3.3 Histogram Processing *(page 142)*

Histogram

$$h(r_k) = n_k$$

- r_k: kth gray level
- n_k: number of pixels of gray level r_k

Normalization $\Rightarrow$ Discrete PDF

$$p(r_k) = n_k/MN$$

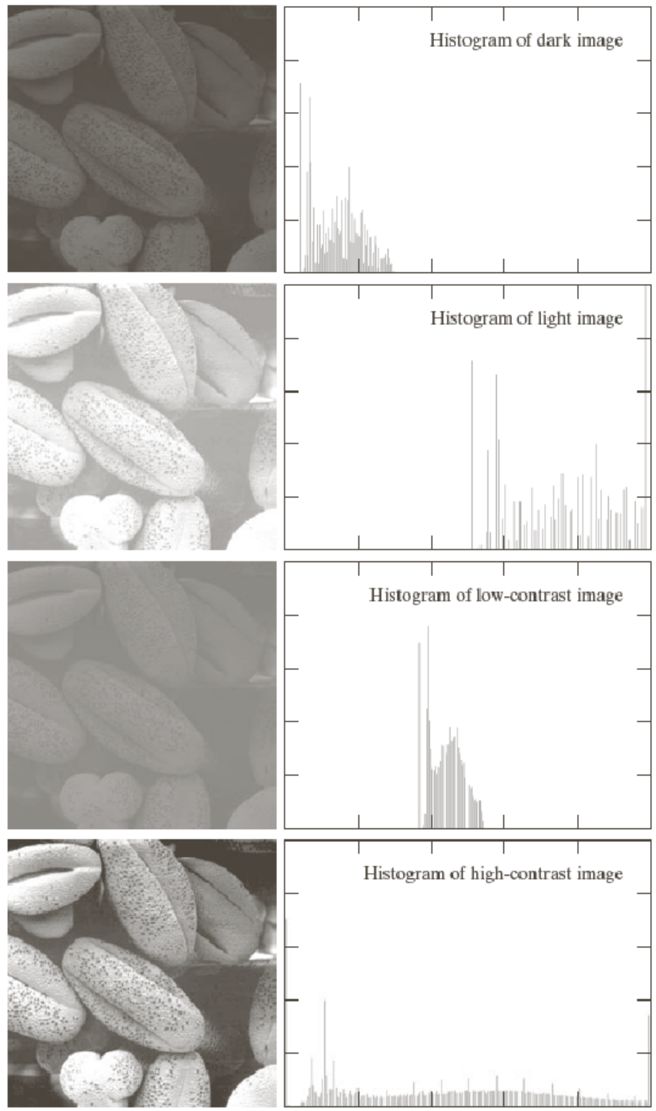
- MN: total number of pixels

$$\sum_{k=0}^{L-1} p(r_k) = 1$$

Histogram equalization: 3.3.1

Histogram specification: 3.3.2 *(Development of method not discussed)*

Examples


Histogram of dark image
Histogram of light image
Histogram of low-contrast image
Histogram of high-contrast image

