ME1020
Mechanical vibrations

Lecture 10

Multi DOF system vibration 2 (Free undamped vibration)

Objectives

- Analyze the free vibration of a 2-DOF undamped system and generalized the results to n-DOF undamped systems
- Describe the characteristics of the principal mode shapes and beat phenomenon in free vibration

Revision of matrix algebra

Consider a matrix $\boldsymbol{A}$ defined by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The inverse of $\boldsymbol{A}$ is (provided det{$\boldsymbol{A}$}≠0):

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\det\{\boldsymbol{A}\}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

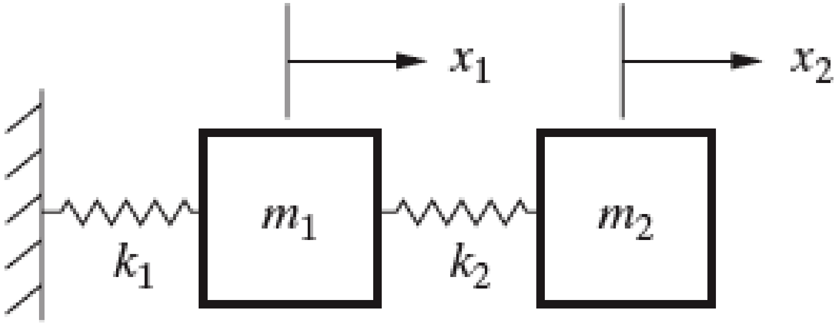
- Where the determinant of matrix $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \det\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} = ad - bc$$

- A square matrix is singular if its determinant is zero

Example 1

Derive the equations of motion for the following 2 DOF system with mass $m_1 = 9$ kg, $m_2 = 1$ kg, spring $k_1 = 24$ N/m, and $k_2 = 3$ N/m



Total kinetic energy:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$T = 4.5 \dot{x}_1^2 + \frac{1}{2} \dot{x}_2^2$$

- Note $q_1 = x_1$, $q_2 = x_2$

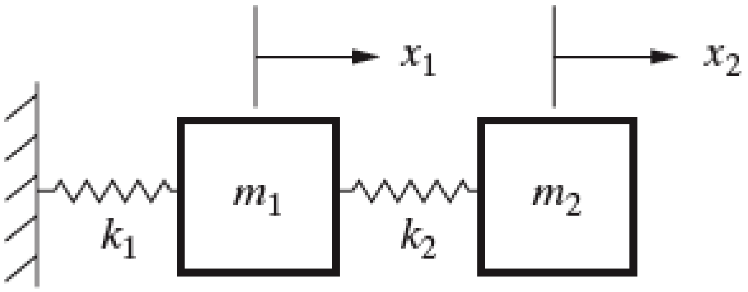
$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}_1} = 9\dot{x}_1$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{x}_2} = \dot{x}_2$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial x_2} = 0$$

Example 1



Potential energy:

$$U = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

$$U = 12x_1^2 + \frac{3}{2}(x_2 - x_1)^2$$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial x_1} = 24x_1 - 3(x_2 - x_1)$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial x_2} = 3(x_2 - x_1)$$

No dissipation function

No generalized forces

Example 1

For $k = 1,2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k$$

❖ For $k = 1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} + \frac{\partial U}{\partial x_1} = Q_1$$

$$\frac{\partial}{\partial t}(9\dot{x}_1) + 24x_1 - 3(x_2 - x_1) = 0$$

❖ For $k = 2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} + \frac{\partial U}{\partial x_2} = Q_2$$

$$\frac{\partial}{\partial t}(\dot{x}_2) + 3(x_2 - x_1) = 0$$

Example 1

$$\frac{\partial}{\partial t}(9\dot{x}_1) + 24x_1 - 3(x_2 - x_1) = 0$$

$$\frac{\partial}{\partial t}(\dot{x}_2) + 3(x_2 - x_1) = 0$$

- ❖ The 2 equations are simplified to:

$$9\ddot{x}_1 + 27x_1 - 3x_2 = 0$$

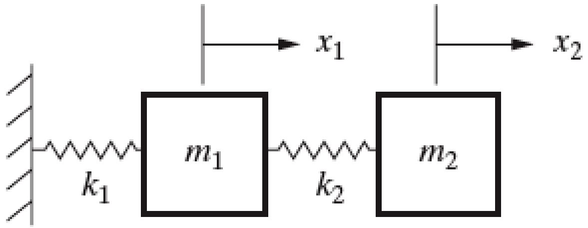
$$\ddot{x}_2 - 3x_1 + 3x_2 = 0$$

- ❖ Express in matrix form:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow [m][\ddot{x}] + [k][x] = [0]$$

- ❖ Note: system is undamped

Free vibration (2-DOF) undamped



Consider the free vibration of the system given in example 1:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow [m][\ddot{x}] + [k][x] = [0]$$

Without damping and forces, the system should vibrate continuously (at the natural frequencies) under initial condition disturbance. Assume a solution of the form

$$[x(t)] = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos(\omega t + \phi) = [u] e^{j\omega t}$$

- $[u]$ is a non zero vector of constants to be determined along with ω
- $j = \sqrt{-1}$ and $e^{j\omega t} = \cos \omega t + j \sin \omega t$ represents harmonic motion

