# ME1020
# Mechanical vibrations

Lecture 9

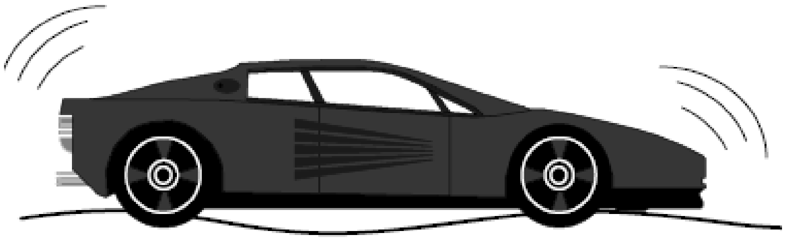
Multi DOF system vibration 1 (modelling)

# Objectives

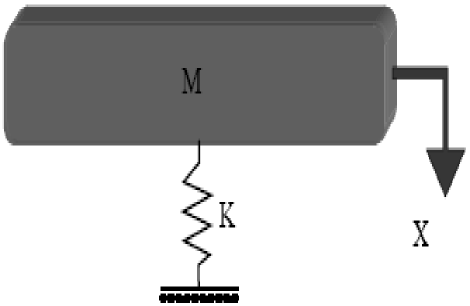
- Derive the $n^{th}$ DOF vibration system model subjected using Lagrange’s equation
- Relate the dynamic and static coupling with the changes in generalized coordinates

# Introduction

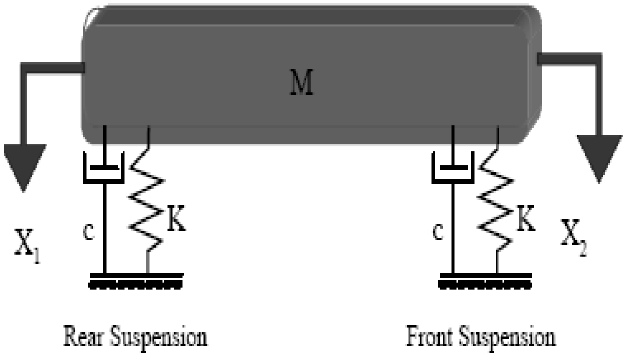
Consider the analysis of a car:



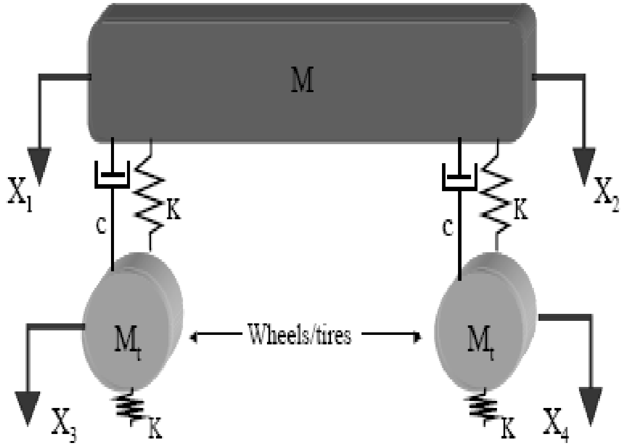
Simplified 1 DOF model




Simplified 2 DOF model




Multi-DOF model



# Lagrange's equation

In terms of generalized coordinate $q_k$, the Lagrange's equation for a $n^{\text{th}}$ DOF system subject to a generalized force has the form for $k = 1,2,\cdots,n$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k$$

- $T$ = Kinetic energy
- $U$ = Potential energy
- $D$ = Rayleigh's damping (or dissipation) function
- $q_k$ = $k^{\text{th}}$ generalized coordinate
- $Q_k = \sum_l F_l \cdot \frac{\partial r_l}{\partial q_k} + \sum_l M_l \cdot \frac{\partial \omega_l}{\partial \dot{q}_k}$ (note dot product) generalized force for the $l$ bodies
- $F_l$ and $M_l$ are the vector representation of the external applied forces and moments on the $l$th body, respectively; $r_l$ and $\omega_l$ are the position and angular velocity vectors due to $F_l$ and $M_l$ respectively