3.3.1 Histogram Equalization

First consider continuous functions and transformations of the form

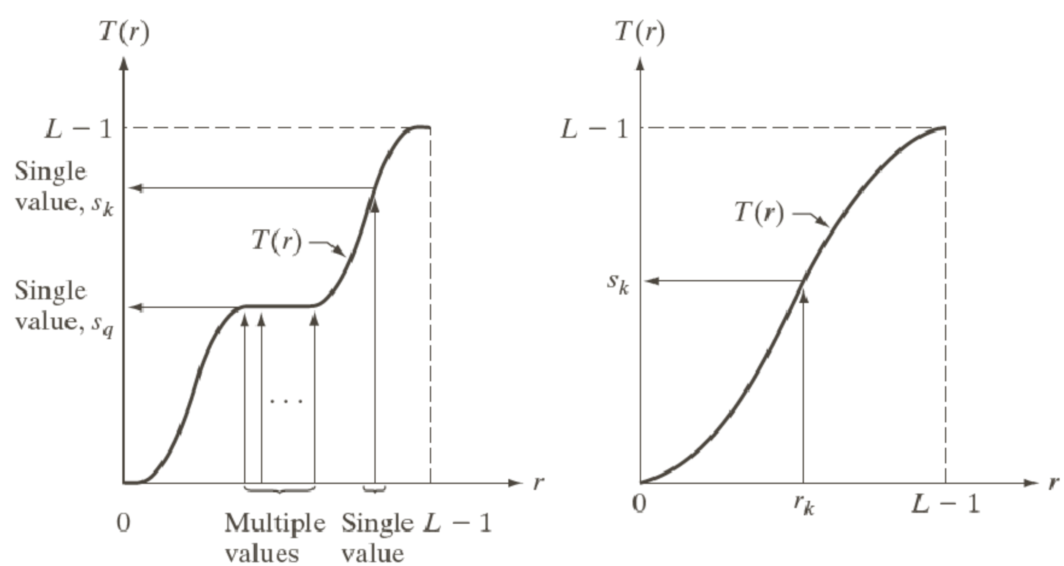
$$s = T(r), \quad r \in [0, L-1]$$

and assume that

(a) $T(r)$ **monotonically increasing for** $r \in [0, L-1]$

(Only requirement for histogram equilization)

(b) $T(r) \in [0, L-1]$ **for** $r \in [0, L-1]$



a b

FIGURE 3.17 (a) Monotonically increasing function, showing how multiple values can map to a single value. (b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.

View gray levels as random variables

- $p_r(r)$: **continuous PDF of** r
- $p_s(s)$: **continuous PDF of** s

If $T(r)$ **continuous and differentiable then** $p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$

Consider the transformation function $s = T(r) = (L-1)\int_0^r p_r(w)\,dw$

RHS is the cumulative distribution function (CDF) of r**, and satisfies conditions (a) and (b). From Leibniz's rule...**

$$\begin{aligned}\frac{ds}{dr} &= \frac{dT(r)}{dr}\\ &= (L-1)\,\frac{d}{dr}\left\{\int_0^r p_r(w)\,dw\right\}\\ &= (L-1)\,p_r(r)\end{aligned}$$

$$\Rightarrow \quad \begin{aligned}p_s(s) &= p_r(r)\left|\frac{dr}{ds}\right|\\ &= p_r(r)\left|\frac{1}{(L-1)\,p_r(r)}\right|\\ &= \frac{1}{L-1}, \quad s\in[0, L-1]\end{aligned}$$

For $T(r) = (L-1)\int_0^r p_r(w)\,dw$**,** $p_s(s)$ **is** **always** **uniform,** **independent** **of** $p_r(r)$

