On the Decidability of Connectedness Constraints in 2D and 3D Euclidean Spaces

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Abstract

We investigate (quantifier-free) spatial constraint languages with equality, contact and connectedness predicates, as well as Boolean operations on regions, interpreted over low-dimensional Euclidean spaces. We show that the complexity of reasoning varies dramatically depending on the dimension of the space and on the type of regions considered. For example, the logic with the interior-connectedness predicate (and without contact) is undecidable over polygons or regular closed sets in $\mathbb{R}^2$, EXPTIME-complete over polyhedra in $\mathbb{R}^3$, and NP-complete over regular closed sets in $\mathbb{R}^3$.

1 Introduction

A central task in Qualitative Spatial Reasoning is that of determining whether some described spatial configuration is geometrically realizable in 2D or 3D Euclidean space. Typically, such a description is given using a spatial logic—a formal language whose variables range over (typed) geometrical entities, and whose non-logical primitives represent geometrical relations and operations involving those entities. Where the geometrical primitives of the language are purely topological in character, we speak of a *topological logic*; and where the logical syntax is confined to that of propositional calculus, we speak of a *topological constraint language*.

Topological constraint languages have been intensively studied in Artificial Intelligence over the last two decades. The best-known of these, $\mathcal{RCC}8$ and $\mathcal{RCC}5$, employ variables ranging over regular closed sets in topological spaces, and a collection of eight (respectively, five) binary predicates standing for some basic topological relations between these sets [Egenhofer and Franzosa, 1991; Randell *et al.*, 1992; Bennett, 1994; Renz and Nebel, 2001]. An important extension of $\mathcal{RCC}8$, known as $\mathcal{BRCC}8$, additionally features standard Boolean operations on regular closed sets [Wolter and Zakharyaschev, 2000].

A remarkable characteristic of these languages is their *in*sensitivity to the underlying interpretation. To show that an $\mathcal{RCC}8$-formula is satisfiable in n-dimensional Euclidean space, it suffices to demonstrate its satisfiability in *any* topological space [Renz, 1998]; for $\mathcal{BRCC}8$-formulas, satisfiability in *any connected* space is enough. This inexpressiveness yields (relatively) low computational complexity: satisfiability of $\mathcal{BRCC}8$-, $\mathcal{RCC}8$- and $\mathcal{RCC}5$-formulas over arbitrary topological spaces is NP-complete; satisfiability of $\mathcal{BRCC}8$-formulas over connected spaces is PSPACE-complete.

However, satisfiability of spatial constraints by *arbitrary* regular closed sets by no means guarantees realizability by practically meaningful geometrical objects, where *connectedness* of regions is typically a minimal requirement [Borgo *et al.*, 1996; Cohn and Renz, 2008]. (A connected region is one which consists of a 'single piece.') It is easy to write constraints in $\mathcal{RCC}8$ that are satisfiable by connected regular closed sets over arbitrary topological spaces but not over $\mathbb{R}^2$; in $\mathcal{BRCC}8$ we can even write formulas satisfiable by connected regular closed sets over arbitrary spaces but not over $\mathbb{R}^n$ for any n. Worse still: there exist very simple collections of spatial constraints (involving connectedness) that are satisfiable in the Euclidean plane, but only by 'pathological' sets that cannot plausibly represent the regions occupied by physical objects [Pratt-Hartmann, 2007]. Unfortunately, little is known about the complexity of topological constraint satisfaction by non-pathological objects in low-dimensional Euclidean spaces. One landmark result [Schaefer *et al.*, r003] in this area shows that satisfiability of $\mathcal{RCC}8$-formulas by *disc-homeomorphs* in $\mathbb{R}^2$ is still NP-complete, though the decision procedure is vastly more intricate than in the general case. In this paper, we investigate the computational properties of more general and flexible spatial logics with connectedness constraints interpreted over $\mathbb{R}^2$ and $\mathbb{R}^3$.

We consider two 'base' topological constraint languages. The language $\mathcal{B}$ features $=$ as its only predicate, but has function symbols $+$, $-$, $\cdot$ denoting the standard operations of fusion, complement and taking common parts defined for regular closed sets, as well as the constants 1 and 0 for the entire space and the empty set. Our second base language, $\mathcal{C}$, additionally features a binary predicate, C, denoting the 'contact' relation (two sets are in *contact* if they share at least one point). The language $\mathcal{C}$ is a notational variant of $\mathcal{BRCC}8$ (and thus an extension of $\mathcal{RCC}8$), while $\mathcal{B}$ is the analogous extension of $\mathcal{RCC}5$. We add to $\mathcal{B}$ and $\mathcal{C}$ one of two new unary predicates: c, representing the property of connectedness, and c°, representing the (stronger) property of having a connected *interior*. We denote the resulting languages by $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$. We are interested in interpretations over (*i*) the regular closed sets of $\mathbb{R}^2$ and $\mathbb{R}^3$, and (*ii*) the regular closed *polyhedral* sets of $\mathbb{R}^2$ and $\mathbb{R}^3$. (A set is polyhedral if it can be defined by finitely many bounding hyperplanes.) By restricting interpretations to polyhedra we rule out satisfaction by pathological sets and use the same 'data structure' as in GISs.

When interpreted over *arbitrary* topological spaces, the complexity of reasoning with these languages is known: satisfiability of $\mathcal{B}c^\circ$-formulas is NP-complete, while for the other three languages, it is EXPTIME-complete. Likewise, the 1D Euclidean case is completely solved. For the spaces $\mathbb{R}^n$ ($n \geq 2$), however, most problems are still open. All four languages contain formulas satisfiable by regular closed sets in $\mathbb{R}^2$, but not by regular closed polygons; in $\mathbb{R}^3$, the analogous result is known only for $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$. The satisfiability problem for $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ is EXPTIME-hard (in both polyhedral and unrestricted cases) for $\mathbb{R}^n$ ($n \geq 2$); however, the only known upper bound is that satisfiability of $\mathcal{B}c^\circ$-formulas by polyhedra in $\mathbb{R}^n$ ($n \geq 3$) is EXPTIME-complete. (See [Kontchakov *et al.*, 2010b] for a summary.)

This paper settles most of these open problems, revealing considerable differences between the computational properties of constraint languages with connectedness predicates when interpreted over $\mathbb{R}^2$ and over abstract topological spaces. Sec. 3 shows that $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ are all sensitive to restriction to polyhedra in $\mathbb{R}^n$ ($n \geq 2$). Sec. 4 establishes an unexpected result: all these languages are *undecidable* in 2D, both in the polyhedral and unrestricted cases ([Dornheim, 1998] proves undecidability of the *first-order* versions of these languages). Sec. 5 resolves the open issue of the complexity of $\mathcal{B}c^\circ$ over regular closed sets (not just polyhedra) in $\mathbb{R}^3$ by establishing an NP upper bound. Thus, Qualitative Spatial Reasoning in Euclidean spaces proves much more challenging if connectedness of regions is to be taken into account. We discuss the obtained results in the context of spatial reasoning in Sec. 6. Omitted proofs can be found in the appendix.

2 Constraint Languages with Connectedness

Let T be a topological space. We denote the closure of any $X \subseteq T$ by X^-, its interior by X° and its boundary by $\delta X = X^- \setminus X^\circ$. We call X *regular closed* if $X = X^{\circ -}$, and denote by $\mathsf{RC}(T)$ the set of regular closed subsets of T. Where T is clear from context, we refer to elements of $\mathsf{RC}(T)$ as *regions*. $\mathsf{RC}(T)$ forms a Boolean algebra under the operations $X + Y = X \cup Y$, $X \cdot Y = (X \cap Y)^{\circ -}$ and $-X = (T \setminus X)^-$. We write $X \leq Y$ for $X \cdot (-Y) = \emptyset$; thus $X \leq Y$ iff $X \subseteq Y$. A subset $X \subseteq T$ is *connected* if it cannot be decomposed into two disjoint, non-empty sets closed in the subspace topology; X is *interior-connected* if X° is connected.

Any $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, $n \geq 1$, bounds two elements of $\mathsf{RC}(\mathbb{R}^n)$ called *half-spaces*. We denote by $\mathsf{RCP}(\mathbb{R}^n)$ the Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ generated by the half-spaces, and call the elements of $\mathsf{RCP}(\mathbb{R}^n)$ (regular closed) *polyhedra*. If $n = 2$, we speak of (regular closed) *polygons*. Polyhedra may be regarded as 'well-behaved' or, in topologists' parlance, '*tame*.' In particular, every polyhedron has finitely many connected components, a property which is not true of regular closed sets in general.

The topological constraint languages considered here all employ a countably infinite collection of variables $r_1, r_2, \ldots$ The language $\mathcal{C}$ features binary predicates $=$ and C, together with the individual constants 0, 1 and the function symbols $+, \cdot, -$. The *terms* τ and *formulas* φ of $\mathcal{C}$ are given by:

$$\begin{array}{lcl}\tau & ::= & r \ \mid\ \tau_1 + \tau_2 \ \mid\ \tau_1 \cdot \tau_2 \ \mid\ -\tau_1 \ \mid\ 1 \ \mid\ 0,\\ \varphi & ::= & \tau_1 = \tau_2 \ \mid\ C(\tau_1, \tau_2) \ \mid\ \varphi_1 \wedge \varphi_2 \ \mid\ \neg\varphi_1.\end{array}$$

The language $\mathcal{B}$ is defined analogously, but without the predicate C. If $S \subseteq \mathsf{RC}(T)$ for some topological space T, an *interpretation over* S is a function $\cdot^{\mathfrak{I}}$ mapping variables r to elements $r^{\mathfrak{I}} \in S$. We extend $\cdot^{\mathfrak{I}}$ to terms τ by setting $0^{\mathfrak{I}} = \emptyset$, $1^{\mathfrak{I}} = T$, $(\tau_1 + \tau_2)^{\mathfrak{I}} = \tau_1^{\mathfrak{I}} + \tau_2^{\mathfrak{I}}$, etc. We write $\mathfrak{I} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathfrak{I}} = \tau_2^{\mathfrak{I}}$, and $\mathfrak{I} \models C(\tau_1, \tau_2)$ iff $\tau_1^{\mathfrak{I}} \cap \tau_2^{\mathfrak{I}} \neq \emptyset$. We read $C(\tau_1, \tau_2)$ as 'τ_1 *contacts* τ_2.' The relation $\models$ is extended to non-atomic formulas in the obvious way. A formula φ is *satisfiable over* S if $\mathfrak{I} \models \varphi$ for some interpretation $\mathfrak{I}$ over S.

Turning to languages with connectedness predicates, we define $\mathcal{B}c$ and $\mathcal{C}c$ to be extensions of $\mathcal{B}$ and $\mathcal{C}$ with the unary predicate c. We set $\mathfrak{I} \models c(\tau)$ iff $\tau^{\mathfrak{I}}$ is connected in the topological space under consideration. Similarly, we define $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$ to be extensions of $\mathcal{B}$ and $\mathcal{C}$ with the predicate c°, setting $\mathfrak{I} \models c^\circ(\tau)$ iff $(\tau^{\mathfrak{I}})^\circ$ is connected. *Sat*$(\mathcal{L}, S)$ is the set of $\mathcal{L}$-formulas satisfiable over S, where $\mathcal{L}$ is one of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$ (the topological space is implicit in this notation, but will always be clear from context). We shall be concerned with *Sat*$(\mathcal{L}, S)$, where S is $\mathsf{RC}(\mathbb{R}^n)$ or $\mathsf{RCP}(\mathbb{R}^n)$ for $n = 2, 3$.

To illustrate, consider the $\mathcal{B}c^\circ$-formulas φ_k given by

$$\bigwedge_{1 \leq i \leq k} \bigl(c^\circ(r_i) \wedge (r_i \neq 0)\bigr) \wedge \bigwedge_{i<j} \bigl(c^\circ(r_i + r_j) \wedge (r_i \cdot r_j = 0)\bigr). \quad (1)$$

One can show that φ_3 is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 2$, but not over $\mathsf{RCP}(\mathbb{R})$, as no three intervals with non-empty, disjoint interiors can be in pairwise contact. Also, φ_5 is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, for $n \geq 3$, but not over $\mathsf{RCP}(\mathbb{R}^2)$, as the graph K_5 is non-planar. Thus, $\mathcal{B}c^\circ$ is sensitive to the dimension of the space. Or again, consider the $\mathcal{B}c^\circ$-formula

$$\bigwedge_{1 \leq i \leq 3} c^\circ(r_i) \ \wedge\ c^\circ(r_1 + r_2 + r_3) \ \wedge \bigwedge_{2 \leq i \leq 3} \neg c^\circ(r_1 + r_i). \quad (2)$$

One can show that (2) is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for any $n \geq 2$ (see, e.g., Fig. 1), but not over $\mathsf{RCP}(\mathbb{R}^n)$. Thus $\mathcal{B}c^\circ$ is sensitive to tameness in Euclidean spaces. It is known [Kontchakov *et al.*, 2010b] that, for the Euclidean *plane*, the same is true of $\mathcal{B}c$ and $\mathcal{C}c$: there is a $\mathcal{B}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but not over $\mathsf{RCP}(\mathbb{R}^2)$. (The example required to show this is far more complicated than the $\mathcal{B}c^\circ$-formula (2).) In the next section, we prove that any of

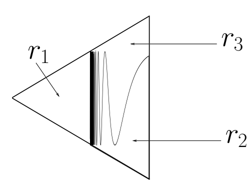

Figure 1: Three regions in $\mathsf{RC}(\mathbb{R}^2)$ satisfying (2).

$\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ contains formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, for every $n \geq 2$, but only by regions with infinitely many components. Thus, all four of our languages are sensitive to tameness in all dimensions greater than one.

3 Regions with Infinitely Many Components

Fix $n \geq 2$ and let d_0, d_1, d_2, d_3 be regions partitioning $\mathbb{R}^n$:

$$\big(\textstyle\sum_{0\leq i\leq 3} d_i = 1\big) \quad \wedge \quad \bigwedge_{0\leq i<j\leq 3}(d_i \cdot d_j = 0). \qquad (3)$$

We construct formulas forcing the d_i to have infinitely many connected components. To this end we require non-empty regions a_i contained in d_i, and a non-empty region t:

$$\bigwedge_{0\leq i\leq 3}\big((a_i \neq 0) \wedge (a_i \leq d_i)\big) \quad \wedge \quad (t \neq 0). \qquad (4)$$

The configuration of regions we have in mind is depicted in Fig. 2, where components of the d_i are arranged like the layers of an onion. The 'innermost' component of d_0 is surrounded by a component of d_1, which in turn is surrounded by a component of d_2, and so on. The region t passes through every layer, but avoids the a_i. To enforce a configuration of this sort, we need the following three formulas, for $0 \leq i \leq 3$:

$$c(a_i + d_{\lfloor i+1\rfloor} + t), \qquad (5)$$

$$\neg C(a_i, d_{\lfloor i+1\rfloor} \cdot (-a_{\lfloor i+1\rfloor})) \ \wedge \ \neg C(a_i, t), \qquad (6)$$

$$\neg C(d_i, d_{\lfloor i+2\rfloor}), \qquad (7)$$

where $\lfloor k\rfloor = k \bmod 4$. Formulas (5) and (6) ensure that each component of a_i is in contact with $a_{\lfloor i+1\rfloor}$, while (7) ensures that no component of d_i can touch any component of $d_{\lfloor i+2\rfloor}$.

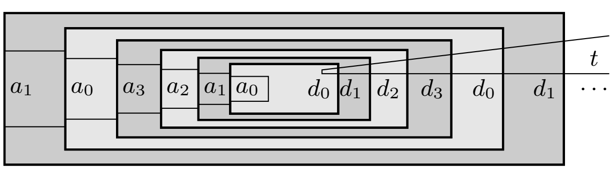

Figure 2: Regions satisfying φ_∞.

Denote by φ_∞ the conjunction of the above constraints. Fig. 2 shows how φ_∞ can be satisfied over $\mathsf{RC}(\mathbb{R}^2)$. By cylindrification, it is also satisfiable over any $\mathsf{RC}(\mathbb{R}^n)$, for $n > 2$.

The arguments of this section are based on the following property of regular closed subsets of Euclidean spaces:

Lemma 1 *If $X \in \mathsf{RC}(\mathbb{R}^n)$ is connected, then every component of $-X$ has a connected boundary.*

The proof of this lemma, which follows from a result in [Newman, 1964], can be found in Appendix A. The result fails for other familiar spaces such as the torus.

Theorem 2 *There is a $\mathcal{C}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

Proof. Let φ_∞ be as above. To simplify the presentation, we ignore the difference between variables and the regions they stand for, writing, for example, a_i instead of $a_i^{\mathfrak{I}}$. We construct a sequence of disjoint components X_i of $d_{\lfloor i\rfloor}$ and open sets V_i connecting X_i to X_{i+1} (Fig. 3). By the first conjunct of (4), let X_0 be a component of d_0 containing points in a_0. Suppose X_i has been constructed. By (5) and (6), X_i is in contact with $a_{\lfloor i+1\rfloor}$. Using (7) and the fact that $\mathbb{R}^n$ is locally connected, one can find a component X_{i+1} of $d_{\lfloor i+1\rfloor}$ which has points in a_{i+1}, and a connected open set V_i such that $V_i \cap X_i$ and $V_i \cap X_{i+1}$ are non-empty, but $V_i \cap d_{\lfloor i+2\rfloor}$ is empty.

