequency for properly designed absorber. This means that the energy of the primary system at resonance is "absorbed" by the tuned dynamic absorber
- The design of the absorber involves determining the mass, damper and spring constant of the secondary system, i.e. m_2, c_2, k_2

Primary system

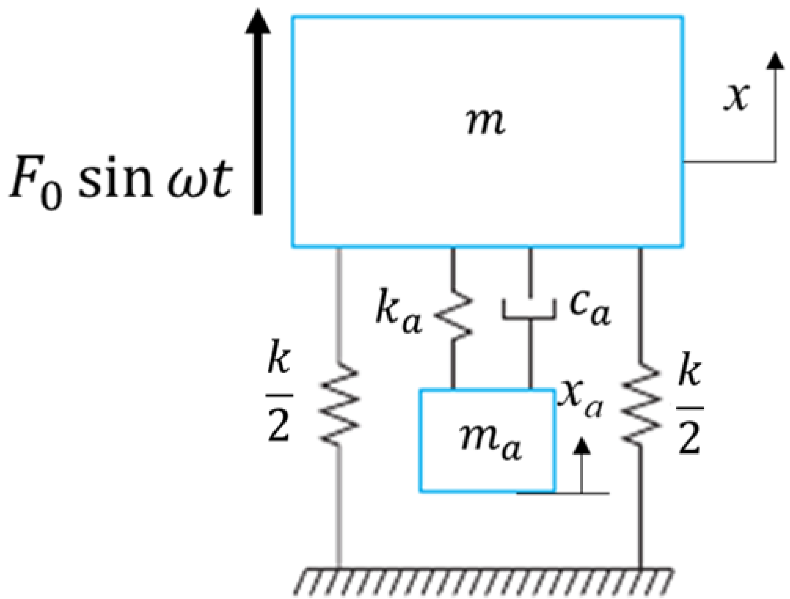


The primary system model is $m\ddot{x} + kx = F_o \sin(\omega t)$

The natural frequency is $\omega_p = \sqrt{k/m}$

When harmonic excitation frequency $\omega = \omega_p$, resonance will occur resulting in large amplitude of vibration

Vibration absorber (damped)



The system model with the absorber in matrix form is

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} c_a & -c_a \\ -c_a & c_a \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} + \begin{bmatrix} (k + k_a) & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix}$$

Vibration absorber (damped)

❖ The amplitude ratio of this system was derived in example 2:

$$\left|\frac{X}{F_0}\right| = \sqrt{\frac{(k_a-\omega^2 m_a)^2+\omega^2 c_a^2}{\{(-m\omega^2+k)(-m_a\omega^2+k_a)-m_a\omega^2 k_a\}^2+\omega^2 c_a^2\{k-m\omega^2-m_a\omega^2\}^2}}$$

$$\left|\frac{Xk}{F_0}\right| = \sqrt{\frac{\left(\frac{k_a}{k}-\frac{\omega^2 m_a}{k}\right)^2+\left(\frac{\omega c_a}{k}\right)^2}{\left\{\frac{(k-m\omega^2)(k_a-m_a\omega^2)}{k^2}-\frac{m_a\omega^2 k_a}{k^2}\right\}^2+\left\{\frac{\omega c_a(k-\omega^2(m+m_a))}{k^2}\right\}^2}}$$

❖ The natural frequency of the primary system without absorber is $\omega_p = \sqrt{k/m}$

❖ The natural frequency of the stand-alone absorber is $\omega_a = \sqrt{k_a/m_a}$

❖ The combined 2-DOF system with absorber will have two natural frequencies ω_1, ω_2 which are different from ω_p and ω_a; let $r_1 = \omega/\omega_p$ and $r_2 = \omega/\omega_a$

❖ Define the frequency ratio as $\beta = \omega_a/\omega_p$ and mass ratio as $\mu = m_a/m$

