ME1020
Mechanical vibrations

Lecture 11

Multi DOF system vibration 3 (Vibration absorber undamped)

Objectives

- Analyze the steady-state vibration of multi-DOF undamped systems to harmonic excitation
- Design of vibration absorber without damping
- Describe the concept of vibration absorbers in engineering applications

Forced vibration (2-DOF) undamped

The equations of motion of a general 2-DOF undamped system under harmonic external forces can be written in matrix form as:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \sin \omega t$$

The steady-state solutions will have the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t)$$

$$\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t)$$

Substituting these back into the equations of motion:

$$-\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t) + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t) = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \sin \omega t$$

$$\left\{ \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \right\} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Forced vibration (2-DOF) undamped

$$\begin{bmatrix} k_{11}-m_{11}\omega^2 & k_{12}-m_{12}\omega^2 \\ k_{21}-m_{21}\omega^2 & k_{22}-m_{22}\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} k_{11}-m_{11}\omega^2 & k_{12}-m_{12}\omega^2 \\ k_{21}-m_{21}\omega^2 & k_{22}-m_{22}\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} = [Z]^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

Note that

$$[Z]^{-1} = \frac{1}{\det([Z])} \begin{bmatrix} k_{22}-m_{22}\omega^2 & -k_{21}+m_{21}\omega^2 \\ -k_{12}+m_{12}\omega^2 & k_{11}-m_{11}\omega^2 \end{bmatrix}$$

$$\det([Z]) = \frac{1}{(k_{11}-m_{11}\omega^2)(k_{22}-m_{22}\omega^2)-(k_{21}-m_{21}\omega^2)(k_{12}-m_{12}\omega^2)}$$

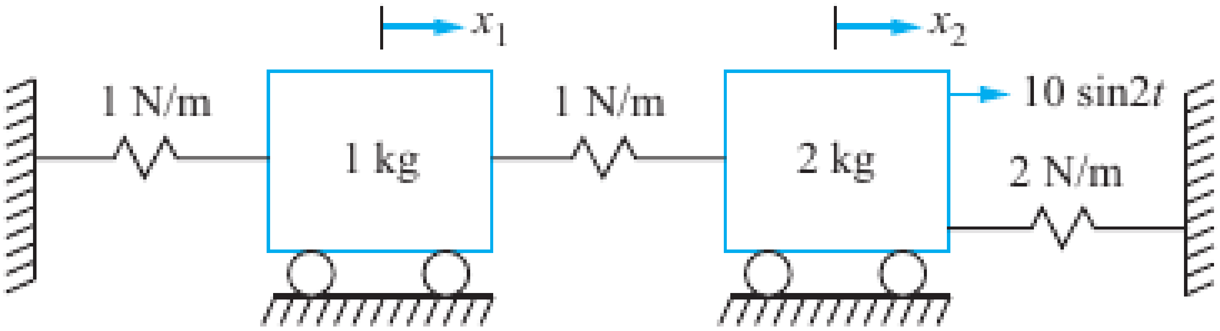
Once we determined $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, the steady state solution is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(\omega t)$$

Example 1

Determine the steady-state responses of the following system

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \sin 2t$$



Example 1

The steady-state solutions will have the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin(2t)$$

Substituting this and its derivative into the matrix equation

$$\left\{ \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Solving the equation give

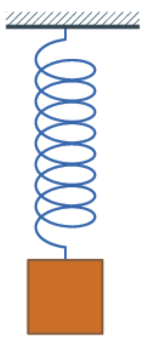
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -5 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} (10/9) \\ (-20/9) \end{bmatrix}$$

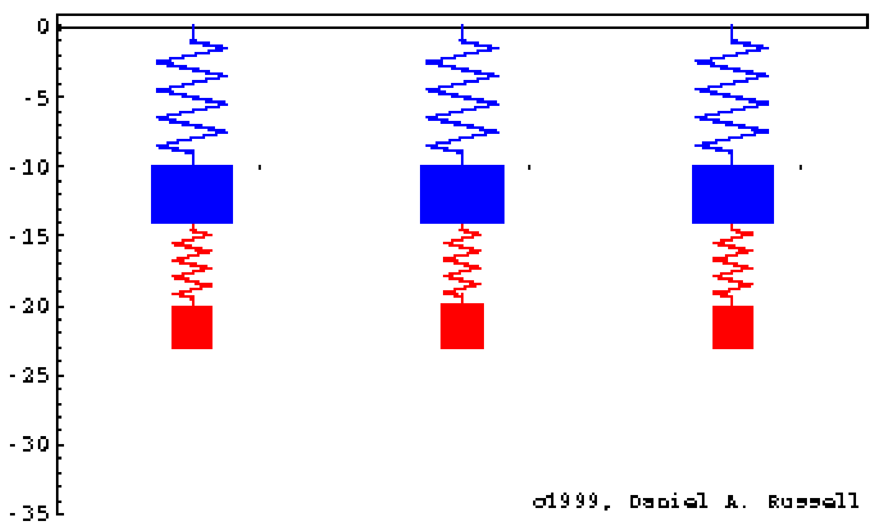
The steady state response of the masses are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (10/9) \\ (-20/9) \end{bmatrix} \sin(2t)$$