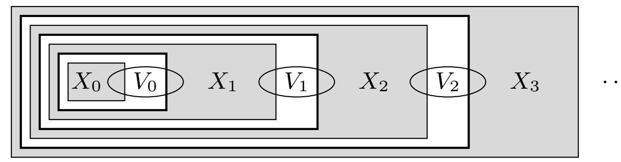

Figure 3: The sequence $\{X_i, V_i\}_{i\geq 0}$ generated by φ_∞. (S_{i+1} and R_{i+1} are the 'holes' of X_{i+1} containing X_i and X_{i+2}.)

To see that the X_i are distinct, let S_{i+1} and R_{i+1} be the components of $-X_{i+1}$ containing X_i and X_{i+2}, respectively. It suffices to show $S_{i+1} \subseteq S_{i+2}^\circ$. Note that the connected set V_i must intersect δS_{i+1}. Evidently, $\delta S_{i+1} \subseteq X_{i+1} \subseteq d_{\lfloor i+1\rfloor}$. Also, $\delta S_{i+1} \subseteq -X_{i+1}$; hence, by (3) and (7), $\delta S_{i+1} \subseteq d_i \cup d_{\lfloor i+2\rfloor}$. By Lemma 1, δS_{i+1} is connected, and therefore, by (7), is entirely contained either in $d_{\lfloor i\rfloor}$ or in $d_{\lfloor i+2\rfloor}$. Since $V_i \cap \delta S_{i+1} \neq \emptyset$ and $V_i \cap d_{\lfloor i+2\rfloor} = \emptyset$, we have $\delta S_{i+1} \not\subseteq d_{\lfloor i+2\rfloor}$, so $\delta S_{i+1} \subseteq d_i$. Similarly, $\delta R_{i+1} \subseteq d_{i+2}$. By (7), then, $\delta S_{i+1} \cap \delta R_{i+1} = \emptyset$, and since S_{i+1} and R_{i+1} are components of the same set, they are disjoint. Hence, $S_{i+1} \subseteq (-R_{i+1})^\circ$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^\circ$. So, S_{i+1} lies in the interior of a component of $-X_{i+2}$, and since $\delta S_{i+1} \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be S_{i+2}. ❑

Now we show how the $\mathcal{C}c$-formula φ_∞ can be transformed to $\mathcal{C}c^\circ$- and $\mathcal{B}c$-formulas with similar properties. Note first that all occurrences of c in φ_∞ have positive polarity. Let φ_∞° be the result of replacing them with the predicate c°. In Fig. 2, the connected regions mentioned in (5) are in fact interior-connected; hence φ_∞° is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$. Since interior-connectedness implies connectedness, φ_∞° entails φ_∞, and we obtain:

Corollary 3 *There is a $\mathcal{C}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

To construct a $\mathcal{B}c$-formula, we observe that all occurrences of C in φ_∞ are negative. We eliminate these using the predicate c. Consider, for example, the formula $\neg C(a_i, t)$ in (6). By inspection of Fig. 2, one can find regions r_1, r_2 satisfying

$$c(r_1) \wedge c(r_2) \wedge (a_i \leq r_1) \wedge (t \leq r_2) \wedge \neg c(r_1 + r_2). \qquad (8)$$

On the other hand, (8) entails $\neg C(a_i, t)$. By treating all other non-contact relations similarly, we obtain a $\mathcal{B}c$-formula ψ_∞ that is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, and that entails φ_∞. Thus:

Corollary 4 *There is a $\mathcal{B}c$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

Obtaining a $\mathcal{B}c^\circ$ analogue is complicated by the fact that we must enforce non-contact constraints using c° (rather than c). In the Euclidean plane, this can be done using *planarity constraints*; see Appendix A.

Theorem 5 *There is a $\mathcal{B}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, but not by regions with finitely many components.*

Theorem 2 and Corollary 4 entail that, if $\mathcal{L}$ is $\mathcal{B}c$ or $\mathcal{C}c$, then $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) \neq Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ for $n \geq 2$. Theorem 5 fails for $\mathsf{RC}(\mathbb{R}^n)$ with $n \geq 3$ (Sec. 5). However, we know from (2) that $Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \neq Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$ for all $n \geq 2$. Theorem 2 fails in the 1D case; moreover, $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R})) = Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}))$ only in the case $\mathcal{L} = \mathcal{B}c$ or $\mathcal{B}c^\circ$ [Kontchakov *et al.*, 2010b].

4 Undecidability in the Plane

Let $\mathcal{L}$ be any of $\mathcal{B}c, \mathcal{C}c, \mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$. In this section, we show, via a reduction of the *Post correspondence problem* (PCP), that $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete. An *instance* of the PCP is a quadruple $\mathbf{w} = (S, T, \mathsf{w}_1, \mathsf{w}_2)$ where S and T are finite alphabets, and each w_i is a word morphism from T^* to S^*. We may assume that $S = \{0, 1\}$ and $\mathsf{w}_i(t)$ is non-empty for any $t \in T$. The instance $\mathbf{w}$ is *positive* if there exists a non-empty $\tau \in T^*$ such that $\mathsf{w}_1(\tau) = \mathsf{w}_2(\tau)$. The set of positive PCP-instances is known to be r.e.-complete. The reduction can only be given in outline here: full details are given in Appendix B.

To deal with arbitrary regular closed subsets of $\mathsf{RC}(\mathbb{R}^2)$, we use the technique of 'wrapping' a region inside two bigger ones. Let us say that a *3-region* is a triple $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ of elements of $\mathsf{RC}(\mathbb{R}^2)$ such that $0 \neq \ddot{a} \ll \dot{a} \ll a$, where $r \ll s$ abbreviates $\neg C(r, -s)$. It helps to think of $\mathfrak{a} = (a, \dot{a}, \ddot{a})$ as consisting of a kernel, $\ddot{a}$, encased in two protective layers of shell. As a simple example, consider the sequence of 3-regions $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ depicted in Fig. 4, where the innermost regions form a sequence of externally touching polygons. When describing arrangements of 3-regions, we use

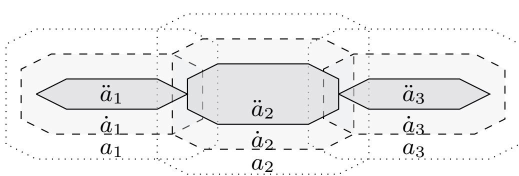

Figure 4: A chain of 3-regions satisfying $\mathsf{stack}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$.

the variable $\mathfrak{r}$ for the triple of variables $(r, \dot{r}, \ddot{r})$, taking the conjuncts $\ddot{r} \neq 0$, $\ddot{r} \ll \dot{r}$ and $\dot{r} \ll r$ to be implicit. As with ordinary variables, we often ignore the difference between 3-region variables and the 3-regions they stand for.

For $k \geq 3$, define the formula $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ by

$$\bigwedge_{1 \leq i \leq k} c(\dot{a}_i + \ddot{a}_{i+1} + \cdots + \ddot{a}_k) \quad \wedge \bigwedge_{j-i>1} \neg C(a_i, a_j).$$

Thus, the triple of 3-regions in Fig. 4 satisfies $\mathsf{stack}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$. This formula plays a crucial role in our proof. If $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ holds, then any point p_0 in the inner shell $\dot{a}_1$ of $\mathfrak{a}_1$ can be connected to any point p_k in the kernel $\ddot{a}_k$ of $\mathfrak{a}_k$ via a Jordan arc $\gamma_1 \cdots \gamma_k$ whose ith segment, γ_i, never leaves the outer shell a_i of $\mathfrak{a}_i$. Moreover, each γ_i intersects the inner shell $\dot{a}_{i+1}$ of $\mathfrak{a}_{i+1}$, for $1 \leq i < k$.

This technique allows us to write $\mathcal{C}c$-formulas whose satisfying regions are guaranteed to contain various networks of arcs, exhibiting almost any desired pattern of intersections. Now recall the construction of Sec. 3, where constraints on the variables $d_0, \ldots, d_3$ were used to enforce 'cyclic' patterns of components. Using $\mathsf{stack}(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$, we can write a formula with the property that the regions in any satisfying assignment are forced to contain the pattern of arcs having the form shown in Fig. 5. These arcs define a 'window,' contain-

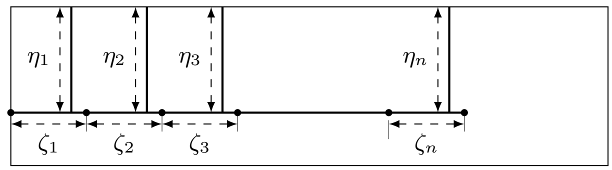

Figure 5: Encoding the PCP: Stage 1.

ing a sequence $\{\zeta_i\}$ of 'horizontal' arcs ($1 \leq i \leq n$), each connected by a corresponding 'vertical arc,' η_i, to some point on the 'top edge.' We can ensure that each ζ_i is included in a region $a_{\lfloor i \rfloor}$, and each η_i ($1 \leq i \leq n$) in a region $b_{\lfloor i \rfloor}$, where $\lfloor i \rfloor$ now indicates i mod 3. By repeating the construction, a second pair of arc-sequences, $\{\zeta'_i\}$ and $\{\eta'_i\}$ ($1 \leq i \leq n'$) can be established, but with each η'_i connecting ζ'_i to the 'bottom edge.' Again, we can ensure each ζ'_i is included in a region $a'_{\lfloor i \rfloor}$ and each η'_i in a region $b'_{\lfloor i \rfloor}$ ($1 \leq i \leq n'$). Further, we can ensure that the final horizontal arcs ζ_n and $\zeta'_{n'}$ (but no others) are joined by an arc ζ^* lying in a region z^*. The cru-

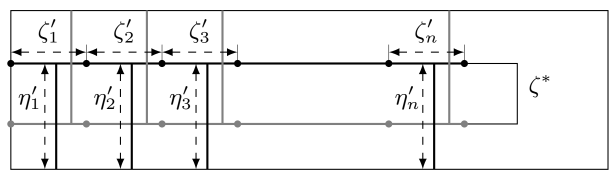

Figure 6: Encoding the PCP: Stage 2.

cial step is to match up these arc-sequences. To do so, we write $\neg C(a'_i, b_j) \wedge \neg C(a_i, b'_j) \wedge \neg C(b_i + b'_i, b_j + b'_j + z^*)$, for all i, j ($0 \leq i, j < 3$, $i \neq j$). A simple argument based on planarity considerations then ensures that the upper and lower sequences of arcs must cross (essentially) as shown in Fig. 6. In particular, we are guaranteed that $n = n'$ (without specifying the value n), and that, for all $1 \leq i \leq n$, ζ_i is connected by η_i (and also by η'_i) to ζ'_i.

Having established the configuration of Fig. 6, we write $(b_i \leq l_0 + l_1) \wedge \neg C(b_i \cdot l_0, b_i \cdot l_1)$, for $0 \leq i < 3$, ensuring that each η_i is included in exactly one of l_0, l_1. These inclusions naturally define a word σ over the alphabet $\{0, 1\}$. Next, we write $\mathcal{C}c$-constraints which organize the sequences of arcs $\{\zeta_i\}$ and $\{\zeta'_i\}$ (independently) into consecutive blocks. These blocks of arcs can then be put in 1–1 correspondence using essentially the same construction used to put the individual arcs in 1–1 correspondence. Each pair of corresponding blocks can now be made to lie in exactly one region from a collection $t_1, \ldots, t_\ell$. We think of the t_j as representing the letters of the alphabet T, so that the labelling of the blocks with these elements defines a word $\tau \in T^*$. It is then straightforward to write non-contact constraints involving the arcs ζ_i ensuring that $\sigma = \mathsf{w}_1(\tau)$ and non-contact constraints involving the arcs ζ_i' ensuring that $\sigma = \mathsf{w}_2(\tau)$. Let $\varphi_\mathbf{w}$ be the conjunction of all the foregoing $\mathcal{C}c$-formulas. Thus, if $\varphi_\mathbf{w}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is a positive instance of the PCP. On the other hand, if $\mathbf{w}$ is a positive instance of the PCP, then one can construct a tuple satisfying $\varphi_\mathbf{w}$ over $\mathsf{RCP}(\mathbb{R}^2)$ by 'thickening' the above collections of arcs into polygons in the obvious way. So, $\mathbf{w}$ is positive iff $\varphi_\mathbf{w}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$ iff $\varphi_\mathbf{w}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. This shows r.e.-hardness of *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$ and *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2))$. Membership of the latter problem in r.e. is immediate because all polygons may be assumed to have vertices with rational coordinates, and so may be effectively enumerated. Using the techniques of Corollaries 3–4 and Theorem 5, we obtain:

Theorem 6 *For $\mathcal{L} \in \{\mathcal{B}c^\circ, \mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$, Sat$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and Sat$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete.*

The complexity of *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^3))$ remains open for the languages $\mathcal{L} \in \{\mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$. However, as we shall see in the next section, for $\mathcal{B}c^\circ$ it drops dramatically.

5 $\mathcal{B}c^\circ$ in 3D

In this section, we consider the complexity of satisfying $\mathcal{B}c^\circ$-constraints by polyhedra and regular closed sets in three-dimensional Euclidean space. Our analysis rests on an important connection between geometrical and graph-theoretic interpretations. We begin by briefly discussing the results of [Kontchakov *et al.*, 2010a] for the *polyhedral* case.

Recall that every partial order (W, R), where R is a transitive and reflexive relation on W, can be regarded as a topological space by taking $X \subseteq W$ to be open just in case $x \in X$ and xRy imply $y \in X$. Such topologies are called *Aleksandrov spaces*. If (W, R) contains no proper paths of length greater than 2, we call (W, R) a *quasi-saw* (Fig. 8). If, in addition, no $x \in W$ has more than two proper R-successors, we call (W, R) a 2-*quasi-saw*. The properties of 2-quasi-saws we need are as follows [Kontchakov *et al.*, 2010a]:

– satisfiability of $\mathcal{B}c$-formulas in arbitrary topological spaces coincides with satisfiability in 2-quasi-saws, and is ExpTime-complete;
– $X \subseteq W$ is connected in a 2-quasi-saw (W, R) iff it is interior-connected in (W, R).

The following construction lets us apply these results to the problem *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3))$. Say that a *connected partition* in $\mathsf{RCP}(\mathbb{R}^3)$ is a tuple $X_1, \dots, X_k$ of non-empty polyhedra having connected and pairwise disjoint interiors, which sum to the entire space $\mathbb{R}^3$. The *neighbourhood graph* (V, E) of this partition has vertices $V = \{X_1, \dots, X_k\}$ and edges $E = \{\{X_i, X_j\} \mid i \neq j \text{ and } (X_i + X_j)^\circ \text{ is connected}\}$ (Fig. 7). One can show that *every* connected graph is the neighbourhood graph of some connected partition in $\mathsf{RCP}(\mathbb{R}^3)$. Furthermore, every neighbourhood graph (V, E) gives rise to a 2-quasi-saw, namely, $(W_0 \cup W_1, R)$, where $W_0 = V$, $W_1 = \{z_{x,y} \mid \{x, y\} \in E\}$, and R is the reflexive closure of $\{(z_{x,y}, x), (z_{x,y}, y) \mid \{x, y\} \in E\}$. From this, we see that (*i*) a $\mathcal{B}c^\circ$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^3)$ iff (*ii*) φ is satisfiable over a connected 2-quasi-saw iff (*iii*) the $\mathcal{B}c$-formula $\varphi^\bullet$, obtained from φ by replacing every occurrence

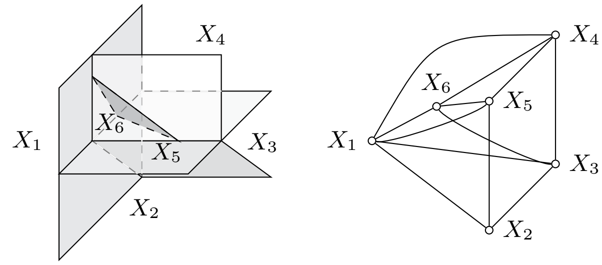

Figure 7: A connected partition and its neighbourhood graph.

of c° with c, is satisfiable over a connected 2-quasi-saw. Thus, *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3))$ is ExpTime-complete.

The picture changes if we allow variables to range over $\mathsf{RC}(\mathbb{R}^3)$ rather than $\mathsf{RCP}(\mathbb{R}^3)$. Note first that the $\mathcal{B}c^\circ$-formula (2) is not satisfiable over 2-quasi-saws, but has a quasi-saw model as in Fig. 8. Some extra geometrical work will show

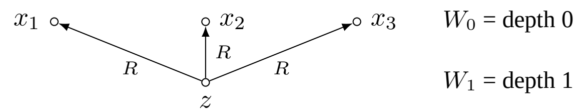

Figure 8: A quasi-saw model $\mathfrak{I}$ of (2): $r_i^\mathfrak{I} = \{x_i, z\}$.

now that (*iv*) a $\mathcal{B}c^\circ$-formula is satisfiable over $\mathsf{RC}(\mathbb{R}^3)$ iff (*v*) it is satisfiable over a connected quasi-saw. And as shown in [Kontchakov *et al.*, 2010a], satisfiability of $\mathcal{B}c^\circ$-formulas in connected spaces coincides with satisfiability over connected quasi-saws, and is NP-complete.

Theorem 7 *The problem Sat$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^3))$ is* NP*-complete.*

Proof. From the preceding discussion, it suffices to show that (*v*) implies (*iv*) for any $\mathcal{B}c^\circ$-formula φ. So suppose $\mathfrak{A} \models \varphi$, with $\mathfrak{A}$ based on a finite connected quasi-saw $(W_0 \cup W_1, R)$, where W_i contains all points of depth $i \in \{0, 1\}$ (Fig. 8). Without loss of generality we will assume that there is a special point z_0 of depth 1 such that $z_0 R x$ for all x of depth 0. We show how $\mathfrak{A}$ can be embedded into $\mathsf{RC}(\mathbb{R}^3)$.

