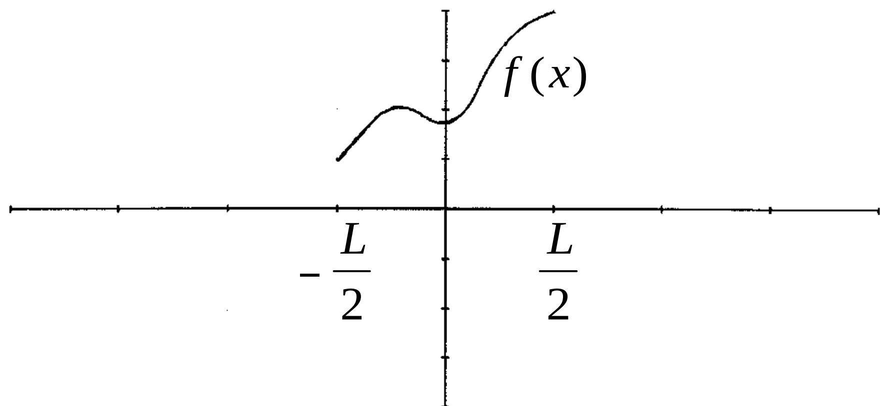
FOURIER SERIES

Consider a function, $f(x)$, defined on the interval $[-\frac{L}{2}, \frac{L}{2}]$:



We now want to approximate $f(x)$ on the interval $[-\frac{L}{2}, \frac{L}{2}]$ with a Fourier series, $\tilde{f}(x)$.

A Fourier series is a linear combination of a set of basis functions, $\{\ \phi_n(x) = e^{2\pi i n x/L},\ n \in \mathbb{Z}\,\}$, that is

$$\boxed{\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}}$$

where c_n are the Fourier coefficients.

These basis functions have two important properties. They are

(1) **PERIODIC with period L, that is $\phi_n(x+L) = \phi_n(x)$, and**

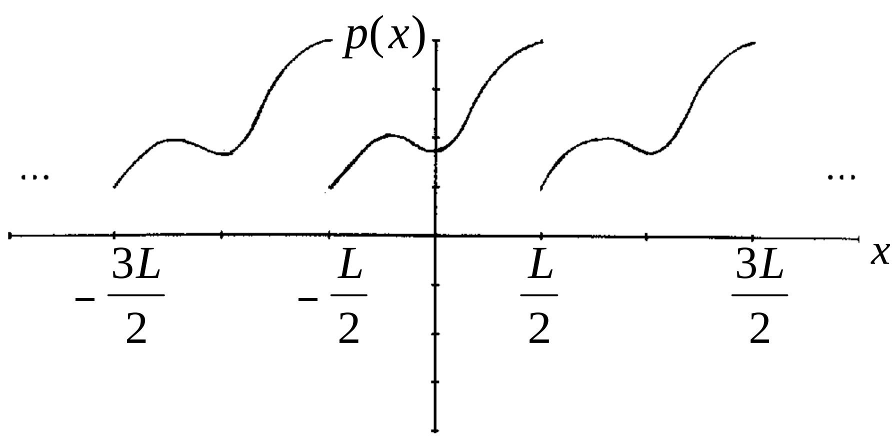
(verify)

(2) **ORTHOGONAL on the interval $[-\frac{L}{2}, \frac{L}{2}]$, that is**

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{2\pi i n x/L} e^{-2\pi i k x/L} dx = \begin{cases} 0, & \textbf{if } n \neq k \\ L, & \textbf{if } n = k \end{cases}$$

(verify)

Property (1) implies that $\tilde{f}(x+L) = \tilde{f}(x)$. Therefore $\tilde{f}(x)$ does not only approximate $f(x)$, but also the periodic continuation of $f(x)$ on the interval $[-\frac{L}{2}, \frac{L}{2}]$, that is $p(x)$:



Property (2) enables one to calculate the Fourier coefficients, c_n, easily and therefore also the Fourier series $\tilde{f}(x)$:

$$\begin{aligned}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-2\pi ikx/L}dx &= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{2\pi inx/L}e^{-2\pi ikx/L}dx \\ &= 0+0+\ldots+c_k\times L+\ldots+0+0\end{aligned}$$

Therefore...

$$\boxed{c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-2\pi inx/L}dx}$$

Example 1: **Find $\tilde{f}(x)$ when $f(x)=|x|$ on the interval $[-1,1]$**

$$\begin{aligned}c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-\pi inx}dx = \frac{1}{2}\int_{-1}^{1} f(x)\left[\cos(n\pi x) - i\sin(n\pi x)\right]dx \\ &= \frac{2}{2}\int_0^1 x\cos(n\pi x)dx = \left(\frac{x}{n\pi}\sin(n\pi x)\right)_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx \\ &= -\frac{1}{n\pi}\left(-\frac{1}{n\pi}\cos(n\pi x)\right)_0^1 = \frac{(-1)^n-1}{n^2\pi^2} = \begin{cases} 0, & \textbf{if } n \textbf{ even } (n\neq 0) \\ -\dfrac{2}{n^2\pi^2}, & \textbf{if } n \textbf{ odd}\end{cases}\end{aligned}$$

$$c_0 = \frac{1}{2}$$ **(verify)**

Therefore

$$\begin{aligned}\tilde{f}(x) &= \frac{1}{2} - \frac{2}{\pi^2} \sum_{\substack{n=-\infty \\ (n\ \textbf{odd})}}^{\infty} \frac{e^{in\pi x}}{n^2} \\ &= \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos(n\pi x)}{n^2}\end{aligned}$$

(verify)

Speed of convergence

How fast does the coefficients $|c_n|$ decrease when $n \to \infty$?

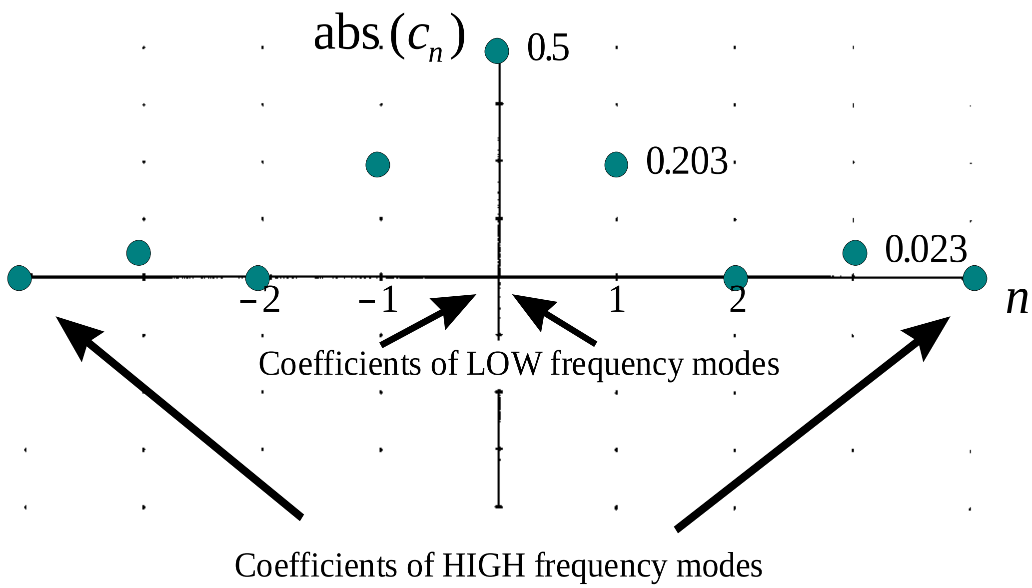
For Example 1 we have:

$$|c_n| = \frac{2}{n^2\pi^2}, \quad n\ \textbf{odd}$$

$$|c_0| = \frac{1}{2}$$

$$|c_n| = 0, \quad n\ \textbf{even}\ (n \neq 0)$$

Therefore the coefficients $|c_n|$ decrease like $\dfrac{1}{n^2}$



Example 2

Find $\tilde{f}(x)$ **when** $f(x) = x$ **on the interval** $[-1, 1]$

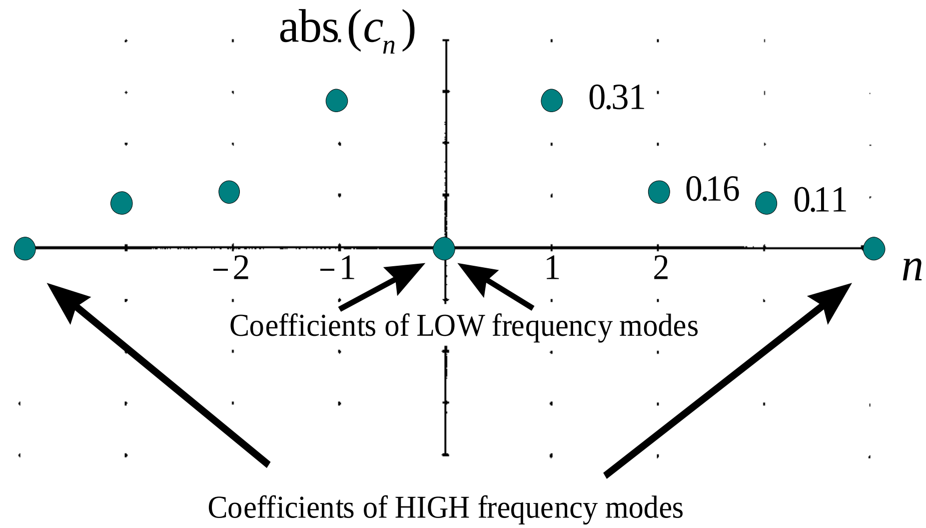
$$c_n = \frac{i(-1)^n}{n\pi}, \quad c_0 = 0$$

$$|c_n| = \frac{1}{n\pi}, \quad |c_0| = 0$$

Therefore the coefficients $|c_n|$ **decrease like** $\dfrac{1}{n}$ **and**

$$\tilde{f}(x) = \frac{i}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{in\pi x}}{n} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n}$$

(verify)



In general we have the following ...

When $p(x)$ is the periodic continuation of $f(x)$ on the interval $[-\frac{L}{2}, \frac{L}{2}]$, then

Periodic continuation	Fourier series
$p(x)$ discontinuous	Converges like $\dfrac{1}{n}$
$p(x)$ continuous $p'(x)$ discontinuous	Converges like $\dfrac{1}{n^2}$
$p(x)$ continuous $p'(x)$ continuous $p''(x)$ discontinuous	Converges like $\dfrac{1}{n^3}$