Vibration absorber undamped

- Consider a harmonic disturbance to a single-degree-of freedom system
- Suppose the disturbance causes large amplitude vibration of the mass in the steady state (e.g. excitation at resonant frequency)
- A second spring mass system is now added to the "primary" mass. The second spring mass system design can cause the primary system to vibrate differently

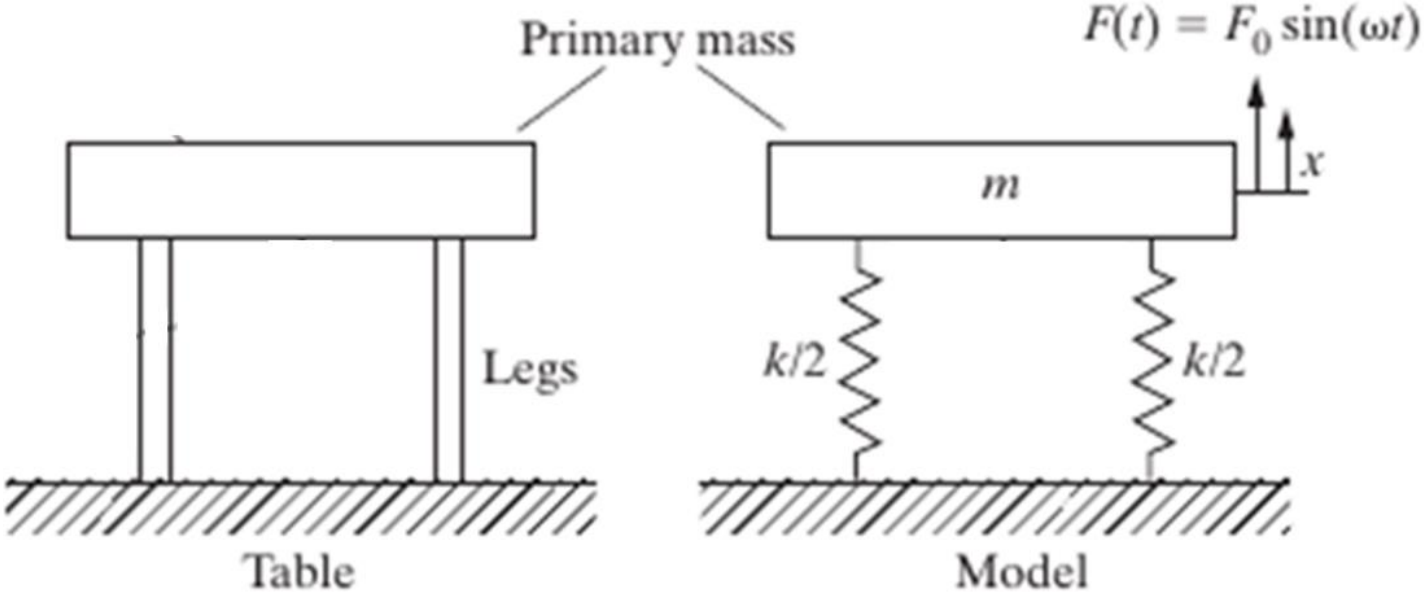




Vibration absorber undamped

- The basic principle is to attached a second spring-mass to the primary spring-mass system which is subjected to harmonic excitation
- The secondary mass-spring system is designed to absorb the vibration of the main system at the specified frequency
- The critical frequency of the primary system is usually its resonance frequency. In this case, the vibration of the primary system can be reduced to zero at its resonance frequency for properly designed absorber. This means that the energy of the primary system at resonance is "absorbed" by the tuned dynamic absorber
- The design of the absorber involves determining the mass and spring constant of the secondary system, i.e. m_2, k_2

