MEMS1045 Automatic control

Lecture 5

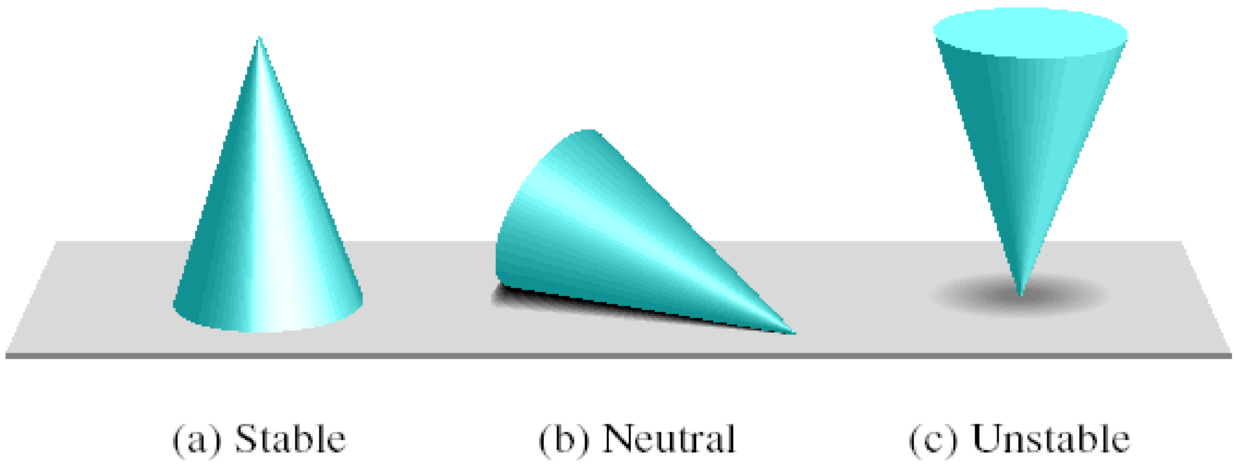
Stability

Objectives

- Explain the concept of stability
- Construct a Routh table based on the transfer function characteristics equation
- Analyze the system stability using the Routh table
- Analyze the system stability of systems represented in state space format

