Impact of Fiscal Policy on Output and Comprehensively Testing Linearity Using the Smooth Transition Regression Model

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Abstract

The current paper re-examines the nonlinear multiplier effect of US government spending by revisiting the empirical study by Auerbach and Gorodnichnko (2012a). In order to draw a large sample inference properly, we first reformulate their vector smooth-transition autoregression (VSTAR) model into the vector smooth-transition error-correction (VSTEC) model so that data nonstationarity can be appropriately handled at the level of stationarity; and we next develop a testing methodology for nonlinearity using the quasi-likelihood ratio (QLR) test statistic applied to a general smooth-transition autoregression model, acquiring an omnibus power for the test. By applying the QLR test statistic to the VSTEC model to the same data as in Auerbach and Gorodnichnko (2012a), we affirm the nonlinear effect of the fiscal policy and also estimate the response functions of the relevant economic variables to a $1 increase of the government spending. Although the estimated response functions are similar to those in the previous literature, the magnitudes of the fiscal multiplier effect measured by the VSTEC model estimation are not so large as estimated by the VSTAR model.

Key Words: Nonlinear fiscal multiplier; QLR test statistic, STAR model, Linearity test, Gaussian process.

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1 Introduction

Estimating the nonlinear effect of government spending has received increasing attention in the literature. For example, Ramey (2011) and Parker (2011) review the debate on the state-dependent effect of the fiscal multiplier. Specifically, new Keynesian economists generally agree on a relatively stable fiscal multiplier effect (*e.g.*, Ramey, 2011; Coenen *et al.*, 2012), whereas traditional Keynesian economists argue for an effect whose strength depends on the state of the economy (*e.g.*, Auerbach and Gorodnichenko, 2012a and 2012b). The aim of this paper is to contribute to the literature by studying effects of the fiscal multiplier in a systematic manner. In particular, this implies testing stability against state-dependence and choosing the latter only when stability is rejected. For this purpose we develop a testing methodology for testing linearity (or stability) against nonlinearity of the smooth transition autoregression (STAR) type. There exist linearity tests for this model, but our aim is to complete existing tests by a test that, unlike its predecessors, is generically comprehensively revealing, which means that the test has non-negligible power against arbitrary nonlinearity. We consider the STAR framework because the vector smooth transition autoregression (VSTAR) model is one of the models that has been applied to estimating nonlinear fiscal multiplier effects. As a by-product, the test we develop can be applied in other fields in which STAR models are typically applied.

As to the present empirical problem, there are two aspects worth mentioning. First, the economic variables typically associated with government spending are nonstationary, and it is not straightforward to draw inferences on the nonlinear effect using such data as nonstationarity renders the standard statistical inference invalid. Second, the Lagrange multiplier (LM) test statistic popularly applied to testing for nonlinearity when the time series are stationary may not have power against every kind of nonlinearity. To the best of our knowledge, there do not exist any studies that would test for nonlinear fiscal policy effects described by a smooth transition model such that they would use a test with power against any conceivable form of nonlinearity.

In this paper we handle nonstationarity of the time series as in Candelon and Lieb (2013) who solved the problem by using the error-correction framework. Following their lead, we re-examine the nonlinear fiscal multiplier specification adopted by Auerbach and Gorodnichenko (2012a and 2012b, AG hereafter). Thus, for a proper inference, instead of their VSTAR model we study the problem using the vector smooth transition error correction (VSTEC) model. As adequacy of any estimated model should be tested, we shall apply other diagnostic tests to the estimated VSTEC model.

Concerning the second aspect, we develop a new test that is readily applicable to testing linearity in the

VSTEC model framework. We begin rather generally by studying the problem of testing for neglected nonlinearity by a quasi-likelihood ratio (QLR) statistic applied to a STAR model. It is well known that for many nonlinear models, the STAR model included, the null hypothesis can be formulated in two different ways, see *e.g.* Davies (1977, 1987). Each of these two ways carries its own identification problem, and this is called the twofold identification problem (*e.g.*, Baek *et al.*, 2015; Cho *et al.*, 2011, 2014). Typically, a linearity test is based on testing one of these two null hypothesis, ignoring the other one. We shall show that the QLR test statistic defined by the likelihood-ratio principle is able to comprehensively handle the twofold identification problem. This means, see Stinchcombe and White (1998), that the test has omnibus power against arbitrary nonlinearity. We shall examine the null limit distribution of the QLR statistic which, as it turns out, is represented as a functional of a multivariate Gaussian stochastic process. This is different from the previous literature in which the twofold identification problem has been studied in the artificial neural networks (ANN) framework.

What makes the QLR test statistic worth considering is in its omnibus power against arbitrary nonlinearity. As Stinchcombe and White (1998) pointed out, a linearity test acquires omnibus power if it is based on an analytic function. The STAR model satisfies this requirement. On the other hand, in the LM test statistic, developed Saikkonen and Luukkonen (1988) and Luukkonen, Saikkonen, and Teräsvirta (1988) and applied by Teräsvirta (1994) and Granger and Teräsvirta (1993) among others, this analytic function is approximated by a polynomial with the result that the omnibus power is not achieved. Nevertheless, the LM test is easy to compute and has an asymptotic χ^2 distribution under the null hypothesis of linearity, which explains its popularity.

The plan of the paper is as follows. Section 2 reviews the literature on the nonlinear fiscal multiplier effect and the linearity test. In Section 3, we derive the null limit distribution of the QLR test statistic. Section 4 contains our results on the multiplier effect of US government spending. Section 5 provides concluding remarks. In the Supplement, we provide three additional topics. First, we examine testing linearity against commonly applied STAR models and also provides simulation evidence of our methodology. We also demonstrate use of Hansen's (1996) weighted bootstrap when the covariance kernel of the multivariate Gaussian process associated with the null limit distribution cannot be exploited. Second, we re-examine the quarterly US unemployment rates and demonstrate different inference results using the QLR and the LM test statistics. We also discuss how to strengthen our inference using the different results. Finally, we contain the mathematical proofs of the main claims.

2 Fiscal Multiplier Effect and Linearity Testing in the Literature

2.1 Fiscal Multiplier Effect in the Literature

It may not be surprising that economists do not agree on the nature and size of the fiscal policy effect because it depends on a number of economic circumstances such as economic openness (Ilzetzki, Mendoza, and Végh, 2013), persistence, forecastability, and sources of government spending (Bachmann and Sims, 2012; Biolsi, 2017; Sims and Wolff, 2018), to mention a few. Various results on the fiscal policy effect have been presented in the literature. As already mentioned, traditional Keynesian economists argue that the fiscal multiplier effect depends on the state of the economy. For example, government expenditure in a recession can be expected to result in a direct employment of idle resources. This in turn leads to an increase in the aggregate output, so that the fiscal policy effect can be expected larger than it would be in an expansion. In particular, AG found this state-dependent multiplier effect by estimating a VSTAR model, and Bachmann and Sims (2012) reported a significantly larger fiscal policy effect in a recession than in an expansion when they estimated a model somewhat similar to the VSTAR model. Canzoneri, Collard, Dellas, and Diba (2016) and Fazzari, Morley, and Panovska (2015), among others, also reported empirical results of this kind.

In contrast, new Keynesian economists generally do not find evidence of such larger fiscal multiplier effect in recessive periods. Specifically, Ramey (2011), Barro and Redlick (2011), Ramey and Zubairy (2018), and Owyang, Ramey, and Zubairy (2013a), to name a few, provide empirical evidence for smaller fiscal multiplier effects that do not exceed unity even in a recession using narrative variables employed to identify exogenous fiscal shocks (*e.g.*, military news shock variable in Ramey, 2011). Christiano, Eichenbaum, and Rebelo (2011) used a dynamic stochastic general equilibrium model and estimated a fiscal multiplier which was larger than what the others had obtained when the nominal interest rate is bound at zero. This assertion was also shared by Coenen *et al.* (2012) and Miyamoto, Nguyen, and Sergeyev (2018). In particular, Shen and Yang (2018) related this view to their study by theoretically examining a channel between the fiscal multiplier effect and downward nominal wage rigidity.

Econometric models on the nonlinear fiscal multiplier effect may be divided into two broad groups: models with stochastically varying parameters and the ones with switching regimes. The paper by Kirchner, Cimadomo, and Hauptmeier (2010) belongs to the first group. They used a vector autoregressive (VAR) model with the vector of stacked VAR coefficients following a random walk with a nondiagonal error covariance matrix. The error covariance structure consisted of a multiplicatively decomposed covariance matrix with coefficients following a random walk. This implies that uncertainty about the structure of the mean increases with time. The authors prevented the variances from exploding by setting the starting values to be sufficiently small when the covariance matrix was estimated recursively. They fitted the model to the Euro area data.

Another study by Pereira and Lopes (2014) made the same model assumptions as Kirchner, Cimadomo, and Hauptmeier (2010) by letting all VAR coefficients and the error covariance matrix be nonstationary. These authors used the US data over 1965Q2–2009Q2 and tested parameter stability by Nyblom's (1989) statistic that tests constancy of parameters against the alternative that the parameters are martingales. The null hypothesis was rejected, and the parameters estimated by Bayesian techniques. Variability of the nonstationary error covariances was restricted as in Kirchner, Cimadomo, and Hauptmeier (2010).

The second group contains several econometric models. First, the VSTAR model in AG is specified as follows:

$$y_t = \sum_{j=1}^{p} \Pi_{jE*} y_{t-j}(1 - f(z_{t-1}, \gamma_*, c_*)) + \sum_{j=1}^{p} \Pi_{jR*} y_{t-j} f(z_{t-1}, \gamma_*, c_*) + u_t \tag{1}$$

where $u_t|(z_t, z_{t-1}, \ldots) \sim N(0, \Omega_t)$, $y_t := (g_t, \tau_t, q_t)'$ with g_t, τ_t, and q_t being log real government spending, log real government net tax receipts, and log real gross domestic product (GDP), respectively. The transition function $f(\cdot)$ is logistic:

$$f(z_{t-1}, \gamma_*, c_*) := \frac{\exp\{\gamma_*(z_{t-1} - c_*)\}}{1 - \exp\{\gamma_*(z_{t-1} - c_*)}, \tag{2}$$

where z_t is a demeaned seven-quarter moving average of the output growth rate and c_* is set to 0. The time-varying covariance matrix Ω_t of the error vector equals

$$\Omega_t := (1 - f(z_{t-1}, \gamma_*, c_*))\Omega_{E*} + f(z_{t-1}, \gamma_*, c_*)\Omega_{R*}. \tag{3}$$

Here, 'E' and 'R' stand for expansion and recession, respectively. The model looks heavily nonlinear because identical transition functions control the mean and the error variance. Nevertheless, no theoretical justification is provided for the assumption that the chosen mean specification and its uncertainty (error variance) depend on the state of the economy exactly in the same way. Neither constancy of the mean nor that of the variance was tested, which would have been complicated by the fact that y_t is nonstationary, although testing the latter would have been possible (*e.g.*, Eklund and Teräsvirta, 2007).

Next, Bachmann and Sims (2012) specified the model with the transition function and the error covariance matrix as (2) and (3), respectively. Their conditional mean, however, was different from (1) and had the following form:

$$y_t = \sum_{j=1}^{p} \Pi_{j1*} y_{t-j} + \sum_{j=1}^{p} \Pi_{j2*} y_{t-j} z_{t-1} + \sum_{j=1}^{p} \Pi_{j3*} y_{t-j} z_{t-1}^2 + u_t. \tag{4}$$

Equation (4) may be viewed as a second-order polynomial approximation to (1). Therefore, testing linearity would have been straightforward, had y_t been stationary, because their model is already linear in parameters. They did not test constancy of the error variance either but assumed, without much explanation, that uncertainty of the mean specification changes monotonically with the state of the economy.

Nonstationarity of y_t[1] received due attention in Candelon and Lieb (2013). In a manner similar to Lo and Zivot (2001), they specified a threshold vector error correction (TVEC) model for ten nonstationary variables:

$$\Delta y_t = \alpha_* \beta_*' y_{t-1} + \sum_{j=1}^{p-1} \Pi_{jE*} \Delta y_{t-j} (1 - \mathbb{I}(x_{t-d} \leq c_*)) + \sum_{j=1}^{p-1} \Pi_{jR*} \Delta y_{t-j} \mathbb{I}(x_{t-d} \leq c_*) + u_t \tag{5}$$

where α_* and β_* are $10 \times r$ matrices, $r < 10$, and $\mathbb{I}(\cdot)$ is the indicator function. Here, the delay $d > 0$, and $\{u_t\}$ is assumed a sequence of identically distributed Gaussian errors. The threshold variable x_t can be chosen from a set of stationary variables. This model was designed to understand the state-dependency of fiscal policy effects that may last for a short period of time, while the long-run relationship among y_t is invariant. Candelon and Lieb (2013) used Bayesian methods to estimate the model as it contained a sign restriction for identification. The estimates of their multiplier effect were different from measure to measure. The so-called present value multiplier yielded a small effect in a recession, but the short-run impact value multiplier which does not take into account dynamic effects of fiscal policy on government spending and outputs, yielded an estimate similar to that in AG. The authors tested linearity using the sup-Wald test statistic that handles Davies' (1977, 1987) identification problem present in the TVEC model. This test does not, however, comprehensively test the linearity hypothesis and its null limit distribution cannot be applied to other smooth-transition models. They also mentioned that the LM test statistic obtained by approximating the transition function might have low power, which is related to the current study.

Fazzari, Morley and Panovska (2015) also employed a vector threshold autoregression (VTAR) model. They assumed independently distributed Gaussian errors with a known structural break point in the covariance matrix. The variables were assumed stationary, at least implicitly, so the model is a standard VTAR model obtained by assuming $\alpha_* = 0$ in (5). Unlike Candelon and Lieb (2013), these authors did not test linearity.

[1]The elements of y_t are not the same in all papers mentioned here, but a common feature in many of them is that y_t is nonstationary.

They provided empirical evidence of the fiscal multiplier being larger in the period of economic slack using the US data running from 1967Q1 to 2012Q4. The focus in the VTAR model by Ferraresi, Roventini, and Fagiolo (2015) is on the state-dependent fiscal multiplier that is dependent on credit market conditions. They used US data over 1984Q1-2010Q4 and rejected linearity based on the test statistic proposed by Tsay (1986). The authors found the multiplier to be larger when the credit market is under stress than it is under normal conditions.

Arin, Koray, and Spagnolo (2015) estimated a Markov regime switching model for the state-dependent fiscal multiplier effect using the US data over 1949Q1–2006Q4. Instead of investigating the structural VAR model directly, they employed the military news shock variable proposed by Ramey (2011) as one of the explanatory variables and identified an exogenous fiscal shock. Estimating their model, they found, similarly to AG, that the fiscal multiplier was larger in recessions than in expansions.

We shall return to econometric modelling the fiscal multiplier problem in Section 4. Instead of simply comparing the coefficients estimated from the linear and nonlinear models as a way to validate the possible nonlinearity, we shall pay proper attention to testing linearity. This is important not only from the economic theory point of view but also because of the twofold identification problem present in the aforementioned regime-switching models. Before doing that we shall in the next section discuss this problem and how to deal with it using the QLR statistic.

2.2 Previous Literature on Testing Linearity Against the STAR Model

The QLR test has hitherto been studied in the ANN framework. For example, Cho, Ishida, and White (2011, 2014), White and Cho (2012), and Baek, Cho, and Phillips (2015), among others, examined the QLR test and twofold identification problem associated with the ANN framework. The current study contributes to this literature particularly by tackling the twofold identification problem in the family of STAR models. This extends the results obtained in the ANN framework because the null limit distribution of the test has to be derived in a multivariate Gaussian process structure, which generalises the results in the aforementioned literature. The standard single-hidden layer (univariate) ANN model is specified for stationary variables and has the following form:

$$y_t = \pi_0 + \widetilde{z}_t' \pi + \sum_{j=1}^{q} \theta_j f(z_t' \gamma_j) + \varepsilon_t \tag{6}$$

where $z_t := (1, \widetilde{z}_t')'$ with $\widetilde{z}_t := (y_{t-1}, ..., y_{t-p})'$, $f(0) =$ constant, and $\pi_0, \pi, \theta_j, \gamma_j, j = 1, ..., q$, are parameters. In many applications, $\pi = 0$. The ANN model (6) thus contains a linear combination of continuous and bounded functions (a hidden layer), typically logistic ones, although other bounded functions are possible. Nowadays, ANN models in applications often contain more than one hidden layer, but the single-hidden layer ANN model serves as a benchmark against which a STAR model may be compared. The twofold identification problem becomes obvious from (6). The model becomes linear by assuming either $\theta_j = 0$ or $\gamma_j = 0$ ($j = 1, ..., q$), so that if $\theta_j = 0$, γ_j disappears from the model; and if $\gamma_j = 0$, π_0 and θ_j are not separably estimable from it. This shows that Davies' (1977, 1987) identification problem arises in two different ways. This makes the Wald test inapplicable, and so the aforementioned papers focussing on the QLR test statistic apply the likelihood-ratio principle.

