MEMS1045 Automatic control

Lecture 2

Modeling

Objectives

- Describe the basic elements used in the modeling of electrical and mechanical systems
- Derive the equations of motion for electrical and mechanical systems based on fundamental principles
- Reduce the equations of motion to obtain the transfer functions for specified input and output

Transfer function

The dynamics of a typical system can be approximated as an *n*th-order time-invariant differential equation of the form:

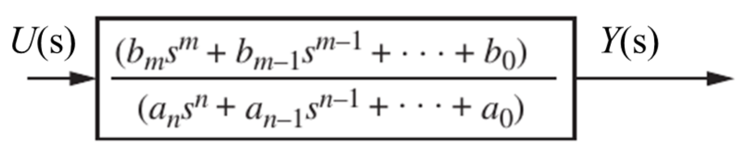
$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_0 u$$

- Output = $y(t)$ and input = $u(t)$
- Laplace transform with zero initial conditions:

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) U(s)$$

$$\frac{Y(s)}{U(s)} = F(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

$$\text{Transfer function} = \frac{Y(s)}{U(s)} = F(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)} = \frac{\text{ouput}}{\text{input}}$$

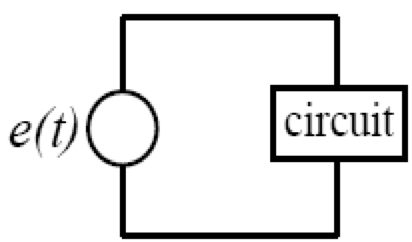


Electrical elements

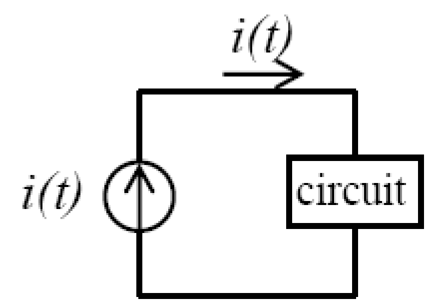
TABLE 2.3 Voltage-current, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	Ls	$\frac{1}{Ls}$

Voltage Source – maintain specified voltage across two points regardless of the required current



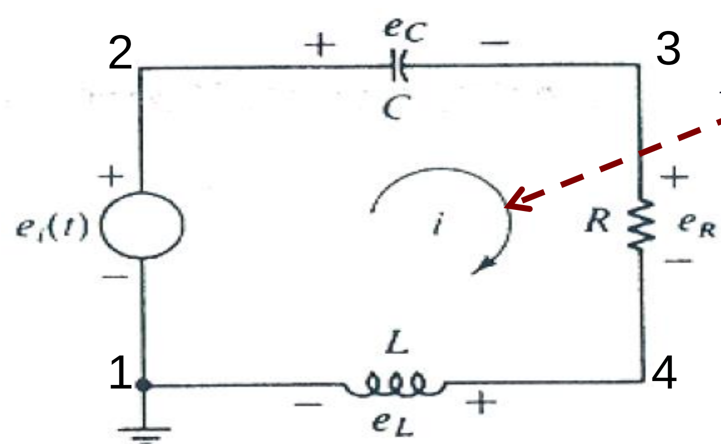
Current Source – maintain specified current regardless of the required voltage



Kirchhoff's voltage law

- The total voltage drop along any closed loop in the circuit is zero

$$\sum_{\substack{\text{Closed}\\\text{Loop}}} (e_i) = 0$$



Ground

$e_0 = 0$

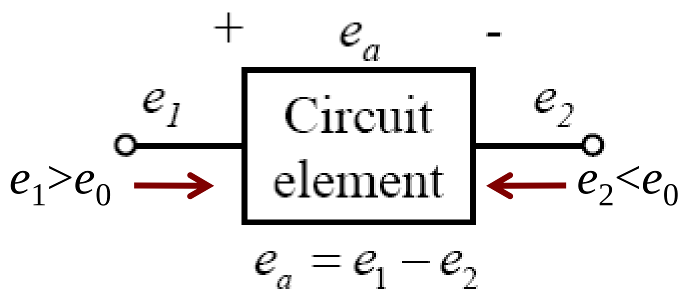
Arrow indicates the flow direction of positive ions
(Current flows from HIGH to LOW potential)

Consider closed loop 1-2-3-4:

$$e_{1-2} + e_{2-3} + e_{3-4} + e_{4-1} = 0$$

$$-e_1 + e_c + e_R + e_L = 0$$

Voltages given in table 2.3

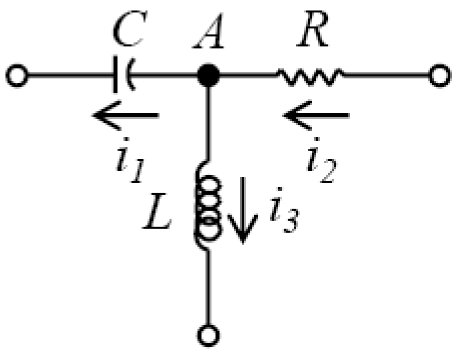


Voltage at a point is the measure electric potential difference from an arbitrary reference

Kirchhoff's current law

- The algebraic sum of the currents at any node in the circuit is zero (i.e. currents in = current out!)

$$\sum_{any\ node} (i_i) = 0$$



Note: away from node is "+"; into the node is "-"

At node 'A':

$$i_1 - i_2 + i_3 = 0$$

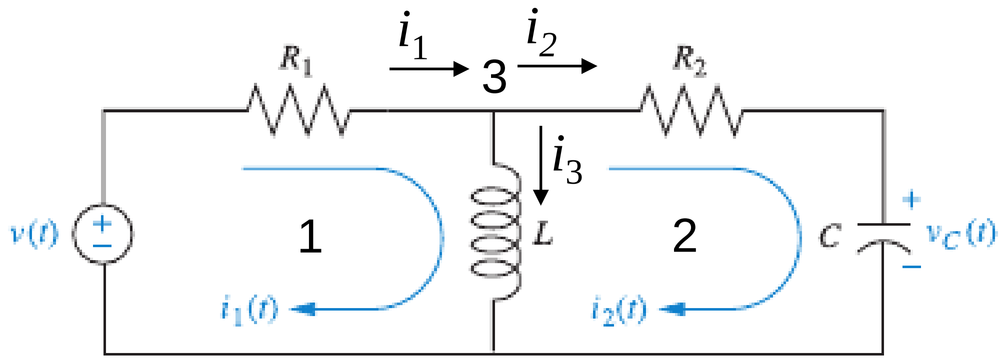
Currents given in table 2.3

Modeling procedure

1. Identify and divide the system into idealized elements. Identify the ground. Label each element, each node and the corresponding voltages and currents
2. Write physical laws to the elements
3. Apply interconnection:
   - For each loop, apply Kirchhoff's Voltage Law
   - At all nodes, apply Kirchhoff's current law
4. Combine the equations to obtain the Equations of Motion (EOM), which are linear time-invariant ordinary differential equations
5. Reduce to input-output form to get the transfer function of desired input and output