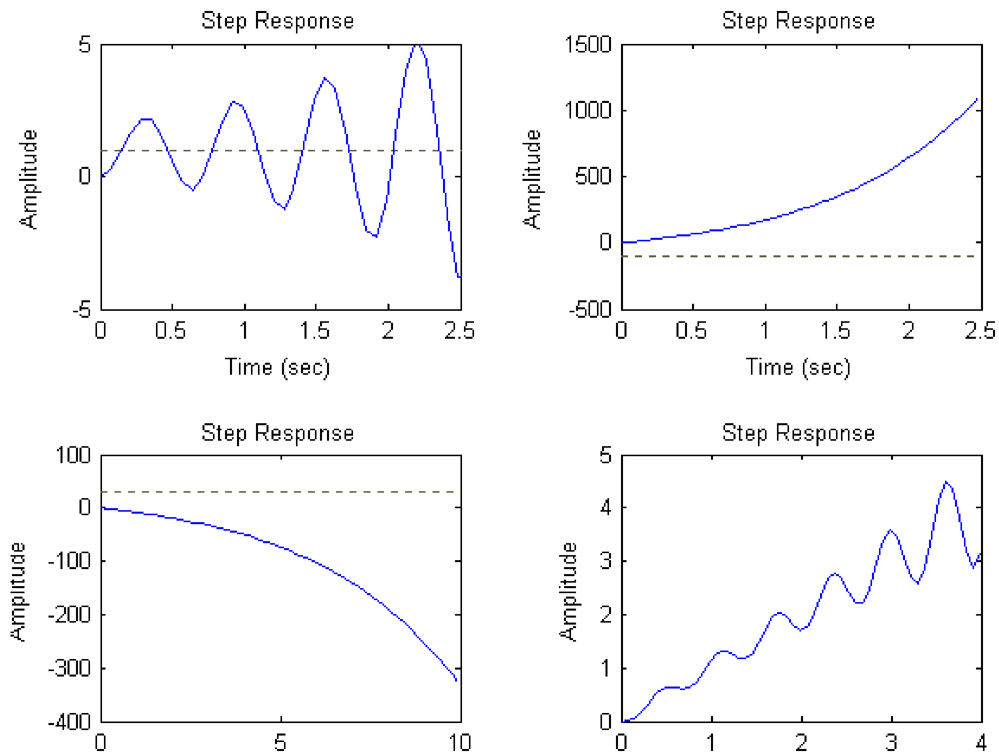
Concept of stability

- Stability is the tendency of a system to return to a condition of static equilibrium after being disturbed
- Stability is a property of the system and is independent of the input or driving function
- Illustration of the concept of stability:



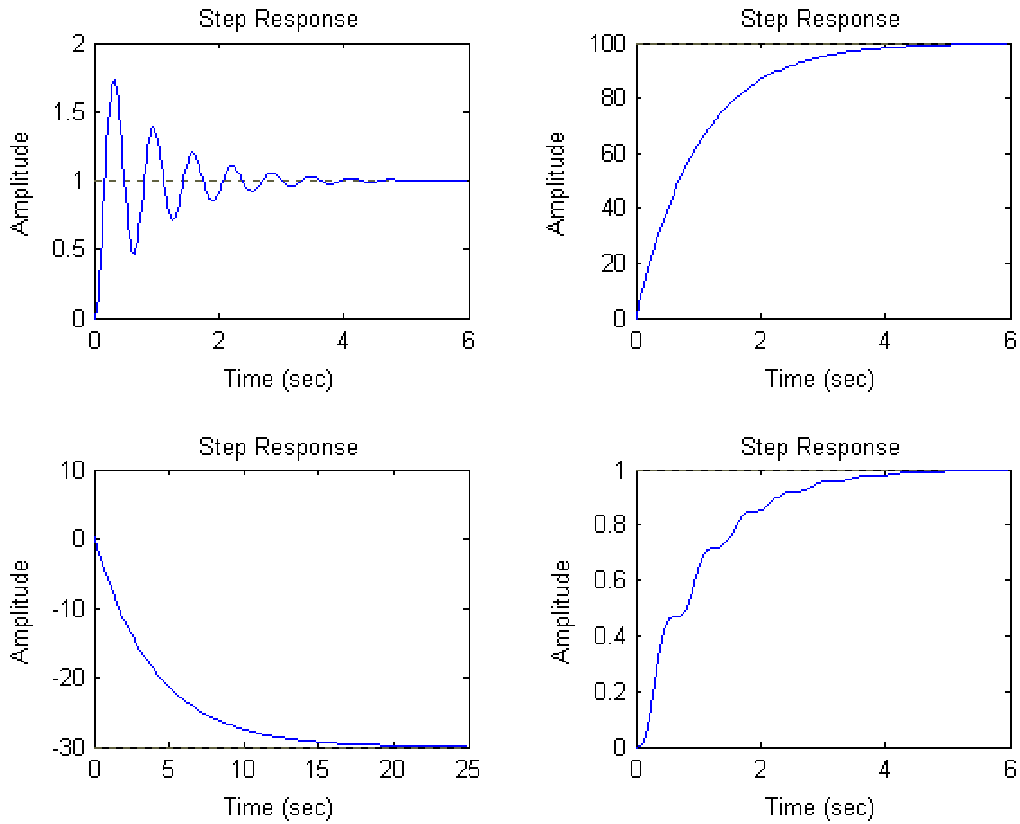
Concept of stability

- A system is stable if every bounded input yields a bounded output (BIBO stability)
- Unstable system has unbounded response to bounded input



Concept of stability

- Stable system has bounded response to bounded input
- A system is considered marginally stable if only certain bounded inputs will result in a bounded output