Primary system

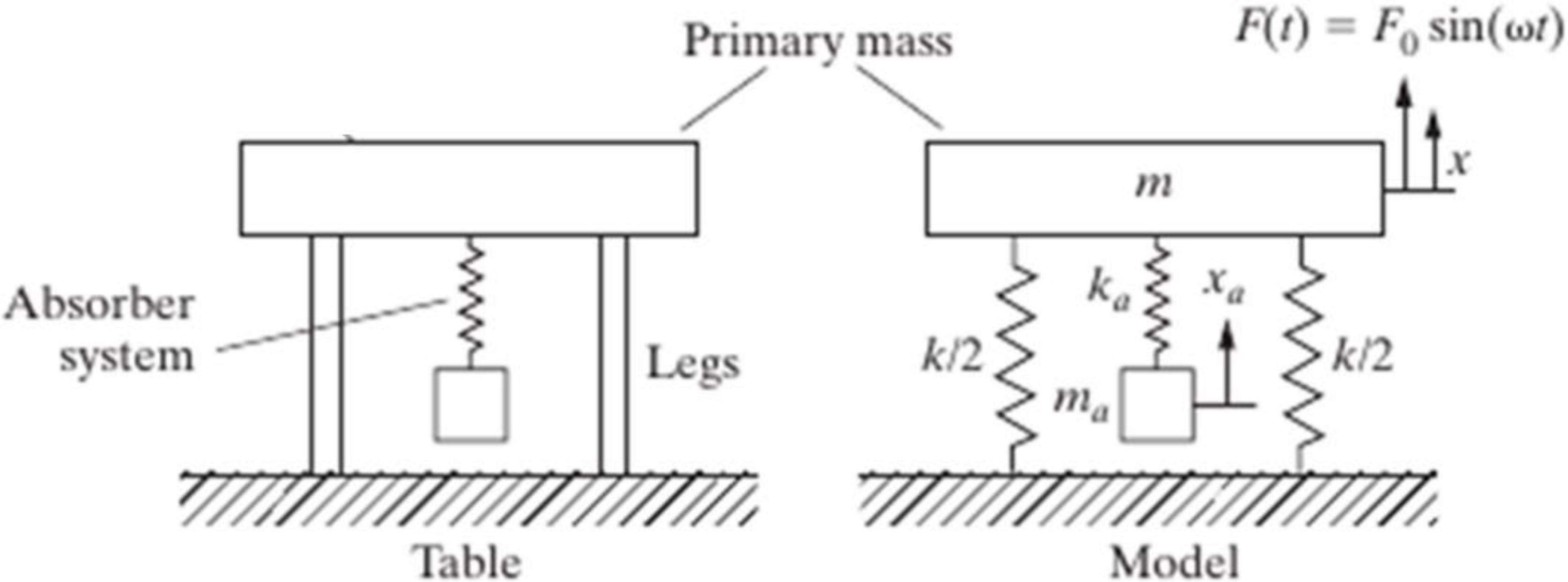


The primary system model is $m\ddot{x} + kx = F_o \sin(\omega t)$

The natural frequency is $\omega_p = \sqrt{k/m}$

What happen when excitation frequency $\omega = \omega_p$?

Vibration absorber (undamped)



The system model in matrix form is

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} (k + k_a) & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix}$$

Vibration absorber (undamped)

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} (k+k_a) & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix}$$

The system is undamped, the steady state solution can be assumed as

$$\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t$$

Differentiating twice

$$\begin{bmatrix} \ddot{x}\,(t) \\ \ddot{x}_a(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t$$

Substitute back into the matrix equation:

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t + \begin{bmatrix} (k+k_a) & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} (k+k_a) - m\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix} \right\} \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t$$

Vibration absorber (undamped)

$$\begin{bmatrix} (k+k_a)-m\omega^2 & -k_a \\ -k_a & k_a-m_a\omega^2 \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \begin{bmatrix} (k+k_a)-m\omega^2 & -k_a \\ -k_a & k_a-m_a\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{1}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2} \begin{bmatrix} k_a-m_a\omega^2 & k_a \\ k_a & (k+k_a)-m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{1}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2} \begin{bmatrix} (k_a-m_a\omega^2)F_0 \\ k_aF_0 \end{bmatrix}$$

$$X = \frac{(k_a-m_a\omega^2)F_0}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}$$

$$X_a = \frac{k_aF_0}{(k+k_a-m\omega^2)(k_a-m_a\omega^2)-k_a^2}$$

Vibration absorber (undamped)

Consider the vibration amplitude of the primary system with absorber attached

$$X = \frac{(k_a - m_a\omega^2)F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2}$$

- Pick m_a and k_a to make $X = 0$, and this will give

$$k_a = m_a\omega^2 \text{ or } \omega^2 = \frac{k_a}{m_a}$$

- Note that ω is the excitation frequency. The vibration is transferred to the absorber. The vibration amplitude of the absorber is now

$$X_a = \frac{k_a F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} = -\frac{F_0}{k_a}$$

- The vibration of the absorber is $x_a(t) = X_a \sin \omega\, t$

Absorber design (undamped)