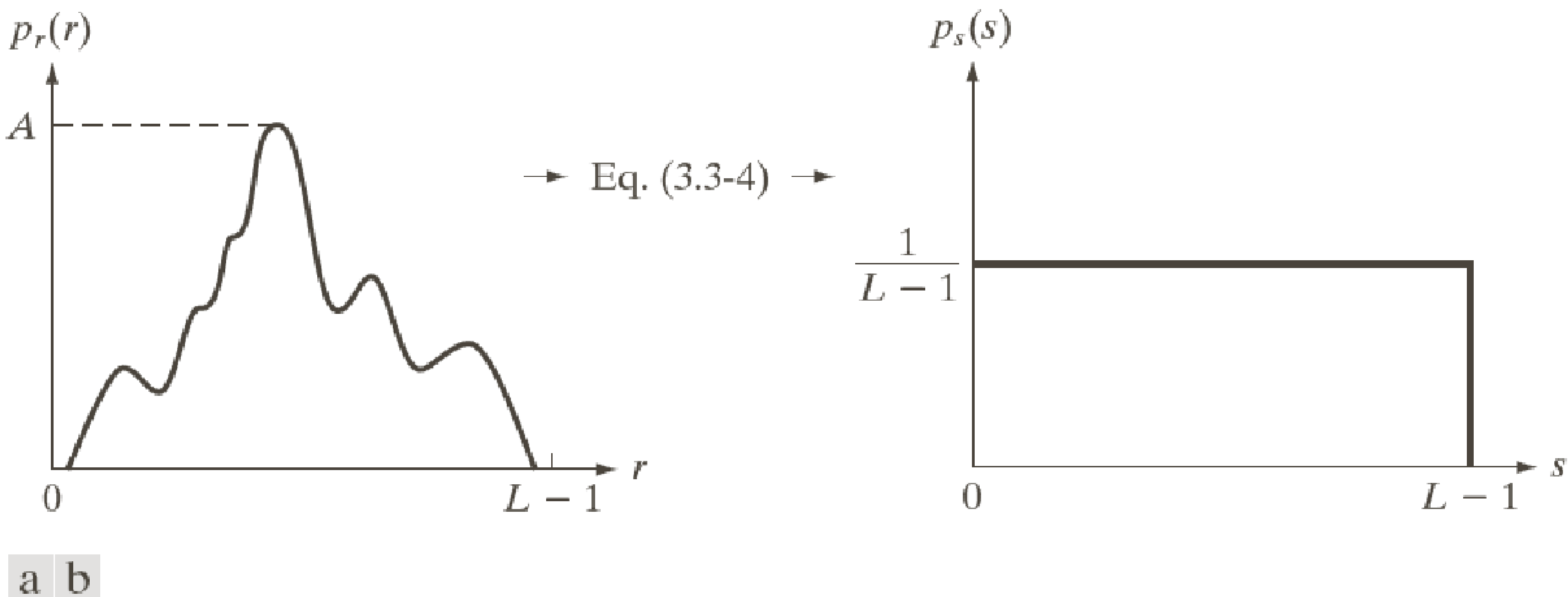


FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

Example 3.4

$$p_r(r) = \begin{cases} \dfrac{2\,r}{(L-1)^2}, & r \in [0, L-1] \\ 0, & \textbf{otherwise} \end{cases}$$

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw = \frac{2}{L-1}\int_0^r w\,dw = \frac{r^2}{L-1}$$

Note: If $L = 10$**, then** $T(3) = 3^2/9 = 1$**.**

$$\begin{aligned} p_s(s) &= p_r(r)\left|\frac{dr}{ds}\right| \\ &= \frac{2\,r}{(L-1)^2}\left|\left(\frac{ds}{dr}\right)^{-1}\right| \\ &= \frac{2\,r}{(L-1)^2}\left|\left(\frac{d}{dr}\frac{r^2}{L-1}\right)^{-1}\right| \\ &= \frac{2\,r}{(L-1)^2}\left|\frac{(L-1)}{2\,r}\right| = \frac{1}{L-1} \end{aligned}$$

Now consider discrete values...

Recall

$$p(r_k) = \frac{n_k}{MN}, \quad k = 0, 1, 2, \ldots, L-1$$

The discrete version of $s = T(r) = (L-1)\int_0^r p_r(w)\,dw$ **is**

$$\begin{aligned} s_k &= T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) \\ &= \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j, \quad k = 0, 1, 2, \ldots, L-1 \end{aligned}$$

and is called histogram equalization

NB: This will not produce a uniform histogram, but will tend to spread out the histogram of the input image

Advantages:

- **Gray-level values cover entire scale (contrast enhancement)**
- **Fully automatic**

Example 3.5

Consider 3-bit image ($L = 8$) of size 64×64 pixels ($MN = 4096$)

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

TABLE 3.1 Intensity distribution and histogram values for a 3-bit, 64×64 digital image.

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33$$

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

Rounding to nearest integer:

$$\begin{aligned} s_0 &= 1.33 \to 1 \\ s_1 &= 3.08 \to 3 \\ s_2 &= 4.55 \to 5 \\ s_3 &= 5.67 \to 6 \end{aligned} \qquad \begin{aligned} s_4 &= 6.23 \to 6 \\ s_5 &= 6.65 \to 7 \\ s_6 &= 6.86 \to 7 \\ s_7 &= 7.00 \to 7 \end{aligned}$$

$$p_s(s_0) = 0; \quad p_s(s_1) = \frac{790}{4096}; \quad p_s(s_2) = 0; \quad p_s(s_3) = \frac{1023}{4096}; \quad p_s(s_4) = 0;$$

$$p_s(s_5) = \frac{850}{4096}; \quad p_s(s_6) = \frac{656 + 329}{4096}; \quad p_s(s_7) = \frac{245 + 122 + 81}{4096}$$