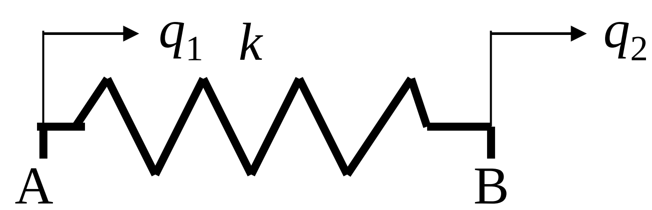
# Lagrange's equation

Kinetic energy $T = \frac{1}{2}\,\mathrm{Inertia}(\dot{q}^2)$

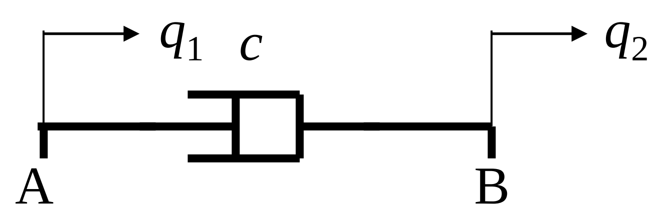
Translational

Rotational

Potential energy $U = \frac{1}{2}k(q_2 - q_1)^2$

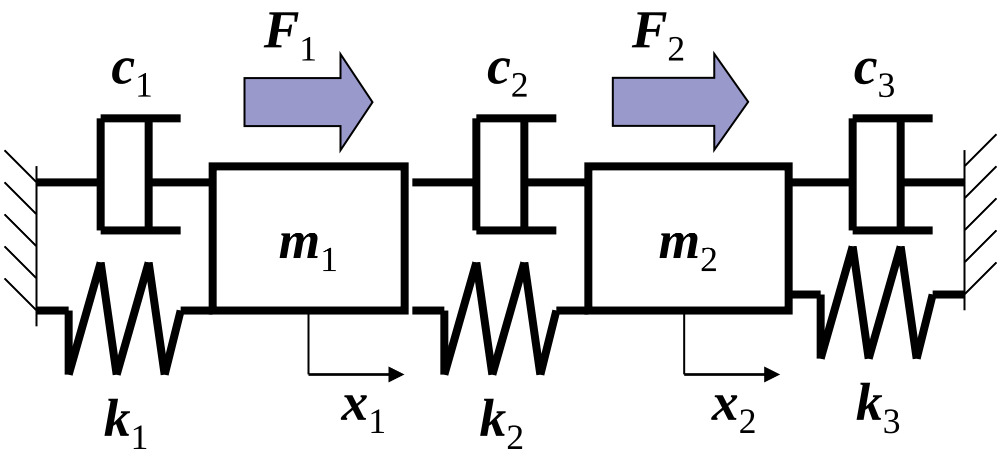



Dissipation function $D = \frac{1}{2}c(\dot{q}_2 - \dot{q}_1)^2$



# Lagrange's equation

Derive the equations of motion for the following 2 DOF system




Total kinetic energy:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

- Note $q_1 = x_1$, $q_2 = x_2$

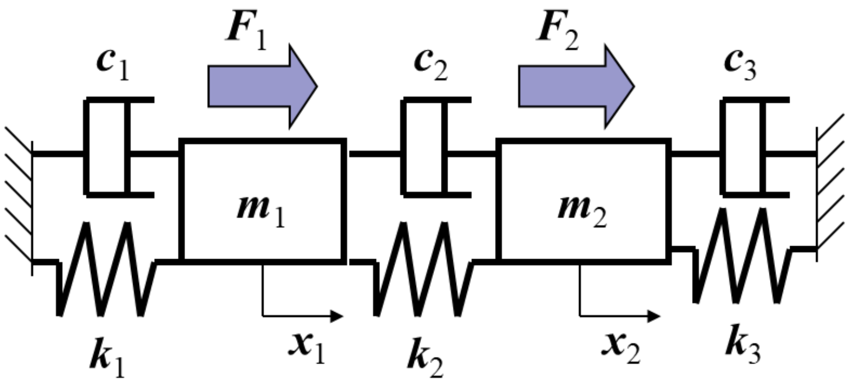
$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial x_2} = 0$$

# Lagrange's equation

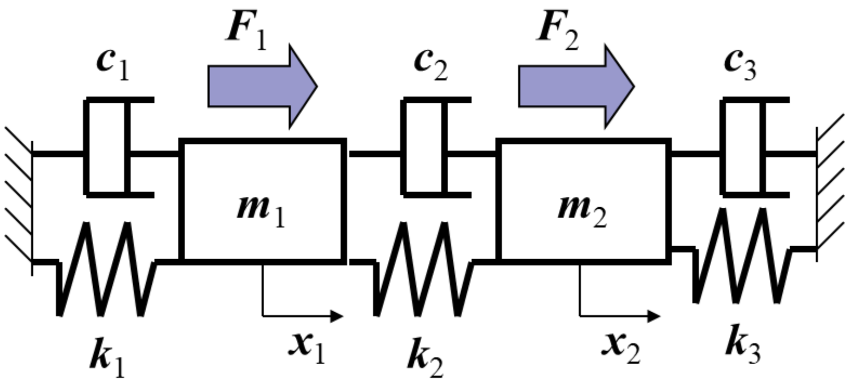



Potential energy: $U = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3 x_2^2$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial x_1} = k_1 x_1 - k_2(x_2 - x_1)$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial x_2} = k_2(x_2 - x_1) + k_3 x_2$$

