MEMS1045 Automatic control

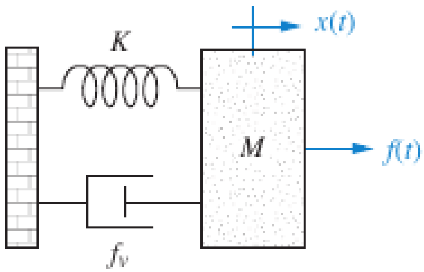
Lecture 3

Modeling

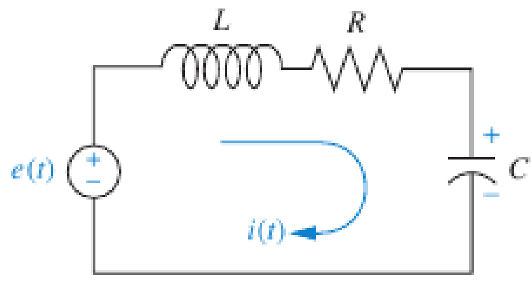
Objectives

- Describe analogous modeling between electrical and mechanical systems
- Derive the equations of motion for electromechanical systems
- Represent the equations of motion in matrix form using state variables
- Apply linearization to approximate the dynamics for small perturbations about operating conditions

Analogous systems



(a)



(b)

Charge = q

$$q = \frac{di}{dt}$$

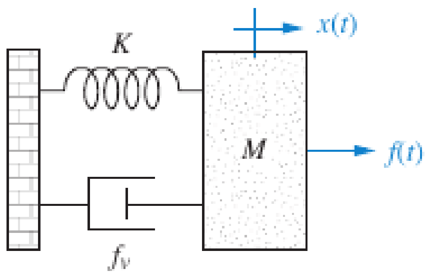
$$m\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} + kx = f(t)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t)$$

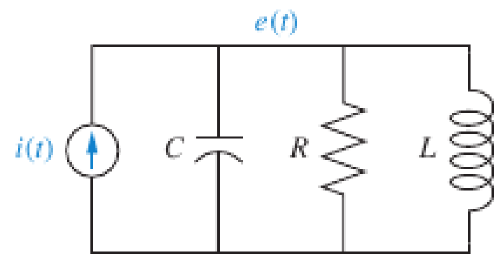
Force–Voltage Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Voltage e
Mass m (moment of inertia J)	Inductance L
Viscous-friction coefficient b	Resistance R
Spring constant k	Reciprocal of capacitance, $1/C$
Displacement x (angular displacement θ)	Charge q
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Current i

Analogous systems



(a)



(b)

Magnetic flux = ψ

$$\frac{d\psi}{dt} = e$$

$$m\frac{d^2x}{dt^2} + f_v\frac{dx}{dt} + kx = f(t)$$

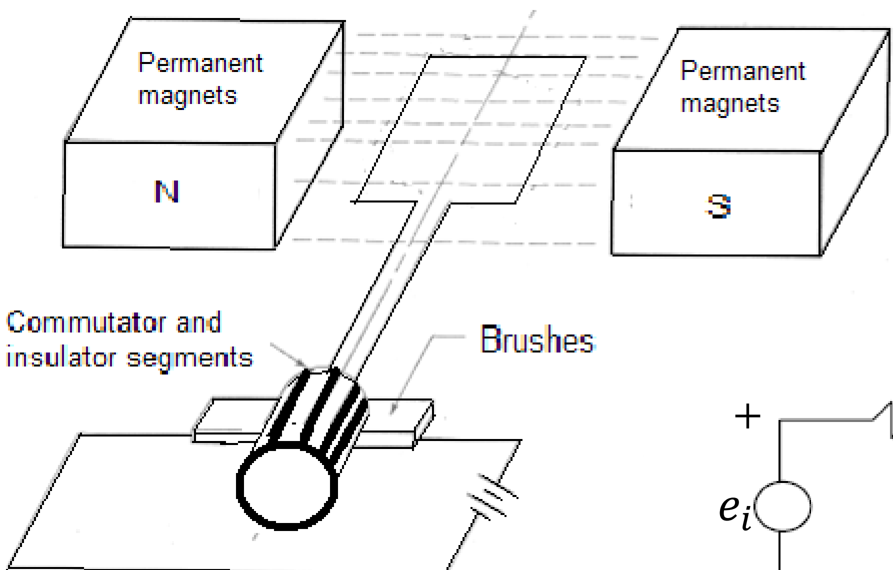
$$C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi = i(t)$$