❖ Define $\zeta = \frac{c_a}{2m_a\omega_p}$

Vibration absorber (damped)

- $\frac{k_a}{k} = \frac{k_a}{m_a}\frac{m_a}{m}\frac{m}{k} = \left(\frac{\omega_a}{\omega_p}\right)^2 \mu = \mu\beta^2$
- $\frac{\omega^2 m_a}{k} = \frac{\omega^2 m}{k}\frac{m_a}{m} = \mu r_1^2$
- $\frac{\omega c_a}{k} = \frac{2c_a\omega_p}{2m_a\omega_p}\frac{m_a}{m}\frac{m\omega}{k} = 2\zeta\mu r_1$
- $\frac{(k-m\omega^2)(k_a-m_a\omega^2)}{k^2} = (1-r_1^2)\left(\frac{k_a}{k}-\frac{m_a\omega^2}{k}\right) = (1-r_1^2)(\beta^2-r_1^2)\mu$
- $\frac{\omega c_a(k-\omega^2(m+m_a))}{k^2} = \left(\frac{\omega c_a}{k}\right)\left(1-\frac{\omega^2}{\omega_p^2}-\frac{\omega^2}{k}\frac{mm_a}{m}\right) = 2\zeta\mu r_1(1-r_1^2-r_1^2\mu)$
- $\frac{m_a\omega^2 k_a}{k^2} = \frac{\omega^2 k_a}{k}\frac{m}{k}\frac{m_a}{m} = r_1^2\beta^2\mu^2$

Vibration absorber (damped)

- Substituting the terms back into the equation

$$\left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(\mu\beta^2 - \mu r_1^2)^2 + (2\zeta\mu r_1)^2}{\{(1 - r_1^2)(\beta^2 - r_1^2)\mu - r_1^2\beta^2\mu^2\}^2 + \{2\zeta\mu r_1(1 - r_1^2 - r_1^2\mu)\}^2}}$$

$$\left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(\beta^2 - r_1^2)^2 + (2\zeta r_1)^2}{\{(1 - r_1^2)(\beta^2 - r_1^2) - r_1^2\beta^2\mu\}^2 + \{2\zeta r_1(1 - r_1^2 - r_1^2\mu)\}^2}}$$

- Case 1: for tuned case for $\beta = 1$ or $\omega_a = \omega_p$, the amplitude X can be minimized respect to r_1 to obtain

$$\zeta^2 = \frac{\mu(\mu + 3)\left(1 + \sqrt{\mu/(\mu + 2)}\right)}{8(1 + \mu)}$$

Vibration absorber (damped)

- Substituting the terms back into the equation

$$\left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(\mu\beta^2 - \mu r_1^2)^2 + (2\zeta\mu r_1)^2}{\{(1 - r_1^2)(\beta^2 - r_1^2)\mu - r_1^2\beta^2\mu^2\}^2 + \{2\zeta\mu r_1(1 - r_1^2 - r_1^2\mu)\}^2}}$$