Concept of stability

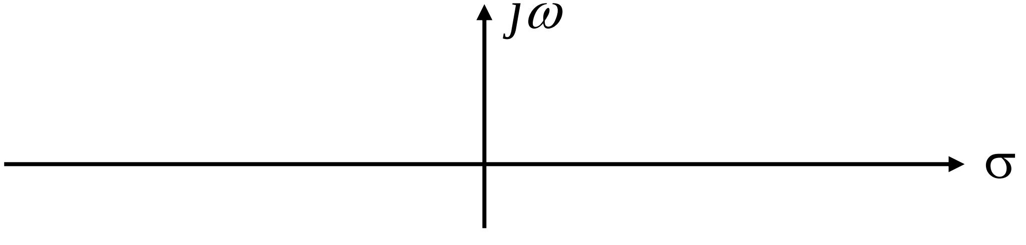
The stability of a feedback system is directly related to the location of the poles of the system transfer function (CLTF):

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

The poles are the roots of the characteristics equation:

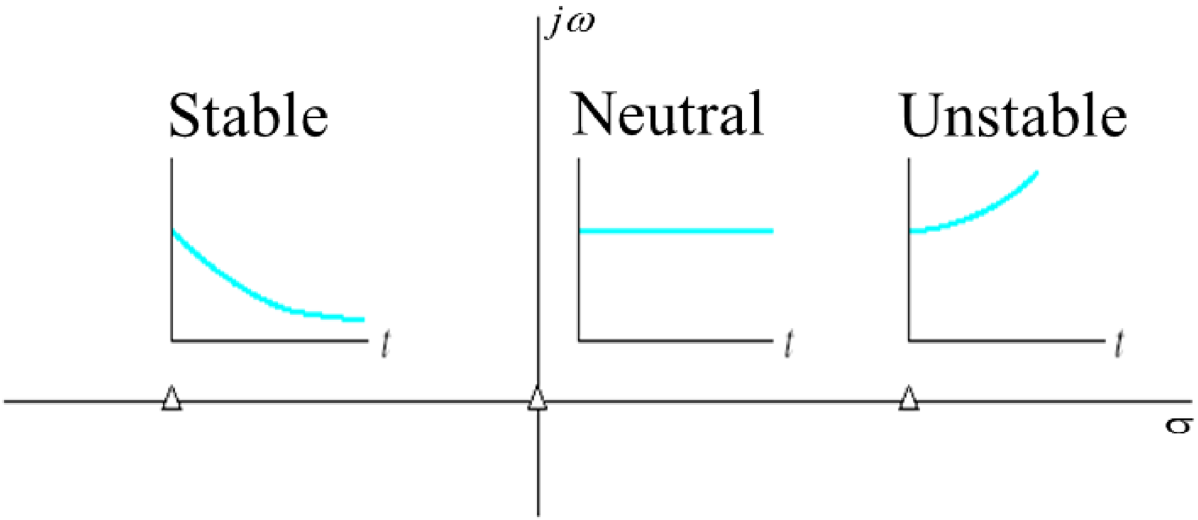
$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

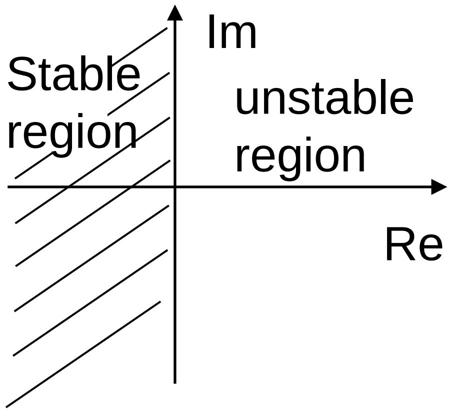
❖ The poles can be real or complex, i.e. $s = \sigma \pm j\omega$



