MEMS1045
Automatic control

Lecture 7

Time response

Objectives

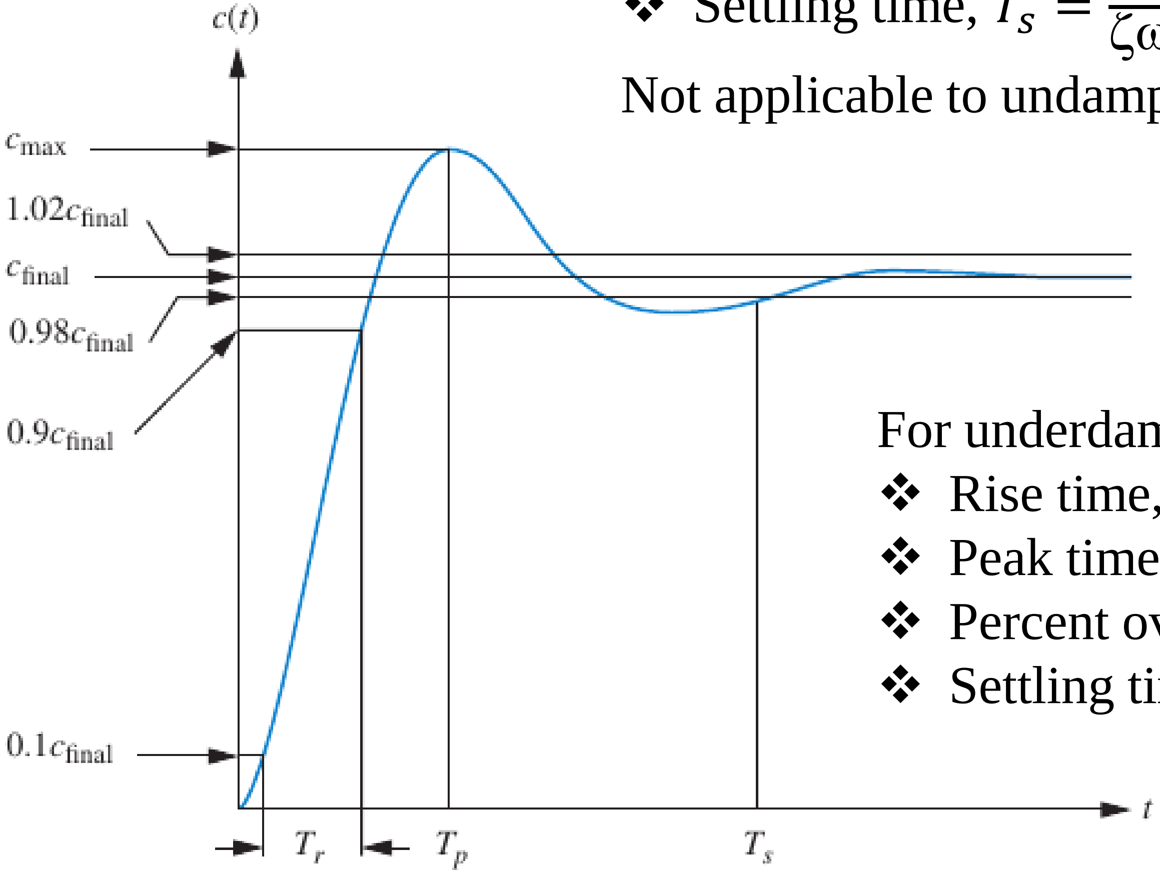
- Describe the step response characteristics of 2nd order underdamped system using performance indicators
- Describe the step responses of 2nd order systems with the addition of more poles and zeros
- Determine the steady state errors to step, ramp and parabola inputs

2^{nd} order response indicators

For over & critically damped system:

- ❖ Settling time, $T_s = \dfrac{4}{\zeta\omega_n}$

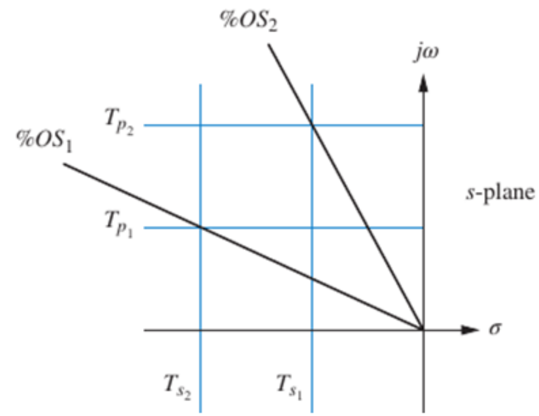
Not applicable to undamped system



For underdamped system:

