高等数学A(上)

第3章　一元函数积分学

本章重点

积分学

不定积分
定积分

微积分基本公式

揭示出定积分与不定积分之间的联系，给出定积分计算的有效而简便的方法

换元法和分部积分

计算定积分的常用方法

第4.1节 不定积分的换元积分法

一、第一类换元法

二、第二类换元法

一、第一类换元法

1. 问题

$$\int e^{2x}\,dx \;\neq\; e^{2x}+C$$

$$\|$$

$$(e^{2x})' = 2e^{2x} \neq e^{2x}$$

解决方法

利用复合函数, 设置中间变量.

e^{2x}由$e^{u}, u=2x$复合而成，且　$dx=\dfrac{1}{2}d(2x)$

过程　令 $u=2x \Rightarrow dx=\dfrac{1}{2}du,$

$$\int e^{2x}dx=\frac{1}{2}\int e^{u}\,du=\frac{1}{2}e^{u}+C=\frac{1}{2}e^{2x}+C$$

$2x$

在一般情况下:

设 $F'(u)=f(u)$,则 $\int f(u)\mathrm{d}u=F(u)+C$.

如果 $u=\varphi(x)$(可微)

$\because\quad \mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$

$\therefore\quad \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C$

$=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$ 由此可得换元法定理

定理1

设$f(u)$具有原函数, $u=\varphi(x)$可导, 则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

即$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f[\varphi(x)]\mathrm{d}\varphi'(x)$

第一类换元公式(凑微分法)

说明: 使用此公式的关键在于将

$$\int g(x)\mathrm{d}x \text{ 化为 } \int f[\varphi(x)]\varphi'(x)\mathrm{d}x.$$

观察重点不同, 所得结论不同.

例1 求 $\int (2x-1)^{10}\,\mathrm{d}x$

$$\int u^k\,\mathrm{d}u = \frac{u^{k+1}}{k+1} + C$$

解 $\int (2x-1)^{10}\,\mathrm{d}x = \int (2x-1)^{10}\,\frac{1}{2}\mathrm{d}(2x-1) \underline{\underline{u=2x-1}} \frac{1}{2}\int u^{10}\,\mathrm{d}u$

$$= \frac{1}{2}\cdot\frac{u^{11}}{11} + C = \frac{(2x-1)^{11}}{22} + C$$

例2 求 $\int \frac{1}{3+2x}\,\mathrm{d}x$

$$\int \frac{1}{u}\,\mathrm{d}u = \mathrm{In}|u| + C$$

解 $\int \frac{1}{3+2x}\,\underline{\mathrm{d}x} = \int \frac{1}{3+2x}\,\frac{1}{2}\,\underline{\mathrm{d}(3+2x)}\ \underline{\underline{u=3+2x}}\ \frac{1}{2}\int \frac{1}{u}\mathrm{d}u$

$$= \frac{1}{2}\mathrm{In}|u| + C = \frac{1}{2}\mathrm{In}|3+2x| + C$$

注 对第一换元积分法熟练后,可以不再写出中间变量.

上面的例题

$$\int u^k \, \mathrm{d}u = \frac{u^{k+1}}{k+1} + C$$

$$\int (2x-1)^{10} \, \mathrm{d}x = \frac{1}{2}\int (2x-1)^{10} \, \mathrm{d}(2x-1) = \frac{1}{2}\frac{(2x-1)^{11}}{11} + C$$

$$\int \frac{1}{u} \mathrm{d}u = \mathrm{In}|u| + C$$

$$\int \frac{1}{3+2x} \, \mathrm{d}x = \frac{1}{2}\int \frac{1}{3+2x} \, \mathrm{d}(3+2x) = \frac{1}{2}\ln|3+2x| + C$$

一般地, $\int f(ax+b) \, \mathrm{d}x = \frac{1}{a}\int f(ax+b) \, \mathrm{d}(ax+b)$

一般 设$a \neq 0, a, b$均为常数, 且$F'(u) = f(u)$

$$\text{则} \int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$$

$$= \frac{1}{a}F(ax+b) + C$$

例3 求 $\int x\sqrt{1-x^2}\,\mathrm{d}x.$

解 $\int x\sqrt{1-x^2}\,\mathrm{d}x$

$= \int \sqrt{1-x^2}\cdot x\mathrm{d}x$

令$u = 1-x^2$，则$\mathrm{d}u = -2x\mathrm{d}x$

$= -\frac{1}{2}\int u^{\frac{1}{2}}\,\mathrm{d}u$

$= -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$

$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$

熟练后,不再写出中间变量

解 $\int x\sqrt{1-x^2}\,\mathrm{d}x$

$= \int \sqrt{1-x^2}\cdot x\mathrm{d}x$

$= -\frac{1}{2}\int (1-x^2)^{\frac{1}{2}}\,\mathrm{d}\,(1-x^2)$

$= -\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}} + C$

$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$

例4 求 $\int 2xe^{-x^2}\,dx$.