Free vibration (2-DOF) undamped

$$[x(t)] = [u]e^{j\omega t}$$

$$[\ddot{x}(t)] = -\omega^2[u]e^{j\omega t}$$

Substitute into

$$[m][\ddot{x}(t)] + [k][x(t)] = [0]$$

$$-\omega^2[m][u]e^{j\omega t} + [k][u]e^{j\omega t} = [0]$$

$$\{-\omega^2[m] + [k]\}[u]e^{j\omega t} = [0]$$

$$\{-\omega^2[m] + [k]\}[u] = [0]$$

- Note: $[u] \neq 0$ and this means that inverse of $\{-\omega^2[m] + [k]\}$ cannot exist, which implies that $\det\{-\omega^2[m] + [k]\} = 0$
- For the 2-DOF system, we can obtain two values for $\pm\omega$ by solving

$$\det\{-\omega^2[m] + [k]\} = 0$$

Free vibration (2-DOF) undamped

To illustrate the process, we will utilize the values for $[m]$ and $[k]$:

$$[m] = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [k] = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

- ❖ $\det\{-\omega^2[m] + [k]\} = 0$ will give the characteristic equation

$$\det\left\{-\omega^2 \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} 27 - 9\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix}\right\} = 0$$

$$(27 - 9\omega^2)(3 - \omega^2) - 9 = 0$$

$$9\omega^4 - 54\omega^2 + 72 = 0$$

$$\omega^4 - 6\omega^2 + 8 = 0 \quad \text{or} \quad (\omega^2 - 2)(\omega^2 - 4) = 0$$

- ❖ We get two values for $\pm\omega$ and these are $\omega_1 = \pm\sqrt{2}$ and $\omega_2 = \pm 2$ rad/s after solving the characteristic equation

Free vibration (2-DOF) undamped

- The two values for $\pm\omega$ are $\omega_1 = \pm\sqrt{2}$ and $\omega_2 = \pm 2$ rad/s
- We can use these values to solve for $[u]$ using

$$\{-\omega^2[m] + [k]\}[u] = [0]$$

- For $\omega_1 = \pm\sqrt{2}$ rad/s:

$$\left\{-2\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\right\}[u]_1 = [0]$$

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$9X_1 - 3X_2 = 0 \quad \text{and} \quad -3X_1 + X_2 = 0$$

- We cannot solve for X_1 and X_2 but can get their ratio $X_1/X_2 = 1/3$
- For $X_2 = 1,\ X_1 = 1/3$ and

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}_1$$

Free vibration (2-DOF) undamped

- The two values for $\pm\omega$ are $\omega_1 = \pm\sqrt{2}$ and $\omega_2 = \pm 2$ rad/s
- We can use these values to solve for $[u]$ using

$$\{-\omega^2[m] + [k]\}[u] = [0]$$

- For $\omega_2 = \pm 2$ rad/s

$$\left\{-4\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\right\}[u]_2 = [0]$$

$$\begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-9X_1 - 3X_2 = 0 \quad \text{and} \quad -3X_1 - X_2 = 0$$

- We cannot solve for X_1 and X_2 but can get their ratio $X_1/X_2 = -1/3$
- For $X_2 = 1,\ X_1 = -1/3$ and

$$[u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}_2$$

Free vibration (2-DOF) undamped

- We have $\omega_1 = \pm\sqrt{2}$ rad/s and $\omega_2 = \pm 2$ rad/s
- We also have

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}_1 \text{ and } [u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}_2$$

- The assumed solution is $[x(t)] = [u]e^{j\omega t}$ which is a combination of

$$[x(t)] = \left(ae^{j\omega_1 t} + be^{-j\omega_1 t}\right)[u]_1 + \left(ce^{j\omega_2 t} + de^{-j\omega_2 t}\right)[u]_2$$

- Where *a*, *b*, *c*, and *d* are to be determined based on initial conditions
- Applying Euler's equation, an alternate simpler form of solution is

$$[x(t)] = A\sin(\omega_1 t + \phi_1)\,[u]_1 + B\sin(\omega_2 t + \phi_2)\,[u]_2$$

- Where *A*, *B*, ϕ_1, and ϕ_2 are to be determined based on initial conditions
- In this form it is clear that the natural frequencies are ω_1 and ω_2
- $[u]_1$ and $[u]_2$ are called the mode shapes

Free vibration (2-DOF) undamped

To illustrate the process, we will continue to obtain the free response for the system with initial conditions $x_1(0) = 1$ mm, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

$$[x(t)] = A\sin(\omega_1 t + \phi_1)\,[u]_1 + B\sin(\omega_2 t + \phi_2)\,[u]_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\sin\left(\sqrt{2}t + \phi_1\right)\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}_1 + B\sin(2t + \phi_2)\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A\sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3}B\sin(2t + \phi_2) \\ A\sin\left(\sqrt{2}t + \phi_1\right) + B\sin(2t + \phi_2) \end{bmatrix}$$

Differentiate to get the velocities

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{3}A\cos\left(\sqrt{2}t + \phi_1\right) - \frac{2}{3}B\cos(2t + \phi_2) \\ \sqrt{2}A\cos\left(\sqrt{2}t + \phi_1\right) + 2B\cos(2t + \phi_2) \end{bmatrix}$$

Free vibration (2-DOF) undamped

A time $t = 0$: $x_1(0) = 1$ mm, $x_2(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A\sin(\phi_1) - \frac{1}{3}B\sin(\phi_2) \\ A\sin(\phi_1) + B\sin(\phi_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{3}A\cos(\phi_1) - \frac{2}{3}B\cos(\phi_2) \\ \sqrt{2}A\cos(\phi_1) + 2B\cos(\phi_2) \end{bmatrix}$$