Example 1

Derive the transfer function for the circuit shown relating output current $i_2(t)$ to input voltage $v(t)$



KCL at node 3:

$$-i_1 + i_2 + i_3 = 0$$

KVL in loop '1':

$$-v + i_1 R_1 + L\frac{di_3}{dt} = 0$$

$$-V(s) + I_1(s)R_1 + Ls(I_1(s) - I_2(s)) = 0$$

KVL in loop '2':

$$i_2 R_2 + \frac{1}{C}\int i_2 dt - L\frac{di_3}{dt} = 0$$

$$I_2(s)R_2 + \frac{1}{Cs}I_2(s) - Ls\big(I_1(s) - I_2(s)\big) = 0$$

$$I_1(s) = \frac{1}{Ls}\left(Ls + \frac{1}{Cs} + R_2\right)I_2(s)$$

$$I_1(s) = \left(1 + \frac{1}{LCs^2} + \frac{R_2}{Ls}\right)I_2(s)$$

Example 1

$$-V(s)+I_1(s)R_1+Ls(I_1(s)-I_2(s))=0$$
$$V(s)=(R_1+Ls)I_1(s)-LsI_2(s)$$

$$V(s)=(R_1+Ls)\left(1+\frac{1}{LCs^2}+\frac{R_2}{Ls}\right)I_2(s)-LsI_2(s)$$

$$V(s)=\left(R_1+\frac{R_1}{LCs^2}+\frac{R_1R_2}{Ls}+\frac{Ls}{LCs^2}+\frac{LsR_2}{Ls}\right)I_2(s)$$

$$V(s)=\left((R_1+\frac{R_1}{LCs^2}+\frac{R_1R_2}{Ls}+\frac{1}{Cs}+R_2\right)I_2(s)$$

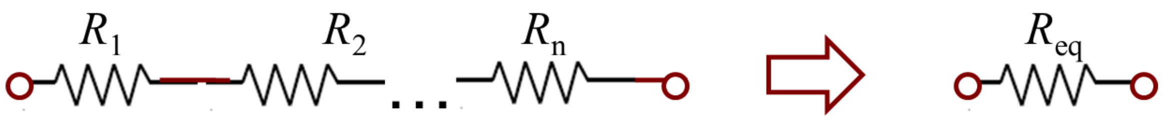
$$V(s)=\left(\frac{(R_1+R_2)LCs^2+R_1R_2Cs+Ls+R_1}{LCs^2}\right)I_2(s)$$

$$\frac{I_2(s)}{V(s)}=\frac{LCs^2}{(R_1+R_2)LCs^2+R_1R_2Cs+Ls+R_1}$$

Resistors in series and parallel

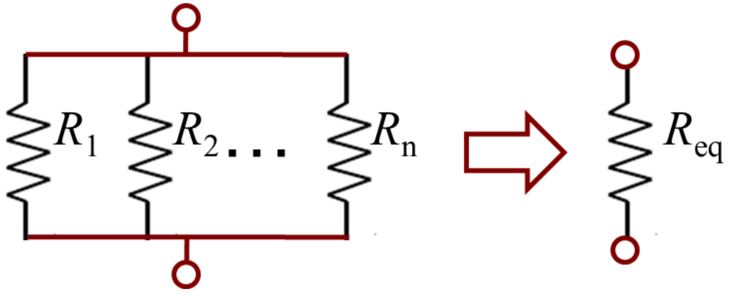
Resistors connected in series:

$$R_{eq} = R_1 + R_2 + \cdots + R_n = \sum_{i=1}^{n} R_i$$



Resistors connected in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$



Note: Resistance can be replaced with impedance

Voltage divider

Voltage divider is used to adjust the output voltage level:

$$-i_{R1} - i_1 + i_{R2} = 0$$

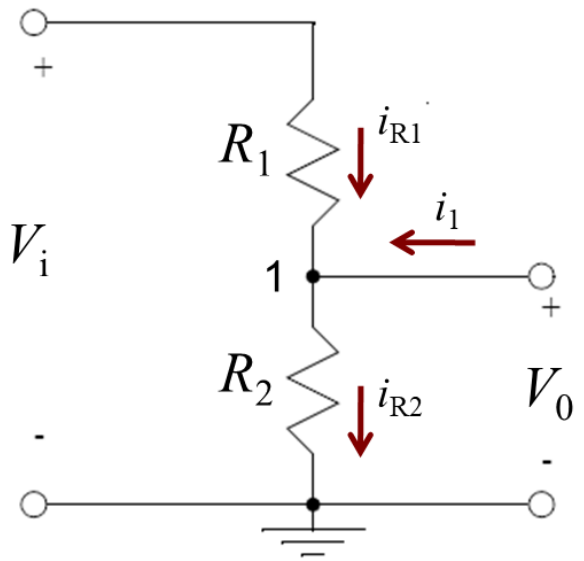
$$-\left(\frac{V_i - V_0}{R_1}\right) - i_1 + \left(\frac{V_0}{R_2}\right) = 0$$

For large output resistance, i_1 is negligible, i.e. $i_1=0$

$$\left(\frac{V_0}{R_2}\right) = \left(\frac{V_i - V_0}{R_1}\right)$$

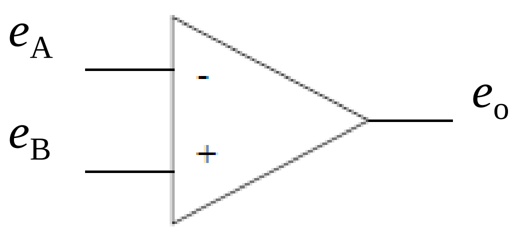
$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_0 = \left(\frac{1}{R_1}\right) V_i$$

$$\frac{V_0}{V_i} = \left(\frac{R_2}{R_1 + R_2}\right)$$



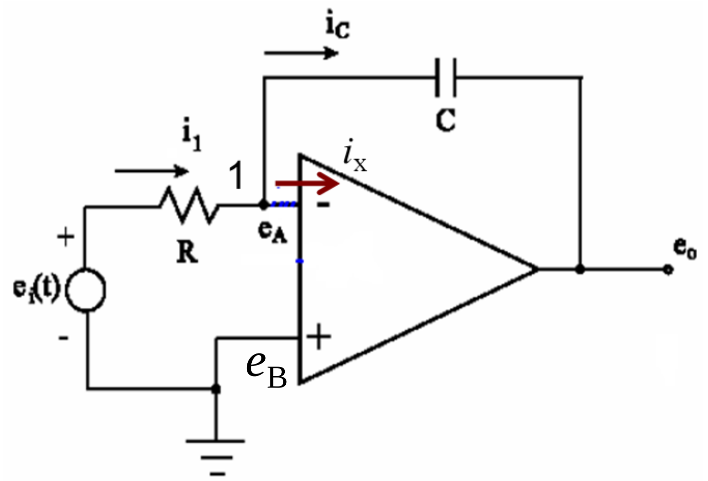
Operational amplifiers

- Also called “op-amps” are used in circuits to amplify, add, remove, integrate, and invert signals
- It has 2 input terminals (- for inverting input terminal e_A and + for non-inverting input terminal e_B)
- It has 1 output terminal e_o
- Current flowing into the amplifier is negligible
- The voltage gain A is very large with $A \approx 10^5$
- $e_o = A(e_B - e_A)$



