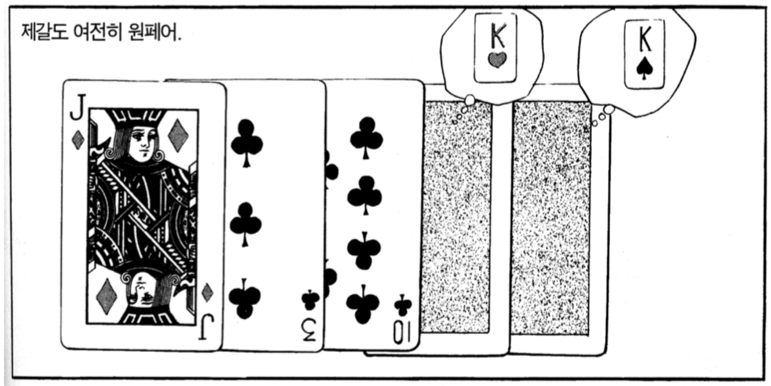
Counting

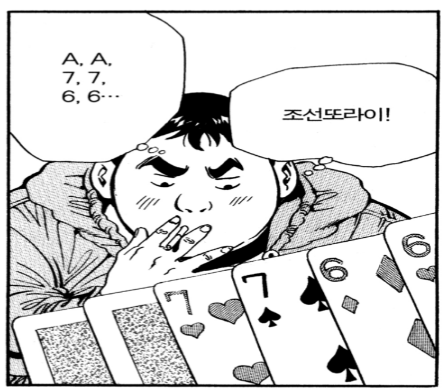
Ranks of cards 1

No pair

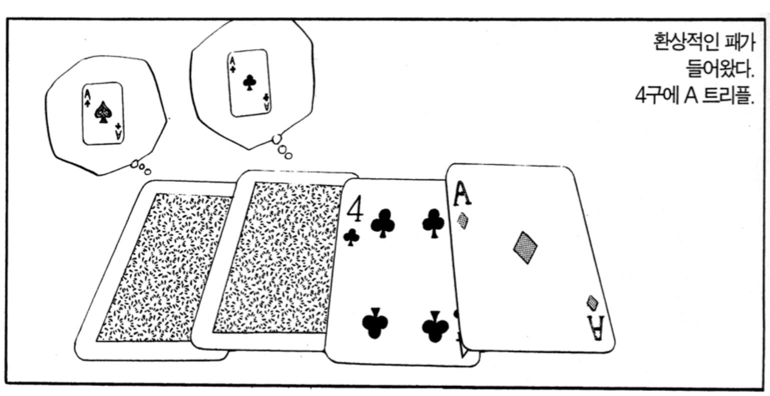
One pair



Two pair

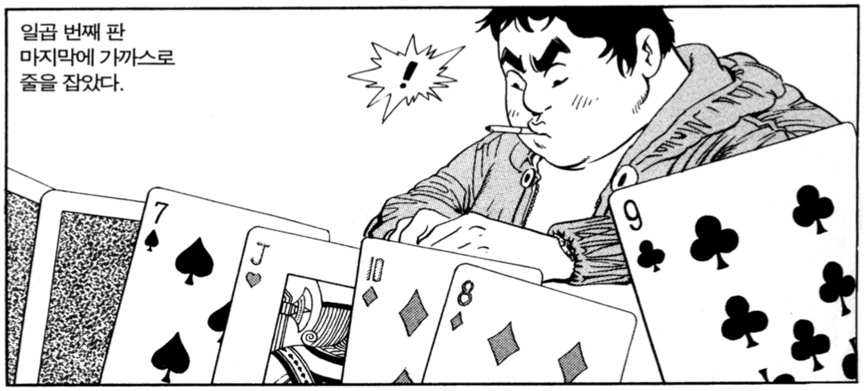