Take pairwise disjoint *closed* balls B_x^1, for x of depth 0, and pairwise disjoint *open* balls D_z, for all z of depth 1 except z_0 (we assume the D_z are disjoint from the B_x^1). Let D_{z_0} be the closure of the complement of all B_x^1 and D_z.

We expand the B_x^1 to sets B_x in such a way that

(A) the B_x form a connected partition in $\mathsf{RC}(\mathbb{R}^3)$, that is, they are regular closed and sum up to $\mathbb{R}^3$, and their interiors are non-empty, connected and pairwise disjoint;

(B) every point in D_z is either in the interior of some B_x with zRx, or on the boundary of *all* of the B_x with zRx.

The required B_x are constructed as follows. Let $q_1, q_2, \dots$ be an enumeration of all the points in the interiors of D_z with *rational* coordinates. For $x \in W_0$, we set B_x to be the closure of the infinite union $\bigcup_{k=1}^\infty (B_x^k)^\circ$, where the regular closed sets B_x^k are defined inductively as follows (Fig. 9). Assuming that the B_x^k are defined, let q_i be the first point in the list $q_1, q_2, \dots$ that is not in any B_x^k yet. So, q_i is in the interior of some D_z. Take an open ball C_{q_i} in the interior of D_z centred in q_i and disjoint from the B_x^k. For each $x \in W_0$ with

zRx, expand B_x^k by a closed ball in C_{q_i} and a closed 'rod' connecting it to B_x^1 in such a way that the ball and the rod are disjoint from the rest of the B_x^k; the result is denoted by B_x^{k+1}. Consider a function f that maps regular closed sets

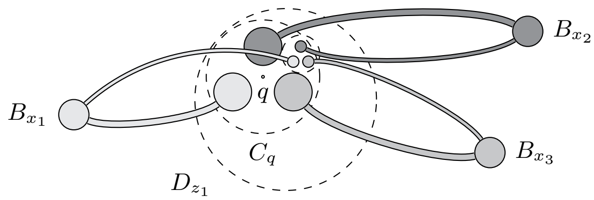

Figure 9: Filling D_{z_1} with B_{x_i}, for z_1Rx_i, $i = 1, 2, 3$.

$X \subseteq W$ to $\mathsf{RC}(\mathbb{R}^3)$ so that $f(X)$ is the union of all B_x, for x of depth 0 in X. By (A), f preserves $+$, $\cdot$, $-$, 0 and 1. Define an interpretation $\mathfrak{I}$ over $\mathsf{RC}(\mathbb{R}^3)$ by $r^{\mathfrak{I}} = f(r^{\mathfrak{A}})$. To show that $\mathfrak{I} \models \varphi$, it remains to prove that X° is connected iff $(f(X))^\circ$ is connected (details are in Appendix C). ❑

The remarkably diverse computational behaviour of $\mathcal{B}c^\circ$ over $\mathsf{RC}(\mathbb{R}^3)$, $\mathsf{RCP}(\mathbb{R}^3)$ and $\mathsf{RCP}(\mathbb{R}^2)$ can be explained as follows. To satisfy a $\mathcal{B}c^\circ$-formula φ in $\mathsf{RC}(\mathbb{R}^3)$, it suffices to find polynomially many points in the regions mentioned in φ (witnessing non-emptiness or non-internal-connectedness constraints), and then to 'inflate' those points to (possibly internally connected) regular closed sets using the technique of Fig. 9. By contrast, over $\mathsf{RCP}(\mathbb{R}^3)$, one can write a $\mathcal{B}c^\circ$-formula analogous to (8) stating that two internally connected polyhedra do not share a 2D face. Such 'face-contact' constraints can be used to generate constellations of exponentially many polyhedra simulating runs of alternating Turing machines on polynomial tapes, leading to EXPTIME-hardness. Finally, over $\mathsf{RCP}(\mathbb{R}^2)$, planarity considerations endow $\mathcal{B}c^\circ$ with the extra expressive power required to enforce full non-contact constructs (not possible in higher dimensions), and thus to encode the PCP as sketched in Sec. 4.

6 Conclusion

This paper investigated topological constraint languages featuring connectedness predicates and Boolean operations on regions. Unlike their less expressive cousins, $\mathcal{RCC}8$ and $\mathcal{RCC}5$, such languages are highly sensitive to the spaces over which they are interpreted, and exhibit more challenging computational behaviour. Specifically, we demonstrated that the languages $\mathcal{C}c$, $\mathcal{C}c^\circ$ and $\mathcal{B}c$ contain formulas satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but only by regions with infinitely many components. Using a related construction, we proved that the satisfiability problem for any of $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$, interpreted either over $\mathsf{RC}(\mathbb{R}^2)$ or over its polygonal subalgebra, $\mathsf{RCP}(\mathbb{R}^2)$, is *undecidable*. Finally, we showed that the satisfiability problem for $\mathcal{B}c^\circ$, interpreted over $\mathsf{RC}(\mathbb{R}^3)$, is NP-complete, which contrasts with EXPTIME-completeness for $\mathsf{RCP}(\mathbb{R}^3)$. The complexity of satisfiability for $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ over $\mathsf{RC}(\mathbb{R}^n)$ or $\mathsf{RCP}(\mathbb{R}^n)$ for $n \geq 3$ remains open. The obtained results rely on certain distinctive topological properties of Euclidean spaces. Thus, for example, the argument of Sec. 3 is based on the property of Lemma 1, while Sec. 4 similarly relies on *planarity* considerations. In both cases, however, the moral is the same: the topological spaces of most interest for Qualitative Spatial Reasoning exhibit special characteristics which any topological constraint language able to express connectedness must take into account.

The results of Sec. 4 pose a challenge for Qualitative Spatial Reasoning in the Euclidean plane. On the one hand, the relatively low complexity of $\mathcal{RCC}8$ over disc-homeomorphs suggests the possibility of usefully extending the expressive power of $\mathcal{RCC}8$ without compromising computational properties. On the other hand, our results impose severe limits on any such extension. We observe, however, that the constructions used in the proofs depend on a strong interaction between the connectedness predicates and the Boolean operations on regular closed sets. We believe that by restricting this interaction one can obtain non-trivial constraint languages with more acceptable complexity. For example, the extension of $\mathcal{RCC}8$ with connectedness constraints is still in NP for both $\mathsf{RC}(\mathbb{R}^2)$ and $\mathsf{RCP}(\mathbb{R}^2)$ [Kontchakov *et al.*, 2010b].

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A Regions with infinitely many components

First we give detailed proofs of Lemma 1 and Theorem 2.

Theorem 8 ([Newman, 1964]) *If X is a connected subset of $\mathbb{R}^n$, then every connected component of $\mathbb{R}^n \setminus X$ has a connected boundary.*

Lemma 1. *If $X \in \mathsf{RC}(\mathbb{R}^n)$ is connected, then every component of $-X$ has a connected boundary.*

Proof. Let Y be a connected component of $-X$. Suppose that the boundary β of Y is not connected, and let β_1 and β_2 be two sets separating β: β_1 and β_2 are disjoint, non-empty, closed subsets of β whose union is β. We will show that Y is not connected. We have $Y = (\bigcup_{i\in I} Z_i)^-$, for some index set I, where the Z_i are distinct connected components of $\mathbb{R}^n \setminus X$. By Theorem 8,'the boundaries α_i of Z_i are connected subsets of β, for each $i \in I$. Hence, either $\alpha_i \subseteq \beta_1$ or $\alpha_i \subseteq \beta_2$, for otherwise $\alpha_i \cap \beta_1$ and $\alpha_i \cap \beta_2$ would separate α_i. Let $I_j = \{i \in I \mid \alpha_i \subseteq \beta_j\}$ and $Y_j = (\bigcup_{i\in I_j} Z_i)^-$, for $j = 1, 2$. Clearly, Y_1 and Y_2 are closed, and $Y = Y_1 \cup Y_2$. Hence, it suffices to show that Y_1 and Y_2 are disjoint. We know that, for $j = 1, 2$,

$$Y_j = \big(\bigcup_{i\in I_j} \alpha_i\big)^- \cup \bigcup_{i\in I_j} Z_i.$$

Clearly, $\bigcup_{i\in I_1} Z_i$ and $\bigcup_{i\in I_2} Z_i$ are disjoint. We also know that $(\bigcup_{i\in I_1} \alpha_i)^-$ and $(\bigcup_{i\in I_2} \alpha_i)^-$ are disjoint, as subsets of β_1 and β_2, respectively. Finally, $(\bigcup_{i\in I_j} \alpha_i)^-$ and $\bigcup_{i\in I_k} Z_i$ are disjoint, for $j, k = 1, 2$, as subsets of the boundary and the interior of Y, respectively. So, Y is not connected, which is a contradiction. ❑

Theorem 2. *If $\mathfrak{I}$ is an interpretation over $\mathsf{RC}(\mathbb{R}^n)$ such that $\mathfrak{I} \models \varphi_\infty$, then every $d_i^{\mathfrak{I}}$ has infinitely many components.*

Proof. To simplify presentation, we ignore the difference between variables and the regions they stand for, writing, for example, a_i instead of $a_i^{\mathfrak{I}}$. We also set $b_i = d_i \cdot (-a_i)$. We construct a sequence of disjoint components X_i of $d_{\lfloor i\rfloor}$ and open sets V_i connecting X_i to X_{i+1} (Fig. 3). By the first conjunct of (4), let X_0 be a component of d_0 containing points in a_0. Suppose X_i has been constructed, for $i \geq 0$. By (5) and (6), there exists a point $q \in X_i \cap a_{\lfloor i+1\rfloor}$. Since $q \notin b_{\lfloor i+1\rfloor} \cup d_{\lfloor i+2\rfloor} \cup d_{\lfloor i+3\rfloor}$, and because $\mathbb{R}^n$ is locally connected, there exists a connected neighbourhood V_i of q such that $V_i \cap (b_{\lfloor i+1\rfloor} \cup d_{\lfloor i+2\rfloor} \cup d_{\lfloor i+3\rfloor}) = \emptyset$, and so, by (3), $V_i \subseteq d_{\lfloor i\rfloor} + a_{\lfloor i+1\rfloor}$. Further, since $q \in a_{\lfloor i+1\rfloor}$, $V_i \cap a_{\lfloor i+1\rfloor}{}^\circ \neq \emptyset$. Take X'_{i+1} to be a component of $a_{\lfloor i+1\rfloor}$ that intersects V_i and X_{i+1} the component of $d_{\lfloor i+1\rfloor}$ containing X'_{i+1}.

To see that the X_i are distinct, let S_{i+1} and R_{i+1} be the components of $-X_{i+1}$ containing X_i and X_{i+2}, respectively. It suffices to show $S_{i+1} \subseteq S^\circ_{i+2}$. Note that the connected set V_i must intersect δS_{i+1}. Evidently, $\delta S_{i+1} \subseteq X_{i+1} \subseteq d_{\lfloor i+1\rfloor}$. Also, $\delta S_{i+1} \subseteq -X_{i+1}$; hence, by (3) and (7), $\delta S_{i+1} \subseteq d_i \cup d_{\lfloor i+2\rfloor}$. By Lemma 1, δS_{i+1} is connected, and therefore, by (7), is entirely contained either in $d_{\lfloor i\rfloor}$ or in $d_{\lfloor i+2\rfloor}$. Since $V_i \cap \delta S_{i+1} \neq \emptyset$ and $V_i \cap d_{\lfloor i+2\rfloor} = \emptyset$, we have $\delta S_{i+1} \not\subseteq d_{\lfloor i+2\rfloor}$, so $\delta S_{i+1} \subseteq d_i$. Similarly, $\delta R_{i+1} \subseteq d_{i+2}$. By (7), then, $\delta S_{i+1} \cap \delta R_{i+1} = \emptyset$, and since S_{i+1} and R_{i+1} are components of the same set, they are disjoint. Hence, $S_{i+1} \subseteq (-R_{i+1})^\circ$, and since $X_{i+2} \subseteq R_{i+1}$, also $S_{i+1} \subseteq (-X_{i+2})^\circ$. So, S_{i+1} lies in the interior of a component of $-X_{i+2}$, and since $\delta S_{i+1} \subseteq X_{i+1} \subseteq S_{i+2}$, that component must be S_{i+2}. ❑

Now we extend the result to the language $\mathcal{C}c^\circ$. All occurrences of c in φ_∞ have positive polarity. Let φ°_∞ be the result of replacing them with the predicate c°. In the configuration of Fig. 2, all connected regions mentioned in φ_∞ are in fact interior-connected; hence φ°_∞ is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$. Since interior-connectedness implies connectedness, φ°_∞ entails φ_∞ in a common extension of $\mathcal{C}c^\circ$ and $\mathcal{C}c$. Hence:

Corollary 3. *There is a $\mathcal{C}c^\circ$-formula satisfiable over $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, but not by regions with finitely many components.*

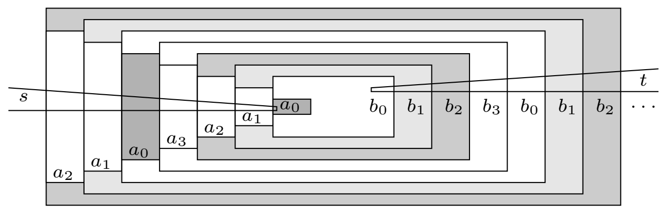


Figure 10: Satisfying $\varphi^c_{\neg C}(a_0, b_1, s, t)$ and $\varphi^c_{\neg C}(a_0, b_2, s, t)$.

To extend Theorem 2 to the language $\mathcal{B}c$, notice that all occurrences of C in φ_∞ are negative. We shall eliminate these using only the predicate c. We use the fact that, if the sum of two connected regions is not connected, then they must be disjoint. Consider the formula

$$\varphi^c_{\neg C}(r, s, r', s') := c(r + r') \wedge c(s + s') \\ \wedge \neg c((r + r') + (s + s')).$$

Note that $\varphi^c_{\neg C}(r, s, r', s')$ implies $\neg C(r, s)$. We replace $\neg C(a_i, t)$ with $\varphi^c_{\neg C}(a_i, t, a_0 + a_1 + a_2 + a_3, t)$, which is clearly satisfiable by the regions on Fig. 2. Further, we replace $\neg C(a_i, b_{\lfloor i+1\rfloor})$ with $\varphi^c_{\neg C}(a_i, b_{\lfloor i+1\rfloor}, s, t)$. As shown on Fig. 10, there exists a region s satisfying this formula. Instead of dealing with $\neg C(d_i, d_{i+2})$, we consider the equivalent:

$$\neg C(a_i, b_{\lfloor i+2\rfloor}) \wedge \neg C(b_i, a_{\lfloor i+2\rfloor}) \wedge \\ \neg C(a_i, a_{\lfloor i+2\rfloor}) \wedge \neg C(b_i, b_{\lfloor i+2\rfloor}).$$

We replace $\neg C(a_i, b_{\lfloor i+2\rfloor})$ by $\varphi^c_{\neg C}(a_i, b_{\lfloor i+2\rfloor}, s, t)$, which is satisfiable by the regions depicted on Fig. 10. We ignore $\neg C(b_i, a_{\lfloor i+2\rfloor})$, because it is logically equivalent to $\neg C(a_i, b_{\lfloor i+2\rfloor})$, for different values of i. We replace $\neg C(a_i, a_{\lfloor i+2\rfloor})$ by $\varphi^c_{\neg C}(a_i, a_{\lfloor i+2\rfloor}, a'_i, a'_{\lfloor i+2\rfloor})$, which is satisfiable by the regions depicted on Fig. 11. The fourth conjunct is then treated symmetrically. Transforming φ_∞ in the way just described, we obtain a $\mathcal{B}c$-formula φ^c_∞, which implies φ_∞ (in the language $\mathcal{C}c$) and which is satisfiable by the arrangement of $\mathsf{RC}(\mathbb{R}^n)$. Hence, we obtain the following:

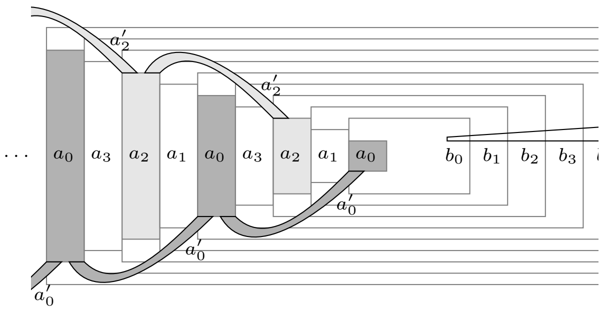


Figure 11: Satisfying $\varphi^c_{\neg C}(a_0, a_2, a_0', a_2')$.