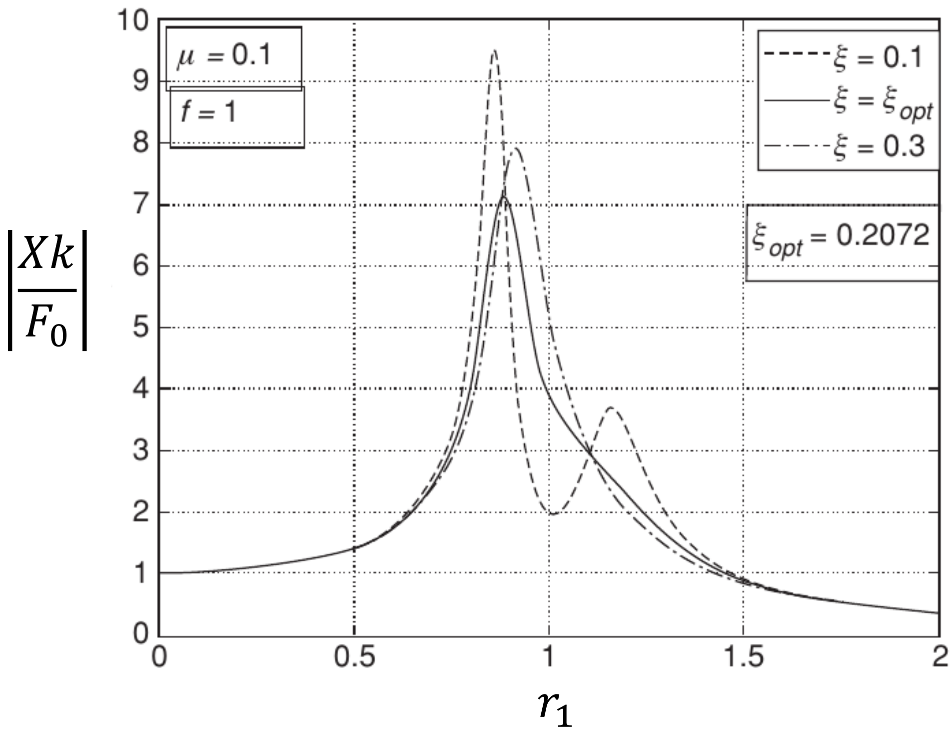
$$\left|\frac{Xk}{F_0}\right| = \sqrt{\frac{(\beta^2 - r_1^2)^2 + (2\zeta r_1)^2}{\{(1 - r_1^2)(\beta^2 - r_1^2) - r_1^2\beta^2\mu\}^2 + \{2\zeta r_1(1 - r_1^2 - r_1^2\mu)\}^2}}$$

- Note that the design for an absorber with damping would involve optimization of the amplitude with respect to some objective function

Absorber design (damped)

- Case 1: for tuned case for $\beta = 1$ or $\omega_a = \omega_p$, the amplitude X can be minimized respect to r_1 to obtain

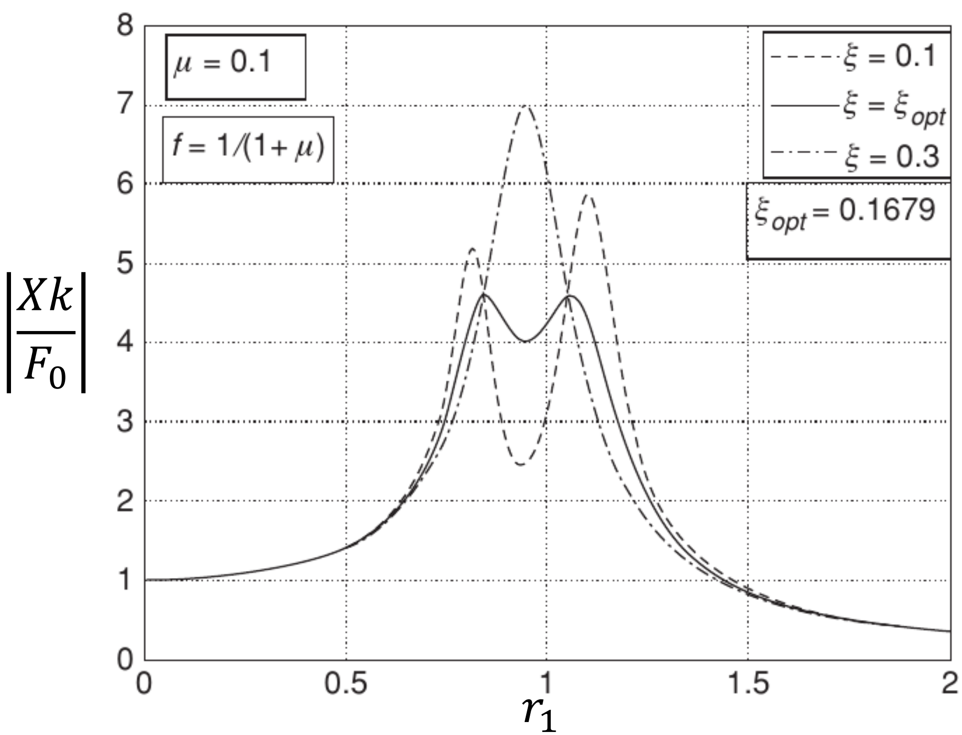
$$\zeta^2 = \frac{\mu(\mu+3)\left(1+\sqrt{\mu/(\mu+2)}\right)}{8(1+\mu)}$$