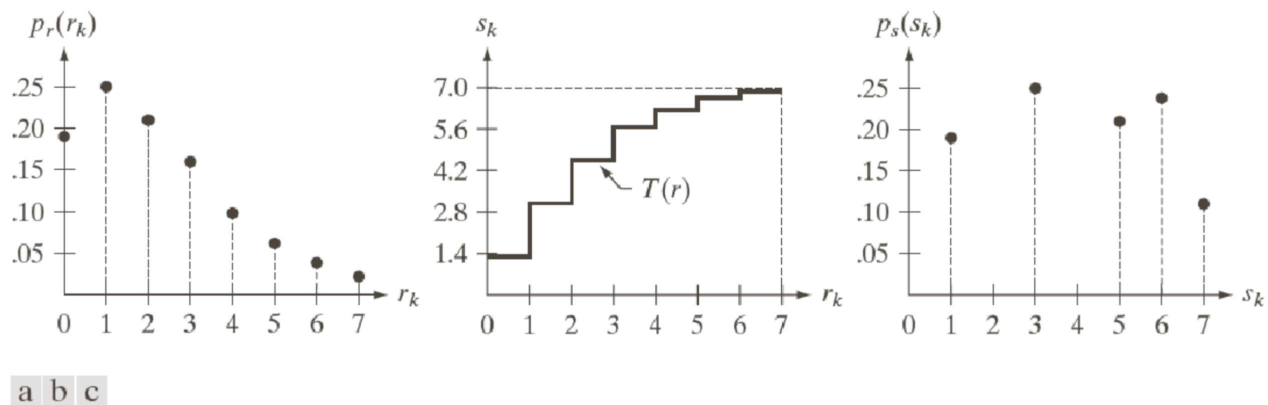
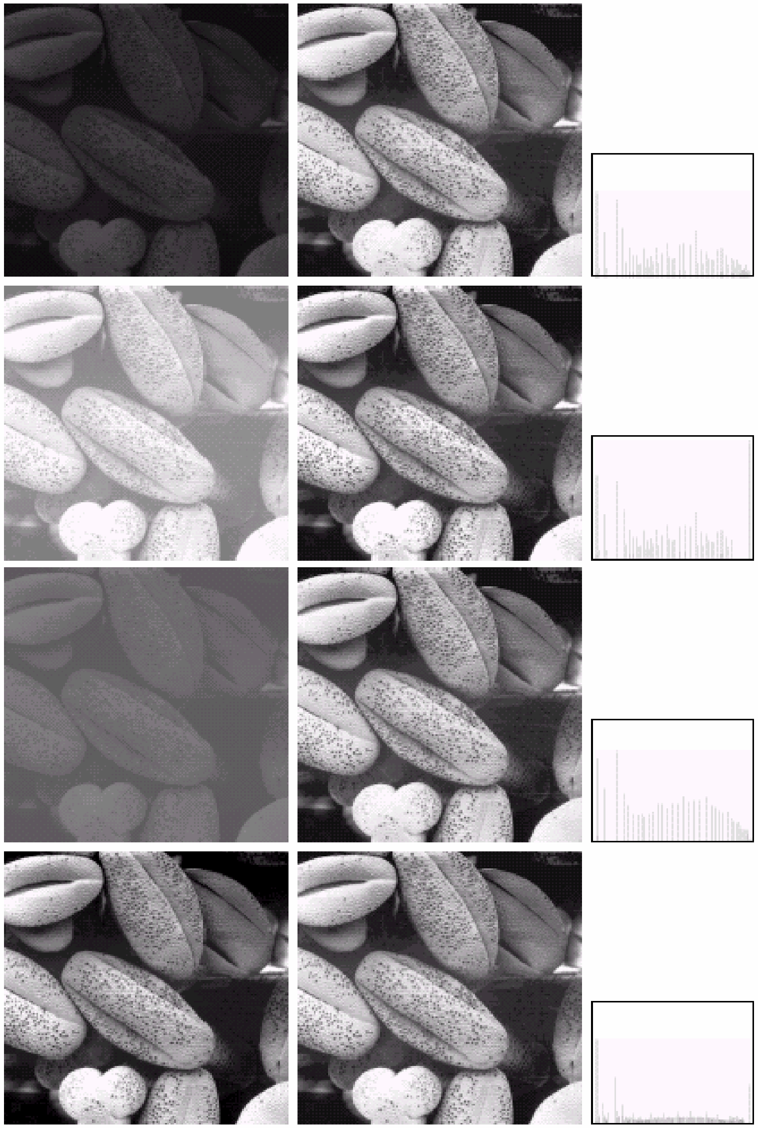
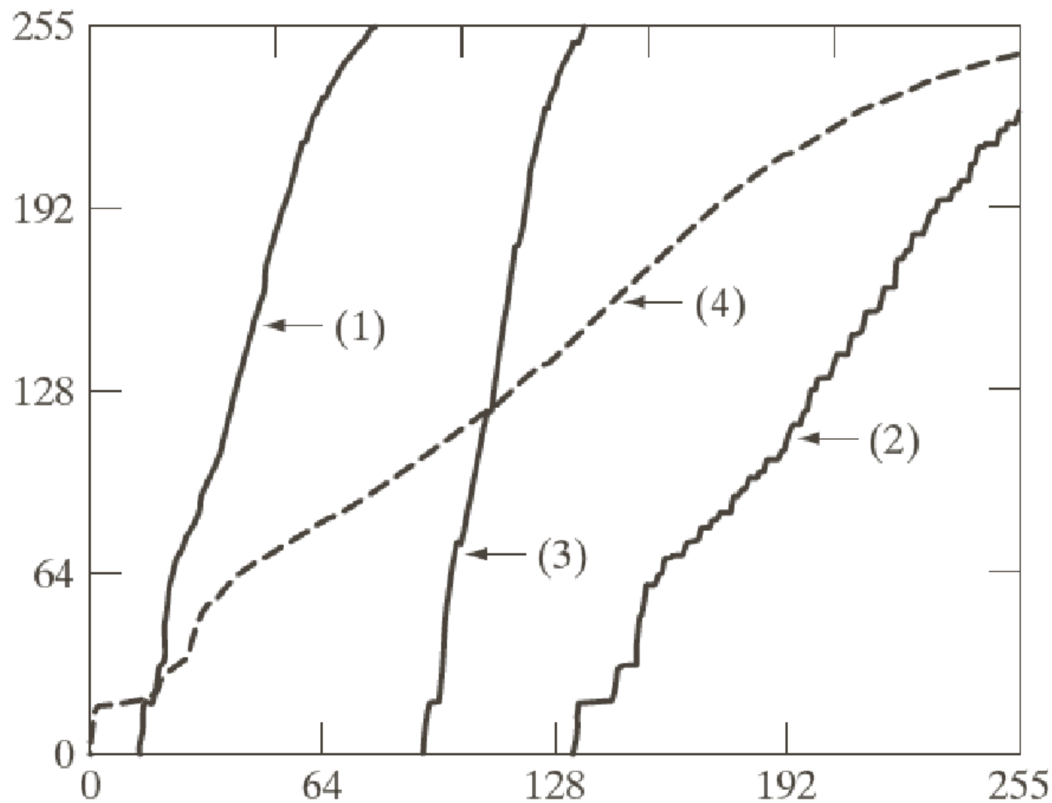


FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

Example 3.6: Histogram equalization



Example 3.6: Transformation functions

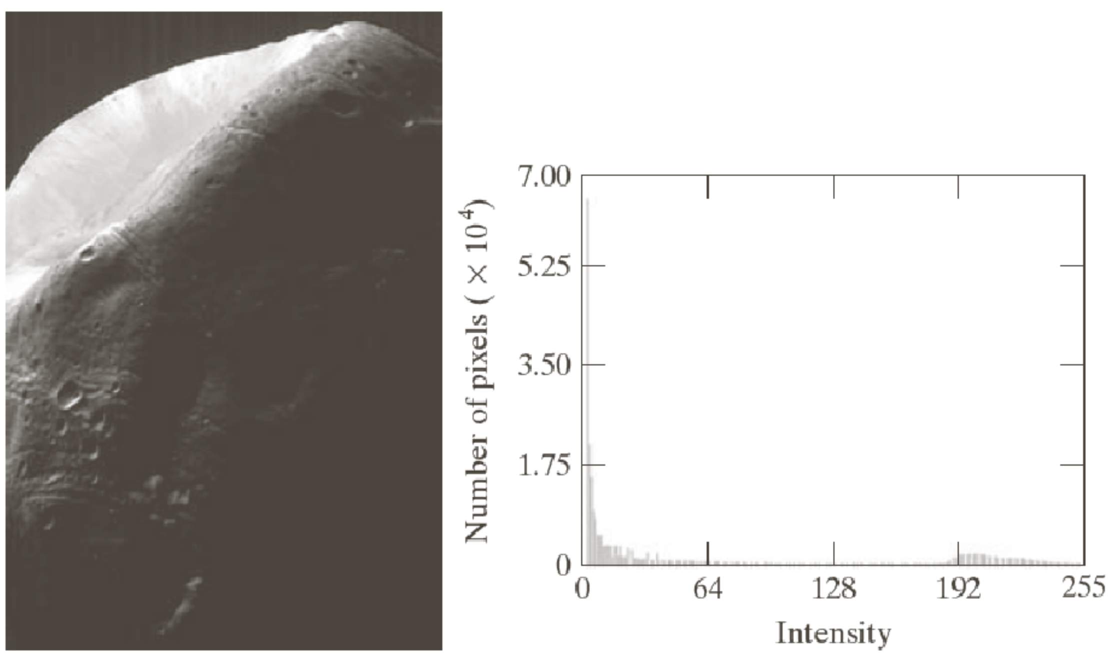


3.3.2 Histogram Matching (Specification)

- **Some applications: hist. equalization not best approach**
- **So, generate processed image with specified histogram**