Triple



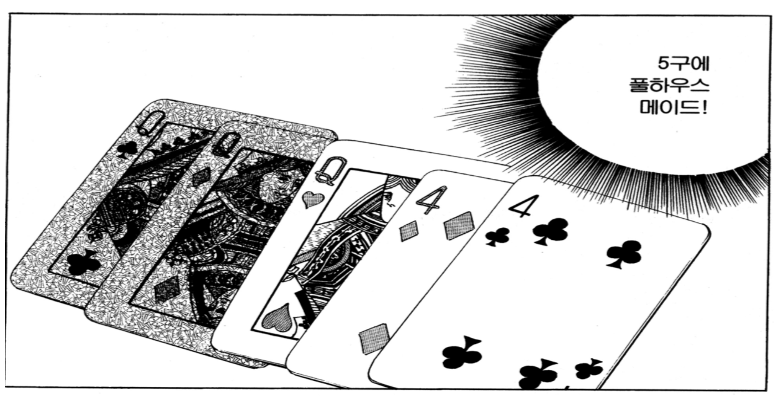
Ranks of cards 2

Straight



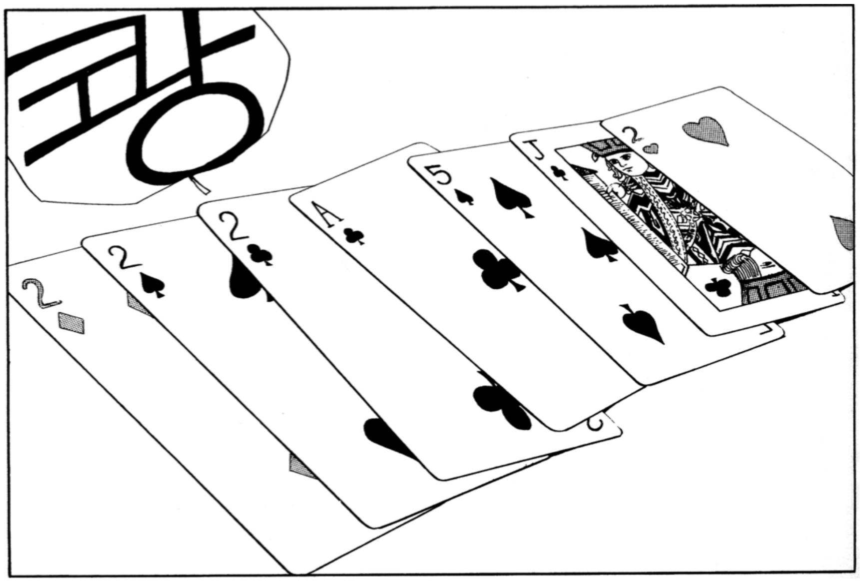
Flush

Full house

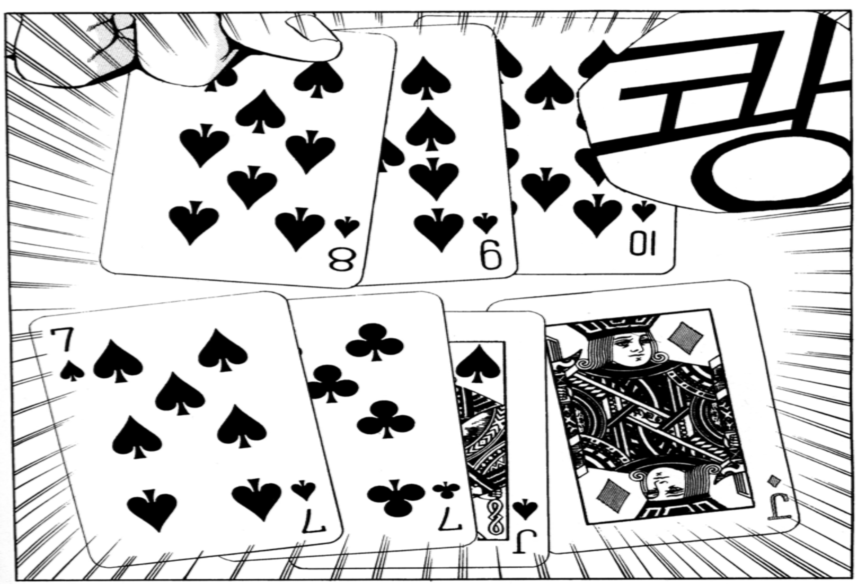