# Lagrange's equation

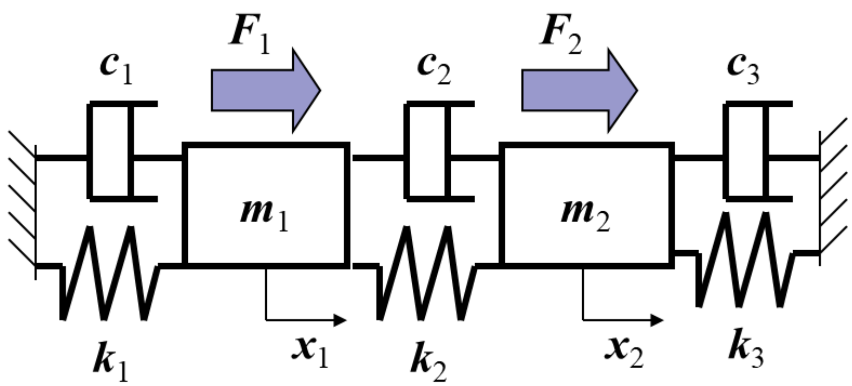



Dissipation function: $D = \frac{1}{2} c_1 \dot{x}_1^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} c_3 \dot{x}_2^2$

$$\frac{\partial D}{\partial \dot{q}_1} = \frac{\partial D}{\partial \dot{x}_1} = c_1 \dot{x}_1 - c_2 (\dot{x}_2 - \dot{x}_1)$$

$$\frac{\partial D}{\partial \dot{q}_2} = \frac{\partial D}{\partial \dot{x}_2} = c_2 (\dot{x}_2 - \dot{x}_1) + c_3 \dot{x}_2$$

# Lagrange's equation




Generalized force:

$$Q_k = \sum_l F_l \cdot \frac{\partial r_l}{\partial q_k} + \sum_l M_l \cdot \frac{\partial \omega_l}{\partial \dot{q}_k}$$

$$Q_1 = F_1 \frac{\partial r_1}{\partial q_1} + F_2 \frac{\partial r_2}{\partial q_1} = F_1 \frac{\partial x_1}{\partial x_1} + F_2 \frac{\partial x_2}{\partial x_1} = F_1$$

$$Q_2 = F_1 \frac{\partial r_1}{\partial q_2} + F_2 \frac{\partial r_2}{\partial q_2} = F_1 \frac{\partial x_1}{\partial x_2} + F_2 \frac{\partial x_2}{\partial x_2} = F_2$$

# Lagrange's equation

For $k = 1,2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k$$

❖ For $k = 1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} + \frac{\partial U}{\partial x_1} = Q_1$$

$$\frac{\partial}{\partial t}(m_1\dot{x}_1) + c_1\dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) + k_1x_1 - k_2(x_2 - x_1) = F_1$$

❖ For $k = 2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} + \frac{\partial U}{\partial x_2} = Q_2$$

$$\frac{\partial}{\partial t}(m_2\dot{x}_2) + c_2(\dot{x}_2 - \dot{x}_1) + c_3\dot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = F_2$$

# Lagrange's equation

$$\frac{\partial}{\partial t}(m_1\dot{x}_1) + c_1\dot{x}_1 - c_2(\dot{x}_2 - \dot{x}_1) + k_1x_1 - k_2(x_2 - x_1) = F_1$$

$$\frac{\partial}{\partial t}(m_2\dot{x}_2) + c_2(\dot{x}_2 - \dot{x}_1) + c_3\dot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = F_2$$

❖ The 2 equations are simplified to:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

❖ The 2 equations can be arranged into matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & (c_2 + c_3) \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

❖ To solve the equations, we need initial conditions $x_1(0), x_2(0), \dot{x}_1(0), \dot{x}_2(0)$ and forcing functions $F_1, F_2$

# Equations of motion

($n\times n$) Stiffness matrix

($n\times n$) Damping matrix

($n\times 1$) Force vector

($n\times n$) Mass matrix

$$[m][\ddot{x}]+[c][\dot{x}]+[k][x]=[F]$$

To solve this system of differential equations we need the initial conditions for $[x(0)]$ and $[\dot{x}(0)]$

