Unified Causal Field Theory: A Proof of Geometric Subsumption and Extension of Causal Inference Methods Into Unified Framework

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Abstract

This document extends my field-theoretic framework in a generalized $(n\mathrm{D}+\mathrm{T})$ spacetime, with signature $(-,+,...,+)$, to formally subsume, reinterpret, and unify a wide range of causal inference methods. I demonstrate its applicability to Difference-in-Differences (DiD), regression interaction effects, potential outcomes, Dynamic Structural Equation Modeling (DSEM), and propensity scoring. Critically, I prove that Regression Discontinuity Design (RDD) and Regression Kink Design (RKD) are field-theoretic boundary phenomena differentiated by the order of causal curvature discontinuity—interpretable as a collision angle within the manifold. This geometric-dynamical paradigm offers a novel mechanism for the detection of unobserved variable bias, leveraging tensor calculus and linear algebra. The framework's superior performance stems from its foundation in Lorentzian spacetime, the inherent mathematical structure of causality, providing a native environment for causal relationships that is inaccessible to methods operating on degraded, flattened projections of reality. True mathematical discontinuities are shown to be approximations arising when the characteristic time scale of a causal transition falls below the observational time step, analogous to the Planck time limit in physics. The empirical success of existing low-dimensional causal inference methods provides a compelling validation for the mathematical necessity and superior performance of this higher-dimensional spacetime framework.

1 Introduction

Traditional approaches to causal analysis often grapple with inherent limitations, particularly concerning the accurate representation of temporal dynamics and the pervasive challenge of unobserved confounding variables. These methods typically operate in flattened, discrete, or Euclidean-like spaces, thereby losing information crucial for capturing the underlying causal structure. I propose a field-theoretic framework operating within a generalized $(n\mathrm{D}+\mathrm{T})$ spacetime. This spacetime is characterized by n spatial dimensions representing system features and a single, irreducible temporal dimension. I will demonstrate how this framework, leveraging the capabilities of tensor calculus and linear algebra, provides a more comprehensive and robust foundation for causal inference, inherently subsuming existing methodologies, and offering a novel mechanism for the detection of unobserved variable bias. By operating in the native Lorentzian spacetime where causality naturally lives, this framework necessarily captures causal relationships more accurately, representing a mathematically correct foundation that existing methods approximate poorly.

2 Theorem: Causal Subsumption and Unobserved Variable Detection in Generalized $(n\mathrm{D}+\mathrm{T})$ Spacetime

Theorem 2.1. *Given a generalized spacetime manifold $(\mathcal{M}, g_{\mu\nu})$ with signature $(-,+,...,+)$, where n spatial dimensions represent system features and one temporal dimension represents the irreducible flow of evolution, a field-theoretic approach, leveraging tensor calculus and linear algebra, allows for the subsumption of traditional causal analysis frameworks and provides a mechanism for detecting unobserved variable bias.*

2.1 Axiomatic Foundations:

Axiom 2.1 (Spacetime Manifold with Causal Structure). *There exists a smooth, connected, $(n+1)$-dimensional differentiable manifold $(\mathcal{M}, g_{\mu\nu})$ called spacetime, where $g_{\mu\nu}$ is a pseudo-Riemannian metric tensor with Lorentzian signature $(-,+,...,+)$ encoding intrinsic "distance" and, critically, defining the causal light cone structure. The coordinates $x^\mu = (f^1, \ldots, f^n, t)$ are defined such that Greek indices $(\mu, \nu, \rho, \ldots)$ run from 0 to n (with $x^0 = t$), and Latin indices $(i, j, k, \ldots)$ run from 1 to n.*

Axiom 2.2 (Physics as the Science of Ground-State Causality). *Physics, while fundamentally concerned with matter and energy, is inherently the science of* ***ground-state causality****. Physical laws describe how states evolve deterministically or probabilistically from prior states, establishing a fundamental relationship between physical dynamics and causal inference. The unidirectional and irreducible nature of time, as described in Axiom 2.6, provides the fundamental arrow of causation for all physical phenomena. This intimate relationship makes the mathematical structure of causality inseparable from the mathematical structure of physics.*

Axiom 2.3 (Field Evolution). *A field $\Psi(x^\mu)$ exists on $\mathcal{M}$ and its evolution is governed by partial differential equations of the form $\frac{\partial \Psi}{\partial t} = \mathcal{O}[\Psi] + J(x^\mu)$. Here, $\Psi(x^\mu)$ can be a scalar field, a vector field $\Psi^\alpha(x^\mu)$, or a tensor field $\Psi^{\alpha\beta}(x^\mu)$. $\mathcal{O}$ is a general spatio-temporal operator (which may itself be a tensor operator) representing internal system dynamics (e.g., a d'Alembertian operator $\Box\Psi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi$, characteristic of wave equations, or a non-linear reaction term), and $J(x^\mu)$ represents external influences or sources. For a causal influence to be meaningful, the field $\Psi(x^\mu)$ must be at least a rank 2 tensor field, $\Psi^{\alpha\beta}$, where the indices α, β capture the interactions between features or concepts necessary for defining a relationship or effect. A rank 1 (vector) field can represent a flow, but a rank 2 field is the minimum required to describe how one component or feature interacts with or influences another, thus establishing a causal connection.*

Axiom 2.4 (Observable Field Strengths). *At any point $x^\mu \in \mathcal{M}$, the observable "field strength" $S(x^\mu)$ from system phenomena can be empirically estimated. This strength is derived from the field's gradients; for a scalar field Ψ, $S(x^\mu) = \sqrt{g^{\mu\nu}(\nabla_\mu\Psi)(\nabla_\nu\Psi)}$. For vector or tensor fields, $S(x^\mu)$ would be a suitable scalar invariant constructed from their covariant derivatives and the metric tensor, e.g., $S(x^\mu) = \sqrt{g_{\mu\nu}(\nabla^\mu\Psi^\alpha)(\nabla^\nu\Psi_\alpha)}$. The metric $g_{\mu\nu}$ is implicitly or explicitly defined, for instance, it could be proportional to the Fisher Information Metric $G_{\mu\nu}$ of the local field distribution, allowing $S(x^\mu)$ to represent curvature-weighted field gradients. Observed actions are explicitly emphasized and highlight the system's dynamics through their manifestations as field strengths.*

Axiom 2.5 (Path Integration and Distance). *The cumulative field strength encountered along any path γ between system states C_a and C_b is given by $D_{ab}(\gamma) = \int_\gamma S(x^\mu)ds$, where $ds = \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$ is the spacetime interval along the path. A distance d^*_{ab} can be defined as the infimum of $D_{ab}(\gamma)$ over all causal paths, $d^*_{ab} = \inf_{\gamma\in\Gamma_{ab}} D_{ab}(\gamma)$. This d^* forms a metric space. These geodesics represent optimal or most likely trajectories for system transitions given the field strengths, reflecting the underlying dynamics of the system. This provides a normative framework for causality: the observed causal path is the one that minimizes the relevant action or cost.*

Axiom 2.6 (Irreducible Unidirectionality of Time). *The temporal dimension $x^0 = t$ in spacetime is strictly unidirectional and irreversible. This is encoded by the negative signature of the temporal component of the metric $g_{00} < 0$. This indefinite metric is not merely analogous; it is the actual mathematical structure that gives rise to causal relationships through the light cone geometry, inherently distinguishing past, future, and spacelike separated events. Critically, time itself is an irreducible, ground-state concept. Furthermore, the higher-order temporal derivatives (velocity, acceleration, $(n+2)$-th derivative, etc.) constitute distinct, yet intrinsically linked, higher* ***temporal dimensions****within a richer temporal structure.*

Axiom 2.7 (Relative Reference Frame and Zero Initial Conditions). *All physical changes and causal interactions are measured relative to the system's own reference frame. Since causality concerns the dynamics of change rather than absolute states, initial conditions for all derivatives are assumed to equal zero in the system's natural reference frame. This axiom ensures that causal effects are measured as deviations from the system's baseline state, making the framework independent of arbitrary coordinate choices or external reference points. Formally, for any field component $\Psi^{\alpha\beta}$ and its derivatives, we set $\frac{D^k\Psi^{\alpha\beta}}{Dt^k}\big|_{t=0} = 0$ for all $k \geq 0$ in the system's intrinsic reference frame.*

Axiom 2.8 (Continuous Derivative Spectrum and Fractional Causality)**.** *The derivative order α may vary continuously over the real numbers $\alpha \in \mathbb{R}^+$, extending beyond integer derivatives to encompass fractional derivatives. For any fractional order α, the causal impact at that order is bounded by B_α, where B_α represents the maximum allowable magnitude of the α-th order fractional derivative. The fractional derivative (using Caputo or Riemann-Liouville definitions) captures non-local causal memory and distributed influence across temporal scales. This continuous spectrum enables modeling of systems with* ***non-instantaneous influence****,* ***viscoelastic memory****, and* ***scale-dependent causal propagation****, where influence is distributed across a continuum of time scales rather than concentrated at discrete derivative orders.*

Definition 2.1 (Causality is Interaction: Time Necessary, Not Sufficient)**.** *Causality is fundamentally* ***interaction****. While time is* ***necessary****for interactions to unfold and for changes to manifest, it is* ***not sufficient****to define causality. Mere temporal precedence does not imply causation. True causality arises from the dynamic interplay and coupling between system features, fields, or external influences, which are formally represented by the evolution of the field $\Psi(x^\mu)$ driven by the operator $\mathcal{O}[\Psi]$ and external sources $J(x^\mu)$, leading to non-zero higher-order temporal derivatives as specified in Theorem 5.3. The minimum requirement for a field Ψ to represent meaningful causal interactions is that it be a rank 2 tensor field $\Psi^{\alpha\beta}$, where the indices α, β explicitly capture these inter-feature relationships.*

2.2 Proof:

Proof. The proof proceeds in two main parts: demonstrating the subsumption of traditional causal analysis and establishing the mechanism for unobserved variable detection.

> **Plain English:** We'll show two things: (1) that existing causal methods are special cases of our framework, and (2) that our framework can detect hidden confounding variables that other methods miss.

2.2.1 Part 1: Subsumption of Causal Analysis

Traditional causal analysis frameworks operate in flattened, discrete, or Euclidean-like spaces, often making assumptions about temporal ordering and struggling with unobserved confounders. My generalized field theory in $(n\mathrm{D} + \mathrm{T})$ spacetime inherently addresses these limitations.

1. **Intrinsic Temporal Ordering and Dynamic Evolution:** The field $\Psi(x^\mu)$ (Axiom 2.3) evolves dynamically, with time t as an explicit and irreversible dimension (Axiom 2.6). This inherent temporal ordering directly reflects Axiom 2.2, making causal inference a natural consequence of the system's dynamics.

   > **Summary:** Time naturally flows in one direction in our mathematical space, automatically giving us the "arrow of causality" without having to assume it.

2. **Commentary: Causality in Lorentzian Spacetime** The choice of a Lorentzian signature for the spacetime metric (Axiom 2.1, Axiom 2.6) is not merely an analogy to physics but is mathematically necessary for modeling causality. Unlike Riemannian space $(+, +, +, ..., +)$, which has no preferred time direction and thus no inherent causal structure, Lorentzian geometry with its indefinite metric naturally defines light cones.

   These light cones delineate absolute past, absolute future, and causally disconnected (spacelike) regions for any event. This structure inherently encodes the principle that effects must follow their causes, and that information cannot propagate instantaneously.

   Therefore, the causal relationships derived within this framework are not merely correlations or statistical inferences, but are directly tied to the fundamental geometric properties of the underlying spacetime. This provides a native environment for causal relationships that is mathematically unavailable to flattened or Euclidean-based approaches.

   > **Summary:** We use Einstein's spacetime geometry because it automatically creates natural boundaries for what can cause what, unlike flat mathematical spaces.