Ranks of cards 3

Four of a kind

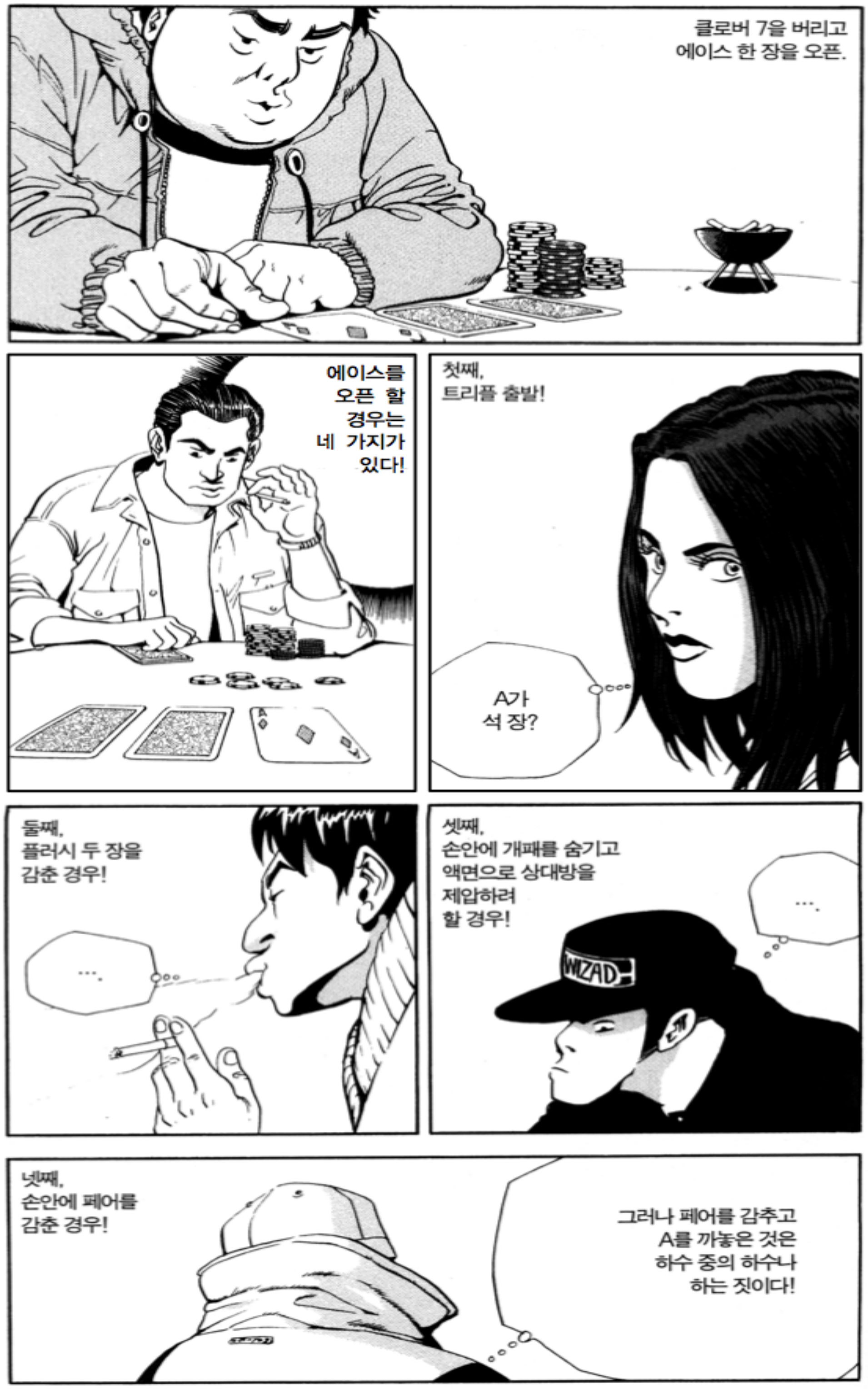


Straight flush



Royal straight flush

Reason for counting



Counting

Enumeration

List all the cases in Ω and count them.