- The frequency of the excitation is ω
- The natural frequency of the primary system without absorber is

$$\omega_p = \sqrt{k/m}$$

- The natural frequency of the stand-alone absorber is

$$\omega_a = \sqrt{k_a/m_a}$$

- After the absorber is attached to the primary system, the combined 2-DOF system will have two natural frequencies ω_1, ω_2 which are different from ω_p and ω_a
- Define the frequency ratio as $\beta = \omega_a/\omega_p$
- Define the mass ratio as $\mu = m_a/m$
- With these definitions, note that

$$\frac{k_a}{k} = \mu \frac{\omega_a^2}{\omega_p^2} = \mu\beta^2$$

Absorber design (undamped)

Let $r_1 = \omega/\omega_p$ and $r_2 = \omega/\omega_a$

The vibration amplitude of the primary system with absorber can be rewritten as

$$X = \frac{(k_a - m_a\omega^2)F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} = \frac{(1 - r_2^2)F_0}{(k + k_a - m\omega^2)(1 - r_2^2) - k_a}$$

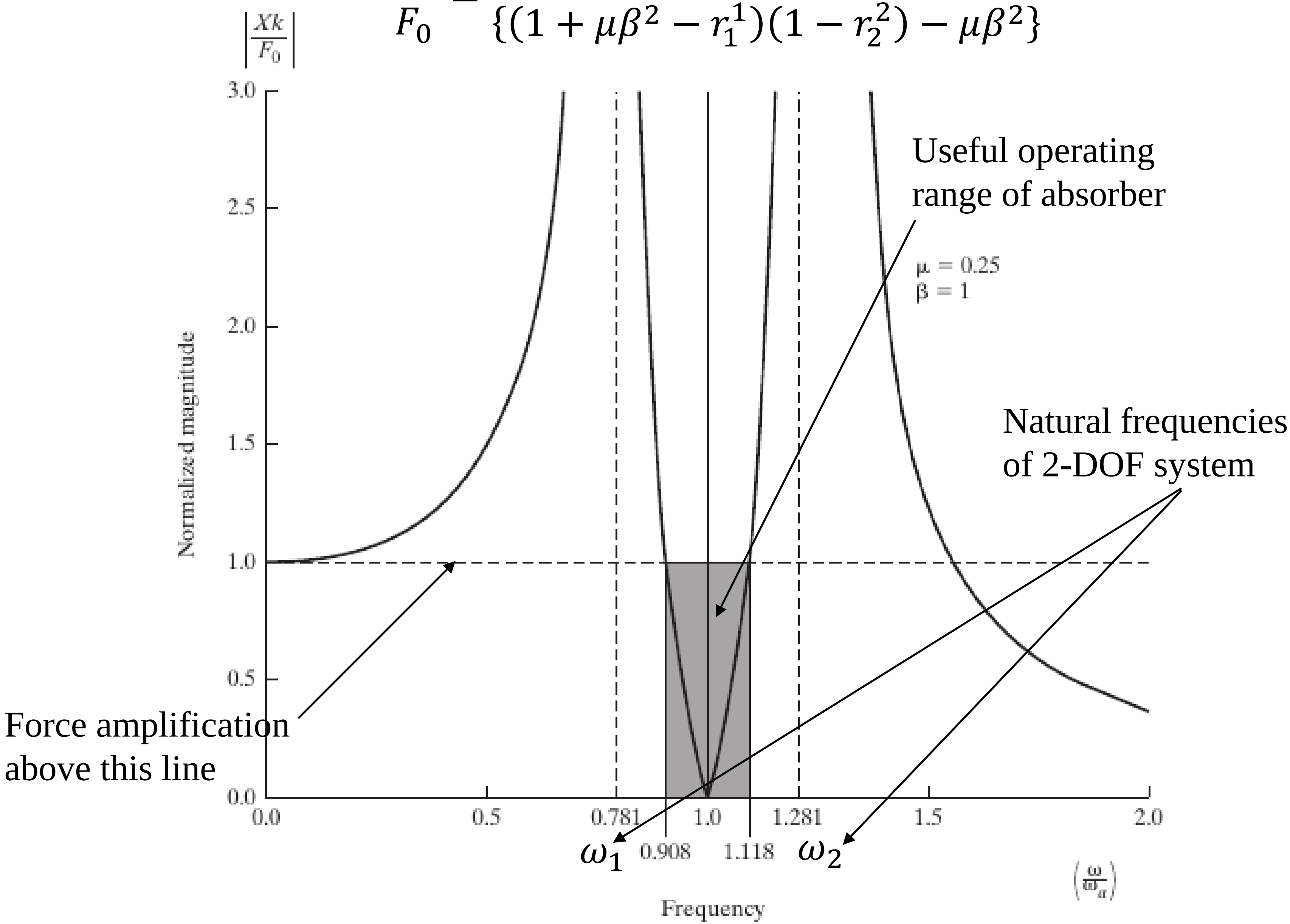
$$X = \frac{(1 - r_2^2)F_0}{k\{(1 + \mu\beta^2 - r_1^1)(1 - r_2^2) - \mu\beta^2\}}$$

$$\frac{Xk}{F_0} = \frac{(1 - r_2^2)}{\{(1 + \mu\beta^2 - r_1^1)(1 - r_2^2) - \mu\beta^2\}}$$

