MEMS1045 Automatic control

Lecture 6

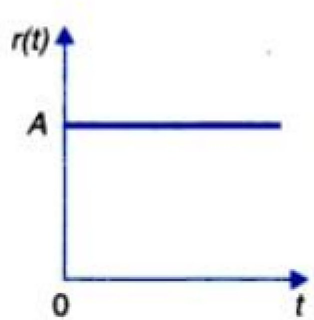
Time response

Objectives

- Determine the step response of first order system
- Describe the changes of the step response characteristics with respect to the pole and zero of the first order system
- Determine the responses of second order systems with different damping ratios
- Describe the changes of the step response characteristics with respect to the poles of the second order system

Introduction

- We have developed equations describing systems. Now we are interested in behavior of those systems under various inputs
- The step input is commonly used to examine the system response with respect t time



$$\frac{C(s)}{R(s)} = G(s)$$

Input signal	$r(t)$	$R(s)$
Step	$r(t) = A \quad \text{for } t > 0$ $r(t) = 0 \quad \text{for } t < 0$	$R(s) = \frac{A}{s}$

Step response: 1st order system

Consider a first order system $\frac{C(s)}{R(s)} = G(s) = \frac{a}{s+a}$

- For a unit step input $R(s) = \frac{1}{s}$ and with zero initial conditions

$$C(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{(s+a)}$$
$$c(t) = 1 - e^{-at}$$

- Note: the pole is at $s = -a$
- Response consists of 2 parts:

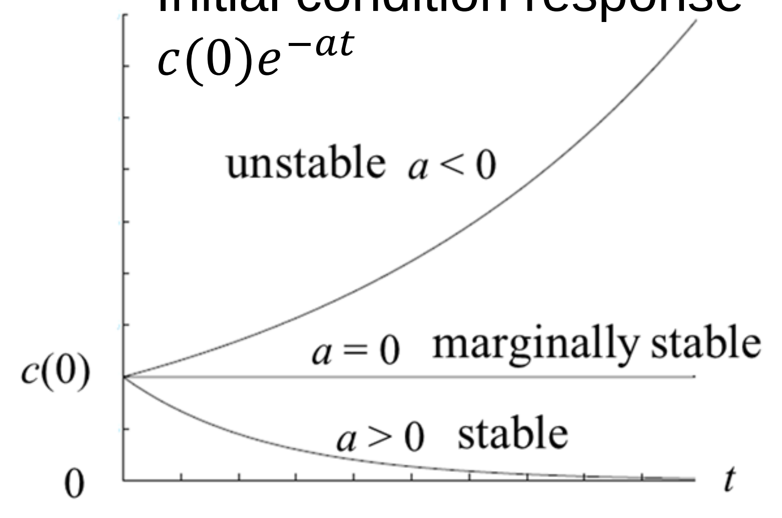
a) Forced response

b) Natural response

- When $a < 0$, unstable
- When $a = 0$, marginally stable
- When $a > 0$, stable

Initial condition response

$c(0)e^{-at}$



Step response: 1st order system

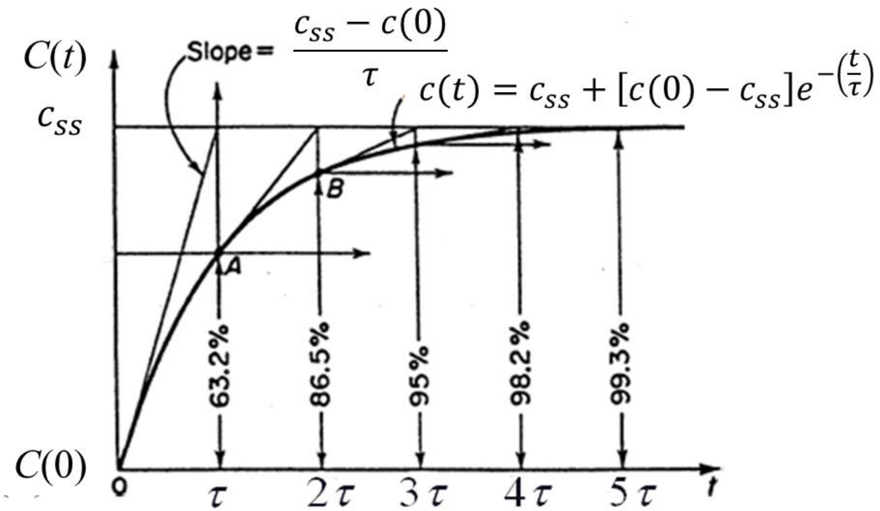
Consider a general first order system $\frac{C(s)}{R(s)} = G(s) = \frac{b}{s+a} = \frac{K}{\tau s+1}$

For a step input $R(s) = \frac{A}{s}$ and zero initial condition

The step response due to a pole $s = -a$ on the real axis in LHP is

$$c(t) = A\frac{b}{a} - A\frac{b}{a}e^{-at} = AK - AKe^{-t/\tau}$$

- The steady state value when $t \to \infty$ is $c_{ss} = A\frac{b}{a} = AK$ (note $\frac{dc}{dt} = 0$ as $t \to \infty$)
- DC gain = K
- Time constant = $\tau = \frac{1}{a}$



Transient & steady state response:

- At time = τ (reach 63.2%)
- Settling time = 4τ (reach 98%)
- Rise time = 2.2τ (1% to 90%)
- No overshoot (no oscillations)