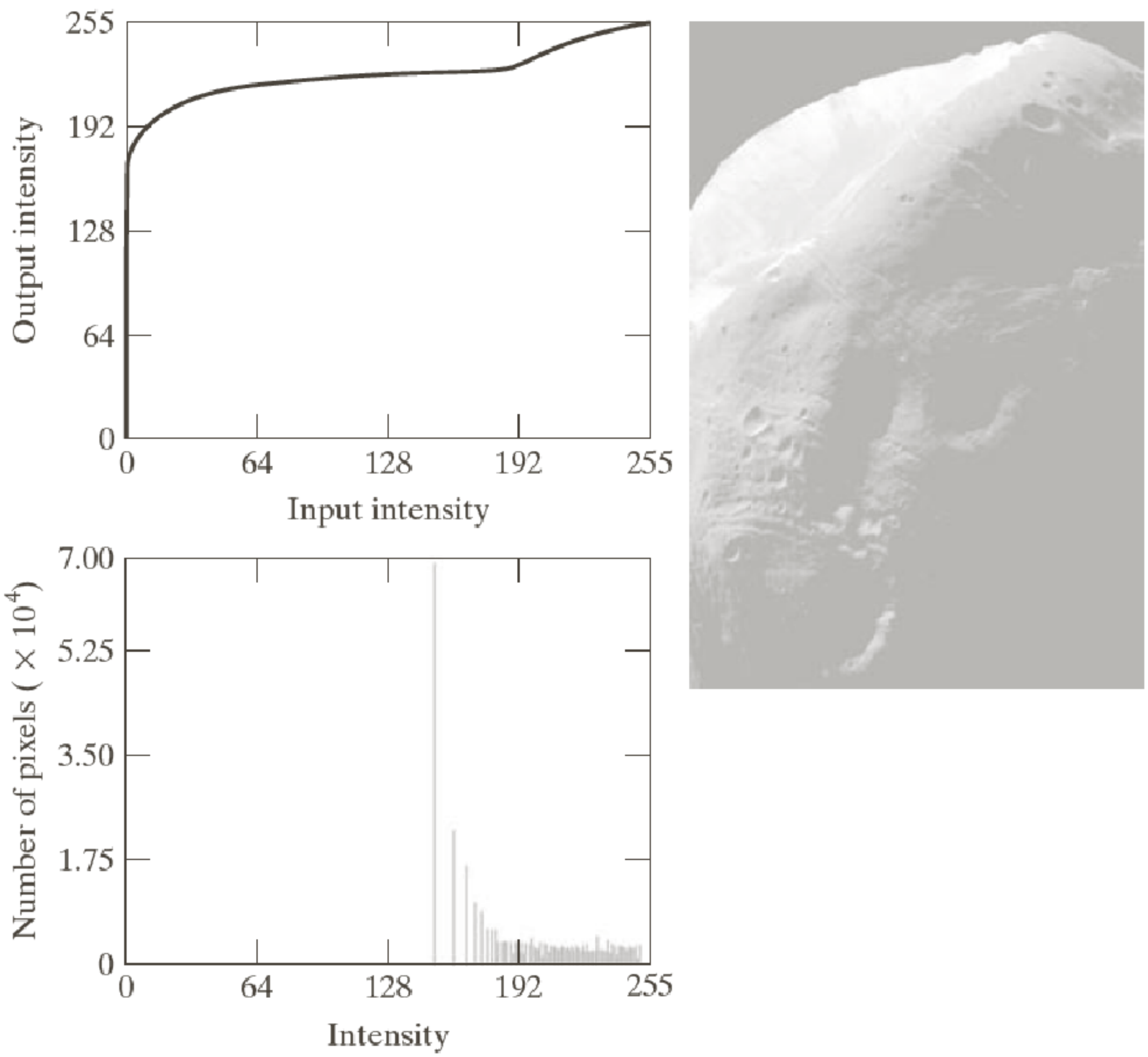
Development of the method: Not discussed