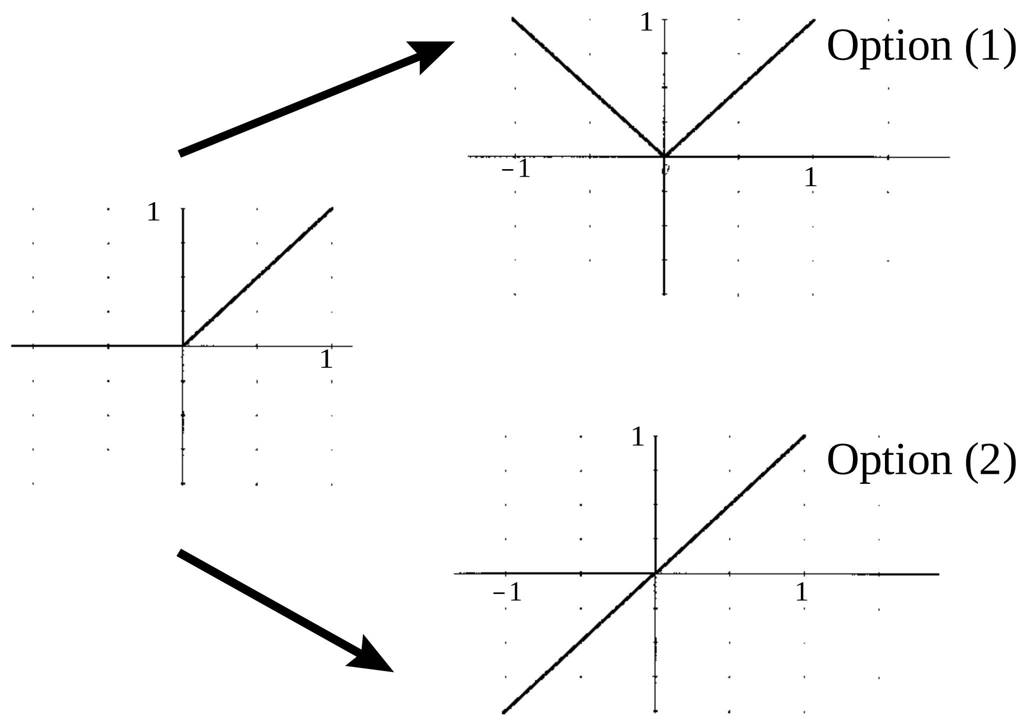
etc.

Therefore, the smoother the periodic continuation of $f(x)$ is, the faster its Fourier series will converge

This property is important when it comes to the compression of images (LATER)