There are four equations:

$$\frac{1}{3}A\sin(\phi_1) - \frac{1}{3}B\sin(\phi_2) = 1$$

$$A\sin(\phi_1) + B\sin(\phi_2) = 0$$

$$\left.\begin{aligned} \frac{\sqrt{2}}{3}A\cos(\phi_1) - \frac{2}{3}B\cos(\phi_2) &= 0 \\ \sqrt{2}A\cos(\phi_1) + 2B\cos(\phi_2) &= 0 \end{aligned}\right\} \quad \begin{gathered} \frac{2\sqrt{2}}{3}A\cos(\phi_1)=0 \\ \phi_1 = \pi/2 \\ \phi_2 = \pi/2 \end{gathered}$$

Free vibration (2-DOF) undamped

Found $\phi_1 = \pi/2$ and $\phi_2 = \pi/2$

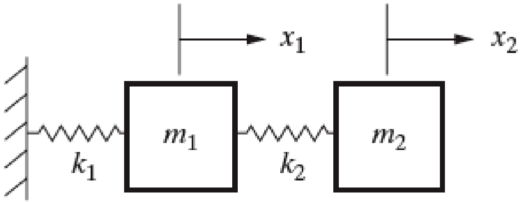
$$\tfrac{1}{3}A\sin(\phi_1) - \tfrac{1}{3}B\sin(\phi_2) = \tfrac{1}{3}A - \tfrac{1}{3}B = 1$$

$$A\sin(\phi_1) + B\sin(\phi_2) = A + B = 0$$

Solve to get: $2A = 3$ or $A = 1.5$ mm and $B = -1.5$ mm

The system is:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



The solution is:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.5\sin\left(\sqrt{2}t + \pi/2\right) + 0.5\sin(2t + \pi/2) \\ 1.5\sin\left(\sqrt{2}t + \pi/2\right) - 1.5\sin(2t + \pi/2) \end{bmatrix}$$

- ❖ The response contains both natural frequencies of the system

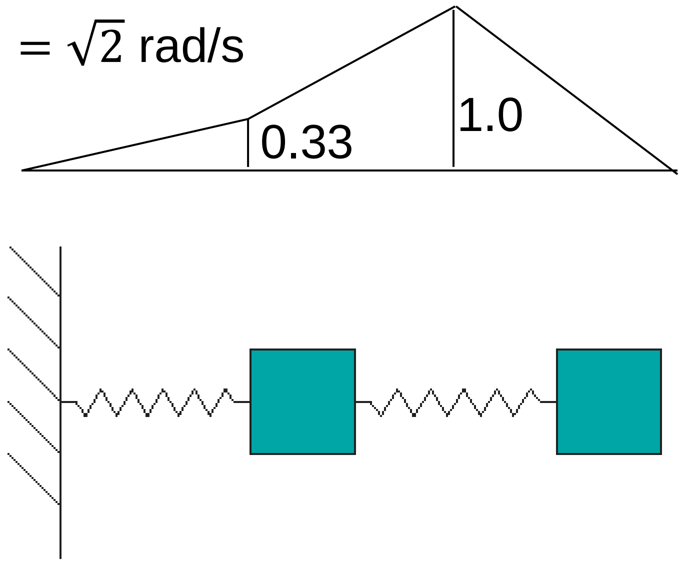
Principal modes

The principal modes for the 2-DOF system discussed are the mode shapes

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}_1 \quad \text{and} \quad [u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}_2$$

Consider the first mode where $[x]_1 = A_1 \sin\left(\sqrt{2}t\right) [u]_1$

$\omega_1 = \sqrt{2}$ rad/s



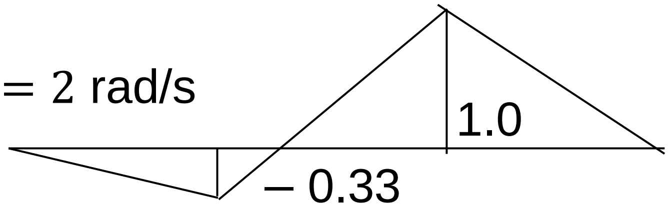
Principal modes

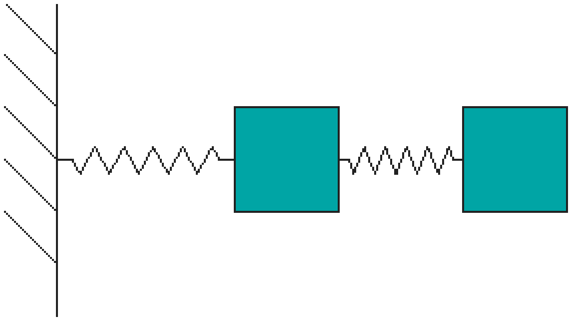
The principal modes for the 2-DOF system discusses are the mode shapes

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}_1 \quad \text{and} \quad [u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}_2$$