Example 2

Derive the transfer function for the circuit shown relating output voltage e_0(t) to input voltage $e_i(t)$ when op-amp gain A is very large



KCL at node '1':

$$-i_1 + i_x + i_C = 0$$

where $i_x \approx 0$

$$i_1 = \frac{(e_i - e_A)}{R}$$

$$i_C = C\frac{d(e_A - e_0)}{dt}$$

Hence

$$C\frac{d(e_A - e_0)}{dt} = \frac{(e_i - e_A)}{R}$$

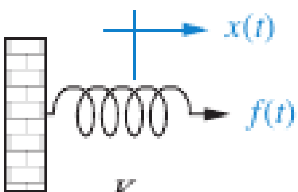
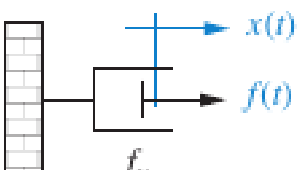
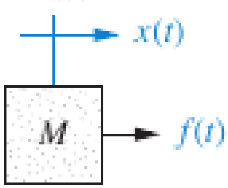
But $e_o = A(e_B - e_A) = A(-e_A)$

Or $e_A = -\frac{1}{A}e_o \approx 0$ for large A

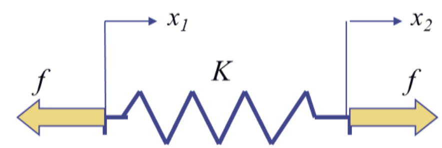
$$-C\dot{e}_0 = \frac{e_i}{R} \text{ or } \frac{E_o(s)}{E_i(s)} = -\frac{1}{RCs}$$

Mechanical elements

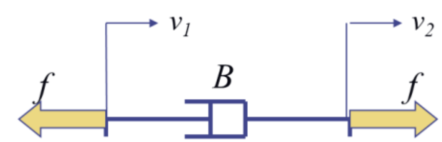
TABLE 2.4 Force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring  K	$f(t) = Kx(t)$	K
Viscous damper  f_v	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Spring can only be tension or compression



$$f = K(x_2 - x_1)$$



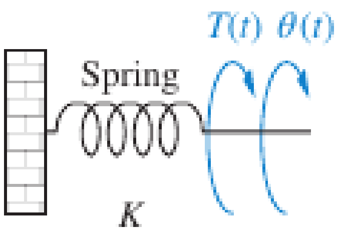
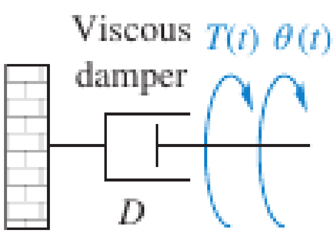
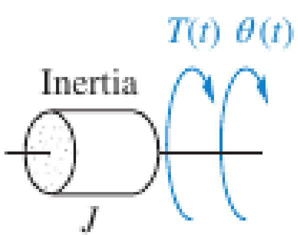
$$f = B(\dot{x}_2 - \dot{x}_1)$$

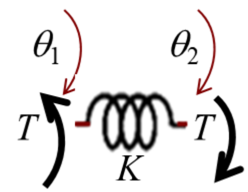
All points on the body must move with the same velocity

Reference arrow indicates positive direction (**does not imply actual direction of motion or force**)

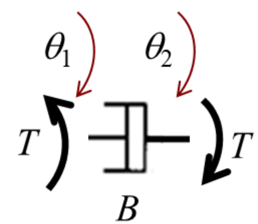
Mechanical elements

TABLE 2.5 Torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\theta(t)$	K
	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2



$$T = K(\theta_2 - \theta_1)$$



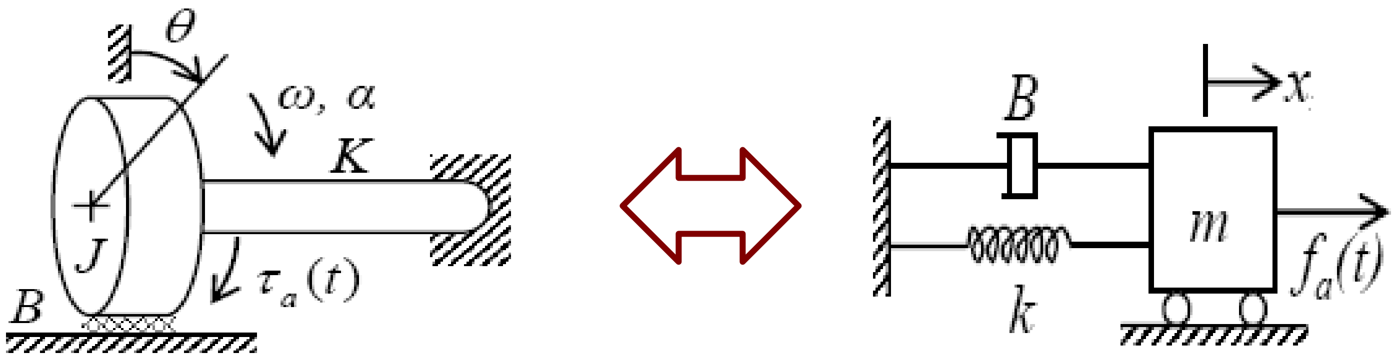
$$T = B(\dot{\theta}_2 - \dot{\theta}_1)$$

Reference arrow indicates positive direction (**does not imply actual direction of motion or torque**)

Rotational vs translational

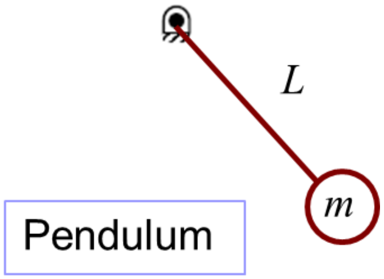
Rotational variables:

Notation	Variable	Units	Translational Analog
θ	Angular displacement	radians (rad)	Displacement (x, m)
$\omega = \dot{\theta}$	Angular velocity	rad/s	Velocity (v, m/s)
$\alpha = \ddot{\theta}$	Angular acceleration	rad/s^2	Acceleration (a, m/s^2)
τ	**Torque**	N-m	**Force** (F, N)
J	Moment of inertia	Kg-m^2	Mass (M, kg)