Concept of stability

- If the real part of any roots of the characteristic equation of a system is positive, then the system is unstable (only one pole needs to be in the right half of the s-plane to indicate an unstable system)
- Any characteristics on the imaginary axis (i.e., with real part = zero) indicate a "neutrally stable" or "marginally stable" system
- For linear systems, stability can be determined by looking at the characteristics of the poles as plotted in the s-plane





Example 1

Study stability of the system given by the following transfer functions:

1) $F(s) = \frac{1}{s+4}$

2) $F(s) = \frac{1}{s(s-2)}$

3) $F(s) = \frac{1}{s^2+4s+4}$

4) $F(s) = \frac{s+2}{s^2-s-2}$

5) $F(s) = \frac{s-2}{s^2+16}$

6) $F(s) = \frac{4}{6s^5-4s^3+2s^2+16}$

Routh-Hurwitz stability criterion

It was discovered that all coefficients of the characteristic polynomial must have the same sign and non-zero if all the roots are to be in the left-hand plane, i.e. for the characteristics equation:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

none of the coefficient is missing and $a_0, a_1, \ldots, a_{n-1}, a_n > 0$

- ❖ These requirements are necessary but not sufficient. If the above requirements are not met, it is known that the system is unstable. But, if the requirements are met, we still must investigate the system further to determine the stability of the system
- ❖ The Routh-Hurwitz criterion is an algebraic method to test whether the roots of closed loop characteristic equation lie in the LHP, on the jω-axis or in the RHP (it is a necessary and sufficient criterion for the stability of linear systems)

Routh-Hurwitz stability criterion

- Routh-Hurwitz stability criterion enables stability testing of a polynomial of any order without calculating roots of the polynomial
- This stability criterion uses only the coefficients of the characteristic equation, to verify that the system is stable or unstable
- It determines only the number of poles in the left and right half plane but not their exact location
- The process involves the following:
- Step 1: Get the characteristics equation:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- Step 2: Create the Routh-Hurwitz table (Routh array) using the coefficients of the characteristics equation
- Step 3: Examine the first column of the Routh array to determine the number of sign changes

Routh-Hurwitz stability criterion

Step 1: Get the characteristics equation:

$a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$

Step 2: Create the basic Routh array using the coefficients of the characteristics equation (note: a_n and a_{n-1} constitutes the first column):

$$\begin{array}{ccccc} s^n & a_n & a_{n-2} & \dots & a_0 \\ s^{n-1} & a_{n-1} & a_{n-3} & \dots & 0 \end{array} \quad \text{or} \quad \begin{array}{ccccc} s^n & a_n & a_{n-2} & \dots & a_1 \\ s^{n-1} & a_{n-1} & a_{n-3} & \dots & a_0 \end{array}$$

❖ Add a new row to the Routh array using the following formula:

$$s^{n-2} \quad \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} \quad \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} \quad \dots \quad \frac{-\begin{vmatrix} a_n & a_0 \\ a_{n-1} & 0 \end{vmatrix}}{a_{n-1}} \quad \frac{-\begin{vmatrix} a_n & 0 \\ a_{n-1} & 0 \end{vmatrix}}{a_{n-1}}$$

where

$$\frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}}$$

Routh-Hurwitz stability criterion

$$
\begin{array}{lcccc}
s^n & a_n & a_{n-2} & \cdots & a_0 \\
s^{n-1} & a_{n-1} & a_{n-3} & \cdots & 0 \\
s^{n-2} & b_1 & b_2 & \cdots & 0
\end{array}
$$

❖ Continue adding another new row based on the pattern:

$$
s^{n-3} \quad \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} \quad \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1} \quad \ldots \quad 0
$$

where

$$
\frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{(b_1)(a_{n-3}) - b_2(a_{n-1})}{b_1}
$$