In contrast, a standard STAR model, see, for example, Teräsvirta (1994), Granger and Teräsvirta (1993), and van Dijk, Teräsvirta, and Franses (2002), can be written as follows: $y_t = z_t'\pi + z_t'\theta f(z_t'\alpha, \gamma) + \varepsilon_t$, where $f(z_t'\alpha, \gamma)$ is another analytic bounded function, $f(z_t'\alpha, 0) = 0$, $\gamma > 0$ is a scalar parameter, and $\alpha = (0, ..., 0, 1, 0, ..., 0)'$ is a known vector. It becomes linear by either setting $\theta = 0$ or $\gamma = 0$, so the same identification problem arises even here. The main difference between these two models is that the single hidden-layer ANN model contains a linear combination of several transitions that are themselves functions of linear combinations of elements of z_t, whereas in the standard STAR model a linear combination of these elements is multiplied by a transition function usually with a single argument.[2] This argument is most often an element of $\widetilde{z}_t$, although it can also be a weighted sum of several variables where the weights are assumed known.

Due to these differences, the analysis of the QLR test statistic needs to be generalised in order to make the QLR test statistic applicable in the STAR framework. As an example, Cho, Ishida, and White (2011, 2014) characterised the null limit distribution of the QLR test statistic in the ANN context as a functional of a univariate Gaussian process. This limit distribution cannot, however, be applied to the STAR case because, as it turns out, a multivariate Gaussian process is required for the null limit distribution. The following STAR model of order p is frequently specified as a prediction model of a time-series data y_t (*e.g.*, Teräsvirta, 1994; Granger and Teräsvirta, 1993): $\mathcal{M}_0 := \{h_0(\cdot, \pi, \theta, \gamma) : (\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma\}$, where $h_0(z_t, \pi, \theta, \gamma) := z_t'\pi + f(\widetilde{z}_t'\alpha, \gamma)(z_t'\theta)$, $z_t := (1, \widetilde{z}_t')'$ is a $(p+1) \times 1$ vector of regressors with a transition variable $\widetilde{z}_t'\alpha$. Here, $\widetilde{z}_t := (y_{t-1}, y_{t-2}, \ldots, y_{t-p})'$, and $\alpha = (0, \ldots, 1, 0, \ldots, 0)'$ denotes a selection vector chosen by the researcher. The

[2]STAR models can also contain more than one additive transition, but this seems to be uncommon in applications.

other parameter vectors $\pi := (\pi_0, \pi_1, \ldots, \pi_p)'$ and $\theta := (\theta_0, \theta_1, \ldots, \theta_p)'$ are the mean transition parameters, and γ is used to describe the smooth transition from one extreme regime to the other. Symbols Π, Θ, and Γ denote the parameter spaces of π, θ, and γ, respectively. The transition function $f(\cdot, \gamma)$ is a nonlinear, continuously differentiable, and uniformly bounded function. It is typically either exponential, $f_E(\widetilde{z}_t'\alpha, \gamma) := 1 - \exp(-\gamma(\widetilde{z}_t'\alpha)^2)$, or logistic, $f_L(\widetilde{z}_t'\alpha, \gamma) := \{1 + \exp(-\gamma \widetilde{z}_t'\alpha)\}^{-1}$. In both cases, $\gamma > 0$. Our STAR model is a special case of the original STAR model in which the transition function $f(\widetilde{z}_t'\alpha - c, \gamma)$ with a constant c is substituted for $f(\widetilde{z}_t'\alpha, \gamma)$ in $\mathcal{M}_0$. We set $c = 0$ in $\mathcal{M}_0$ as in the regular exponential autoregressive model in Haggan and Ozaki (1981) and AG because the essential property in testing linearity is that $f(\widetilde{z}_t'\alpha, \cdot)$ is an analytic function. As we detail below, if c is estimated along with the other parameters π and θ, the inference becomes more complicated than ours, and this complexity limits its applicability.

The STAR model has a continuum of regimes defined by transition functions obtaining values between 0 to 1. This feature makes the model an appealing alternative in empirical studies because the behaviour of economic agents can often be best described by multiple regimes and smooth transitions between them. For more discussion on the STAR model the reader is referred to van Dijk, Teräsvirta, and Franses (2002), Teräsvirta (1994), Granger and Teräsvirta (1993), and Teräsvirta, Tjøstheim, and Granger (2010), among others.

The ESTAR and LSTAR models are specified by transforming the exponential function that is analytic, so it is generically comprehensively revealing for model misspecification as pointed out by Stinchcombe and White (1998). Therefore, the estimated parameters in the transition function become statistically significant such that the nonlinear component necessarily reduces the mean squared error of the model even when the assumed STAR model is misspecified. This implies that if the linear model is misspecified, the mean square error obtained from estimating the corresponding STAR model becomes smaller than that from the linear model. This in turn motivates testing linearity by comparing the estimated mean squared errors from the STAR and the linear model nested in the STAR. This process delivers an omnibus testing procedure for nonlinearity.

Similar arguments can be found in the previous literature on the QLR test such as Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). As will soon be apparent, however, the nonlinear structure of the STAR model being different from that of the ANN model leads to a different null limit distribution for the QLR test statistic.

As we have mentioned earlier, the LM test statistic, which is widely used for testing linearity in the STAR framework, concerns only one of the two hypotheses that make the STAR model linear. The QLR statistic extends the LM test by handling both null hypotheses simultaneously. In the Supplement we show how the

QLR and LM test statistics can complement each other in the empirical analysis.

3 Testing Linearity Using the STAR Model

3.1 DGP and QLR Test Statistic

We consider a general STAR model that treats each univariate VSTEC model as a special case and study the null limit behavior of the QLR test statistic in this framework. We apply this null limit distribution in Section 4. In order to proceed, we make the following assumptions:

Assumption 1. $\{(y_t, \tilde{z}_t')' \in \mathbb{R}^{1+p} : t = 1, 2, \ldots\}$ $(p \in \mathbb{N})$ *is a strictly stationary and absolutely regular process defined on the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with* $\mathbb{E}[|y_t|] < \infty$ *and mixing coefficient* β_τ *such that for some* $\rho > 1$, $\sum_{\tau=1}^{\infty} \tau^{1/(\rho-1)} \beta_\tau < \infty$. □

Here, the mixing coefficient is defined as $\beta_\tau := \sup_{s\in\mathbb{N}} \mathbb{E}[\sup_{A\in\mathcal{F}_{s+\tau}^{\infty}} |\mathbb{P}(A|\mathcal{F}_{-\infty}^{s}) - \mathbb{P}(A)|]$, where $\mathcal{F}_\tau^s$ is the σ-field generated by $(y_t, \ldots, y_{t+s})$. Many time series models satisfy this condition, and the autoregressive process is one of them. It is general enough to cover the stationary time series we are interested in. We impose the following regular STAR model condition:

Assumption 2. *Let* $f(\tilde{z}_t'\alpha, \cdot) : \Gamma \mapsto [0, 1]$ *be a non-polynomial analytic function with probability 1. Let* $\Pi \in \mathbb{R}^{p+1}$, $\Theta \in \mathbb{R}^{p+1}$*, and* $\Gamma \in \mathbb{R}$ *be non-empty convex and compact sets such that* $0 \in \Gamma$*. Let* $h(z_t, \pi, \theta, \gamma) := z_t'\pi + \{f(\tilde{z}_t'\alpha, \gamma) - f(\tilde{z}_t'\alpha, 0)\}(z_t'\theta)$*, and let* $\mathcal{M} := \{h(\,\cdot\,, \pi, \theta, \gamma) : (\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma\}$ *be the model specified for* $\mathbb{E}[y_t|z_t]$. □

Note that $\mathcal{M}$ differs from $\mathcal{M}_0$. The transition function is centered at $f(\tilde{z}_t'\alpha, 0)$ for analytical convenience. As $f(\tilde{z}_t'\alpha, 0)$ is constant, the nonlinearity of the STAR model is not modified by centering. For example, we have $f_E(\tilde{z}_t'\alpha, 0) = 0$ and $f_L(\tilde{z}_t'\alpha, 0) = 1/2$, so f_L will be centered to have value zero. Furthermore, centering reduces the dimension of the identification problem as detailed below. The parameters to be estimated are π, θ, and γ, as α is defined by the researcher.

Using Assumption 2, the linearity hypothesis and the alternative are specified as follows: $\mathcal{H}_0 : \exists \pi \in \mathbb{R}^{p+1}$ such that $\mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) = 1$; vs. $\mathcal{H}_1 : \forall \pi \in \mathbb{R}^{p+1}$, $\mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) < 1$. These hypotheses are the same as the ones in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). As in the previous literature, the focus is on developing an omnibus test statistic but now against STAR, and the QLR test statistic will be a vehicle for reaching this goal. The QLR test statistic is formally defined as $QLR_n :=$

$n(1-\widehat{\sigma}^2_{n,A}/\widehat{\sigma}^2_{n,0})$, where $\widehat{\sigma}^2_{n,0} := \min_\pi \frac{1}{n}\sum_{t=1}^n (y_t - z_t'\pi)^2$, $\widehat{\sigma}^2_{n,A} := \min_{\pi,\theta,\gamma} \frac{1}{n}\sum_{t=1}^n \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$, and $f_t(\gamma) := f(\widetilde{z}_t'\alpha, \gamma) - f(\widetilde{z}_t'\alpha, 0)$. We let the nonlinear least squares (NLS) estimator $(\widehat{\pi}_n, \widehat{\theta}_n, \widehat{\gamma}_n)$ minimise the squared errors with respect to (π, θ, γ). Furthermore, $(\pi_*, \theta_*, \gamma_*)$ denotes the probability limits of the NLS estimator: $(\pi_*, \theta_*, \gamma_*) := \arg\min_{\pi,\theta,\gamma} \mathbb{E}[\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2]$ is the pseudo-true parameter.

The main reason for proceeding with the QLR statistic is that linearity leads to a twofold identification problem, and this statistic is able to handle both parts of it. To be more specific, the twofold identification problem can be described by the parameter set $(\pi_*, \theta_*, \gamma_*)$. If $\mathbb{E}[y_t|z_t]$ is linear with respect to z_t with coefficient π_*, we can generate a linear function from $h(\cdot, \pi_*, \theta_*, \gamma_*)$ in two different ways, either by letting $\theta_* = 0$ or by assuming $\gamma_* = 0$. Because of this, $(\pi_*, \theta_*, \gamma_*)$ is not uniquely determined. If $\theta_* = 0$, $h(\cdot, \pi_*, 0, \gamma_*) = z_t'\pi_*$, so that γ_* is not identified. We call this problem type I identification problem, under which $(\pi_*, \theta_*, \gamma_*)$ becomes any element in $\{(\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma : \pi = \pi_*, \theta = 0\}$. If we employed $f(\widetilde{z}_t'\alpha - c, \gamma)$ instead of $f(\widetilde{z}_t'\alpha, \gamma)$ for $\mathcal{M}$ as in the original STAR model, neither γ_* nor the additional c_* is identified under $\theta_* = 0$, which leads to a more complicated identification problem. We fix our interest in the current derivative model $\mathcal{M}$ that excludes c_*. Alternatively, if $\gamma_* = 0$, $h(\cdot, \pi_*, \theta_*, 0) = z_t'\pi_*$, so that θ_* is not identified. This leads to a type II identification problem, in which $(\pi_*, \theta_*, \gamma_*)$ becomes any element in $\{(\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma : \pi = \pi_*, \gamma = 0\}$. On the other hand, if the transition function is not centered at $f(\widetilde{z}_t'\alpha, 0)$, letting $\gamma_* = 0$ leads to $h_0(z_t, \pi_*, \theta_*, 0) = z_t'(\pi_* + f(\widetilde{z}_t'\alpha, 0)\theta_*)$. This implies that the type II identification problem becomes more complicated as π_* and θ_* are not separately identified. Centering thus transforms this complication into a relatively straightforward identification problem. Besides, mainly due to the invariance principle the null limit distribution does not change by this centering. Note that π in $\mathcal{M}_0$ is reparameterised to $\pi - f(\widetilde{z}_t'\alpha, 0)\theta$ in $\mathcal{M}$, so that the QLR test obtained by this reparameterisation becomes identical to that before the reparameterisation. Without it, the null model investigation has to be separately conducted by discerning the parameters with type II identification problem. So, we avoid the involved complication by the centering and obtain the null limit distribution of the QLR statistic efficiently. This centering is also indirectly applied in the literature when the null limit distribution of the LM test statistic is being derived. If z_t contains a constant, this limit distribution is not affected by centering because the centering parameter is merged into the linear component in the Taylor expansion that forms the basis of the LM statistic.

As described above, the null holds for the following two sub-hypotheses: $\mathcal{H}_{01} : \theta_* = 0$ and $\mathcal{H}_{02} : \gamma_* = 0$. The limit distribution of the QLR test statistic can be derived both under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$, leading to different null limit distributions even for the same statistic. We call these derivations type I and type II analysis, respectively, and show below that they yield two different null weak limits. The null hypothesis of linearity against STAR is properly tested by tackling both $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ simultaneously, and we shall demonstrate that the QLR test statistic has the capability of doing that. For this purpose, we derive its null limit distribution from the two null weak limits obtained under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ in the spirit of the aforementioned literature on testing linearity by the likelihood-ratio principle. We will examine how the weak limits are related to the null limit distribution of the QLR statistic.

Our view to testing linearity by accommodating type I and II analyses differs from the other tests in the literature. For example, the aforementioned LM test statistic does not accommodate the twofold identification problem. The main argument for the LM test is that its asymptotic null distribution is chi-squared, which makes the test easily applicable. As another example, Cheng (2015) assumed the standard single-hidden layer ANN model by letting $\pi_0 = 0$ and $\pi = 0$, in and the vein of type I analysis developed a Wald statistic for testing whether some or all of θ_{j*}'s are equal to zero or not .

3.2 The Null Limit Distribution of the QLR Test

We now derive the null limit distribution of the QLR test and highlight the difference between the STAR-based approach and the ANN-based one. We first study the limit distributions of the QLR test under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ separately, combine them and, finally, obtain the limit distribution under $\mathcal{H}_0$. For this, we let our quasi-likelihood (QL) function be $\mathcal{L}_n(\pi, \theta, \gamma) := -\sum_{t=1}^{n}\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$.

3.2.1 Type I Analysis: Testing $\mathcal{H}_{01} : \theta_* = 0$

In this subsection, we discuss the limit distribution of the QLR test under $\mathcal{H}_{01} : \theta_* = 0$. The problem is that γ_* is not identified under this hypothesis. We obtain the NLS estimator by maximizing the QL function with respect to γ in the final stage for the purpose of testing $\mathcal{H}_{01}$: $\mathcal{L}_n^{(1)} := \max_\gamma \max_\theta \max_\pi -\sum_{t=1}^{n}\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$ and let $QLR_n^{(1)}$ be the QLR statistic obtained by this optimization process. That is, $\mathcal{L}_n^{(1)} := \max_{\gamma\in\Gamma}\{-u'Mu + u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'\ F(\gamma)Mu\}$, where $u := [u_1, u_2, \ldots, u_n]'$, $u_t := y_t - \mathbb{E}[y_t|z_t]$, $Z := [Z_1, Z_2, \ldots, Z_n]'$, $M := I - Z(Z'Z)^{-1}Z'$, $F(\gamma) := \mathrm{diag}[f_1(\gamma),\ f_2(\gamma), \ldots,\ f_n(\gamma)]$, and $QLR_n^{(1)} := \max_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu$ under $\mathcal{H}_{01}$ using the fact that $y_t = \mathbb{E}[y_t|z_t] + u_t = z_t'\pi_* + u_t$. Note that the numerator of $QLR_n^{(1)}$ is identical to $n(\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2)$ under $\mathcal{H}_{01} : \theta_* = 0$, so that the QLR test accords with $QLR_n^{(1)}$. Furthermore, we cannot let $\gamma = 0$ when $QLR_n^{(1)}$ is derived. If $\gamma = 0$, the alternative model reduces to the linear model, so that the QLR statistic cannot test the null model by letting

$\gamma = 0$. We therefore examine its null limit distribution by supposing $\gamma \neq 0$.