Consider the second mode where $[x]_2 = A_2 \sin(2t)\,[u]_2$

$\omega_2 = 2$ rad/s

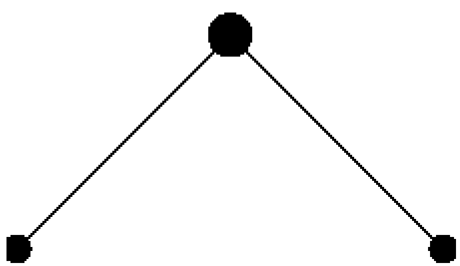


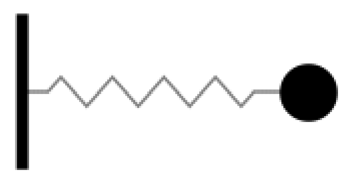


Mode shapes

A 1 DOF system has one mode of vibration and 1 natural frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

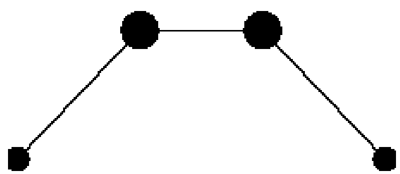


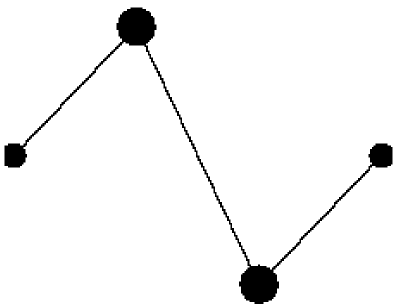


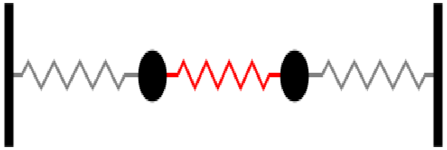
Mode shapes

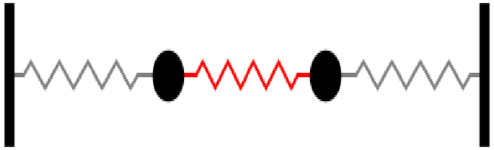
A 2 DOF system has two modes of vibration and 2 natural frequencies