Absorber design (damped)

❖ Case 2: absorber not tuned to the primary system. The objective function to be minimized is $\frac{F_0}{k}\sqrt{1+2/\mu}$; The minimal occurs at $\beta = \frac{1}{1+\mu}$ where

$$\zeta^2 = \frac{3\mu}{8(1+\mu)^3}$$



Absorber design (damped)

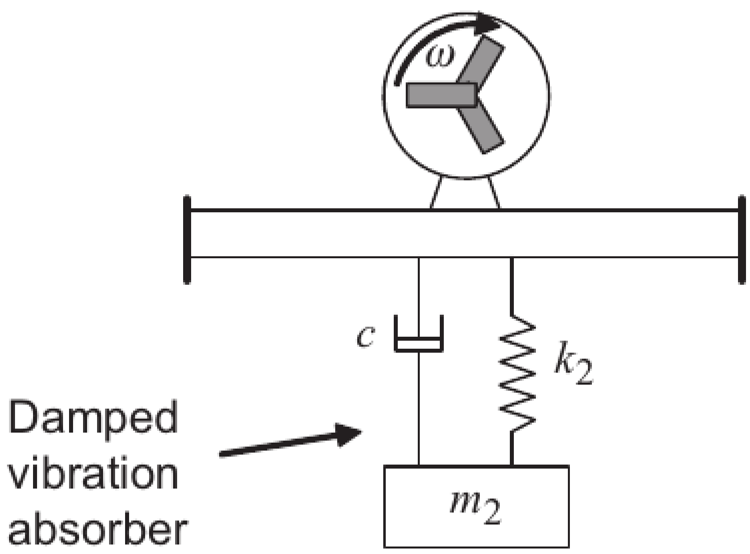
Note that for case 2, the operating range requires that the two peaks have approximately the same amplitude. The 2 peaks would occur at

$$r_1 = \sqrt{\frac{1 + (1+\mu)\beta^2 \pm \sqrt{1 - 2\beta^2 + (1+\mu)^2\beta^4}}{2+\mu}}$$

Case 2 is the preferred method if a wide operating range is needed

Example 3

When a fan with 1,000 kg mass operates at a speed of 2,400 rpm on the roof of a room, there is a large amount of vibration. Design an optimally damped vibration absorber with the mass ratio equal to 0.2025 (assume that the absorber not tuned to the primary system)



Example 3

Note the primary system is the fan with $\omega_p = 2400\frac{2\pi}{60} = 8\pi$ rad/s

Given mass ratio $\mu = 0.2$

For observer not tuned to the primary system, frequency ratio

$$\beta = \omega_a/\omega_p = \frac{1}{1+\mu} = 0.83$$

Note that

$$\frac{k_a}{m_a} = \omega_a^2 = \beta^2\omega_p^2 = (0.83 \times 8\pi)^2$$

The mass of the fan is 1000 kg

The mass of the absorber is $m_a = \mu m = 200$ kg

The absorber should have a spring with stiffness $k_a = 8.85(10^6)$ N/m

$$\zeta^2 = \frac{3\mu}{8(1+\mu)^3} \text{ or } \zeta = 0.21$$

The absorber damping coefficient is $c_a = 2\zeta m_a\omega_p = 2.1(10^4)$ Ns/m