Example 3.9: Histogram specification



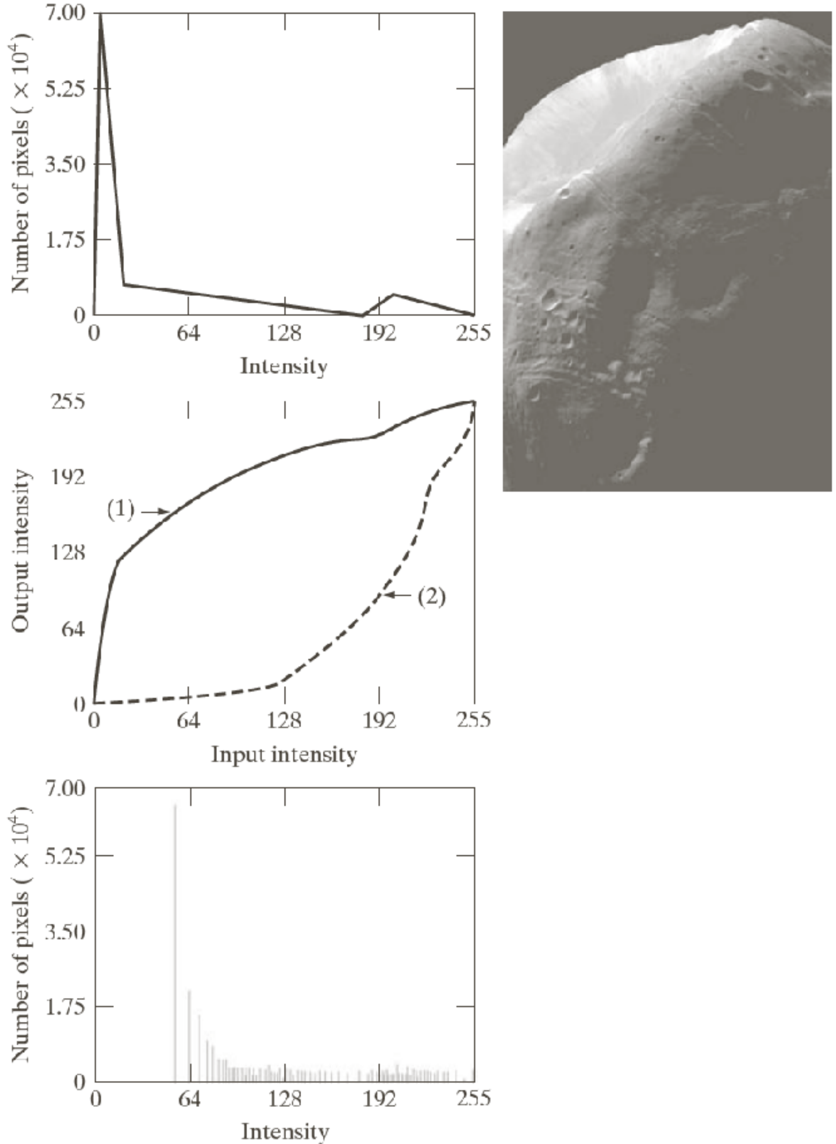
a b

FIGURE 3.23 (a) Image of the Mars moon Phobos taken by NASA's *Mars Global Surveyor.* (b) Histogram. (Original image courtesy of NASA.)



a b
c

FIGURE 3.24
(a) Transformation function for histogram equalization.
(b) Histogram-equalized image (note the washed-out appearance).
(c) Histogram of (b).



a c
b
d

FIGURE 3.25
(a) Specified histogram.
(b) Transformations.
(c) Enhanced image using mappings from curve (2).
(d) Histogram of (c).

3.3.3 Local Enhancement (Previous methods (3.3.1 and 3.3.2) were global)

- Define square or rectangular neighbourhood (mask) and move the center from pixel to pixel
- For each neighbourhood...
  - Calculate histogram of the points in the neighbourhood
  - Obtain histogram equalization/specification function
  - Map gray level of pixel centered in neighbourhood
- Can use new pixel values and previous hist to calculate next hist

Example 3.10: Enhancement using local histograms

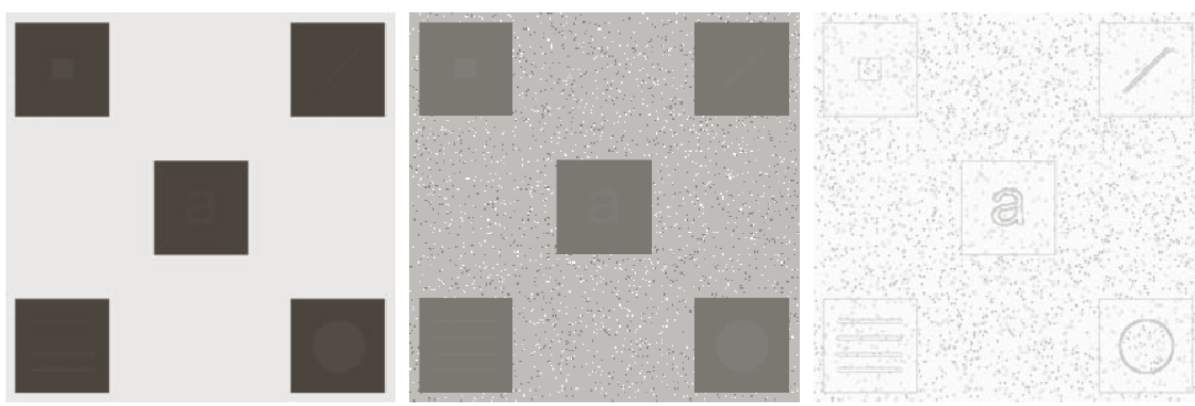


FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3×3.