Corollary 4. *There is a* $\mathcal{B}c$*-formula satisfiable over* $\mathsf{RC}(\mathbb{R}^n)$*,* $n \geq 2$*, but not by regions with finitely many components.*

The only remaining task in this section is to prove Theorem 5. The construction is similar to the one developed in Sec. 4, and as such uses similar techniques. We employ the following notation. If α is a Jordan arc, and p, q are points on α such that q occurs after p, we denote by $\alpha[p, q]$ the segment of α from p to q. Consider the formula $\mathsf{stack}^\circ(a_1, \dots, a_n)$ given by:

$$\bigwedge_{1 \leq i < n} (c^\circ(a_i + \cdots + a_n) \wedge a_i \cdot a_{i+1} = 0) \wedge \bigwedge_{j-i>1} \neg C(a_i, a_j)$$

This formula allows us to construct sequences of arcs in the following sense:

Lemma 9 *Suppose that the condition* $\mathsf{stack}^\circ(a_1, \dots, a_n)$ *obtains,* $n > 1$*. Then every point* $p_1 \in a_1^\circ$ *can be connected to every point* $p_n \in a_n^\circ$ *by a Jordan arc* $\alpha = \alpha_1 \cdots \alpha_{n-1}$ *such that for all* i $(1 \leq i < n)$*, each segment* $\alpha_i \subseteq (a_i + a_{i+1})^\circ$ *is a non-degenerate Jordan arc starting at some point* $p_i \in a_i^\circ$*.*

Proof. By $c^\circ(a_1 + \cdots + a_n)$, let $\alpha_1' \subseteq (a_1 + \cdots + a_n)^\circ$ be a Jordan arc connecting p_1 to p_n (Fig. 12). By the non-contact constraints, α_1' has to contain points in a_2°. Let p_2' be one such point. For $2 \leq i < n$ we suppose $\alpha_1, \dots, \alpha_{i-2}$, α_{i-1}' and p_i' to have been defined, and proceed as follows. By $c^\circ(a_i + \cdots + a_n)$, let $\alpha_i'' \subseteq (a_i + \cdots + a_n)^\circ$ be a Jordan arc connecting p_i' to p_n. By the non-contact constraints, α_i'' can intersect $\alpha_1 \cdots \alpha_{i-2}\alpha_{i-1}'$ only in its final segment α_{i-1}'. Let p_{i-1} be the first point of α_{i-1}' lying on α_i'; let α_{i-1} be the initial segment of α_{i-1}' ending at p_{i-1}; and let α_i' be the final segment of α_i'' starting at p_{i-1}. It remains only to define α_{n-1}, and to this end, we simply set $\alpha_{n-1} := \alpha_{n-1}'$. To see that p_i, $2 \leq i < n$, are as required, note that $p_i \in \alpha_i \cap \alpha_{i-1}$. By the disjoint constraints p_i must be in a_i. If p_i was in $\delta(a_i)$, it would also have to be in $\delta(a_{i-1})$ and $\delta(a_{i+1})$, which is forbidden by the disjoint constraints. Hence $p_i \in a_i^\circ$, $1 \leq i \leq n$. Given $a_i \cdot a_{i+1} = 0$, $1 \leq i < n$, this also guarantees that the arcs α_i are non-degenerate. ❑

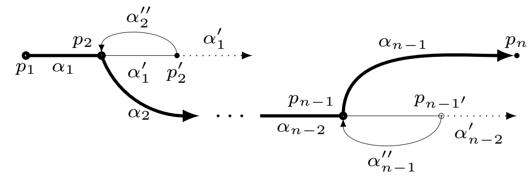


Figure 12: The constraint $\mathsf{stack}^\circ(a_1, \dots, a_n)$ ensures the existence of a Jordan arc $\alpha = \alpha_1 \cdots \alpha_{n-1}$ which connects a point $p_1 \in a_1^\circ$ to a point $p_n \in a_n^\circ$.

Consider now the formula $\mathsf{frame}^\circ(a_0, \dots, a_{n-1})$ given by:

$$\bigwedge_{0 \leq i < n} \left(c^\circ(a_i) \wedge c^\circ(a_i + a_{\lfloor i+1 \rfloor}) \wedge a_i \neq 0\right) \wedge \bigwedge_{j-i>1} a_i \cdot a_j = 0,$$

where $\lfloor k \rfloor$ denotes k mod n. This formula allows us to construct Jordan curves in the plane, in the following sense:

Lemma 10 *Let* $n \geq 3$*, and suppose* $\mathsf{frame}^\circ(a_0, \dots, a_{n-1})$*. Then there exist Jordan arcs* $\alpha_0, \dots, \alpha_{n-1}$ *such that* $\alpha_0 \dots \alpha_{n-1}$ *is a Jordan curve lying in the interior of* $a_0 + \cdots + a_{n-1}$*, and* $\alpha_i \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$*, for all* i*,* $0 \leq i < n$*.*

Proof. For all i $(0 \leq i < n)$, pick $p_i' \in a_i^\circ$, and pick a Jordan arc $\alpha_i' \subseteq (a_i + a_{\lfloor i+1 \rfloor})^\circ$ from p_i to $p_{\lfloor i+1 \rfloor}$. For all i $(2 \leq i \leq n)$, let $p_{\lfloor i \rfloor}$ be the first point of α_{i-1} lying on $\alpha_{\lfloor i \rfloor}$, and let p_1'' be the first point of α_0' lying on α_1'. For all i $(2 \leq i < n)$, let $\alpha_i = \alpha_i'[p_i, p_{i+1}]$, let $\alpha_1'' = \alpha_1'[p_1'', p_2]$, and let α_0'' denote the section of α_0' (in the appropriate direction) from p_0 to p_1''. Now let p_1 be the first point of α_0'' lying on α_1'', let $\alpha_0 = \alpha_0''[p_0, p_1]$, and let $\alpha_1 = \alpha_1''[p_1, p_2]$. It is routine to verify that the arcs $\alpha_0, \dots, \alpha_{n-1}$ have the required properties. ❑

We will now show how to separate certain types of regions in the language $\mathcal{B}c^\circ$. We make use of Lemma 10 and the following fact.

Lemma 11 *[Newman, 1964, p. 137] Let* F*,* G *be disjoint, closed subsets of* $\mathbb{R}^2$ *such that* $\mathbb{R}^2 \setminus F$ *and* $\mathbb{R}^2 \setminus G$ *are connected. Then* $\mathbb{R}^2 \setminus (F \cup G)$ *is connected.*

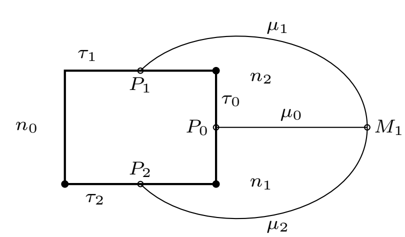


Figure 13: The Jordan curve $\Gamma = \tau_0\tau_1\tau_2$ separating m_1 from m_2.

We say that a region r is *quasi-bounded* if either r or $-r$ is bounded. We can now prove the following.

Lemma 12 *There exists a $\mathcal{B}c^\circ$-formula $\eta^*(r,s,\bar{v})$ with the following properties:* (*i*) *$\eta^*(r,s,\bar{v})$ entails $\neg C(r,s)$ over* $\mathsf{RC}(\mathbb{R}^2)$*;* (*ii*) *if the regions r and s can be separated by a Jordan curve, then there exist polygons $\bar{v}$ such that $\eta^*(\tau_1,\tau_2,\bar{v})$;* (*iii*) *if r, s are disjoint polygons such that r is quasi-bounded and $\mathbb{R}^2 \setminus (r+s)$ is connected, then there exist polygons $\bar{v}$ such that $\eta^*(\tau_1,\tau_2,\bar{v})$.*

Proof. Let $\bar{v}$ be the tuple of variables $(t_0,\ldots,t_5,m_1,m_2)$, and let $\eta^*(r,s,\bar{v})$ be the formula

$$\mathsf{frame}^\circ(t_0,\ldots,t_5) \wedge r \leq m_1 \wedge s \leq m_2 \wedge$$
$$(t_0+\ldots+t_5)\cdot(m_1+m_2) = 0 \wedge \bigwedge_{\substack{i=1,3,5\\ j=1,2}} c^\circ(t_i+m_j).$$

Property (*i*) follows by a simple planarity argument. By $\mathsf{frame}^\circ(t_0,\ldots,t_5)$ and Lemma 10, let α_i, for $0 \leq i \leq 5$, be such that $\Gamma = \alpha_0 \cdots \alpha_5$ is a Jordan curve included in $(t_0+\cdots+t_5)^\circ$. Further, let $\tau_i = \alpha_{2i}\alpha_{2i+1}$, $0 \leq i \leq 2$ (Fig.13). Note that all points in a_{2i+1}, $0 \leq i \leq 2$, that are on Γ are on τ_i. By $c^\circ(t_{2i+1}+m_1)$, $0 \leq i \leq 2$, let $\mu_i \subseteq (m_1+t_{2i+1})^\circ$ be a Jordan arc with endpoints $M_1 \in m_1^\circ$ and $T_i \in \tau_i \cap t_{2i+1}^\circ$. We may assume that these arcs intersect only at their common endpoint M_1, so that they divide the residual domain of Γ which contains M_1 into three subdomains n_i, for $0 \leq i \leq 2$. The existence of a point $M_2 \in m_2$ in any n_i, $0 \leq i \leq 2$, will contradict $c^\circ(t_{2i+1}+m_2)$. So, m_2 must be contained entirely in the residual domain of Γ not containing M_1. Similarly, all points in m_1 must lie in the residual domain of Γ containing M_1. It follows that m_1 and m_2 are disjoint, and by $r \leq m_1$ and $s \leq m_2$, that r and s are disjoint as well. For Property (*ii*), let Γ be a Jordan curve separating r and s. Now thicken Γ to form an annular element of $\mathsf{RCP}(\mathbb{R}^2)$, still disjoint from r and s, and divide this annulus into the three regions $t_0,\ldots,t_5$ as shown (up to similar situation) in Fig. 14. Choose m_1 and m_2 to be the connected components of $-(t_0+\cdots+t_5)$ containing r and s, respectively. For Property (*iii*), it is routine using Lemma 11 to show that there exists a piecewise linear Jordan curve Γ in $\mathbb{R}^2 \setminus (r+s)$ separating r and s. ❑

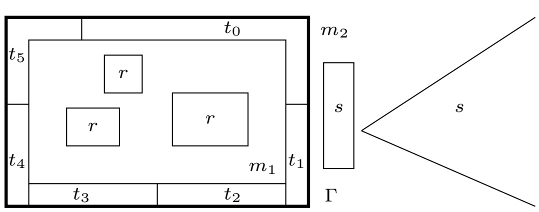


Figure 14: Separating disjoint polygons by an annulus.

Lemma 13 *There exists a $\mathcal{B}c^\circ$-formula $\eta(r,s,\bar{v})$ with the following properties:* (*i*) *$\eta(r,s,\bar{v})$ entails $\neg C(r,s)$ over* $\mathsf{RC}(\mathbb{R}^2)$*;* (*ii*) *if r, s are disjoint quasi-bounded polygons, then there exist polygons $\bar{v}$ such that $\eta(\tau_1,\tau_2,\bar{v})$.*

Proof. Let $\eta(r,s,\bar{v})$ be the formula

$$r = r_1 + r_2 \wedge s = s_1 + s_2 \wedge \bigwedge_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}} \eta^*(r_i,s_j,\bar{u}_{i,j}),$$

where η^* is the formula given in Lemma 12. Property (*i*) is then immediate. For Property (*ii*), it is routine to show that there exist polygons r_1, r_2 such that $r = r_1 + r_2$ and $\mathbb{R}^2 \setminus r_i$ is connected for $i = 1,2$; let s_1, s_2 be chosen analogously. Then for all i ($1 \leq i \leq 2$) and j ($1 \leq j \leq 2$) we have $r_i \cap s_j = \emptyset$ and, by Lemma 11, $\mathbb{R}^2 \setminus (r_i + s_j)$ connected. By Lemma 12, let $\bar{u}_{i,j}$ be such that $\eta^*(r_i,s_j,\bar{u}_{i,j})$. ❑

We are now ready to prove:

Theorem 5. *There is a $\mathcal{B}c^\circ$-formula satisfiable over* $\mathsf{RC}(\mathbb{R}^2)$*, but only by regions with infinitely many components.*

Proof. We first write a $\mathcal{C}c^\circ$-formula, φ^*_∞ with the required properties, and then show that all occurrences of C can be eliminated. Note that φ^*_∞ is not the same as the formula φ°_∞ constructed for the proof of Corollary 3.

Let s, s', a, a', b, b', $a_{i,j}$ and $b_{i,j}$ ($0 \leq i < 2$, $1 \leq j \leq 3$) be variables. The constraints

$$\mathsf{frame}^\circ(s,s',b,b',a,a') \tag{9}$$
$$\mathsf{stack}^\circ(s,b_{i,1},b_{i,2},b_{i,3},b) \tag{10}$$
$$\mathsf{stack}^\circ(b_{\lfloor i-1\rfloor,2},a_{i,1},a_{i,2},a_{i,3},a) \tag{11}$$
$$\mathsf{stack}^\circ(a_{\lfloor i-1\rfloor,2},b_{i,1},b_{i,2},b_{i,3},b) \tag{12}$$

are evidently satisfied by the arrangement of Fig. 15.

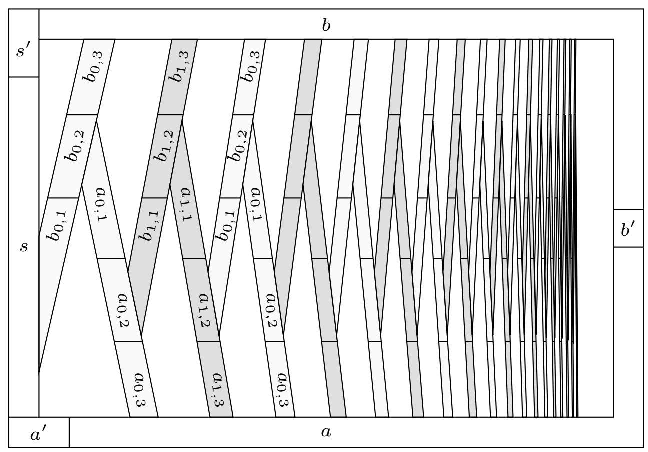


Figure 15: A tuple of regions satisfying (9)–(12): the pattern of components of the $a_{i,j}$ and $b_{i,j}$ repeats forever.

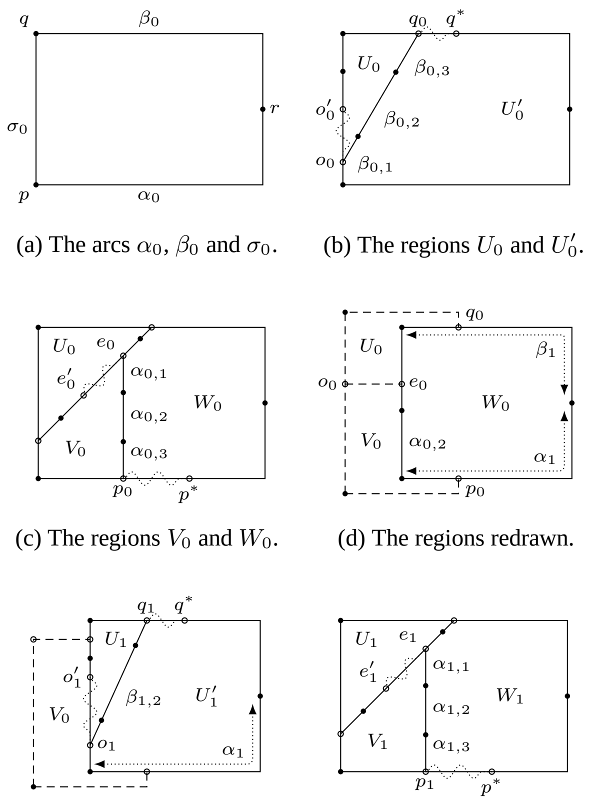
Let φ^*_∞ be the conjunction of (9)–(12) as well as all conjuncts

$$r \cdot r' = 0, \tag{13}$$

where r and r' are any two distinct regions depicted on Fig. 15. Note that the regions $a_{i,j}$ and $b_{i,j}$ have infinitely many connected components. We will now show that this is true for every satisfying tuple of φ^*_∞.

By (9), we can use Lemma 10 to construct a Jordan curve $\Gamma = \sigma\sigma'\beta\beta'\alpha\alpha'$ whose segments are Jordan arcs lying in the respective sets $(s+s')^\circ$, $(s'+b)^\circ$, $(b+b')^\circ$, $(b'+a)^\circ$, $(a+a')^\circ$, $(a'+s)^\circ$. Further, let $\sigma_0 = \sigma\sigma'$, $\beta_0 = \beta\beta'$ and $\alpha_0 = \alpha\alpha'$ (Fig. 16a). Note that all points in s, a and b that are on Γ are on σ_0, α_0 and β_0, respectively. Let $o'_0 \in \sigma_0 \cap$

s°, and let $q^* \in \beta_0 \cap b^\circ$. By (10) and Lemma 9 we can connect o_0' to q^* by a Jordan arc $\beta_{0,1}'\beta_{0,2}\beta_{0,3}'$ whose segments lie in the respective sets $(s + b_{0,1})^\circ$, $(b_{0,1} + b_{0,2} + b_{0,3})^\circ$ and $(b + b_{0,3})^\circ$ (Fig. 16b). Let o_0 be the last point on $\beta_{0,1}'$ that is on σ_0 and let $\beta_{0,1}$ be the final segment of $\beta_{0,1}'$ starting at o_0. Similarly, let q_0 be the first point on $\beta_{0,3}'$ that is on β_0 and let $\beta_{0,3}$ be the initial segment of $\beta_{0,3}'$ ending at q_0. Hence, the arc $\beta_{0,1}\beta_{0,2}\beta_{0,3}$ divides one of the regions bounded by Γ into two sub-regions. We denote the sub-region whose boundary is disjoint from α_0 by U_0, and the other sub-region we denote by U_0'. Let $\beta_1 := \beta_{0,3}\beta_0[q_0, r] \subseteq (b + b_{0,3} + b_{1,3})^\circ$.



(a) The arcs α_0, β_0 and σ_0.

(b) The regions U_0 and U_0'.

(c) The regions V_0 and W_0.

(d) The regions redrawn.

(e) The regions U_1 and U_1'.

(f) The regions V_1 and W_1.

Figure 16: Establishing infinite sequences of arcs.