We now derive the limit distribution of $QLR_n^{(1)}$ under $\mathcal{H}_{01}$. For this and to guarantee regular behaviour of the null limit distribution, we impose the following conditions:

Assumption 3. *(i)* $\mathbb{E}[u_t|z_t, u_{t-1}, z_{t-1}, \ldots] = 0$*; and (ii)* $\mathbb{E}[u_t^2|z_t, u_{t-1}, z_{t-1}, \ldots] = \sigma_*^2$. $\square$

Assumption 4. $\sup_{\gamma\in\Gamma} |(\partial/\partial\gamma) f_t(\gamma)| \leq m_t$. $\square$

Assumption 5. *There exists a sequence of stationary ergodic random variables* m_t *such that for* $i = 1, 2, \ldots, p$, $|\widetilde{z}_{t,i}| \leq m_t$, $|u_t| \leq m_t$, $|y_t| \leq m_t$, *and for some* $\omega \geq 2(\rho - 1), \mathbb{E}[m_t^{6+3\omega}] < \infty$, *where* ρ *is in Assumption 1, and* $z_{t,i}$ *is the* i*-th row element of* z_t. $\square$

Assumption 6. *For each* $\gamma \neq 0$, $V_1(\gamma)$ *and* $V_2(\gamma)$ *are positive definite, where for each* γ, $V_1(\gamma) := \mathbb{E}[u_t^2\widetilde{r}_t(\gamma)\widetilde{r}_t(\gamma)']$ *and* $V_2(\gamma) := \mathbb{E}[\widetilde{r}_t(\gamma)\widetilde{r}_t(\gamma)']$ *with* $\widetilde{r}_t(\gamma) := (f_t(\gamma)z_t', z_t')'$. $\square$

Assumption 3(*i*) implies that the model in Assumption 2 is not dynamically misspecified, and Assumption 3(*ii*) implies that the errors are conditionally homoskedastic. Here, conditional homoskedasticity is not needed in proving the main theorems in this study, but this assumption will be made whenever it facilitates understanding the theoretical results. Assumption 4 plays an integral role in applying the tightness condition in Doukhan, Massart, and Rio (1995) to the QLR test statistic. Here it can be easily verified for the ESTAR and LSTAR models by noting that $|(\partial/\partial\gamma)f_E(\widetilde{z}_t'\alpha, \gamma)| = (1 - f_E(\widetilde{z}_t'\alpha, \gamma))(\widetilde{z}_t'\alpha)^2 \leq (\widetilde{z}_t'\alpha)^2$ and $|(\partial/\partial\gamma)f_L(\widetilde{z}_t'\alpha, \gamma)| = f_L(\widetilde{z}_t'\alpha, \gamma)(1 - f_L(\widetilde{z}_t'\alpha, \gamma))|(\widetilde{z}_t'\alpha)| \leq |(\widetilde{z}_t'\alpha)|$, so that we can let m_t in Assumption 4 be $(\widetilde{z}_t'\alpha)^2$ and $|(\widetilde{z}_t'\alpha)|$, respectively. The moment condition in Assumption 5 is stronger than those in Cho, Ishida, and White (2011, 2014), and it also implies that $\mathbb{E}[u_t^6]$ and $\mathbb{E}[y_t^6]$ are finite. The multiplicative component $f_t(\gamma)z_t'\theta$ in the STAR model makes the stronger moment condition necessary in deriving the regular null limit distribution of the QLR test statistic. Assumption 6 is imposed for the invertibility of the limit covariance matrix. This makes our test statistic non-degenerate.

Given these assumptions, we have the following lemma:

Lemma 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and* $\mathcal{H}_{01}$, *(i)* $\widehat{\sigma}_{n,0}^2 \stackrel{\text{a.s.}}{\to} \sigma_*^2 := \mathbb{E}[u_t^2]$*; (ii)* $\{n^{-1/2}Z'\, F(\cdot)Mu, \widehat{\sigma}_{n,0}^2 n^{-1}Z'F(\cdot)MF(\cdot)Z\} \Rightarrow \{\mathcal{Z}_1(\cdot),\, A_1(\cdot,\, \cdot)\}$ *on* $\Gamma(\epsilon)$ *and* $\Gamma(\epsilon) \times \Gamma(\epsilon)$, *respectively, where* $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$, $\mathcal{Z}_1(\cdot)$ *is a continuous Gaussian process with* $\mathbb{E}[\mathcal{Z}_1(\gamma)] = 0$, *and for each* γ *and* $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{Z}_1(\gamma)\mathcal{Z}_1(\widetilde{\gamma})'] = B_1(\gamma, \widetilde{\gamma})$ *such that* $B_1(\gamma, \widetilde{\gamma}) := \mathbb{E}[u_t^2 f_t^*(\gamma) f_t^*(\widetilde{\gamma})']$ *and* $A_1(\gamma, \widetilde{\gamma}) := \sigma_*^2\mathbb{E}[f_t^*(\gamma)f_t^*(\widetilde{\gamma})']$ *with* $f_t^*(\gamma) = f_t(\gamma)z_t - \mathbb{E}[f_t(\gamma)z_tz_t']\,\mathbb{E}[z_tz_t']^{-1}z_t$*; (iii) if, in addition, Assumption 3(ii) holds,* $B_1(\gamma, \widetilde{\gamma}) = A_1(\gamma, \widetilde{\gamma})$. $\square$

There is a caveat to Lemma 1. It is clear from $QLR_n^{(1)}$ that its limit distribution is determined by the limit behaviour under $\mathcal{H}_{01}$ of both $n^{-1/2}Z'F(\cdot)Mu$ and $n^{-1}Z'F(\cdot)M\,F(\cdot)Z$. Furthermore, $\lim_{\gamma\to 0} Z'F(\gamma)Mu \overset{\text{a.s.}}{=} Z'F(0)Mu = 0$ and $\lim_{\gamma\to 0} Z'\,F(\gamma)MF(\gamma)Z \overset{\text{a.s.}}{=} Z'F(0)MF(0)Z = 0$. This implies that it is not straightforward to obtain the limit distribution of $QLR_n^{(1)}$ around $\gamma = 0$. We therefore assume for the moment that 0 is not included in Γ by considering $\Gamma(\epsilon)$ instead of Γ and accommodate this effect by restricting the QLR test statistic to $QLR_n^{(1)}(\epsilon) := \max_{\gamma\in\Gamma(\epsilon)}(1/\widehat{\sigma}^2_{n,0})u'MF(\gamma)Z[Z'\,F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu$. We relax this restriction when the limit distribution is examined under $\mathcal{H}_0$.

Lemma 1 plays a central role in deriving the null limit distribution of $QLR_n^{(1)}(\epsilon)$ and corresponds to lemma 1 of Cho, Ishida, and White (2011). Despite being similar, the two lemmas are not identical. Note that $\mathcal{Z}_1(\cdot)$ is mapped to $\mathbb{R}^{p+1}$, whereas their lemma obtains a univariate Gaussian process. The multivariate Gaussian process $\mathcal{Z}_1(\cdot)$ distinguishes the STAR model-based testing from the ANN-based approach. By this, the STAR model has a different null limit distribution, and the QLR test based upon the STAR model has power over alternatives in directions different from those of the ANN-based approach.

Theorem 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and $\mathcal{H}_{01}$, for each $\epsilon > 0$, (i) $QLR_n^{(1)}(\epsilon) \Rightarrow \sup_{\gamma\in\Gamma(\epsilon)} \mathcal{G}_1(\gamma)'\mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian process such that for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{G}_1(\gamma)] = 0$ and $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}B_1(\gamma,\widetilde{\gamma})\,A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$; (ii) if, in addition, Assumption 3 (ii) holds, then $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}A_1(\gamma,\widetilde{\gamma})\,A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$.* $\square$

As continuous mapping makes proving Theorem 1 trivial, no proof is given.

Theorem 1 implies that $QLR_n^{(1)}(\epsilon)$ does not asymptotically follow a chi-squared distribution under $\mathcal{H}_{01}$ as does the LM statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993). The difficulty here is that the null limit distribution contains the unidentified nuisance parameter γ.

3.2.2 Type II Analysis: Testing $\mathcal{H}_{02} : \gamma_* = 0$

Here the focus is on the limit distribution under $\mathcal{H}_{02} : \gamma_* = 0$. This hypothesis is tested using the LM statistic. As we know, θ_* is not identified under $\mathcal{H}_{02}$. We therefore maximise the QL function with respect to θ at the final stage: $\mathcal{L}_n^{(2)} := \sup_\theta \sup_\gamma \sup_\pi -\sum_{t=1}^n\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$, and denote the QLR test defined by this maximization process by $QLR_n^{(2)}$.

Several remarks are in order. First, maximizing the QL with respect to π is relatively simple due to linearity. We let the concentrated QL (CQL) function be $\mathcal{L}_n^{(2)}(\gamma,\theta) := \sup_\pi \mathcal{L}_n(\pi,\theta,\gamma) = -[y - F(\gamma)Z\theta]'M[y - F(\gamma)Z\theta]$, where $y := [y_1, y_2, \ldots, y_n]$. Here, we have to assume $\theta \neq 0$. If $\theta = 0$, the STAR model becomes linear, so the QLR test statistic cannot compare the null model with the alternative. Second, $\mathcal{L}_n^{(2)}(\cdot)$ is not linear with respect to γ, so that the next stage CQL function with respect to γ cannot be analytically derived. We approximate the CQL function with respect to γ around $\gamma_* = 0$ and capture its limit behaviour under $\mathcal{H}_{02}$. The first-order derivative of $\mathcal{L}_n^{(2)}(\gamma,\theta)$ with respect to γ is $(d/d\gamma)\,\mathcal{L}_n^{(2)}(\gamma,\theta) = 2[y - F(\gamma)Z\theta]'M(\partial F(\gamma)/\partial\gamma)Z\theta$, where $(\partial F(\gamma)/\partial\gamma) := (\partial/\partial\gamma)\,(f(\widetilde{z}_1'\alpha,\gamma), ..., f(\widetilde{z}_n'\alpha,\gamma))$. For the LSTAR model, $\partial f_L(\widetilde{z}_t'\alpha,\gamma)/\,\partial\gamma = f_L(\widetilde{z}_t'\alpha, \gamma)(1 - f_L(\widetilde{z}_t'\alpha,\gamma))\widetilde{z}_t'\alpha$ and $\partial F(0)/\partial\gamma = (1/4)(\widetilde{z}_1'\alpha, ..., \widetilde{z}_n'\alpha)'$, whereas for ESTAR, it follows that $\partial f_E(\widetilde{z}_t'\alpha,\gamma)/\,\partial\gamma = (\widetilde{z}_t'\alpha)^2(1 - f_E(\widetilde{z}_t'\alpha,\gamma))$, so $\partial F(0)/\partial\gamma = ((\widetilde{z}_1'\alpha)^2, ..., (\widetilde{z}_n'\alpha)^2)'$, implying that we can approximate the CQL function by a second-order approximation. Nevertheless, as Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011, 2014) pointed out, the first-order derivative of the CQL is often zero for many other models. For example, in $\mathcal{M}_A := \{\pi y_{t-1} + \theta\{1 + \exp(\gamma y_{t-1})\}^{-1} : (\pi,\theta,\gamma) \in \Pi \times \Theta \times \Gamma\}$, the first-order derivative of the CQL is zero when $\gamma_* = 0$. Due to this, we need a higher-order approximation. Cho, Ishida, and White (2014) adopt a sixth-order Taylor expansion, whereas Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011) use fourth-order Taylor expansions to obtain the null limit distributions of their tests. The order of expansion is determined by the functional form of $f(\widetilde{z}_t'\alpha,\cdot)$.

As we do not assume a specific form for our STAR model, we simply let κ ($\kappa \in \mathbb{N}$) be the smallest order such that the κ-th order partial derivative with respect to γ is different from zero at $\gamma = 0$, so that for all $j < \kappa$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\cdot) \equiv 0$. For example, $\kappa = 3$ for $\mathcal{M}_A$. Then, the CQL function is expanded as

$$\mathcal{L}_n^{(2)}(\gamma,\theta) = \mathcal{L}_n^{(2)}(0,\theta) + \frac{1}{\kappa!}\frac{\partial^\kappa}{\partial\gamma^\kappa}\mathcal{L}_n^{(2)}(0,\theta)\gamma^\kappa + \ldots + \frac{1}{(2\kappa)!}\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0,\theta)\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{7}$$

Note that for $j = 1, 2, \ldots, \kappa - 1$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\theta) = 0$ by the definition of κ. If $\kappa = 1$, the first-order derivative differs from zero, so that none of the derivatives is zero, meaning that $j = 0$. The partial derivatives in (7) are given in the following lemma:

Lemma 2. *Given Assumption 2, the definition of* κ*, and* $\mathcal{H}_{02}$*, for each* $\theta \neq 0$*,* $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0,\theta) = 2\theta'Z'H_j(0)Mu$ *for* $\kappa \leq j < 2\kappa$*; and* $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0,\theta) = 2\theta'Z'H_{2\kappa}(0)Mu - \binom{2\kappa}{\kappa}\theta'Z'H_\kappa(0)M\ H_\kappa(0)Z\theta$ *for* $j = 2\kappa$*, where* $H_j(\gamma) := (\partial^j/\partial\gamma^j)F(\gamma)$*.* □

Using Lemma 2 we can specifically write (7) as $\mathcal{L}_n^{(2)}(\gamma,\theta) - \mathcal{L}_n^{(2)}(0,\theta) = \sum_{j=\kappa}^{2\kappa}\frac{2}{j!}\{\theta'Z'\ H_j(0)Mu\}\gamma^j -$

$\frac{1}{(2\kappa)!}\binom{2\kappa}{\kappa}\theta' Z' H_\kappa(0) M H_\kappa(0) Z\theta\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa})$. To reduce notational clutter, we further let $G_j := [g_{j,1}, g_{j,2}, \ldots, g_{j,n}]' := MH_j(0)Z$, where $g_{j,t} := h_{j,t}(0)z_t - Z'H_j(0)Z\ (Z'Z)^{-1}Z'z_t$ and $\varsigma_n := n^{1/2\kappa}\gamma$ with $h_{j,t}(0)$ being the t-th diagonal element of $H_j(0)$. Then,

$$\mathcal{L}_n^{(2)}(\gamma,\theta) - \mathcal{L}_n^{(2)}(0,\theta) = \sum_{j=\kappa}^{2\kappa} \frac{2\{\theta' G_j' u\}}{j!\, n^{j/2\kappa}} \varsigma_n^j - \frac{1}{(2\kappa)!\, n}\binom{2\kappa}{\kappa}\{\theta' G_\kappa' G_\kappa \theta\}\varsigma_n^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{8}$$

We note that if $j = \kappa$, $n^{-j/2\kappa}G_j' u = O_{\mathbb{P}}(1)$ by applying the central limit theorem. Furthermore, for $j = \kappa+1, \ldots, 2\kappa-1$, $n^{-j/(2\kappa)}(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(\gamma,\theta) = o_{\mathbb{P}}(1)$ and $\theta' G_{2\kappa} u = o_{\mathbb{P}}(n)$ by the ergodic theorem, so that they become asymptotically negligible, implying that the smallest j-th component greater than κ and surviving at the limit becomes the second-final term in the right side of (8). Note that $n^{-1}G_\kappa' G_\kappa = O_{\mathbb{P}}(1)$, if the ergodic theorem applies, and the terms with $j > 2\kappa$ belong to $o_{\mathbb{P}}(\gamma^{2\kappa})$ by Taylor's theorem, so that they are asymptotically negligible under the null at any rate. Due to this fact, $\mathcal{L}_n^{(2)}(\cdot,\theta)$ is approximated by the 2κ-th degree polynomial function in (8), and we provide the following condition for the asymptotic analysis of the polynomial function:

Assumption 7. *For $j = \kappa, \kappa+1, \ldots, 2\kappa$ and $i = 0, 1, \ldots, p$, (i) $\mathbb{E}[|u_t|^8] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^4] < \infty$; or (ii) $\mathbb{E}[|u_t|^4] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^8] < \infty$.* □

Using Assumption 7, we can apply the CLT to $n^{-1/2}G_j' u$ for $j = \kappa, \kappa+1, \ldots, 2\kappa$. Note that $G_j' u = \sum_{t=1}^n (u_t g_{j,t})$, and $\mathbb{E}[(u_t g_{j,t})^2] < \infty$ by the moment conditions in Assumption 7 and Cauchy-Schwarz inequality, so that for $j = \kappa+1, \ldots, 2\kappa-1$, $n^{-j/2\kappa}G_j' u = o_{\mathbb{P}}(1)$. Although the QLR test statistic is approximated by the 2κ-th degree polynomial function, the moment conditions in Assumption 7 are sufficient to apply the CLT to the first term in (8).