3.3.4 Use of Histogram Statistics for Image Enhancement

With $p(r_i)$ a normalized histogram, the nth moment of r (discrete) about its mean is defined as

$$\mu_n(r) = \sum_{i=0}^{L-1} (r_i - m)^n p(r_i)$$

where m is the mean value of r:

$$m = \sum_{i=0}^{L-1} r_i p(r_i)$$

Note that $\mu_0 = 1$ and $\mu_1 = 0$, and that μ_2 is the variance $\sigma^2(r)$:

$$\mu_2(r) = \sum_{i=0}^{L-1} (r_i - m)^2 p(r_i)$$

Mean: measure of average gray level
Variance: measure of average contrast

Direct estimates from sample values $\Rightarrow$ sample mean and variance:

$$m = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$\sigma^2 = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} [f(x,y)-m]^2$$

Example 3.11: **Shows that m and σ^2 obtained from histogram and sample values are the same**

Local mean and variance: Let (x,y) be the coordinates of a pixel in an image and S_{xy} denote a subimage centered at (x,y), with histogram $p_{S_{xy}}$, then

$$m_{S_{xy}} = \sum_{i=0}^{L-1} r_i\, p_{S_{xy}}(r_i)$$

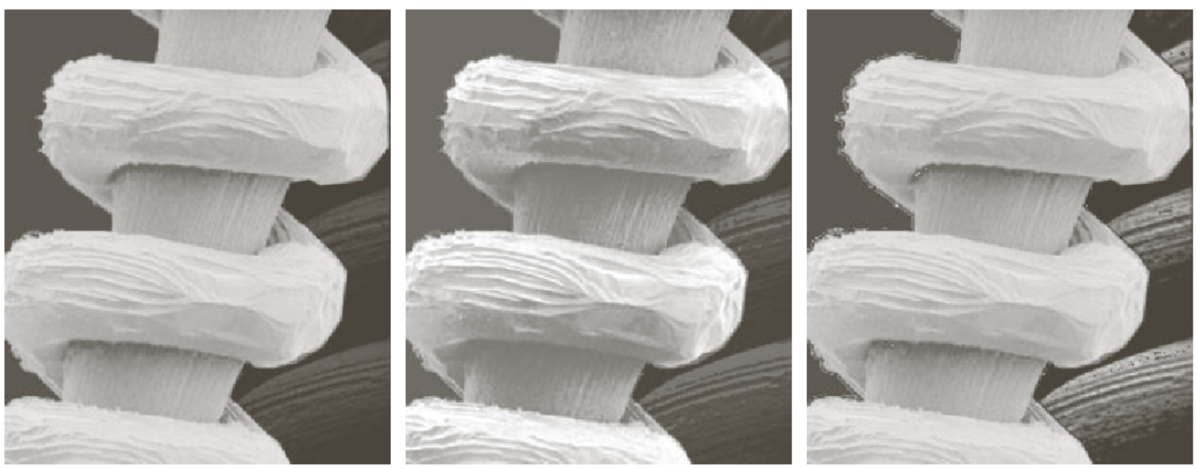
$$\sigma^2_{S_{xy}} = \sum_{i=0}^{L-1} \left\{r_i - m_{S_{xy}}\right\}^2 p_{S_{xy}}(r_i)$$

Example 3.12: Enhancement based on local statistics

$$g(x,y) = \begin{cases} E \cdot f(x,y) & \textbf{if } m_{S_{xy}} \in [0, k_0 m_G] \textbf{ AND } \sigma_{S_{xy}} \in [k_1\sigma_G, k_2\sigma_G] \\ f(x,y) & \textbf{otherwise} \end{cases}$$

m_G: **Global mean;** σ_G: **Global standard deviation**

$E = 4.0; \quad k_0 = 0.4; \quad k_1 = 0.02; \quad k_2 = 0.4; \quad (3 \times 3)$ **local region**



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)