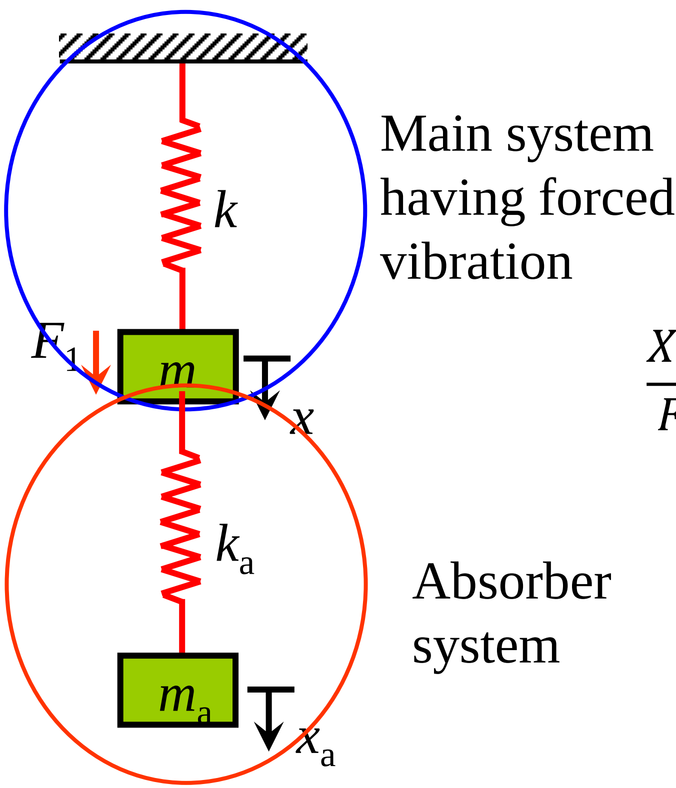
Similarly

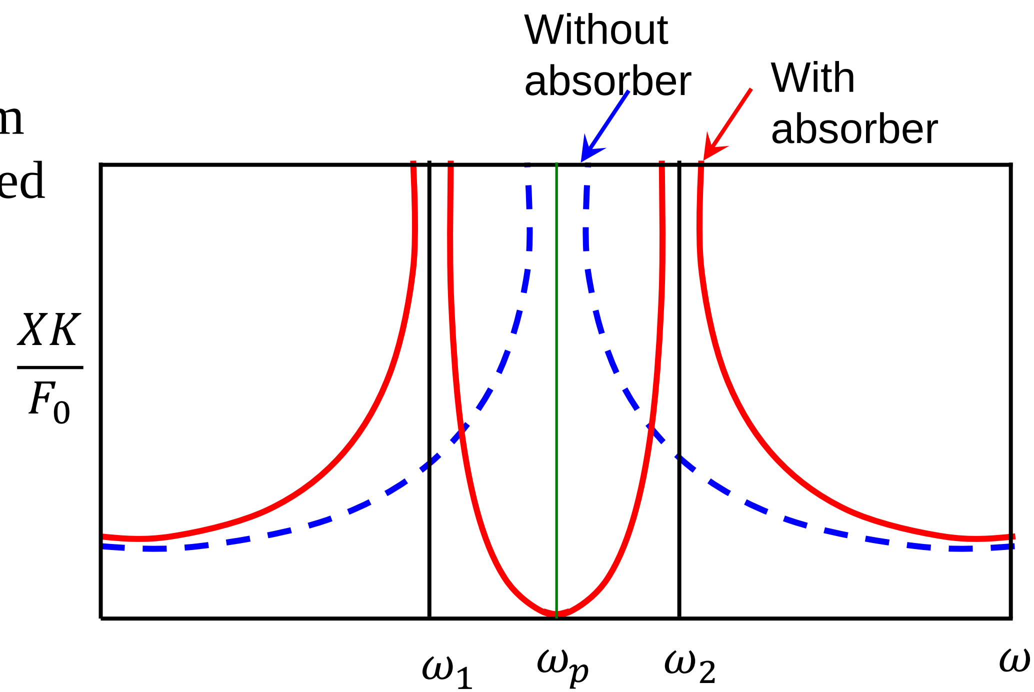
$$\frac{X_a k}{F_0} = \frac{1}{\{(1 + \mu\beta^2 - r_1^1)(1 - r_2^2) - \mu\beta^2\}}$$

$$\frac{Xk}{F_0} = \frac{(1 - r_2^2)}{\{(1 + \mu\beta^2 - r_1^1)(1 - r_2^2) - \mu\beta^2\}}$$