解 $\int 2xe^{-x^2}\,dx$

$= \int e^{-x^2} \cdot (2x)\,dx$

$\xlongequal[\text{则} du = -2xdx]{\text{令} u = -x^2} -\int e^u\,du$

$\boxed{\int e^u\,du = e^u + C}$

$= -e^u + C$

$= -e^{-x^2} + C$

熟练后，不再写出中间变量

解 $\int 2xe^{-x^2}\,dx$

$= \int e^{-x^2} \cdot (2x)\,dx$

$= -\int e^{-x^2}\,d(-x^2)$

$= -e^{-x^2} + C$

一般地 设$a \neq 0, a, b$均为常数，且$F'(u) = f(u)$

$$\int f(ax^2 + b)\, x\mathrm{d}x \frac{1}{2a} = \int f(ax^2 + b)\, \mathrm{d}(ax^2 + b)$$

$$= \frac{1}{2a} F(ax^2 + b) + C$$

$$\int f(x^n) x^{n-1}\, \mathrm{d}x = \frac{1}{n} \int f(x^n)\, \mathrm{d}x^n$$

例5 求 $\displaystyle\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}dx.$

$$d\sqrt{x} = \frac{1}{2\sqrt{x}}dx$$

解 原式 $= \displaystyle\int e^{3\sqrt{x}} \cdot \frac{1}{\sqrt{x}}dx$

$$\xlongequal[\text{则} du = \frac{3}{2\sqrt{x}}dx]{\text{令} u = 3\sqrt{x}} \frac{2}{3}\int e^{u}\,du$$

$$= \frac{2}{3}e^{u} + C$$

$$= \frac{2}{3}e^{3\sqrt{x}} + C$$

熟练后，不再写出中间变量

解 $\displaystyle\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}dx$

$$= \int e^{3\sqrt{x}} \cdot \frac{1}{\sqrt{x}}dx$$

$$= \frac{2}{3}\int e^{3\sqrt{x}}\,d(3\sqrt{x})$$

$$= \frac{2}{3}e^{3\sqrt{x}} + C$$

一般　设$a \neq 0, a, b$均为常数，且$F'(u) = f(u)$

$$\text{则} \int f(a\sqrt{x}) \frac{1}{\sqrt{x}} \mathrm{d}x = \frac{2}{a} \int f(a\sqrt{x})\, \mathrm{d}(a\sqrt{x})$$

$$= \frac{2}{a} F(a\sqrt{x}) + C$$

$$\int \mathrm{e}^{f(x)} f'(x) \mathrm{d}x = \int \mathrm{e}^{f(x)} \mathrm{d}f(x)$$

例6　求 $\int \frac{x^2}{(x+2)^3}\mathrm{d}x$

解　$\int \frac{x^2}{(x+2)^3}\mathrm{d}x \xlongequal[\quad]{u=x+2} \int \frac{(u-2)^2}{u^3}\mathrm{d}u = \int \frac{u^2-4u+4}{u^3}\mathrm{d}u$

$$= \int (u^{-1} - 4u^{-2} + 4u^{-3})\mathrm{d}u$$

$$= \ln|u| + 4u^{-1} - 2u^{-2} + C$$

$$= \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$$

例7 $\displaystyle\int \frac{\ln x}{x}\mathrm{d}x$

解 $\displaystyle\int \frac{\ln x}{x}\mathrm{d}x = \int \ln x \mathrm{d}\ (\ln x) = \frac{(\ln x)^2}{2} + C$

例8 $\displaystyle\int \frac{1}{x(1+2\ln x)}\mathrm{d}x$

解 $\displaystyle\int \frac{1}{x(1+2\ln x)}\mathrm{d}x = \frac{1}{2}\int \frac{1}{1+2\ln x}\mathrm{d}(1+2\ln x)$

$$= \frac{1}{2}\ln|1+2\ln x| + C$$

一般, 设$F'(u) = f(u)$

$$\int \frac{f(\ln x)}{x} \mathrm{d}x = \int f(\ln x) \mathrm{d} \ln x = F(\ln x) + C$$

一般 $\displaystyle\int \frac{\varphi'(x)}{\varphi(x)} \mathrm{d}x = \int \frac{1}{\varphi(x)} \mathrm{d}[\varphi(x)] = \ln|\varphi(x)| + C$

课堂练习

求 $\int \tan x\mathrm{d}x$.

答案 $\int \tan x\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x} = -\ln|\cos x| + C$

类似 $\int \cot x\mathrm{d}x = \int \frac{\cos x}{\sin x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{\sin x} = \ln|\sin x| + C$

常用的几种配元形式:

(1) $\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$

(2) $\int f(x^n)x^{n-1}\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}x^n$

(3) $\int f(x^n)\frac{1}{x}\mathrm{d}x = \frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}x^n$

万能凑幂法

(4) $\int f(\sin x)\cos x\mathrm{d}x = \int f(\sin x)\,\mathrm{d}\sin x$

(5) $\int f(\cos x)\sin x\mathrm{d}x = -\int f(\cos x)\,\mathrm{d}\cos x$

$$(6)\quad \int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\,\mathrm{d}\tan x$$

$$(7)\quad \int f(\mathrm{e}^x)\mathrm{e}^x\mathrm{d}x = \int f(\mathrm{e}^x)\,\mathrm{d}\mathrm{e}^x$$

$$(8)\quad \int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$$

例9 求 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

解 $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u=\frac{1}{a}\,\mathrm{d}x$

$= \frac{1}{a}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{a}\arctan u + C$

$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

想到公式

$\int \frac{\mathrm{d}u}{1+u^2}$

$= \arctan u + C$

例10　求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\ (a>0)$.

解

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}x}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$$= \arcsin\frac{x}{a} + C$$

想到 $\displaystyle\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C$

$$\int f\left[\varphi(x)\right]\varphi'(x)\,\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$ （直接配元）

例11　求 $\int \frac{\mathrm{d}x}{x^2-a^2}$.