($n\times 1$) Displacement vector

($n\times 1$) Velocity vector

($n\times 1$) Acceleration vector

# Revision of matrix algebra

Consider a matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Transpose of matrix $A$ is formed by interchanging the rows and columns:

$$\boldsymbol{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- For matrix $\boldsymbol{A}$ to be symmetry, $\boldsymbol{A} = \boldsymbol{A}^T$
- The inverse of a square matrix $\boldsymbol{A}$ is a matrix of the same dimension denoted by $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity matrix

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# Coupling of matrixes

$$[m][\ddot{x}]+[c][\dot{x}]+[k][x]=[F]$$

- A diagonal stiffness matrix $[k]$ indicates the system is static uncoupled. Example

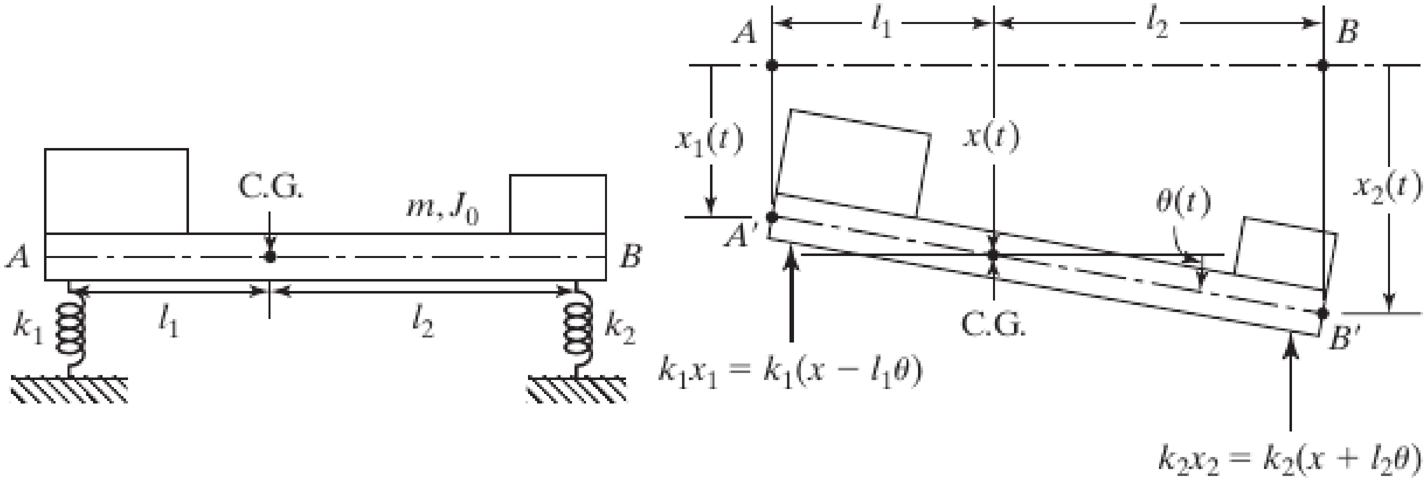
$[k] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ is static uncoupled but $[k] = \begin{bmatrix} k_1 & k_3 \\ k_3 & k_2 \end{bmatrix}$ has static coupling

- A diagonal damping matrix $[c]$ indicates the system is velocity uncoupled. Example

$[c] = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ is velocity uncoupled