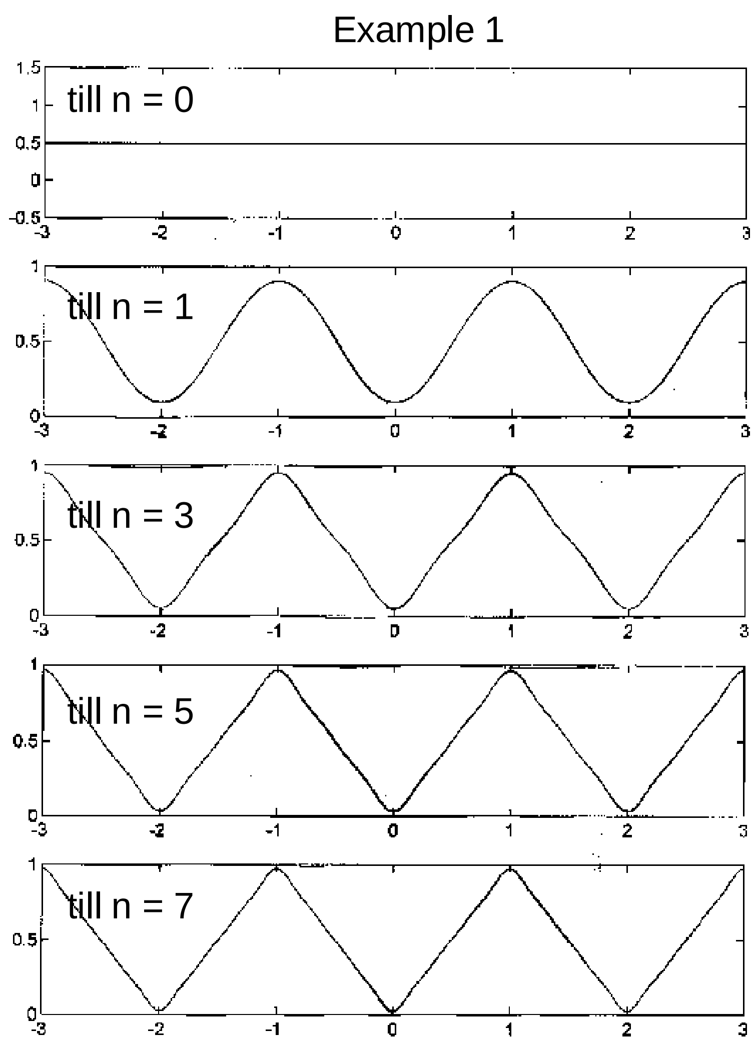
Example 3

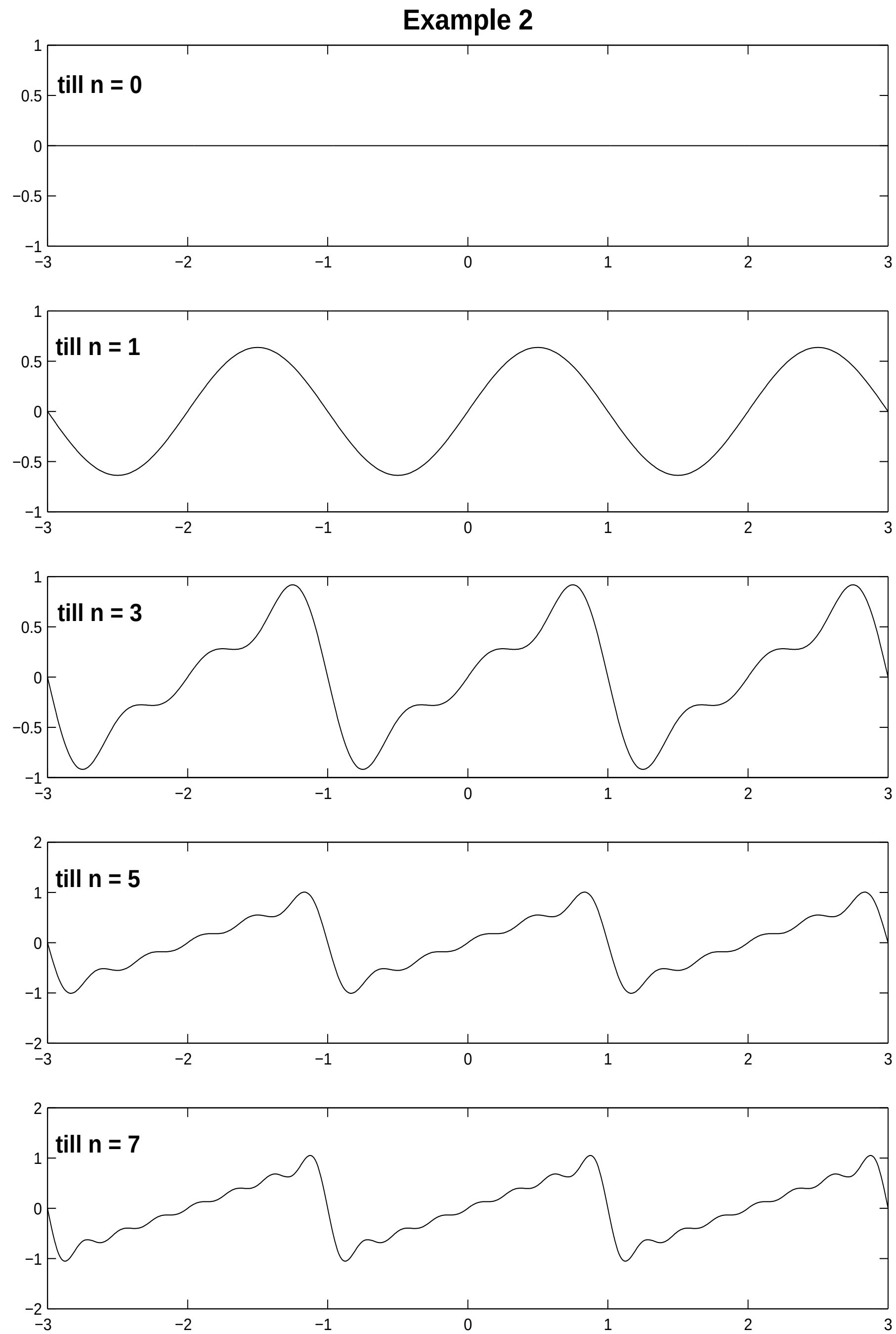
Find $\tilde{f}(x)$ when $f(x) = x,\ x \in [0,1]$