- ❖ Rise time, T_r
- ❖ Peak time, T_p
- ❖ Percent overshoot, $\%OS$
- ❖ Settling time, T_s

2^{nd} order response indicators



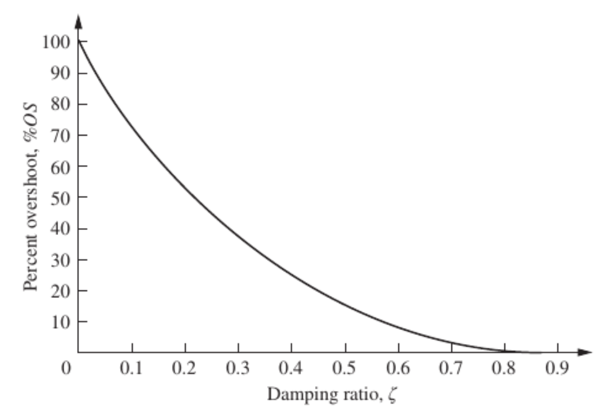
- Peak time, $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$

- Settling time, $T_s = \frac{4}{\zeta\omega_n}$

- Percent overshoot, $\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = \frac{c(T_p)-c(\infty)}{c(\infty)} \times 100$

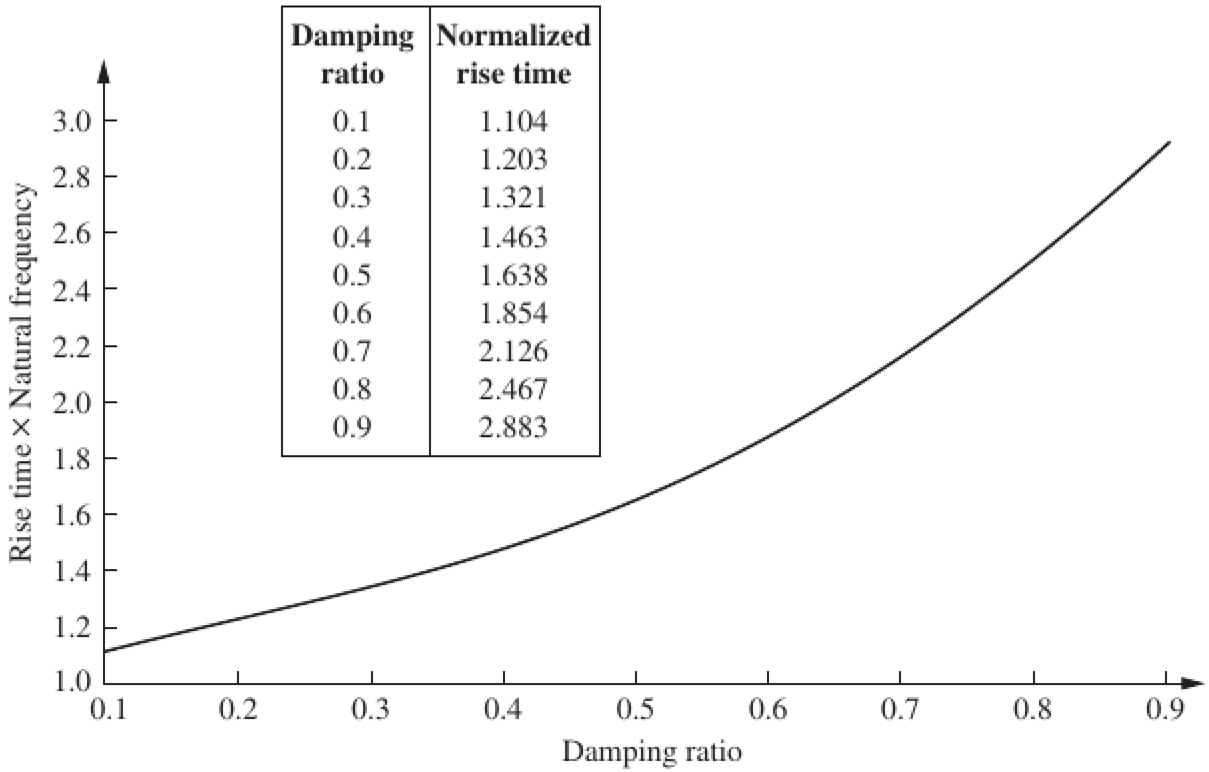
Note: We can determine ζ by measuring the $\%OS$:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



2nd order response indicators

❖ Normalized rise time $= \omega_n T_r$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