- A diagonal mass matrix $[m]$ indicates the system is dynamically uncoupled

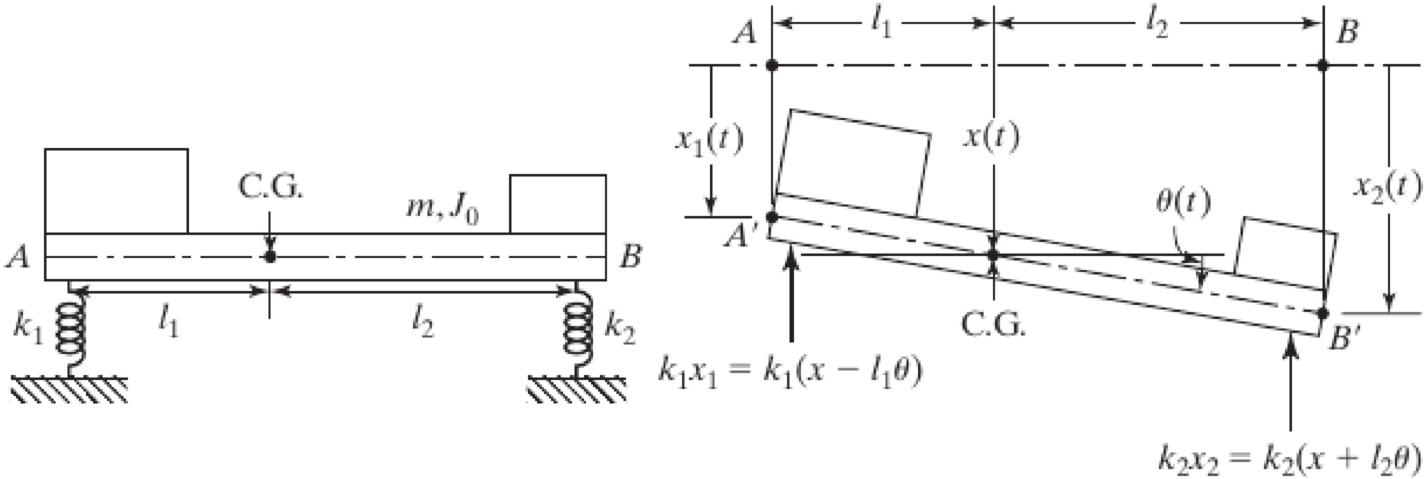
# Example 1




Derive the equations of motion for the 2 DOF system using generalized coordinates $(x, \theta)$

- ❖ Note: no applied loads
- ❖ No dampers

# Example 1



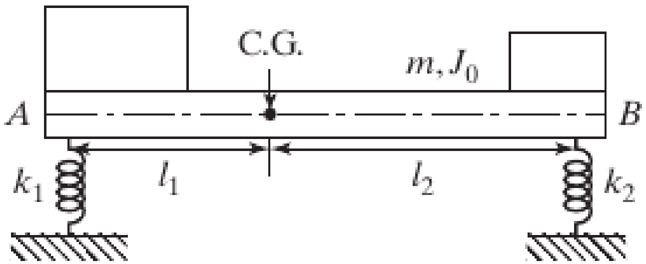

Total kinetic energy (Note $q_1 = x$, $q_2 = \theta$):

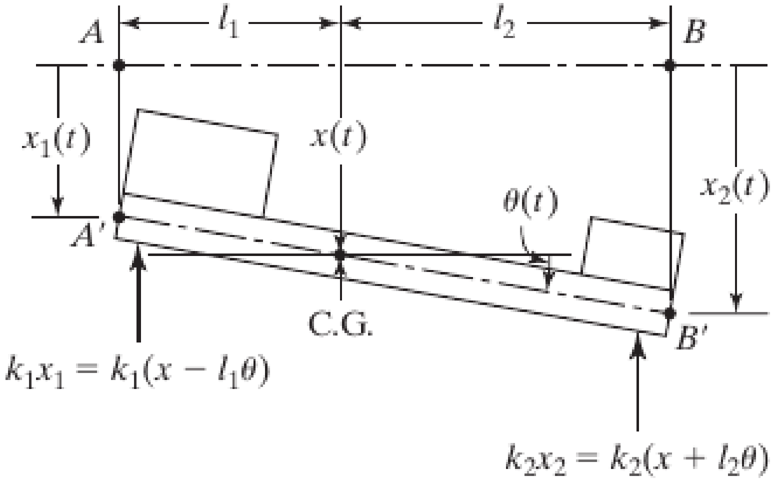
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{\theta}} = J_0\dot{\theta} \quad \text{and} \quad \frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial \theta} = 0$$

# Example 1




Potential energy: $U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 = \frac{1}{2}k_1(x - l_1\theta)^2 + \frac{1}{2}k_2(x + l_2\theta)^2$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial x} = k_1(x - l_1\theta) + k_2(x + l_2\theta)$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial \theta} = -k_1l_1(x - l_1\theta) + k_2l_2(x + l_2\theta)$$

# Example 1

- For $k = 1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} + \frac{\partial U}{\partial q_1} = Q_1$$

$$\frac{\partial}{\partial t}(m\dot{x}) + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0$$

$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0$$

- For $k = 2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} + \frac{\partial U}{\partial q_2} = Q_2$$

$$\frac{\partial}{\partial t}\left(J_0\dot{\theta}\right) - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0$$