Example 1

For the following transfer functions, identify the pole locations, time constants and sketch their responses in terms of settling time and steady state value to a unit step input assuming zero initial conditions (which one response fastest?):

a) $G(s) = \frac{10}{s+2}$

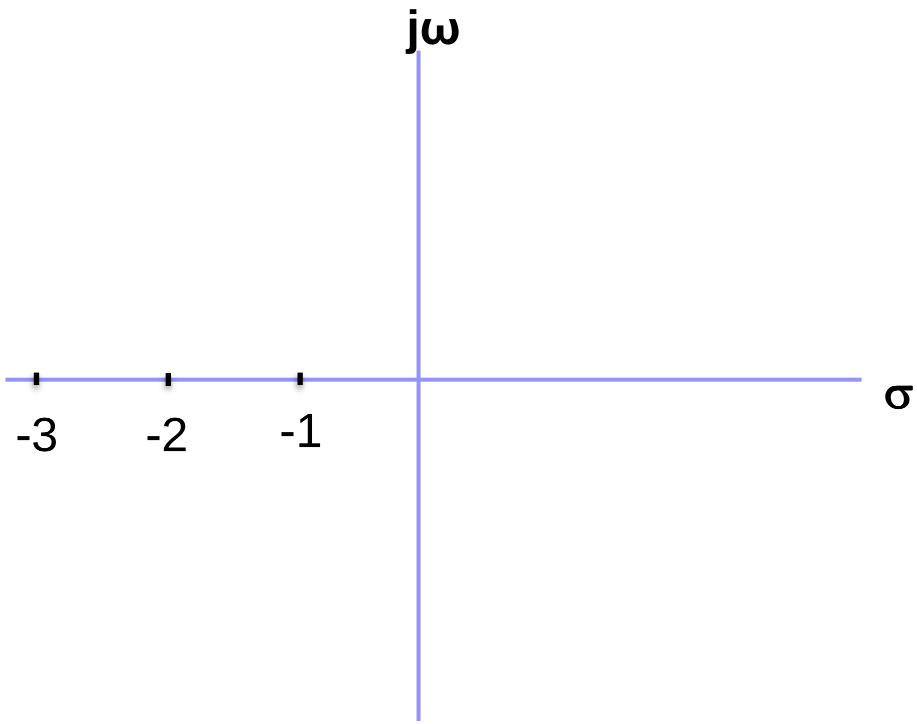
b) $G(s) = \frac{5}{0.5s+1}$

c) $G(s) = \frac{10}{s+1}$

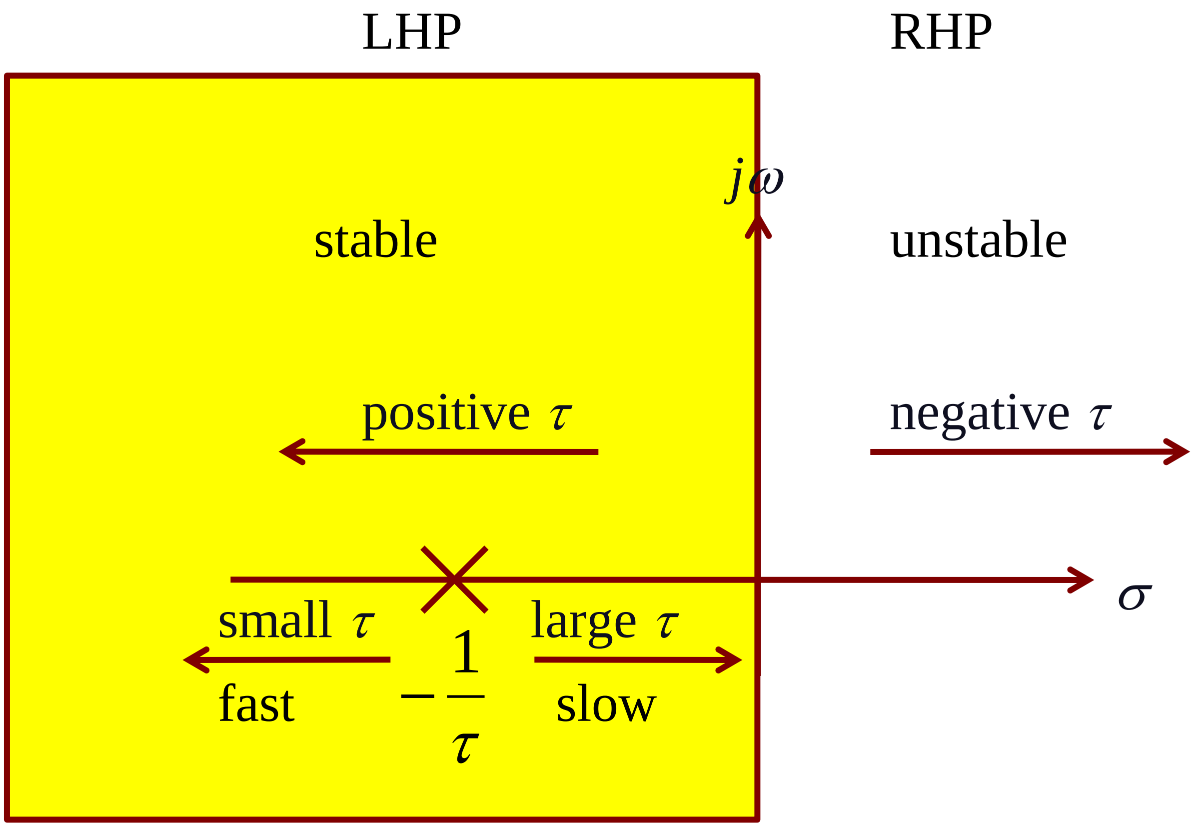
d) $G(s) = \frac{10}{s+3}$

e) $G(s) = \frac{10}{s-2}$

f) $G(s) = \frac{1}{s}$



Effect of single pole



1^{st} Order System with a Zero

Consider a general first order system $\frac{C(s)}{R(s)} = G(s) = \frac{c(s+b)}{s+a}$