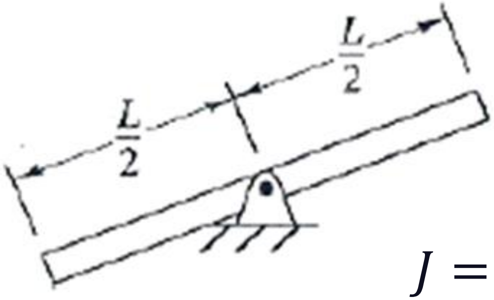


Moment of inertia

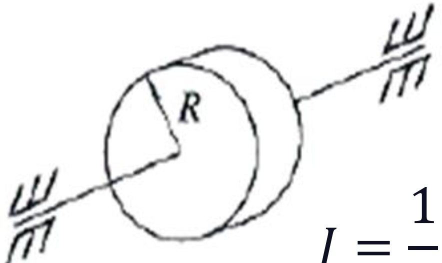
- Moment of inertia for a point mass m (example pendulum) is $J = mL^2$



- Moment of inertia J for many regular shapes can be found in books. Examples:

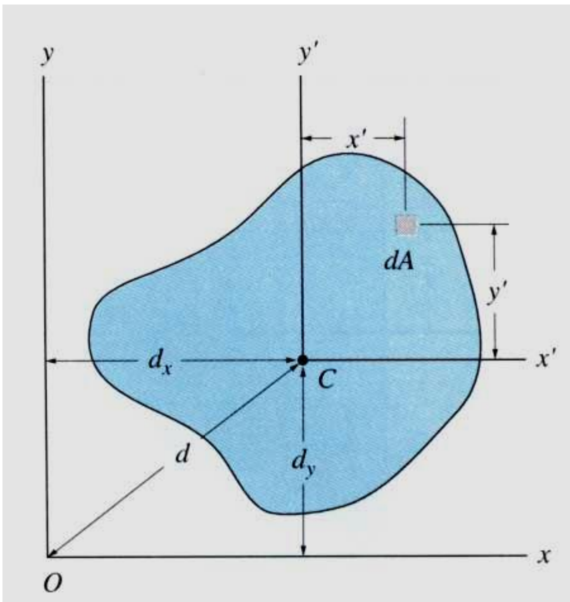


$$J = \frac{1}{12} mL^2$$



$$J = \frac{1}{2} mR^2$$

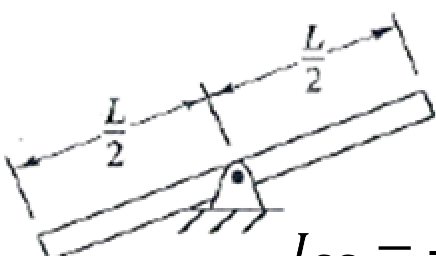
Parallel-axis theorem



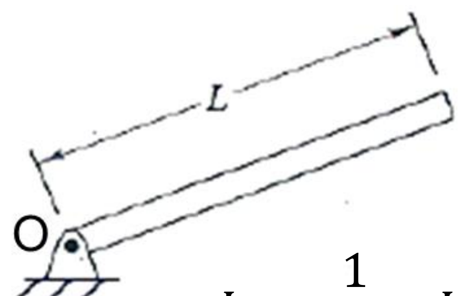
This theorem relates the moment of inertia of an area about an axis passing through the area's center of mass to the moment of inertia of the area about a corresponding parallel axis. This theorem has many practical applications, especially when working with composite areas.

$$J_O = \bar{J}_{CG} + md^2$$

Example



$$J_{CG} = \frac{1}{12}mL^2$$



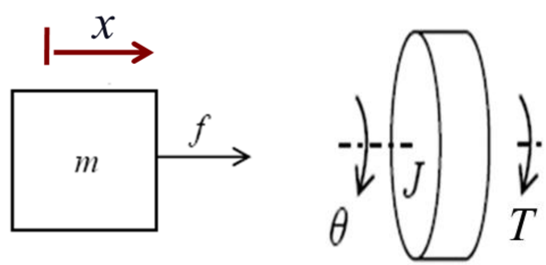
$$d = \frac{L}{2}$$

$$J_O = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$$

Newton's laws

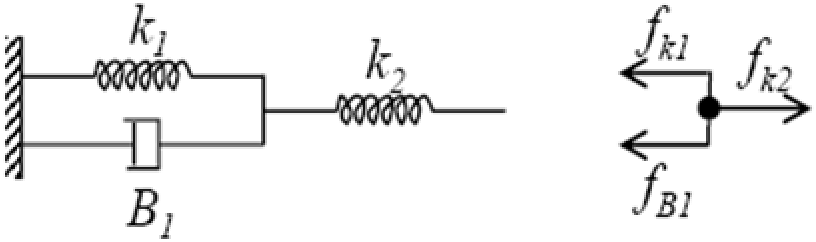
- Newton's 2nd law (at a mass):

$m\ddot{x} = \sum f$ or $m\ddot{\theta} = \sum T$

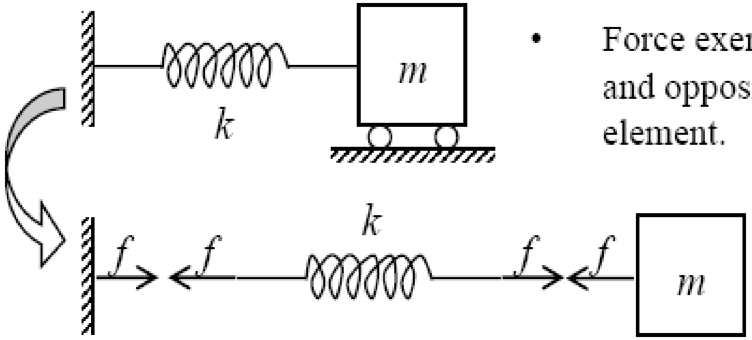


- Newton's 2nd law (at a connection point with no mass):

$\sum f = 0$ or $\sum T = 0$



- Newton's 3rd law: All forces occur in equal and opposite pairs (action/reaction)



- Force exerted by an element is equal and opposite to the force on the element.