$$J_0\ddot{\theta} - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0$$

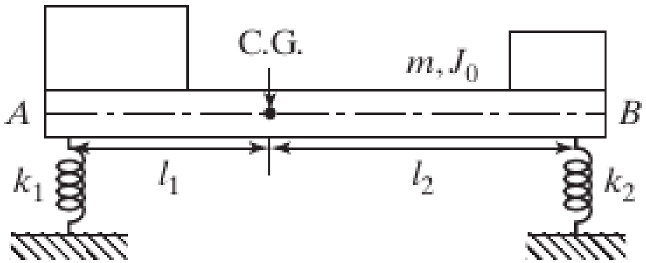
# Example 1

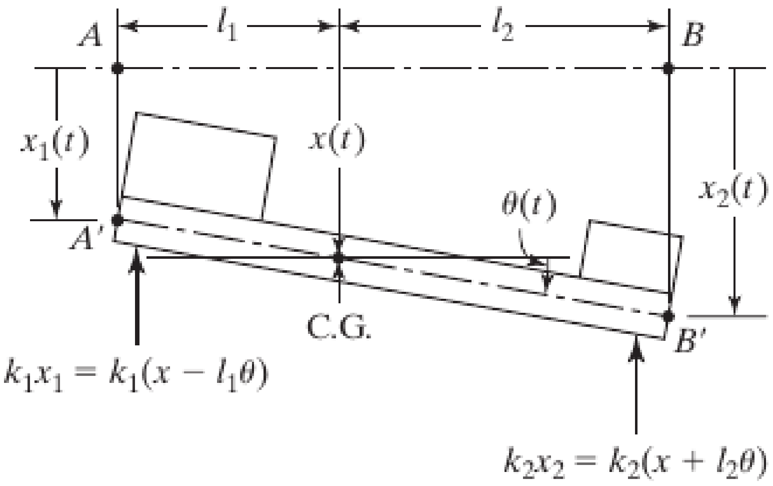
$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0$$
$$J_0\ddot{\theta} - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0$$

❖ Rearrange into matrix form

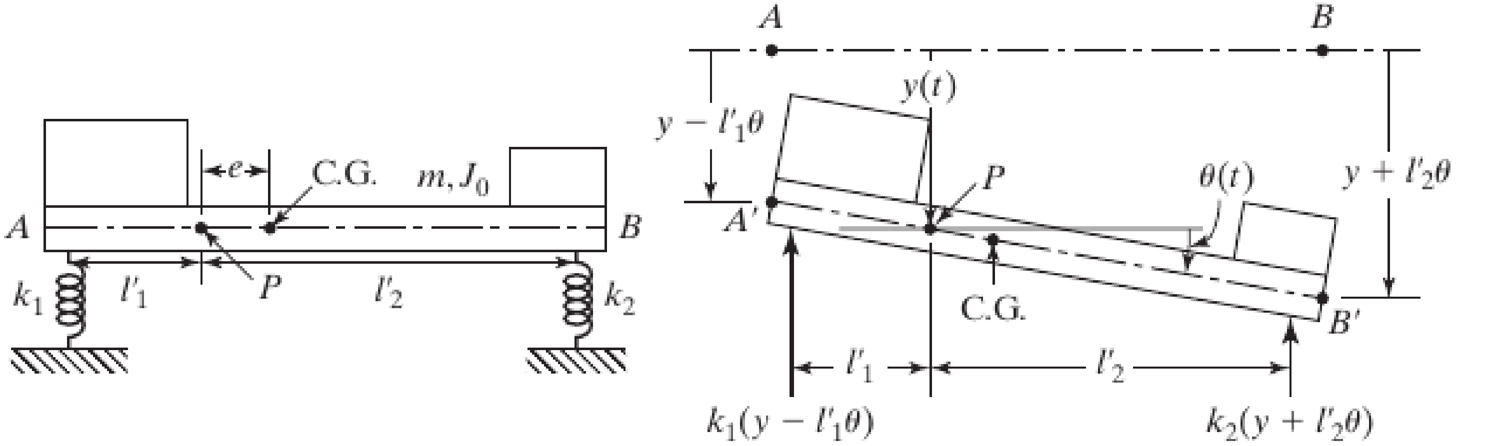
$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