$$\omega_n \neq \sqrt{\frac{k}{m}}$$





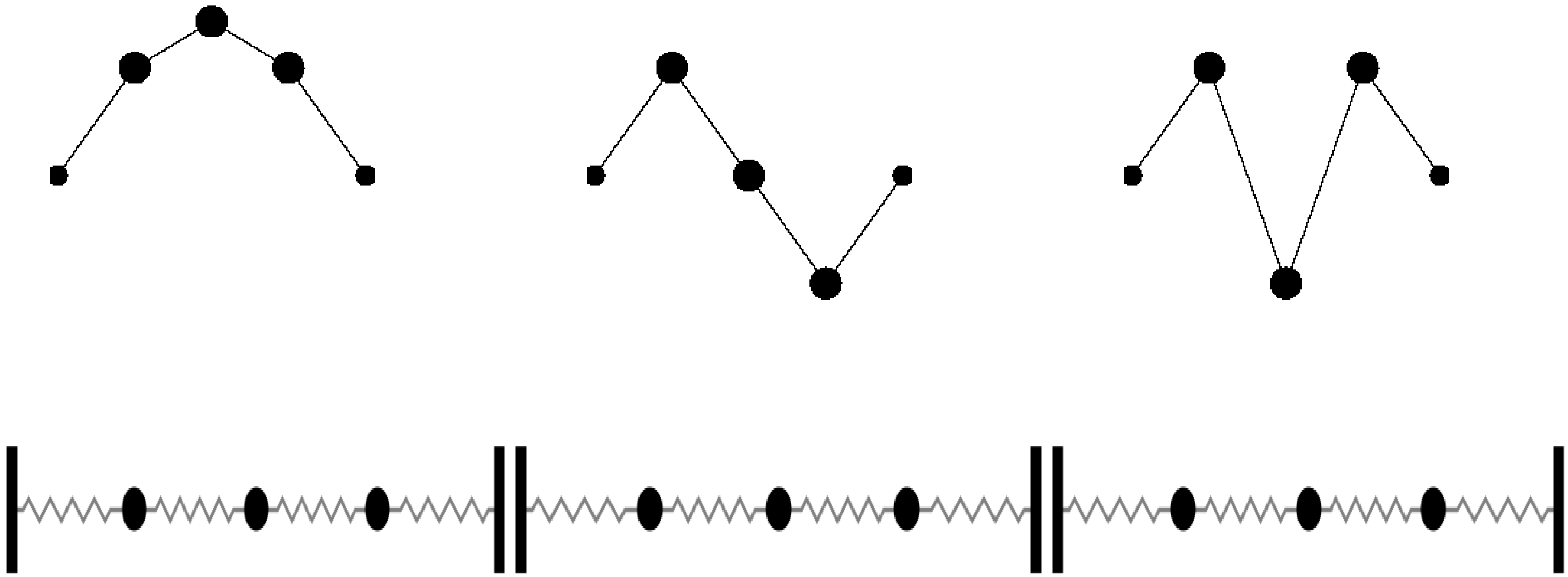




Mode shapes

A 3 DOF system has three modes of vibration and 3 natural frequencies

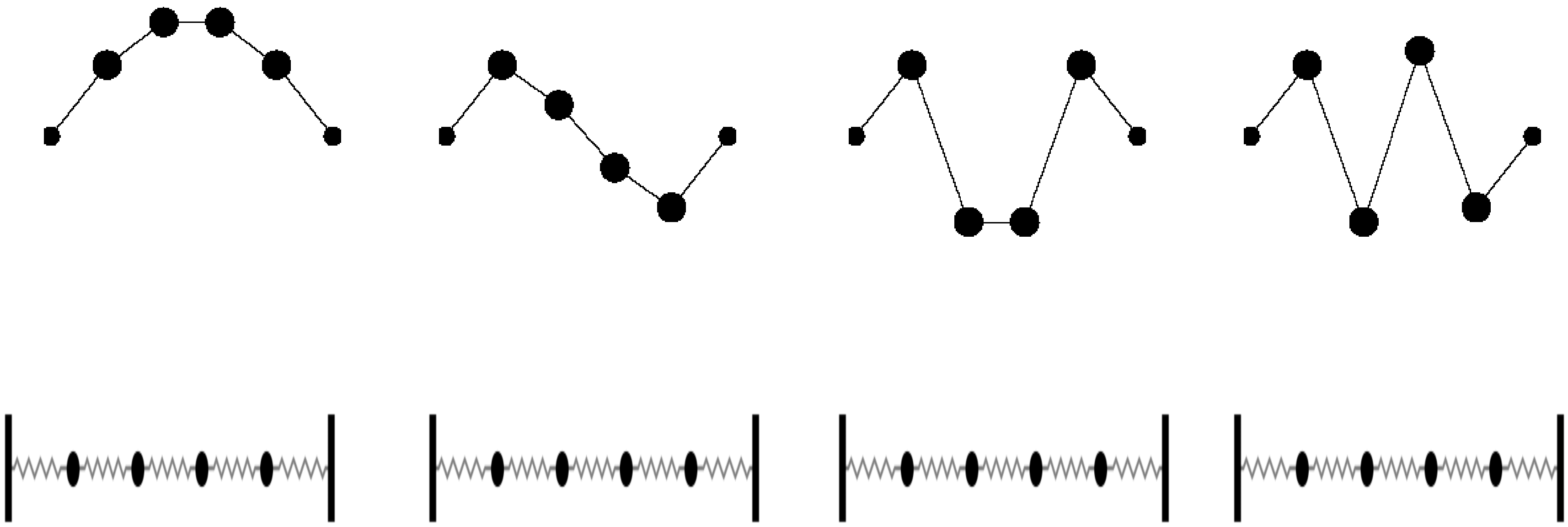
$$\omega_n \neq \sqrt{\frac{k}{m}}$$



Mode shapes

A 4 DOF system has four modes of vibration and 4 natural frequencies

$$\omega_n \neq \sqrt{\frac{k}{m}}$$

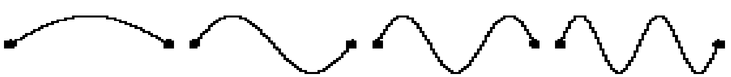


Mode shapes

A Discrete system consists of a finite number of masses or rigid bodies. The DOF of a discrete system can be viewed as:

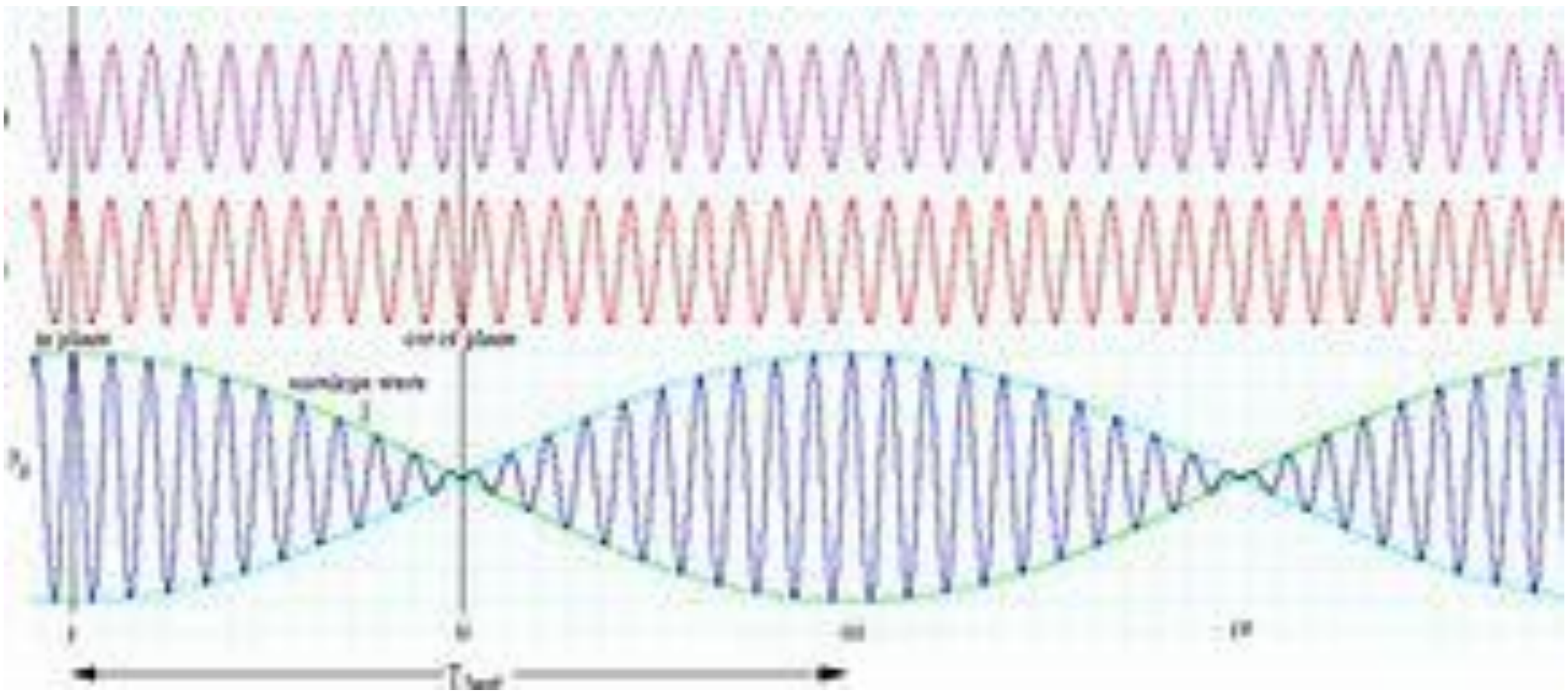
DOF=∑(number of possible motion of mass *i*)