Divide and conquer

[Step 1] Divide all the cases into several disjoint categories A_i.

[Step 2] Count all the cases in each category A_i.

[Step 3] $|\Omega| = \sum_i |A_i|$.

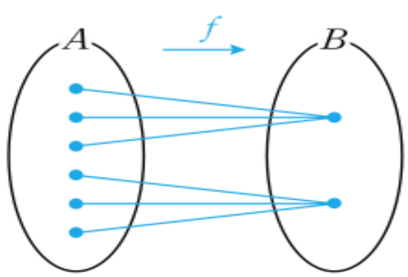
Tree diagram

[Step 1] Divide all the cases into several disjoint categories A_i using tree diagram.

[Step 2] Count all the cases in each category A_i.

[Step 3] $|\Omega| = \sum_i |A_i|$.

Many-to-one



$$\Rightarrow \quad |\boldsymbol{A}| : |\boldsymbol{B}| = \boldsymbol{k} : \boldsymbol{1} \quad \Rightarrow \quad |\boldsymbol{B}| = \frac{|\boldsymbol{A}|}{\boldsymbol{k}}$$

Inclusion-exclusion principle

$$|\cup_{i=1}^n A_i| = \sum_{i=1}^n |A_i| - \sum_{1\le i<j\le n} |A_iA_j| + \cdots + (-1)^{n+1} |A_1A_2\cdots A_n|$$

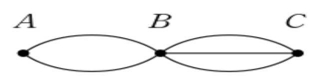
Complement

$$|A| = |\Omega| - |A^c|$$

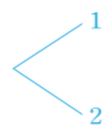
Tree diagram

Example - Number of paths from A to C

Count number of paths from A to C, where there are paths from A to B and from B to C as follows:

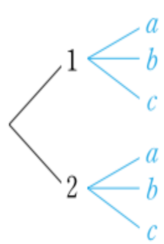


Branching of paths from A to B



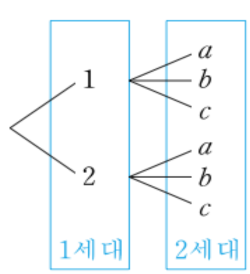
A에서 B로 가는 경로
1
2

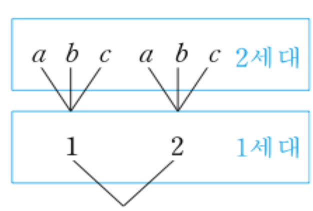
Branching of paths from B to C



A에서 B로 가는 경로	B에서 C로 가는 경로	A에서 B를 거쳐 C로 가는 경로
1	a	$1a$
	b	$1b$
	c	$1c$
2	a	$2a$
	b	$2b$
	c	$2c$

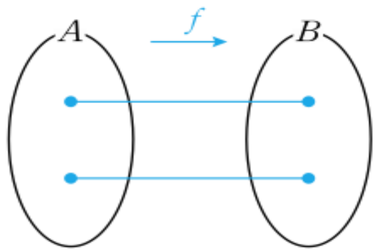
Tree diagram of paths from A to C





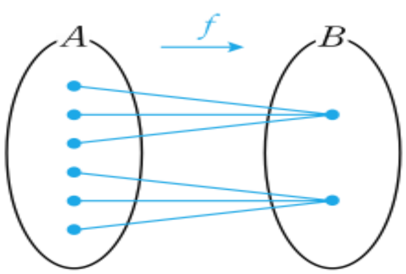
Many-to-one

One-to-one



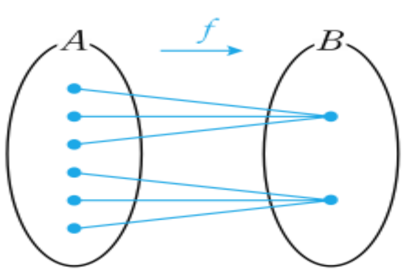
$$\Rightarrow \quad |B| = |A|$$

Many-to-one



$$\Rightarrow \quad |A| : |B| = k : 1 \quad \Rightarrow \quad |B| = \frac{|A|}{k}$$