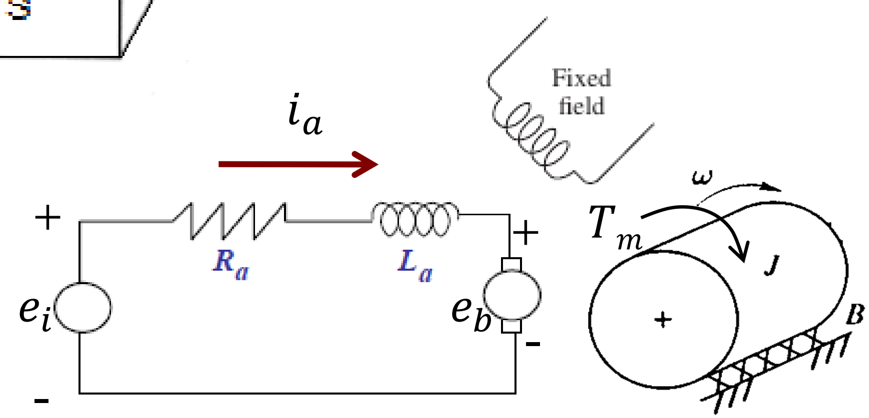
Force–Current Analogy

Mechanical Systems	Electrical Systems
Force p (torque T)	Current i
Mass m (moment of inertia J)	Capacitance C
Viscous-friction coefficient b	Reciprocal of resistance, $1/R$
Spring constant k	Reciprocal of inductance, $1/L$
Displacement x (angular displacement θ)	Magnetic flux linkage ψ
Velocity $\dot{x}$ (angular velocity $\dot{\theta}$)	Voltage e

DC motor

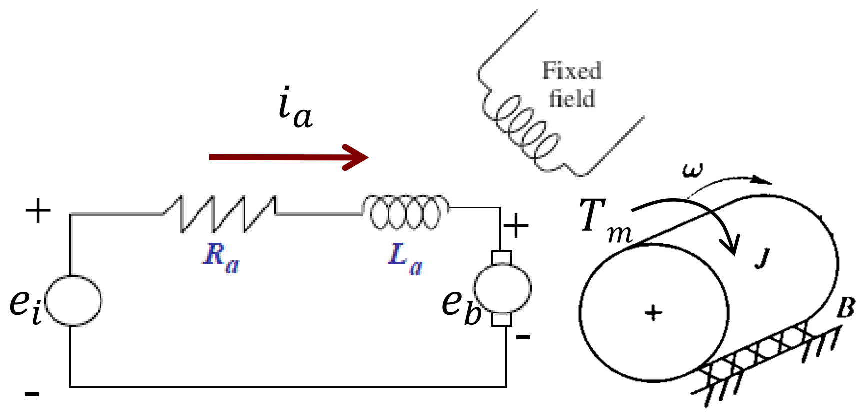
Simplified diagram for permanent DC motor analysis:





- R_a= armature resistance
- L_a= armature inductance
- T_m= motor torque

DC motor



$$T_m = K_T i_a$$
$$T_m(s) = K_T I_a(s)$$
K_T= Torque constant

$$e_b = K_b \omega$$
$$E_b(s) = K_b s \theta_m(s)$$
K_b= Back EMF constant

$$e_i = i_a R_a + L_a \frac{di_a}{dt} + e_b$$
$$E_i(s) = R_a I_a(s) + L_a s I_a(s) + E_b(s)$$
$$E_i(s) = R_a I_a(s) + L_a s I_a(s) + K_b s \theta_m(s)$$
$$E_i(s) - K_b s \theta_m(s) = (R_a + L_a s) I_a(s)$$
$$I_a(s) = \frac{E_i(s)}{(R_a + L_a s)} - \frac{K_b s \theta_m(s)}{(R_a + L_a s)}$$

$$J\ddot{\theta}_m = -B\dot{\theta}_m + T_m$$
$$Js^2\theta_m(s) = -Bs\theta_m(s) + T_m(s)$$
$$Js^2\theta_m(s) = -Bs\theta_m(s) + K_T I_a(s)$$

DC motor

$$Js^2\theta_m(s) = -Bs\theta_m(s) + K_T I_a(s)$$

$$Js^2\theta_m(s) = -Bs\theta_m(s) + \frac{K_T E_i(s)}{(R_a + L_a s)} - \frac{K_b K_T s\theta_m(s)}{(R_a + L_a s)}$$

$$Js^2\theta_m(s) + Bs\theta_m(s) + \frac{K_b K_T s\theta_m(s)}{(R_a + L_a s)} = \frac{K_T E_i(s)}{(R_a + L_a s)}$$

For small L_a:

$$Js^2\theta_m(s) + Bs\theta_m(s) + \frac{K_b K_T s\theta_m(s)}{(R_a)} = \frac{K_T E_i(s)}{(R_a)}$$

$$s\left[s + \frac{1}{J}\left(B + \frac{K_b K_T}{(R_a)}\right)\right]\theta_m(s) = \frac{K_T}{J(R_a)} E_i(s)$$

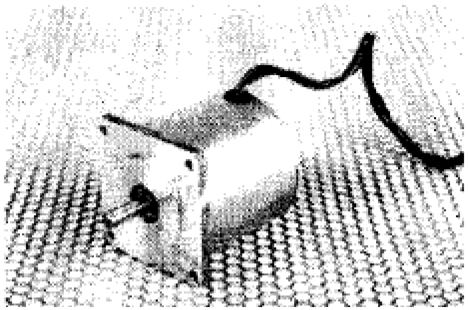
Motor transfer function

$$\frac{\theta_{m(s)}}{E_i(s)} = \frac{K_T/(R_a J)}{s\left[s + \frac{1}{J}\left(B + \frac{K_b K_T}{(R_a)}\right)\right]} = \frac{K_m}{s\left(s + \frac{1}{\tau_m}\right)}$$

K_m = motor gain constant

τ_m = motor time constant

Commercial DC motor



Brushless DC motor

SPECIFICATIONS

PARAMETER	UNITS	$\boxed{\text{Value}}$	
Torque Constant	oz-in/amp	1.65	K_T = 0.01164 Nm/A
Back EMF Constant	V/Krpm	1.22	K_b = 0.01164 Vs/rad
D.C. Resistance	ohms	0.12	
Inductance	mH	0.11	Very small inductance
Max Speed	rpm	15,000	
Cont. Stall Torque	oz-in	25	
Motor Constant	oz-in/sq.rt.W	5.22	
Max. Winding Temp.	Deg.C	155	

Commercial DC motor

For small L_a:

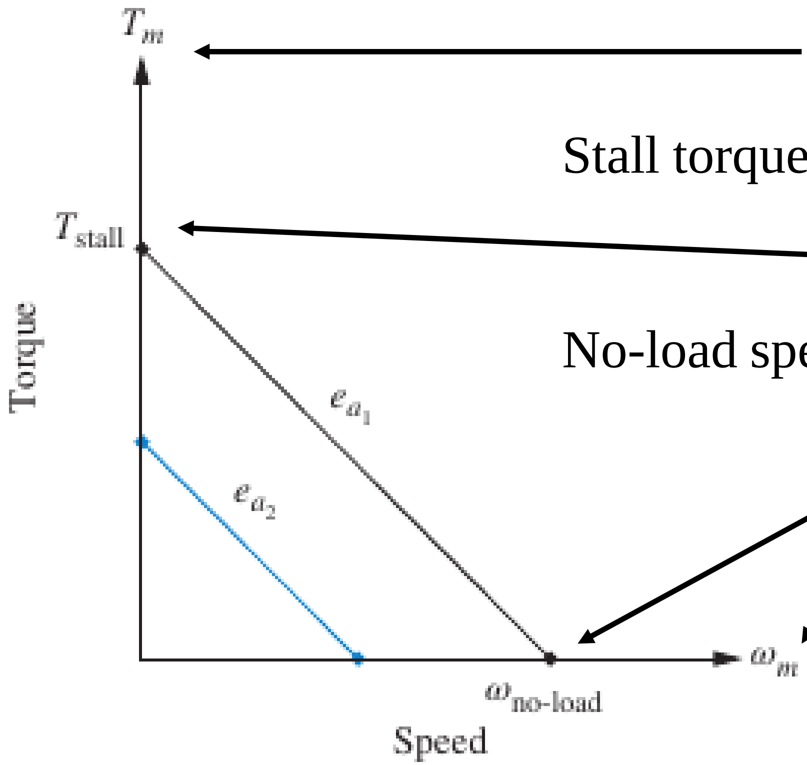
$$T_m = -\frac{K_b K_T}{R_a}\omega + \frac{K_T}{R_a}e_a$$

Stall torque:

$$T_{stall} = \frac{K_T}{R_a}e_a$$

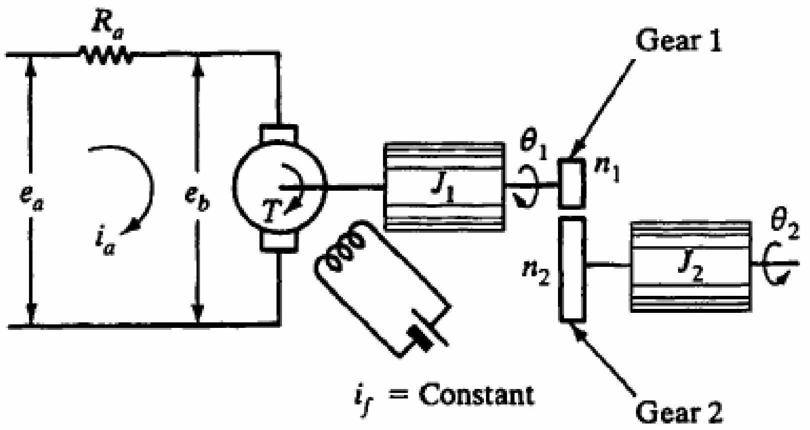
No-load speed:

$$\omega_{no-load} = \frac{e_a}{K_b}$$



Example 1

Consider the dc servomotor system shown. The armature inductance is negligible and is not shown in the circuit. Obtain the transfer function between the output θ_2 and the input e_a.



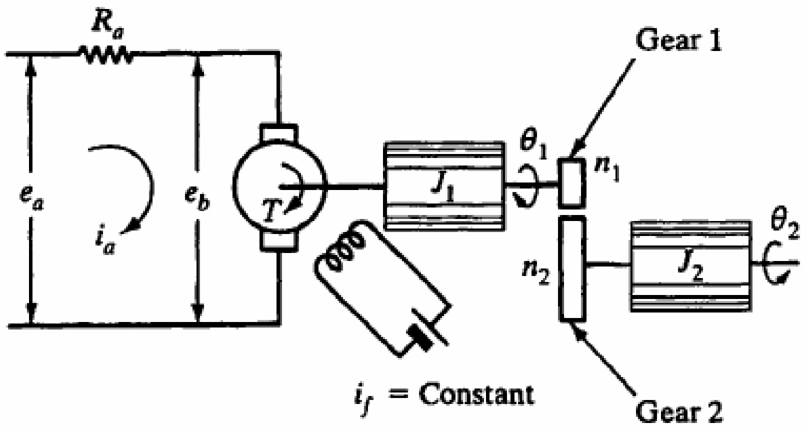
n_1 = number of teeth on gear 1
n_2 = number of teeth on gear 2