解　$\because \frac{1}{x^2-a^2} = \frac{1}{2a} \frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore$ 原式 $= \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)\mathrm{d}x$

$$\int \frac{1}{u}\mathrm{d}u = \ln|u| + C$$

$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$

$= \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

第一换元积分法是不定积分的基础，且有很大的灵活性，可通过三角恒等变换、代数运算、加一项减一项、上,下同除以一个因子等方法，使积分变得易求.

某些三角函数积分举例

例12 求 $\int \frac{1}{1+\cos x}\mathrm{d}x$.

解 $$\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\mathrm{d}x$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\mathrm{d}x = \int \frac{1-\cos x}{\sin^2 x}\mathrm{d}x$$

$$= \int \frac{1}{\sin^2 x}\mathrm{d}x - \int \frac{1}{\sin^2 x}\mathrm{d}(\sin x) = -\cot x + \frac{1}{\sin x} + C.$$

例13 求 $\int \csc x \, dx$.

解法一 $\int \csc x \, dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$

$= \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C.$

（使用了三角函数恒等变形）

解法二

$$\int \csc x \, dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \quad (u = \cos x)$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C.$$

类似地可推出 $\int \sec x dx = \ln|\sec x + \tan x| + C$.

例14　设$f'(\sin^2 x) = \cos^2 x$，求$f(x)$.

解　令 $u = \sin^2 x \Rightarrow \cos^2 x = 1 - u$,

$$f'(u) = 1 - u,$$

$$f(u) = \int (1-u)\,\mathrm{d}u = u - \frac{1}{2}u^2 + C,$$

$$f(x) = x - \frac{1}{2}x^2 + C.$$

例15 求 $\int \sin^2 x \cdot \cos^5 x \, dx$.

解 $\int \sin^2 x \cdot \cos^5 x \, dx = \int \sin^2 x \cdot \cos^4 x \, d(\sin x)$

$= \int \sin^2 x \cdot (1 - \sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

例16 求 $\int \sin^2 x \cos^2 3x\mathrm{d}x.$

解 $\because \quad \sin^2 x\cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x\sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x\cos 2x + \frac{1}{8}(1-\cos 4x)$$

$$= \frac{1}{4} - \frac{1}{8}\cos 8x - \frac{1}{8}\cos 4x - \sin^2 2x\cos 2x$$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x\mathrm{d}\,(8x) - \frac{1}{32}\int \cos 4x\,\mathrm{d}(4x) - \frac{1}{2}\int \sin^2 2x\mathrm{d}\,(\sin 2x)$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{32}\sin 4x - \frac{1}{6}\sin^3 2x + C$$

对 $\int \sin^m x \cdot \cos^n x \mathrm{d}x$ 型的积分

当m,n中有奇数时,拆开奇次幂的项去凑微分.

当m,n均为偶数时,先利用倍角公式降幂.

例17 求 $\int \tan^5 x \sec^3 x \, dx$.

解 原式 $= \int \tan^4 x \sec^2 x \cdot (\sec x \tan x) dx$　　$\sec^2 x = \tan^2 x + 1$

$= \int (\sec^2 x - 1)^2 \sec^2 x \, d\sec x$　　$d(\sec x) = (\sec x \tan x) dx$

$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x) d\sec x$

$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.$

例18 求 $\int \tan^2 x \sec^4 x \, dx$.

解 原式 $= \int \tan^2 x \sec^2 x \cdot (\sec^2 x) dx$ $\quad \sec^2 x = \tan^2 x + 1$

$= \int \tan^2 x (\tan^2 x + 1)^2 d(\tan x)$ $\quad d(\tan x) = (\sec^2 x) dx$

$= \int (\tan^6 x + 2\tan^4 x + \tan^2 x) d(\tan x)$

$= \frac{1}{7}\tan^7 x + \frac{2}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C.$

对 $\int \tan^{2k-1} x \sec^{n} x \,\mathrm{d}x$ 型的积分,将积分表达式改写成

$$\tan^{2k-1} x \sec^{n} x \,\mathrm{d}x = (\sec^2 x - 1)^k \sec^{n-1} x \mathrm{d} \sec x$$

对 $\int \tan^{n} x \sec^{2k} x \,\mathrm{d}x$ 型的积分,将积分表达式改写成

$$\tan^{n} x \sec^{2k} x \,\mathrm{d}x = \tan^{n} x \,(\tan^2 x + 1)^{k-1} x \mathrm{d} \tan x$$

例19 求 $\int \cos 3x \cos 2x \mathrm{d}x$.

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$,

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x \mathrm{d}x = \frac{1}{2}\int (\cos x + \cos 5x)\mathrm{d}x$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

对
$$\boxed{\begin{aligned}&\int \sin mx\cdot\cos nx\mathrm{d}x,\\&\int \sin mx\cdot\sin nx\mathrm{d}x,\\&\int \cos mx\cdot\cos nx\mathrm{d}x\end{aligned}}$$
其中 $m,n\in\mathbf{N},\ m\neq n.$

型的积分,将被积函数先利用三角函数积化和差公式化简.

例20　求 $\displaystyle\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}\mathrm{d}x.$

解

$$\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}\mathrm{d}x = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}\arcsin\frac{x}{2}}\mathrm{d}\frac{x}{2}$$

$$= \int \frac{1}{\arcsin\frac{x}{2}}\mathrm{d}\left(\arcsin\frac{x}{2}\right) = \ln\arcsin\frac{x}{2} + C.$$

二、第二类换元法

1. 问题 $\int \sqrt{1-x^2}\mathrm{d}x$

解决方法 改变中间变量的设置方法.