3. **Continuous Field Dynamics vs. Discrete Variables:** This framework models system phenomena as continuous fields $\Psi(x^\mu)$ rather than relying on discrete variables or observations. This continuous nature allows for the capture of subtle, non-linear interactions and emergent phenomena that discrete models might oversimplify or miss.

   Changes in system state are described by smooth field evolution, offering a richer causal landscape than simple variable correlations.

   > **Summary:** Instead of looking at separate variables jumping between values, we study smooth flowing fields that can capture subtle interactions.

4. **Geodesic Paths and Optimal Transitions:** The functional $D_{ab}(\gamma)$ (Axiom 2.5) represents a "cost" or "effort" associated with transitioning between system states along a path. Minimizing this functional, $\inf_\gamma D_{ab}(\gamma)$, yields geodesics within an effective metric space.

   These geodesics represent optimal or most likely trajectories for system transitions given the field strengths, reflecting the underlying dynamics of the system. This provides a normative framework for causality: the observed causal path is the one that minimizes the relevant action or cost.

   > **Summary:** Systems naturally follow the "path of least resistance" through our mathematical space, giving us a principled way to predict causal trajectories.

2.2.2 Part 2: Addressing the Unobserved Variable Bias Problem Through Detection

The problem of unobserved variable bias arises when a confounding variable, not accounted for in the analysis, influences both the "cause" and the "effect," leading to spurious correlations. My field-theoretic approach provides a mechanism to detect such hidden influences.

1. **Non-Commutativity of Spacetime Operators as an Indicator:** In a multidimensional spacetime, the interaction between spatial and temporal processes is critical. The presence of an unobserved variable can be detected by examining the non-commutativity of fundamental operators.

   This is rigorously expressed via the exterior derivative on differential forms, particularly for vector or tensor fields.

   Specifically, consider a 1-form $\omega = A_\mu dx^\mu$ (where A_μ are components of a covariant vector field representing flow or potential). The exterior derivative $d\omega$ is given by: $(d\omega)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

   If $d\omega \neq 0$ for components involving both spatial and temporal indices (e.g., $(\partial_i A_0 - \partial_0 A_i) \neq 0$), it implies a non-exact differential form, suggesting a "rotational" component or a failure of the vector field to be a pure gradient of a scalar potential.

   This non-integrability can be interpreted as an unseen dependency or a hidden source/sink distorting the expected flow. This provides a formal detection mechanism for potential unobserved confounders by identifying such non-integrable structures.

   For a scalar field Ψ, while partial derivatives ∂_i, ∂_0 inherently commute (i.e., $[\partial_i, \partial_0]\Psi = 0$), the effect of an unobserved variable may manifest as non-commutativity if the operators themselves are more complex than simple partial derivatives, perhaps incorporating dependencies on hidden coordinates, or by inducing torsion or non-metricity that affects the covariant derivatives in a complex way. The presence of non-commuting operations that are *expected* to commute based on the observed variables can rigorously indicate the influence of unmodeled factors.

   > **Summary:** When mathematical operations that should give the same result in any order suddenly don't, it's a smoking gun for hidden variables twisting the system.

2. **Discrepancies in System Behavior and Predicted Consistency:** A comprehensive field theory predicts consistent relationships across different observed manifestations of the field. For example, I define spectral similarity as the L^2 inner product of normalized frequency-domain representations of $\Psi(x^\mu)$ over fixed spatial slices, and spatial distance as Euclidean or geodesic distance in the manifold embedding of observed field configurations. A robust theory would predict a strong correlation ($r > 0.7$) between these measures.

- If a hidden variable influences the system, it would introduce subtle or significant deviations from these predicted consistencies. For instance, an unobserved factor might alter the effective metric, leading to changes in distances that are not reflected in corresponding spectral changes, or vice-versa.
- The absence of the predicted strong correlation (e.g., correlation below a statistically significant threshold, such as 0.5) would indicate that the current model, based solely on observed field strengths and their dynamics, is incomplete. Such a deviation would compel me to hypothesize the existence of unobserved variables or interactions that are causing the inconsistency, thereby providing a detection mechanism for bias. Linear algebra, specifically vector space analysis and matrix operations (e.g., Fourier transforms to compute spectral content, matrix norms for distance calculations, covariance matrices for correlation), would be crucial for computing and comparing these multi-dimensional spatial distances and spectral similarities, and for determining their correlation.

3. **Commentary: Degraded Projections and Information Loss** Traditional causal inference methods often operate in flattened, discrete, or Euclidean-like subspaces. By projecting the full $(n\mathrm{D} + \mathrm{T})$ Lorentzian spacetime onto such degraded representations, critical information about the underlying causal structure is lost.

   For instance, the light cone structure, which inherently defines causal precedence and independence, is absent in Euclidean spaces. Similarly, discretizing continuous field dynamics introduces approximations that can obscure subtle non-linear interactions and emergent phenomena.

   Crucially, by collapsing the rich, multi-dimensional structure of **Hypertime** (where each derivative order represents a distinct temporal dimension, as discussed in Axiom 2.6 and Section 5.3) onto a single, linear time axis, vast amounts of causal information about the dynamics of change itself are lost.

   The detection mechanisms proposed in this framework are capable of identifying the presence of these "hidden" or "lost" dimensions and their causal implications, thereby accessing information that is mathematically unavailable to approaches limited to such degraded projections.

   Summary: Traditional methods work by flattening our rich multi-dimensional causal space into simpler forms, but this throws away crucial information we can recover.

□

2.3 Mathematical Necessity of Superior Performance

The empirical success of traditional causal inference methods, despite their reliance on simplified models, provides a compelling argument for the inherent validity and superior performance of my proposed higher-dimensional field-theoretic framework. The logical chain is as follows:

1. **Empirical Validity of Traditional Methods:** Decades of empirical success across various scientific disciplines demonstrate that traditional causal inference methods (e.g., DiD, regression) effectively identify causal relationships in many real-world scenarios.

2. **Traditional Methods as Degraded Projections:** As I established in this paper, traditional methods operate on simplified, flattened, or discrete projections of the full $(n\mathrm{D} + \mathrm{T})$ Lorentzian spacetime and its multi-dimensional Hypertime structure. These projections inherently discard or approximate information present in the complete manifold (see Commentary: Degraded Projections and Information Loss).

3. **Information Loss in Projections:** A fundamental principle of information theory and geometry dictates that a projection from a higher-dimensional space to a lower-dimensional subspace necessarily involves a loss of information, unless the projection is an isometry onto an isomorphic subspace. Given the simplified nature of traditional models compared to a full field theory in spacetime, such an isometry is not generally expected.

4. **The Contradiction Argument and Implied Superiority:** If my higher-dimensional field-theoretic framework did *not* accurately capture the underlying causal structure, then its lower-dimensional projections (i.e., traditional methods) could not consistently yield empirically valid

causal inferences. However, since I observe that these low-dimensional projections *do* work, it implies, by mathematical necessity, that the full framework must be accurately representing the true causal structure from which these successful projections are derived. Consequently, by operating in the full, native Lorentzian spacetime and its rich Hypertime, my proposed framework must work at least as well as, and indeed, likely better than, its information-degraded projections.

Remark 2.1 (Scaling with Dimensions: DiDiD and Beyond)**.** *This argument extends directly to the concept of scaling causal differentiation with dimensions. Just as 2D DiD (Difference-in-Differences) empirically works, a 3D extension, often termed DiDiD (Difference-in-Difference-in-Differences), must work better by enabling control for more complex confounding patterns across an additional feature dimension. This pattern logically extends:*

- *2D DiD works $\implies$ 3D DiDiD must work better*
- *3D DiDiD works $\implies$ 4D DiDiDiD must work better*
- *And so on...*

Each additional dimension allows for another layer of causal differentiation, effectively enabling successively more refined control groups across multiple feature dimensions simultaneously.

Remark 2.2 (Bounded Performance Guarantees)**.** *A significant practical implication of this mathematical necessity is that the detection mechanisms for unobserved variables and the causal inference capabilities of this framework have bounded performance guarantees. They cannot perform worse than the traditional methods, which are their own degraded projections. By geometric necessity, and the restoration of lost information from the full spacetime, the performance should be strictly superior, offering more precise identification of causal effects and confounders.*

3 Extensions to Higher Dimensions for Specific Causal Inference Methods

My generalized spacetime $(\mathcal{M}, g_{\mu\nu})$ with n spatial dimensions and 1 temporal dimension provides a rich manifold within which various causal inference methods can be interpreted as specific configurations, projections, or analyses of the evolving field $\Psi(x^\mu)$.

3.1 Difference-in-Differences (DiD)

DiD models aim to estimate the causal effect of an intervention by comparing the changes in outcomes over time between a treatment group and a control group. In my framework, DiD can be seen as analyzing the impact of a localized external influence $J(x^\mu)$ on $\Psi(x^\mu)$ within a specific subspace and over a defined temporal interval.

1. **Proof of Group and Time Representation:** I formally define the treatment and control groups as distinct, spatially segregated field regions within the nD spatial dimensions of $\mathcal{M}$. Let the spatial coordinates be $x^i = (f^1, \dots, f^n)$, where f^k is a binary indicator for group membership (e.g., $f^k = 1$ for treated, $f^k = 0$ for control). The intervention itself is represented by an external source term $J(x^\mu)$ in the field evolution equation $\frac{\partial \Psi}{\partial t} = \mathcal{O}[\Psi] + J(x^\mu)$, such that $J(x^\mu) > 0$ for $x \in \mathcal{R}_T$ (treatment region) and $t \geq t_{\text{intervention}}$. This setup allows for the direct application of field dynamics to group-level and temporal variations.
2. **Proof of DiD as a Mixed Covariant Derivative:** The fundamental DiD estimator, $\Delta\Delta Y = (Y_{T,\text{post}} - Y_{T,\text{pre}}) - (Y_{C,\text{post}} - Y_{C,\text{pre}})$, measures a difference of differences across group and time. In my continuous field framework, this corresponds to a mixed second-order covariant derivative of the field $\Psi(x^\mu)$. Let f^s represent the spatial dimension corresponding to group assignment (e.g., $f^s \in \{\text{control}, \text{treated}\}$). The DiD effect is mathematically captured by the component $\nabla_0 \nabla_s \Psi$, where ∇_0 denotes the covariant derivative with respect to time t, and ∇_s denotes the covariant derivative with respect to the group dimension f^s. A non-zero DiD effect $\Delta\Delta\Psi \neq 0$ implies that the temporal evolution of the field in the treated region (governed by $J(x^\mu)$) deviates significantly from the counterfactual evolution in the control region. This deviation is a localized spatio-temporal gradient that cannot be explained by the inherent dynamics of $\mathcal{O}[\Psi]$ alone, demonstrating a causal impact.

3. **Mathematical Illustration (Rank 2 Field):** Consider a simplified (2D + T) spacetime where coordinates are (f^1, f^2, t). Let f^1 be an outcome feature and f^2 be a spatial dimension distinguishing treatment ($f^2 = 1$) and control ($f^2 = 0$) groups. Let $\Psi^{\alpha\beta}$ be a rank 2 tensor field, where indices α, β can represent components related to outcomes, covariates, or their interactions. For instance, Ψ^{11} could represent the value of the outcome feature f^1 (scalar outcome embedded in a tensor field), or $\Psi^{1\text{cov}}$ could represent an interaction between f^1 and some covariate. The essence of the DiD effect on $\Psi^{\alpha\beta}$ can be expressed by examining specific components of its second covariant derivative:
$$\Delta_{DiD}\Psi^{\alpha\beta} = \left(\nabla_t \nabla_{f^2} \Psi^{\alpha\beta}\right)_{\text{treated group}} - \left(\nabla_t \nabla_{f^2} \Psi^{\alpha\beta}\right)_{\text{control group}}$$
Here, ∇_t is the covariant derivative with respect to time, and ∇_{f^2} is the covariant derivative with respect to the group dimension. A non-zero value for this difference indicates that the "change in the temporal rate of change across groups" is different between the observed treated group and the observed control group, signaling the causal impact of the treatment. For example, if Ψ^{11} is the primary outcome, the DiD effect is observed if $(\nabla_t \nabla_{f^2} \Psi^{11}) \neq 0$ due to the localized $J(x^\mu)$ in the treated region, representing a specific type of spatio-temporal curvature induced by the intervention. This effectively measures the twist in the field across the treatment boundary over time.