We establish the following lemma by collecting the asymptotically surviving terms:

Lemma 3. *Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, $QLR_n^{(2)} = \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(n)$, where for given $\theta \neq 0$, $\overline{QLR}_n^{(2)}(\theta) := \sup_{\varsigma_n} \frac{1}{\widehat{\sigma}_{n,0}^2}\left\{\frac{2}{\kappa! n^{1/2}}\{\theta' G_\kappa' u\}\varsigma_n^\kappa - \frac{1}{(2\kappa)! n}\binom{2\kappa}{\kappa}\theta' G_\kappa' G_\kappa \theta \varsigma_n^{2\kappa}\right\}$ and $\widehat{\varsigma}_n^\kappa(\theta)$ maximises the given objective function, so that $\widehat{\varsigma}_n^\kappa(\theta) = W_n(\theta)$, if κ is odd; and $\widehat{\varsigma}_n^\kappa(\theta) = \max[0, W_n(\theta)]$, if κ is even, where $W_n(\theta) := \kappa! n^{1/2}\{\theta' G_\kappa' u\}/\{\theta' G_\kappa' G_\kappa \theta\}$.* □

Lemma 3 implies that the functional form of $\overline{QLR}_n^{(2)}(\cdot)$ depends on κ: for each $\theta \neq 0$, $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}\frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}$, if κ is odd; and $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}\max\left[0, \frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}\right]$, if κ is even. If θ is a scalar as in the previous literature, θ cancels out, so maximization with respect to θ does not matter any longer. This implies that $QLR_n^{(2)}$ and $\overline{QLR}_n^{(2)}(\cdot)$ are asymptotically equivalent under $\mathcal{H}_{02}$. On the other hand, if θ is a vector, the asymptotic null distribution of the test statistic has to be determined by further maximizing $\overline{QLR}_n^{(2)}(\cdot)$ with respect to θ.

We now derive the regular asymptotic distribution of QLR test statistic under $\mathcal{H}_{02}$. The following additional condition is sufficient for doing it:

Assumption 8. *$V_3(0)$ and $V_4(0)$ are positive definite, where for each γ, $V_3(\gamma) := \mathbb{E}[u_t^2 \bar{r}_t(\gamma)\ \bar{r}_t(\gamma)']$ and $V_4(\gamma) := \mathbb{E}[\bar{r}_t(\gamma)\bar{r}_t(\gamma)']$ with $\bar{r}_t(\gamma) := (h_{t,\kappa}(\gamma)z_t', z_t')'$.* □

We note that the nuisance parameter γ does not play a significant role in Assumption 8 as it does in the previous case, because $\overline{QLR}_n(\cdot)$ has already concentrated the QL function with respect to γ. Given these regularity conditions, the key limit results of the components that constitute $\overline{QLR}_n^{(2)}(\cdot)$ appear in the following lemma:

Lemma 4. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i) $n^{-1/2}G_\kappa' u \Rightarrow \mathcal{Z}_2$, where $\mathbb{E}[\mathcal{Z}_2] = 0$ and $\mathbb{E}[\mathcal{Z}_2\mathcal{Z}_2'] = \mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}']$; (ii) $n^{-1}G_\kappa' G_\kappa \stackrel{\text{a.s.}}{\to} A_2$, where $A_2 := \mathbb{E}[g_{t,\kappa} g_{t,\kappa}']$; and (iii) if, additionally, Assumption 3(iii) holds, $\mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}'] = \sigma_*^2 \mathbb{E}[g_{t,\kappa} g_{t,\kappa}']$.* □

Using Lemma 4, Theorem 2 describes the limit distribution of $QLR_n^{(2)}$ under $\mathcal{H}_{02}$:

Theorem 2. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i) $QLR_n^{(2)} \Rightarrow \max_{\theta\in\Theta} \mathcal{G}_2(\theta)^2$ if κ is odd; and if κ is even, $QLR_n^{(2)} \Rightarrow \max_{\theta\in\Theta} \max[0, \mathcal{G}_2(\theta)]^2$, where $\mathcal{G}_2(\cdot)$ is a Gaussian process such that for each θ, $\mathbb{E}[\mathcal{G}_2(\theta)] = 0$ and $\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = A_2(\theta,\theta)^{-1/2}B_2(\theta,\widetilde{\theta})A_2(\widetilde{\theta},\ \widetilde{\theta})^{-1/2}$, where $B_2(\theta,\widetilde{\theta}) := \theta'\mathbb{E}[u_t^2 g_{t,\kappa}\ g_{t,\kappa}']\widetilde{\theta}$ and $A_2(\theta,\widetilde{\theta}) := \sigma_*^2\theta'\mathbb{E}[g_{t,\kappa}\ g_{t,\kappa}']\widetilde{\theta}$; (ii) if, additionally, Assumption 3(iii) holds, $\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = A_2(\theta,\theta)^{-1/2}A_2(\theta,\widetilde{\theta})A_2(\widetilde{\theta},\widetilde{\theta})^{-1/2}$.* □

As Theorem 2 follows from Lemma 4 and continuous mapping, its proof is omitted.

Several remarks are in order. First, the covariance kernel of $\mathcal{G}_2(\cdot)$ is bilinear with respect to θ and $\widetilde{\theta}$. This implies that $\mathcal{G}_2(\theta)$ is a linear Gaussian process with respect to θ. Therefore, if $z \sim N(0, \mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}'])$, $z'\theta$ as a function of θ is distributionally equivalent to $\mathcal{G}_2(\cdot)$. This fact relates the null limit distribution to the chi-squared distribution. Corollary 1 of Cho and White (2018) shows that $\max_{\theta\in\Theta} \mathcal{G}_2(\theta)^2 \stackrel{\text{d}}{=} \mathcal{X}_{p+1}^2$ if $\mathcal{G}_2(\cdot)$ is a linear Gaussian process and $\mathbb{E}[u_t^2 g_{t,\kappa} g_{t,\kappa}'] = \sigma_*^2\mathbb{E}[g_{t,\kappa} g_{t,\kappa}']$, where $\mathcal{X}_{p+1}^2$ is a chi-squared distribution with $p+1$ degrees of freedom . Second, the chi-squared null limit distributions of the LM test statistics in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) follow from the fact that the LM test statistic is equivalent to the QLR test statistic under $\mathcal{H}_{02}$. Finally, if $\theta = 0$, $\mathcal{G}_2(\theta)$ is not well defined as the weak limit in Theorem 2 is obtained by assuming that $\theta \neq 0$. Nevertheless, the null limit distribution of the QLR test is well represented by Theorem 2 as it obtains the alternative model by letting $\theta \neq 0$.

3.2.3 Null Limit Distribution of the QLR Test Statistic under $\mathcal{H}_0$

In this subsection, we derive the limit distribution of the QLR test under $\mathcal{H}_0$ by examining the relationship between $QLR_n^{(1)}$ and $QLR_n^{(2)}$. Specifically, we show that $QLR_n^{(1)} \geq QLR_n^{(2)}$, which means the limit distribution under $\mathcal{H}_0$ equals that of $QLR_n^{(1)}$. Although this idea is the same as the one in Cho, Ishida, and White (2011, 2014), their approach is not applicable here because the associated Gaussian process $\mathcal{G}_1(\cdot)$ is multivariate.

The following lemma generalises the approach in Cho, Ishida, and White (2011, 2014):

Lemma 5. *Let* $n(\gamma) := Z'F(\gamma)Mu$ *and* $D(\gamma) := Z'F(\gamma)MF(\gamma)Z'$ *with* $n^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)n(\gamma)$*, and* $D^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)D(\gamma)$*. Under Assumptions 1, 2 and 3, (i) for* $j < \kappa$*,* $\lim_{\gamma\to 0} n^{(j)}(\gamma) \overset{\text{a.s.}}{=} 0$ *and* $\lim_{\gamma\to 0} D^{(j)}(\gamma) \overset{\text{a.s.}}{=} 0$*; (ii)* $\lim_{\gamma\to 0} n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G'_\kappa u$*; and (iii)* $\lim_{\gamma\to 0} D^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G'_\kappa G_\kappa$*.* □

The limit obtained by letting $\gamma \to 0$ under $\mathcal{H}_{01}$ can be compared with that obtained under $\mathcal{H}_{02}$. More specifically, using Lemma 5 and L'Hôpital's rule, we obtain that $\lim_{\gamma\to 0} n(\gamma)'D(\gamma)^{-1}n(\gamma) \overset{\text{a.s.}}{=} \lim_{\gamma\to 0} n^{(\kappa)}(\gamma)'D^{(\kappa)}(\gamma)^{-1}n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} u'G_\kappa(G'_\kappa G_\kappa)^{-1}G'_\kappa u$. From this, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta)$ as $QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}^2_{n,0}} n(\gamma)'D(\gamma)^{-1}n(\gamma) \geq \lim_{\gamma\to 0} \frac{1}{\widehat{\sigma}^2_{n,0}} n(\gamma)'D(\gamma)^{-1} n(\gamma) \overset{\text{a.s.}}{=} \frac{1}{\widehat{\sigma}^2_{n,0}} u'G_\kappa(G'_\kappa G_\kappa)^{-1}G'_\kappa u$. Furthermore, $\overline{QLR}_n^{(2)}(\theta)$ is asymptotically equal to $\frac{1}{\widehat{\sigma}^2_{n,0}} u'G_\kappa\theta\ (\theta'G'_\kappa G_\kappa\theta)^{-1}\ \theta'G'_\kappa u$. Thus, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(1)$, if $G_\kappa(G'_\kappa G_\kappa)^{-1}G'_\kappa - G_\kappa\theta(\theta'G'_\kappa G_\kappa\theta)^{-1}\theta'G'_\kappa$ is positive semidefinite irrespective of θ. To show this we first note that the two terms are idempotent and symmetric matrices. To do this, we make use of Exercise 8.58 in Abadir and Magnus (2005, p. 233). Then, $\{G_\kappa(G'_\kappa G_\kappa)^{-1}G'_\kappa\}\{G_\kappa\theta(\theta'G'_\kappa G_\kappa\theta)^{-1}\theta'G'_\kappa\} = G_\kappa\theta(\theta'G'_\kappa G_\kappa\theta)^{-1}\theta'G'_\kappa$, so that it is positive semidefinite. This implies $QLR_n = \max[QLR_n^{(1)}, QLR_n^{(2)}] + o_{\mathbb{P}}(1) = \max[QLR_n^{(1)}, \sup_\theta \overline{QLR}_n^{(2)}(\theta)] + o_{\mathbb{P}}(1) = QLR_n^{(1)} + o_{\mathbb{P}}(1)$. Given that $\Gamma(\epsilon)$ was considered in Theorem 1 to remove $\gamma = 0$ from Γ, if ϵ is selected as small as possible to have $QLR_n = QLR_n(\epsilon) + o_{\mathbb{P}}(1)$ and we can let $\gamma \to 0$ as posited in Lemma 5, the null limit distribution of the QLR test is characterised by the Gaussian process in Theorem 1. Therefore, if the conditions in Theorems 1 and 2 hold simultaneously, the null limit distribution of the QLR test statistic can be derived by combining Theorems 1 and 2. For this purpose, we combine Assumptions 6 and 8 into a new assumption as follows:

Assumption 9. *For each $\gamma \neq 0$, $V_5(\gamma)$ and $V_6(\gamma)$ are positive definite, where $V_5(\gamma) := \mathbb{E}[u_t^2 \ddot{r}_t(\gamma)\ddot{r}_t(\gamma)']$, $V_6(\gamma) := \mathbb{E}[\ddot{r}_t(\gamma)\ddot{r}_t(\gamma)']$, and $\ddot{r}_t(\gamma) := (h_{t,\kappa}(0)z_t', f_t(\gamma)z_t', z_t')'$.* □

The following theorem now yields the limit distribution of the QLR test under $\mathcal{H}_0$:

Theorem 3. *Given Assumptions 1, 2, 3(i), 4, 5, 7, 9, and $\mathcal{H}_0$, (i) $QLR_n \Rightarrow \sup_{\gamma\in\Gamma} \mathcal{G}_1(\gamma)'\mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian process such that for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{G}_1(\gamma)] = 0$ with $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}B_1(\gamma,\widetilde{\gamma})$ $A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$; (ii) if Assumption 3(ii) additionally holds, then $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2}A_1(\gamma,\widetilde{\gamma})A_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$.* □

Theorem 3 immediately follows from Theorems 1 and 2 and from our earlier argument that $QLR_n = QLR_n^{(1)} + o_{\mathbb{P}}(1)$, which is why we do not prove it in the Supplement. Note that the consequence of Theorem 3 is the same as that of Theorem 1, although the null hypothesis is extended from $\mathcal{H}_{01}$ to $\mathcal{H}_0$ by enlarging the parameter space from $\Gamma(\epsilon)$ to Γ with sufficiently small ϵ. Also note that the Gaussian process $\mathcal{G}_1(\cdot)$ is obtained by supposing that $\gamma \neq 0$. Otherwise, a meaningful QLR test statistic is not properly defined.

The null limit distribution in Theorem 3 is derived as in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). Nevertheless, our proofs generalise theirs due to the existence of the multivariate Gaussian process. Furthermore, this null limit distribution extends the scope of the LM test statistics in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) who only test $\mathcal{H}_{02}$. Finally, this null limit distribution differs from that of the sup-Wald test statistics testing for a structural break. The associated null limit distribution depends on the joint distribution of the data and the smooth-transition function used for the STAR model because the covariance kernel structure of the Gaussian process is affected by these factors. We illustrate this fact in the Supplement by simulation using different DGPs and model assumptions.

4 Empirical Inference on the Nonlinear Fiscal Multiplier Effect

Having derived the test statistic, we return to fiscal policy effects on other macroeconomic variables. As already discussed, we revisit the empirical study in AG in a systematic manner while focusing on revealing the nonlinear fiscal policy effect. In addition, we study the validity of the other assumptions made by AG. In case the VSTAR model adequately describes the nonlinearity in the data, we exploit the model estimates to examine the state-dependent multiplier effect on the macroeconomic variables.

The plan of our empirical analysis is as follows. As discussed in Section 2.1, we first note that AG based their VSTAR model on the VAR model in Blanchard and Perotti (2002). It has the form

$$y_t = (1 - f_t(\gamma_*))\Pi_{E*}(L)y_{t-1} + f_t(\gamma_*)\Pi_{R*}(L)y_{t-1} + u_t$$

such that $u_t|(y_t, y_{t-1}, \ldots) \sim N(0, \Omega_t)$, where for some Ω_{E*} and Ω_{R*}, $\Omega_t := (1 - f_t(\gamma_*))\Omega_{E*} + f_t(\gamma_*)\Omega_{R*}$, and $y_t := (g_t, \tau_t, q_t)'$ with g_t, τ_t, and q_t being log real government spending, log real government net tax receipts, and log real GDP deflated by the 2012 GDP deflator, respectively. Here, $\Pi_{E*}(L)$ and $\Pi_{R*}(L)$ are the lag operators employed to capture the lag coefficients under economic expansion and recession periods, respectively. AG chose the transition function $f_t(\gamma)$ to be logistic and the transition variable to be a seven-quarter moving average of the output growth rate centered at the sample average. They assumed $c_* = 0$ in the logistic function.