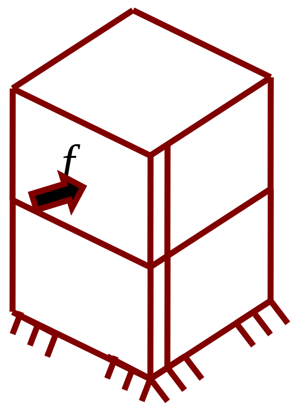
Modeling procedure

1. Identify and divide the system into idealized elements along with reference point and positive direction
2. Draw the free-body-diagram for each element
3. Write physical laws to the elements
4. Apply interconnection laws between elements
5. Combine the equations to obtain the Equations of Motion (EOM), which are linear time-invariant ordinary differential equations
6. Reduce the equations to input-output form to get the transfer function of desired input and output

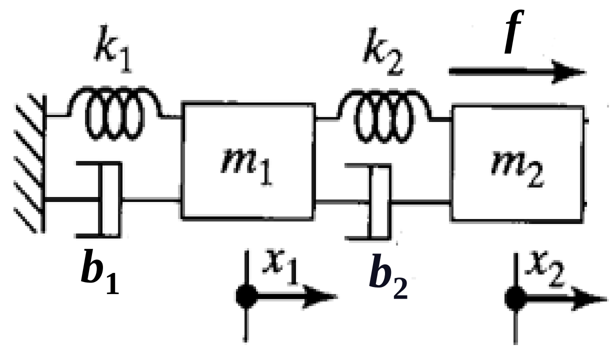
Example 3

Derive the equations of motion for the given idealized model of the 2-storey container building subjected to input wind load f and output x_2.

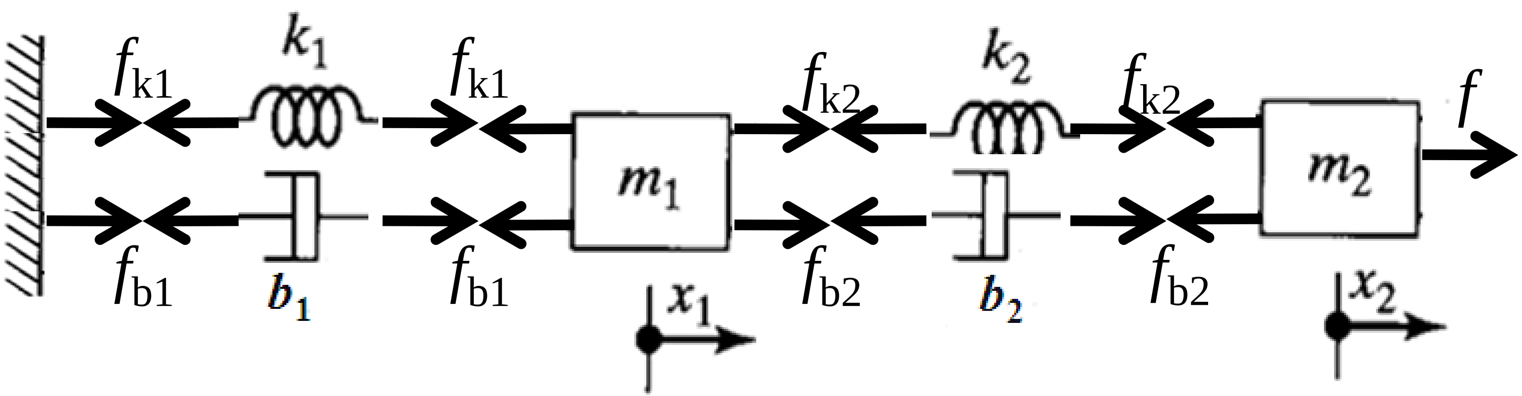
Input = f and
Outputs = x_1, x_2



Idealized model

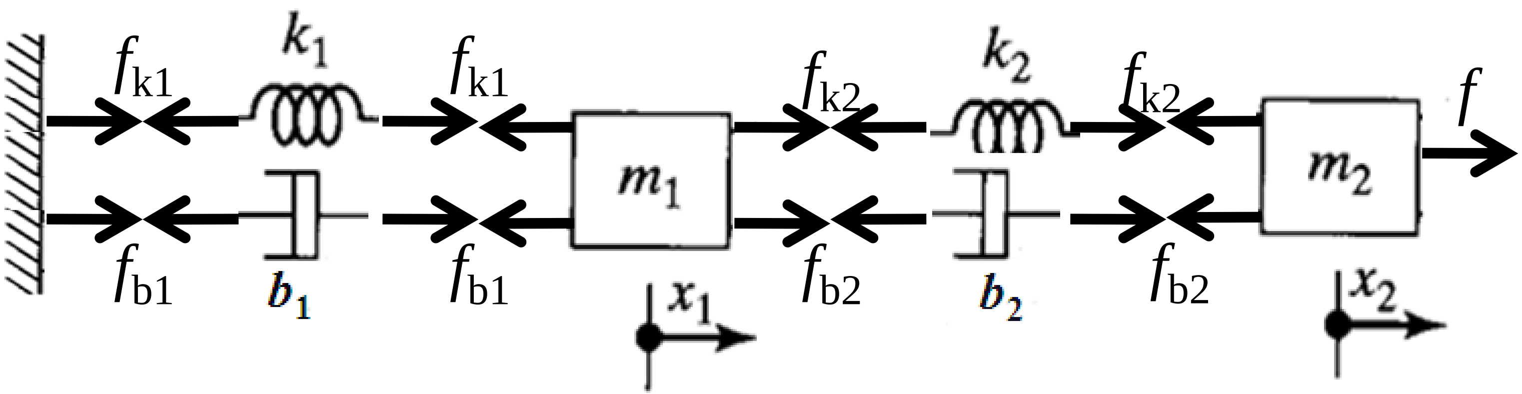


FBD:



Example 3

FBD:



$$f_{k1} = k_1 x_1 \qquad f_{k2} = k_2 (x_2 - x_1) \qquad m_1 \ddot{x}_1 = -f_{k1} - f_{b1} + f_{k2} + f_{b2}$$

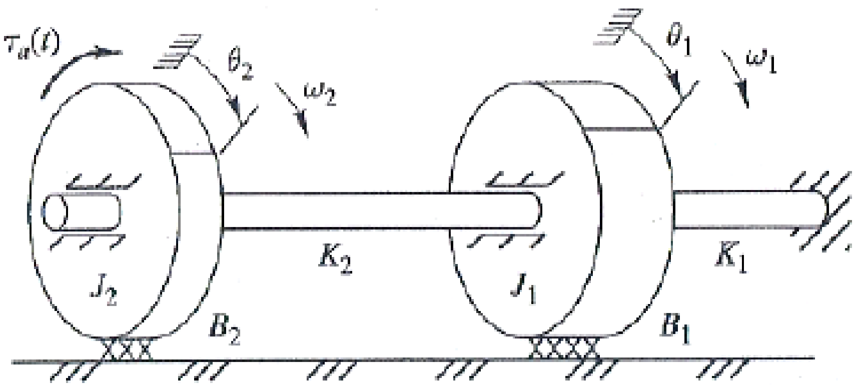
$$f_{b1} = b_1 \dot{x}_1 \qquad f_{b2} = b_2 (\dot{x}_2 - \dot{x}_1) \qquad m_2 \ddot{x}_2 = -f_{k2} - f_{b2} + f$$

$$m_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 - b_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