FIGURE 4.16 Normalized rise time versus damping ratio for a second-order underdamped response

Example 1

Find the damping ratio, natural frequency, rise time, peak time, settling time, and percent overshoot to a unit step response for a system whose transfer function is

$$\frac{C(s)}{R(s)} = \frac{361}{s^2 + 16s + 361}$$

Natural frequency $\omega_n = \sqrt{361} = 19$

$$2\zeta\omega_n = 16$$

Damping ratio $\zeta = 0.421$

Settling time $T_s = \frac{4}{\zeta\omega_n} = 0.5\text{sec.}$

Peak time $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.182\text{sec.}$

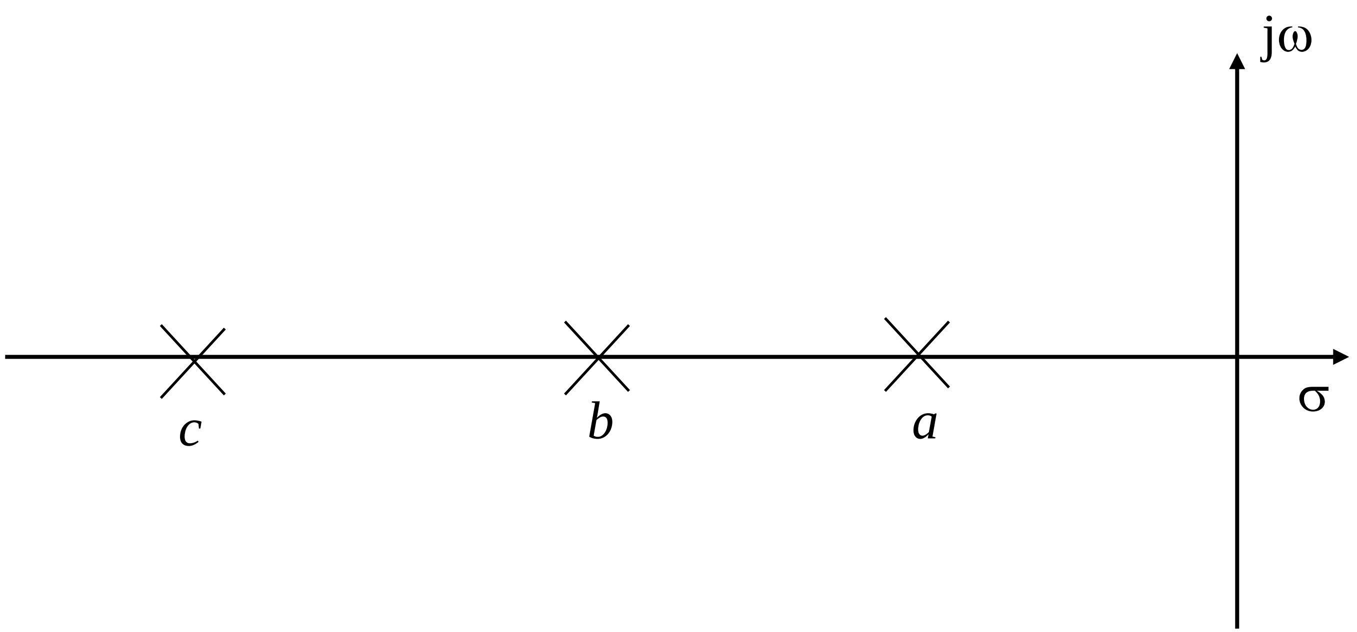
Percent overshoot $\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 23.3\%$

For $\zeta = 0.421$, from fig. 4.16 $\omega_n T_r = 1.5$ or $T_r = 0.08\text{sec.}$

Addition of poles

Illustration: step response of system with 3 real poles in the LHP

a) What is the dominating (worst case) settling time?
b) What is the effect on the settling time if the 3 poles are far away from each other? And if the poles are near each other?
c) Can this be generalized to more than 3 real poles in the LHP?