过程 令 $x=\sin t \Rightarrow \mathrm{d}x=\cos t\,\mathrm{d}t$,

$$\text{则} \int \sqrt{1-x^2}\mathrm{d}x = \int \sqrt{1-\sin^2 t}\cos t\,\mathrm{d}t = \int \cos^2 t\,\mathrm{d}t$$

$$= \int \frac{1+\cos 2t}{2}\mathrm{d}t = \frac{1}{2}t + \frac{1}{2}\int \cos 2t\,\mathrm{d}t \cdots$$

（应用“凑微分”即可求出结果）

定理2 设$x=\psi(t)$是单调可导的函数，并且$\psi'(t)\neq 0$. 又设$f[\psi(t)]\psi'(t)$具有原函数，则有换元公式

$$\int f(x)\mathrm{d}x=\left[\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\right]_{t=\psi^{-1}(x)}$$

第二类积分换元公式

其中$t=\psi^{-1}(x)$是$x=\psi(t)$的反函数.

证 设$f[\psi(t)]\psi'(t)$的原函数为$\Phi(t)$，令$F(x)=\Phi[\psi^{-1}(x)]$

则 $F'(x)=\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=f[\psi(t)]\psi'(t)\cdot\dfrac{1}{\psi'(t)}=f(x)$

$\therefore \int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C=\int f[\psi(t)]\psi'(t)\mathrm{d}t\,|_{t=\psi^{-1}(x)}$

证毕

例21 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x \quad (a>0)$.

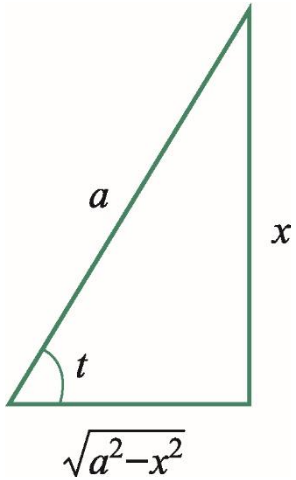
解 令 $x=a\sin t \Rightarrow \mathrm{d}x=a\cos t\mathrm{d}t \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

则 $\int\sqrt{a^2-x^2}\,\mathrm{d}x$

$$=\int a\cos t\cdot a\cos t\mathrm{d}t \ =a^2\int\cos^2 t\mathrm{d}t$$

$$=a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

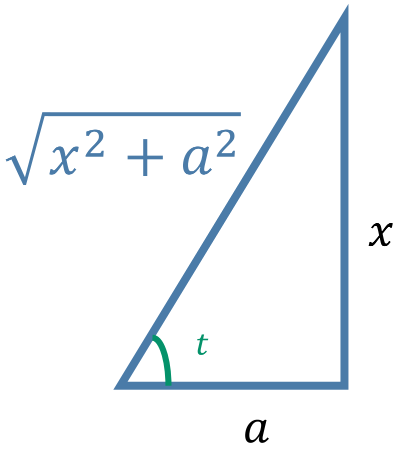
$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$



例22 求 $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x \quad (a>0)$.

解 令 $x = a\tan t \Rightarrow \mathrm{d}x = a\sec^2 t\mathrm{d}t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

则 $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \int \frac{1}{a\sec t}\cdot a\sec^2 t\mathrm{d}t$



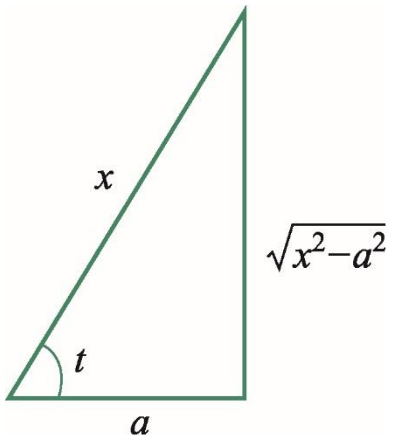
$= \int \sec t\mathrm{d}t \;= \ln|\sec t + \tan t| + C_1$

$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C_1 = \ln\left(x + \sqrt{x^2+a^2}\right) + C.$

例23　求 $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x \quad (a>0)$.

解　令 $x = a\sec t \Rightarrow \mathrm{d}x = a\sec t\tan t\,\mathrm{d}t,\ t\in\left(0,\frac{\pi}{2}\right)$,

$$\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x = \int \frac{a\sec t\cdot\tan t}{a\tan t}\mathrm{d}t$$



$$= \int \sec t\mathrm{d}t = \ln(\sec t+\tan t)+C$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1 = \ln\left|x+\sqrt{x^2-a^2}\right|+C.$$

说明(1)

以上几例所使用的均为三角代换. 三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2 - x^2}$ 可令 $x = a\sin t$;

(2) $\sqrt{a^2 + x^2}$ 可令 $x = a\tan t$;

(3) $\sqrt{x^2 - a^2}$ 可令 $x = a\sec t$.

说明(2)

积分中为了化掉根式除采用三角代换外还可用双曲代换.

$\because \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1 \quad \therefore x = a \operatorname{sh} t,\ x = a \operatorname{ch} t$ 也可以化掉根式.

例 $$\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x \xlongequal[\text{则}\mathrm{d}x = a\operatorname{ch} t\,\mathrm{d}t]{\text{令}x = a\operatorname{sh}t} \int \frac{a\operatorname{ch} t}{a\operatorname{ch} t}\mathrm{d}t = \int \mathrm{d}t = t + C_1$$

$$= \operatorname{arsh}\frac{x}{a} + C_1 = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C_1 = \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$

说明(3)

积分中为了化掉根式是否一定采用三角代换(或双曲代换)并不是绝对的,需根据被积函数的情况来定.