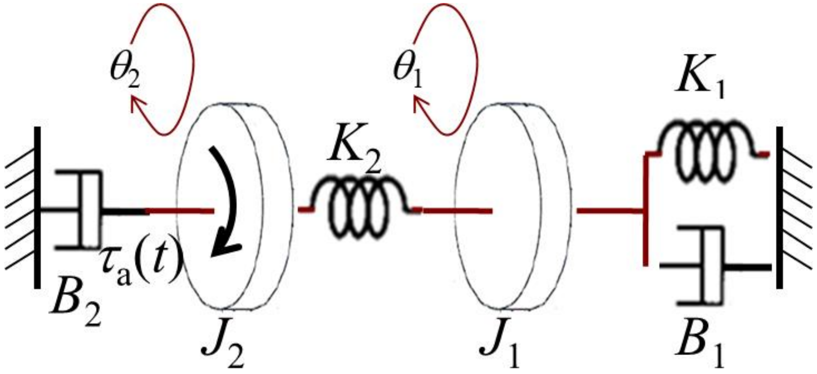
$$m_2 \ddot{x}_2 - b_2 \dot{x}_1 + b_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = f$$

Example 4

Derive the equations of motion for the system shown

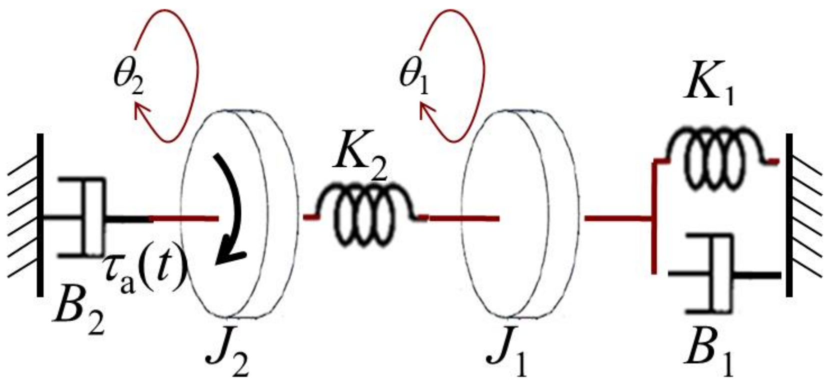


Input torque = τ_a
Outputs = θ_1, θ_2



Idealized model

Example 4



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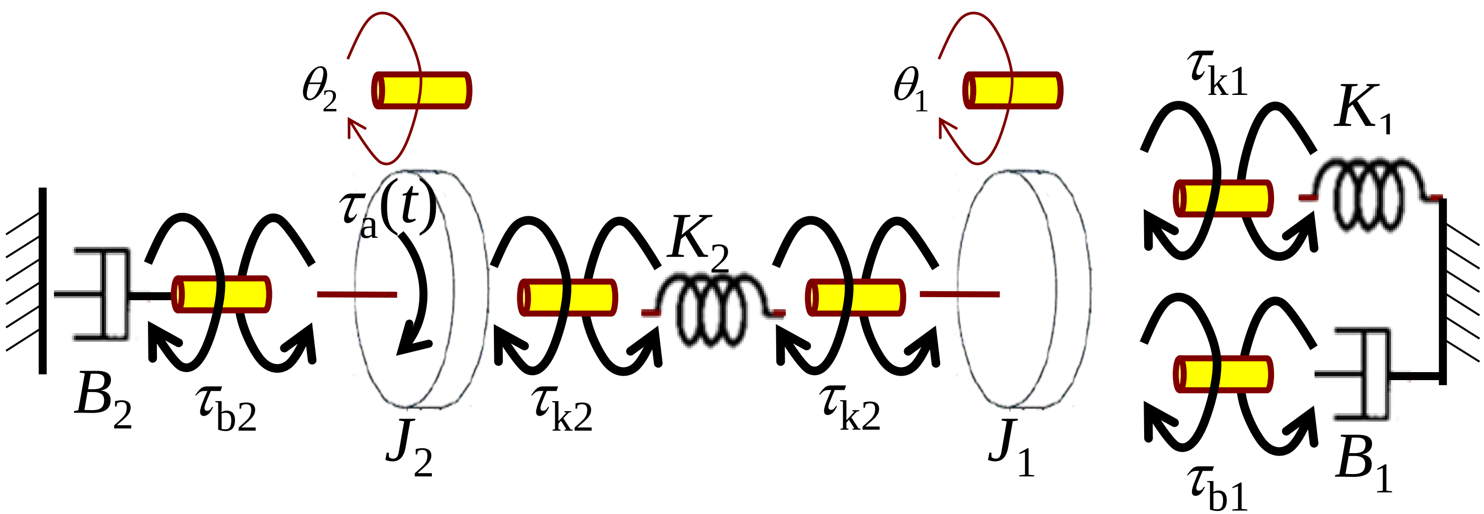
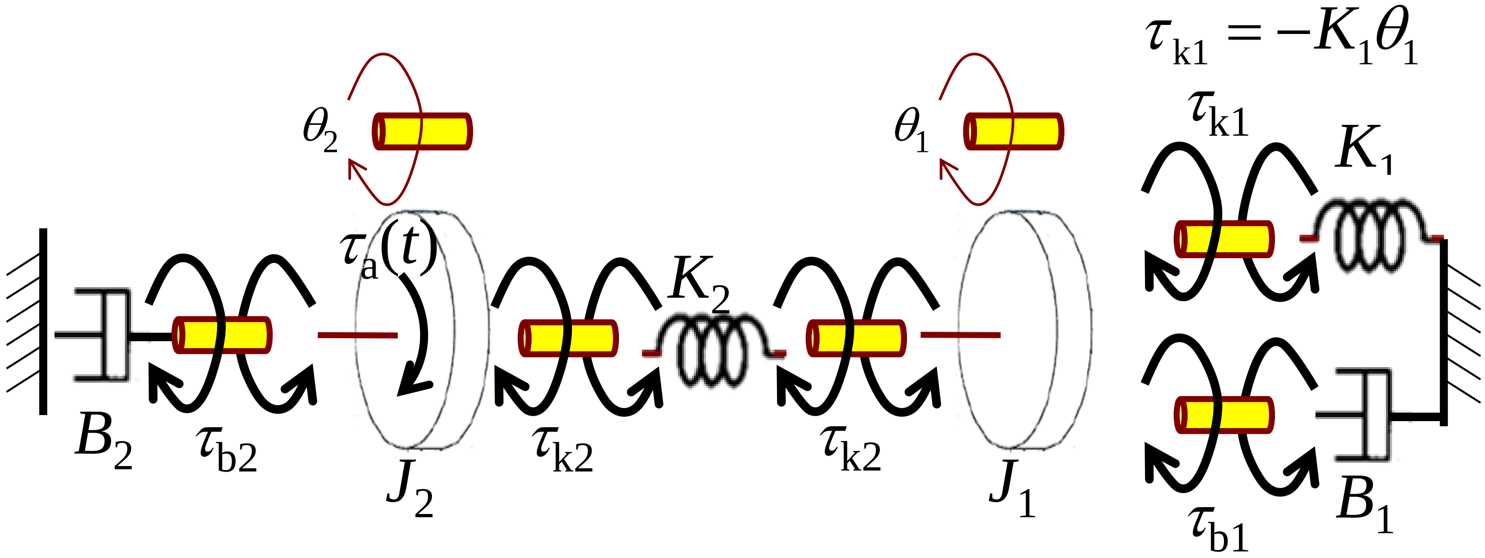