Addition of poles

Illustration: step response of system with 1 real pole and 2 complex poles in the LHP

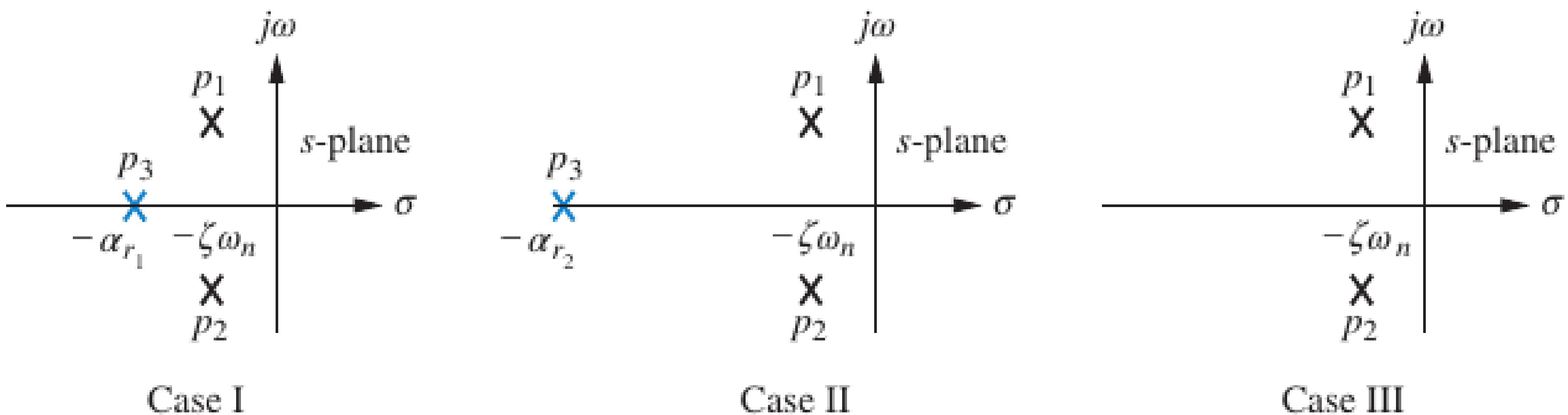
Case 1: real pole close to $\zeta\omega_n$

Case 2: real pole much larger than $\zeta\omega_n$

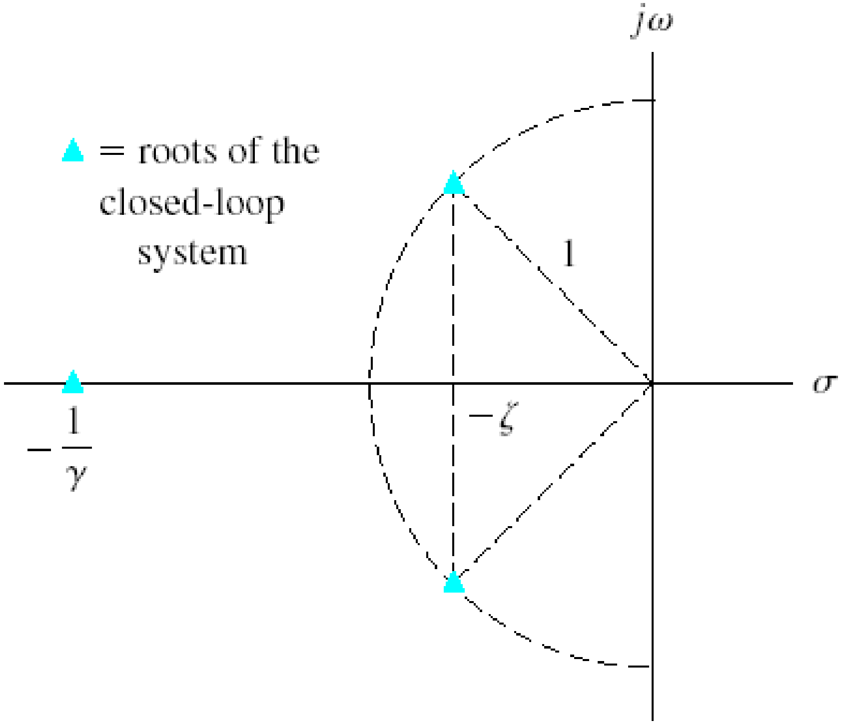
Case 3: real pole very far away from $\zeta\omega_n$