One-to-many



$$\Rightarrow \quad |A| : |B| = k : 1 \quad \Rightarrow \quad |A| = k\,|B|$$

Many-to-many

Example - Football

Example - Number of words made from BOB

Attach indices

Number of words made from B_1OB_2 $\qquad 3 \times 2 \times 1 = 3!$

Remove indices

$$\begin{array}{ccc} B_1B_2O & & BBO \\ B_1OB_2 & & BOB \\ B_2B_1O & \xrightarrow{\text{remove indices}} & BBO \\ B_2OB_1 & & BOB \\ OB_1B_2 & & OBB \\ OB_1B_2 & & OBB \end{array}$$

This is 2-to-1.

Many-to-one

Number of words made from BOB $\qquad \dfrac{3!}{2}$

Example - Number of words made from $BBOOO$

Attach indices

Number of words made from $B_1B_2O_1O_2O_3$ $5!$

Remove indices

Before removing indices		**After removing indices**
$B_1B_2O_1O_2O_3$	$\to$	$BBOOO$
$B_1B_2O_1O_3O_2$	$\to$	$BBOOO$
$B_1B_2O_2O_1O_3$	$\to$	$BBOOO$
$B_1B_2O_2O_3O_1$	$\to$	$BBOOO$
$B_1B_2O_3O_1O_2$	$\to$	$BBOOO$
$B_1B_2O_3O_2O_1$	$\to$	$BBOOO$
$B_2B_1O_1O_2O_3$	$\to$	$BBOOO$
$B_2B_1O_1O_3O_2$	$\to$	$BBOOO$
$B_2B_1O_2O_1O_3$	$\to$	$BBOOO$
$B_2B_1O_2O_3O_1$	$\to$	$BBOOO$
$B_2B_1O_3O_1O_2$	$\to$	$BBOOO$
$B_2B_1O_3O_2O_1$	$\to$	$BBOOO$

This is (2!3!)-to-1.

Many-to-one

Number of words made from $BBOOO$ $\frac{5!}{2!3!}$

Permutation and combination

Number of choosing president, vice-president, secretary

$$\begin{aligned} & \text{There are } n \text{ people} \\ & \text{Choose president - Number of branching } n \\ & \text{Choose vice-president - Number of branching } n-1 \\ & \text{Choose secretary - Number of branching } n-2 \\ \text{Tree diagram} \quad \Rightarrow \quad & \text{Number of choosing president, vice-president, secretary is} \\ & n \times (n-1) \times (n-2) \end{aligned}$$

Number of choosing three committee members

$$\begin{aligned} & \text{There are } n \text{ people} \\ & \text{Choose president - Number of branching } n \\ & \text{Choose vice-president - Number of branching } n-1 \\ & \text{Choose secretary - Number of branching } n-2 \\ \text{Tree diagram} \quad \Rightarrow \quad & \text{Number of choosing president, vice-president, secretary is} \\ & n \times (n-1) \times (n-2) \\ & \text{Remove rank or indices} \\ 3!\text{-to-1} \quad \Rightarrow \quad & \text{Number of choosing three committee members is} \\ & \frac{n \times (n-1) \times (n-2)}{3!} \end{aligned}$$

Permutation (n choose k with order)

The number of ways of choosing from n people, k people to form a committee of president, vice-president,$\cdots$, secretary is

$$n \times (n-1) \times (n-2) \cdots \times (n-(k-1))$$

Combination (n choose k without order)

The number of ways of choosing from n people, k people to form a committee is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Two different ways of counting

Number of ways of forming a committee of size k from n people

Choose k people without order and form a committee $\binom{n}{k}$

Choose $n-k$ people without order and remove them $\binom{n}{n-k}$

Form a committee with remaining k people

$$\binom{n}{k} = \binom{n}{n-k}$$

Number of ways of forming a committee of size k with president from n people

Choose k people without order and form a committee $\binom{n}{k}$

Elect one as president from k members $k\binom{n}{k}$

Elect one as president from n people n

Choose remaining $k-1$ members to form a committee $n\binom{n-1}{k-1}$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Number of ways of forming a committee of size k from m men and n women