- ❖ Zero of the system lie at $s = -b$ and pole at $s = -a$

For a unit step input of magnitude $R(s) = \frac{1}{s}$ and zero initial condition

The step response due to a pole and a zero on the real axis in LHP is

$$C(s) = \frac{c(s+b)}{s(s+a)} = \frac{bc}{as} - \frac{c(b-a)}{a(s+a)}$$

$$c(t) = \frac{bc}{a} - \frac{c(b-a)}{a} e^{-at}$$

- ❖ Compare with a first order system without zero: $\frac{C(s)}{R(s)} = G(s) = \frac{c}{s+a}$

$$c(t) = \frac{c}{a} - \frac{c}{a} e^{-at}$$

- ❖ Setting time affected only by pole and not by zero

1^{st} Order System with a Zero

$\frac{C(s)}{R(s)} = \frac{c}{s+a}$ for $R(s) = \frac{1}{s}$ the response is $c(t) = \frac{c}{a} - \frac{c}{a}e^{-at}$

- At time = 0, offset is $\frac{c}{a} - \frac{c}{a} = 0$
- $c_{ss} = \frac{c}{a}$

$\frac{C(s)}{R(s)} = \frac{c(s+b)}{s+a}$ for $R(s) = \frac{1}{s}$ the response is $c(t) = \frac{bc}{a} - \frac{c(b-a)}{a}e^{-at}$

- At time = 0, offset is $\frac{bc}{a} - \frac{c(b-a)}{a} = c$
- $c_{ss} = \frac{bc}{a}$
- The zeros and poles generate the amplitudes for both the forced and natural responses
- Both pole and zero affect the steady state value c_{ss}
- Both pole and zero affect the offset in the initial magnitude

1^{st} Order System with a Zero

- What happen if we add a zero at the pole location, i.e. when $b = a$?

$$\frac{C(s)}{R(s)} = \frac{c(s+b)}{s+a} = \frac{c(s+a)}{s+a} = c$$

- Mathematically, we have pole-zero cancellation, i.e. the effect of the zero cancel out the effect of the pole
- Pole-zero cancellation will not work if the pole is in the RHP, i.e. never use a zero cancel the unstable pole. This is because any disturbance in the system will not be cancelled
- Apply pole-cancellation only for minimum phase system, i.e. system which has all of the poles and zeroes of its transfer function in the left half of the s-plane representation (not recommended)

Initial state & final value

- The initial value theorem gives the value of signal *f*(*t*) as *t* approaches zero. It does not give the value at exactly *t* = 0 but at a time slightly greater than 0 (indicated by 0+):

$$f(0+) = \lim_{s\to\infty} s\,F(s)$$

- The final value theorem relates the steady state behavior of signal *f*(*t*) as *t* approaches infinity:

$$f(\infty) = \lim_{s\to 0} s\,F(s)$$

- Example

$$C(s) = \frac{c(s+b)}{s(s+a)}$$

$$c(0+) = \lim_{s\to\infty} s\,C(s) = \lim_{s\to\infty} \frac{sc(s+b)}{s(s+a)} = \lim_{s\to\infty} \frac{c(1+b/s)}{(1+a/s)} = c$$

$$c(\infty) = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} \frac{sc(s+b)}{s(s+a)} = \frac{bc}{a}$$

2^{nd} order system

Consider a second order system without zero: $\frac{C(s)}{R(s)} = G(s) = \frac{1}{(s^2+a_1)(s+a_0)}$

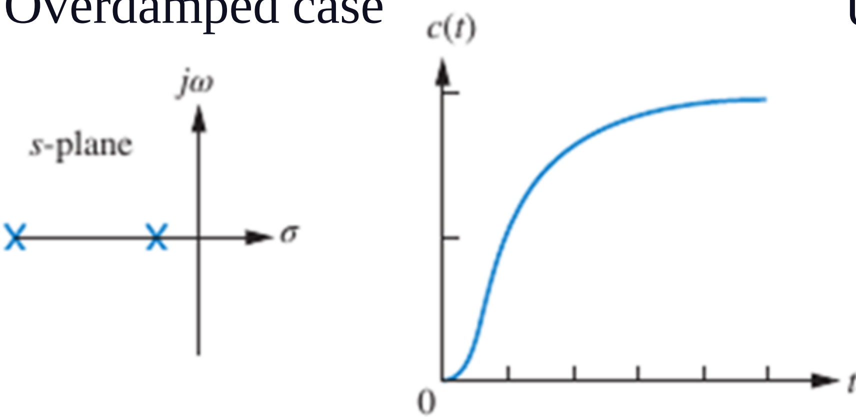
- ❖ The system has 2 poles
- ❖ The 2^{nd} order system can be classified based on the locations of the 2 poles in the LHP:

1) Overdamped case: 2 distinct poles on the real axis, i.e. $s_1 = -a$ and $s_2 = -b$
2) Critically damped case: 2 poles at the same location on the real axis, i.e. $s_1 = s_2 = -a$
3) Underdamped case: 2 complex poles $s_1 = -a + j\omega$ and $s_2 = -a - j\omega$
4) Undamped case: 2 poles on the jω-axis, i.e. $s_1 = j\omega$ and $s_2 = -j\omega$

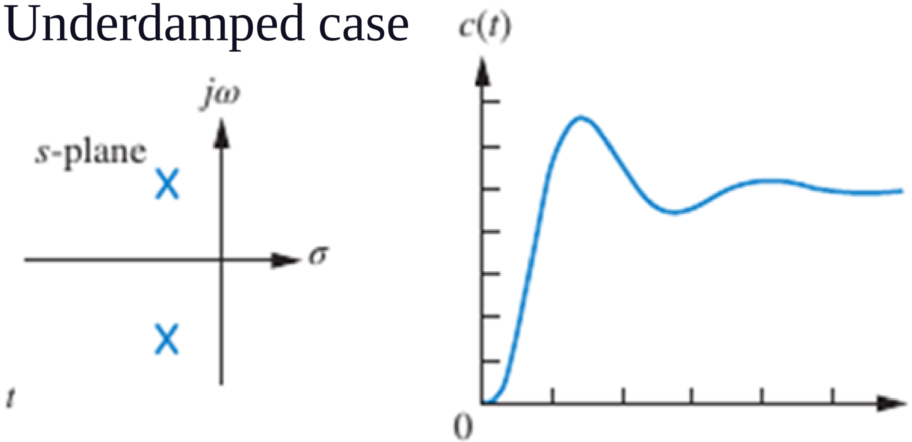
- ❖ Note: The step response of these 4 cases are different due to their poles location

2nd order system

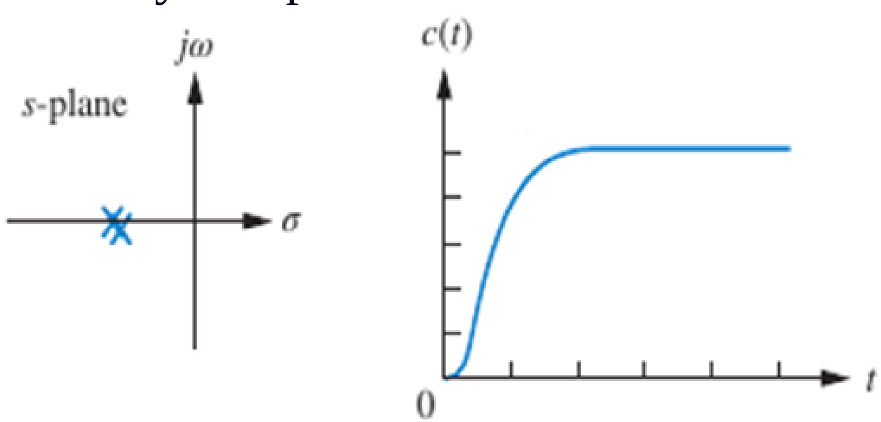
Overdamped case



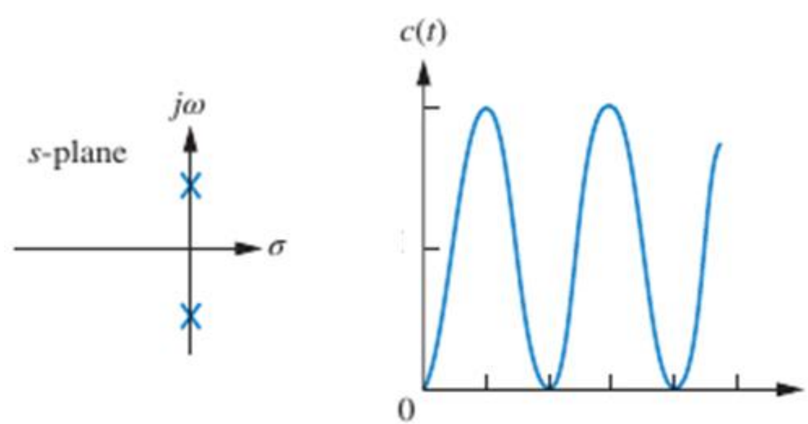
Underdamped case



Critically damped case



Undamped case



2nd order system

The second order system can be put into the form:

$$\frac{C(s)}{R(s)} = G(s) = \frac{1}{s^2+a_1 s+a_0} = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$$