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$



Addition of poles



For the 3rd order system

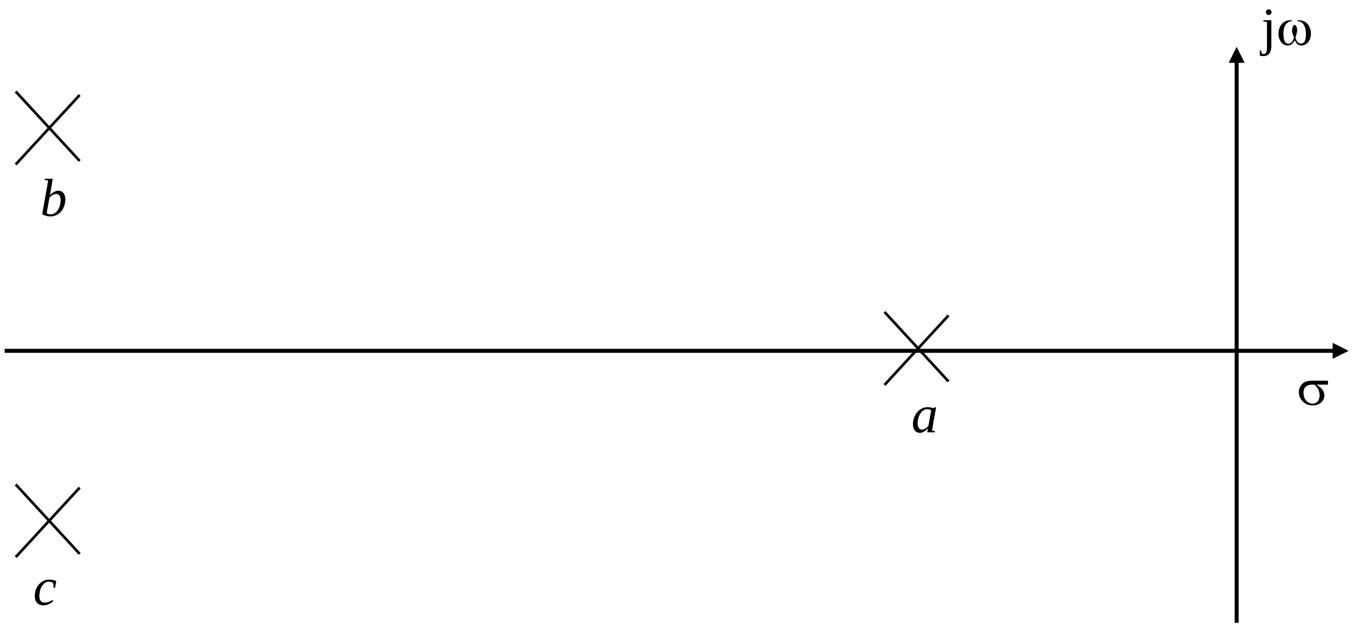
$$G(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}$$

If the real pole is far away from the complex poles, the step response can be approximated by the response of the 2 dominating complex poles

Some step responses of higher order systems can be approximated as first or second order system based on the dominant poles

Example 2

How would you approximate the step response of system with 1 real pole and 2 complex poles in the LHP as follows:



Beware that the steady state using the approximation must match that of the original!

Example 3

Determine the validity of a second-order approximation for each of these two transfer functions:

$$\frac{C(s)}{R(s)} = \frac{700}{(s+15)(s^2+4s+100)}$$

$$\frac{C(s)}{R(s)} = \frac{360}{(s+4)(s^2+2s+90)}$$

Note: we are only approximating the step responses. All the poles and zeros will have to be considered in the design of the controllers

Addition of zeros

- Mathematically speaking, the presence of zeros in a transfer function is to modify the coefficients of the exponential terms in the transient response
- Consider the two transfer functions with same poles but different zeros:

$$G_1(s) = \frac{2}{(s+1)(s+2)} \qquad G_2(s) = \frac{2(s+1.1)}{1.1(s+1)(s+2)}$$

$$G_1(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)} \qquad G_2(s) = \frac{0.18}{(s+1)} + \frac{1.64}{(s+2)}$$

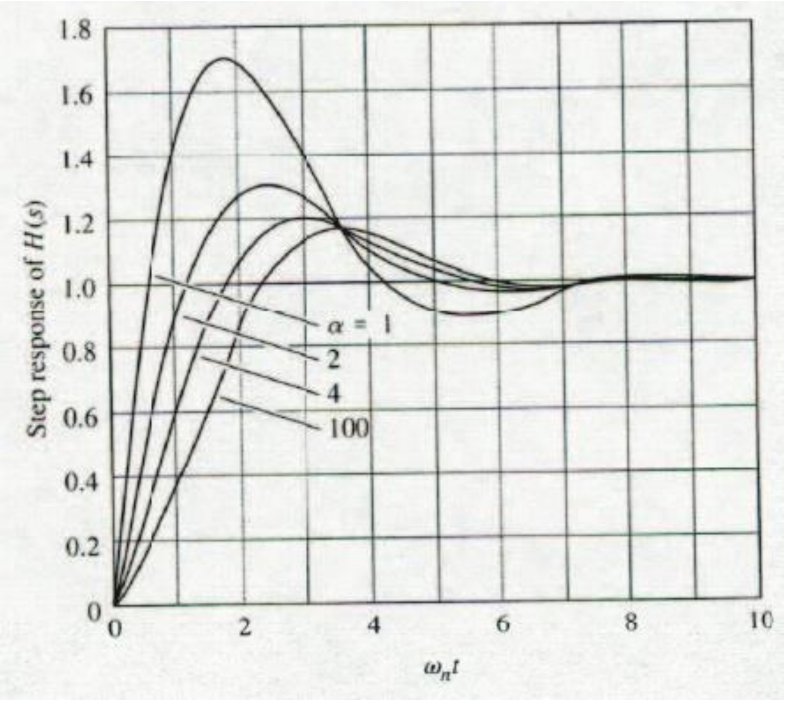
- The two transfer functions are normalized to have the same DC gain
- The zero at -1.1 reduces the pole at -1
- When the zero approaches -1, pole-zero cancellation will occur
- Note: pole-zero cancellation only for minimum phase systems