- A Continuous system has distributed mass properties.
- So far we have modeled the system using discrete masses. The mode shape concept can be extended to a continuous system

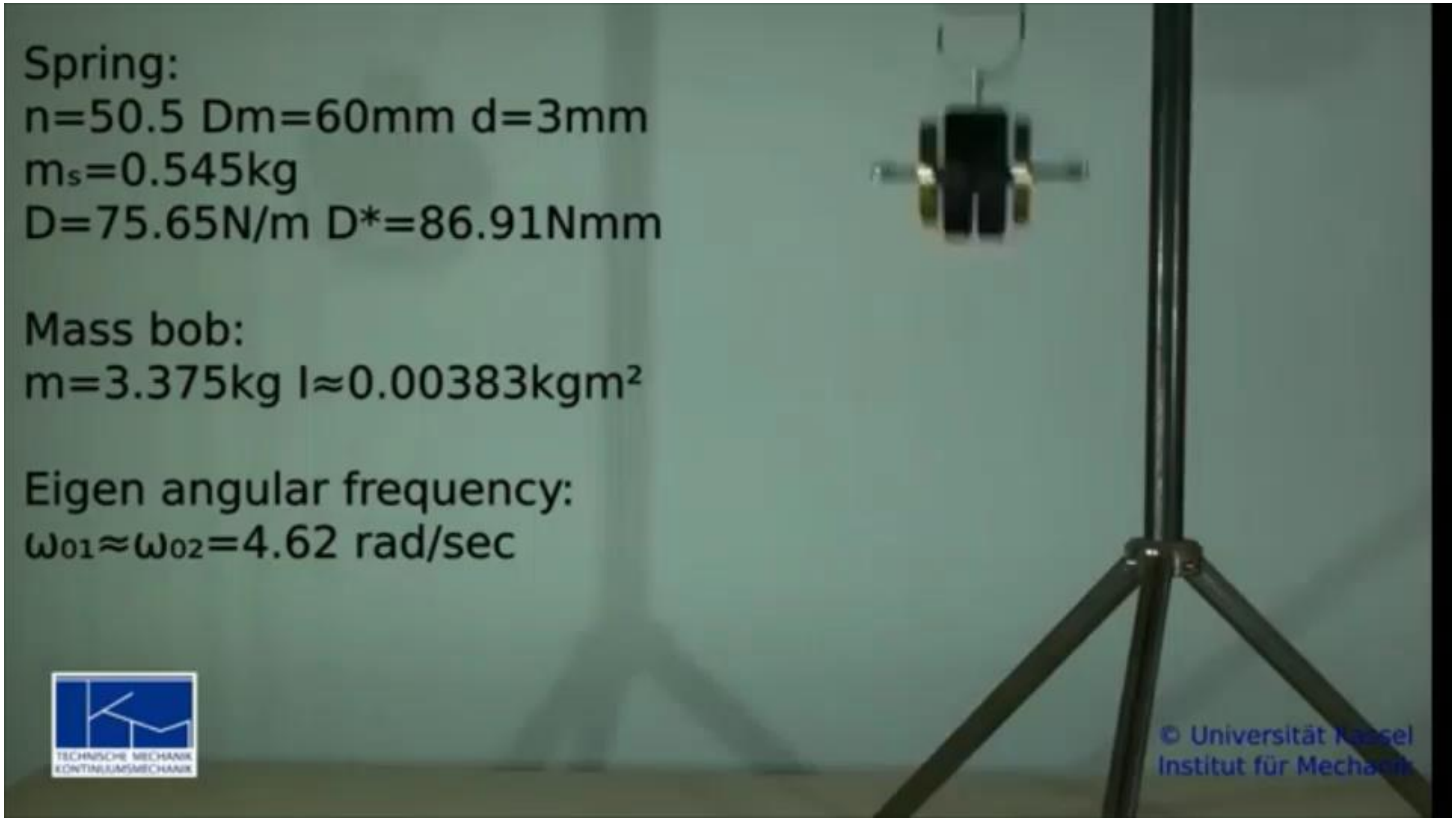


Beats

For a 2-DOF free undamped vibration system, the beats phenomenon occurs when the 2 natural frequencies are near each other



Beats



Spring:
n=50.5 Dm=60mm d=3mm
ms=0.545kg
D=75.65N/m D*=86.91Nmm
Mass bob:
m=3.375kg I≈0.00383kgm²
Eigen angular frequency:
ω01≈ω02=4.62 rad/sec