Despite the appealing features of the VSTAR model detailed in AG, nonstationarity of their time series makes it difficult to infer the associated parameters by the standard approach. Their estimation method is technically complex as it employs the Markov-Chain Monte Carlo method. The complexity may be due to the fact that exactly the same transition function is used both in the mean and the variance components of the model. This restriction was not motivated in the paper.

We avoid this difficulty by converting the VSTAR model into a VEC form as in Candelon and Lieb (2013). Doing this to the VAR form in Blanchard and Perotti (2002) yields the following model: $\Delta y_t = \Phi_* y_{t-1} + \widetilde{\Pi}_*(L)\Delta y_{t-1} + u_t$. The first term on the right-hand side captures the long-run relationship between the variables. It is assumed invariant to the economic states, which means that we may rewrite the VEC form as $\Delta y_t = \alpha_* \beta_*' y_{t-1} + \widetilde{\Pi}_*(L)\Delta y_{t-1} + u_t$, where α_* and β_*' are $3 \times m$ and $m \times 3$ matrices, respectively, and m is the cointegration rank. This conversion allows us to rewrite VSTAR model as a VSTEC model: $\Delta y_t = (1 - f_t(\gamma_*))[\alpha_{E*} y_{t-1}^* + \widetilde{\Pi}_{E*}(L)\Delta y_{t-1}] + f_t(\gamma_*)[\alpha_{R*} y_{t-1}^* + \widetilde{\Pi}_{R*}(L)\Delta y_{t-1}] + u_t$, where $y_{t-1}^* := \beta_*' y_{t-1}$ is an $m \times 1$ vector of stationary variables (*e.g.*, Rothman, van Dijk, and Franses, 2001; Hubrich and Teräsvirta, 2013). Note that the cointegration vector β_* is invariant to the economic state, whereas the adjustment coefficient $\alpha_* = (1 - f_t(\gamma_*))\alpha_{E*} + f_t(\gamma_*)\alpha_{R*}$, where α_{E*} is not necessarily equal to α_{R*}. The model thus has a continuum of states, which is different from Candelon and Lieb (2013) whose TAR model contains two states with the restriction $\alpha_{E*} = \alpha_{R*}$. We define the error covariance matrix as $\Omega_t = D_t P D_t$, where D_t is the diagonal matrix of conditionally heteroskedastic standard deviations, and P is the corresponding correlation matrix assumed constant over time, see Eklund and Teräsvirta (2007). This decomposition allows us to test homoskedasticity before assuming heteroskedastic errors. We specifically assume that, under the alternative, $D_t := (1 - f_t(\varphi_*))D_{E*} + f_t(\varphi_*)D_{R*}$, where D_{E*} and D_{R*} have diagonal elements σ_{i*}^2 and $\sigma_{i*}^2 + \lambda_{i*}$, respectively for $i = 1, 2$, and 3, so that D_t becomes a diagonal matrix with $\sigma_{i*}^2 + \lambda_{i*} f_t(\varphi_*)$. Under the null hypothesis of conditional homoskedasticity, $D_{E*} = D_{R*}$, and we test this hypothesis by the test statistic provided by Eklund and Teräsvirta (2007). Admittedly, there is no theoretical reason for believing that correlations remain constant over time while the variances exhibit conditional heteroskedasticity. Nevertheless, we believe that the conditional error variances are likely to dominate the conditional heteroskedastic correlations for many empirical data exhibiting conditional heteroskedasticity. Using this belief and the fact that there are fewer parameters in our model than that of AG, we test for conditional heteroskedasticity by employing this specification test.

The data of AG comprised US quarterly macroeconomic variables, and their sample ran from 1947Q1 to 2008Q4. We use the same dataset and begin by fitting a linear VEC model it. In order to do that we specify the cointegration rank m and estimate β_* following Johansen (1995). As the maximum likelihood estimator $\widehat{\beta}_n$ for β_* is super-consistent, we replace y_{t-1}^* in the VSTEC model with $\widehat{y}_{t-1} := \widehat{\beta}_n' y_{t-1}$ and estimate the other parameters by NLS.

Having estimated the VEC model, we test linearity against the VSTEC model using the QLR test and employ other diagnostic tests to assess the model adequacy. In order to do this we modify the VSTEC model such that testing it becomes possible. Note that the VSTEC model can be rewritten as $\Delta y_t = \Psi_{1*}(L)w_{t-1} + f_t(\gamma_*)\Psi_{2*}(L)w_{t-1} + u_t$, where $u_t = (u_{1t}, u_{2t}, u_{3t})'$, $w_{t-1} := [y_{t-1}^{*\prime}, \Delta y_{t-1}']'$, $\Psi_{1*}(L) := [\alpha_{R*}, \widetilde{\Pi}_{R*}(L)]$, and $\Psi_{2*}(L) := [\alpha_{D*}, \widetilde{\Pi}_{D*}(L)]$ with $\alpha_{D*} := \alpha_{R*} - \alpha_{E*}$ and $\widetilde{\Pi}_{D*}(L) := \widetilde{\Pi}_{R*}(L) - \widetilde{\Pi}_{E*}(L)$. Given the normality assumption for u_t, each equation can be marginalised as follows: for $j = 1, 2, 3$, $\Delta y_{jt} = \theta_{j*}' \Delta y_{-jt} + \xi_{j1*}(L)' w_{t-1} + f_t(\gamma_*)\xi_{j2*}(L)' w_{t-1} + \epsilon_{jt}$, where for $j = 1, 2, 3$, $\theta_{j*}' := \mathbb{E}[u_{jt} u_{-jt}]\mathbb{E}[u_{-jt} u_{-jt}']^{-1}$, and, further, $\xi_{j1*}(L)' := \psi_{j1*}(L)' - \theta_{j*}' \psi_{-j1*}(L)'$ and $\xi_{j2*}(L)' := \psi_{j2*}(L)' - \theta_{j*}' \psi_{-j2*}(L)'$. Here, u_{jt} and u_{-jt} denote the j-th row element of u_t and the 2×1 vector obtained by removing u_{jt} from u_t, respectively. Furthermore, for each $i = 1$ and 2, $\psi_{ji*}(L)'$ and $\psi_{-ji*}(L)'$ are the j-th row vector of $\Psi_{i*}(L)$ and $2 \times (m + 3)$ matrix obtained by removing the j-th row from $\Psi_{i*}(L)$, respectively. We use the marginal model as our baseline model for testing for nonlinearity, where we replace w_{t-1} in the marginal model with $\widehat{w}_{t-1} := [\widehat{y}_{t-1}', \Delta y_{t-1}']'$ to again exploit the super-consistency of $\widehat{\beta}_n$. We apply the QLR test for each $j = 1, 2, 3$. Note that rejecting the linearity hypothesis of this marginalized individual equation is sufficient for rejecting the linearity hypothesis of the VEC equation. In addition to the QLR test, we test for serial correlation and conditional heteroskedasticity in the errors. Serially uncorrelated errors are a necessary condition for the adequacy of the model. If the errors are serially correlated, the nonlinear model is misspecified in the sense that the dynamic structure of the model is incorrect. In addition, as detailed above, the test statistic by Eklund and Teräsvirta (2007) is employed to test for conditional heteroskedasticity. If it turns out to be conditionally homoskedastic, we do not have to specify a nonlinear model for error covariances as AG do.

Finally, based on the estimated nonlinearity associated with y_t, we examine the impact of government spending by separately estimating the impulse response functions of the variables in y_t under the expansion and recession states. This is done in order to compare our empirical findings based upon the VSTEC model with those given by AG. They estimate the corresponding impulse response functions by estimating the VSTAR model directly. In order to generate the impulse response functions, we use the VAR structure given in Blanchard and Perotti (2002) by ordering the variables in y_t as above and relating it to Cholesky decomposition. By this ordering, the shocks in τ_t and q_t do not affect g_t in the same quarter, as they assumed for model identification. AG also adopted this ordering. More specifically, after estimating the expansion and recession regime parameters in the VSTEC model, we convert them into their respective VSTAR parameters and next convert them again into the VAR parameters in expansion and recession by Cholesky decomposition. Given the constant long-run relationship $\beta_*' y_t$, we suppose that the identification strategy is sensible, although it may be further improved by employing sign restrictions (*e.g.* Candelon and Lieb, 2013) or the narrative approach (*e.g.*, Owyang *et al.*, 2013a; Ramey and Zubairy, 2018). Using the separate expansion and recession VAR estimates, we finally estimate their corresponding impulse response functions of the variables in y_t in response to a \$1 increase of government spending under expansion and recession. Note that this approach limits our attention to the extreme economic states because the estimated impulse response functions are obtained by letting $f_t(\gamma_*) = 0$ and $f_t(\gamma_*) = 1$ for expansion and recession states, respectively, so that it ignores the path of exogenous shock and its interactions with other variables. In this regard, the generalised impulse response functions proposed by Koop, Pesaran and Potter (1996) may be a better alternative. Nevertheless, by adopting this approach, we can make a reasonable comparison of our fiscal multiplier estimates with those in other macroeconomic literature.

We now report our empirical findings. In order to estimate the cointegration rank m, we apply Johansen's (1988, 1991) trace testing procedure with lag equal to 3 selected by AIC and BIC[3] and cannot reject the hypothesis that $m = 2$ at the 5% significance level. Using this rank, we estimate the cointegration coefficient

[3]The lag order is also identical to that selected by AG.

β_* to obtain $\widehat{y}_{t-1}$.

Next, we apply several diagnostic testing procedures to validate our assumptions. Our first diagnostic test is for the nonlinearity assumption made for the VSTEC model. As explained above, we replace w_{t-1} in the marginal model with $\widehat{w}_{t-1} := [\widehat{y}'_{t-1}, \Delta y'_{t-1}]'$ and test the linearity hypothesis by the QLR tests based upon LSTAR model. We report the p-values of the QLR tests in Table 1. They are obtained by applying Hansen's (1996) weighted bootstrap with 20,000 replications, and they strongly reject the linearity hypothesis. This implies that the linear VEC model is not adequate and the VSTEC model can better capture the dynamic interrelationship among the variables. Next we test for serial correlation in the errors using the multivariate Ljung-Box test and fail to reject the null of no serial correlation of the errors at 5% level of significance as reported in Table 1. As the final diagnostic test, we apply the test statistic in Eklund and Teräsvirta (2007) to test for conditional heteroskedasticity. The p-value of the test can be found in Table 1, indicating that the null of conditional homoskedasticity is not rejected at the 5% level of significance.[4]

We compare our estimates with those in AG. After converting the estimated expansive and recessive parameters in the VSTEC model into their respective VAR parameters by Cholesky decomposition, we estimate the impulse response functions of the three variables. The results appear in Figure 1. The impulse responses are plotted following AG. That is, we show how g_t, τ_t, and q_t respond to the government spending over 10 quarters under expansion and recession states and depict their 90% confidence bands obtained from 50,000 Monte Carlo replications. We also plot the impulse response functions obtained from the linear VEC model. The impulse response functions of g_t, τ_t, and q_t are shown from the top, to the middle and bottom rows in the figure, respectively.

Our empirical findings are summarised as follows. First, the overall shapes of the functions agree with those in AG. Specifically, the response of government purchases under the expansion state achieves its maximum after a short delay, as argued in AG. Furthermore, the immediate effect of the increase in government spending on q_t is less than unity, irrespective of the economic states. The estimated immediate effect is about 0.114 for both expansion and recession states, which is slightly greater than that obtained from the linear VEC model, 0.106. This also agrees with AG. Second, the details of the impulse response functions, nevertheless, are not exactly the same as those in AG. For instance, the bottom row of Figure 1 shows a smaller response of q_t to the government spending in both states compared to that in AG; it reaches only around 0.2 in the recession state,

[4] In addition to Eklund and Teräsvirta's (2007) test statistic, we also applied Breusch-Pagan's conditional heteroskedasticity and multivariate Ljung-Box test statistic to detect conditional heteroskedasticity but failed to detect. Testing results are available from the authors upon request.

which is about one tenth of that in AG. In addition, over time, the estimated response of q_t in both states is not severely divergent from each other.

As to the numerical results on the impulse response functions, we report the estimated spending multipliers in Table 2. They are estimated following AG. That is, we estimate them by the maximum output after increasing the government spending, $\max_{h=1,\ldots,H} Y_h$, and the sum of outputs relative to the sum of government purchases over H quarters after increasing the government spending, $\sum_{h=1}^{H} Y_h / \sum_{h=1}^{H} G_h$, where we let H be 8 and 20 quarters. To do this, we convert the estimated percentage changes of government spending and output into their dollar equivalents over the same sample period and report the point estimates along with standard errors. Table 2 reports the estimates in detail, and the figures in parentheses are their standard errors. Compared to the estimated multipliers in Table 1 of AG, most of our estimates have larger standard errors, especially under the expansion and recession states, and thus we cannot rule out the possibility that they are of smaller magnitude than those in AG if they are nonzero. The multiplier effect estimated by AG for the recession state exceeds unity and has been doubted in a number of studies (*e.g.*, Owyang *et al.*, 2013a; Ramey and Zubairy, 2018), whereas our model estimates are free from this doubt. Furthermore, Table 2 reports that a larger but insignificant fiscal multiplier effect over the longer horizon, implying that the fiscal policy is effective especially in recessions for a short period of time. The overall effect is smaller than that estimated by AG.

5 Conclusion

We reconsider the neglected nonlinearity in the multiplier effect of US government spending by revisiting the empirical study made by AG. In order to draw proper inferences, we tackle the nonlinearity of the fiscal multiplier effect in two ways. First, we reformulate the VSTAR model in AG into the VSTEC model so that the nonstationarity present in the variables can be handled appropriately. We develop a methodology for testing linearity in the STAR framework and apply it to testing thr VEC model against the VSTEC one. As the frequently used LM test statistic does not comprehensively test the linearity hypothesis, we examine the QLR test statistic that does have this property. We show that the QLR test statistic weakly converges to a functional of a multivariate Gaussian process under the null of linearity. By applying the QLR test statistic to the VSTEC model using the same data as in AG, we affirm the nonlinear fiscal multiplier effect and also estimate the response functions of the economic variables relevant to a $1 increase of the government spending, whose overall forms are similar to those in AG. Nevertheless, the magnitudes of the fiscal multiplier effect measured by the VSTEC model estimation are not so large as those given in AG.

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QLR tests \ variables	Δg_t	$\Delta \tau_t$	Δq_t
QLR_n	**0.000**	**0.004**	**0.000**
Eklund and Teräsvirta's (2007) conditional heteroskedasticity test			0.158
Multivariate Ljung-Box test			0.175

Table 1: p-VALUES OF THE DIAGNOSTIC TEST STATISTICS. Notes: The figures in QLR_n row show the p-values of the QLR test statistics for linearity based upon the LSTAR VSTEC model, and they are obtained using 20,000 bootstrap replications. The variables in the first row denote the dependent variables in the marginal models. In addition, we repot the p-values of the multivariate Ljung-Box test statistic and Eklund and Teräsvirta (2007) test statistic for conditional heteroskedasticity. Boldface p-values indicate significance levels less than or equal to 0.05.

	$\max_{h=1,\ldots,H} Y_h$		$\sum_{h=1}^{H} Y_h / \sum_{h=1}^{H} G_h$	
	$H = 8$	$H = 20$	$H = 8$	$H = 20$
Linear	1.299	1.299	0.664	0.926
	(0.762)	(1.530)	(0.849)	(1.550)
Expansive	0.666	0.666	0.223	0.272
	(0.440)	(1.906)	(0.848)	(1.617)
Recessive	0.961	1.125	0.587	1.001
	(0.248)	(0.525)	(0.331)	(0.569)

Table 2: ESTIMATED GOVERNMENT SPENDING MULTIPLIERS UNDER EACH LINEAR, RECESSIVE, AND EXPANSIVE STATES. This table shows the estimated output multipliers for a \$1 increase in government spending, measured by maximum output increase and total output increase relative to total government spending over 8 and 20 quarters. The figures in parentheses are the standard errors of the estimates.