We will now construct a cross-cut $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ in U_0'. Let $e_0' \in \beta_{0,2} \cap {b_{0,2}}^\circ$ and $p^* \in \alpha_0 \cap a^\circ$. By (11) and Lemma 9 we can connect e_0' to p^* by a Jordan arc $\alpha_{0,1}'\alpha_{0,2}\alpha_{0,3}'$ whose segments lie in the respective sets $(b_{0,2} + a_{0,1})^\circ$, $(a_{0,1} + a_{0,2} + a_{0,3})^\circ$ and $(a + a_{0,3})^\circ$ (Fig. 16c). Let e_0 be the last point on $\alpha_{0,1}'$ that is on $\beta_{0,2}$ and let $\alpha_{0,1}$ be the final segment of $\alpha_{0,1}'$ starting at e_0. Similarly, let p_0 be the first point on $\alpha_{0,3}'$ that is on α_0 and let $\alpha_{0,3}$ be the initial segment of $\alpha_{0,3}'$ ending at p_0. By the non-overlapping constraints, $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ does not intersect the boundaries of U_0 and U_0' except at its endpoints, and hence it is a cross-cut in one of these regions. Moreover, that region has to be U_0' since the boundary of U_0 is disjoint from α_0. So, $\alpha_{0,1}\alpha_{0,2}\alpha_{0,3}$ divides U_0' into two sub-regions. We denote the sub-region whose boundary contains β_1 by W_0, and the other sub-region we denote by V_0. Let $\alpha_1 := \alpha_{0,3}\alpha_0[p_0, r]$ (Fig 16d). Note that $\alpha_1 \subseteq (a + a_{0,3} + a_{1,3})^\circ$.

We can now forget about the region U_0, and start constructing a cross-cut $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ in W_0. As before, let $\beta_{1,1}'\beta_{1,2}\beta_{1,3}'$ be a Jordan arc connecting a point $o_1' \in \alpha_{0,2} \cap a_{0,2}^\circ$ to a point $q^* \in \beta_1 \cap b_i^\circ$ such that its segments are contained in the respective sets $(a_{0,2} + b_{1,1})^\circ$, $(b_{1,1} + b_{1,2} + b_{1,3})^\circ$ and $(b + b_{1,3})^\circ$. As before, we choose $\beta_{1,1} \subseteq \beta_{1,1}'$ and $\beta_{1,3} \subseteq \beta_{1,3}'$ so that the Jordan arc $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ with its endpoints removed is disjoint from the boundaries of V_0 and W_0. Hence $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ has to be a cross-cut in V_0 or W_0, and since the boundary of V_0 is disjoint from β_1 it has to be a cross-cut in W_0 (Fig. 16e). So, $\beta_{1,1}\beta_{1,2}\beta_{1,3}$ separates W_0 into two regions U_1 and U_1' so that the boundary of U_1 is disjoint from α_1. Let $\beta_2 := \beta_{1,3}\beta_1[q_1, r] \subseteq (b + b_{0,3} + b_{1,3})^\circ$. Now, we can ignore the region V_0, and reasoning as before we can construct a cross-cut $\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}$ in U_1' dividing it into two sub-regions V_1 and W_1.

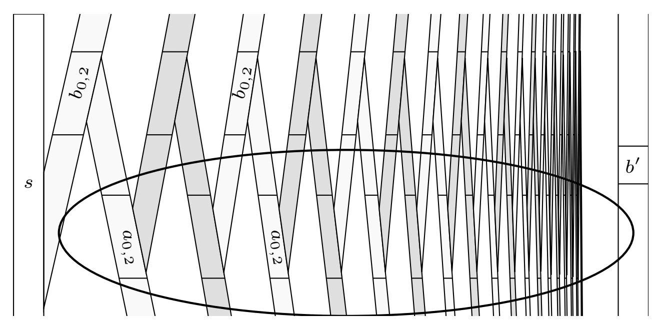


Figure 17: Separating $a_{0,2}$ from $b_{0,2}$ by a Jordan curve.

Evidently, this process continues forever. Now, note that by construction and (13), W_{2i} contains in its interior $\beta_{2i+1,2}$ together with the connected component c of $b_{1,2}$ which contains $\beta_{2i+1,2}$. On the other hand, W_{2i+2} is disjoint from c, and since $W_i \subseteq W_j$, $i > j$, $b_{1,2}$ has to have infinitely many connected components.

So far we know that the $\mathcal{C}c^\circ$-formula φ_∞^* forces infinitely many components. Now we replace every conjunct in φ_∞^* of the form $\neg C(r, s)$ by $\eta^*(r, s, \bar{v})$, where $\bar{v}$ are fresh variables each time. The resulting formula entails φ_∞^*, so we only have to show that it is still satisfiable. By Lemma 12 (*ii*), it suffices to separate by Jordan curves every two regions on Fig. 15 that are required to be disjoint. It is shown on Fig. 17 that there exists a curve which separates the regions $b_{0,2}$ and $a_{0,2}$. All other non-contact constraints are treated analogously. ❑

B Undecidability of $\mathcal{B}c$ and $\mathcal{C}c$ in the Euclidean plane

In this section, we prove the undecidability of the problems *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and *Sat*$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$, for $\mathcal{L}$ any of $\mathcal{B}c$, $\mathcal{C}c$,

$\mathcal{B}c^\circ$ or $\mathcal{C}c^\circ$. We begin with some technical preliminaries, again employing the notation from the proof of Theorem 5: if α is a Jordan arc, and p, q are points on α such that q occurs after p, we denote by $\alpha[p,q]$ the segment of α from p to q. For brevity of exposition, we allow the case $p = q$, treating $\alpha[p,q]$ as a (degenerate) Jordan arc.

Our first technical preliminary is to formalize our earlier observations concerning the formula $\mathsf{stack}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$, defined by:

$$\bigwedge_{1\leq i\leq n} c(\dot{a}_i + \ddot{a}_{i+1} + \cdots + \ddot{a}_n) \wedge \bigwedge_{j-i>1} \neg C(a_i,a_j).$$

Lemma 14 *Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ be 3-regions satisfying $\mathsf{stack}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$, for $n \geq 3$. Then, for every point $p_0 \in \dot{a}_1$ and every point $p_n \in \ddot{a}_n$, there exist points $p_1,\ldots,p_{n-1}$ and Jordan arcs $\alpha_1,\ldots,\alpha_n$ such that*:

(*i*) *$\alpha = \alpha_1\cdots\alpha_n$ is a Jordan arc from p_0 to p_n;*

(*ii*) *for all i $(0 \leq i < n)$, $p_i \in \dot{a}_{i+1} \cap \alpha_i$; and*

(*iii*) *for all i $(1 \leq i \leq n)$, $\alpha_i \subseteq a_i$.*

Proof. Since $\dot{a}_1 + \ddot{a}_2 + \cdots + \ddot{a}_n$ is a connected subset of $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)^\circ$, let β_1 be a Jordan arc connecting p_0 to p_n in $(a_1 + \dot{a}_2 + \cdots + \dot{a}_n)^\circ$. Since a_1 is disjoint from all the a_i except a_2, let p_1 be the first point of β_1 lying in $\dot{a}_2$, so $\beta_1[p_0,p_1] \subseteq a_1^\circ \cup \{p_1\}$, i.e., the arc $\beta_1[p_0,p_1]$ is either included in a_1°, or is an end-cut of a_1°. (We do not rule out $p_0 = p_1$.) Similarly, let β_2' be a Jordan arc connecting p_1 to p_n in $(a_2 + \dot{a}_3 + \cdots + \dot{a}_n)^\circ$, and let q_1 be the last point of β_2' lying on $\beta_1[p_0,p_1]$. If $q_1 = p_1$, then set $v_1 = p_1$, $\alpha_1 = \beta_1[p_0,p_1]$, and $\beta_2 = \beta_2'$. so that the endpoints of β_2 are v_1 and p_n. Otherwise, we have $q_1 \in a_1^\circ$. We can now construct an arc $\gamma_1 \subseteq a_1^\circ \cup \{p_1\}$ from p_1 to a point v_1 on $\beta_2'[q_1,p_n]$, such that γ_1 intersects $\beta_1[p_0,p_1]$ and $\beta_2'[q_1,p_n]$ only at its endpoints, p_1 and v_1 (upper diagram in Fig. 18). Let $\alpha_1 = \beta_1[p_0,p_1]\gamma_1$, and let $\beta_2 = \beta_2'[v_1,p_n]$.

Since β_2 contains a point $p_2 \in \dot{a}_3$, we may iterate this procedure, obtaining $\alpha_2,\alpha_3,\ldots\alpha_{n-1},\beta_n$. We remark that α_i and α_{i+1} have a single point of contact by construction, while α_i and α_j $(i < j-1)$ are disjoint by the constraint $\neg C(a_i,a_j)$. Finally, we let $\alpha_n = \beta_n$ (lower diagram in Fig. 18). ❑

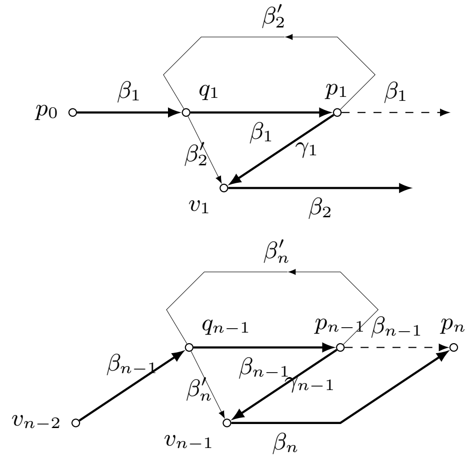


Figure 18: Proof of Lemma 14.

In fact, we can add a 'switch' w to the formula $\mathsf{stack}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$, in the following sense. If w is a region variable, consider the formula $\mathsf{stack}_w(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$

$$\neg C(w\cdot\dot{a}_1, (-w)\cdot\dot{a}_1) \;\wedge\; \mathsf{stack}((-w)\cdot\mathfrak{a}_1,\mathfrak{a}_2,\ldots,\mathfrak{a}_n),$$

where $w\cdot\mathfrak{a}$ denotes the 3-region $(w\cdot a, w\cdot\dot{a}, w\cdot\ddot{a})$. The first conjunct of $\mathsf{stack}_w(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$ ensures that any component of $\dot{a}_1$ is either included in w or included in $-w$. The second conjunct then has the same effect as $\mathsf{stack}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$ for those components of $\dot{a}_1$ included in $-w$. That is, if $p \in \dot{a}_1\cdot(-w)$, we can find an arc $\alpha_1\cdots\alpha_n$ starting at p, with the properties of Lemma 14. However, if $p \in \dot{a}\cdot w$, no such arc need exist. Thus, w functions so as to 'de-activate' the formula $\mathsf{stack}_w(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$ for any component of $\dot{a}_1$ included in it.

As a further application of Lemma 14, consider the formula $\mathsf{frame}(\mathfrak{a}_0,\ldots,\mathfrak{a}_n)$ given by:

$$\begin{aligned}\mathsf{stack}(\mathfrak{a}_0,\ldots,\mathfrak{a}_{n-1}) \wedge \neg C(a_n, a_1 + \ldots + a_{n-2}) \wedge \\ c(\dot{a}_n) \wedge \dot{a}_0\cdot\dot{a}_n \neq 0 \wedge \ddot{a}_{n-1}\cdot\dot{a}_n \neq 0. \end{aligned} \tag{14}$$

This formula allows us to construct Jordan curves in the plane, in the following sense:

Lemma 15 *Let $n \geq 3$, and suppose $\mathsf{frame}(\mathfrak{a}_0,\ldots,\mathfrak{a}_n)$. Then there exist Jordan arcs γ_0, ..., γ_n such that $\gamma_0\ldots\gamma_n$ is a Jordan curve, and $\gamma_i \subseteq a_i$, for all i, $0 \leq i \leq n$.*

Proof. By $\mathsf{stack}(\mathfrak{a}_0,\ldots,\mathfrak{a}_{n-1})$, let $\alpha_0,\ldots,\alpha_{n-1}$ be Jordan arcs in the respective regions $a_0,\ldots,a_{n-1}$ such that, $\alpha = \alpha_0\cdots\alpha_{n-1}$ is a Jordan arc connecting a point $p' \in \dot{a}_0\cdot\dot{a}_n$ to a point $q' \in \ddot{a}_{n-1}\cdot\dot{a}_n$ (see Fig. 19). Because $\dot{a}_n$ is a connected subset of the interior of a_n, let $\alpha_n \subseteq a_n^\circ$ be an arc connecting p' and q'. Note that α_n does not intersect α_i, for $1 \leq i < n-1$. Let p be the last point on α_0 that is on α_n (possibly p'), and q be the first point on α_{n-1} that is on α_n (possibly q'). Let γ_0 be the final segment of α_0 starting at p. Let $\gamma_i := \alpha_i$, for $1 \leq i \leq n-2$. Let γ_{n-1} be the initial segment of α_{n-1} ending at q. Finally, take γ_n to be the segment of α_n between p and q. Evidently, the arcs γ_i, $0 \leq i \leq n$, are as required. ❑

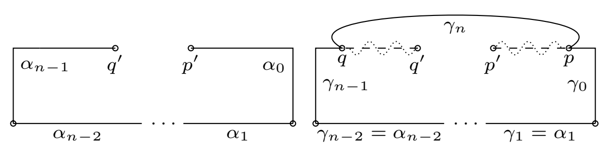


Figure 19: Establishing a Jordan curve.

Our final technical preliminary is a simple device for labelling arcs in diagrams.

Lemma 16 *Suppose r, $t_1,\ldots,t_\ell$ are regions such that*

$$(r \leq t_1 + \cdots + t_\ell) \wedge \bigwedge_{1\leq i<j\leq\ell} \neg C(r\cdot t_i, r\cdot t_j), \tag{15}$$

and let X be a connected subset of r. Then X is included in exactly one of the t_i, $1 \leq i \leq \ell$.

Proof. If $X \cap t_1$ and $X \cap t_2$ are non-empty, then $X \cap t_1$ and $X \cap (t_2 + \cdots + t_\ell)$ partition X into non-empty, non-intersecting sets, closed in X. ❑

When (15) holds, we may think of the regions $t_1, \ldots, t_\ell$ as 'labels' for any connected $X \subseteq r$—and, in particular, for any Jordan arc $\alpha \subseteq r$. Hence, any sequence $\alpha_1, \ldots, \alpha_n$ of such arcs encodes a word over the alphabet $\{t_1, \ldots, t_\ell\}$.

The remainder of this section is given over to a proof of

Theorem 6. *For $\mathcal{L} \in \{\mathcal{B}c^\circ, \mathcal{B}c, \mathcal{C}c^\circ, \mathcal{C}c\}$, Sat$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ is r.e.-hard, and Sat$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ is r.e.-complete.*

We have already established the upper bounds; we consider here only the lower bounds, beginning with an outline of our proof strategy. Let a PCP-instance $\mathbf{w} = (\{0,1\}, T, \mathsf{w}_1, \mathsf{w}_2)$ be given, where T is a finite alphabet, and $\mathsf{w}_i \colon T^* \to \{0,1\}^*$ a word-morphism ($i = 1, 2$). We call the elements of T *tiles*, and, for each tile t, we call $\mathsf{w}_1(t)$ the *lower word* of t, and $\mathsf{w}_2(t)$ the *upper word* of t. Thus, $\mathbf{w}$ asks whether there is a sequence of tiles (repeats allowed) such that the concatenation of their upper words is the same as the concatenation of their lower words. We shall henceforth restrict all (upper and lower) words on tiles to be non-empty. This restriction simplifies the encoding below, and does not affect the undecidability of the PCP.

We define a formula $\varphi_{\mathbf{w}}$ consisting of a large conjunction of $\mathcal{C}c$-literals, which, for ease of understanding, we introduce in groups. Whenever conjuncts are introduced, it can be readily checked that—provided $\mathbf{w}$ is positive—they are satisfiable by elements of $\mathsf{RCP}(\mathbb{R}^2)$. (Figs. 20 and 22 depict *part* of a satisfying assignment; this drawing is additionally useful as an aid to intuition throughout the course of the proof.) The main object of the proof is to show that, conversely, if $\varphi_{\mathbf{w}}$ is satisfied by any tuple in $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ must be positive. Thus, the following are equivalent:

1. $\mathbf{w}$ is positive;
2. $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$;
3. $\varphi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$.

This establishes the r.e.-hardness of *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and *Sat*$(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ for $\mathcal{L} = \mathcal{C}c$; we then extend the result to the languages $\mathcal{B}c$, $\mathcal{C}c^\circ$ and $\mathcal{B}c^\circ$.

The proof proceeds in five stages.

Stage 1. In the first stage, we define an assemblage of arcs that will serve as a scaffolding for the ensuing construction. Consider the arrangement of polygonal 3-regions depicted in Fig. 20, assigned to the 3-region variables $\mathfrak{s}_0, \ldots, \mathfrak{s}_9$, $\mathfrak{s}'_8, \ldots, \mathfrak{s}'_1$, $\mathfrak{d}_0, \ldots, \mathfrak{d}_6$ as indicated. It is easy to verify that this arrangement can be made to satisfy the following formulas:

$$\mathsf{frame}(\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_8, \mathfrak{s}_9, \mathfrak{s}'_8, \ldots, \mathfrak{s}'_1), \tag{16}$$

$$(s_0 \leq \dot{t}_0) \wedge (s_9 \leq \ddot{t}_6), \tag{17}$$

$$\mathsf{stack}(\mathfrak{d}_0, \ldots, \mathfrak{d}_6). \tag{18}$$

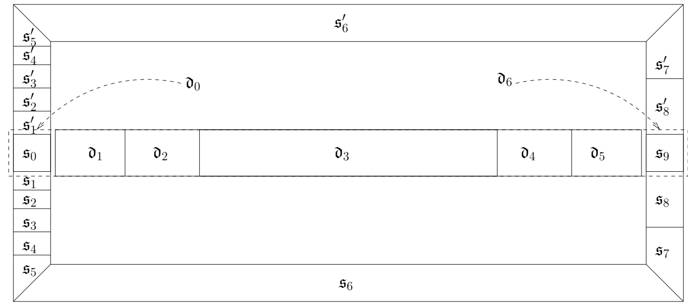


Figure 20: A tuple of 3-regions satisfying (16)–(18). The 3-regions $\mathfrak{d}_0$ and $\mathfrak{d}_6$ are shown in dotted lines.