Absorber design (undamped)





If ω hits ω_1 or ω_2 resonance occurs for the 2-DOF system

Absorber design (undamped)

Note that ω_1 and ω_2 are the natural frequencies of the 2-DOF system and these values can be obtained by solving

$$\det\{-\omega^2[m] + [k]\} = 0$$

$$(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2 = 0$$

$$m m_a \omega^4 - \{(k + k_a)m_a + k_a m\}\omega^2 + k k_a = 0$$

$$\frac{m m_a}{k k_a}\omega^4 - \left\{\left(\frac{k + k_a}{k}\right)\frac{m_a}{k_a} + \frac{m}{k}\right\}\omega^2 + 1 = 0$$

$$\frac{\omega^4}{\omega_p^2\omega_a^2} - \left\{\left(\frac{k + k_a}{k}\right)\frac{1}{\omega_a^2} + \frac{1}{\omega_p^2}\right\}\omega^2 + 1 = 0$$

$$\frac{\omega^4}{\omega_p^2\omega_a^2}\left(\frac{\omega_p^2}{\omega_a^2}\right) - \left\{\left(1 + \frac{k_a}{k}\right)\frac{1}{\omega_a^2} + \frac{1}{\omega_p^2}\right\}\left(\frac{\omega_p^2}{\omega_a^2}\right)\omega^2 + \left(\frac{\omega_p^2}{\omega_a^2}\right) = 0$$

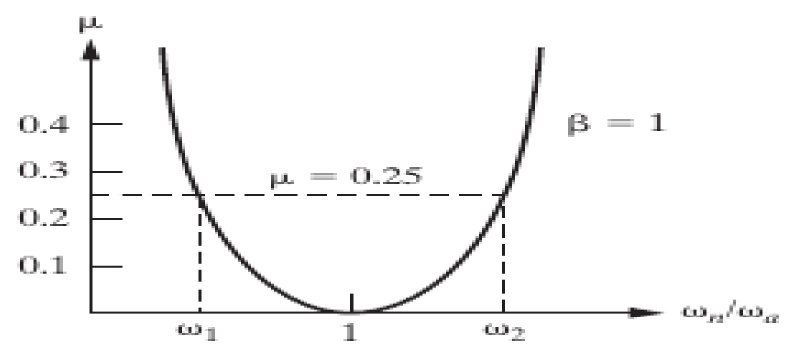
Absorber design (undamped)

$$\frac{\omega^4}{\omega_p^2\omega_a^2}\left(\frac{\omega_p^2}{\omega_a^2}\right)-\left\{\left(1+\frac{k_a}{k}\right)\frac{1}{\omega_a^2}+\frac{1}{\omega_p^2}\right\}\left(\frac{\omega_p^2}{\omega_a^2}\right)\omega^2+\left(\frac{\omega_p^2}{\omega_a^2}\right)=0$$

$$\frac{\omega^4}{\omega_a^4}-\left\{(1+\mu\beta^2)\frac{1}{\beta^2}+1\right\}\left(\frac{\omega^2}{\omega_a^2}\right)+\left(\frac{1}{\beta^2}\right)=0$$

$$\beta^2\left(\frac{\omega^2}{\omega_a^2}\right)^2-\{(1+\mu\beta^2)+\beta^2\}\left(\frac{\omega^2}{\omega_a^2}\right)+1=0$$

- Note $\mu = m_a/m$ affects ω_1 and ω_2
- Recommended $0.05 \leq \mu \leq 0.25$



Absorber design (undamped)

Note that for $\beta = 1$ (i.e. $\omega_a = \omega_p$)

$$\left(\frac{\omega^2}{\omega_a^2}\right)^2 - \{(1+\mu)+1\}\left(\frac{\omega^2}{\omega_a^2}\right) + 1 = \left(\frac{\omega^2}{\omega_a^2}\right)^2 - \{2+\mu\}\left(\frac{\omega^2}{\omega_a^2}\right) + 1 = 0$$

The solution to the quadratic equation is

$$\left(\frac{\omega^2}{\omega_a^2}\right) = \{2+\mu\} \pm \frac{\sqrt{\{2+\mu\}^2 - 4}}{2}$$