例24　求 $\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$　　(三角代换很繁琐)

解　令 $t=\sqrt{1+x^2}\ \Rightarrow x^2=t^2-1,\ x\mathrm{d}x=t\mathrm{d}t,$

$$\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x=\int \frac{(t^2-1)^2}{t}t\mathrm{d}t=\int (t^4-2t^2+1)\,\mathrm{d}t$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例25 求 $\displaystyle\int \frac{1}{\sqrt{1+\mathrm{e}^x}}\mathrm{d}x$.

解 令 $t=\sqrt{1+\mathrm{e}^x} \Rightarrow \mathrm{e}^x=t^2-1, x=\ln(t^2-1)$, $\mathrm{d}x=\dfrac{2t}{t^2-1}\mathrm{d}t$,

$$\int \frac{1}{\sqrt{1+\mathrm{e}^x}}\mathrm{d}x=\int \frac{2}{t^2-1}\mathrm{d}t=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\mathrm{d}t$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=2\ln\left(\sqrt{1+\mathrm{e}^x}-1\right)-x+C.$$

说明(4) 当分母的阶较高时, 可采用**倒代换** $x=\dfrac{1}{t}$.

例26　求 $\displaystyle\int \frac{1}{x(x^7+2)}\mathrm{d}x$

解　令 $x=\dfrac{1}{t}\Rightarrow \mathrm{d}x=-\dfrac{1}{t^2}\mathrm{d}t,$

$$\int \frac{1}{x(x^7+2)}\mathrm{d}x=\int \frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\int \frac{t^6}{1+2t^7}\mathrm{d}t$$

$$=-\frac{1}{14}\ln|1+2t^7|+C=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C.$$

例27　求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

当分母次幂较高时，可利用倒代换 $x=\frac{1}{t}$

解　令 $x=\frac{1}{t}$，则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}t$

$$\text{原式}=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}t=-\int (a^2t^2-1)^{\frac{1}{2}}|t|\,\mathrm{d}t$$

当 $x>0$时，$\text{原式}=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}\,(a^2t^2-1)$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当$x<0$时，类似可得同样结果.

说明(5)

当被积函数含有两种或两种以上的根式$\sqrt[k]{x}, \cdots, \sqrt[l]{x}$ 时,

可采用令$x = t^n$(其中n为各根指数的最小公倍数).

例28 求 $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}\mathrm{d}x.$

解 令 $x=t^6 \Rightarrow \mathrm{d}x=6t^5\mathrm{d}t,$

原式 $=\int \frac{6t^5}{t^3(1+t^2)}\mathrm{d}t=\int \frac{6t^2}{1+t^2}\mathrm{d}t=6\int \frac{t^2+1-1}{1+t^2}\mathrm{d}t$

$=6\int\left(1-\frac{1}{1+t^2}\right)\mathrm{d}t$

$=6[t-\arctan t]+C=6[\sqrt[6]{x}-\arctan \sqrt[6]{x}]+C.$

1. 第二类换元法常见类型:

(1) $\int f(x,\sqrt[n]{ax+b})\mathrm{d}x$, 令 $t=\sqrt[n]{ax+b}$

(2) $\int f\left(x,\sqrt[n]{\frac{ax+b}{cx+\mathrm{d}}}\right)\mathrm{d}x$, 令 $t=\sqrt[n]{\frac{ax+b}{cx+\mathrm{d}}}$

(3) $\int f(x,\sqrt{a^2-x^2})\mathrm{d}x$, 令 $x=a\sin t$ 或 $x=a\cos t$

(4) $\int f(x,\sqrt{a^2+x^2})\mathrm{d}x$, 令 $x=a\tan t$ 或 $x=a\,\mathrm{sh}\,t$

(5) $\int f(x,\sqrt{x^2-a^2})\mathrm{d}x$, 令 $x=a\sec t$ 或 $x=a\,\mathrm{ch}\,t$

(6) $\int f(a^x)\mathrm{d}x$ ， 令 $t = a^x$

(7) 分母中因子次数较高时, 可试用倒代换

2. 常用基本积分公式的补充

(16) $\int \tan x \mathrm{d}x = -\ln|\cos x| + C$

(17) $\int \cot x \mathrm{d}x = \ln|\sin x| + C$

(18) $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$

(19) $\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C$

(20) $$\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a}+C$$

(21) $$\int \frac{1}{x^2-a^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

(22) $$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a}+C$$

(23) $$\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \ln(x+\sqrt{x^2+a^2})+C$$

(24) $$\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x = \ln|x+\sqrt{x^2-a^2}|+C$$

例29 求$I=\int\frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解 $I=\frac{1}{2}\int\frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}}=\frac{1}{2}\ln(2x+\sqrt{4x^2+9})+C$

例30　求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解　原式 $\displaystyle= \int \frac{\mathrm{d}\left(x-\frac{1}{2}\right)}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \arcsin\frac{2x-1}{\sqrt{5}}+C$

例31　求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}-1}}$.