Addition of zeros

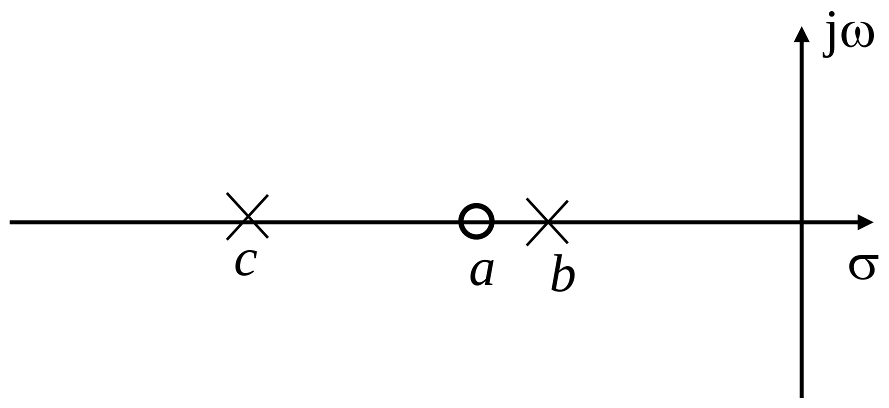
The effect of a zero on the transient response of an underdamped 2nd order system is examined using a transfer function with a zero and two complex poles:

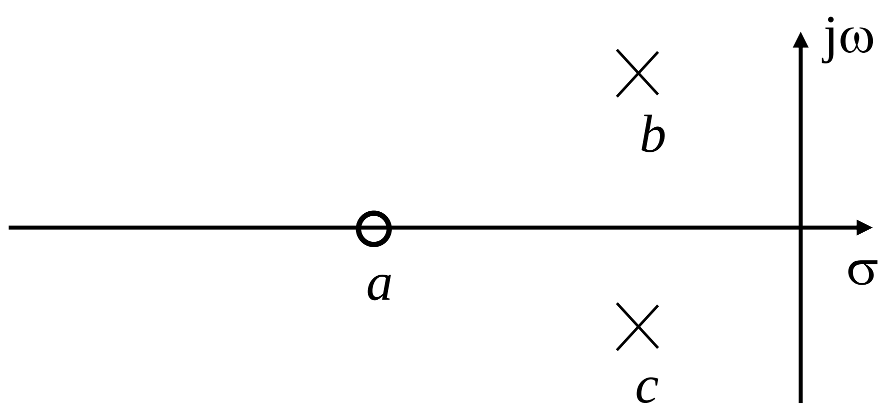
$$G(s) = \frac{(s/\alpha\zeta\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

- ❖ 2 complex poles at $\zeta\omega_n \pm j\omega_d$
- ❖ If α is large then the zero will be far away from the real part of the poles and its impact will be minimal
- ❖ If $\alpha \approx 1$, then the zero will be close to the real part of the poles and its impact will be significant (Step response with 0.5 damping ratio and for different α is shown - the zero impacts the overshoot, but settling time is not affected)



Addition of zeros





Example 4

For the two given transfer functions, determine the effect of cancellation between the zero and the pole closest to the zero

$$G_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$G_2(s) = \frac{26.25(s+4)}{s(s+4.1)(s+5)(s+6)}$$

$$G_1(s) = \frac{1}{s} - \frac{1}{(s+3.5)} - \frac{3.5}{(s+5)} + \frac{3.5}{(s+6)}$$

$$G_2(s) = \frac{0.87}{s} + \frac{0.033}{(s+4.1)} - \frac{5.3}{(s+5)} + \frac{4.4}{(s+6)}$$