3.2 Regression Interaction Effects

Interaction effects in regression models imply that the effect of one variable on an outcome depends on the level of another variable. In my spacetime, this is inherently modeled by non-linearities in the field evolution operator $\mathcal{O}[\Psi]$ or the external influence $J(x^\mu)$.

1. **Proof of Interaction as Non-Linear Field Coupling:** I define regression interaction effects as arising from non-linear field couplings within $\mathcal{O}[\Psi]$ or in the external influence $J(x^\mu)$. If $\Psi(x^\mu)$ is influenced by multiple spatial dimensions (features) $f^1, f^2, \ldots, f^n$, an interaction effect between, say, f^i and f^j means that the response of Ψ to changes in f^i is explicitly modulated by the values of f^j. This is formally captured by non-linear terms in the field equation that involve products of field components, or products of fields and coordinates. For example, a term in $\mathcal{O}[\Psi]$ of the form $K\Psi^2$ or $K\Psi f^i f^j$ would represent such a coupling, where K is a coupling coefficient. These terms fundamentally create higher-order structure in the field's dynamics, which is naturally described by tensor products in differential geometry.

2. **Proof of Geometric Interpretation:** A non-zero interaction term implies an anisotropy in the field's response, meaning the field's behavior with respect to one spatial dimension changes as a function of another. Geometrically, this manifests as varying curvature components in the effective metric $\tilde{g}_{\mu\nu}$ (which is derived from the field strengths), or a non-separable structure within the metric itself. This is analogous to gauge coupling terms in fundamental physics, where interaction strengths between fields depend on the values of other fields, creating a richer, interdependent causal landscape within the spacetime manifold.

3. **Mathematical Illustration (Rank 2 Field):** Let $\Psi^{\alpha\beta}(x^\mu)$ be a rank 2 tensor field. Consider a (2D + T) spacetime with coordinates (f^1, f^2, t). An interaction between f^1 and f^2 in the context of the field evolution can be represented by coupling terms in the operator $\mathcal{O}[\Psi]$ such as:
$$\frac{\partial \Psi^{\alpha\beta}}{\partial t} = \cdots + K^{\alpha\beta}_{\gamma\delta} \Psi^{\gamma 1} \Psi^{\delta 2} + \ldots$$
where $K^{\alpha\beta}_{\gamma\delta}$ is a coupling tensor coefficient. This term shows how the temporal evolution of $\Psi^{\alpha\beta}$ is influenced by the product of field components related to features f^1 and f^2. The non-linearity captured by this product term $\Psi^{\gamma 1}\Psi^{\delta 2}$ represents the interaction effect: the influence of feature f^1 (via $\Psi^{\gamma 1}$) on the field's evolution depends on the state of feature f^2 (via $\Psi^{\delta 2}$), and vice versa. Geometrically, this corresponds to a non-separable, anisotropic response where the field's curvature in one direction is modulated by its values in another direction, creating a richer, interdependent causal structure.

3.3 Potential Outcomes Framework

The potential outcomes framework posits that for each unit, there exist multiple potential outcomes, one for each possible treatment assignment, only one of which is observed. The causal effect is the difference between these potential outcomes.

1. **Proof of Counterfactuals as Parallel Field Sheets:** I formalize potential outcomes as distinct, parallel "field sheets" within the spacetime manifold $\mathcal{M}$. Let f^T be an unobserved spatial dimension representing treatment assignment (e.g., $f^T = 0$ for the control sheet, $f^T = 1$ for the treated sheet). The observed outcome for a unit i corresponds to the field $\Psi(x_i^\mu)$ on one specific sheet (e.g., the $f^T = 1$ sheet), while the counterfactual outcome $\Psi'(x_i^\mu)$ exists on the other ($f^T = 0$ sheet). The causal effect for unit i is then the difference between the field states on these two sheets at the specific event x_i^μ: $\Psi(x_i^\mu|_{f^T=1}) - \Psi(x_i^\mu|_{f^T=0})$. This explicitly embeds counterfactuals within the manifold structure, rather than as abstract possibilities.

2. **Proof of The Fundamental Problem of Causal Inference:** The challenge of observing only one potential outcome corresponds to a fundamental observational limitation: I am restricted to a single slice (hyperplane) of the full spacetime manifold $\mathcal{M}$ determined by the observed treatment assignment f^T. This is formalized as a projection operator $P_{f^T=k}$ that collapses the full field $\Psi(x^\mu)$ to a scalar realization on the observed sheet. Causal inference methods then become strategies to estimate the properties of the unobserved counterfactual field sheet ($\Psi(\cdot|_{f^T=0})$ if observed $f^T = 1$, or vice versa) by leveraging observable information from the chosen sheet and inferring "geodesically similar" units across the f^T dimension based on other observable spatial dimensions.

3. **Mathematical Illustration (Rank 2 Field):** Consider a (3D + T) spacetime with coordinates (f^1, f^2, f^T, t), where f^1, f^2 are observed covariates and f^T is the unobserved treatment assignment dimension. Let $\Psi^{\alpha\beta}$ be a rank 2 field. For example, Ψ^{11} could represent the outcome for f^1, and Ψ^{12} could represent the interaction between f^1 and f^2. The potential outcomes are represented by the field values on these two distinct sheets. For a given unit, the causal effect on a specific field component Ψ^{ab} (e.g., $\Psi^{\text{outcome,outcome}}$):

$$\text{Causal Effect for } \Psi_i^{ab} = \Psi^{ab}(f_i^1, f_i^2, f^T = 1, t_i) - \Psi^{ab}(f_i^1, f_i^2, f^T = 0, t_i)$$

   where the first term is observed and the second is counterfactual. The goal is to estimate the second term. This estimation involves extrapolating the field $\Psi^{\alpha\beta}$ from the observed sheet to the counterfactual sheet. This can be conceptualized as constructing a path integral along the f^T dimension, or solving the field equation across the treatment boundary ($f^T = 0.5$ hyperplane) to predict the unobserved field state. The problem is therefore framed as a boundary value problem in higher-dimensional spacetime, seeking to define $\Psi^{\alpha\beta}$ on the unobserved sheet by imposing continuity or smoothness conditions derived from observed similarities on the other features.

3.4 Dynamic Structural Equation Modeling (DSEM)

DSEM extends traditional SEM to incorporate dynamic processes and time-series data, modeling how latent variables evolve and influence observed variables over time.

1. **Proof of Latent Variables as Hidden Spatial Dimensions:** I formally define latent variables in DSEM as unobserved spatial dimensions within my nD feature space of $\mathcal{M}$. Let f^L denote such a latent dimension. The temporal evolution of these latent variables, as modeled in DSEM (e.g., autoregressive paths), directly corresponds to the temporal evolution of the field $\Psi(x^\mu)$ along these hidden dimensions. This means that Ψ itself has components or dynamics that are not directly observable but contribute to the overall field behavior.

2. **Proof of Pathways as Field Gradients and Curvatures:** The structural equations in DSEM (e.g., $L \to Y$) are rigorous mappings between dimensions within the spacetime manifold. Causal paths from latent to observed variables are direct geometric relationships: they are represented by components of field gradients ($\nabla_L \Psi$) or curvatures induced by the field in the mixed latent-observed dimensions ($\nabla_L \nabla_Y \Psi$).

   These pathways reflect how the change in the latent field influences the observed field, mediated by the field Ψ and its underlying dynamics $\mathcal{O}[\Psi]$. Time-varying dynamics (e.g., autoregressive processes for latent variables) are explicitly captured by the temporal components of $\mathcal{O}[\Psi]$ involving time derivatives ($\frac{\partial}{\partial t}$) or integral kernels over past states, reflecting the field's memory.

This aligns with fiber bundles, where latent dynamics form connections in the fibers over the observed manifold.

> **Summary:** Hidden variables influence visible ones through geometric relationships that can be calculated using field calculus. Think of invisible forces creating visible effects through mathematical connections.

3. **Mathematical Illustration (Rank 2 Field):** Let's consider a (2D + T) spacetime with coordinates (f^O, f^L, t), where f^O is an observed feature and f^L is a latent feature. Let $\Psi^{\alpha\beta}$ be a rank 2 field. The components Ψ^{OO} could represent observed-observed interactions, Ψ^{LL} latent-latent dynamics, and Ψ^{OL} (or Ψ^{LO}) observed-latent coupling. A DSEM pathway $f^L \to f^O$ is represented by the influence of the field's behavior in the latent dimension f^L on its behavior in the observed dimension f^O. This translates to terms in the field evolution equation that couple these components:
$$\frac{\partial \Psi^{OO}}{\partial t} = \cdots + \mathcal{K}^{OO}_{\gamma\delta\epsilon\zeta}\Psi^{\gamma L}\Psi^{L\delta} + \ldots$$
where $\mathcal{K}$ is a coupling tensor. This term shows how interactions involving the latent field ($\Psi^{\gamma L}$, $\Psi^{L\delta}$) drive the dynamics of the observed field Ψ^{OO}. Autoregressive processes on latent variables are modeled by the temporal derivative of Ψ^{LL} depending on past values of Ψ^{LL} itself, effectively a time-delayed self-interaction captured by the operator $\mathcal{O}[\Psi]$:
$$\frac{\partial \Psi^{LL}(t)}{\partial t} = \int_{-\infty}^{t} G(t-t')\Psi^{LL}(t')dt' + \ldots$$
where $G(t-t')$ is a kernel representing memory or persistence. The causal influence $f^L \to f^O$ is directly given by the component $\nabla_L \Psi^{O\alpha}$, representing how the field gradient related to the observed outcome changes with respect to the latent dimension, or more complex relationships involving interactions between Ψ^{OL} components and other field parts.

3.5 Propensity Scoring

Propensity scoring aims to balance confounders between treated and control groups by estimating the probability of treatment assignment given observed covariates. This creates a "balanced" subspace where direct comparisons are more valid.

1. **Proof of Propensity Score as a Scalar Field Projection:** I define the propensity score $e(x^\mu)$ as a scalar field $P : \mathcal{M} \to [0, 1]$ that is a non-linear projection from the multi-dimensional observed covariate space $(f^1, \ldots, f^k)$ onto a single dimension representing the likelihood of treatment assignment. This means P is a scalar function of x^μ that quantifies the probability of receiving treatment based on observed features. This is a dimension reduction operation within the spacetime manifold.

2. **Proof of Matching/Weighting as Geodesic Equivalence:** Matching or weighting based on propensity scores is a procedure to create a statistically balanced (in covariates) subspace for causal comparison. In my framework, this means identifying units that are "geodesically close" within the effective metric induced by the covariates. Formally, this involves finding a sub-manifold $\mathcal{S}_e \subset \mathcal{M}$ where the spatial metric induced by the observed covariates (excluding the outcome and treatment) is effectively "flat" or "homogeneous" with respect to treatment assignment for units within the same propensity score stratum. This ensures that any observed differences in the field Ψ after intervention are primarily attributable to the treatment $J(x^\mu)$, rather than pre-existing imbalances along the k observed covariate dimensions. This is analogous to a metric-preserving dimensionality reduction, creating a locally isometric embedding that facilitates direct comparison.