Free vibration (n-DOF) undamped

The free vibration analysis for the 2-DOF system can be extended to a *n*-DOF:

- ❖ Determine the n number of natural frequencies ω_i for $i = 1, \cdots, n$ using

$$\det\{-\omega^2[m] + [k]\} = 0$$

- ❖ For for $i = 1, \cdots, n$ use natural frequency ω_i to determine the mode shape $[u]_i$

$$\{-\omega_i^2[m] + [k]\}[u]_i = [0]$$

- ❖ The general solution is of the form

$$[x(t)] = \sum_{i=1}^{n} [u]_i A_i \sin(\omega_i t + \phi_i)$$

- ❖ Initial conditions can be specified for $[x(0)] = \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$ and $[\dot{x}\ (0)] = \begin{bmatrix} \dot{x}_1(0) \\ \vdots \\ \dot{x}_n(0) \end{bmatrix}$
- ❖ Solve for the integration constants A_i and ϕ_i using

$$[x(0)] = \textstyle\sum_{i=1}^{n} [u]_i A_i \sin(\phi_i) \text{ and } [\dot{x}(0)] = \textstyle\sum_{i=1}^{n} [u]_i \omega_i A_i \cos(\phi_i)$$

Repeated frequencies

- If the characteristic equation has m repeated roots, then there are m natural frequencies with modes that coincide. This means that only $n - m$ of the linear algebraic equations from which the mode shape is calculated are independent. Thus the m elements of the mode shape can be arbitrary chosen and the most general mode shape involves m arbitrary constants. These are used to specify the m linearly independent mode shape

$$[u]_i, [u]_{i+1}, \cdots, [u]_{i+m}$$

  and the general solution can be applied
- Note: the method used to compute the natural frequencies and mode shapes presented is not the most efficient way to solve vibration problems

Example 2

Determine the natural frequencies and mode shapes for the following system:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Determine the 2 natural frequencies ω_i for $i = 1, 2$ using

$$\det\{-\omega^2[m] + [k]\} = 0$$

$$\det\left\{-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} 2k - m\omega^2 & 0 \\ 0 & 2k - m\omega^2 \end{bmatrix}\right\} = 0$$

$$(2k - m\omega^2)(2k - m\omega^2) = 0$$

We get two values for $\pm\omega$ and these are $\omega_1 = \omega_2 = \pm\sqrt{2k/m}$ rad/s

The natural frequencies are repeated

Example 2

- We found the 2 natural frequencies are $\omega_1 = \omega_2 = \pm\sqrt{2k/m}$ rad/s
- For $\omega = \sqrt{2k/m}$ rad/s:

$$\{-\omega^2[m] + [k]\}[u] = [0]$$

$$\begin{bmatrix} 2k - m\omega^2 & 0 \\ 0 & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = [0]$$

$$0(X_1) + 0(X_2) = 0$$

- This implies that X_1 and X_2 can take on any values
- Two convenient pairs of mode shapes (perpendicular to each other) are

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 \text{ and } [u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_2$$

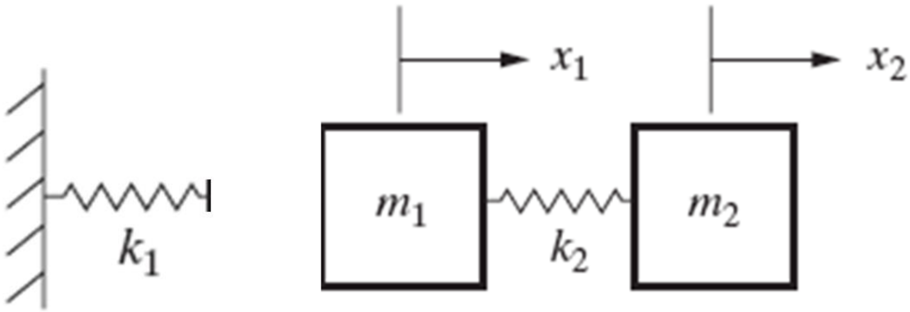
These can be used to solved for the free vibration when given the initial conditions

Unrestrained systems

- Note that the natural frequencies are obtained after solving the characteristics equation
- The characteristic equation only have real roots when mass and stiffness matrices are symmetry
- The system is unstable if the characteristics equation has complex roots
- If the stiffness matrix is singular (i.e. $\det\{[k]\} = 0$), then one of the natural frequency is zero
- The system is unrestrained when one of the natural frequency is zero

Example 3

Analyze the free vibration of example 1 if the spring k_1 is now disconnected from mass m_1 as shown (note: mass = 9 kg, $m_2 = 1$ kg, spring $k_2 = 3$ N/m)



Total kinetic energy:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$T = 4.5\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2$$

- Note $q_1 = x_1$, $q_2 = x_2$

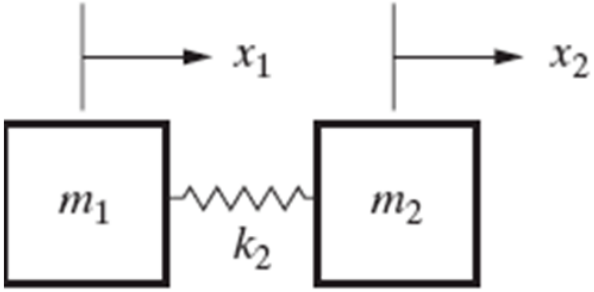
$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}_1} = 9\dot{x}_1$$