Example 4



$$\tau_{b2} = B_2\dot{\theta}_2 \qquad \tau_{k2} = K_2\left(\theta_1 - \theta_2\right) \qquad \tau_{b1} = -B_1\dot{\theta}_1$$

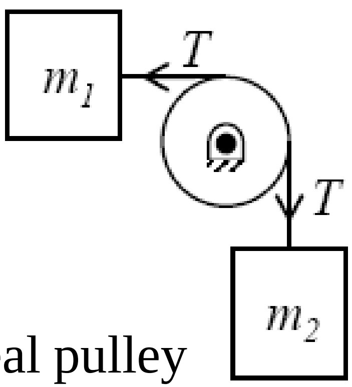
$$J_2\ddot{\theta}_2 = \tau_a + \tau_{k2} - \tau_{b2} \qquad J_1\ddot{\theta}_1 = \tau_{k1} + \tau_{b1} - \tau_{k2}$$

$$J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 - K_2\theta_1 + K_2\theta_2 = \tau_a \qquad J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \left(K_1 + K_2\right)\theta_1 - K_2\theta_2 = 0$$

Ideal pulley element

- ❖ The pulley element is used to change the direction of force (or motion)
- ❖ The assumptions for ideal pulley are:

1. No mass, no friction, no slippage between cable and cylinder
2. Cable is always in **tension**

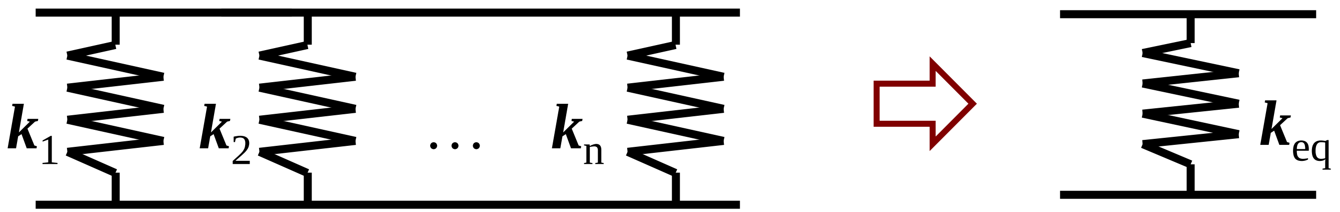


Ideal pulley

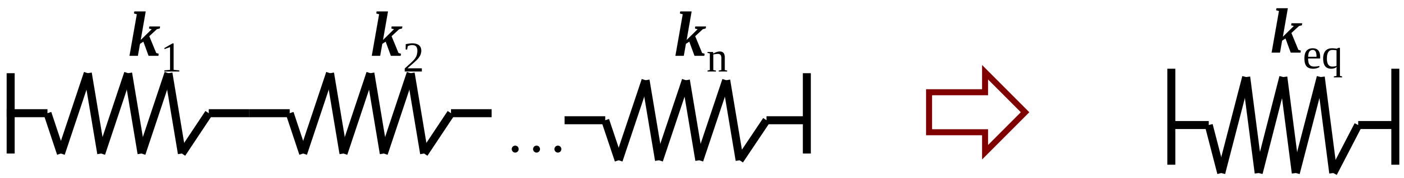
If the pulley is NOT ideal, its mass and frictional effects must be considered

Combining springs

For springs in parallel: $$k_{eq} = k_1 + k_2 + \cdots + k_n = \sum_{i=1}^{n} k_i$$

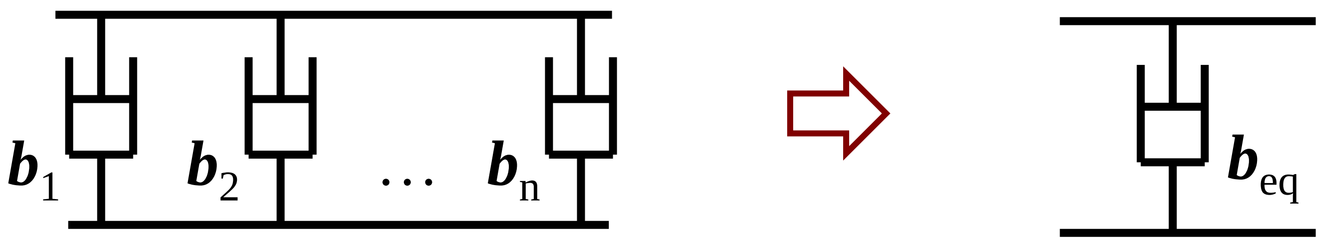


For springs in series: $$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = \sum_{i=1}^{n} \frac{1}{k_i}$$

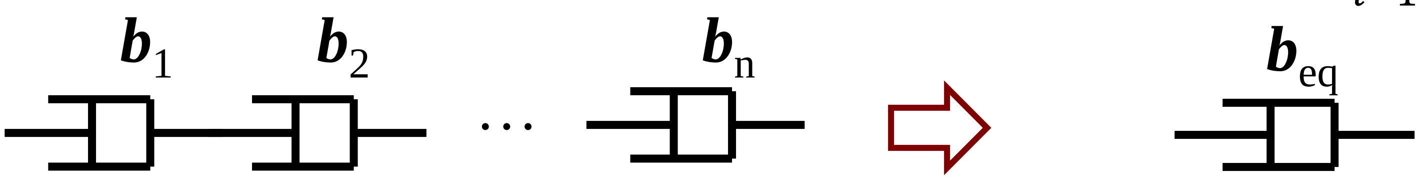


Combining dampers

For dampers in parallel: $$b_{eq} = b_1 + b_2 + \cdots + b_n = \sum_{i=1}^{n} b_i$$