3. **Mathematical Illustration (Rank 2 Field):** Consider a (3D + T) spacetime with coordinates (f^1, f^2, f^3, t), where f^1, f^2, f^3 are observed covariates. The propensity score $e(f^1, f^2, f^3, t)$ is a scalar field derived from these covariates. Let $\Psi^{\alpha\beta}$ be a rank 2 field. For example, Ψ^{12} could represent the covariance or interaction between f^1 and f^2. The process of matching or weighting by propensity score aims to ensure that, for any given value of the scalar field e, the distribution of

$\Psi^{\alpha\beta}$ components (related to covariates) is similar across treatment assignments. This can be viewed as identifying a "propensity hypersurface" in $\mathcal{M}$ where variations in the $\Psi^{\alpha\beta}$ components related to covariates (e.g., $\Psi^{11}, \Psi^{22}, \Psi^{12}$) are minimized conditional on the propensity score. The goal is to make the comparison of $\Psi^{\alpha\beta}$ across treatment groups as if they were on geodesically equivalent paths in covariate space. This implies constructing a local coordinate system or a mapping where the effective metric components involving covariate dimensions, $\tilde{g}_{ij}$, become nearly uniform for units with the same propensity score, ensuring that deviations in $\Psi^{\alpha\beta}$ are not driven by covariate imbalances but by the treatment effect.

4 RDD and RKD: The Same Phenomenon at Different Collision Angles

Regression Discontinuity Design (RDD) and Regression Kink Design (RKD) are both quasi-experimental designs that exploit an arbitrary threshold in an assignment variable to infer causal effects. I prove that they represent the same underlying phenomenon—a localized causal intervention—differing only in the smoothness or "collision angle" of the field's response to that intervention within the spacetime manifold.

4.1 Shared Principle: Localized Causal Intervention at a Boundary

Both RDD and RKD rely on a sharp (RDD) or kinked (RKD) change in the probability of receiving treatment or the treatment intensity as a function of an arbitrary assignment variable A.

- In RDD, units just above or below a threshold A_0 have significantly different probabilities of treatment. This represents a discontinuous jump in treatment status.
- In RKD, the *slope* of the treatment intensity changes discontinuously at A_0. This represents a continuous change in treatment status, but with a discontinuous derivative.

In my field-theoretic framework, the assignment variable A corresponds to one of the spatial dimensions, say f^A. The threshold A_0 defines a boundary hypersurface $\mathcal{S} = \{x^\mu \in \mathcal{M} : f^A(x^\mu) = A_0\}$ within the spatial manifold. The causal effect is observed as a change in the field $\Psi(x^\mu)$ as it crosses this boundary.

4.2 Proof of Equivalence via Collision Angle Analogy

Proof. Consider the field $\Psi(x^\mu)$ and its response to the treatment, which can be seen as an external influence $J(x^\mu)$. The intervention is determined by the assignment variable f^A.

Plain English: We're looking at how a field (representing system outcomes) responds when we apply a treatment at a specific boundary or threshold in one of our dimensions.

1. **RDD as an Apparent Discontinuity / Normal Collision:** In RDD, the treatment (and thus $J(x^\mu)$) is a function that changes abruptly at $f^A = A_0$. While mathematically, true discontinuity is an idealization, in physical and observable systems, this represents a transition occurring within a time scale Δt_{causal} that is significantly smaller than the characteristic observation or modeling time step Δt_{obs} (i.e., $\Delta t_{\text{causal}} \ll \Delta t_{\text{obs}}$). As the angle of interaction approaches criticality, this causal transition time can even approach fundamental limits, analogous to the Planck time in physics, giving the macroscopic appearance of a discontinuity. Let $J(x^\mu) = j(t) \cdot H(f^A - A_0)$, where H is the Heaviside step function. The field $\Psi(x^\mu)$ is expected to exhibit an apparent jump discontinuity (a "discontinuity in levels") at the boundary $\mathcal{S}$. The change in Ψ across the boundary can be represented by a very steep gradient or a Dirac delta function derivative normal to the boundary. In terms of a "collision," the field Ψ experiences an abrupt, direct impact as it crosses the boundary $\mathcal{S}$. This is analogous to a particle colliding head-on (a "normal collision") with an impenetrable barrier, where its velocity (rate of change) immediately flips or significantly alters magnitude, leading to a sharp change in position or state. The "collision angle" is effectively $90°$ relative to the boundary, implying a direct, unfiltered impact.
2. **RKD as a Discontinuous Kink / Oblique Collision:** In RKD, the *rate of change* of the treatment (and thus $J(x^\mu)$) is a function that changes abruptly at A_0. Here, the transition occurs within $\Delta t_{\text{causal}} \ll \Delta t_{\text{obs}}$, but it is the derivative of the treatment effect that exhibits the sharp

change. Let $J(x^\mu) = j(t) \cdot (f^A - A_0) \cdot H(f^A - A_0)$. Here, the field $\Psi(x^\mu)$ itself remains continuous across the boundary $\mathcal{S}$, but its first derivative with respect to f^A (or the derivative of the treatment effect) exhibits an apparent jump discontinuity (a "kink"). This means that while the field value does not abruptly change at the boundary, the slope of the field does. In the "collision" analogy, this is like a particle grazing a surface or undergoing an oblique collision. Its position remains continuous, but its trajectory (velocity vector) changes direction abruptly, resulting in a "kink" in its path. The "collision angle" is acute (e.g., 45°), allowing the particle to smoothly pass the boundary while its dynamics are sharply altered.

Proof of Unification: Both RDD and RKD investigate the causal effect of an intervention that is *structured by a sharp boundary* in the assignment variable. The difference lies in the *order of the apparent discontinuity* in the treatment function or its effect on the field Ψ.

- RDD corresponds to an apparent C^0 discontinuity in $J(x^\mu)$ (the treatment field itself), leading to an apparent C^0 discontinuity in the estimated causal effect on Ψ. This creates a "sharp corner" or "step" in the field response.
- RKD corresponds to an apparent C^1 discontinuity in $J(x^\mu)$ (the derivative of the treatment field), leading to an apparent C^1 discontinuity in the estimated causal effect on Ψ. This creates a "smooth corner" or "kink" in the field response, where the value is continuous but the slope is not.

From a geometric perspective, both are degenerate projections of $(\nabla_A \Psi)^2$. The "collision angle" precisely captures this difference in the order of discontinuity: a normal collision (90°) yields an RDD-like jump, while an oblique collision (acute angle) yields an RKD-like kink. The non-zero value I am looking for in both cases is the effect of the intervention, either on the value of Ψ (RDD) or its first derivative (RKD), localized at the boundary. Both indicate a causal effect manifesting as a deviation from smoothness that is not due to background field fluctuations but rather to the localized intervention. □

5 Subsumption Mapping to Structural Causal Models (SCMs)

Traditional causal models, such as Structural Causal Models (SCMs) or Directed Acyclic Graphs (DAGs), can be seen as degenerate projections of my field-theoretic framework under specific limiting conditions.

Consider a simple SCM where variables X and Y are connected by a causal arrow $X \to Y$, and U is an unobserved confounder $U \to X, U \to Y$. In my field-theoretic framework, this can be represented as follows:

1. **Discrete Variables as Localized Field Discontinuities:** In the limit where the field $\Psi(x^\mu)$ becomes piecewise constant over distinct spatial regions, and its evolution operator $\mathcal{O}$ simplifies (e.g., $\mathcal{O} = 0$, implying no continuous internal dynamics between distinct states), the continuous field can approximate discrete variables. A variable V_k transitioning from state v_1 to v_2 at time t_j can be viewed as an apparent discontinuity or sharp gradient in $\Psi(x^\mu)$ at (x_k^μ), where the causal transition time Δt_{causal} is below the observation time step.
2. **Causal Edges as Temporal Ordering of Discontinuities:** A causal link $X \to Y$ implies that a change in X precedes and influences a change in Y. In my framework, this corresponds to the temporal ordering of field discontinuities. If an apparent discontinuity in Ψ corresponding to X occurs at t_1, and a subsequent apparent discontinuity corresponding to Y occurs at $t_2 > t_1$, such that the field evolution $\Psi(x^\mu)$ connecting these two events is consistent with $J(x^\mu)$ originating from X, then a causal edge is established. The unidirectional nature of time (Axiom 2.6) fundamentally enforces this temporal precedence.
3. **Unobserved Confounders as Non-Orthogonal Parameters in Flattened Subspace:** When traditional causal models operate on "flattened" subspaces, they often omit dimensions corresponding to unobserved variables. In my (nD+T) spacetime, these unobserved variables might correspond to additional spatial dimensions or complex interactions that are not projected onto the observed subspace.

   If these unobserved variables (e.g., represented by coordinates u^k) are not orthogonal to the observed variables (f^i) in the full nD spatial manifold, their influence on the field $\Psi(x^\mu)$ (where $x^\mu = (f^i, u^k, t)$) would naturally induce correlations between f^i that are not direct causal links in the observed subspace. When one then projects or "flattened" this full spacetime onto the observed

(f^i, t) subspace, the non-orthogonality of u^k relative to f^i means that changes in u^k appear as correlated variations in f^i without an explicit causal path *within the flattened subspace.* The detection mechanisms discussed in Part 2 (non-commutativity of operators, cross-modal inconsistencies) precisely identify the presence of these non-orthogonal, unobserved influences. Thus, the unobserved variable bias in SCMs corresponds to the failure of the field's dynamics to be decomposable into purely orthogonal observed components in my higher-dimensional spacetime.

5.1 Fubini's Theorem as a Null Hypothesis for Absence of Hidden Variables

My framework leverages Fubini's Theorem to provide a rigorous test for the presence of unobserved, non-orthogonal variables that induce confounding. The commutativity of integration orders serves as a null hypothesis for the absence of such hidden influences.

Proposition 5.1 (Fubini's Test for Hidden Variables)**.** *Given a field $\Psi(x^\mu)$ defined on $\mathcal{M}$ (or a relevant observable derived from Ψ that is absolutely integrable over the chosen domains), the null hypothesis of no unobserved, non-orthogonal variables inducing confounding is that the mixed integrals of Ψ commute for any choice of distinct integration variables that include time. Formally, for any spatial coordinate f^i and the temporal coordinate t:*

$$\int_{T_1}^{T_2} \left(\int_{F_1^i}^{F_2^i} \Psi(x^\mu) df^i \right) dt = \int_{F_1^i}^{F_2^i} \left(\int_{T_1}^{T_2} \Psi(x^\mu) dt \right) df^i$$

Rejection of this null hypothesis (i.e., the integrals do not commute) implies the presence of a hidden, non-orthogonal variable coupling the integrated dimensions, thus causing unobserved confounding.

Proof. Fubini's Theorem states that if a function $g(x, y)$ is absolutely integrable over a rectangle $[a, b] \times [c, d]$, then the iterated integrals are equal regardless of the order of integration: $\int_c^d \int_a^b g(x, y) dx dy = \int_a^b \int_c^d g(x, y) dy dx$. My null hypothesis posits that the observed field dynamics, when projected onto the (f^i, t) subspace, can be described by a function that satisfies Fubini's conditions for separability. If a hidden variable u^k exists such that it couples f^i and t in a non-separable way (e.g., $\Psi(f^i, t, u^k)$ cannot be simply decomposed into a product or sum of functions dependent only on subsets of variables), then its influence would introduce a non-integrable component or a 'twist' into the observed $\Psi(f^i, t)$ field in the projected subspace. This non-integrability would cause the iterated integrals to differ, thus rejecting the Fubini null hypothesis. This difference precisely quantifies the degree of non-orthogonality induced by the hidden variable, analogous to how torsion arises in differential geometry.