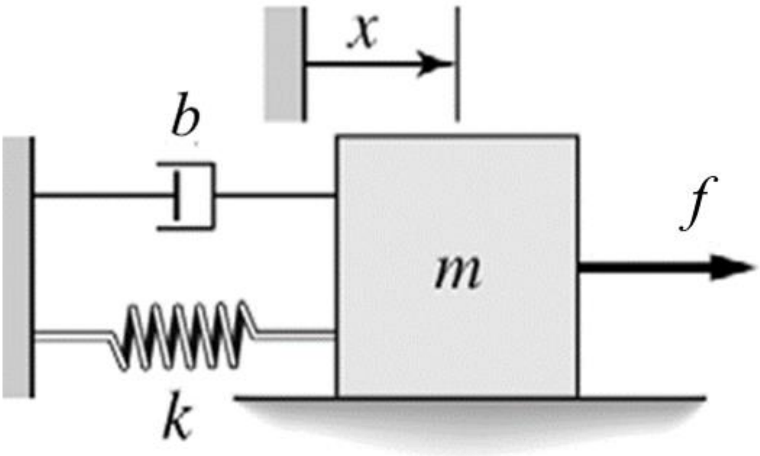
- ζ = damping ratio
- ω_n = natural frequency
- The poles can be found from:

$$s_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

1) Overdamped case: 2 distinct poles on the real axis means that ζ >1 where s_1 and s_2 are at $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
2) Critically damped case: 2 poles at the same location on the real axis means that ζ =1 $s_1 = s_2$ are both at $-\zeta\omega_n$
3) Underdamped case: 2 complex poles means that 0<ζ <1 where s_1 and s_2 are a $-\zeta\omega_n \pm j\omega_d$ where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ = damped natural frequency
4) Undamped case: 2 poles on the jω-axis means that ζ =0 where s_1 and s_2 are at $\pm j\omega_n$

Example 2

Derive the EOM for the system shown and determine the expressions for the natural frequency and damping ratio in terms of the *m*, *b*, and *k*.



EOM: $m\ddot{x} + b\dot{x} + kx = f(t)$

Put into the form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f(t)$$

Note:

$$2\zeta\omega_n = \frac{b}{m} \quad \text{and} \quad \omega_n^2 = \frac{k}{m}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{b}{2m\omega_n} = \frac{b}{2m\omega_n} = \frac{b}{2\sqrt{km}}$$

Transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Overdamped case

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s+a)(s+b)}$$

- Damping ratio $\zeta > 1$
- 2 real distinct poles at $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$
- For a unit step input $R(s) = \frac{1}{s}$ and with zero initial conditions:

$$C(s) = \frac{1}{s(s+a)(s+b)} = \frac{(1/ab)}{s} - \frac{(1/a[b-a])}{(s+a)} - \frac{(1/b[a-b])}{(s+b)}$$

$$c(t) = (1/ab) - (1/a[b-a])e^{-at} - (1/b[a-b])e^{-bt}$$

- Steady state is $c_{ss} = \frac{1}{ab}$; What if input is $\frac{A}{s}$? What if numerator of TF is b_0?
- Natural response of 2 poles are decaying exponentially. What is the response settling time? What if the 2 poles are far apart? What if the 2 poles are near each other?

Critically damped case

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s+a)(s+a)}$$

- Damping ratio $\zeta = 1$
- 2 real poles at same location $-\zeta\omega_n$
- For a unit step input $R(s) = \frac{1}{s}$ and with zero initial conditions:

$$C(s) = \frac{1}{s(s+a)^2} = \frac{(1/a^2)}{s} - \frac{(1/a^2)}{(s+a)} - \frac{(1/a)}{(s+a)^2}$$

$$c(t) = (1/a^2) - (1/a^2)e^{-at} - (1/a)te^{-bt}$$

- Steady state is $c_{ss} = \frac{1}{a^2}$; What if input is $\frac{A}{s}$? What if numerator of TF is b_0?
- Natural response of poles are decaying but rates differ. What is the response settling time?

Underdamped case

- Damping ratio $0 < \zeta < 1$
- 2 complex poles at locations $-\zeta\omega_n \pm j\omega_d$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

- For a unit step input $R(s) = \frac{1}{s}$ and with zero initial conditions:

$$C(s) = \frac{1}{s[(s+\zeta\omega_n)^2 + \omega_d^2]}$$

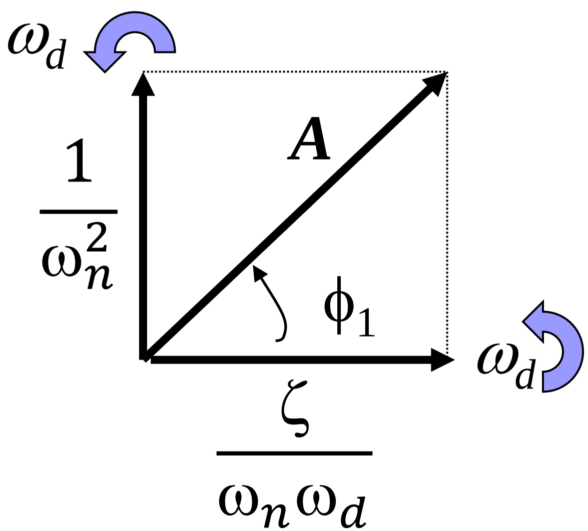
$$= \frac{(1/\omega_n^2)}{s} - \frac{(1/\omega_n^2)(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{(\zeta/\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = \frac{1}{\omega_n^2} - \left\{\frac{1}{\omega_n^2}\cos(\omega_d t)\right\} e^{-\zeta\omega_n t} - \left\{\frac{\zeta}{\omega_n\omega_d}\sin(\omega_d t)\right\} e^{-\zeta\omega_n t}$$

Underdamped case

$$c(t) = \frac{1}{\omega_n^2} - \left\{ \frac{\zeta}{\omega_n \omega_d} \sin(\omega_d t) + \frac{1}{\omega_n^2} \cos(\omega_d t) \right\} e^{-\zeta \omega_n t}$$