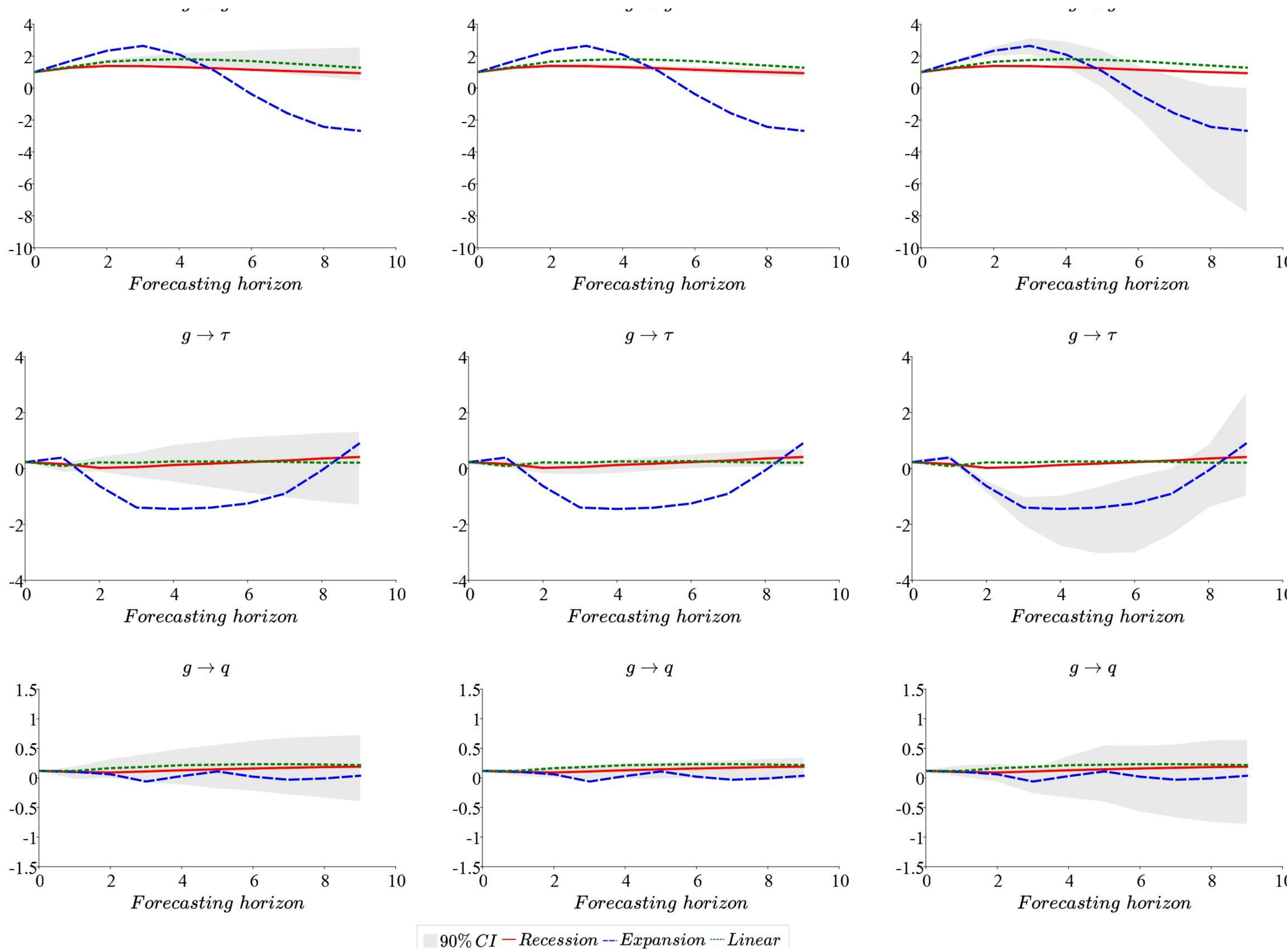

Figure 1: IMPULSE RESPONSES IN LINEAR, EXPANSION, AND RECESSION STATES Notes: The first, second, and third columns show the impulse and response functions of g_t, τ_t, and q_t in response to a \$1 increase of government spending obtained by linear model, recession and expansion state parameters, respectively. Shaded region is the 90 percent confidence band. Long dashed and solid lines show the responses in expansion and recession. Dotted line shows the responses using the linear model.

A Supplement

In this Supplement we examine testing linearity against commonly applied STAR models and also provides simulation evidence of our methodology. We also demonstrate how Hansen's (1996) weighted bootstrap is applied to enhance the applicability of our methodology. We additionally re-examine the quarterly US unemployment rate series that has been previously studied by van Dijk, Teräsvirta, and Franses (2002) to demonstrate how different inference results obtained from the QLR and LM test statistics can supplement each other. Finally, we provide the proofs of the theoretical results in the paper

A.1 Monte Carlo Experiments and Application of the Weighted Bootstrap

A.1.1 Monte Carlo Experiments Using the ESTAR Model

To simplify our illustration, we assume that for all $t = 1, 2, \ldots$, $u_t \sim$ IID $N(0, \sigma_*^2)$ and $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$. Under this DGP, we specify the following first-order ESTAR model: $\mathcal{M}_{ESTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 - \exp[-\gamma y_{t-1}^2]\} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma\}$. The model does not contain an intercept, and the transition variable is y_{t-1}. The nonlinear function $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ is defined on Γ which is compact and convex, and the exponential function is analytic. This means that the QLR test statistic is generically comprehensively revealing. To identify the model it is assumed that $\gamma_* > 0$. In our model set-up, we allow 0 to be included in Γ. The nonlinear function $f_t(\cdot)$ satisfies $f_t(0) = 0$. Given this model, the following hypotheses are of interest: $\mathcal{H}_0' : \exists \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) = 1$; vs. $\mathcal{H}_1' : \forall \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) < 1$. Two parameter restrictions make $\mathcal{H}_0'$ valid: either $\theta_* = 0$ or $\gamma_* = 0$. The sub-hypotheses are thus $\mathcal{H}_{01}' : \theta_* = 0$ and $\mathcal{H}_{02}' : \gamma_* = 0$.

We first examine the null distribution of the QLR test under $\mathcal{H}_{01}'$. By Theorem 1, the null limit distribution of this test statistic is given as $QLR_n^{(1)} = \sup_{\gamma \in \Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} (u'MF(\gamma)Z)^2 /\ Z'F(\gamma)MF(\gamma)Z \Rightarrow \sup_{\gamma \in \Gamma} \widetilde{\mathcal{G}}_1(\gamma)^2$, where $\widetilde{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure $\widetilde{\rho}_1(\gamma, \widetilde{\gamma}) = c_1(\gamma, \gamma)^{-1/2} \widetilde{k}_1(\gamma, \widetilde{\gamma})$ $\widetilde{c}_1(\widetilde{\gamma}, \widetilde{\gamma})^{-1/2}$ with $\widetilde{k}_1(\gamma, \widetilde{\gamma}) = \widetilde{c}_1(\gamma, \widetilde{\gamma}) = \sigma_*^2 \{\mathbb{E}[y_t^2 \exp(-(\gamma + \widetilde{\gamma}) y_t^2)] - \mathbb{E}[y_t^2 \exp(-\gamma y_t^2)] \mathbb{E}[y_t^2 \exp(-\widetilde{\gamma} y_t^2)] /$ $\mathbb{E}[y_t^2]\}$. Furthermore, under $\mathcal{H}_{01}'$, y_t is normally distributed with $\mathbb{E}[y_t] = 0$ and $\text{var}[y_t] = \sigma_y^2 := \sigma_*^2/(1 - \pi_*^2)$, so that y_t^2 follows the gamma distribution with shape parameter $1/2$ and scale parameter $2\sigma_*^2/(1 - \pi_*^2)$. Define $\widetilde{m}(\gamma) := (1 + 2\sigma_*^2/1 - \pi_*^2 \gamma)^{-\frac{1}{2}}$, and $\widetilde{h}(\gamma, \widetilde{\gamma}) := \frac{1}{\sigma_y^2}([(1 + 2\sigma_y^2 \gamma)(1 + 2\sigma_y^2 \widetilde{\gamma})/\{1 + 2\sigma_y^2(\gamma + \widetilde{\gamma})\}]^{3/2} - 1)$. Note that $\widetilde{m}(\gamma) = \mathbb{E}[\exp(-\gamma y_t^2)]$, so that $\mathbb{E}[y_t^2 \exp(-\gamma y_t^2)] = -\widetilde{m}'(\gamma)$. As a result, $\widetilde{\rho}_1(\gamma, \widetilde{\gamma})$ is further simplified to $\widetilde{k}_1(\gamma, \widetilde{\gamma}) = \sigma_*^2 \widetilde{m}'(\gamma) \widetilde{m}'(\widetilde{\gamma}) \widetilde{h}(\gamma, \widetilde{\gamma})$, and $\widetilde{\rho}_1(\gamma, \widetilde{\gamma}) = \widetilde{c}_1(\gamma, \gamma)^{-1/2} \widetilde{k}_1(\gamma,\ \widetilde{\gamma}) \widetilde{c}_1(\widetilde{\gamma}, \widetilde{\gamma})^{-1/2} =$

$\widetilde{h}(\gamma,\gamma)^{-1/2}\widetilde{h}(\gamma,\widetilde{\gamma})\widetilde{h}(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$.

We next examine the limit distribution of the QLR test statistic under $\mathcal{H}_{02}'$: $\gamma_* = 0$. The first-order derivative $(\partial/\partial\gamma)f_t(\gamma) = y_{t-1}^2\exp(-\gamma y_{t-1}^2)$, which is different from zero even when $\gamma = 0$, so that in this case $\kappa = 1$. Thus, we can apply the second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}_{02}'$. As a result, $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}(\theta' G_\kappa' u)^2/\theta' G_\kappa' G_\kappa\theta$, where $\theta' G_\kappa' u = \theta[\sum y_{t-1}^3 u_t - \sum y_{t-1}^4 \sum y_{t-1}u_t / \sum y_{t-1}^2]$ and $\theta' G_\kappa' G_\kappa \theta = \theta^2[\sum y_{t-1}^6 - (\sum y_{t-1}^4\ )^2/\sum y_{t-1}^2]$. Here, θ is a scalar, so that cancels out, and it follows that $QLR_n^{(2)} \Rightarrow \widetilde{\mathcal{G}}_2^2$, where $\widetilde{\mathcal{G}}_2 \sim N(0,1)$.

These two separate results can be combined, which means that we can examine the limit distribution of the QLR test under $\mathcal{H}_0'$. We have $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\widetilde{\mathcal{G}}(\gamma)^2$, where $\widetilde{\mathcal{G}}(\gamma) = \widetilde{\mathcal{G}}_1(\gamma)$, if $\gamma \neq 0$; and $\widetilde{\mathcal{G}}(\gamma) = \widetilde{\mathcal{G}}_2$, otherwise, and $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_1(\gamma,\widetilde{\gamma})$, if $\gamma \neq 0, \widetilde{\gamma} \neq 0$; $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = 1$, if $\gamma = 0, \widetilde{\gamma} = 0$; and $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_3(\gamma)$, if $\gamma \neq 0, \widetilde{\gamma} = 0$ with $\widetilde{\rho}_3(\gamma) := \mathbb{E}[\widetilde{\mathcal{G}}_1(\gamma)\widetilde{\mathcal{G}}_2] = \sqrt{6}\sigma_y^2\gamma/\{\widetilde{h}(\gamma,\gamma)^{1/2}(1+2\sigma_y^2\gamma)\}$ such that $\widetilde{\rho}_3(\gamma)^2 = \lim_{\widetilde{\gamma}\to 0}\widetilde{\rho}_1(\gamma,\widetilde{\gamma})^2 = (\sqrt{6}\sigma_y^2\gamma/\{\widetilde{h}(\gamma,\gamma)^{1/2}(1+2\sigma_y^2\gamma)\})^2$. Thus, we conclude that $QLR_n \Rightarrow \sup_\gamma \widetilde{\mathcal{G}}(\gamma)^2$, which agrees with Theorem 3.

The null limit distribution can be approximated numerically by simulating a distributionally equivalent Gaussian process. To do this we present the following lemma:

Lemma A.1. *If $\{z_k : k = 0,1,2,\ldots\}$ is an IID sequence of standard normal random variables, $\widetilde{\mathcal{G}}(\cdot) \overset{\mathrm{d}}{=} \overline{\mathcal{G}}(\cdot)$, where for each $\gamma \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$, $\overline{\mathcal{G}}(\gamma) := \sum_{k=1}^\infty c(\gamma)a(\gamma)^k\ [(-1)^k\binom{-3/2}{k}]^{1/2}z_k$, $c(\gamma) := \{\sum_{k=1}^\infty(-1)^k a(\gamma)^{2k}\binom{-3/2}{k}\ \}^{-1/2}$, and $a(\gamma) := 2\sigma_y^2\gamma/(1+2\sigma_y^2\ \gamma)$.* □

Note that the term $(-1)^k\binom{-3/2}{k}$ in Lemma A.1 is always positive irrespective of k, and for any γ,

$$\lim_{k\to\infty}\mathrm{var}\left[a(\gamma)^k\left((-1)^k\binom{-3/2}{k}\right)^{1/2}z_k\right] = \lim_{k\to\infty}a(\gamma)^{2k}(-1)^k\binom{-3/2}{k} = 0 \qquad \text{(A.1)}$$

and $\widetilde{h}(\gamma,\gamma) = \sum_{k=1}^\infty a(\gamma)^{2k}(-1)^k\binom{-3/2}{k}$. Using these facts Lemma A.1 shows that for any γ and $\widetilde{\gamma} \neq 0$, $\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_1(\gamma,\widetilde{\gamma})$. Here, the non-negative parameter space condition for Γ is necessary for $\overline{\mathcal{G}}(\cdot)$ to be properly defined on Γ. Without this condition, $\overline{\mathcal{G}}(\gamma)$ cannot be properly generated. We note that $\lim_{\gamma\downarrow 0}\overline{\mathcal{G}}(\gamma) \overset{\mathrm{a.s.}}{=} z_1$, so that if we let $z_1 = \overline{\mathcal{G}}_2$, $\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}_2] = \sqrt{6}\sigma_y^2\gamma\widetilde{h}(\gamma,\gamma)^{-1/2}(1+2\sigma_y^2\gamma)^{-1} = \widetilde{\rho}_3(\gamma)$. It follows that the distribution of $\widetilde{\mathcal{G}}(\cdot)$ can be simulated by iteratively generating $\overline{\mathcal{G}}(\cdot)$. In practice,

$$\overline{\mathcal{G}}(\gamma;K) := \sum_{k=1}^K a(\gamma)^k\left[(-1)^k\binom{-3/2}{k}\right]^{1/2}z_k \Big/ \sqrt{\sum_{k=1}^K a(\gamma)^{2k}(-1)^k\binom{-3/2}{k}}$$

is generated by choosing K to be sufficiently large . By (A.1), if this is the case, the difference between the distributions of $\overline{\mathcal{G}}(\cdot)$ and $\overline{\mathcal{G}}(\cdot; K)$ becomes negligible.

We now conduct Monte Carlo experiment and examine the empirical distributions of the QLR statistic under several different environments. First, we consider four different parameter spaces: $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, and $\Gamma_4 = [0, 5]$. They are selected to examine how the null limit distribution of the QLR test is influenced by the choice of Γ. We obtain the limit distribution by simulating $\sup_{\gamma \in \Gamma} \bar{\mathcal{G}}(\gamma; K)^2$ 5,000 times with $K = 2,000$, where Γ is in turn Γ_1, Γ_2, Γ_3, and Γ_4. Second, we study how the empirical distribution of the QLR test statistic changes with the sample size. We consider the sample sizes $n = 100$, $1,000$, $2,000$, and $5,000$.

Figure A.1 summarises the simulation results and shows that the empirical distribution approaches the null limit distribution under different parameter space conditions. We also provide the estimates of the probability density functions next to the empirical distributions. For every parameter space considered, the empirical rejection rates of the QLR test statistics are most accurate when $n = 2,000$. The empirical rejection rates are closer to the nominal levels when the parameter space is small. This result is significant when $n = 100$: the empirical rejection rates for $\Gamma = \Gamma_1$ are closer to the nominal ones than when $\Gamma = \Gamma_4$. Nonetheless, this difference becomes negligible as the sample size increases. The empirical rejection rates obtained using $n = 2,000$ are already satisfactorily close to the nominal levels, and this result is more or less similar to that from 5,000 observations. This suggests that the theory in Section 3 is effective for the ESTAR model. Considering even larger parameter spaces for γ yielded similar results, so they are not reported here.