❖ Repeat adding new rows until you complete the row s^0

Routh-Hurwitz stability criterion

$$
\begin{array}{cccccc}
s^n & a_n & a_{n-2} & a_{n-4} & \dots & a_0 \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\
s^{n-2} & b_1 = \dfrac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} & b_2 = \dfrac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} & b_3 = \dfrac{-\begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix}}{a_{n-1}} & \dots & 0 \\
s^{n-3} & c_1 = \dfrac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} & c_2 = \dfrac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1} & \dots & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
s^0 & j_1 = \dfrac{-\begin{vmatrix} h_1 & h_2 \\ i_1 & i_2 \end{vmatrix}}{i_1} & j_2 = \dfrac{-\begin{vmatrix} h_1 & h_3 \\ i_1 & i_3 \end{vmatrix}}{i_1} & \dots & \dots & 0
\end{array}
$$

Step 3: Count the number of sign changes in the first column. The criterion states that the number of roots of with positive real parts is equal to the number of sign changes in the first column of the Routh array

Special case 1

❖ If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ε to continue the process. and the rest of the array is evaluated. The value ε is then allowed to approach zero to determine the signs of the entries in the first column

❖ Example characteristics equation: $s^3 + 2s^2 + s + 2 = 0$

s^3	1	1
s^2	2	2
s^1	$\cancel{0}\varepsilon$	
s^0	2	0

No sign changes in first column (no poles in RHP)

The sign of the coefficient above the zero (ε) is the same as that below it, it indicates that there are a pair of imaginary roots $s = \pm j\omega$

System is marginally stable

Special case 2

- If the entire row all the derived coefficients are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s plane, i.e. two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots
- The rows of zeros is replaced by taking the derivative of an auxiliary polynomial (i.e. the row above the zeros)
- Example characteristics equation: $s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0$

s^5	1	6	8	
s^4	7	42	56	$\leftarrow$ auxiliary equation $P(s) = 7s^4 + 42s^2 + 56$
s^3	$\cancel{0}$	$\cancel{0}$	0	$\leftarrow$ Replace with $\frac{dP(s)}{ds} = 28s^3 + 84s$ and continue
s^3	28	84	0	Can also use $\frac{dP(s)}{ds} = s^3 + 3s$ (factor out 28)
s^2	21	56		
s^1	9.3	0		
s^0	56	0		No sign changes in first column (no poles in RHP)

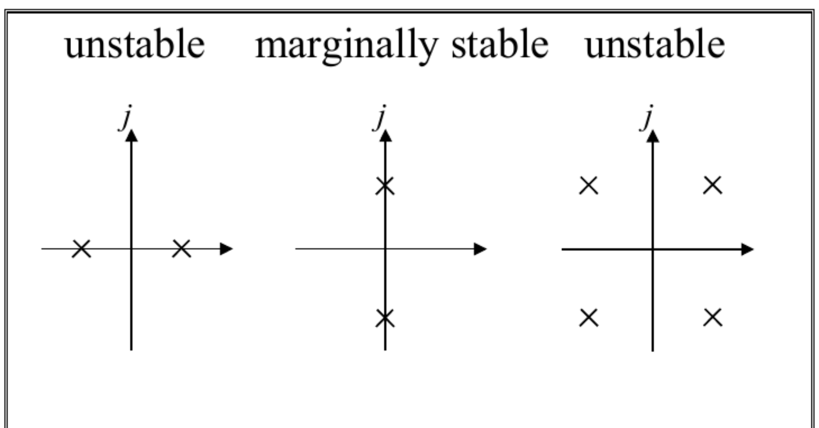
Special case 2

❖ The system with characteristics equation:

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0$$

has no pole in the RHP but the system is marginally stable. Why?