Choose k people and form a committee $\binom{m+n}{k}$

Choose l male, $k-l$ female, form a committee $\binom{m}{l}\binom{n}{k-l}$

Choose male, female, and form a committee $\sum_{max(0,k-n)\le l\le min(m,k)} \binom{m}{l}\binom{n}{k-l}$

Vandermonde's identity $$\binom{m+n}{k} = \sum_{max(0,k-n)\le l\le min(m,k)} \binom{m}{l}\binom{n}{k-l}$$

Binomial expansion

Binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where

$$\text{Binomial coefficient} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

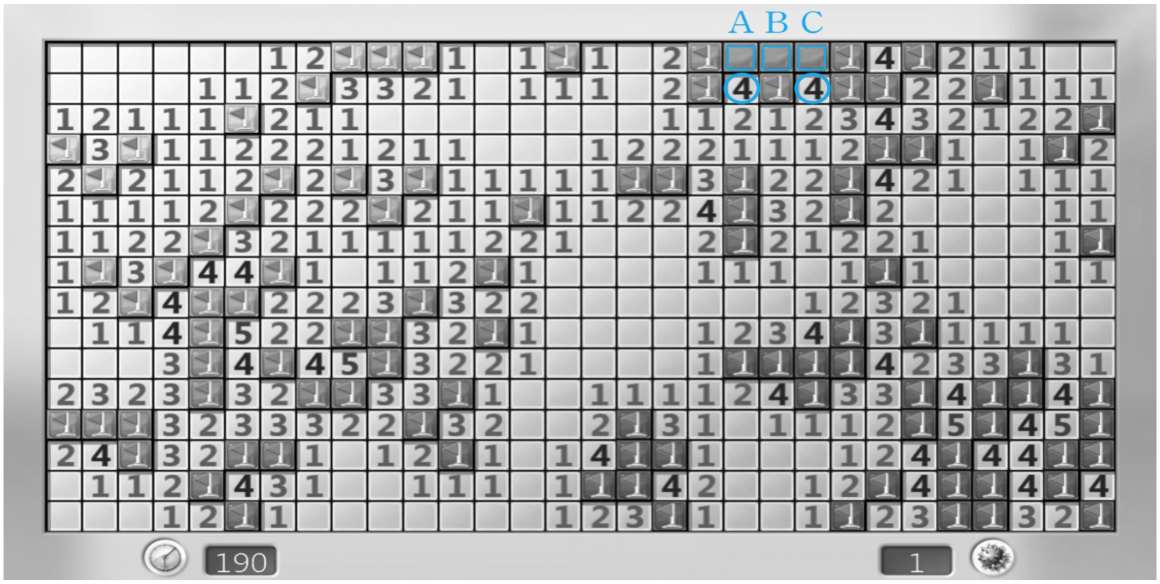
Multinomial expansion

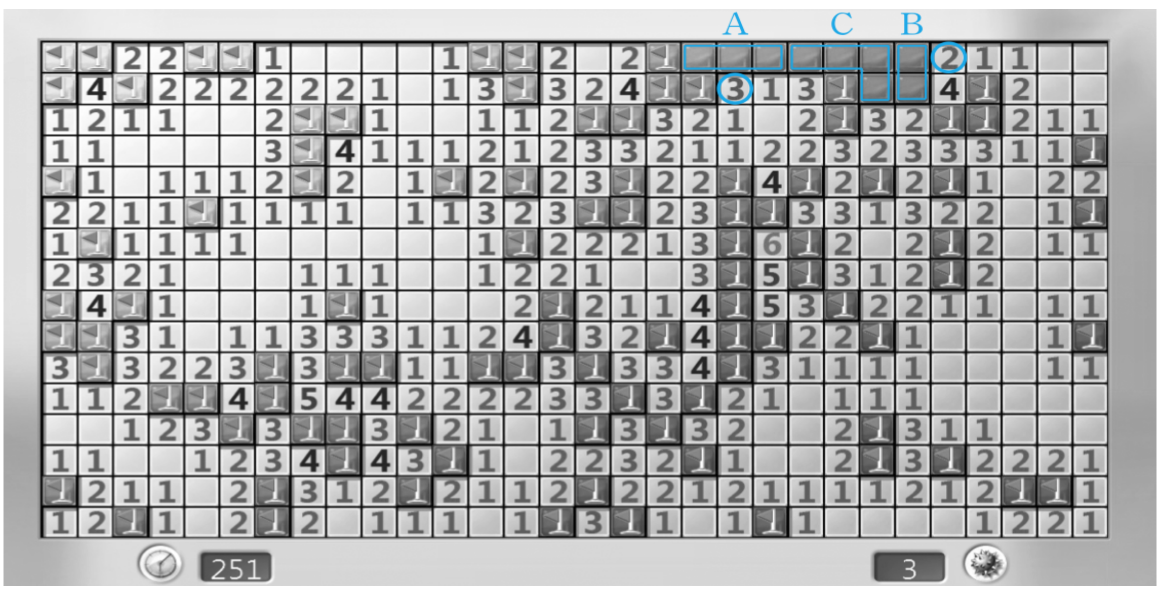
$$(x_1 + \cdots + x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1 \cdots k_m} x_1^{k_1} \cdots x_m^{k_m}$$

where

$$\text{Multinomial coefficient} \qquad \binom{n}{k_1 \cdots k_m} = \frac{n!}{k_1! \cdots k_m!}$$

Minesweeper - Introductory example of inclusion-exclusion principle





Inclusion-exclusion principle

Two sets

$$\begin{aligned} |A \cup B| &\leq |A| + |B| \\ |A \cup B| &= |A| + |B| - |A \cap B| \end{aligned}$$

Three sets

$$\begin{aligned} |A \cup B \cup C| &\leq |A| + |B| + |C| \\ |A \cup B \cup C| &\geq |A| + |B| + |C| - |AB| - |BC| - |CA| \\ |A \cup B \cup C| &= |A| + |B| + |C| - |AB| - |BC| - |CA| + |ABC| \end{aligned}$$

Many sets