The two natural frequencies are

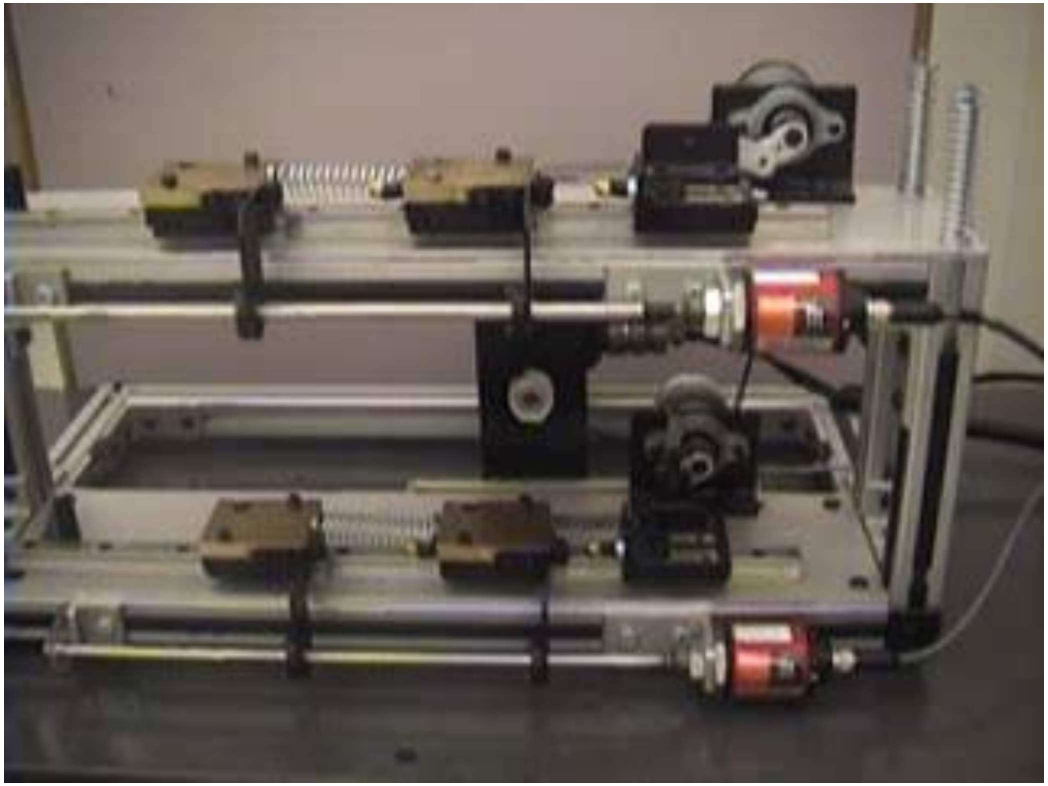
$$\left(\frac{\omega_1^2}{\omega_a^2}\right) = \left\{1+\frac{\mu}{2}\right\} + \sqrt{\left\{1+\frac{\mu}{2}\right\}^2 - 1}$$

$$\left(\frac{\omega_2^2}{\omega_a^2}\right) = \left\{1+\frac{\mu}{2}\right\} - \sqrt{\left\{1+\frac{\mu}{2}\right\}^2 - 1}$$

Absorber design (undamped)

- The absorber design depends on knowing ω exactly, i.e. the resonance frequency of the primary spring-mass system must be known
- The design only absorbs vibration at a single frequency
- If we excite the primary systems at another frequency, it could end up exciting the 2-DOF system natural frequency (resonance)
- Damping, which always exists to some degree, spoils the absorption
- For $\left(\frac{\omega^2}{\omega_a^2}\right) = \left\{1 + \frac{\mu}{2}\right\} + \sqrt{\left\{1 + \frac{\mu}{2}\right\}^2 - 1}$ note that if the mass ratio μ is very small, the effect of the absorber mass is insignificant and the resonance frequencies is close to the natural frequency of the primary system. When the mass ratio μ is of the order of 0.6 and 0.8, ω_1 and ω_2 are quite far apart. Effectiveness of the absorber demands that these frequencies should be at least 20 per cent away from the impressed frequency. Rule of thumb 0.05< μ <0.25