❖ Note that there is static coupling and symmetry of the matrices



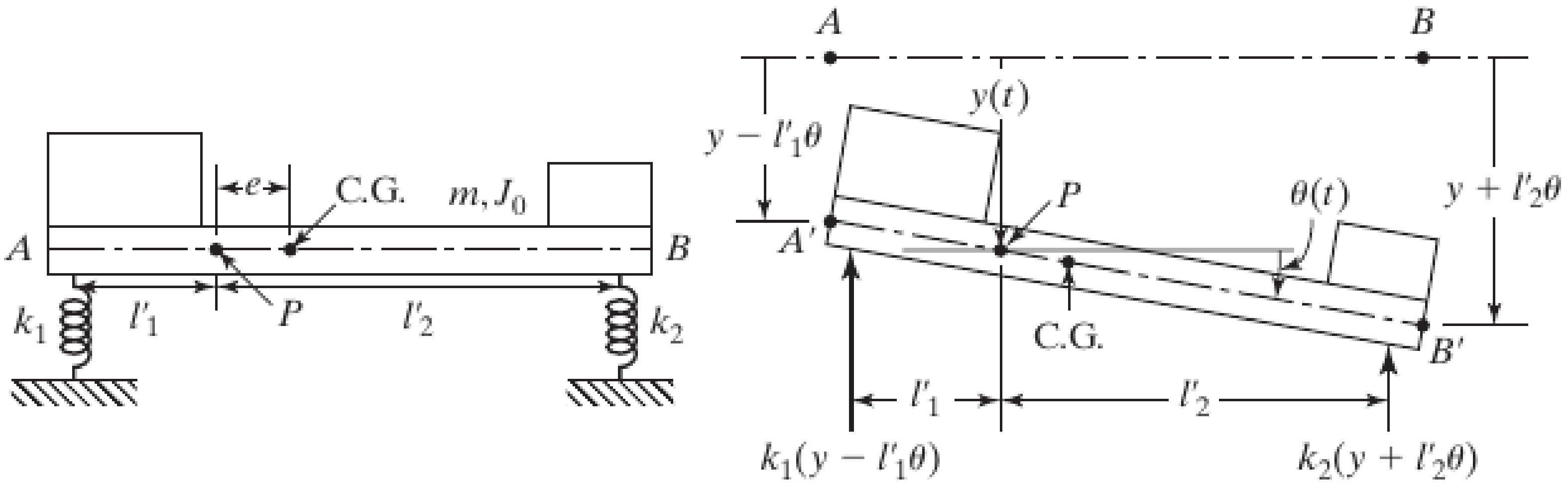
# Example 2




Derive the equations of motion for the 2 DOF system using generalized coordinates $(y, \theta)$ and note that $x = y + e\theta$ or $\dot{x} = \dot{y} + e\dot{\theta}$

- ❖ Same as example 1 but now use different generalized coordinates
- ❖ If the body moves up and down in the *y* direction, the inertia force $m\ddot{y}$, which acts through the center of gravity of the body, induces a rotation in the $\theta$ direction, by virtue of the moment $m\ddot{y}e$
- ❖ Similarly, a motion in the $\theta$ direction induces a motion in the *y* direction due to the force $me\ddot{\theta}$

# Example 2



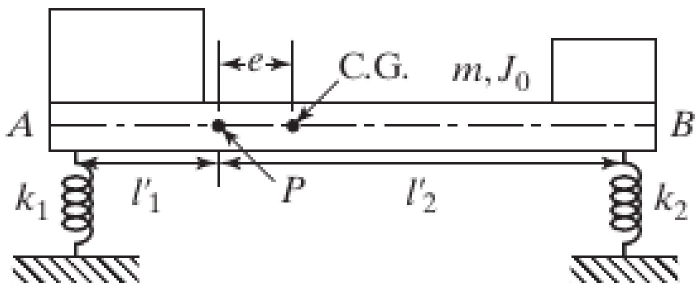

Total kinetic energy (Note $q_1 = y$, $q_2 = \theta$):

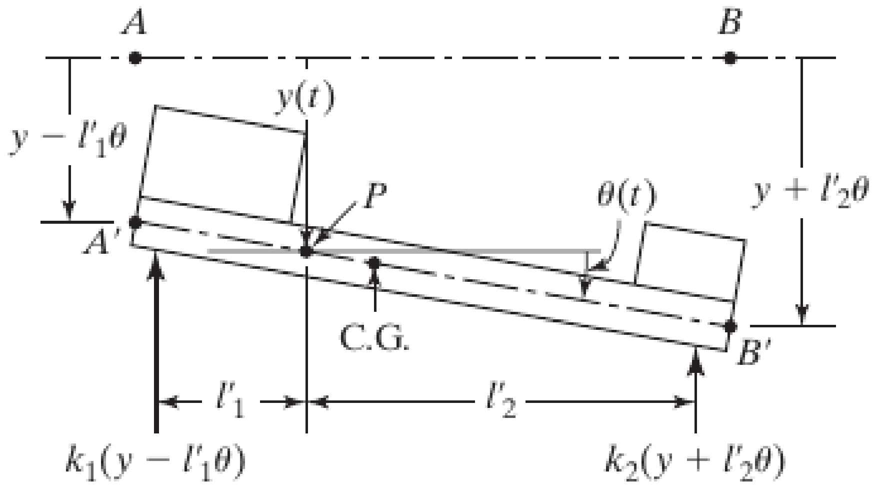
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_p\dot{\theta}^2 = \frac{1}{2}m\left(\dot{y} + e\dot{\theta}\right)^2 + \frac{1}{2}J_p\dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{y}} = m\left(\dot{y} + e\dot{\theta}\right) \quad \text{and} \quad \frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial y} = 0$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{\theta}} = me\left(\dot{y} + e\dot{\theta}\right) + J_p\dot{\theta} \quad \text{and} \quad \frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial \theta} = 0$$