解　原式 $\displaystyle= -\int \frac{\mathrm{d}\mathrm{e}^{-x}}{\sqrt{1-\mathrm{e}^{-2x}}} = -\arcsin \mathrm{e}^{-x}+C$

例32 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解 令 $x=\frac{1}{t}$,

$$\text{原式} = -\int \frac{t}{\sqrt{a^2t^2+1}}\mathrm{d}t$$

$$= -\frac{1}{2a^2}\int \frac{\mathrm{d}(a^2t^2+1)}{\sqrt{a^2t^2+1}} = -\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$= -\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例33　求 $\displaystyle\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解　原式 $\displaystyle= \int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$　$\boxed{令\ x+1=\dfrac{1}{t}}$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t = -\int \frac{t^2}{\sqrt{1-t^2}}\mathrm{d}t$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t \boxed{= \int \sqrt{1-t^2}\,\mathrm{d}t} - \int \frac{1}{\sqrt{1-t^2}}\mathrm{d}t$$

$$= \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t - \arcsin t + C$$

$$= \frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2}\arcsin\frac{1}{x+1} + C$$

第4.2节 定积分的换元积分法

一、第一类换元法

二、第二类换元法

一、换元公式

定理1　假设函数$f(x)$在$[a,b]$上连续, 函数$x=\varphi(t)$满足条件

(1)　$\varphi(\alpha)=a, \varphi(\beta)=b$;

(2)　$\varphi(t)$在$[\alpha,\beta]$(或$[\beta,\alpha]$)上具有连续导数,且其值域$R_{\varphi}=[a,b]$,

则有

$$\int_a^b f(x)\mathrm{d}x=\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)\mathrm{d}t.$$

定积分换元公式

证　设$F(x)$是$f(x)$的一个原函数,则

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

记$\Phi(t) = F[\varphi(t)]$,　　$\because$　$\Phi'(t) = F'(\varphi(t)) \cdot \varphi'(t) = f[\varphi(t)]\varphi'(t)$,

$\therefore$　$\Phi(t)$是$f[\varphi(t)]\varphi'(t)$的一个原函数,

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t = \Phi(b) - \Phi(a) = F[\varphi(\beta)] - F[\varphi(\alpha)]$$

$$= F(b) - F(a).$$

$$\therefore \quad \int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t.$$

证毕

注 定积分换元公式 $\int_{\alpha}^{b} f(x)\,\mathrm{d}x = \int_{\alpha}^{\beta} f[\varphi(t)]\,\varphi'(t)\,\mathrm{d}t$

（1）当$\alpha > \beta$时，换元公式仍成立；

（2）换元必换限，原函数中的变量不必代回.

（3）换元公式也可反过来使用，即

$$\int_{\alpha}^{\beta} f[\varphi(t)]\,\varphi'(t)\mathrm{d}t \xlongequal[a=\varphi(\alpha),\,b=\varphi(\beta)]{\text{令}x=\varphi(t)} \int_{a}^{b} f(x)\mathrm{d}x$$

或配元 $\int_{\alpha}^{\beta} f[\varphi(t)]\,\underline{\varphi'(t)\,\mathrm{d}t} = \int_{\alpha}^{\beta} f[\varphi(t)]\,\mathrm{d}\varphi(t)$

配元不换限!

例1 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}}\mathrm{d}x$.

解 令$t=\sqrt{2x+1}$, 则$x=\frac{t^2-1}{2}$, $\mathrm{d}x=t\mathrm{d}t$, 且

当 $x=0$ 时, $t=1$; 当$x=4$ 时,$t=3$.

$$\therefore \quad 原式=\int_1^3 \frac{\frac{t^2-1}{2}+2}{t}t\mathrm{d}t=\frac{1}{2}\int_1^3 (t^2+3)\mathrm{d}t$$

$$=\frac{1}{2}\left(\frac{1}{3}t^3+3t\right)\Big|_1^3=\frac{22}{3}.$$

例2 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x\mathrm{d}x$.

解 令$t=\cos x$, $\mathrm{d}t=-\sin x\,\mathrm{d}x$,

$x=\frac{\pi}{2} \Rightarrow t=0,$

$x=0 \Rightarrow t=1.$

$$原式=-\int_1^0 t^5\mathrm{d}t$$

$$=\left.\frac{t^6}{6}\right|_0^1=\frac{1}{6}.$$

不写出换元,则不需要换限

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x\mathrm{d}x$$

$$=-\int_0^{\frac{\pi}{2}} \cos^5 x\,\mathrm{d}(\cos x)$$

$$=-\frac{1}{6}\cos^6 x\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{6}.$$

课堂练习.

计算 $\int_1^2 \frac{e^x}{e^x - 1}dx$.

答案 令$e^x = t$, 则$dx = \frac{1}{t}dt$,

$$原式 = \int_e^{e^2} \frac{1}{t-1} \cdot \frac{1}{t}dt$$

$$= \int_e^{e^2} \frac{1}{t-1}dt = [\ln|t-1|]_e^{e^2}$$

$$= \ln(e^2 - 1) - \ln(e - 1)$$

$$= \ln(e + 1).$$

不写出换元,则不需要换限

$$\int_1^2 \frac{e^x}{e^x - 1}dx$$

$$= \int_1^2 \frac{e^x}{e^x - 1}d(e^x - 1)$$

$$= \ln|e^x - 1|\Big|_1^2$$

$$= \ln(e + 1).$$

例3　计算 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx$.