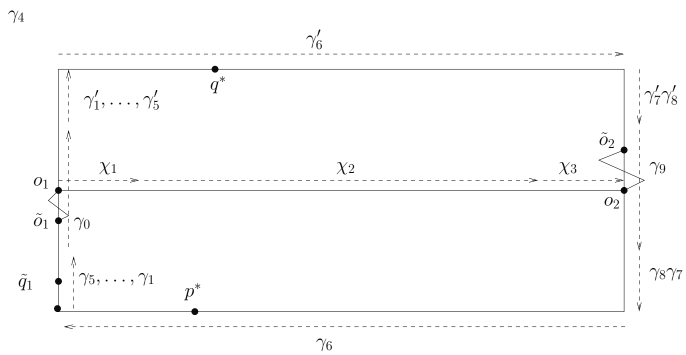


Figure 21: The arcs $\gamma_0, \ldots, \gamma_9$ and $\chi_1, \ldots \chi_3$.

And trivially, the arrangement can be made to satisfy any formula

$$\neg C(r, r') \tag{19}$$

for which the corresponding 3-regions $\mathfrak{r}$ and $\mathfrak{r}'$ are drawn as not being in contact. (Remember, r is the outer-most shell of the 3-region $\mathfrak{r}$, and similarly for r'.) Thus, for example, (19) includes $\neg C(s_0, d_1)$, but not $\neg C(s_0, d_0)$ of $\neg C(d_0, d_1)$.

Now suppose $\mathfrak{s}_0, \ldots, \mathfrak{s}_9$, $\mathfrak{s}'_8, \ldots, \mathfrak{s}'_1$, $\mathfrak{d}_0, \ldots, \mathfrak{d}_6$ is *any* collection of 3-regions (not necessarily polygonal) satisfying (16)–(19). By Lemma 15 and (16), let $\gamma_0, \ldots, \gamma_9, \gamma'_8, \ldots, \gamma'_1$ be Jordan arcs included in the respective regions $s_0, \ldots, s_9, s'_8, \ldots, s'_1$, such that $\Gamma = \gamma_0 \cdots \gamma_9 \cdot \gamma'_8 \cdots \gamma'_1$ is a Jordan curve (note that γ'_i and γ_i have opposite directions). We select points $\tilde{o}_1$ on γ_0 and $\tilde{o}_2$ on γ_9 (see Fig. 21). By (17), $\tilde{o}_1 \in \dot{t}_0$ and $\tilde{o}_2 \in \ddot{t}_6$. By Lemma 14 and (18), let $\tilde{\chi}_1$, χ_2, $\tilde{\chi}_3$ be Jordan arcs in the respective regions

$$(d_0 + d_1), \qquad (d_2 + d_3 + d_4), \qquad (d_5 + d_6)$$

such that $\tilde{\chi}_1 \chi_2 \tilde{\chi}_3$ is a Jordan arc from $\tilde{o}_1$ to $\tilde{o}_2$. Let o_1 be the last point of $\tilde{\chi}_1$ lying on Γ, and let χ_1 be the final segment of $\tilde{\chi}_1$, starting at o_1. Let o_2 be the first point of $\tilde{\chi}_3$ lying on Γ, and let χ_3 be the initial segment of $\tilde{\chi}_3$, ending at o_2. By (19), we see that the arc $\chi = \chi_1 \chi_2 \chi_3$ intersects Γ only in its endpoints, and is thus a chord of Γ, as shown in Fig. 21.

A word is required concerning the generality of this diagram. The reader is to imagine the figure drawn on a *spherical* canvas, of which the sheet of paper or computer screen in front of him is simply a small part. This sphere represents the

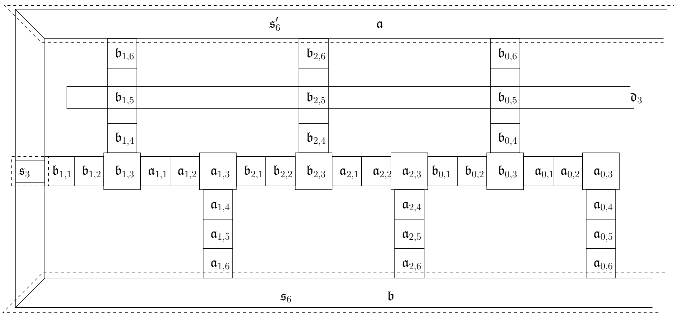


Figure 22: A tuple of 3-regions satisfying (20)–(22). The arrangement of components of the $\mathfrak{a}_{i,j}$ and $\mathfrak{b}_{i,j}$ repeats an indeterminate number of times. The 3-regions $\mathfrak{a}$, $\mathfrak{b}$ and one component of $\mathfrak{a}_{0,3}$ are shown in dotted lines. The 3-regions $\mathfrak{s}_3$, $\mathfrak{s}_6$, $\mathfrak{s}_6'$ and $\mathfrak{d}_3$ are as in Fig 22, but not drawn to scale.

plane with a 'point' at infinity, under the usual stereographic projection. We do not say where this point at infinity is, other than that it never lies on a drawn arc. In this way, a diagram in which the spherical canvas is divided into n cells represents n different configurations in the plane—one for each of the cells in which the point at infinity may be located. For example, Fig .21 represents three topologically distinct configurations in $\mathbb{R}^2$, and, as such, depicts the arcs $\gamma_0, \ldots, \gamma_9$, $\gamma_1', \ldots, \gamma_8'$, χ_1, χ_2, χ_3 and points o_1, o_2 in full generality. All diagrams in this proof are to be interpreted in this way. We stress that our 'spherical diagrams' are simply a convenient device for using one drawing to represent several possible configurations in the Euclidean plane: in particular, we are interested only in the satisfiability of of $\mathcal{C}c$-formulas over $\mathsf{RCP}(\mathbb{R}^2)$ and $\mathsf{RC}(\mathbb{R}^2)$, not over the regular closed algebra of any other space! For ease of reference, we refer to the the two rectangles in Fig .21 as the 'upper window' and 'lower window', it being understood that these are simply handy labels: in particular, either of these 'windows' (but not both) may be unbounded.

Stage 2. In this stage, we we construct two sequences of arcs, $\{\zeta_i\}$, $\{\eta_i\}$ of indeterminate length $n \geq 1$, such that the members of the former sequence all lie in the lower window. Here and in the sequel, we write $\lfloor k \rfloor$ to denote k modulo 3. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}_{i,j}$ and $\mathfrak{b}_{i,j}$ ($0 \leq i < 3$, $1 \leq j \leq 6$) be 3-region variables, let z be an ordinary region-variable, and consider the formulas

$$(s_6 \leq \ddot{a}) \wedge (s_6' \leq \ddot{b}) \wedge (s_3 \leq \dot{a}_{0,3}), \tag{20}$$

$$\mathsf{stack}_z(\mathfrak{a}_{\lfloor i-1 \rfloor,3}, \mathfrak{b}_{i,1}, \ldots, \mathfrak{b}_{i,6}, \mathfrak{b}), \tag{21}$$

$$\mathsf{stack}(\mathfrak{b}_{i,3}, \mathfrak{a}_{i,1}, \ldots, \mathfrak{a}_{i,6}, \mathfrak{a}). \tag{22}$$

The arrangement of polygonal 3-regions depicted in Fig. 22 (with z assigned appropriately) is one such satisfying assignment. We stipulate that (19) applies now to all regions depicted in either Fig 20 or Fig 22. Again, these additional constraints are evidently satisfiable.

It will be convenient in this stage to rename the arcs γ_6 and γ_6' as λ_0 and μ_0, respectively. Thus, λ_0 forms the bottom edge of the lower window, and μ_0 the top edge of the upper window. Likewise, we rename γ_3 as α_0, forming part of the left-hand side of the lower window. Let $\tilde{q}_{1,1}$ be any point of α_0, p^* any point of λ_0, and q^* any point of μ_0 (see Fig. 21). By (20), then, $\tilde{q}_{1,1} \in \dot{a}_{0,3}$, $p^* \in \ddot{a}$, and $q^* \in \ddot{b}$. Adding the constraint

$$\neg C(s_3, z),$$

further ensures that $\tilde{q}_{1,1} \in -z$. By Lemma 14 and (21), we may draw an arc $\tilde{\beta}_1$ from $\tilde{q}_{1,1}$ to q^*, with successive segments $\tilde{\beta}_{1,1}, \beta_{1,2}, \ldots, \beta_{1,5}, \tilde{\beta}_{1,6}$ lying in the respective regions $a_{0,3} + b_{1,1}, b_{1,2}, \ldots, b_{1,5}, b_{1,6} + b$; further, we can guarantee that $\beta_{1,2}$ contains a point $\tilde{p}_{1,1} \in \dot{b}_{1,3}$. Denote the last point of $\beta_{1,5}$ by $q_{1,2}$. Also, let $q_{1,1}$ be the last point of $\tilde{\beta}_1$ lying on α_0, and $q_{1,3}$ the first point of $\tilde{\beta}_1$ lying on μ_0 Finally, let β_1 be the segment of $\tilde{\beta}_1$ between $q_{1,1}$ and $q_{1,2}$; and we let μ_1 be the segment of $\tilde{\beta}_1$ from $q_{1,2}$ to $q_{1,3}$ followed by the final segment of μ_0 from $q_{1,3}$. (Fig. 23a). By repeatedly using the constraints in (19), it is easy to see that that β_1 together with the initial segment of μ_1 up to $q_{1,3}$ form a chord of Γ. Adding the constraints

$$c(b_{0,5} + d_3),$$

and taking into account the constraints in (19) ensures that β_1 and χ lie in the same residual domain of Γ, as shown. The wiggly lines indicate that we do not care about the exact positions of $\tilde{q}_{1,1}$ or q^*; otherwise, Fig. 23a) is again completely general. Note that μ_1 lies entirely in $b_{1,6} + b$, and hence certainly in the region

$$b^* = b + b_{0,6} + b_{1,6} + b_{2,6}.$$

Recall that $\tilde{p}_{1,1} \in \dot{b}_{1,3}$, and $p^* \in \ddot{a}$. By Lemma 14 and (22), we may draw an arc $\tilde{\alpha}_1$ from $\tilde{p}_{1,1}$ to p^*, with successive segments $\tilde{\alpha}_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,5}, \tilde{\alpha}_{1,6}$ lying in the respective regions $b_{1,3} + a_{1,1}, a_{1,2}, \ldots, a_{1,5}$ $a_{1,6} + a$; further, we can guarantee that the segment lying in $a_{1,3}$ contains a point $\tilde{q}_{2,1} \in \dot{a}_{1,3}$. Denote the last point of $\alpha_{1,5}$ by $p_{1,2}$. Also, let $p_{1,1}$ be the last point of $\tilde{\alpha}_1$ lying on β_1, and $p_{1,3}$ the first point of $\tilde{\alpha}_1$ lying on λ_0. From (19), these points must be arranged as shown in Fig. 23b. Let α_1 be the segment of $\tilde{\alpha}_1$ between $p_{1,1}$ and $p_{1,2}$. Noting that (19) entails

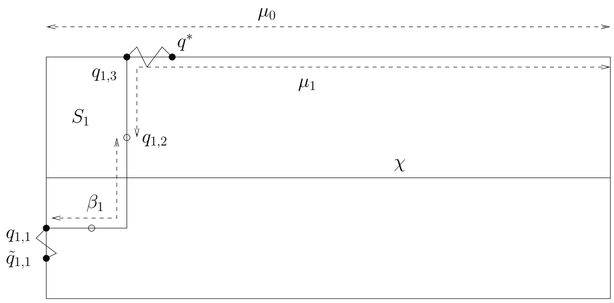
$$\neg C(a_{1,k}, s_0 + s_9 + d_0 + \cdots + d_5) \qquad 1 \leq k \leq 6,$$

we can be sure that α_1 lies entirely in the 'lower' window, whence β_1 crosses the central chord, χ, at least once. Let o_1 be the first such point (measured along χ from left to right). Finally, let λ_1 be the segment of $\tilde{\alpha}_1$ between $p_{1,2}$ and $p_{1,3}$, followed by the final segment of λ_0 from $p_{1,3}$. Note that λ_1 lies entirely in $a_{1,6} + a$, and hence certainly in the region

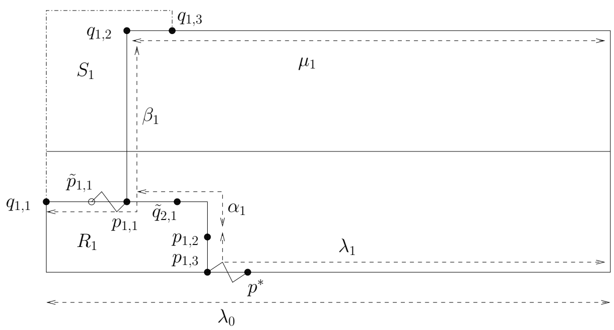
$$a^* = a + a_{0,6} + a_{1,6} + a_{2,6}.$$

We remark that, in Fig. 23b, the arcs β_1 and μ_1 have been slightly re-drawn, for clarity. The region marked S_1 may now be forgotten, and is suppressed in Figs. 23c and 23d.

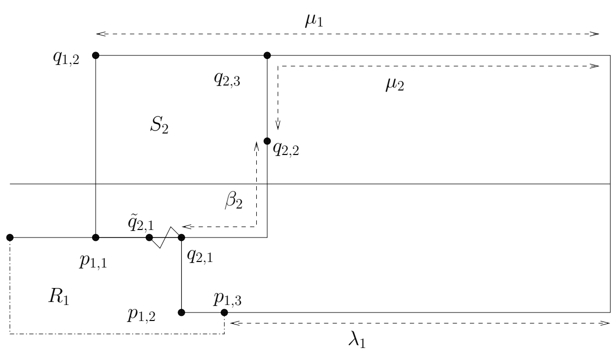
By construction, the point $\tilde{q}_{2,1}$ lies in some component of $\dot{a}_{1,3}$, and, from the presence of the 'switching' variable z in (22), that component is either included in z or included in $-z$. Suppose the latter. Then we can repeat the above construction to obtain an arc $\tilde{\beta}_2$ from $\tilde{q}_{2,1}$ to q^*, with successive segments $\tilde{\beta}_{2,1}, \beta_{2,2}, \ldots, \beta_{2,5}, \tilde{\beta}_{2,6}$ lying in the respective



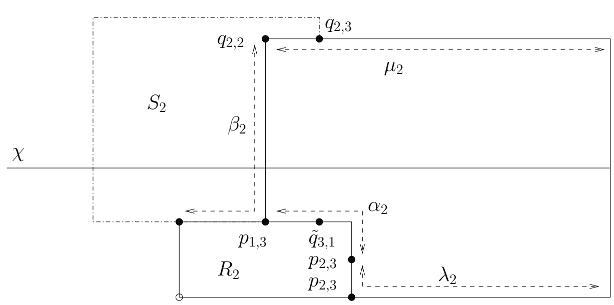
(a) The arc β_1.



(b) The arc α_1.



(c) The arc β_2.



(d) The arc α_2.

Figure 23: Construction of the arcs $\{\alpha_i\}$ and $\{\beta_i\}$

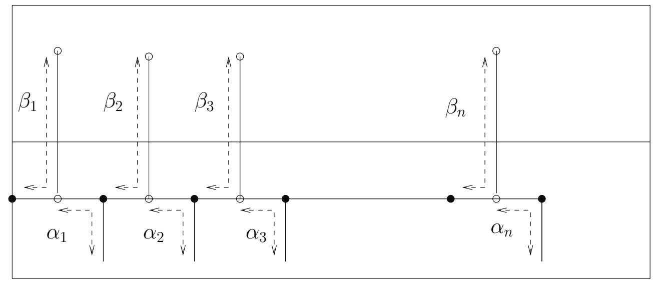


Figure 24: The sequences of arcs $\{\alpha_i\}$ and $\{\beta_i\}$.

regions $a_{1,3} + b_{2,1}$, $b_{2,2}$, $\ldots$, $b_{2,5}$, $b_{2,6} + b$; further, we can guarantee that $\beta_{2,2}$ contains a point $\tilde{p}_{2,1} \in \dot{b}_{2,3}$. Denote the last point of $\beta_{2,5}$ by $q_{2,2}$. Also, let $q_{2,1}$ be the last point of $\tilde{\beta}_2$ lying on α_1, and $q_{2,3}$ the first point of $\tilde{\beta}_2$ lying on μ_1. Again, we let β_2 be the segment of $\tilde{\beta}_2$ between $q_{2,1}$ and $q_{2,2}$; and we let μ_2 be the segment of $\tilde{\beta}_2$ from $q_{2,1}$ to $q_{2,3}$, followed by the final segment of μ_1 from $q_{2,3}$. Note that μ_2 lies in the set b^*. It is easy to see that β_2 must be drawn as shown in Fig. 23c: in particular, β_2 cannot enter the interior of the region marked R_1. For, by construction, β_2 can have only one point of contact with α_1, and the constraints (19) ensure that β_2 cannot intersect any other part of δR_1; since $q^* \in a$ is guaranteed to lie outside R_1, we evidently have $\beta_2 \subseteq -R_1$. This observation having been made, R_1 may now be forgotten.