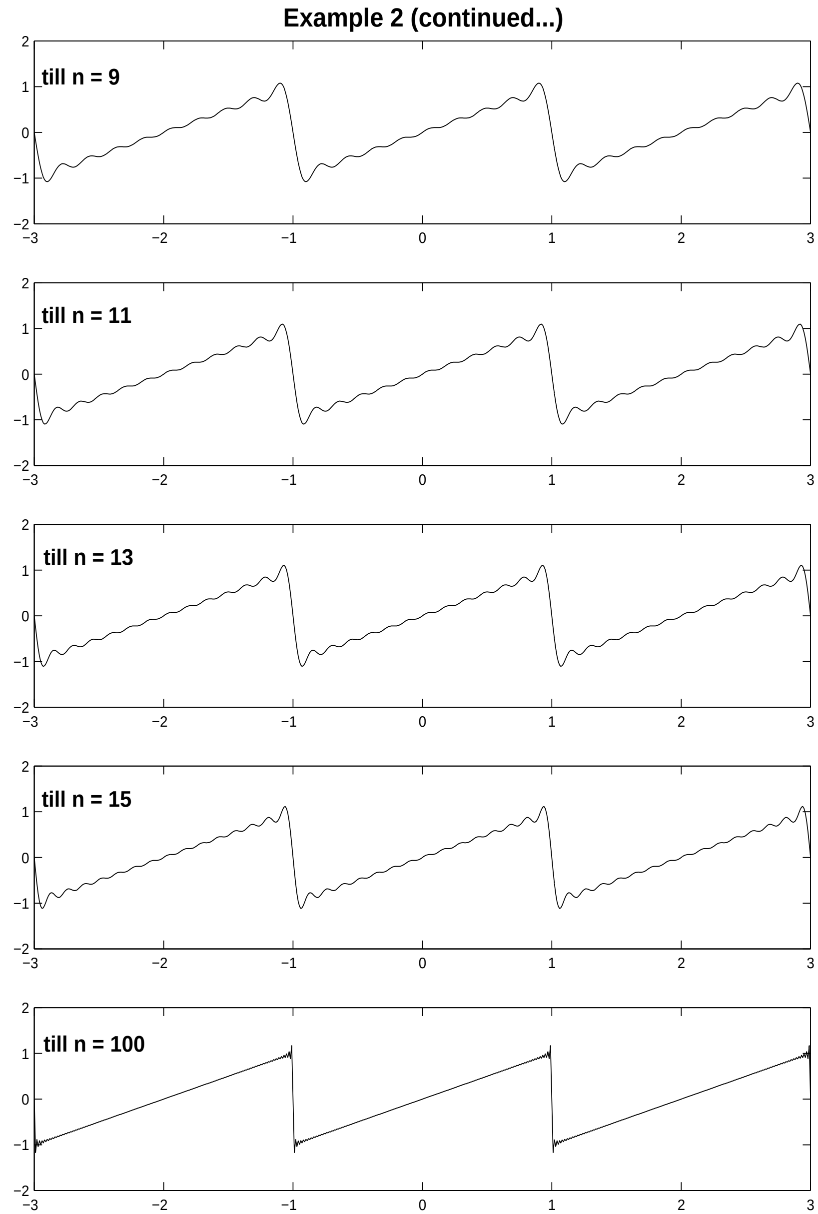


Why is option (1) better?