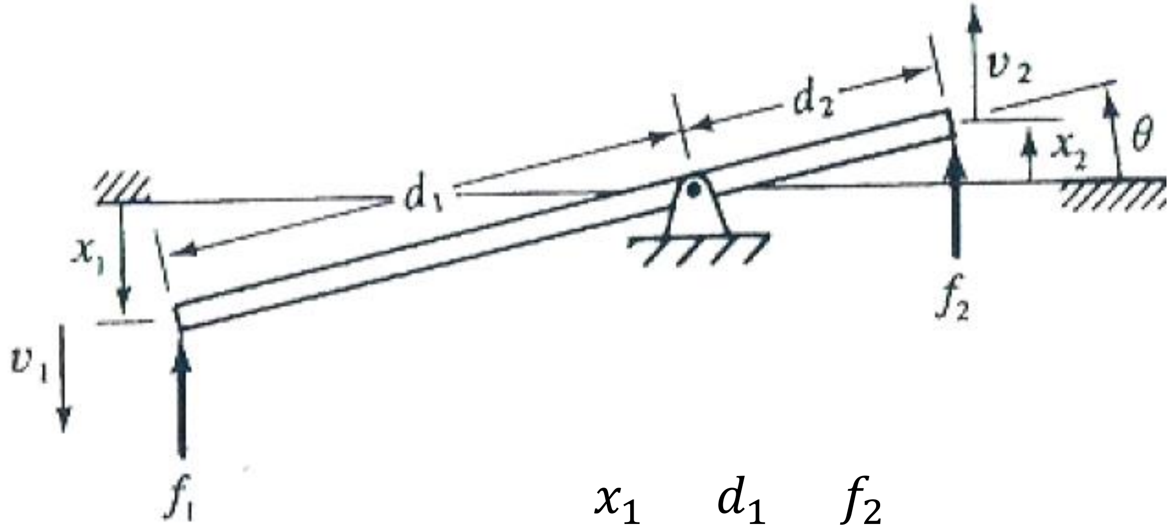


For dampers in series: $$\frac{1}{b_{eq}} = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} = \sum_{i=1}^{n} \frac{1}{b_i}$$



Levers

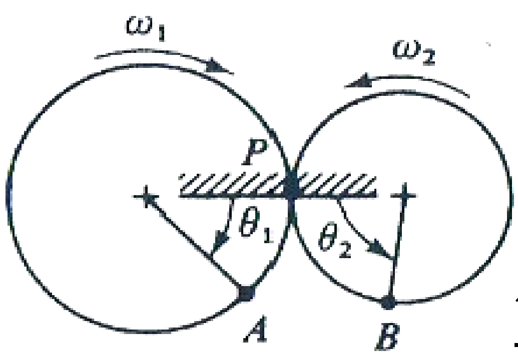
- The lever is a motion transformer element
- The assumptions for ideal lever are: No mass, no friction, and rigid (i.e. angle of rotation θ is small)



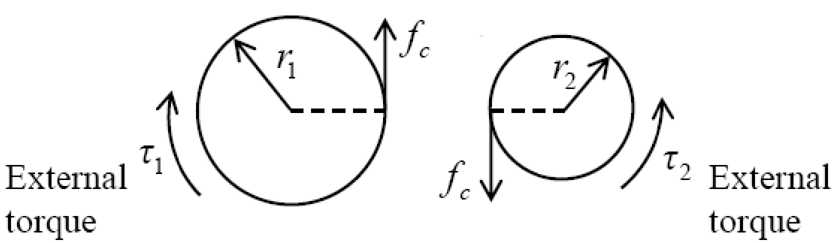
$$\frac{x_1}{x_2} = \frac{d_1}{d_2} = \frac{f_2}{f_1}$$

Gears

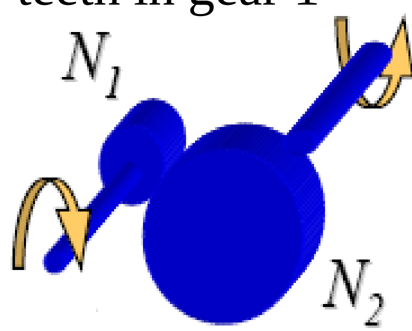
- Gears are motion transformer elements used to change direction of rotational motion usually to decrease angular velocity and increase torque
- The assumptions for ideal gears are: No inertia, no friction, rigid and perfect meshing



$$\frac{\tau_2}{\tau_1} = -\frac{r_2}{r_1} = -N$$



Number of teeth in gear 1



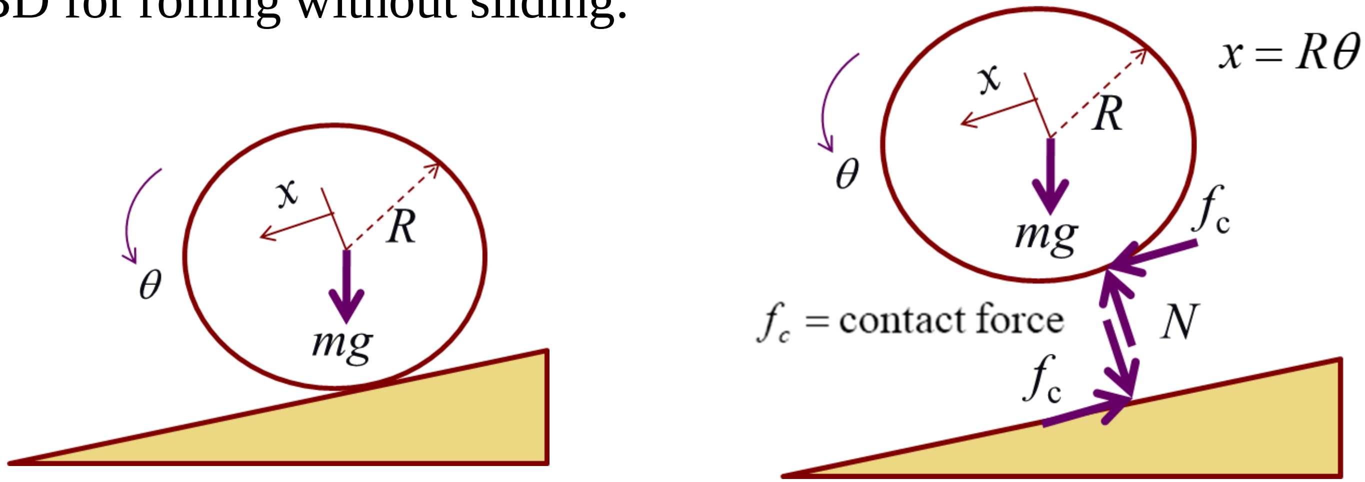
Number of teeth in gear 2

$$\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} = \frac{N_2}{N_1} = N$$

N = gear ratio

Rolling and sliding motion

FBD for rolling without sliding:



- Without friction the cylinder would slide: the translational motion of CG and the rotational motion about the CG are independent of each other
- If the cylinder rolls without sliding, a static friction acts at all times such that $x=R\theta$