R_a = armature resistance,
i_a = armature current,
i_f = field current,
e_a = applied armature voltage, e_b = back emf,
θ_1 = angular displacement of the motor shaft,
θ_2 = angular displacement of the load element,
T = torque developed by the motor
J_1 = moment of inertia of motor rotor,
J_2 = moment of inertia of the load

Example 1

Torque developed by the motor is $T_m = K_T i_a$ or $T_m(s) = K_T I_a(\mathrm{s})$ where
K_T= Torque constant
The induced voltage e_b is $e_b = K_b\omega$ or $E_b(s) = K_b s\theta_1(s)$ where
K_b= Back EMF constant
The equation for the armature circuit is $e_a = i_a R_a + e_b$
$E_a(s) = R_a I_a(s) + E_b(s)$ or $E_a(s) = R_a I_a(s) + K_b s\theta_1(s)$

$$I_a(s) = \frac{E_a(s)}{R_a} - \frac{K_b s\theta_1(s)}{R_a}$$



Example 1

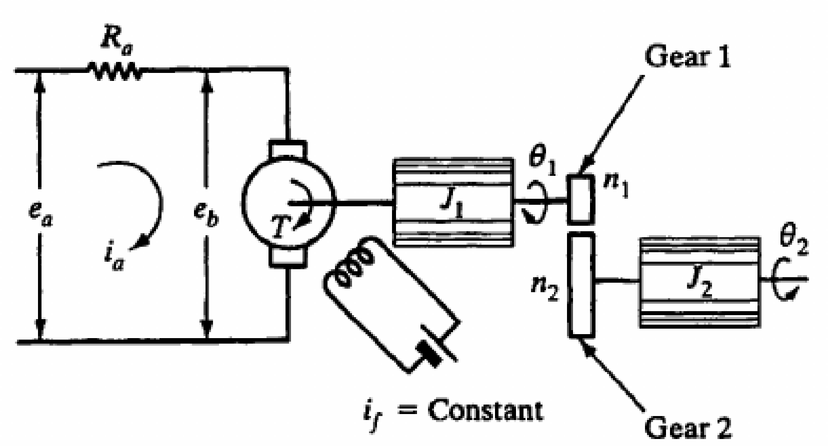
The total kinetic energy of the load inertias:

$$KE = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\left(\frac{\dot{\theta}_2^2}{\dot{\theta}_1^2}\right)\dot{\theta}_1^2$$

$$KE = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\left(\frac{n_1^2}{n_2^2}\right)\dot{\theta}_1^2 = \frac{1}{2}\left(J_1 + J_2\left(\frac{n_1}{n_2}\right)^2\right)\dot{\theta}_1^2$$

The equivalent moment of inertia of the motor rotor plus the load inertia referred to the motor shaft is

$$J_{1eq} = J_1 + N^2 J_2 = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2$$



Example 1

$$J_{1eq} = J_1 + N^2 J_2 = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2$$