- The row of zeros can be due to 3 possible cases based on symmetry about the origin
- Roots can be obtained by solving $P(s)$



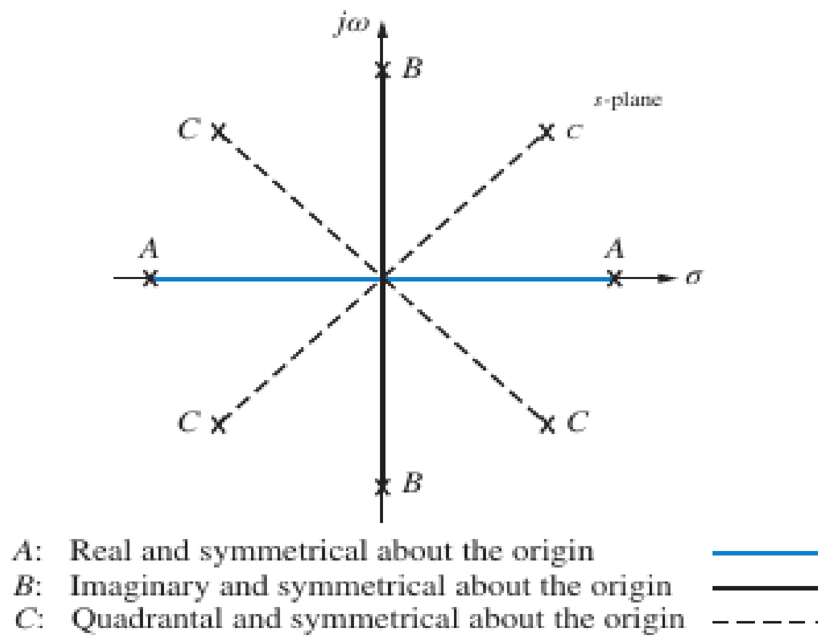


FIGURE 6.5 Root positions to generate even polynomials: A, B, C, or any combination

Example 2

Analyse stability of the following system:

$$F(s) = \frac{1}{s^4 + s^3 - 3s^2 - s + 2}$$

$$\begin{array}{lcccc}
s^4 & 1 & -3 & 2 & 0 \\
s^3 & 1 & -1 & 0 & 0 \\
s^2 & -2 & 2 & 0 & 0 \\
s^1 & (\cancel{0})\varepsilon & (\cancel{0})0 & (\cancel{0})0 & (\cancel{0})0 \\
s^0 & 2 & 0 & 0 & 0
\end{array}$$

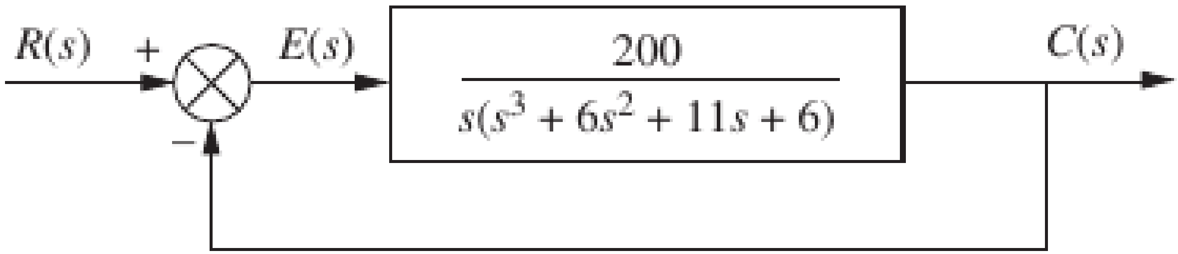
Two sign changes in first column. System is unstable with two poles in the right half plane. This is confirmed if the characteristic polynomial is written as