A.1.2 Illustration Using the LSTAR Model

As another illustration, we consider testing against the first-order LSTAR model. We assume that the data-generating process is $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$ and $u_t = \ell_t$ with probability $1 - \pi_*^2$; and $u_t = 0$ with probability π_*^2, where $\{\ell_t\}_{t=1}^n$ follows the Laplace distribution with mean 0 and variance 2. Under this assumption, y_t follows the same distribution as ℓ_t that makes the algebra associated with the LSTAR model straightforward. For example, the covariance kernel of the Gaussian process associated with the null limit distribution of the QLR test statistic is analytically obtained thanks to this distributional assumption. This data-generating process is a variation of the exponential autoregressive model in Lawrence and Lewis (1980). Their exponential distribution is replaced by the Laplace distribution to allow y_t to obtain negative values.

Given this DGP, the first-order LSTAR model for $\mathbb{E}[y_t|y_{t-1}, y_{t-2}, \ldots]$ is defined as follows: $\mathcal{M}^0_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 + \exp(-\gamma y_{t-1})\}^{-1} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma := [0, \bar{\gamma}]\}$. The nonlinear logistic function $\{1+\exp(-\gamma y_{t-1})\}^{-1}$ contains an exponential function. It is therefore analytic, and this fact delivers a consistent power for the QLR test statistic. Note, however, that for $\gamma = 0$ the value of the logistic function equals 1/2. This difficulty is avoided by subtracting 1/2 from the logistic function when carrying out the test, viz., $\mathcal{M}_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{[1+\exp(-\gamma y_{t-1})]^{-1} - 1/2\} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma := [0, \bar{\gamma}]\}$. By the invariance principle, this shift does not affect the null limit distribution of the QLR test statistic. We here let $\gamma \geq 0$ so that transition function is bounded, which modifies the limit space of ς_n into $\mathbb{R}^+$. The null and the alternative hypotheses are identical to those in the ESTAR case.

Before proceeding, note that $\{1+\exp(-\gamma y_{t-1})\}^{-1} - \frac{1}{2} = \frac{1}{2}\tanh\left(\frac{\gamma y_{t-1}}{2}\right)$. Using the hyperbolic tangent function as in Bacon and Watts (1971) makes it easy to find a Gaussian process that is distributionally equivalent to the Gaussian process obtained under the null.

Using this fact, the limit distribution of QLR test statistic under $\mathcal{H}'_{01}$ is derived as in before. By Theorem 1, $QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}^2_{n,0}}(u'MF(\gamma)Z)^2/Z'F(\gamma)MF(\gamma)Z \Rightarrow \sup_{\gamma\in\Gamma} \ddot{\mathcal{G}}_1(\gamma)^2$, where $\ddot{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure $\ddot{\rho}_1(\gamma,\widetilde{\gamma}) := \ddot{c}_1(\gamma,\gamma)^{-1/2}\ddot{k}_1(\gamma,\widetilde{\gamma})\ddot{c}_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2}$. The function $\ddot{k}_1(\gamma,\tilde{\gamma})$ is equivalent to $\ddot{c}_1(\gamma,\tilde{\gamma})$ by the conditional homoskedasticity condition, and for each nonzero γ and $\widetilde{\gamma}$, we now obtain that $\ddot{k}_1(\gamma,\tilde{\gamma}) = \frac{1}{4}\mathbb{E}[y_t^2\tanh(\frac{\gamma y_t}{2})\tanh(\frac{\widetilde{\gamma} y_t}{2})] - \frac{1}{4}\mathbb{E}[y_t^2\tanh(\frac{\gamma y_t}{2})]\mathbb{E}[y_t^2\tanh(\frac{\widetilde{\gamma} y_t}{2})]/\,\mathbb{E}[y_t^2]$. In the proof of Lemma A.2 given below, we further show that $\ddot{k}_1(\gamma,\widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$, where $b_1(\gamma) := \frac{1}{\sqrt{2}}(1 - 2a(\gamma))$ with $a(\gamma) := \sum_{k=1}^{\infty}(-1)^{k-1}/(1+\gamma k)^3$ and for $n = 2, 3, \ldots$, $b_n(\gamma) := \sqrt{n(n+1)}\sum_{k=1}^{\infty}(-1)^{k-1}$ $(\gamma k)^{n-1}/(1+\gamma k)^{n+2}$.

Next, we derive the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$. Note that for $\gamma = 0$, $(\partial/\partial\gamma)f_t(\gamma)$ $= y_{t-1}\exp(\gamma y_{t-1})/[1+\exp(-\gamma y_{t-1})]^2 \neq 0$, implying that κ is unity as for the ESTAR case, so that we can apply a second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$: $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}^2_{n,0}}(\theta' G'_\kappa u)^2/(\theta' G'_\kappa G_\kappa \theta)$, where, similarly to the ESTAR case, $\theta' G'_\kappa u = \frac{\theta}{4}[\sum y_{t-1}^2 u_t - \sum y_{t-1}^3 \sum y_{t-1}u_t/\sum y_{t-1}^2]$ and $\theta' G'_\kappa G_\kappa\theta = \frac{\theta^2}{16}[\sum y_{t-1}^4 - (\sum y_{t-1}^3)^2/\sum y_{t-1}^2]$. From this, it follows that $QLR_n^{(2)} \Rightarrow \ddot{\mathcal{G}}_2^2$, where $\ddot{\mathcal{G}}_2 \sim N(0,1)$.

Therefore, we conclude that $QLR_n \Rightarrow \sup_\gamma \ddot{\mathcal{G}}(\gamma)^2$, where $\ddot{\mathcal{G}}(\gamma) := \ddot{\mathcal{G}}_1(\gamma)$, if $\gamma \neq 0$; and $\ddot{\mathcal{G}}(\gamma) := \mathcal{G}_2$, otherwise. The limit variance of $\ddot{\mathcal{G}}(\gamma)$ is given as $\ddot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})]$ such that $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \ddot{\rho}_1(\gamma,\widetilde{\gamma})$ if $\gamma \neq 0$ and $\widetilde{\gamma} \neq 0$; $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = 1$, if $\gamma = 0$ and $\widetilde{\gamma} = 0$; and $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \ddot{\rho}_3(\gamma)$, if $\gamma \neq 0$ and $\widetilde{\gamma} = 0$, where $\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}_2] = \ddot{k}_1(\gamma,\gamma)^{-1/2}\ddot{r}_1(\gamma)\ddot{q}^{-1/2}$ with $\ddot{r}_1(\gamma) := \frac{1}{2}\mathbb{E}\left[y_{t-1}^3\tanh\left(\frac{\gamma y_{t-1}}{2}\right)\right]$ and $\ddot{q} := \mathbb{E}[y_t^4] - \mathbb{E}[y_t^3]^2/\mathbb{E}[y_t^2]$. From this it follows that $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}(\gamma)^2$. Furthermore, $\mathbb{E}[y_t^3] = 0$ and

$\mathbb{E}[y_t^4] = 24$ given our DGP, so that

$$\ddot{\rho}_3(\gamma) = \frac{\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)]}{4\sqrt{6}\ddot{k}_1(\gamma,\gamma)^{1/2}}. \tag{A.2}$$

Here, we note that

$$\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)] = \frac{1}{8\gamma^4}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right] \tag{A.3}$$

by some tedious algebra assisted by Mathematica, where $P_G(n,x)$ is the polygamma function: $P_G(n,x) := d^{n+1}/d\,x^{n+1}\log(\Gamma(x))$. Inserting (A.3) into (A.2) yields

$$\ddot{\rho}_3(\gamma) = \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]. \tag{A.4}$$

In addition, we show in Lemma A.3 given below that applying L'Hôpital's rule iteratively yields that

$$\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 = \left[\frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]\right]^2. \tag{A.5}$$

This fact implies that $\mathrm{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1(\gamma)^2 = \ddot{\mathcal{G}}_2^2$. That is, the weak limit of the QLR test statistic under $\mathcal{H}_{02}'$ can be obtained from $\ddot{\mathcal{G}}_1(\cdot)^2$ by letting γ converging to zero, so that $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1(\gamma)^2$ under $\mathcal{H}_0'$.

Next, we derive another Gaussian process that is distributionally equivalent to $\ddot{\mathcal{G}}(\cdot)$ and conduct Monte Carlo simulations using it. The process is presented in the following lemma.

Lemma A.2. *If $\{z_k\}_{k=1}^{\infty}$ is an IID sequences of standard normal random variables, then for each γ and $\widetilde{\gamma} \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$, $\ddot{\mathcal{G}}(\cdot) \overset{\mathrm{d}}{=} \dot{\mathcal{G}}(\cdot)$, where $\ddot{\mathcal{Z}}_1(\gamma) := \sum_{n=1}^{\infty} b_n(\gamma)z_n$ and $\dot{\mathcal{G}}(\gamma) := \left(\sum_{n=1}^{\infty} b_n(\gamma)^2\right)^{-1/2}\ddot{\mathcal{Z}}_1(\gamma)$.* □

We prove Lemma A.2 by showing that the Gaussian process $\dot{\mathcal{G}}(\cdot)$ given in Lemma A.2 has the same covariance structure as $\ddot{\mathcal{G}}(\cdot)$, and for this purpose, we focus on proving that for all γ, $\widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$. If γ, $\widetilde{\gamma} > 0$, the desired equality trivially follows from the definition of $\dot{\mathcal{G}}(\cdot)$. On the other hand, applying L'Hôpital's rule iterative shows that $\mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2 \sim N(0,1)$, so that if we let $\dot{\mathcal{G}}_2 := \lim_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma)$, then for $\gamma \neq 0$, $\mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] = \left[\sqrt{3}b_1(\gamma) + b_2(\gamma)\right]/\{2\ddot{k}_1(\gamma,\gamma)^{1/2}\}$. We show in the proof of Lemma A.2 that the term on the right side is identical to $\ddot{\rho}_3(\gamma)$ in (A.4), so that the covariance kernel of $\dot{\mathcal{G}}(\cdot)$ is identical to $\ddot{\rho}(\cdot,\cdot)$. This fact implies that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$, and $\dot{\mathcal{G}}_2$ can be regarded as the weak limit obtained under $\mathcal{H}_{02}'$.

Lemma A.2 can be used to obtain the approximate null limit distribution of the QLR test statistic. We cannot generate $\dot{\mathcal{G}}(\cdot)$ using the infinite number of $b_n(\cdot)$, but we can simulate the following process to approximate the distribution of $\dot{\mathcal{G}}(\cdot)$: $\dot{\mathcal{G}}(\gamma; K) := (\sum_{n=1}^{K} b_{K,n}(\gamma)^2)^{-1/2} \sum_{n=1}^{K} b_{K,n}(\gamma) z_k$, where for $n = 2, 3, \ldots, b_{K,1}(\gamma) := (1-2a_K(\gamma))/\sqrt{2}$, $a_K(\gamma) := \sum_{k=1}^{K} (-1)^{k-1}/(1+\gamma k)^3$ and $b_{K,n}(\gamma) := \sqrt{n(n+1)} \sum_{k=1}^{K} (-1)^{k-1}(\gamma k)^{n-1}/ (1+\gamma k)^3$. If K is sufficiently large, the distribution of $\dot{\mathcal{G}}(\cdot; K)$ is close to that of $\dot{\mathcal{G}}(\cdot)$ as can be easily affirmed by simulations.

We conduct Monte Carlo Simulations for the LSTAR case as in the ESTAR case. The only aspect different from the ESTAR case is that the DGP is the one defined in the beginning of this section. Simulation results are summarised into Figure A.2. We use the same parameter spaces $\Gamma = \Gamma_i$, $i = 1, ..., 4$, as before, and we can see that the empirical distribution and PDF estimate of the QLR test approach the null limit distribution and its PDF that are obtained using $\dot{\mathcal{G}}(\cdot; K)$ with $K = 2,500$. This shows that the theory in Section 3 is also valid for the LSTAR model. When the parameter space Γ for γ becomes even larger, we obtain similar results. To save space, they are not reported.

A.1.3 Application of the Weighted Bootstrap

The standard approach to obtaining the null limit distribution of the QLR test is not applicable for empirical analysis because it requires knowledge of the error distribution. Without this information it is not possible in practice to obtain a distributionally equivalent Gaussian process. Hansen's (1996) weighted bootstrap is useful for this case. We apply it to our models as in Cho and White (2010), Cho, Ishida, and White (2011, 2014), and Cho, Cheong, and White (2011).

Although the relevant weighted bootstrap is available in Cho, Cheong, and White (2011), we provide here a version adapted to the structure of the STAR model. We consider the previously studied ESTAR and LSTAR models and proceed as follows. First, we compute the following score for each grid point of $\gamma \in \Gamma$: $\widetilde{W}_n(\gamma) := n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t^2(\gamma) z_t z_t' - n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' [n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t']^{-1} n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t'$ and $\widetilde{d}_{n,t}(\gamma) := z_t f_t(\gamma)\, \widetilde{u}_{n,t} - n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' [n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t']^{-1} z_t \widetilde{u}_{n,t}$, where $\widetilde{u}_{n,t} := y_t - y_{t-1}\widetilde{\theta}_n$ and $\widetilde{\theta}_n$ is the least squares estimator of θ from the null model. Here, $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ for ESTAR and $f_t(\gamma) = \{1 + \exp(\gamma y_{t-1})\}^{-1} - 1/2$ for the LSTAR model. Second, given these functions, we construct the following score function and pseudo-QLR test statistic: $\overline{QLR}_{b,n} := \sup_{\gamma \in \Gamma} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma)' \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma)$ and $\widetilde{s}_{n,t}(\gamma) := \{\widetilde{W}_n(\gamma)\}^{-1/2}\, \widetilde{d}_{n,t}(\gamma) z_{b,t}$, where $z_{b,t} \sim$ IID $(0, 1)$ with respect to b and t, $b = 1, 2, \ldots, B$, and B is the number of bootstrap replications. For example, we can resample $z_{b,t}$ from the standard normal distribution. For possible two-point distributions, see Davidson *et al.* (2007). Third, we estimate the empirical p-value by $\widehat{p}_n := B^{-1}\sum_{b=1}^{B}\mathbb{I}[QLR_n < \overline{QLR}_{b,n}]$, where $\mathbb{I}[\cdot]$ is the indicator function. We set $B = 300$ to obtain $\widehat{p}_n^{(i)}$ with $i = 1, 2, \ldots, 2,000$. Finally, for a specified nominal value of α, we compute $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$. When the null hypothesis holds, this proportion converges to α.

The intuition of the weighted bootstrap is straightforward. Note that if the null hypothesis is valid, the QLR test statistic is bounded in probability, and its null limit distribution can be revealed by the covariance structure of $\widetilde{s}_{n,t}(\cdot)$ asymptotically. That is, for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\widetilde{s}_{n,t}(\gamma)\widetilde{s}_{n,t}(\widetilde{\gamma})']$ converges to $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})']$ from the fact that $z_{b,t}$ is independent of $\widetilde{d}_{n,t}(\cdot)$ such that its population mean is zero and variance is unity. This means that $\mathbb{E}[\widetilde{s}_{n,t}(\gamma)\widetilde{s}_{n,t}(\widetilde{\gamma})']$ is asymptotically equivalent to $\widetilde{W}_n(\gamma)^{-1/2}\mathbb{E}[\widetilde{d}_{n,t}(\gamma)\widetilde{d}_{n,t}(\widetilde{\gamma})']\widetilde{W}_n(\widetilde{\gamma})^{-1/2}$, which is asymptotically equivalent to $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})']$ as given in Theorem 3. Therefore, the null limit distribution can be asymptotically revealed by the resampling distribution of $\overline{QLR}_{b,n}$. On the contrary, if the alternative hypothesis is valid, the QLR test statistic is not bounded in probability, but $\overline{QLR}_{b,n}$ is bounded in probability from the fact that $z_{b,t}$ is distributed around zero, so that the chance for the QLR test statistic to be bounded by the critical value obtained by the resampling distribution of $\overline{QLR}_{b,n}$ gets smaller, as n increases. This aspect implies that the weighted bootstrap is asymptotically consistent.