The armature current produces the torque that is applied to the equivalent moment of inertia J_{1eq}. Thus

$$J_{1eq}\ddot{\theta}_1 = T_m = K_T i_a$$

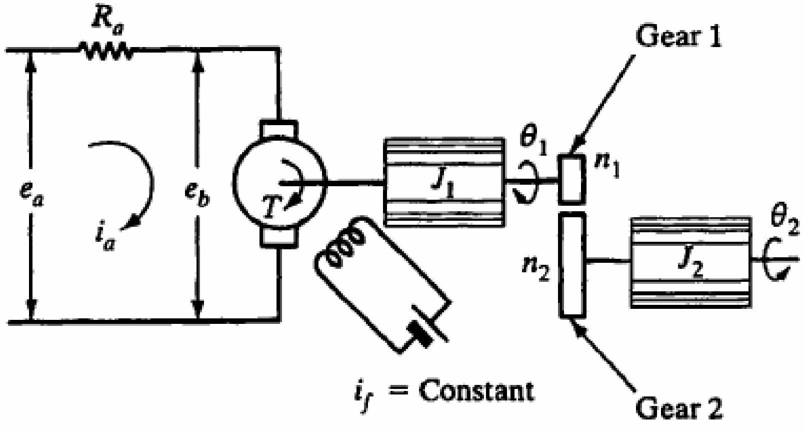
$$J_{1eq} s^2 \theta_1(s) = K_T I_a(s)$$

But $I_a(s) = \frac{E_a(s)}{R_a} - \frac{K_b s \theta_1(s)}{R_a}$ or $J_{1eq} s^2 \theta_1(s) = K_T \left(\frac{E_a(s)}{R_a} - \frac{K_b s \theta_1(s)}{R_a}\right)$

$$s\left(J_{1eq} s + \frac{K_T K_b}{R_a}\right)\theta_1(s) = \frac{K_T E_a(s)}{R_a}$$

$$s\left(J_{1eq} s + \frac{K_T K_b}{R_a}\right)\left(\frac{n_2}{n_1}\right)\theta_2(s) = \frac{K_T E_a(s)}{R_a}$$

$$\frac{\theta_2(s)}{E_a(s)} = \frac{\left(\frac{n_1}{n_2}\right)K_T}{s\left[R_a\left\{J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\right\}s + K_T K_b\right]}$$



State Variable Equations

- Transfer function relates the dynamic relationship between the input and output
- For a system with **many** inputs $u_1(t)$, $u_2(t)$, … and **many** outputs y_1, y_2, …, we need to reduce a set of differential equations to a transfer function for each input and output pair
- An alternative is to represent the system using the state space approach
- A system of linear state equations (i.e. with n state variables, m inputs, and p outputs) can be expressed in a general matrix form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

where $\boldsymbol{x}$ = ($n\times1$) state vector; $\dot{\boldsymbol{x}}$ = derivative of the ($n\times1$) state vector w.r.t. time;
$\boldsymbol{u}$ = ($m\times1$) input vector; $\boldsymbol{y}$ = ($p\times1$) output vector;
$\boldsymbol{A}$ = ($n \times n$) system matrix; $\boldsymbol{B}$ = ($n \times m$) input matrix;
$\boldsymbol{C}$ = ($p \times n$) output matrix; $\boldsymbol{D}$ = ($p \times m$) feedforward matrix;

State Variable Equations

- State variables: The smallest set of linearly independent system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t > t_0$
- Typically, the minimum number required equals the order of the differential equation describing the system
- Example: $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$

  To solve the ODE we need to know the values of the coefficients, the inputs and the 2 initial conditions/states at $y(t_0)$ and $\frac{dy}{dt}(t_0)$. Hence there are 2 state variables and they can be y and $\frac{dy}{dt}$
- For a nth-order differential equation, the state variables (i.e. $x_1, x_2, \ldots x_n$) reduce the nth-order differential equation into a set of first order ODEs

Example 2

Consider the ODE: $m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$. Obtain a state representation of the system with output y

Define state variables as $x_1 = y$ and $x_2 = \frac{dy}{dt}$

Derivatives of state variables $\dot{x}_1 = \frac{dy}{dt} = x_2$

and $\dot{x}_2 = \frac{d^2y}{dt^2} = \frac{1}{m}u - \frac{b}{m}\frac{dy}{dt} - \frac{k}{m}y = \frac{1}{m}u - \frac{b}{m}x_2 - \frac{k}{m}x_1$