Symmetrically, we construct the arc $\tilde{\alpha}_2 \subseteq b_{1,3} + a_{2,1} + \cdots + a_{2,6} + a$, and points $p_{2,1}$, $p_{2,2}$, $p_{2,3}$, together with the arcs arcs α_2 and λ_2, as shown in Fig. 23d (where the region R_1 has been suppressed and the region S_2 slightly re-drawn). Again, we know from (19) that α_2 lies entirely in the 'lower' window, whence β_2 must cross the central chord, χ, at least once. Let o_2 be the first such point (measured along χ from left to right).

This process continues, generating arcs $\beta_i \subseteq a_{\lfloor i-1 \rfloor,3} + b_{\lfloor i \rfloor,1} + \cdots + b_{\lfloor i \rfloor,5}$ and $\alpha_i \subseteq b_{\lfloor i \rfloor,3} + a_{\lfloor i \rfloor,1} + \cdots + a_{\lfloor i \rfloor,5}$, as long as α_i contains a point $\tilde{q}_{i,1} \in -z$. That we eventually reach a value $i = n$ for which no such point exists follows from (19). For the conjuncts $\neg C(b_{i,j}, d_k)$ $(j \neq 5)$ together entail $o_i \in b_{\lfloor i \rfloor,5}$, for every i such that β_i is defined; and these points cycle on χ through the regions $b_{0,5}$, $b_{1,5}$ and $b_{2,5}$. If there were infinitely many β_i, the o_i would have an accumulation point, lying in all three regions, contradicting, say, $\neg C(b_{0,5}, b_{1,5})$. The resulting sequence of arcs and points is shown, schematically, in Fig. 24.

We finish this stage in the construction by 're-packaging' the arcs $\{\alpha_i\}$ and $\{\beta_i\}$, as illustrated in Fig. 25. Specifically, for all i $(1 \leq i \leq n)$, let ζ_i be the initial segment of β_i up to the point $p_{i,1}$ followed by the initial segment of α_i up to the point $q_{i+1,1}$; and let η_i be the final segment of β_i from the point $p_{i,1}$:

$$\begin{aligned} \zeta_i &= \beta_i[q_{i,1}, p_{i,1}]\alpha_i[p_{i,2}, q_{i+1,1}] \\ \eta_i &= \beta_i[p_{i,1}, q_{i,2}]. \end{aligned}$$

The final segment of α_i from the point q_{i+1} may be forgotten. Defining, for $0 \leq i < 3$,

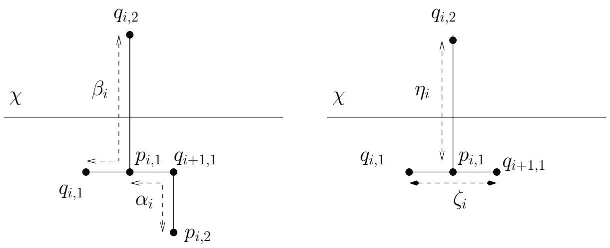


Figure 25: ‘Re-packaging’ of α_i and β_i into ζ_i and η_i: before and after.

$$\begin{aligned} a_i &= a_{1-i,3} + b_{i,1} + \cdots + b_{i,4} + a_{i,1} + \cdots + a_{i,4} \\ b_i &= b_{i,2} + \cdots + b_{i,5}, \end{aligned}$$

the constraints (19) guarantee that, for $1 \le i \le n$,

$$\begin{aligned} \zeta_i &\subseteq a_{\lfloor i \rfloor} \\ \eta_i &\subseteq b_{\lfloor i \rfloor}. \end{aligned}$$

Observe that the arcs ζ_i are located entirely in the ‘lower window’, and that each arc η_i connects ζ_i to some point $q_{i,2}$, which in turn is connected to a point $q^* \in \lambda_0$ by an arc in b^*.

Stage 3. We now repeat Stage 2 symmetrically, with the ‘upper’ and ‘lower’ windows exchanged. Let $\mathfrak{a}'_{i,j}$, $\mathfrak{b}'_{i,j}$ be 3-region variables (with indices in the same ranges as for $\mathfrak{a}_{i,j}$, $\mathfrak{b}_{i,j}$). Let $\mathfrak{a}' = \mathfrak{b}$, $\mathfrak{b}' = \mathfrak{a}$; and let

$$\begin{aligned} a'_i &= a'_{1-i,3} + b'_{i,1} + \cdots + b'_{i,4} + a'_{i,1} + \cdots + a'_{i,4} \\ b'_i &= b'_{i,2} + \cdots + b'_{i,5}, \end{aligned}$$

for $0 \le i < 2$. The constraints

$$\begin{aligned} &(s'_3 \le \dot{a}'_{0,3}) \\ &\mathsf{stack}_z(\mathfrak{a}'_{\lfloor i-1 \rfloor,3}, \mathfrak{b}'_{i,1}, \ldots, \mathfrak{b}'_{i,6}, \mathfrak{b}'), \\ &\mathsf{stack}(\mathfrak{b}'_{i,3}, \mathfrak{a}'_{k,1}, \ldots, \mathfrak{a}'_{i,6}, \mathfrak{a}') \\ &c(b'_{0,5} + d_3) \end{aligned}$$

then establish sequences of arcs $\{\zeta'_i\}$, $\{\eta'_i\}$, $(1 \le i \le n')$ satisfying

$$\begin{aligned} \zeta'_i &\subseteq a'_{\lfloor i \rfloor} \\ \eta'_i &\subseteq b'_{\lfloor i \rfloor} \end{aligned}$$

for $1 \le i \le n'$. The arcs ζ'_i are located entirely in the ‘upper window’, and each arc η'_i connects ζ'_i to a point $p_{i,2}$, which in turn is connected to a point p^* by an arc in the region

$$b^{*\prime} = b' + b'_{0,6} + b'_{1,6} + b'_{2,6}.$$

Our next task is to write constraints to ensure that $n = n'$, and that, furthermore, each η_i (also each η'_i) connects ζ_i to ζ'_i, for $1 \le i \le n = n'$. Let z^* be a new region-variable, and write

$$\neg C(z^*, s_0 + \cdots + s_9 + s'_1 + \cdots + s'_8 + d_1 + \cdots + d_4 + d_6).$$

Note that d_5 does not appear in this constraint, which ensures that the only arc depicted in Fig. 21 which z may intersect is

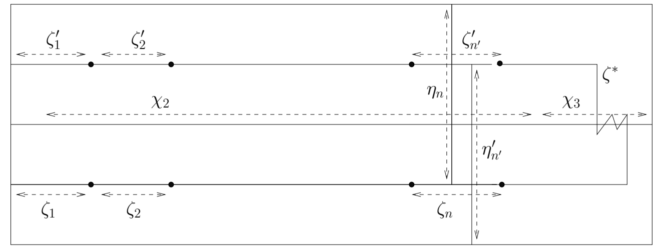


Figure 26: The arc ζ^*.

χ_3. Recalling that α_n and $\alpha'_{n'}$ contain points $q_{n,1}$ and $q'_{n',1}$, respectively, both lying in z, the constraints

$$c(z) \wedge \neg C(z, -z^*)$$

ensure that $q_{n,1}$ and $q'_{n',1}$ may be joined by an arc, say ζ^*, lying in $(z^*)^\circ$, and also lying entirely in the upper and lower windows, crossing χ only in χ_3. Without loss of generality, we may assume that ζ^* contacts ζ_n and $\zeta'_{n'}$ in just one point. Bearing in mind that the constraints (19) force η_n and $\eta'_{n'}$ to cross χ in its central section, χ_2, writing

$$\neg C(b_{i,j}, z) \wedge \neg C(b'_{i,j}, z) \tag{23}$$

for all i $(0 \le i < 3)$ and j $(1 \le i \le 6)$ ensures that ζ^* is (essentially) as shown in Fig. 26. Now consider the arc η_1. Recalling that $\eta_1\mu_1$ joins ζ_1 to the point q^* (on the upper edge of the upper window), crossing χ_2, we see by inspection of Fig. 26 that (23) together with

$$\neg C(a'_i, b^*)$$

for $0 \le i < 3$ forces η_1 to cross one of the arcs $\zeta'_{j'}$ $(1 \le j' \le n')$; and the constraints

$$\neg C(a'_i, b_j)$$

for $0 \le i < 3$, $0 \le j < 3$, $i \ne j$, ensure that $j' \equiv 1$ modulo 3. We write the symmetric constraints

$$\neg C(a_i, b'_j) \tag{24}$$

for $0 \le i < 3$, $0 \le j < 3$, $i \ne j$, together with

$$\neg C(b_i, b'_j) \tag{25}$$

for $0 \le i < j \le 3$. Now suppose $j' \ge 4$. The arc $\eta'_2\lambda'_2$ must connect ζ'_2 to the point p^* on the bottom edge of the lower window, which is now impossible without η'_2 crossing either ζ_1 or η_1—both forbidden by (24)–(25). Thus, η_1 intersects ζ'_j if and only if $j = 1$. Symmetrically, η'_1 intersects ζ_j if and only if $j = 1$. And the reasoning can now be repeated for η_2, η'_2, η_3, η'_3 …, leading to the 1–1 correspondence depicted in Fig. 27. In particular, we are guaranteed that $n = n'$.

Stage 4. Recall the given PCP-instance, $\mathbf{w} = (\{0,1\}, T, \mathsf{w}_1, \mathsf{w}_2)$. We think of T as a set of ‘tiles’, and the morphisms w_1, w_2 as specifying, respectively, the ‘lower’ and ‘upper’ strings of each tile. In this stage, we shall ‘label’ the arcs $\zeta_1, \ldots, \zeta_n$, with elements of $\{0,1\}$, thus defining a word σ over this alphabet. Using a slightly more complicated labelling scheme, we shall label the arcs $\eta_1, \ldots, \eta_n$ so as to

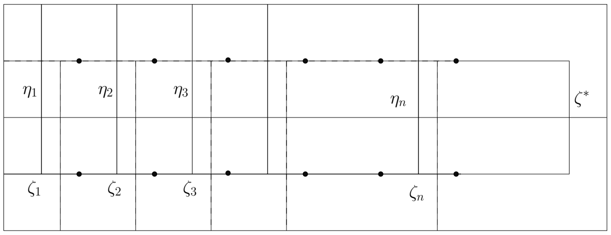


Figure 27: The 1–1 correspondence between the ζ_i and the ζ'_i established by the η_i and the η'_i.

define a word τ (of length $m \leq n$) over the alphabet T; likewise we shall label the arcs $\eta'_1, \ldots, \eta'_n$ so as to define another word τ' (of length $m' \leq n$) over T.

We begin with the ζ_i. Consider the constraints

$$b_i \leq l_0 + l_1 \wedge \neg C(b_i \cdot l_0, b_i \cdot l_1) \qquad (i = 0,\ 1).$$

By Lemma 16, in any satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$, every arc η_i $(1 \leq i \leq n)$ is included in ('labelled with') exactly one of the regions l_0 or l_1, so that the sequence of arcs $\eta_1, \ldots, \eta_n$ defines a word $\sigma \in \{0,1\}^*$, with $|w| = n$.

Turning our attention now to the ζ_i, let us write $T = \{t_1, \ldots, t_\ell\}$. For all j $(1 \leq j \leq \ell)$, we shall write $\sigma_j = \mathsf{w}_1(t_j)$ and $\sigma'_j = \mathsf{w}_2(t_j)$; further, we denote $|\sigma_j|$ by $u(j)$ and $|\sigma'_j|$ by $u'(j)$. (Thus, by assumption, the $u(j)$ and $u'(j)$ are all positive.)

Now let $t_{j,k}$ $(1 \leq j \leq \ell,\ 1 \leq k \leq u(j))$ and $t'_{j,k}$ $(1 \leq j \leq \ell,\ 1 \leq k \leq u'(j))$ be fresh region variables. We think of $t_{j,k}$ as standing for the kth letter in the word σ_j, and likewise think of $t'_{j,k}$ as standing for the kth letter in the word σ'_j. By Lemma 16, we may write constraints ensuring that each component of either a_0, a_1 or a_2—and hence each of the arcs $\zeta_1, \ldots, \zeta_n$—is 'labelled with' one of the $t_{j,k}$, in the by-now familiar sense. Further, we can ensure that these labels are organized into (contiguous) blocks, $E_1, \ldots, E_m$ such that, in the hth block, E_h, the sequence of labels reads $t_{j,1}, \ldots, t_{j,u(j)}$, for some fixed j $(1 \leq j \leq \ell)$. This amounts to insisting that: (*i*) the very first arc, ζ_1, must be labelled with $t_{j,1}$ for some j; (*ii*) if, ζ_i is labelled with $t_{j,k}$, where $i < n$ and $k < u(j)$, then the next arc, namely ζ_{i+1}, must be labelled with the next letter of σ_j, namely $t_{j,k+1}$; (*iii*) if ζ_i $(i < n)$ is labelled with the final letter of w_j, then the next arc must be labelled with the initial letter of some possibly different word $\sigma_{j'}$; and (*iv*) ζ_n must be labelled with the final letter of some word. To do this we simply write:

$$\begin{array}{ll}
\neg C(t_{j,i}, s_3) & (\text{if } i \neq 1) \\
\neg C(a_k \cdot t_{j,i}, a_{\lfloor k+1 \rfloor} \cdot t_{j',i'}) & \begin{array}{l}(i < u(j) \text{ and either} \\ j' \neq j \text{ or } i' \neq i+1)\end{array} \\
\neg C(a_k \cdot t_{j,u(j)}, a_{\lfloor k+1 \rfloor} \cdot t_{j',i'}) & (\text{if } i' \neq 1) \\
\neg C(t_{j,i}, z^*) & (\text{if } i \neq u(j)),
\end{array}$$

where $1 \leq j, j' \leq \ell$, $1 \leq i \leq u(j)$ and $1 \leq i' \leq u(j')$.

Thus, within each block E_h, the labels read $t'_{j,1}, \ldots, t'_{j,u'(j)}$, for some fixed j; we write $j(h)$ to denote the common subscript j. The sequence of indices $j(1), \ldots, j(m)$ corresponding to the successive blocks thus defines a word $\tau = t_{j(1)}, \ldots t_{j(m)} \in T^*$.

Using corresponding formulas, we label the arcs ζ'_i $(1 \leq i \leq n)$ with the alphabet $\{t'_{j,k} \mid 1 \leq j \leq \ell, 1 \leq k \leq u'(j)\}$, so that, in any satisfying assignment over $\mathsf{RC}(\mathbb{R}^2)$, every arc ζ'_i $(1 \leq i \leq n)$ is labelled with exactly one of the regions $t'_{j,k}$. Further, we can ensure that these labels are organized into (say) m' contiguous blocks, $E'_1, \ldots, E'_{m'}$ such that in the hth block, E'_h, the sequence of labels reads $t'_{j,1}, \ldots, t'_{j,u'(j)}$, for some fixed j. Again, writing $j'(h)$ for the common value of j, the sequence of of indices $j'(1), \ldots, j'(m')$ corresponding to the successive blocks defines a word $\tau' = t_{j'(1)}, \ldots t_{j'(m')} \in T^*$.

Stage 5. The basic job of the foregoing stages was to define the words $\sigma \in \{0,1\}^*$ and $\tau, \tau' \in T^*$. In this stage, we enforce the equations $\sigma = \mathsf{w}_1(\tau)$, $\sigma = \mathsf{w}_2(\tau')$ and $\tau = \tau'$. That is: the PCP-instance $\mathbf{w} = (\{0,1\}, T, \mathsf{w}_1, \mathsf{w}_2)$ is positive.

We first add the constraints

$$\begin{array}{ll}
\neg C(l_h, t_{j,k}) & \text{the } k\text{'th letter of } \sigma_j \text{ is not } h \\
\neg C(l_h, t'_{j,k}) & \text{the } k\text{'th letter of } \sigma'_j \text{ is not } h.
\end{array}$$

Since η_i is in contact with ζ_i for all i $(1 \leq i \leq n)$, the string $\sigma \in \{0,1\}^*$ defined by the arcs η_i must be identical to the string $\sigma_{j(1)} \cdots \sigma_{j(m)}$. But this is just to say that $\sigma = \mathsf{w}_1(\tau)$. The equation $\mathsf{w}_2(\tau') = \sigma$ may be secured similarly.

It remains only to show that $\tau = \tau'$. That is, we must show that $m = m'$ and that, for all h $(1 \leq h \leq m)$, $j(h) = j'(h)$. The techniques required have in fact already been encountered in Stage 3. We first introduce a new pair of variables, f_0, f_1, which we refer to as 'block colours', and with which we label the arcs ζ_i in the fashion of Lemma 16, using the constraints:

$$\begin{array}{ll}
(a_0 + a_1 + a_2) \leq (f_0 + f_1) & \\
\neg C(f_0 \cdot a_i, f_1 \cdot a_i), & (0 \leq i < 3).
\end{array}$$

We force all arcs in each block E_j to have a uniform block colour, and we force the block colours to alternate by writing, for $0 \leq h < 2$, $1 \leq j, j' \leq \ell$, $1 \leq k < u(j)$ and $0 \leq i < 3$:

$$\begin{array}{l}
\neg C(f_h \cdot t_{j,k}, f_{\lfloor h+1 \rfloor} \cdot t_{j,k+1}), \\
\neg C(f_h \cdot t_{j,u(j)} \cdot a_i, f_h \cdot t'_{j',1} \cdot a_{\lfloor i+1 \rfloor})
\end{array}$$

Thus, we may speak unambiguously of the colour (f_0 or f_1) of a block: if E_1 is coloured f_0, then E_2 will be coloured f_1, E_3 coloured f_0, and so on. Using the the *same* variables f_0 and f_1, we similarly establish a block structure $E'_1, \ldots, E'_{m'}$ on the arcs η'_i. (Note that there is no need for primed versions of f_0 and f_1.)