# Example 2




Potential energy: $U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 = \frac{1}{2}k_1(y - l_1^*\theta)^2 + \frac{1}{2}k_2(y + l_2^*\theta)^2$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial y} = k_1(y - l_1^*\theta) + k_2(y + l_2^*\theta)$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial \theta} = -k_1l_1^*(y - l_1^*\theta) + k_2l_2^*(y + l_2^*\theta)$$

# Example 2

- For $k = 1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial D}{\partial \dot{q}_1} + \frac{\partial U}{\partial q_1} = Q_1$$

$$\frac{\partial}{\partial t}\left(m\left(\dot{y} + e\dot{\theta}\right)\right) + k_1(y - l_1^*\theta) + k_2(y + l_2^*\theta) = 0$$

$$m\ddot{y} + me\ddot{\theta} + k_1(y - l_1^*\theta) + k_2(y + l_2^*\theta) = 0$$

- For $k = 2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial D}{\partial \dot{q}_2} + \frac{\partial U}{\partial q_2} = Q_2$$

$$\frac{\partial}{\partial t}\left(me\left(\dot{y} + e\dot{\theta}\right) + J_p\dot{\theta}\right) - k_1 l_1^*(y - l_1^*\theta) + k_2 l_2^*(y + l_2^*\theta) = 0$$

$$me\ddot{y} + J_p\ddot{\theta} - k_1 l_1^*(y - l_1^*\theta) + k_2 l_2^*(y + l_2^*\theta) = 0$$

# Example 2

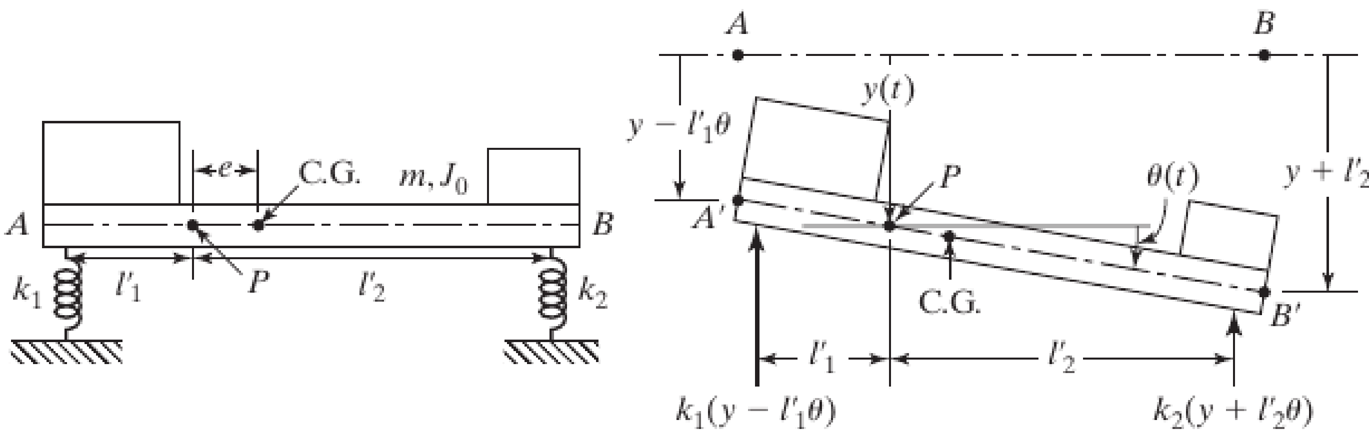
$$m\ddot{y} + me\ddot{\theta} + k_1(y - l_1^*\theta) + k_2(y + l_2^*\theta) = 0$$

$$me\ddot{y} + J_p\ddot{\theta} - k_1 l_1^*(y - l_1^*\theta) + k_2 l_2^*(y + l_2^*\theta) = 0$$

❖ Rearrange into matrix form

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1^* - k_2 l_2^*) \\ -(k_1 l_1^* - k_2 l_2^*) & (k_1 l_1^{*2} + k_2 l_2^{*2}) \end{bmatrix}\begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

❖ Note that there is now static and dynamic coupling and symmetry of the matrices



# Coupling and symmetry

$$[m][\ddot{x}]+[c][\dot{x}]+[k][x]=[F]$$

- The nature of the coupling depends on the coordinates used
- It may be possible to choose a system of coordinates which give equations of motion that are uncoupled both statically and dynamically. Such coordinates are called principal or natural coordinates
- The symmetry of the mass, damping and stiffness matrices is an inherent property of linear vibrating systems. Conservative system (i.e. with no damping or externally applied forces) will give symmetrical mass and stiffness matrices