$$s^4 + s^3 - 3s^2 - s + 2 = (s-1)^2(s+1)(s+2)$$

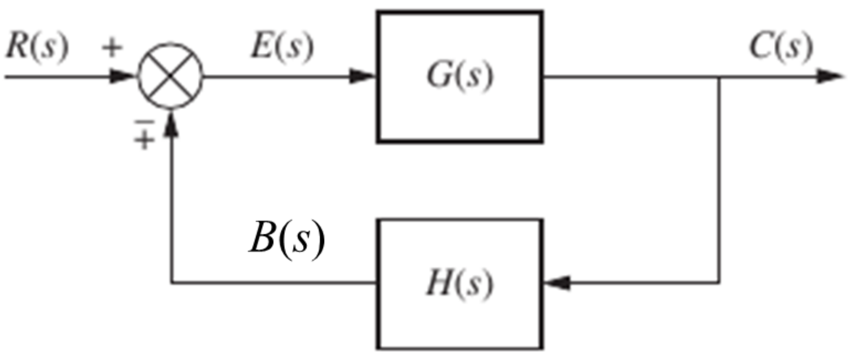
Therefore poles are located at: s_1=1, s_2=1, s_3=-1, s_4=-2

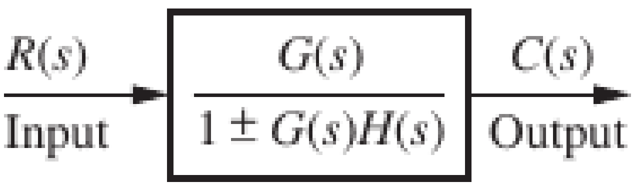
Example 3

Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the following system



Find the closed-loop transfer function: $F(s) = \frac{200}{s^4+6s^3+11s^2+6s+200}$





Example 3

$$\begin{array}{cccccc}
s^4 & 1 & 11 & 200 & 0 \\
s^3 & 6 & 6 & 0 & 0 \\
s^2 & 10 & 200 & 0 & 0 \\
s^1 & -114 & 0 & 0 & 0 \\
s^0 & 200 & 0 & 0 & 0
\end{array}$$

Two sign changes ⇒ 2 poles in RHP
No rows of zeros. No poles on the jω-axis
Two remaining poles in the LHP.
The closed loop system is unstable

Stability in State Space

Given the state and output equations (assuming zero initial conditions):

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \text{ or } \boldsymbol{s}\boldsymbol{X}(\boldsymbol{s}) = \boldsymbol{A}\boldsymbol{X}(\boldsymbol{s}) + \boldsymbol{B}\boldsymbol{U}(\boldsymbol{s})$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \text{ or } \boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{C}\boldsymbol{X}(\boldsymbol{s}) + \boldsymbol{D}\boldsymbol{U}(\boldsymbol{s})$$

The characteristics equation can be found from:

$$\det[\boldsymbol{s}\boldsymbol{I} - \boldsymbol{A}] = 0$$

Note: $\boldsymbol{I}$ = identity matrix

Once the characteristics equation is obtained, we can proceed the analysis with Routh-Hurwitz stability criterion

Example 4

Describe the locations of the poles in the following state-space system:

$$[\dot{\boldsymbol{x}}] = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} [\boldsymbol{x}] + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} [\boldsymbol{x}]$$

The characteristics equation can be found from:

$$\det[\boldsymbol{sI} - \boldsymbol{A}] = 0$$

$$\det\left[\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix}\right] = 0$$

$$\det\left[\begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}\right] = 0$$

Example 4

$$\det\left[\begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix}\right] = 0$$

$$\mathrm{s}\begin{vmatrix} s-8 & -1 \\ 5 & s+2 \end{vmatrix} - (-3)\begin{vmatrix} -2 & -1 \\ 10 & s+2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & s-8 \\ 10 & 5 \end{vmatrix} = 0$$

$$\mathrm{s}[(s-8)(s+2)+5] + 3[-2(s+2)+10] - [-10-10(s-8)] = 0$$

$$\mathrm{s}[s^2-6s-11] + 3[-2s+6] - [-10s+70] = 0$$

$$[s^3-6s^2-11s] + [-6s+18] + [10s-70] = 0$$

$$s^3-6s^2-7s-52 = 0$$

System is unstable (why?)

s^3	1	-7
s^2	-6	-52
s^1	-15.6	0
s^0	-52	0

One sign change in first column (one pole in RHP); 2 poles in LHP (unstable as predicted)