$$\begin{aligned} |\cup_{i=1}^{n} A_i| &\leq \sum_{i=1}^{n} |A_i| \\ |\cup_{i=1}^{n} A_i| &\geq \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i A_j| \\ |\cup_{i=1}^{n} A_i| &\leq \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i A_j| + \sum_{1 \leq i < j < k \leq n} |A_i A_j A_k| \\ &\cdots \\ |\cup_{i=1}^{n} A_i| &= \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i A_j| + \cdots + (-1)^{n+1} |A_1 A_2 \cdots A_n| \end{aligned}$$

Example - Matching problem

For a bijective function f on $\{1, 2, \cdots, n\}$, x is called a fixed point of f if $f(x) = x$. Count the number of the bijective functions f on $\{1, 2, \cdots, n\}$ with no fixed points.

Ω	Set of all the bijective functions f on $\{1, 2, \cdots, n\}$
A_i	Set of the bijective functions f which fixes i
$\cup_{i=1}^n A_i$	Set of the bijective functions f which fixes some i
$B = \Omega \setminus \cup_{i=1}^n A_i$	Set of the bijective functions f with no fixed points

$\cup_i A_i$ bijective functions that fix some i

By the inclusion-exclusion principle

$$\begin{aligned}
|\cup_{i=1}^n A_i| &= \sum_{i=1}^n |A_i| - \sum_{1 \le i < j \le n} |A_i A_j| + \cdots + (-1)^{n+1} |A_1 A_2 \cdots A_n| \\
&= \binom{n}{1} \times (n-1)! - \binom{n}{2} \times (n-2)! + \cdots + (-1)^{n+1} \binom{n}{n} \times (n-n)! \\
&= n! \left(\frac{1}{1!} - \frac{1}{2!} + \cdots + (-1)^{n+1} \frac{1}{n!} \right)
\end{aligned}$$

$B = \Omega \setminus \cup_{i=1}^n A_i$ bijective functions with no fixed points

$$|B| = |\Omega| - |\cup_{i=1}^n A_i| = n! - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right)$$