The residue of the pole at 4.01, which is closest to the zero at 4, is equal to 0.033, about two orders of magnitude below any of the other residues

Example 5

The following system transfer functions experience a unit step input. Describe the speed of response and the steady state value:

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 6s + 8}$$

$$\frac{C(s)}{R(s)} = \frac{12}{(s+3)(s^2+6s+8)}$$

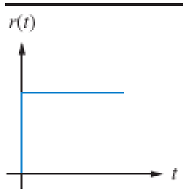
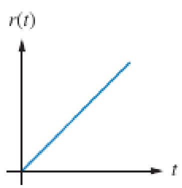
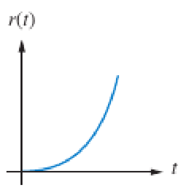
$$\frac{C(s)}{R(s)} = \frac{12}{(s+3)(s^2+6s+10)}$$

$$\frac{C(s)}{R(s)} = \frac{12(s+1)}{(s+3)(s^2+6s+8)}$$

Steady state error

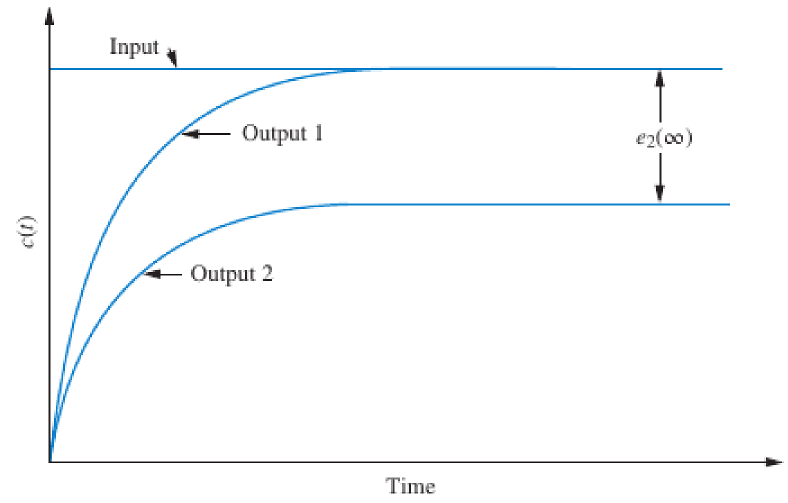
- Steady-state error is the difference between the input and the output for a prescribed test input as time $t \to \infty$
- Test inputs used for steady-state error analysis include the step, ramp and parabola functions

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

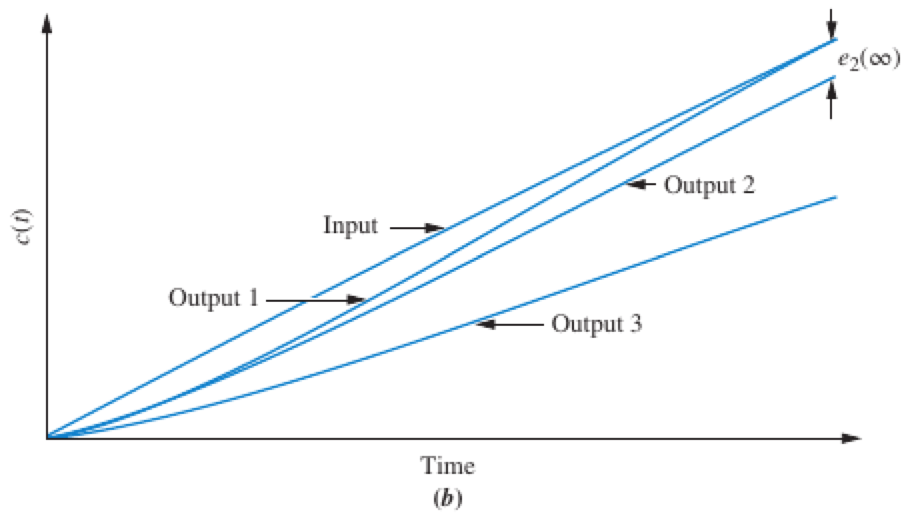
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Steady state error

Step response and steady state error:

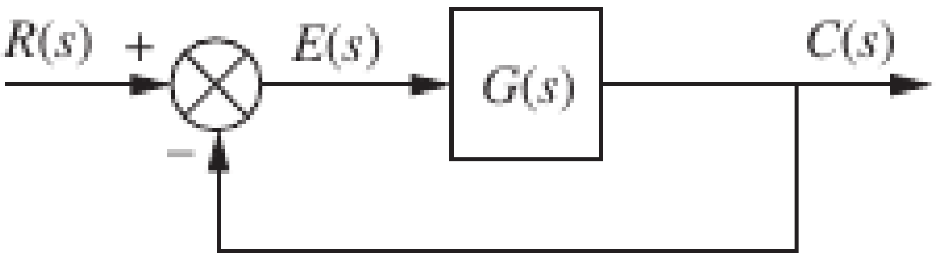


Ramp response and steady state error:



Steady state error

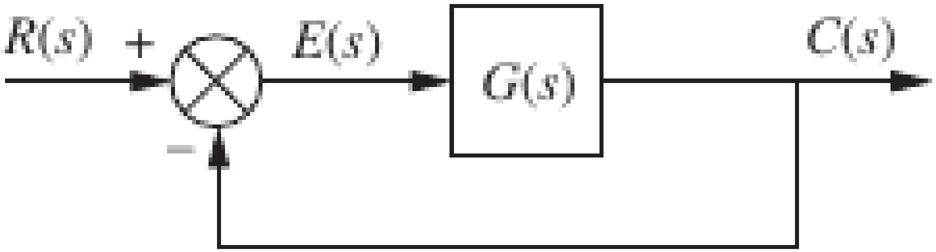
The block diagram of a unity feedback system:



- $R(s)$ = reference input
- $C(s)$ = control output
- $G(s)$ = forward transfer function
- $E(s)$ = error = $R(s) - C(s)$
- The steady state error is defined as the error when $t \rightarrow \infty$
- The closed-loop transfer function is $T(s)$

$$\frac{C(s)}{R(s)} = T(s) = \frac{G(s)}{1 + G(s)}$$

Steady state error



$$C(s) = T(s)R(s) = \frac{G(s)}{1+G(s)}R(s)$$

But

$$E(s) = R(s) - C(s) = R(s) - T(s)R(s) = [1 - T(s)]R(s)$$

$$E(s) = \frac{R(s)}{1+G(s)}$$

Using the final value theorem, the steady state error can be found from

$$e_{ss}(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Steady state error

The steady state error can be found using

$$e_{ss}(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

- For step input $R(s) = k/s$

$$e_{step}(\infty) = \lim_{s\to 0} \frac{k}{1+G(s)} = \frac{k}{1+\lim_{s\to 0} G(s)}$$

- For ramp input $R(s) = k/s^2$

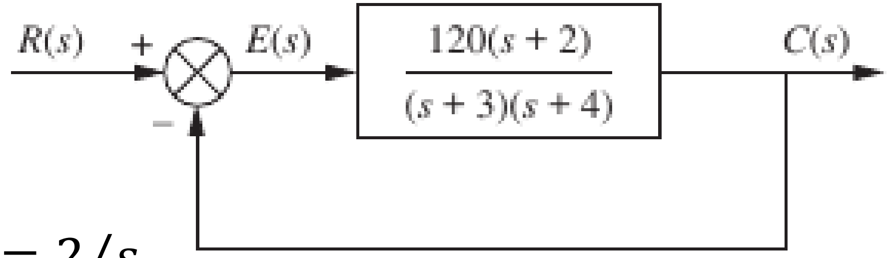
$$e_{ramp}(\infty) = \lim_{s\to 0} \frac{k}{s+sG(s)} = \lim_{s\to 0} \frac{k}{sG(s)} = \frac{k}{\lim_{s\to 0} sG(s)}$$

- For parabola input $R(s) = k/s^3$

$$e_{parabola}(\infty) = \lim_{s\to 0} \frac{k}{s^2+s^2G(s)} = \lim_{s\to 0} \frac{k}{s^2G(s)} = \frac{k}{\lim_{s\to 0} s^2G(s)}$$

Example 6

Find the steady-state errors for inputs of $2u(t)$, $2tu(t)$, and $2t^2u(t)$ to the system, where $u(t)$ is the unit step function



- For $r(t) = 2u(t)$, step input $R(s) = 2/s$

$$e_{step}(\infty) = \frac{2}{1 + \lim\limits_{s \to 0} G(s)} = \frac{2}{1 + 20} = \frac{2}{21}$$

- For $r(t) = 2tu(t)$, ramp input $R(s) = 2/s^2$

$$e_{ramp}(\infty) = \frac{2}{\lim\limits_{s \to 0} sG(s)} = \frac{2}{0} = \infty$$

- For $r(t) = 2t^2u(t)$, parabola input $R(s) = 4/s^3$

$$e_{parabola}(\infty) = \frac{4}{\lim\limits_{s \to 0} s^2G(s)} = \frac{4}{0} = \infty$$