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{x}_2} = \dot{x}_2$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial x_2} = 0$$

Example 3



Potential energy:

$$U = \frac{1}{2}k_2(x_2 - x_1)^2$$

$$U = \frac{3}{2}(x_2 - x_1)^2$$

$$\frac{\partial U}{\partial q_1} = \frac{\partial U}{\partial x_1} = -3(x_2 - x_1)$$

$$\frac{\partial U}{\partial q_2} = \frac{\partial U}{\partial x_2} = 3(x_2 - x_1)$$

No dissipation function

No generalized forces

Example 3

For $k = 1,2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k$$

- For $k = 1$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} + \frac{\partial U}{\partial x_1} = Q_1$$

$$\frac{\partial}{\partial t}(9\dot{x}_1) - 3(x_2 - x_1) = 9\ddot{x}_1 - 3(x_2 - x_1) = 0$$

- For $k = 2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} + \frac{\partial U}{\partial x_2} = Q_2$$

$$\frac{\partial}{\partial t}(\dot{x}_2) + 3(x_2 - x_1) = \ddot{x}_2 + 3(x_2 - x_1) = 0$$

Example 3

$$9\ddot{x}_1 - 3(x_2 - x_1) = 0$$
$$\ddot{x}_2 + 3(x_2 - x_1) = 0$$

- Express in matrix form:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow [m][\ddot{x}] + [k][x] = [0]$$

- Determine the *2* natural frequencies ω_i for $i = 1, 2$ using

$$\det\{-\omega^2[m] + [k]\} = 0$$

$$\det\left\{-\omega^2 \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} 3 - 9\omega^2 & -3 \\ -3 & 3 - \omega^2 \end{bmatrix}\right\} = 0$$

$$(3 - 9\omega^2)(3 - \omega^2) - 9 = 0$$

$$9\omega^4 - 30\omega^2 = 0 \quad \text{or} \quad \omega^2(9\omega^2 - 30) = 0$$

We get two values for $\pm\omega$ and these are $\omega_1 = 0$ and $\omega_2 = \pm 1.826$ rad/s

Note the system is restrained

Example 3

- The two values for $\pm\omega$ are $\omega_1 = 0$ and $\omega_2 = \pm 1.826$ rad/s
- We can use these values to solve for $[u]$ using

$$\{-\omega^2[m] + [k]\}[u] = [0]$$

- For $\omega_1 = 0$ rad/s:

$$\left\{\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\right\}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = [0]$$

$$[u]_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1$$

- For $\omega_2 = \pm 1.826$ rad/s

$$\left\{-3.334\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\right\}[u]_2 = [0]$$

$$\begin{bmatrix} -27 & -3 \\ -3 & -0.334 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad [u]_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_2 = \begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

Example 3

To illustrate the process, we will continue to obtain the free response for the system with initial conditions $x_1(0) = 1$ mm, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

- The solution of the free vibration of the undamped and unrestrained system is a combination of the modes

$$[x(t)] = A\sin(\omega_1 t + \phi_1)\,[u]_1 + B\sin(\omega_2 t + \phi_2)\,[u]_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\sin(\phi_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 + B\sin(1.826t + \phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = 1.826B\cos(1.826t + \phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

Applying the initial conditions at $t = 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\sin(\phi_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 + B\sin(\phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1.826B\cos(\phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

Example 3

$$\begin{bmatrix}1\\0\end{bmatrix} = A\sin(\phi_1)\begin{bmatrix}1\\1\end{bmatrix}_1 + B\sin(\phi_2)\begin{bmatrix}-0.111\\1\end{bmatrix}_2$$

$$\begin{bmatrix}0\\0\end{bmatrix} = 1.826B\cos(\phi_2)\begin{bmatrix}-0.111\\1\end{bmatrix}_2$$

These consist of four equations

$$1 = A\sin(\phi_1) - 0.111B\sin(\phi_2)$$

$$0 = A\sin(\phi_1) + B\sin(\phi_2)$$

$$\left.\begin{aligned}0 &= -0.2027B\cos(\phi_2)\\ 0 &= 1.826B\cos(\phi_2)\end{aligned}\right\} \quad \begin{aligned}&\phi_2 = \pi/2\\ &\text{or } \ 3\pi/2\end{aligned}$$

Using $\phi_2 = \pi/2$ or $3\pi/2$

$$\left.\begin{aligned}1 &= A\sin(\phi_1) \mp 0.111B\\ 0 &= A\sin(\phi_1) \pm B\end{aligned}\right\} \quad \begin{aligned}1 &= \pm 1.111B\\ B &= \pm 0.9\end{aligned}$$

Using $B = \pm 0.9$, we get $A\sin(\phi_1) = \mp 0.9$

Example 3

- The solution of the free vibration of the undamped and unrestrained system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\sin(\phi_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 + B\sin(1.826t + \phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mp 0.9\begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 \pm 0.9\sin(1.826t + \phi_2)\begin{bmatrix} -0.111 \\ 1 \end{bmatrix}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mp 0.9 \pm \sin(1.826t + \phi_2)\begin{bmatrix} -0.1 \\ 0.9 \end{bmatrix}$$

Where plus (minus) sign is to be used when ϕ_2 is taken as $\pi/2$ $(3\pi/2)$