The difference in the speed of convergence of the Fourier series in Examples 1 and 2 is clear when the truncated Fourier series are plotted for different values of n




Example 2
till n = 0
till n = 1
till n = 3
till n = 5
till n = 7
1
0.5
0
−0.5
−1
2
−2
−3
3



The Gibbs phenomenon occurs in Example 2, since the periodic continuation has step discontinuities

We now set out to develop the Fourier transform (Fourier integral) of a function $f(x)$, since Fourier series have two important deficiencies:

(1) They utilize only periodic functions $\phi(x) = e^{2\pi inx/L}$ with frequencies of $\frac{n}{L}$, where $n \in \mathbb{Z}$, and no periodic functions with frequencies between these discrete values

(2) They can only approximate periodic functions, that is L has to be finite

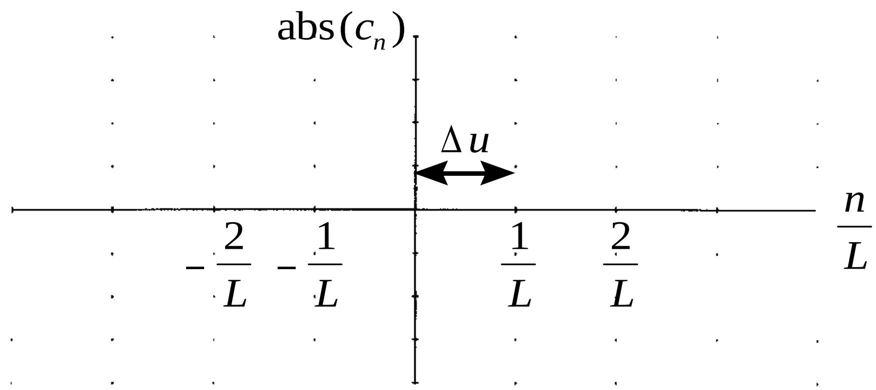
To go from a Fourier series to a Fourier transform, we need to let $L \to \infty$

THE FOURIER TRANSFORM (Derivation)

$$\begin{aligned}
\tilde{f}(x) &= \frac{1}{L}\sum_{n=-\infty}^{\infty}\left[\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)e^{-2\pi int/L}dt\right] e^{2\pi inx/L} \\
&= \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)e^{2\pi in(x-t)/L}dt \\
&= \frac{2}{L}\sum_{n=1}^{\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\cos(2\pi n(x-t)/L)dt + \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)dt
\end{aligned}$$

Let $u_n = \dfrac{n}{L}$ **then** $u_{n+1} = \dfrac{n+1}{L} = u_n + \Delta u$, **or** $u_{n+1} - u_n = \Delta u = \dfrac{1}{L}\ \forall\ n$,

then: $$\tilde{f}(x) = 2\sum_{n=1}^{\infty} \Delta u \int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\cos(2\pi u_n(x-t))dt + \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(t)\,dt$$



Let $L \to \infty \Rightarrow \Delta u \to 0 \Rightarrow$ **the above becomes continuous** $\Rightarrow$ **deficiencies (1) and (2) are addressed**

If we assume that $\int_{-\infty}^{\infty} f(t)\,dt < \infty$ **(**$f(t) \to 0$ **when** $t \to \pm\infty$**), then ...**

$$\begin{aligned}
\tilde{f}(x) = f(x) &= 2\lim_{\Delta u \to 0}\sum_{n=1}^{\infty} \Delta u \overbrace{\int_{-\infty}^{\infty} f(t)\cos(2\pi u_n(x-t))\,dt}^{H(u_n)} \\
&= 2\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos(2\pi u(x-t))\,dt\,du \quad \textbf{(continuous!)} \\
&= \frac{1}{2}(2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos(2\pi u(x-t))\,dt\,du \quad \textbf{.....} \quad \boxed{\mathbf{1}}
\end{aligned}$$

Also $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin(2\pi u(x-t))\,dt\,du = 0 \;\;\ldots\ldots\;\; \boxed{2}$$

$\boxed{1}+i\,\boxed{2}$:
$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{2\pi ui(x-t)}dtdu \\ &= \int_{-\infty}^{\infty} e^{2\pi iux} \underbrace{\int_{-\infty}^{\infty} f(t)e^{-2\pi iut}dt}_{F(u)}\,du \end{aligned}$$

In summary, we have the following for the one dimensional continuous case:

$$\textbf{FT}\,\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx$$

$$\textbf{IFT}\,\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)e^{2\pi iux}du$$