- We can simplify the sine and cosine terms by treating them as rotating phasors:



$$A = \sqrt{\left(\frac{1}{\omega_n^2}\right)^2 + \left(\frac{\zeta}{\omega_n \omega_d}\right)^2} = \frac{1}{\omega_n \omega_d}$$

$$\phi_1 = \tan^{-1}\left(\frac{\omega_n \omega_d}{\zeta \omega_n^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\frac{1}{\omega_n^2} \cos(\omega_d t) + \frac{\zeta}{\omega_n \omega_d} \sin(\omega_d t) = \frac{1}{\omega_n \omega_d} \sin(\omega_d t + \phi_1)$$

Underdamped case

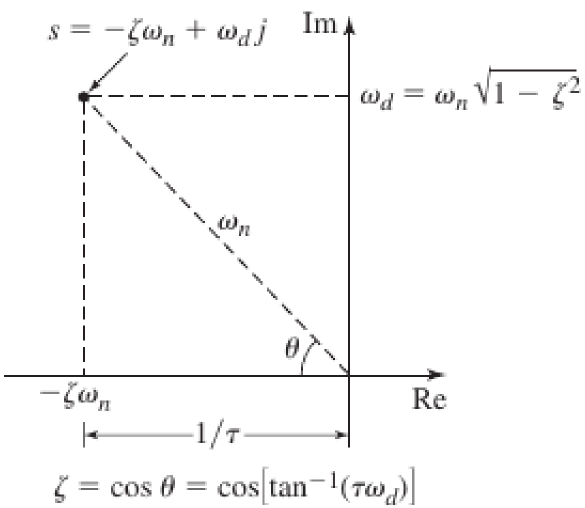
$$c(t) = \frac{1}{\omega_n^2} - \left\{ \frac{1}{\omega_n \omega_d} \sin(\omega_d t + \phi_1) \right\} e^{-\zeta \omega_n t}$$

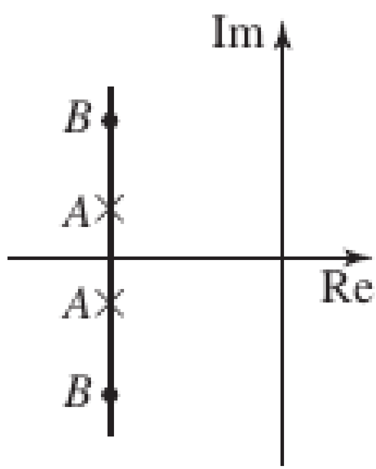
where: $\phi_1 = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

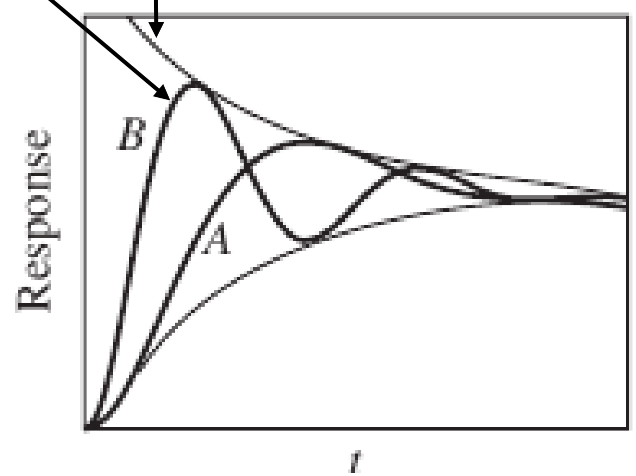
- This is the response for a unit step input to $\frac{C(s)}{R(s)} = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$
- What is the steady state? What if input is $\frac{A}{s}$? What if numerator of TF is b_0?
- Natural response of 2 poles are oscillating while decaying exponentially. What is the response settling time?
- Note that the 2 poles are at $-\zeta\omega_n \pm j\omega_d$ where $\omega_d = \omega_n\sqrt{1-\zeta^2}$; How would the response change when ζ, $\zeta\omega_n$ and ω_d changes?

Underdamped case

$$c(t) = \frac{1}{\omega_n^2} - \{\sin(\omega_d t + \phi_1)\}\left[\frac{1}{\omega_n \omega_d} e^{-\zeta\omega_n t}\right]$$