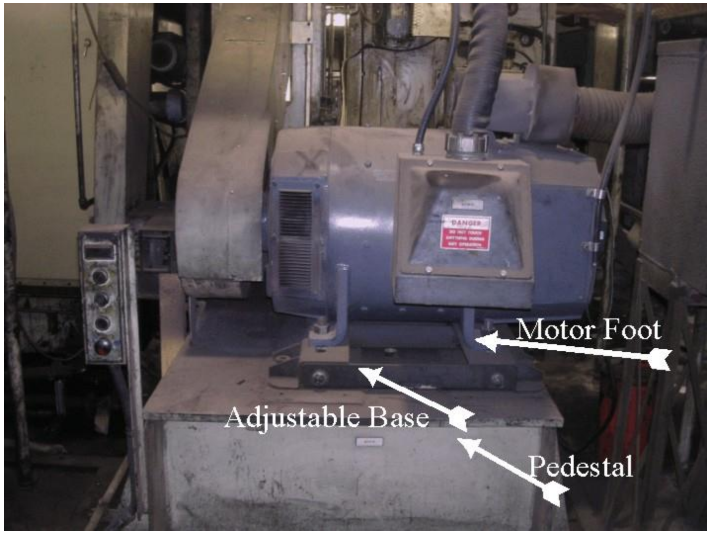
Absorber design (undamped)



Absorber design (undamped)

Example 2

A diesel motor is supported on a pedestal mount. It has been observed that the motor induces vibration into the surrounding area through its pedestal at an operating speed of 6000 rpm. Determine the parameters of the vibration absorber that will reduce the vibration when mounted on the pedestal. The magnitude of the exciting force is 250 N, and the amplitude of motion of the auxiliary mass is to be limited to 2 mm.



Example 2

Forcing frequency is 6000 rpm or $\omega = 6000\frac{2\pi}{60} = 628.3$ rad/s

Given $F_0 = 250$ N; motor weight is 3000 N and absorber vibration amplitude is limited to $\|X_a\| = 2 \times 10^{-3}$ m;

Since $\|X_a\| = \frac{F_0}{k_a}$ the absorber spring constant is

$$k_a = \frac{F_0}{\|X_a\|} = 125000 \text{ N/m}$$

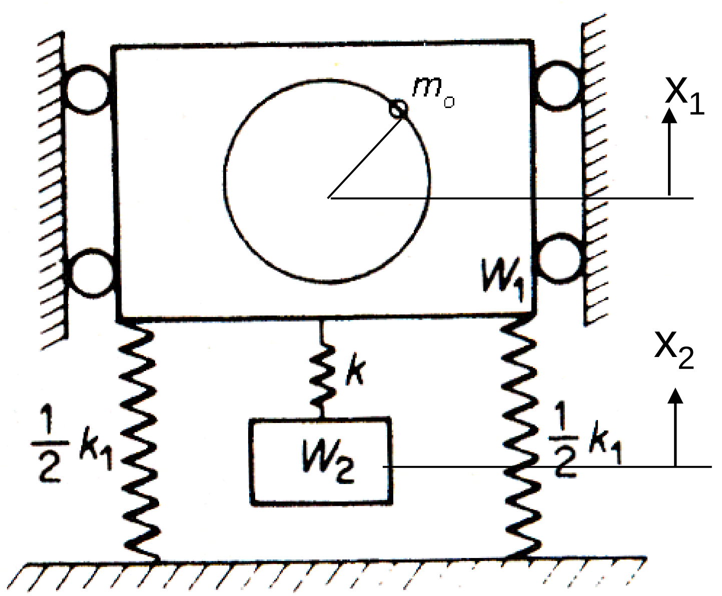
The absorber is to be designed so that at this frequency, the vibration will be transferred to the absorber so that $X = 0$; hence

$$\omega^2 = \frac{k_a}{m_a} \quad \text{or} \quad m_a = \frac{k_a}{\omega^2} = 0.3166 \text{ kg}$$

The absorber should have a mass $m_a = 0.3166$ kg and spring with stiffness $k_a = 125000$ N/m

Example 3

For the system shown below, W_1 = 883 N and the absorber weight W_2 = 225 N. If W_1 is excited by a 23 kg.mm unbalanced rotating mass at 1800 rpm, determine the proper value of the absorber spring k_a. What will be the amplitude of W_2?



Example 3

- Given primary system mass $m = \frac{W_1}{9.81} = 90$ kg;
- Mass of absorber $m_a = \frac{W_2}{9.81} = 23$ kg;
- Rotating unbalanced $m_o e = 0.023$ kg-m
- Note that the harmonic force due to the unbalanced rotating mass is

$$F = m_0 e \omega^2 \sin \omega t$$

Where $\omega = 2\pi \frac{1800}{60} = 188.5$ rad/s

Design the absorber mass and stiffness using $\frac{k_a}{m_a} = \omega^2$

Therefore $k_a = m_a \omega^2 = 8.17 \times 10^5$ N/m

At this frequency with absorber, the primary system displacement = 0 and

$$|X_a| = \frac{F_0}{k_a} = \frac{m_0 e \omega^2}{k_a} = \frac{0.023(188.5)^2}{8.17 \times 10^5} = 0.001 \text{ m}$$