解　原式 $=\int_0^{\pi}\sqrt{\sin^3 x\,(1-\sin^2 x)}dx=\int_0^{\pi}|\cos x|(\sin x)^{\frac{3}{2}}dx$

$$=\int_0^{\frac{\pi}{2}}\cos x\,(\sin x)^{\frac{3}{2}}dx-\int_{\frac{\pi}{2}}^{\pi}\cos x\,(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}d\sin x-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}d\sin x$$

$$=\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}-\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4}{5}.$$

例4 计算 $\displaystyle\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\mathrm{d}x. \qquad (a>0)$

解 令 $x = a\sin t,\ \mathrm{d}x = a\cos t\,\mathrm{d}t,\ x = a \Rightarrow t = \frac{\pi}{2},\ x = 0 \Rightarrow t = 0.$

$$原式 = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + \sqrt{a^2(1-\sin^2 t)}}\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t}\mathrm{d}t = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1 + \frac{\cos t - \sin t}{\sin t + \cos t}\right)\mathrm{d}t$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\left[\ln|\sin t + \cos t|\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

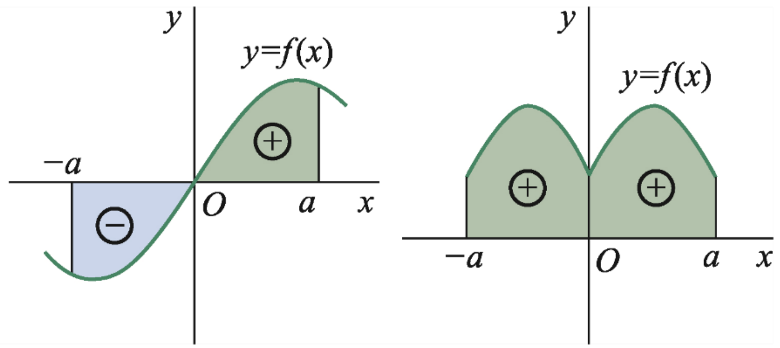
例5 设$f(x)$为$[-a, a]$上的连续函数.

（1）若$f(-x)=f(x)$，则$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$；

（2）若$f(-x)=-f(x)$，则$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.

偶倍奇零

由定积分的几何意义(面积的代数和)可得.



证 $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$

令$x=-t$

$$= \int_{0}^{a} f(-t)\,\mathrm{d}t + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} [f(-x)+f(x)]\,\mathrm{d}x$$

$$= \begin{cases} 2\int_{0}^{a} f(x)\,\mathrm{d}x, & f(-x)=f(x), \\ 0, & f(-x)=-f(x). \end{cases}$$

奇、偶函数在对称区间上的定积分性质 偶倍奇零

例6 计算 $\displaystyle\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}\mathrm{d}x$.

解 原式 $\displaystyle=\int_{-1}^{1}\underbrace{\frac{2x^2}{1+\sqrt{1-x^2}}}_{\text{偶函数}}\mathrm{d}x+\int_{-1}^{1}\underbrace{\frac{x\cos x}{1+\sqrt{1-x^2}}}_{\text{奇函数}}\mathrm{d}x$

$$=4\int_{0}^{1}\frac{x^2}{1+\sqrt{1-x^2}}\mathrm{d}x=4\int_{0}^{1}\frac{x^2\left(1-\sqrt{1-x^2}\right)}{1-(1-x^2)}\mathrm{d}x$$

$$=4\int_{0}^{1}\left(1-\sqrt{1-x^2}\right)\mathrm{d}x=4-\underbrace{4\int_{0}^{1}\sqrt{1-x^2}\mathrm{d}x}_{\text{单位圆的面积}}$$

$$=4-\pi.$$

例7 若$f(x)$在$[0,1]$上连续，证明

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$;

(2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x$. 由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$.

证 (1) $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x \overset{\text{设}x=\frac{\pi}{2}-t,\ \mathrm{d}x=-\mathrm{d}t}{=\!=\!=\!=\!=} -\int_{\frac{\pi}{2}}^{0} f(\cos t)\mathrm{d}t$

$= \int_0^{\frac{\pi}{2}} f(\cos t)\mathrm{d}t = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$. 证毕

（2） $\int_0^{\pi} xf(\sin x)\mathrm{d}x \xlongequal[\text{则 } \mathrm{d}x = -\mathrm{d}t]{\text{设} x = \pi - t} -\int_0^{\pi}(\pi - t)f(\sin(\pi - t))\mathrm{d}t$

$$= \int_0^{\pi}(\pi - t)f(\sin t)\mathrm{d}t$$

$$= \pi\int_0^{\pi} f(\sin t)\mathrm{d}t - \int_0^{\pi} tf(\sin t)\mathrm{d}t$$

$$= \pi\int_0^{\pi} f(\sin x)\mathrm{d}x - \int_0^{\pi} xf(\sin x)\mathrm{d}x,$$

$$\therefore \int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$$ 证毕

$$\because \int_0^{\pi} x f(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x.$$

$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\mathrm{d}x$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\mathrm{d}(\cos x)$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

例8 设$f(x)$是连续的周期函数，周期为T，证明：

(1) $\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$;

(2) $\int_a^{a+nT} f(x)\mathrm{d}x = n\int_0^T f(x)\mathrm{d}x \ (n \in \mathbf{N})$，由此计算 $\int_0^{n\pi} \sqrt{1+\sin 2x}\mathrm{d}x$.

证（1）方法1.

记 $\Phi(a) = \int_a^{a+T} f(x)\mathrm{d}x$,

则 $\because \ \Phi'(a) = f(a+T) - f(a) = 0$, $\therefore \Phi(a) = C$.

$\therefore \ \Phi(a) = \Phi(0) = \int_0^T f(x)\mathrm{d}x$，即 $\int_a^{a+T} f(x)\mathrm{d}x = \int_a^T f(x)\mathrm{d}x$.