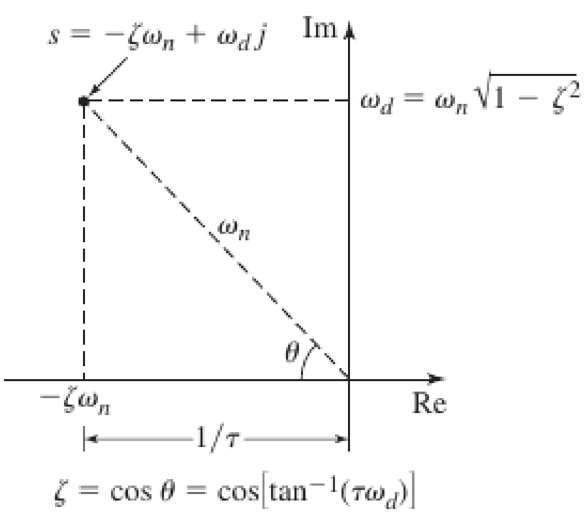




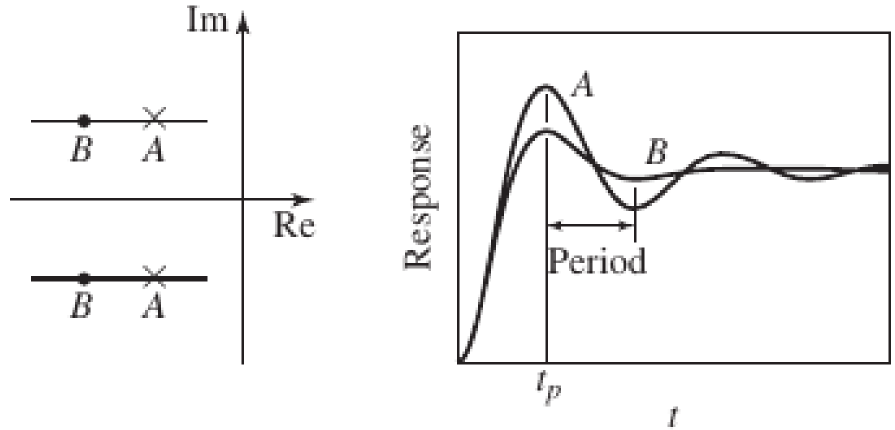


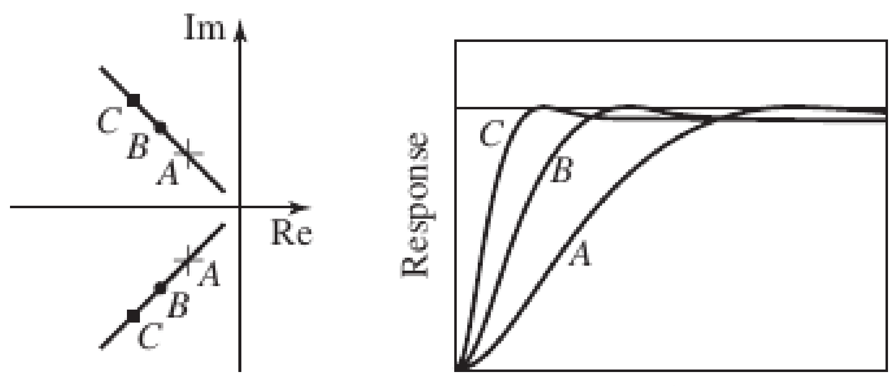
Constant $\zeta\omega_n$ but changing ω_d

Underdamped case



Constant ω_d but changing settling time





Constant ζ but changing settling time

Undamped case

- Damping ratio ζ = 0
- 2 complex poles on the jω-axis means at $\pm j\omega_n$

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + \omega_n^2}$$

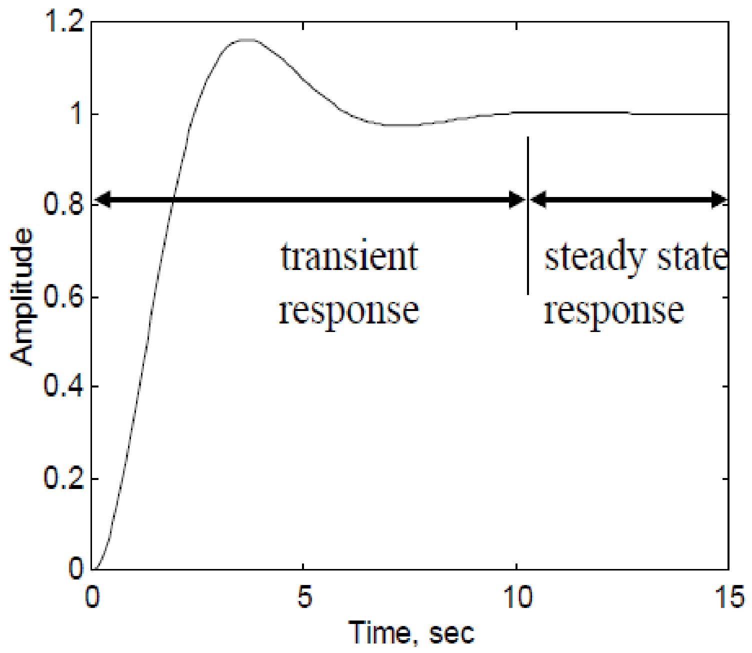
- For a unit step input $R(s) = \frac{1}{s}$ and with zero initial conditions:

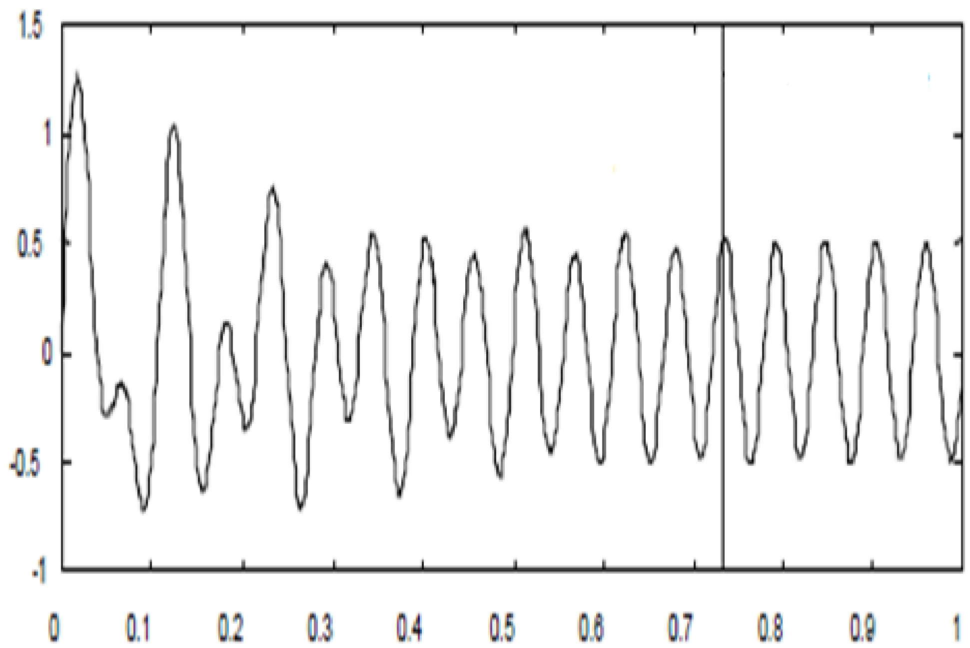
$$C(s) = \frac{1}{s(s^2 + \omega_n^2)} = \frac{(1/\omega_n^2)}{s} - \frac{(1/\omega_n^2)s}{s^2 + \omega_n^2}$$

$$c(t) = \frac{1}{\omega_n^2} - \frac{1}{\omega_n^2}\cos(\omega_n t)$$

- What is the steady state? What if input is $\frac{A}{s}$? What if numerator of TF is b_0?
- Natural response of 2 poles are oscillating without decaying. What is the response settling time? How would the response change when ω_n changes?

Transient and steady-state





- ❖ Transient responses decay to zero as t approaches ∞
- ❖ Steady-state responses may or may not reach a finite value as t approaches ∞ (the steady state value c_{ss} is meaningless)

Example 3

Sketch the step responses for the following systems:

a) $\frac{C(s)}{R(s)} = \frac{2}{s^2+3s+2}$;

b) $\frac{C(s)}{R(s)} = \frac{9}{s^2+3s+9}$;

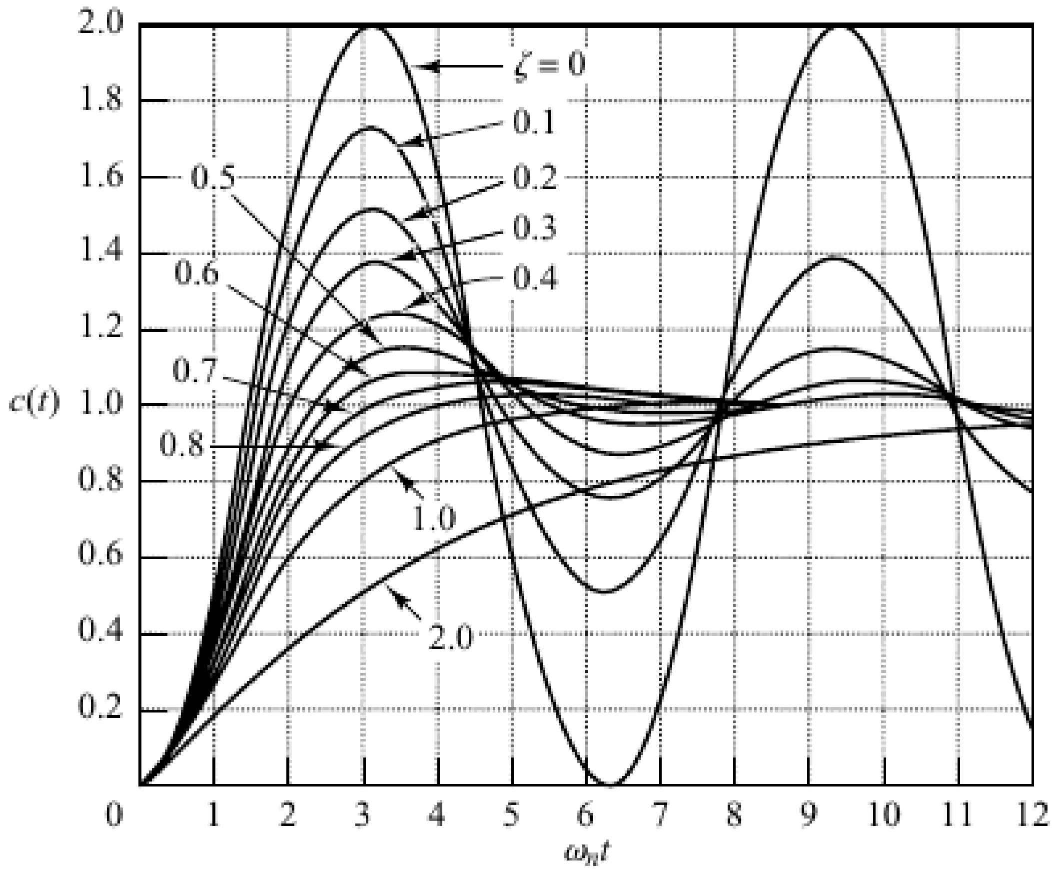
c) $\frac{C(s)}{R(s)} = \frac{1}{s^2+2s+1}$;

d) $\frac{C(s)}{R(s)} = \frac{9}{s^2-1}$;

e) $\frac{C(s)}{R(s)} = \frac{9}{s^2+9}$;

Changes in damping ratio

Effect on the response due to changes in damping ratio ζ



Changes in pole locations