Remark 5.1 (Simpson's Paradox as a Fubini Failure)**.** *Simpson's Paradox occurs when a trend appearing in several different groups of data disappears or reverses when these groups are combined. This can be interpreted as a failure of Fubini's commutativity when integrating over a confounding "group" dimension. If combining (integrating out) a hidden group variable changes the observed relationship between two other variables, it implies that the order of integration (e.g., integrating within groups then across time, versus integrating across time then across groups) does not commute. My Fubini test can therefore identify when such paradoxical behavior is due to an unobserved confounder, guiding the researcher to either identify the proper conditioning variable (the "group" in Simpson's) or acknowledge that the causal relationship is fundamentally non-integrable in the observed subspace. This provides a formal criterion for selecting the appropriate level of analysis or identifying irreducible confounding.*

Remark 5.2 (Extension to Higher Dimensions)**.** *This principle naturally extends to higher dimensions. For any triplet of variables (f^i, f^j, t), one can test whether $\int(\int(\int \Psi df^i) df^j) dt = \int(\int(\int \Psi dt) df^j) df^i$, and so on for all permutations. A non-commuting integral involving the time dimension t and any spatial dimension f^i is particularly indicative of a *causal* confounding, as it signifies that the time evolution of f^i is non-trivially coupled to other factors not accounted for by the explicit ordering of processes. The test can be applied to any k-tuple of dimensions, providing a comprehensive diagnostic for detecting hidden multi-dimensional causal dependencies.*

□

5.2 Formalizing Causal State Space and Hypertime

My framework rigorously defines causality not merely as correlation or temporal precedence, but as a specific kind of non-zero change in the field, relating to higher-order derivatives and introducing a novel "derivative dimension."

Definition 5.1 (Causal State Vector and Space)**.** *At each point $x^\mu \in \mathcal{M}$, I define a* ***Causal State Vector*** $\mathbf{C}(x^\mu)$ *as a vector in an infinite-dimensional* ***Causal State Space*** $\mathcal{C}$*. The components of* $\mathbf{C}(x^\mu)$ *represent the field* $\Psi(x^\mu)$ *and its successive total time derivatives:*

$$\mathbf{C}(x^\mu) = \left(\Psi(x^\mu), \frac{D\Psi}{Dt}(x^\mu), \frac{D^2\Psi}{Dt^2}(x^\mu), \dots, \frac{D^k\Psi}{Dt^k}(x^\mu), \dots\right)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + V^\nu \nabla_\nu$ *is the covariant time derivative along a generalized flow vector* V^ν*, capturing the intrinsic time evolution of the field and its components. This vector serves as my formal "causal position vector" in* $\mathcal{C}$*. For a tensor field* $\Psi^{\alpha\beta}$*, each component* $(\Psi^{\alpha\beta})_k = \frac{D^k\Psi^{\alpha\beta}}{Dt^k}$ *would be an entry in the causal state vector, effectively expanding* $\mathcal{C}$ *to include components for all indices of* Ψ*.*

Definition 5.2 (Derivative-Degree Spacetime and Basis)**.** *I define the* ***Derivative-Degree Spacetime*** $\mathcal{D}$ *as an infinite-dimensional vector space whose basis vectors* $\mathbf{e}_k$ *correspond to the* k*-th order of temporal differentiation* ($k \in \{0, 1, 2, \dots\}$)*. Specifically:*

- $\mathbf{e}_0$*: corresponds to the 0-th derivative (position or field value).*
- $\mathbf{e}_1$*: corresponds to the 1st derivative (velocity or rate of change).*
- $\mathbf{e}_2$*: corresponds to the 2nd derivative (acceleration or rate of change of velocity).*
- $\dots$
- $\mathbf{e}_k$*: corresponds to the* k*-th derivative.*

Any element in $\mathcal{D}$ *can be written as a linear combination* $\sum_k c_k \mathbf{e}_k$*. In this sense, these derivative orders represent distinct, hierarchical* ***temporal dimensions****, forming a* ***temporal spacetime****where* ***time is changing each dimension****, and this change represents the* ***magnitude of causal impact geometrically****.*

Definition 5.3 (Hypertime Coordinate System and Manifold)**.** *I define the* ***Hypertime manifold*** $\mathbb{H}$ *as a manifold whose coordinates at each point* $x^\mu \in \mathcal{M}$ *are given by the components of the Causal State Vector* $\mathbf{C}(x^\mu)$*. A point in Hypertime is denoted* $\tau = (\tau_0, \tau_1, \tau_2, \dots)$*, where* $\tau_k = \frac{D^k\Psi}{Dt^k}(x^\mu)$*. Thus, Hypertime is a phase space that captures the entire history of a field's rate of change, making the state of "change" itself a coordinate. The basis vectors for Hypertime can be denoted* ∂_{τ_k}*. This formalizes the concept that successive temporal derivatives constitute the fundamental* ***temporal dimensions****within which causal dynamics unfold.*

Remark 5.3 (Directional Possibility Space in Hypertime)**.** *The structure of Hypertime, through iterated integration over constant intervals, reveals an exponential expansion of* ***directional possibility space****for causal influences. Given a unit influence over a time interval* $[0, T]$*, the integrated magnitudes at successive derivative levels are:*

- k***-th derivative level:*** $\int_0^T 1\, dt = T$ *(linear possibilities for influencing the* $(k-1)$*-th derivative).*
- *(*$k-1$***)-th derivative level:*** $\int_0^T T\, dt = T^2$ *(a* $T \times T$ *grid of possibilities for influencing the* $(k-2)$*-th derivative).*
- *(*$k-2$***)-th derivative level:*** $\int_0^T T^2\, dt = T^3$ *(a* $T \times T \times T$ *cube of possibilities for influencing the* $(k-3)$*-th derivative).*
- ***Position level (0th derivative, if*** $k = 3$ ***initial source):*** $\int_0^T T^3\, dt = T^4$ *(a* T^4 *hypercube of possibilities for final position).*

This demonstrates that the total ***directional possibility space volume*** *available at derivative level m (from an initial k-th order source) grows as T^{k-m+1}. However, the* ***actual trajectory magnitude*** *generated by a constant unit influence over time T up to derivative level m (from a k-th order initial cause) is $\int_0^T \cdots \int_0^T 1\, dt^{k-m+1} = \frac{T^{k-m+1}}{(k-m+1)!}$.*

For a $(3D+T)$ spacetime, a k-th order tensor representing causal influence (e.g., $(n+2)$-th derivative where n is the positional dimension, $k = n+2$) would operate in a directional possibility space where each component runs over the spacetime indices, such as $\mathbf{T}^{(k)}_{\mu_1\mu_2\ldots\mu_k} \in \mathbb{R}^{4^k}$. The "unused" space, representing the difference between the exponentially growing possibility space and the polynomially growing trajectory magnitude, does not imply ongoing causal freedom within a chain. Instead, it represents the ***unchosen directions****in the causal possibility manifold at the point of causal initiation. Once a causal chain begins, the ripple effects propagate deterministically, preserving directional coherence across all derivative levels.*

[1]

Remark 5.4 (The Causal Sphere of Possibility and Its Geodesics)**.** *The interplay between the vast directional possibility spaces defined in Hypertime and the actual trajectories traced within them forms a core geometric metaphor for causality:* ***"The sphere of possibility and the causal geodesic through it."*** *The "sphere of possibility" is the effective volume of reachable states (like the T^k directional possibility space) that a causal influence can affect, bounded by the maximum causal forces (e.g., $K^{(k)}_{\max}$) acting over time T. The "causal geodesic" is the specific deterministic path actually taken through this sphere, defined as the path that minimizes the generalized action or effort $D_{ab}(\gamma)$ (Axiom 2.5). Because causal chains propagate deterministically with directional coherence, the possibility space is not uniformly filled. Instead, the physically realized paths are constrained to specific geodesics, forming a 'ring' or 'shell' within the larger theoretical volume. In the absence of an explicit scale and for a point-like object (occupying one space), this 'ring' can be conceptualized as* ***one unit wide****, representing the narrow band of optimal or maximally efficient causal evolutions. Observed deviations from this *predicted minimal path* (as computed by the current, observable model) serve as empirical evidence for the presence of an* ***unobserved variable****. This unobserved variable acts as a hidden causal force, providing the "energy" or influence that perturbs the system, effectively* ***altering the underlying geometry of the true causal landscape****and leading to a trajectory that appears non-minimal from the perspective of the incomplete, observed subspace.*

Definition 5.4 (Causal Possibility Shell)**.** *Let $x^{(k)}(t)$ be the k-th temporal derivative of position $x(t)$, and let $K^{(k)}_{max}$ be its maximal bounded value. The* ***Causal Possibility Shell*** *$\mathcal{S}^{(k)}(t)$ at time t for the position variable, initiated by the k-th order derivative, is defined as the set of all positions reachable from an initial state under admissible trajectories where the k-th derivative is bounded: $\mathcal{S}^{(k)}(t) = \left\{ x(t) \;\middle|\; \left| \frac{d^k x}{dt^k} \right| \leq K^{(k)}_{max},\ t \in [0,T] \right\}$ For a constant maximal bounded k-th derivative $K^{(k)}_{max}$ applied over a time interval T, the range of displacement for $x(T)$ (assuming zero initial conditions for all derivatives below k per Axiom 2.7) is given by: $x(T) \in \left[-\frac{1}{k!} K^{(k)}_{max} T^k,\ +\frac{1}{k!} K^{(k)}_{max} T^k \right]$ This shell forms a bounded region whose volume scales proportionally to T^k, representing the maximum reachable displacement from the influence of the k-th derivative.*

Definition 5.5 (Causal Geodesic)**.** *Given a field $\Psi(x^\mu)$ and its associated covariant derivatives, the cost functional $D_{ab}(\gamma) = \int_\gamma S(x^\mu) ds$ from Axiom 2.5 defines an action. The* ***Causal Geodesic*** *$x^*(t)$ is the path that minimizes this cumulative action, i.e., the optimal or most likely causal trajectory through the Causal Possibility Shell $\mathcal{S}^{(k)}(t)$: $x^*(t) = \arg\inf_{\gamma \in \Gamma_{ab}} D_{ab}(\gamma)$ More specifically, for a causal influence primarily characterized by a k-th order derivative, the causal geodesic can be derived from the extremization of an action functional based on that derivative. For example, minimizing cumulative kinetic energy associated with the k-th derivative of position: $x^*(t) = \arg\min_{x(t)} \int_0^T \left\| \frac{d^k x}{dt^k} \right\|^2 dt$ This variational principle identifies the specific trajectory taken through the space of possibilities defined by the Causal Possibility Shell.*

Definition 5.6 (Derivative Lattice of Reachability)**.** *The* ***Derivative Lattice of Reachability*** *is a layered combinatorial structure representing the number of discrete states reachable at time T, given an*

[1]Hypertime is not a second temporal metric—our singular time t remains the only temporal flow. Rather, Hypertime represents how **iterated integration** of our singular time dimension creates nested causal possibility spaces of exponentially increasing complexity. We observe these spaces through derivatives from our $(3D+T)$ vantage point, but the exponential structure emerges from the mathematical compounding of temporal integration across derivative levels.

N-ary (e.g., ternary, $N = 3$) step for causal influence per unit time at each derivative order. For a causal process initiated by an N-ary step from the (k)-th derivative, the number of discrete states reachable at the (m)-th derivative level ($m < k$) over time T (where T represents units of time) is: $N^{(m)}_{states}(T) = T^{k-m} \times N^{(k-m)}$ More precisely, if each derivative level j has a discrete choice set of size N_j for its range over time T, the total possibility space across k derivative levels would be a product of these ranges. If $N_j = T$ (as in the T^k discussions of directional possibility space), the volume of possibility for a k-th order source influencing the m-th order state over T time units is: Volume of Possibility Space$^{(m)}(T) = T^{k-m+1}$ For the example of position being the 0th derivative, and a (k)-th order derivative (like the $(n+2)$-th derivative where n is the positional dimension, $k = n+2$) as the highest causal source over time T:

- *Velocity ($m = 1$): $T^{(n+2)-1+1} = T^{n+2}$ total volume of possibility for its range of motion.*
- *Position ($m = 0$): $T^{(n+2)-0+1} = T^{n+3}$ total volume of possibility for its range of motion.*

The total phase space volume for these derivative states, considering all cross-derivative combinations up to the k-th order, can be seen as proportional to $\prod_{j=0}^{k} T^{k-j+1} = T^{(k+1)(k+2)/2}$.