State equation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}[u]$$

Output equation:

$$[y] = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0][u]$$

Example 3

Find the state-space representation in phase-variable form for the transfer function

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

Rewrite the transfer function as

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r(t)$$

Define state variables as $x_1 = c; x_2 = \dot{c}; x_3 = \ddot{c}$

Derivatives of state variables $\dot{x}_1 = \dot{c} = x_2, \dot{x}_2 = \ddot{c} = x_3$

and $\dot{x}_3 = \dddot{c} = 24r - 9\ddot{c} - 26\dot{c} - 24c = 24r - 9x_3 - 26x_2 - 24x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} [r]$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][u]$$

State space to transfer function

Given the state and output equations (assuming zero initial conditions):

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu} \text{ or } \boldsymbol{sX}(\boldsymbol{s}) = \boldsymbol{AX}(\boldsymbol{s}) + \boldsymbol{BU}(\boldsymbol{s})$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du} \text{ or } \boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{CX}(\boldsymbol{s}) + \boldsymbol{DU}(\boldsymbol{s})$$

$$\boldsymbol{sX}(\boldsymbol{s}) = \boldsymbol{AX}(\boldsymbol{s}) + \boldsymbol{BU}(\boldsymbol{s})$$

$$[\boldsymbol{s} - \boldsymbol{A}]\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{BU}(\boldsymbol{s})$$

$$\boldsymbol{X}(\boldsymbol{s}) = [\boldsymbol{s} - \boldsymbol{A}]^{-\boldsymbol{1}}\boldsymbol{BU}(\boldsymbol{s})$$

$$\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{CX}(\boldsymbol{s}) + \boldsymbol{DU}(\boldsymbol{s}) = \boldsymbol{C}[\boldsymbol{s} - \boldsymbol{A}]^{-\boldsymbol{1}}\boldsymbol{BU}(\boldsymbol{s}) + \boldsymbol{DU}(\boldsymbol{s})$$

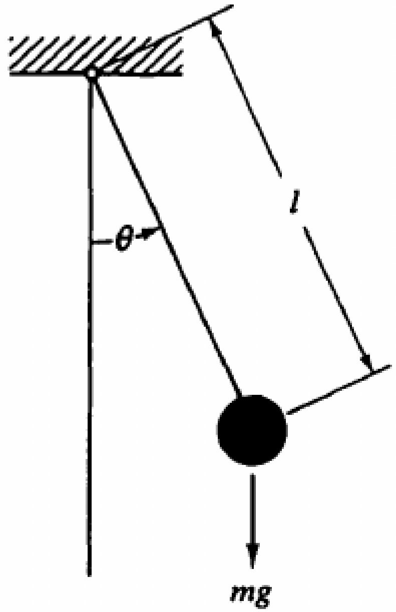
$$\boldsymbol{Y}(\boldsymbol{s}) = \left\{\boldsymbol{C}[\boldsymbol{s} - \boldsymbol{A}]^{-\boldsymbol{1}}\boldsymbol{B} + \boldsymbol{D}\right\}\boldsymbol{U}(\boldsymbol{s})$$

Hence

$$\frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{U}(\boldsymbol{s})} = \left\{\boldsymbol{C}[\boldsymbol{s} - \boldsymbol{A}]^{-\boldsymbol{1}}\boldsymbol{B} + \boldsymbol{D}\right\}$$

Linearization

- All systems have some non-linearities
- We usually linearized the dynamics for small perturbations about the operating conditions



EOM:

$$J\ddot{\theta} = mgL\sin\theta$$

where $J = mL^2$

For small θ, $\sin\theta \approx \theta$ and the EOM simplifies to

$$J\ddot{\theta} = mgL\theta$$

System and models