方法2 $\Phi(a)=\int_a^0 f(x)\mathrm{d}x+\int_0^T f(x)\mathrm{d}x+\int_T^{a+T} f(x)\mathrm{d}x$

$$\because \int_T^{a+T} f(x)\mathrm{d}x \xlongequal{x=T+t} \int_0^a f(T+t)\mathrm{d}t=\int_0^a f(t)\mathrm{d}t=-\int_a^0 f(x)\mathrm{d}x$$

$$\therefore \quad \Phi(a)=\int_0^T f(x)\mathrm{d}x \text{ 即 } \int_a^{a+T} f(x)\mathrm{d}x=\int_a^T f(x)\mathrm{d}x.$$

（2） $\int_a^{a+nT} f(x)\mathrm{d}x$

$$=\int_a^{a+T} f(x)\mathrm{d}x+\int_{a+T}^{a+2T} f(x)\mathrm{d}x+\cdots+\int_{a+(n-1)T}^{a+nT} f(x)\mathrm{d}x$$

$$=\int_0^T f(x)\mathrm{d}x+\int_0^T f(x)\mathrm{d}x+\cdots+\int_0^T f(x)\mathrm{d}x=n\int_0^T f(x)\mathrm{d}x$$ 证毕

$\because \sqrt{1+\sin 2x}$是周期为π的连续函数

$$\therefore \int_0^{n\pi}\sqrt{1+\sin 2x}\mathrm{d}x = n\int_0^{\pi}\sqrt{1+\sin 2x}\mathrm{d}x = n\int_0^{\pi}|\sin x+\cos x|\mathrm{d}x$$

$$=\sqrt{2}n\int_0^{\pi}|\sin(x+\frac{\pi}{4})|\mathrm{d}x$$

$$\underline{\underline{x+\frac{\pi}{4}=t}}\ \sqrt{2}n\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}|\sin t|\mathrm{d}t$$

$$=\sqrt{2}n\int_0^{\pi}|\sin t|\mathrm{d}t=\sqrt{2}n\int_0^{\pi}\sin t\,\mathrm{d}t=2\sqrt{2}n$$

例9 计算 $\displaystyle\int_0^3 \frac{x^2}{(x^2-3x+3)^2}\mathrm{d}x.$

解　原式 $\displaystyle=\int_0^3 \frac{x^2}{[(x-\frac{3}{2})^2+\frac{3}{4}]^2}\mathrm{d}x$

令 $x-\dfrac{3}{2}=\dfrac{\sqrt{3}}{2}\tan t$，

则 $\mathrm{d}x=\dfrac{\sqrt{3}}{2}\sec^2 t\,\mathrm{d}t$,

$[(x-\frac{3}{2})^2+\frac{3}{4}]^2=\dfrac{9}{16}\sec^4 t$

$x=0\Rightarrow t=-\dfrac{\pi}{3}$,

$x=3\Rightarrow t=\dfrac{\pi}{3}$.

$$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{(\frac{3}{2}+\frac{\sqrt{3}}{2}\tan t)^2}{\frac{9}{16}\sec^4 t}\cdot\frac{\sqrt{3}}{2}\sec^2 t\,\mathrm{d}t$$

$$=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(\frac{9}{4}+\frac{3\sqrt{3}}{2}\tan t+\frac{3}{4}\tan^2 t)\cdot\frac{8\sqrt{3}}{9}\cos^2 t\mathrm{d}t$$

$$=\frac{2}{\sqrt{3}}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(3\cos^2 t+2\sqrt{3}\tan t\cos^2 t+\sin^2 t)\,\mathrm{d}t$$

$$= \frac{2}{\sqrt{3}}\left[\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(3\cos^2 t + \sin^2 t)\,\mathrm{d}t + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2\sqrt{3}\tan t\cos^2 t\,\mathrm{d}t\right]$$

$$= \frac{2}{\sqrt{3}}\cdot 2\int_{0}^{\frac{\pi}{3}}(1 + 2\cos^2 t)\,\mathrm{d}t + 0$$

偶倍奇零

$$= \frac{4}{\sqrt{3}}\int_{0}^{\frac{\pi}{3}}(2 + \cos 2t)\mathrm{d}t = \frac{4}{\sqrt{3}}\left(2t + \frac{1}{2}\sin 2t\right)\Bigg|_{0}^{\frac{\pi}{3}}$$

$$= \frac{8\pi}{3\sqrt{3}} + 1$$

例10 设函数$f(x)=\begin{cases}\dfrac{1}{1+\cos x}, & -\pi<x<0,\\ xe^{-x^2}, & x\geqslant 0.\end{cases}$ 计算$\int_1^4 f(x-2)\,dx$.

解 令$x-2=t$, 则$dx=dt$, $x=1\Rightarrow t=-1$, $x=4\Rightarrow t=2$.

$$\int_1^4 f(x-2)\,dx=\int_{-1}^2 f(t)\,dt=\int_{-1}^0 f(t)\,dt+\int_0^2 f(t)\,dt$$

$$=\int_{-1}^0\frac{1}{1+\cos t}dt+\int_0^2 te^{-t^2}dt=\int_{-1}^0\frac{1}{\cos^2\frac{t}{2}}d\left(\frac{t}{2}\right)-\frac{1}{2}\int_0^2 e^{-t^2}d(-t^2)$$

$$=\tan\frac{t}{2}\bigg|_{-1}^0-\frac{1}{2}e^{-t^2}\bigg|_0^2=\tan\frac{1}{2}-\frac{1}{2}e^{-4}+\frac{1}{2}.$$