Theorem 5.2 (Dimensional Growth of Causal Possibility Volume)**.** *Under bounded linear influence from the k-th order derivative, the volume of the maximum reachable possibility space for the position (0th derivative) grows as T^{k+1} with respect to time T. Specifically, if the k-th derivative is bounded by $K^{(k)}_{\max}$, the maximal displacement for position $x(T)$ from a constant input of $K^{(k)}_{\max}$ over time T (assuming zero initial conditions for all lower derivatives per Axiom 2.7) defines a range such that the overall possibility volume scales proportionally to T^{k+1}. This power-law scaling directly defines the higher-dimensional geometry of causal influence and demonstrates why T^k (representing sequential range compounding, for a fixed initial k-th derivative input) and T^{k+1} (for total possibility volume) rather than $\int dt^k$ (representing specific trajectory) are the appropriate metrics for the volume of causal possibility.*

Definition 5.7 (Hypertime Operator $\mathcal{H}_\Psi$)**.** *I define the* ***Hypertime Operator*** *$\mathcal{H}_\Psi$ as a formal operator that encapsulates the recursive generation and interaction across derivative degrees within the Causal State Space. Acting on a component τ_k of the Hypertime coordinates, $\mathcal{H}_\Psi$ generates the next higher-order derivative component, or more generally, describes the evolution within Hypertime: $\mathcal{H}_\Psi(\tau_k) = \tau_{k+1}$ This operator formally represents the process of successive covariant time differentiation, $\frac{D}{Dt}$, within the Hypertime manifold, formalizing the recursive nature of causal dynamics across its temporal dimensions.*

Definition 5.8 (Causal Origin Principle (COP))**.** *In a recursive derivative manifold (Hypertime), the causal force acting on a system at derivative order n originates at derivative order $n+1$. That is, a causal influence on the n-th temporal dimension is fundamentally driven by the state of the $(n+1)$-th temporal dimension: Cause$_n \equiv \frac{D^{n+1}\Psi}{Dt^{n+1}}$ for field Ψ Thus, each higher derivative acts as the source or "coordinate of causal origin" for the dynamics observed at the level below. This principle reframes the structure of influence not as a flat chain, but as a recursive address space, where, for instance, velocity (1st derivative) is caused by acceleration (2nd derivative), acceleration by the $(n+2)$-th derivative (where n is the positional dimension), and so on. Each layer is driven by the next—but visible only via projection into lower-dimensional observation. This is how hidden variables "ripple" downward: they're higher-derivative displacements folded into lower-order state changes.*

Remark 5.5 (Causal Flow and Quantitative Cascades)**.** *Given the Causal State Vector $\mathbf{C}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \\ x^{(n+2)}(t) \\ \vdots \end{bmatrix}$, the causal flow is not merely in the state vector itself, but is formally indicated by a non-zero higher-order derivative driving changes in the component below it in the derivative hierarchy. A* ***Causal jump at level n iff*** *$\frac{D}{Dt}x^{(n)} \neq 0$, which is equivalent to $x^{(n+1)} \neq 0$. This implies that a causal input is always manifested as a non-zero higher-order derivative that drives changes in the component below it in the derivative hierarchy. For example, if a constant k-th derivative $X^{(k)}$ acts for time T, its accumulated effect on position (x) is proportional to $X^{(k)} \times T^k$. This illustrates how a cause originating in a higher temporal dimension (k-th derivative) cascades its influence down, manifesting with increasing magnitude across lower temporal dimensions (e.g., position, 0th derivative), effectively mapping the magnitude of change across the multi-dimensional structure of Hypertime. This propagation exemplifies the directionality of causal influence within the Hypertime manifold.*

Theorem 5.3 (Causality as Non-Zero $(n+2)$-th Derivative)**.** *Causality, defined as the direct generative influence of one event or field configuration on another, is formally identified when the $(n+2)$-th component of the Causal State Vector is non-zero, corresponding to the $(n+2)$-th derivative of the relevant field component with respect to time, evaluated along a causal path. That is, a causal event for a base field value at the n-th derivative level occurs if $\frac{D^{n+2}\Psi}{Dt^{n+2}} \neq 0$. More generally, for a tensor field $\Psi^{\alpha\beta}$ (at minimum rank 2, as per Axiom 2.3 and Definition 2.1), causality is implied by a non-zero component $(\Psi^{\alpha\beta})_{(n+2)} = \frac{D^{n+2}\Psi^{\alpha\beta}}{Dt^{n+2}} \neq 0$, where n is the derivative degree of the observed "causal position" component, in accordance with the Causal Origin Principle (Definition 5.8).*

Proof. Let's consider the general case where the observed "causal position" is at the n-th derivative level, $\tau_n = \frac{D^n\Psi}{Dt^n}$.

- Position (n-th derivative, τ_n): $\frac{D^n\Psi}{Dt^n}$
- Velocity ($n+1$-th derivative, τ_{n+1}): $\frac{D^{n+1}\Psi}{Dt^{n+1}}$
- Acceleration ($n+2$-th derivative, τ_{n+2}): $\frac{D^{n+2}\Psi}{Dt^{n+2}}$

If the observed causal position is $\frac{D^n\Psi}{Dt^n}$, then causality is defined by $\tau_{n+2} = \frac{D^{n+2}\Psi}{Dt^{n+2}} \neq 0$. This generalizes Newton's second law, where force (the cause) directly produces acceleration (a non-zero second derivative of position). For any derivative level n, a system only experiences a true "causal push" if its rate of change is itself changing at the $(n+2)$-th derivative level. Constant velocity at the $(n+1)$-th level (zero $(n+2)$-th derivative) implies no ongoing external causal input at that level. This is a direct consequence of the Causal Origin Principle, where the "Cause" for the n-th derivative level is $\frac{D^{n+1}\Psi}{Dt^{n+1}}$, and the effect on the n-th level is the $(n+2)$-th derivative $\frac{D^{n+2}\Psi}{Dt^{n+2}}$.

For the specific case of position ($n=0$): If the "causal position" is the field itself, Ψ ($n=0$), then causality is defined by $\tau_{0+2} = \tau_2 = \frac{D^2\Psi}{Dt^2} \neq 0$, which aligns with the classical understanding of acceleration as the signature of causal force.

This framework implies that visible "ripple effects" in lower derivative dimensions (e.g., changes in position or velocity) are ultimately consequences of these higher-order causal events. A non-zero $(n+2)$-th derivative in the spacetime field Ψ generates cascades of non-zero lower-order derivatives down to the base level, creating observable changes that propagate through the system.

Remark 5.6 (Mathematical Illustration (Rank 2 Field):)**.** *For a rank 2 field $\Psi^{\alpha\beta}(x^\mu)$, causality is manifest when a specific component of its causal state vector corresponding to the $(n+2)$-th derivative is non-zero. For instance, if the observed causal position is a field component Ψ^{01} (e.g., coupling between time and feature 1, at the $n=0$ level), then causality is indicated by: $\frac{D^{0+2}\Psi^{01}}{Dt^{0+2}} = \frac{D^2\Psi^{01}}{Dt^2} \neq 0$ This indicates an "acceleration" in the time-feature coupling. If the causal position is a spatial gradient component, e.g., $\nabla_1\Psi^{01}$ (which itself is a first derivative along f^1, so we can consider this as a new "position" at $n=0$ for this derived quantity), then causality on this gradient is indicated by its $(0+2)$-th total time derivative: $\frac{D^{0+2}}{Dt^{0+2}}\left(\nabla_1\Psi^{01}\right) = \frac{D^2}{Dt^2}\left(\nabla_1\Psi^{01}\right) \neq 0$ This signifies an "acceleration" in the spatial gradient itself. This recursive structure allows for a precise localization and quantification of causal events within the complex dynamics of the tensor field across the temporal dimensions.*

□

Proposition 5.4 (Tensor Extension: Causality Operator $\mathcal{K}_{\text{event}}$)**.** *The conceptual product "velocity × derivative$^{(n+2)\prime} formally yields a component of the Causality Operator that identifies non-zero (n+2)-th derivatives. I def$ as a tensor field whose components capture the higher-order causal influences. Its structure emerges from contractions or exterior products of fundamental causal state elements.*

Proof. I formalize "velocity" and "derivative$^{(n+2)\prime} in terms of elements of the Causal State Vector$ C(x$^\mu$) and the Derivative-Degree Space $\mathcal{D}$.

- **Causal Velocity Component**: Let $V_t = \frac{D\Psi}{Dt}$ (the component of $\mathbf{C}$ at derivative degree 1, τ_1). More generally, for a specific field component Ψ^{ab}, the causal velocity is $\frac{D\Psi^{ab}}{Dt}$.
- **Recursive Derivative Component** τ_{n+2}: This is the $(n+2)$-th component of the Hypertime coordinates, $\tau_{n+2} = \frac{D^{n+2}\Psi}{Dt^{n+2}}$. The "derivative$^{(n+2)\prime} represents the recursive application of the$ D$\overline{Dt}$ operator $(n+2)$ times, leading to the $(n+2)$-th order derivative component τ_{n+2}.

The phrase “velocity $\times$ derivative$^{(n+2)}$ $\rightarrow$ causality operator” implies that the Causality Operator $\mathcal{K}_{\text{event}}$ can be thought of as identifying the specific component τ_{n+2} of the Causal State Vector that becomes non-zero due to a causal event. This $\mathcal{K}_{\text{event}}$ operates by recursively advancing through the derivative degrees. The “product” refers to the conceptual composition that results in this identification, which is formally achieved by the repeated application of the Hypertime Operator $\mathcal{H}_\Psi$.

For a general field $\Psi^{\alpha\beta}$, the Causality Operator $\mathcal{K}_{\text{event}}$ that quantifies the $(n{+}2)$-th derivative takes the form of a generalized differential operator acting on $\Psi^{\alpha\beta}$. The “velocity $\times$ derivative$^{(n+2)\prime} pointstotheprocessofapplyingsu$
$\mathrm{K}_{\text{event}}(\Psi^{\alpha\beta}) = \mathcal{H}_\Psi^{n+1}\left(\frac{D\Psi^{\alpha\beta}}{Dt}\right) = \frac{D^{n+2}\Psi^{\alpha\beta}}{Dt^{n+2}}$ This identifies the specific $(n+2)$-th derivative component in the Causal State Vector that becomes non-zero, indicating the causal origin in that Hypertime dimension.

Remark 5.7 (Interaction Order Alters Causality Class:)**.** *The “interaction order” in the field evolution equation* $\frac{\partial\Psi}{\partial t} = \mathcal{O}[\Psi] + J(x^\mu)$ *refers to the non-linear terms in* $\mathcal{O}[\Psi]$ *(e.g., terms proportional to* Ψ^2*,* Ψ^3*, or products of different* Ψ *components) or in* $J(x^\mu)$*. These non-linearities can alter the “causality class,” meaning they can change the minimum* n *for which* $\frac{D^{n+2}\Psi}{Dt^{n+2}} \neq 0$*. For instance, a linear causal interaction might primarily manifest as a non-zero* $\frac{D^{0+2}\Psi}{Dt^{0+2}} = \frac{D^2\Psi}{Dt^2}$ *(causality class* $n = 0$*). However, a strong quadratic interaction (e.g.,* Ψ^2 *term in* $\mathcal{O}[\Psi]$*) might generate a primary causal signature at the level of* $\frac{D^{1+2}\Psi}{Dt^{1+2}} = \frac{D^3\Psi}{Dt^3}$ *(causality class* $n = 1$*), or even higher. This happens because non-linear terms can recursively generate higher-order changes. These non-linear couplings mean that the very dynamics of the field dictate at which derivative degree the causal influence becomes most prominent, effectively shifting the “causality class” across Hypertime’s temporal dimensions. This is formalized by how the components of the Causality Operator* $\mathcal{K}_{event}$ *acquire non-zero values due to these specific non-linear interactions.*

□

Remark 5.8 (Causality is Curvature in Derivative Space)**.** *The elegant identity that* ***Causality is curvature in derivative space*** *unifies the geometric, causal, and recursive derivative aspects of my theory. This implies that a causal event is a localized “bending” or “distortion” within the* ***Derivative-Degree Spacetime (Hypertime)****. Just as mass-energy causes spacetime to curve in General Relativity, a causal force induces curvature in this abstract space of successive rates of change (i.e., its temporal dimensions). This curvature in Hypertime corresponds directly to a non-zero higher-order derivative, indicating the active generative force. Fubini’s rejection then detects when that curvature is non-separable — i.e., when the order of integration fails to commute, implying influence is not orthogonal and not contained within the observed coordinate patch. This identity serves as a powerful conceptual anchor, highlighting the deep geometric nature of causality within my framework.*

> **Plain English:** Just as heavy objects bend spacetime in Einstein’s theory, causal forces ”bend” the space of derivatives (Hypertime). When we detect this bending through non-commuting integrals, we’ve found evidence of causality or hidden influences.
>
> **Math Translation:** Causality manifests as curvature in the infinite-dimensional space spanned by all orders of derivatives, detectable through the failure of Fubini’s theorem (non-commuting integrals).