Now we can match up the blocks in a 1–1 fashion just as we matched up the individual arcs. Let $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}'_0$ and $\mathfrak{g}'_1$ be new 3-regions variables. We may assume that every arc ζ_i contains some point of $\dot{b}_{\lfloor i \rfloor,1}$. We wish to connect any such arc that starts a block E_h (i.e. any ζ_i labelled by $t_{j,1}$ for some j) to the top edge of the upper window, with the connecting arc depending on the block colour. Setting $w_k = -(f_k \cdot \sum_{i=1}^{i=\ell} t_{j,1})$ $(0 \leq k < 2)$, we can do this using the constraints:

$$\mathsf{stack}_{w_k}(\mathfrak{b}_{i,1}, \mathfrak{g}_k, \mathfrak{a}) \qquad (1 \leq k < 2, 0 \leq i < 3).$$

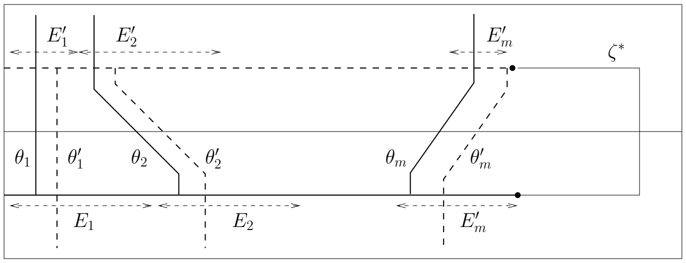


Figure 28: The 1–1 correspondence between the E_h and the E'_h established by the θ_i and the θ'_i.

Specifically, the first arc in each block E_h ($1 \leq h \leq m$) is connected by an arc $\theta_h\tilde{\theta}_h$ to some point on the upper edge of the upper window, where $\theta_h \subseteq b_{i,1} + g_i$ and $\tilde{\theta}_h \subseteq a$. Similarly, setting $w'_k = -(f_k \cdot \sum_{i=1}^{i=\ell} t'_{j,1})$ ($0 \leq k < 2$), the constraints

$$\mathsf{stack}_{w'_k}(\mathfrak{b}'_{i,1}, \mathfrak{g}'_k, \mathfrak{b}) \qquad (1 \leq k < 2, 0 \leq i < 3)$$

ensure that the first arc in each block $E_{h'}$ ($1 \leq h' \leq m'$) is connected by an arc $\theta'_{h'}\tilde{\theta}'_{h'}$ to some point on the bottom edge of the lower window, where $\theta_{h'} \subseteq b'_{i,1} + g'_i$ and $\tilde{\theta}'_{h'} \subseteq b$. Furthermore, from the arrangement of the ζ_i, ζ'_i and ζ^* (Fig. 26) we can easily write non-contact constraints forcing each θ_h to intersect one of the arcs ζ'_i ($1 \leq i \leq n$), and each θ'_h to intersect one of the arcs $\zeta_{i'}$ ($1 \leq i' \leq n$).

We now write the constraints

$$\neg C(g_k, f_{1-k}) \wedge \neg C(g'_k, f_{1-k}) \qquad (0 \leq k < 2).$$

Thus, any θ_h included in g_k must join some arc ζ_i in a block with colour f_k to some arc $\zeta'_{i'}$ also in a block with colour f_k; and similarly for the θ'_h. Adding

$$\neg C(g_0 + g'_0, g_1 + g'_1)$$

then ensures, via reasoning exactly similar to that employed in Stage 3, that θ_1 connects the block E_1 to the block E'_1, θ_2 connects E_2 to E'_2, and so on; and similarly for the θ'_h (as shown, schematically, in Fig. 28). Thus, we have a 1–1 correspondence between the two sets of blocks, whence $m = m'$.

Finally, we let $d_1, \ldots, d_\ell$ be new regions variables labelling the components of g_0 and of g_1, and hence the arcs $\theta_1, \ldots, \theta_m$:

$$g_i \leq \sum_{1 \leq j \leq \ell} d_j \wedge \bigwedge_{1 \leq j \leq \ell} C(d_j \cdot g_i, (-d_j) \cdot g_i)$$

for $0 \leq i < 2$. Adding the constraints

$$\neg C(p_{j,k}, d_{j'}) \qquad (j \neq j')$$
$$\neg C(p'_{j,k}, d_{j'}) \qquad (j \neq j')$$

where $1 \leq j \leq \ell$, $1 \leq k \leq u(j)$ and $1 \leq j' \leq \ell$, instantly ensures that the sequences of tile indices $j(1), \ldots, j(m)$ and $j'(1), \ldots, j'(m)$ are identical. In other words, $\tau = \tau'$. This completes the proof that $\mathbf{w}$ is a positive instance of the PCP.

We have established the r.e.-hardness of *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$ and *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2))$. We must now extend these results to the other languages considered here. We deal with the languages $\mathcal{C}c^\circ$ and $\mathcal{B}c$ as in Sec. 3. Let $\varphi^\circ_{\mathbf{w}}$ be the $\mathcal{C}c^\circ$ formula obtained by replacing all of occurrences of c in $\varphi_{\mathbf{w}}$ with c°. Since all occurrences of c in $\varphi_{\mathbf{w}}$ are positive, $\varphi^\circ_{\mathbf{w}}$ entails $\varphi_{\mathbf{w}}$. On the other hand, the connected regions satisfying $\varphi_{\mathbf{w}}$ are also interior-connected, and thus satisfy $\varphi^\circ_{\mathbf{w}}$ as well.

For the language $\mathcal{B}c$, observe that, as in Sec. 3, all conjuncts of $\varphi_{\mathbf{w}}$ featuring the predicate C are *negative*. (Remember that there are additional such literals implicit in the use of 3-region variables; but let us ignore these for the moment.) Recall from Sec. A that

$$\varphi^c_{\neg C}(r, s, r', s') := c(r + r') \wedge c(s + s') \wedge \neg c((r + r') + (s + s')),$$

and consider the effect of replacing any literal $\neg C(r, s)$ from (19) with the $\mathcal{B}c$-formula $\varphi^c_{\neg C}(r, s, r', s')$ where r' and s' are fresh variables, and let the formula obtained be ψ. It is easy to see that ψ entails $\varphi_{\mathbf{w}}$; hence if ψ is satisfiable, then $\mathbf{w}$ is a positive instance of the PCP. To see that ψ is satisfiable, consider the satisfying tuple of $\varphi_{\mathbf{w}}$. Note that if $\mathfrak{r}$ and $\mathfrak{s}$ are 3-regions whose outer-most elements r and s are disjoint (for example: $\mathfrak{r} = \mathfrak{a}_{0,1}$, $\mathfrak{s} = \mathfrak{a}_{0,3}$), then r and s have finitely many connected components and have connected complements. Hence, it is easy to find r' and s' in $\mathsf{RCP}(\mathbb{R}^2)$ satisfying the corresponding formula $\varphi^c_{\neg C}(r, s, r', s')$. Fig. 29 represents the situation in full generality. (As usual, we assume a spherical canvas, with the point at infinity not lying on the boundary of any of the depicted regions.) We may therefore assume, that all such literals involving C have been eliminated from $\varphi_{\mathbf{w}}$.

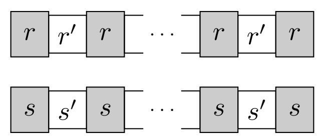
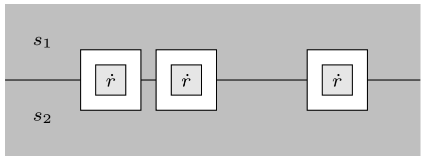


Figure 29: Satisfying $\varphi^c_{\neg C}(r, s, r', s')$

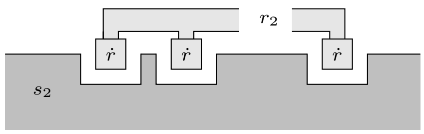
We are not quite done, however. We must show that we can replace the *implicit* non-contact constraints that come with the use of 3-region variables by suitable $\mathcal{B}c$-formulas. For example, a 3-region variable $\mathfrak{r}$ involves the implicit constraints $\neg C(\ddot{r}, -\dot{r})$ and $\neg C(\dot{r}, -r)$. Since the two conjuncts are identical in form, we only show how to deal with $\neg C(\dot{r}, -r)$. Because the complement of $-r$ is in general not connected, a direct use of $\varphi^c_{\neg C}$ will result in a formula which is not satisfiable. Instead, we represent $-r$ as the sum of two regions s_1 and s_2 with connected complements, and then proceed as before. In particular, we replace $\neg C(\dot{r}, -r)$ by:

$$-r = s_1 + s_2 \wedge \varphi^c_{\neg C}(\dot{r}, s_1, r_1, s_1) \wedge \varphi^c_{\neg C}(\dot{r}, s_2, r_2, s_2).$$

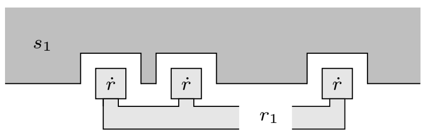
For $i = 1, 2$, $\dot{r} + r_i$ is a connected region that is disjoint from s_i. So, $\dot{r}$ is disjoint from s_1 and s_2, and hence disjoint from their sum $-r := s_1 + s_2$. Fig 30 shows regions s_i, r_i, for $i = 1, 2$, which satisfy the above formula. Let $\psi_{\mathbf{w}}$ be the result of replacing all the conjuncts (explicit or implicit) containing the predicate C, as just described. We have thus shown that, if $\psi_{\mathbf{w}}$ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then $\mathbf{w}$ is positive, and that,



(a) The region $-r$ is the sum of s_1 and s_2.



(b) The mutually disjoint connected regions $\dot r + r_2$ and s_2.



(c) The mutually disjoint connected regions $\dot r + r_1$ and s_1.

Figure 30: Eliminating the conjuncts of the form $\neg C(-r, \dot r)$.

if $\mathbf{w}$ is positive, then $\psi_{\mathbf{w}}$ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. This completes the proof.

The final case we must deal with is that of $\mathcal{B}c^\circ$. We use the r.e.-hardness results already established for $\mathcal{C}c^\circ$, and proceed, as before, to eliminate occurrences of C. Since all the polygons in the tuple satisfying $\varphi^\circ_{\mathbf{w}}$ are quasi-bounded, we can eliminate all occurrences of C from $\varphi^\circ_{\mathbf{w}}$ using Lemma 12 (*iii*). This completes the proof of Theorem 6.

C $\mathcal{B}c^\circ$ in 3D

Denote by ConRC the class of all connected topological spaces with regular closed regions. As shown in [Kontchakov *et al.*, 2010b], every $\mathcal{B}c^\circ$-formula satisfiable over ConRC can be satisfied in a finite connected quasi-saw model and the problem *Sat*$(\mathcal{B}c^\circ, \mathsf{ConRC})$ is NP-complete.

Theorem 17 *The problems Sat*$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n))$, $n \geq 3$, *coincide with Sat*$(\mathcal{B}c^\circ, \mathsf{ConRC})$, *and so are all* NP-*complete.*

Proof. It suffices to show that every $\mathcal{B}c^\circ$-formula φ satisfiable over connected quasi-saws can also be satisfied over any of $\mathsf{RC}(\mathbb{R}^n)$, for $n \geq 3$. So suppose that φ is satisfied in a model $\mathfrak{A}$ based on a finite connected quasi-saw (W, R). Denote by W_i the set of points of depth i in (W, R), for $i = 0, 1$. Without loss of generality we may assume that there exists a point $z_0 \in W_1$ with $z_0 R x$ for all $x \in W_0$. Indeed, if this is not the case, take the interpretation $\mathfrak{B}$ obtained by extending $\mathfrak{A}$ with such a point z_0 and setting $z_0 \in r^{\mathfrak{B}}$ iff $x \in r^{\mathfrak{A}}$ for some $x \in W_0$. Clearly, we have $\mathfrak{A} \models (\tau = \tau')$ iff $\mathfrak{B} \models (\tau = \tau')$, for any terms τ, τ'. To see that $\mathfrak{A} \models c^\circ(\tau)$ iff $\mathfrak{B} \models c^\circ(\tau)$, recall that (W, R) is connected, and so τ° is disconnected in $\mathfrak{A}$ iff there are two distinct points $x, y \in \tau^{\mathfrak{A}} \cap W_0$ connected by at least one path in (W, R) and such that no such path lies entirely in $(\tau^{\mathfrak{A}})^\circ$. It follows that if $(\tau^{\mathfrak{A}})^\circ$ is disconnected then $W_0 \setminus \tau^{\mathfrak{A}} \neq \emptyset$, and so $z_0 \notin (\tau^{\mathfrak{B}})^\circ$. Thus, by adding z_0 to (W, R) we cannot make a disconnected open set in $\mathfrak{A}$ connected in $\mathfrak{B}$.

We show now how $\mathfrak{A}$ can be embedded into $\mathbb{R}^n$, for any $n \geq 3$. First we take pairwise disjoint *closed* balls B^1_x for all $x \in W_0$. We also select pairwise disjoint *open* balls D_z for $z \in W_1 \setminus \{z_0\}$, which are disjoint from all of the B^1_x, and take D_{z_0} to be the complement of

$$\bigcup_{x \in W_0} (B^1_x)^\circ \ \cup \bigcup_{z \in W_1 \setminus \{z_0\}} D_z.$$

(Note that ${D_z}^\circ$ is connected for each $z \in W_1$; all D_z, for $z \in W_1 \setminus \{z_0\}$, are open, while D_{z_0} is closed). We then expand every B^1_x to a set B_x in such a way that the following two properties are satisfied:

(A) the B_x, for $x \in W_0$, form a *connected partition in* $\mathsf{RC}(\mathbb{R}^n)$ in the sense that the B_x are regular closed sets in $\mathbb{R}^n$, whose interiors are non-empty, connected and pairwise disjoint, and which sum up to the entire space;

(B) every point in D_z, $z \in W_1$, is either
  - in the interior of some B_x with zRx, or
  - on the boundary of *all* of the B_x for which zRx.

The required sets B_x are constructed as follows. Let $q_1, q_2, \dots$ be an enumeration of all the points in $\bigcup_{z \in W_1} {D_z}^\circ$ with *rational* coordinates. For $x \in W_0$, we set B_x to be the closure of the infinite union $\bigcup_{k \in \omega} (B^k_x)^\circ$, where the regular closed sets B^k_x are defined inductively as follows (see Fig. 31):

- Assuming that the B^k_x are already defined, let q_i be the first point in the list $q_1, q_2, \dots$ such that $q_i \notin B^k_x$, for all $x \in W_0$. Suppose $q_i \in {D_z}^\circ$ for $z \in W_1$. Take an open ball $C_{q_i} \subsetneq {D_z}^\circ$ of radius $< 1/k$ centred in q_i and disjoint from the B^k_x. For each $x \in W_0$ with zRx, expand B^k_x by a closed ball in C_{q_i} and a closed rod connecting it to B^1_x in such a way that the ball and the rod are disjoint from the rest of the B^k_x. The resulting set is denoted by B^{k+1}_x.

Let $\mathsf{RC}(W, R)$ be the Boolean algebra of regular closed sets in (W, R) and let $\mathsf{RC}(\mathbb{R}^n)$ be the Boolean algebra of regular closed sets in $\mathbb{R}^n$. Define a map f from $\mathsf{RC}(W, R)$ to $\mathsf{RC}(\mathbb{R}^n)$ by taking

$$f(X) \ = \bigcup_{x \in X \cap W_0} B_x, \qquad \text{for } X \in \mathsf{RC}(W, R).$$

By (A), f is an isomorphic embedding of $\mathsf{RC}(W, R)$ into $\mathsf{RC}(\mathbb{R}^n)$, that is, f preserves the operations $+, \cdot$ and $-$ and the constants 0 and 1. Define an interpretation $\mathfrak{I}$ over $\mathsf{RC}(\mathbb{R}^n)$ by taking $r^{\mathfrak{I}} = f(r^{\mathfrak{A}})$. To show that $\mathfrak{I} \models \varphi$, it remains to prove that, for every $X \in \mathsf{RC}(W, R)$, X° is connected if, and only if, $(f(X))^\circ$ is connected. This equivalence follows from the fact that

$$(f(X))^\circ \ = \bigcup_{x \in X \cap W_0} B^\circ_x \ \cup \bigcup_{z \in X \cap W_1,\ V_z \subseteq X} D_z,$$

where $V_z \subseteq W_0$ is the set of all R-successors of z of depth 0, which in turn is an immediate consequence of (B). ❑

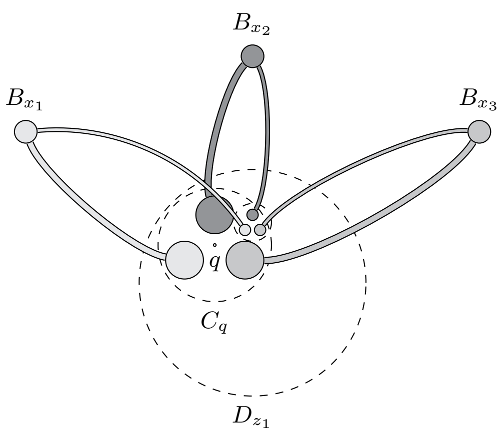


Figure 31: The first two stages of filling D_{z_1} with B_{x_i}, for $z_1 R x_i$, $i = 1, 2, 3$. (In $\mathbb{R}^3$, the sets B_{x_1} and B_{x_2} would not intersect.)