Conjecture 5.5 (Hypertime: The Arena for Consciousness and Cognition, and The Elusive Origin of Free Will)**.** *The concepts of a “derivative dimension” and “velocity dimensional space” implicitly suggest the existence of* ***Hypertime****. Hypertime is a richer temporal structure that includes not only the standard unidirectional flow of events (t) but also higher-order temporal dimensions that characterize the dynamics of change itself (e.g., derivatives of derivatives). The interaction between the level of derivative and velocity (a ratio of derivatives) creates a recursively complex space where consciousness and cognition, with their inherently recursive, meta-cognitive, and multi-layered processing, emerge as consequences of dynamics unfolding in these higher-order dimensions of Hypertime. This is formally represented by the relationship where higher-order temporal dimensions recursively act as the causal origin for lower-order ones, enabling complex, emergent phenomena. This can be conceptually summarized by a generative principle for quantitative concepts: Quantitative Concept* $\sim$ *Idea Concept* $\times$ $($*Generative Metric Operator*$)^n$ *where* n *signifies the iteration of applying the generative metric operator (related to* $\mathcal{H}_\Psi$*), directly mapping to the depth within the derivative dimensions (Hypertime). For example, “length of a 10-foot pole” is not merely “pole”* $\times$ *“10 feet,” but “pole”* $\times$ *(“foot”* $\times$ *“iteration operator”)*^{10}*, where each iteration generates further quantifiable complexity across dimensions. This recursive construction generates complexity and defines the unique interaction space of cognitive phenomena.*

Furthermore, within this deterministic framework, the existence of free will presents a profound challenge. If it exists, its origin likely resides at a level beyond our current capacity for systemic analysis, akin to a "rigged quantum collapse." This suggests a possibility where the answer to free will lies one dimension higher in the causal hierarchy, but that higher dimension itself is mutually constitutive with (creates or was created by) the emergence of new causal dimensions. This fundamental conundrum means that, like free will, the ultimate origin of causality within Hypertime remains elusive, resulting in the answer to the free will debate being just ***Gödel's ghost laughing****.*

6 Fractional Causal Spectrum and Future Work

6.1 Generalization via Fractional Derivatives

We define causal influence not as a function of a single derivative order, but as a continuous distribution over a spectrum of derivative orders $\alpha \in \mathbb{R}^+$. This leads to the **Fractional Action Functional**: $\mathcal{A}[x] = \int_0^T \int_0^{\alpha_{\max}} w(\alpha) \, |{}_a\mathcal{D}_t^\alpha x(t)|^p \; d\alpha \, dt$ Here, $w(\alpha)$ serves as the **causal weighting function**, representing the system's tendency to privilege specific derivative orders.

- When $w(\alpha) = \delta(\alpha - n)$, we recover the classical n-th order model.
- When $w(\alpha)$ is smooth or multimodal, we capture systems with memory, fatigue, or hybridized dynamics.

6.2 Interpretation: Causal Shell Deformation

Causal shells deform under different $w(\alpha)$ profiles:

- **Sharp shells**: Highly localized $w(\alpha)$ (e.g., classical systems) produce thin, sharply bounded reach sets.
- **Diffused shells**: Broad $w(\alpha)$ spectra yield expanded, memory-inclusive causal regions.

This implies that the *geometry* of future possibility is directly tied to the system's underlying derivative cost function.

6.3 Future Work: Inverse Spectral Inference

A promising direction is **inverse inference**: recovering the implicit causal spectrum $w(\alpha)$ from observed trajectories. This would enable:

- **Empirical system identification**: Characterizing unknown systems through their causal memory signatures
- **Cognitive or physical profiling**: Understanding biological or artificial systems via their temporal processing characteristics
- **Adaptive policy design**: Developing control systems that adapt to the discovered causal spectrum

Such techniques may involve spectral regularization, entropy-constrained inference, or neural mapping of trajectories to fractional support profiles.

> **Summary:** By watching how systems behave, we can reverse-engineer their "causal DNA" - the unique signature of how they process influence across time scales.

6.4 Computational Frontiers

Several computational challenges emerge from this framework:

1. **Numerical methods for fractional geodesics**: Developing efficient algorithms for computing optimal paths under fractional action functionals
2. **Real-time fractional derivative computation**: Implementing fast fractional calculus for dynamic systems

3. **Spectral optimization**: Finding optimal $w(\alpha)$ profiles for specific causal inference tasks
4. **Multi-scale causal detection**: Identifying causal effects across different temporal scales simultaneously

7 Conclusion

In summary, by operating in a dynamic (nD + T) spacetime with **continuous fractional derivative spectra**, this generalized field theory offers a direct representation of causal flow through the irreversible nature of time and the principle of least effort. The **continuous nature**of the framework, enhanced by fractional calculus, enables modeling of **non-local causal memory**, **distributed temporal influence**, and **scale-dependent causal propagation**—capturing phenomena that discrete approaches fundamentally cannot access. Furthermore, it provides concrete, measurable indicators—such as non-integrability detected by the exterior derivative and deviations from predicted cross-modal consistencies—that serve as powerful detection mechanisms for unobserved variable bias, moving beyond merely statistical correlation to a more profound, physically interpretable understanding of causality. The inherent capabilities of tensor calculus and linear algebra are integral to formulating these concepts and executing the necessary analytical tests within this framework. My framework fundamentally clarifies the nature of feedback loops by showing how they are inherently acyclic in true spacetime, resolving traditional ambiguities and allowing for a more complete causal analysis. The introduction of Fubini's theorem as a powerful diagnostic for hidden variables, coupled with my novel definition of causality based on **continuous fractional derivatives**and the emerging concept of Hypertime (where derivatives *are* temporal dimensions), lays a foundational mathematical language for understanding the complex dynamics of consciousness and cognition itself. The **fractional generalization**from discrete derivative orders to the $(n+\alpha)$-th derivative framework provides a unified approach that scales naturally with the dimensionality of the system under study and captures **temporal texture**through continuous causal gradation, while the incorporation of relative reference frames ensures that all causal measurements are made with respect to the system's intrinsic dynamics rather than arbitrary external coordinates. This **continuous fractional framework**is particularly powerful for modeling biological systems, economic dynamics, neural networks, and any system exhibiting **memory effects**and **non-instantaneous influence propagation**.

References

Appendix A: Glossary of Terms

This appendix provides brief conceptual overviews of mathematical and theoretical concepts fundamental to the framework presented in this document.

7.1 Differential Geometry and Manifolds

Manifolds

A manifold is a mathematical space that locally resembles Euclidean space but can be globally curved or twisted. For example, the surface of a sphere is a 2D manifold, as any small patch on it looks flat like a piece of paper, but globally it is curved. Manifolds are essential for representing spaces (like spacetime) where coordinates can be locally defined but global flatness is not assumed.

Metric Tensor

On a manifold, a metric tensor $g_{\mu\nu}$ provides a way to measure distances, angles, and volumes. It is a mathematical object that, at each point, defines an inner product on tangent vectors. For a curve parametrized by λ, the infinitesimal square of arc length is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The specific signs in its diagonal form (its "signature") are crucial.

Riemannian vs. Pseudo-Riemannian/Lorentzian Geometry

- **Riemannian Geometry**: All diagonal elements of the metric tensor are positive (e.g., $(+,+,+)$ for 3D space). This corresponds to familiar Euclidean-like geometry, where distances are always positive and there is no inherent distinction between "time" and "space."
- **Pseudo-Riemannian Geometry**: The diagonal elements of the metric tensor can have mixed signs (e.g., $(-,+,+)$ for 2D space + 1D time). This allows for "timelike" intervals (where $ds^2 < 0$), "spacelike" intervals (where $ds^2 > 0$), and "null" or "lightlike" intervals (where $ds^2 = 0$).
- **Lorentzian Geometry**: A specific type of pseudo-Riemannian geometry with exactly one negative sign (e.g., $(-,+,+,+)$ for 3D space + 1D time). This is the mathematical framework for Einstein's theory of General Relativity, where the negative sign on the time component gives time its unique character and defines the causal structure through light cones.

Covariant Derivative and Christoffel Symbols

In curved spacetime, the "direction" of a vector changes as it moves from point to point due to the curvature. The covariant derivative ∇_μ is a modification of the ordinary partial derivative that accounts for this effect, ensuring that tensor equations hold consistently across the manifold. It incorporates Christoffel symbols, which are components derived from the metric tensor that describe how the basis vectors change across the manifold.

Geodesics

Geodesics are the "straightest possible paths" in a curved manifold. In flat Euclidean space, these are straight lines. On a sphere, they are great circles. In spacetime, timelike geodesics represent the paths of freely moving particles. In this framework, they signify paths of "least effort" or optimal transitions between system states.

7.2 Field Theory

Fields

In physics, a field is a physical quantity that has a value at every point in space and time. Examples include the electric field, magnetic field, or gravitational field. Fields can be scalars (a single number at each point), vectors (a magnitude and direction), or tensors (more complex mathematical objects). The dynamics of fields are typically described by partial differential equations.

Partial Differential Equations (PDEs)

PDEs are equations that involve an unknown function of multiple independent variables and their partial derivatives. They are used to describe how physical quantities (like fields) evolve in space and time. Examples include the wave equation, diffusion equation, or Maxwell's equations.

D'Alembertian Operator

The d'Alembertian operator, often written as $\Box$, is a fundamental differential operator in relativistic physics. It is the spacetime generalization of the Laplacian operator (which describes diffusion or steady-states in space). The d'Alembertian appears in wave equations, describing how disturbances propagate at finite speed.

7.3 Tensor Calculus and Linear Algebra

Tensors

Tensors are mathematical objects that generalize scalars (0-order tensors), vectors (1st-order tensors), and linear operators (2nd-order tensors). They provide a framework for describing physical quantities that have different components in different coordinate systems but represent the same underlying reality. The metric tensor $g_{\mu\nu}$ is a key example.

Lie Bracket $[V, W]$

For vector fields, the Lie bracket $[V, W]$ measures the degree to which applying two vector field transformations in different orders (i.e., VW versus WV) produces a different result. If $[V, W] = 0$, the transformations commute. A non-zero Lie bracket between coordinate basis vector fields can indicate torsion or curvature, or a hidden dependency in the system.

Exterior Derivative d

The exterior derivative d is an operator in differential geometry that takes an n-form to an $(n+1)$-form. It generalizes common vector calculus operations: for a scalar function, df is its gradient; for a vector field (represented as a 1-form), $d\omega$ captures its "curl." If $d\omega = 0$, the 1-form is "exact" (meaning it can be expressed as the gradient of a scalar potential), implying path independence or conservative forces. A non-zero exterior derivative indicates non-conservative behavior or rotational components.

Linear Algebra

The fundamental operations of linear algebra, such as vector space analysis, matrix operations (e.g., for transformations, norms, and inner products), and eigenvalue problems, are essential for computing quantities like spectral content (via Fourier transforms), spatial distances (via matrix norms), and correlations between multidimensional data. They provide the computational tools to work with tensors and fields numerically.

7.4 Causal Inference Concepts

Difference-in-Differences (DiD)

A quasi-experimental method used to estimate the causal effect of an intervention. It compares the change in outcomes over time for a group exposed to the intervention (treatment group) to the change in outcomes over time for a group not exposed (control group), assuming "parallel trends" in the absence of treatment.

Regression Interaction Effects

In a regression model, an interaction effect occurs when the effect of one independent variable on the dependent variable changes depending on the level of another independent variable. For instance, the effect of education on income might differ for men and women.

Potential Outcomes Framework

A formal approach to causal inference that defines causal effects by comparing "potential outcomes" under different treatment assignments for the same unit. The fundamental problem is that only one potential outcome (the one corresponding to the observed treatment) can ever be realized.

Dynamic Structural Equation Modeling (DSEM)

An extension of Structural Equation Modeling (SEM) that explicitly incorporates time and dynamic relationships. It models how latent (unobserved) and observed variables influence each other over time, often using time-series data.

Propensity Scoring

A statistical method used to reduce confounding bias in observational studies. It estimates the probability of receiving a treatment given a set of observed covariates (the propensity score), and then uses this score to balance treatment and control groups, making them more comparable.

Regression Discontinuity Design (RDD)

A quasi-experimental design that identifies causal effects by exploiting a "sharp" (discontinuous) threshold in an assignment variable. Units just above and just below the threshold are assumed to be comparable, differing essentially only in their treatment status. The causal effect is estimated by comparing outcomes at the discontinuity.

Regression Kink Design (RKD)

Similar to RDD, but instead of a discontinuous jump in treatment probability, RKD exploits a "kink" or a discontinuous change in the *slope* of the treatment intensity as a function of an assignment variable. The causal effect is estimated by comparing the change in the slope of the outcome variable at the kink.

7.5 Fractional Calculus

Fractional Derivatives

Fractional derivatives extend ordinary derivatives to non-integer orders $\alpha \in \mathbb{R}^+$. The most common definitions are:

- **Riemann-Liouville derivative**: ${}^{RL}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-1-\alpha} f(\tau) d\tau$
- **Caputo derivative**: ${}^{C}_{a}\mathcal{D}^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau$

where $n-1 < \alpha < n$ and Γ is the gamma function.

Memory and Non-locality

Fractional derivatives inherently capture **memory effects**because they involve integrals over the entire history of the function from the starting point a to the current time t. This makes them ideal for modeling systems where past events have a continuously decaying influence on present behavior, such as viscoelastic materials, financial markets, or biological adaptation.

Physical Interpretation

Different fractional orders capture different types of temporal behavior:

- $0 < \alpha < 1$: Sub-diffusive processes with strong memory
- $\alpha = 1$: Classical first derivative (velocity)
- $1 < \alpha < 2$: Super-diffusive processes with intermediate memory
- $\alpha = 2$: Classical second derivative (acceleration)

- **Spacetime Manifold** $(\mathcal{M}, g_{\mu\nu})$: A differentiable manifold equipped with a pseudo-Riemannian metric tensor $g_{\mu\nu}$ of Lorentzian signature $(-,+,...,+)$. It is the geometric setting for events, combining n spatial dimensions and one temporal dimension.
- **Lorentzian Signature**: The specific choice of signs for the diagonal elements of the metric tensor, here $(-,+,...,+)$. The negative sign for the temporal component g_{00} is crucial for defining a causal structure with light cones, distinguishing timelike, null (lightlike), and spacelike intervals.
- **Field** $\Psi(x^\mu)$: A physical quantity that has a value at every point x^μ in spacetime. It can be a scalar, vector, or tensor, and its evolution is governed by differential equations. In this context, it represents system phenomena.
- **Metric Tensor** $g_{\mu\nu}$: A fundamental tensor field that defines distances and angles within the spacetime manifold. In a pseudo-Riemannian manifold, it also defines causal relationships through its signature.
- **Covariant Derivative** ∇_μ: A generalization of the partial derivative for tensor fields on a curved manifold. It accounts for the changing basis vectors in curved space, ensuring that tensor equations transform correctly under coordinate changes.
- **D'Alembertian Operator** $\Box$: Also known as the wave operator, it is a differential operator that appears in wave equations and other hyperbolic partial differential equations. In curved spacetime, it is given by $\Box\Psi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Psi$.
- **Spacetime Interval** ds: The invariant "distance" between two infinitesimally separated events in spacetime, given by $ds = \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$. For timelike paths, it represents proper time.

- **Geodesic**: The path of shortest (or extremal) length between two points in a curved manifold. In Lorentzian spacetime, timelike geodesics represent the paths of freely moving particles. In this framework, they signify paths of "least effort" or optimal transitions between system states.
- **Lie Bracket** $[V, W]$: For two vector fields V and W, the Lie bracket $[V, W] = VW - WV$ is another vector field that measures the extent to which the vector fields fail to commute. A non-zero Lie bracket between coordinate basis vector fields can indicate torsion or curvature, or a hidden dependency in the system.
- **Exterior Derivative** $d\omega$: An operator in differential geometry that generalizes the gradient, curl, and divergence. For a 1-form $\omega = A_\mu dx^\mu$, its exterior derivative $d\omega = (\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu$ measures its "curl-like" behavior. If $d\omega = 0$, the 1-form is "exact" (meaning it can be expressed as the gradient of a scalar potential), implying path independence or conservative forces. A non-zero exterior derivative indicates non-conservative behavior or rotational components.
- **Heaviside Step Function** $H(x)$: A discontinuous function that is 0 for $x < 0$ and 1 for $x \geq 0$. Used to model sharp transitions.
- **Dirac Delta Function** $\delta(x)$: A generalized function (or distribution) that is zero everywhere except at $x = 0$, and its integral over all real numbers is 1. Often used to model impulses or sharp discontinuities in derivatives.
- **Light Cone**: In Lorentzian spacetime, the structure defined by null (lightlike) paths emanating from an event. It separates events into absolute past, absolute future, and causally disconnected (spacelike) regions.
- **Riemannian Manifold**: A manifold equipped with a Riemannian metric tensor, which has a positive definite signature $(+, +, ..., +)$. It defines distances and angles in a way analogous to Euclidean space but lacks the inherent causal structure of Lorentzian spacetime.
- **Covariant Tensor**: A tensor whose components transform inversely to coordinate changes. Its indices are typically subscripts.
- **Contravariant Tensor**: A tensor whose components transform in the same way as coordinate changes. Its indices are typically superscripts.
- **Pseudo-Riemannian Metric**: A generalization of a Riemannian metric where the quadratic form defined by the metric tensor is not necessarily positive definite (it can have mixed signs), allowing for concepts like timelike and spacelike intervals. Lorentzian metrics are a specific type of pseudo-Riemannian metric.
- **Fisher Information Metric ($G_{\mu\nu}$)**: A Riemannian metric on the space of probability distributions, used to measure distances between probability distributions. Here, it could be used to define the metric for observable field strengths related to local field distributions.
- **Causal State Vector $\mathbf{C}(x^\mu)$**: A vector in the infinite-dimensional Causal State Space $\mathcal{C}$ at each point x^μ, whose components are the field $\Psi(x^\mu)$ and its successive total time derivatives. It acts as the formal "causal position vector."
- **Causal State Space** $\mathcal{C}$: An infinite-dimensional vector space whose points are the Causal State Vectors, representing all orders of time derivatives of the field Ψ.
- **Derivative-Degree Spacetime** $\mathcal{D}$: An infinite-dimensional vector space whose basis vectors $\mathbf{e}_k$ correspond to the k-th order of temporal differentiation. These derivative orders are interpreted as distinct temporal dimensions, forming a temporal spacetime where time is changing each dimension, representing the magnitude of causal impact geometrically.
- **Hypertime Manifold** $\mathbb{H}$: A manifold whose coordinates τ_k are given by the components of the Causal State Vector $\mathbf{C}(x^\mu)$, representing the state of change itself. This manifold formally embodies the concept of temporal dimensions defined by successive derivatives.
- **Hypertime Operator** $\mathcal{H}_\Psi$: An operator that encapsulates the recursive generation and interaction across derivative degrees within the Causal State Space, formalizing successive covariant time differentiation and evolution within Hypertime.

- **Causal Origin Principle (COP)**: States that in a recursive derivative manifold (Hypertime), the causal force acting on a system at derivative order n originates at derivative order $n+1$, meaning $\text{Cause}_n \equiv \frac{D^{n+1}\Psi}{Dt^{n+1}}$.
- **Causality Operator $\mathcal{K}_{\textbf{event}}$**: A tensor field whose components quantify higher-order causal influences by identifying non-zero $(n+2)$-th derivatives, constructed by repeated application of the Hypertime Operator.
- **Causal Possibility Shell**: The set of all positions reachable from an initial state under admissible, bounded higher-order derivative trajectories over a given time interval. Its volume scales with T^k where k is the order of the highest derivative influence.
- **Causal Geodesic**: The path that minimizes the generalized action or effort functional ($D_{ab}(\gamma)$) between system states, representing the optimal or most likely causal trajectory through the Causal Possibility Shell.
- **Derivative Lattice of Reachability**: A layered combinatorial structure representing the number of discrete states reachable at different derivative levels over time, based on discrete causal steps.
- **Fractional Derivative ${}_a\mathcal{D}_t^\alpha$**: A generalization of integer-order derivatives to real or complex orders α. The Caputo fractional derivative ${}_a\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau) d\tau$ where $n-1 < \alpha < n$. Fractional derivatives capture **non-local temporal memory**and **distributed influence**across time scales.
- **Fractional Causal Possibility Shell**: A generalization of the causal possibility shell where derivative orders vary continuously rather than discretely, bounded by B_α for each fractional order α, modeling systems with **non-instantaneous influence**and **scale-dependent causal propagation**.
- **Fractional Action Functional**: An action functional $A_{\text{frac}}[x(t)] = \int_0^T \int_0^{\alpha_{\max}} w(\beta) \left| {}_a\mathcal{D}_t^\beta x(t) \right|^p d\beta\, dt$ that incorporates the continuous spectrum of fractional derivatives, where $w(\beta)$ weights the contribution of each fractional order.
- **Non-Local Causal Memory**: The property of fractional derivative systems where past events influence present dynamics through a continuously weighted temporal kernel, rather than through discrete time steps. This enables modeling of **viscoelastic memory**, **neural adaptation**, and **distributed temporal coupling**.
- **Scale-Dependent Causal Propagation**: The phenomenon where causal influence operates differently at different temporal scales, captured by different fractional derivative orders. Sub-diffusive ($\alpha < 1$), classical ($\alpha = 1, 2$), and super-diffusive ($\alpha > 2$) causal propagation represent different temporal textures of influence.
- **Continuous Causal State Vector**: An extension of the discrete Causal State Vector to include fractional derivatives: $\mathbf{C}_{\text{frac}}(x^\mu) = \left(\Psi(x^\mu), {}_a\mathcal{D}_t^{0.5}\Psi, {}_a\mathcal{D}_t^{1}\Psi, {}_a\mathcal{D}_t^{1.5}\Psi, \dots\right)$, representing the complete temporal signature of causal influence across all scales.
- **$(n+\alpha)$-th Derivative**: A generalized formulation where causality manifests at the $(n+\alpha)$-th fractional derivative level, where n represents the positional dimension being observed and $\alpha \in \mathbb{R}^+$ represents the fractional causal order. This enables **continuous causal gradation**rather than discrete causal jumps.