The simulation results are displayed in the percentile-percentile (PP) plots for the ESTAR and LSTAR models in Figures A.3 (ESTAR) and A.4 (LSTAR). The horizonal unit interval stands for α, and the vertical unit interval is the space of p-values. As a function of α, the aforementioned proportion should converge to the 45-degree line under the null hypothesis. As before, the four parameter spaces are considered: $\Gamma = \Gamma_i$, $i = 1, ..., 4$. The results are summarised as follows. First, as a function of α, the proportion $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$ does converge to the 45-degree line. Second, the empirical rejection rates estimated by the weighted bootstrap are closest to the nominal levels when the parameter space is small. Although the overall finite sample level distortions are smaller for the ESTAR model than the LSTAR model, the empirical rejection rate is close to the nominal significance level if α is close to zero. Finally, as the size of the parameter space increases, more observations are needed to better approximate the 45-degree line in the PP plots. We have conducted simulations using even larger parameter spaces and obtained similar results. We omit reporting them for brevity.

A.2 Comparison of the QLR and LM test statistics via the US Unemployment Rates

The main purpose of this section is to compare the results using the QLR test statistic with those from the LM test statistic. Through this comparison, we desire to demonstrate that the QLR and LM tests can complement each other when the p-values of the QLR and LM tests are computed by the weighted bootstrap and the methodology for the F-test statistic in Teräsvirta (1994), respectively.

Before examining another empirical example, a US unemployment series, we briefly review the model framework for the LM test statistics. The following auxiliary model is first estimated for the LM test statistics: $\mathcal{M}_{AUX} := \{h_{AUX}(\cdot, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) : (\alpha_0', \alpha_1', \alpha_1', \alpha_1', \alpha_1')' \in \Theta\}$, where $h_{AUX}(z_t, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $:= \alpha_0' z_t + \alpha_1'(\tilde{z}_t t_t) + \alpha_2'(\tilde{z}_t t_t^2) + \alpha_3'(\tilde{z}_t t_t^3) + \alpha_4'(\tilde{z}_t t_t^4)$, and t_t is the transition variable, viz., $\tilde{z}_t'\alpha$. This model is obtained by applying a fourth-order Taylor expansion to the analytic function as an intermediate step to compute the LM test statistics conveniently. Although it is different from the STAR model, testing the coefficients of nonlinear components by the LM test statistics turns out to be equivalent to computing the LM test statistics that test the STAR model assumption under $\mathcal{H}_{02}$. Luukkonen, Saikkonen, and Teräsvirta (1988) and Teräsvirta (1994) provide detailed rationales of this equivalence.

The following four sets of hypotheses are considered as common hypotheses of the empirical example: $\mathcal{H}_{0,1} : \alpha_{1*} = \alpha_{2*} = \alpha_{3*} = 0 | \alpha_{4*} = 0$; vs. $\mathcal{H}_{1,1} : \alpha_{1*} \neq 0, \alpha_{2*} \neq 0,$ or $\alpha_{3*} \neq 0 | \alpha_{4*} = 0$. $\mathcal{H}_{0,2} : \alpha_{1*} = \alpha_{2*} = \alpha_{3*} = \alpha_{4*} = 0$; vs. $\mathcal{H}_{1,2} : \alpha_{1*} \neq 0, \alpha_{2*} \neq 0, \alpha_{3*} \neq 0,$ or $\alpha_{4*} \neq 0$. $\mathcal{H}_{0,3} : \alpha_{1*} = \alpha_{3*} = 0$; vs. $\mathcal{H}_{1,3} : \alpha_{1*} \neq 0$ or $\alpha_{3*} \neq 0$. $\mathcal{H}_{0,4} : \alpha_{2*} = \alpha_{4*} = 0$; vs. $\mathcal{H}_{1,4} : \alpha_{2*} \neq 0$ or $\alpha_{4*} \neq 0$. These hypotheses are specified by following Teräsvirta (1994) and Escribano and Jordà (1999). We denote the LM test statistics testing $\mathcal{H}_{0,i}$ as $LM_{i,n}$, $i = 1, ..., 4$. $LM_{1,n}$ and $LM_{2,n}$ are general tests against STAR. On the other hand, $LM_{3,n}$ and $LM_{4,n}$ are tests against the LSTAR and ESTAR models, respectively. We also denote the QLR tests against ESTAR and LSTAR models as QLR_n^E and QLR_n^L, respectively.

We now examine the performance of the tests when applied to the monthly US unemployment rate. van Dijk, Teräsvirta, and Franses (2002) tested linearity of the series running from June 1968 to December 1999. We perform the tests both with their time series and the same series extended to August 2015.[1]

van Dijk, Teräsvirta, and Franses (2002) point out that the US unemployment rate is a persistent series with an asymmetric adjustment process and strong seasonality. They specify a STAR model with monthly dummy variables mainly because first differences of the seasonally unadjusted unemployment rate of males aged 20 and over is used for Δy_t. They test linearity against STAR assuming that the transition variable is a lagged twelve-month difference of the unemployment rate. The alternative (STAR) model has the following form (the lag length has been determined by AIC): $\Delta y_t = \pi_0 + \pi_1 y_{t-1} + \sum_{p=1}^{15} \pi_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \pi_{17+k} d_{t,k} + [\theta_0 +$

[1]The data set used by van Dijk, Teräsvirta, and Franses (2002) is available at `<http://swopec.hhs.se/hastef/abs/hastef0380.htm>` that was originally retrieved from the Bureau of Labor Statistics.

$\theta_1 y_{t-1} + \sum_{p=1}^{15} \theta_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \theta_{17+k} d_{t,k}] f(\Delta_{12} y_{t-d}, \gamma) + u_t$, where y_t is the monthly US unemployment rate in question, Δy_t is the first difference of y_t, $f(\cdot, \cdot)$ is a nonlinear transition function, $\Delta_{12} y_t$ is the twelve-month difference of y_t, $d_{t,k}$ is the dummy for month k, and $u_t \sim$ IID $(0, \sigma^2)$. The twelve-month difference $\Delta_{12} y_{t-d}$ is not included as an explanatory variable in the null (linear) model. The theory in Section 3 can nonetheless be used without modification as a null model including $\Delta_{12} y_{t-d}$ can be thought of having a zero coefficient for this variable. Following van Dijk, Teräsvirta, and Franses (2002), we test linearity by letting $\Delta_{12} y_{t-d}$, $d = 1, 2, \ldots, 6$, be the transition variable.

Our test results using the same series as van Dijk, Teräsvirta, and Franses (2002) are reported in the top panel of Table A.1. Both the LM tests and QLR_n^L reject linearity when $d = 2$, and, besides, $LM_{3,n}$ that has power against LSTAR yields $p = 0.037$ for $d = 2$. The p-values of QLR_n^L, however, lie at or below 0.05 for all six lags, suggesting that at least in this particular case this QLR test is more powerful than the LM tests. The smallest p-value is even here attained for $d = 2$. The results from QLR_n^E are quite different in that they reject the null only for $d = 1, 2$, but not for other lags. This makes sense as this statistic is designed for ESTAR , and asymmetry in the unemployment rate is best described by an LSTAR model.

The bottom panel of Table A.1 contains the results from the series extended to August 2015.[2] Now there seems to be plenty of evidence of asymmetry: all p-values of $LM_{1,n}$ are rather small. $LM_{3,n}$ also has small values for the first three lags, as has $LM_{2,n}$. The p-values from QLR_n^L are smallest of all, which is in line with the results in the top panel. Even QLR_n^E rejects the null of linearity at the 5% level for $d = 1, 2, 3, 4, 5$. This outcome may be expected as the QLR statistics are omnibus tests and as such respond to any deviation from the null hypothesis as the sample size increases. Note, however, that even $LM_{4,n}$ now yields two p-values ($d = 2, 3$) that lie below 0.05, although the test does not have the omnibus property. The behaviour of the unemployment rate during and after the financial crisis (a quick upswing and slow decrease) has probably contributed to these results.

A.3 Proofs

Proof of Lemma 1. (*i*) Given Assumptions 1, 2, 3, and 5, it is trivial to show that $\widehat{\sigma}_{n,0}^2 \stackrel{\text{a.s.}}{\to} \sigma_*^2$ by the ergodic theorem.

(*ii*) The null limit distribution of $QLR_n^{(1)}$ is determined by the two terms in $QLR_n^{(1)}$: $Z'F(\cdot)Mu$ and $Z'F(\cdot)M$

[2] The recent observations of the monthly US unemployment rate are available at <`http://beta.bls.gov/dataViewer/view/timeseries/LNU04000025`>.

$F(\cdot)Z$. We examine their null limit behaviour one by one and combine the limit results using the converging-together lemma in Billingsley (1999, p. 39).

(a) We show the weak convergence part of $n^{-1/2}Z'F(\cdot)Mu$. Using the definition of $M := I - Z(Z'Z)^{-1}Z'$ we have $Z'F(\gamma)Mu = Z'F(\gamma)u - Z'F(\gamma)Z(Z'Z)^{-1}Z'u$, and we now examine the components on the right-hand side of this equation separately. For each $\gamma \in \Gamma$, we define $\widehat{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - (\sum_{t=1}^n f_t(\gamma)z_t z_t')(\sum_{t=1}^n z_t z_t')^{-1}\sum_{t=1}^n z_t u_t$, $\widetilde{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - \mathbb{E}[f_t(\gamma)z_t z_t']\mathbb{E}[z_t z_t']^{-1}\sum_{t=1}^n z_t u_t$ and show that

$$\sup_{\gamma\in\Gamma(\epsilon)} \left\| n^{-1/2}\sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right]\right\|_\infty = o_{\mathbb{P}(1)}, \tag{A.6}$$

where $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$ and $\|\cdot\|_\infty$ is the uniform matrix norm. We have

$$\begin{aligned}
&\sup_{\gamma\in\Gamma(\epsilon)} \left\| \frac{1}{\sqrt{n}}\sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right]\right\|_\infty \\
&\quad\leq \sup_{\gamma\in\Gamma(\epsilon)} \left\| \left(\frac{1}{n}\sum_{t=1}^n f_t(\gamma)z_t z_t'\right)\left\{\left(\frac{1}{n}\sum_{t=1}^n z_t z_t'\right)^{-1} - \mathbb{E}[z_t z_t']^{-1}\right\} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty \\
&\quad + \sup_{\gamma\in\Gamma(\epsilon)} \left\| \left\{\left(n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t'\right) - \mathbb{E}[f_t(\gamma)z_t z_t']\right\}\mathbb{E}[z_t z_t']^{-1} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty .
\end{aligned} \tag{A.7}$$

We show that each term on the right-hand side of (A.7) is $o_{\mathbb{P}}(1)$. Now, $\{z_t u_t, \mathcal{F}_t\}$ is a martingale difference sequence, where $\mathcal{F}_t$ is the smallest sigma-field generated by $\{z_t u_t, z_{t-1}u_{t-1}, \ldots\}$. Therefore, $\mathbb{E}[z_t u_t|\mathcal{F}_{t-1}] = 0$, $\mathbb{E}[|Z_{t,j}u_t|^2] = \mathbb{E}[u_t^4]^{1/2}\mathbb{E}[|Z_{t,j}|^4]^{1/2} \leq \mathbb{E}[m_t^4]^{1/2}\mathbb{E}[Z_{t,j}^4]^{1/2} < \infty$, and $\mathbb{E}[u_t^2 z_t z_t']$ is positive definite. Thus, $n^{-1/2}\sum_{t=1}^n z_t u_t$ is asymptotically normal. Next, we note that $n^{-1/2}\sum_{t=1}^n f_t(\gamma)u_t z_t$ is also asymptotically normal. This follows from the fact that $\{f_t(\gamma)u_t z_t, \mathcal{F}_t\}$ is a martingale difference sequence, and for each j, $|f_t(\gamma)u_t z_{t,j}|^2 \leq m_t^6$, and $\mathbb{E}[m_t^6] < \infty$ by Assumptions 4 and 5. Furthermore, $\sup_{\gamma\in\Gamma}\|n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t' - \mathbb{E}[f_t(\gamma)z_t z_t']\|_\infty = o_{\mathbb{P}}(1)$ by Ranga Rao's (1962) uniform law of large numbers. Thus,

$$\sup_{\gamma\in\Gamma(\epsilon)} \left\| \left\{n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t' - \mathbb{E}[f_t(\gamma)z_t z_t']\right\}\mathbb{E}[z_t z_t']^{-1} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty = o_{\mathbb{P}}(1). \tag{A.8}$$

This shows that the second term of (A.7) is $o_{\mathbb{P}}(1)$. We now demonstrate that the first term of (A.7) is also $o_{\mathbb{P}}(1)$. By Assumption 4 and the ergodic theorem, we note that $\|n^{-1}\sum_{t=1}^n z_t z_t' - \mathbb{E}[z_t z_t']\|_\infty = o_{\mathbb{P}}(1)$, and $|\sum_{t=1}^n f_t(\gamma)z_{t,j}z_{t,i}| \leq \sum_{t=1}^n m_t^3 = O_{\mathbb{P}}(n)$, so that (A.8) follows, leading to (A.6). Therefore, $n^{-1/2}Z'F(\gamma)Mu \overset{\text{A}}{\sim} N[0, B_1(\gamma,\gamma)]$ by noting that $\mathbb{E}[\widetilde{f}_{n,t}(\gamma)\widetilde{f}_{n,t}(\gamma)'] = B_1(\gamma,\gamma)$. Using the same methodology, we can show that for each $\gamma, \widetilde{\gamma} \in \Gamma(\epsilon)$,

$$\frac{1}{\sqrt{n}}\left[\begin{array}{c} Z'F(\gamma)Mu \\ Z'F(\widetilde{\gamma})Mu \end{array}\right] \overset{\text{A}}{\sim} N\left[\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left[\begin{array}{cc} B_1(\gamma,\gamma) & B_1(\gamma,\widetilde{\gamma}) \\ B_1(\widetilde{\gamma},\gamma) & B_1(\widetilde{\gamma},\widetilde{\gamma}) \end{array}\right]\right].$$

Finally, we have to show that $\{\widetilde{f}_{n,t}(\cdot)\}$ is tight. First note that by Assumptions 1, 2, and 4, it follows that $|f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t| \leq m_t\,|z_{t,j}u_t|\;\;|\gamma-\widetilde{\gamma}|$ for each j. From this we obtain that $\sup_{|\gamma-\widetilde{\gamma}|<\eta}|f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t|^{2+\omega} \leq m_t^{2+\omega}|z_{t,j}u_t|^{2+\omega}\eta^{2+\omega} \leq m_t^{6+3\omega}\eta^{2+\omega}$, so that it follows that $\mathbb{E}[\sup_{|\gamma-\widetilde{\gamma}|<\eta}|f_t(\gamma)z_{t,j}u_t - f_t(\widetilde{\gamma})z_{t,j}u_t|^{2+\omega}]^{\frac{1}{2+\omega}} \leq \mathbb{E}[m_t^{6+3\omega}]^{\frac{1}{2+\omega}}\eta$ for each j. This implies that $\{n^{-1/2}f_t(\cdot)z_{t,j}u_t\}$ is tight because Ossiander's $L^{2+\omega}$ entropy is finite.

Next, for some $c>0$, it holds that $\|\mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t - \mathbb{E}[f_t(\widetilde{\gamma})z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\|_\infty = \|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\}z_tz_t']\mathbb{E}[z_t\;z_t']^{-1}z_tu_t\|_\infty \leq cm_t^2\left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty\|\mathbb{E}[\{f_t(\gamma)-f_t(\widetilde{\gamma})\}z_tz_t']\|_\infty$ by the property of the uniform norm and Assumption 5. Also note that $\|\mathbb{E}[f_t(\gamma)z_tz_t' - f_t(\widetilde{\gamma})z_tz_t']\|_\infty \leq \|\mathbb{E}[\{f_t(\gamma)-f_t(\widetilde{\gamma})\}z_tz_t']\|_1$ and by Assumption 4, for each $i,j = 1,2,\ldots,m+1$, $|z_{t,j}z_{t,i}[f_t(\gamma)-f_t(\widetilde{\gamma})]| \leq m_t^3|\gamma-\widetilde{\gamma}|$, where $\|[g_{i,j}]\|_1 := \sum_i\sum_j|g_{i,j}|$. Therefore,

$$\begin{aligned}&\left\|\mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t - \mathbb{E}[f_t(\widetilde{\gamma})z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\right\|_\infty \\ &\quad\leq cm_t^2\left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty\left\|\mathbb{E}[\{f_t(\gamma)-f_t(\widetilde{\gamma})\}z_tz_t']\right\|_\infty \leq c(m+1)^2m_t^2\left\|\mathbb{E}[z_tz_t']^{-1}\right\|_\infty\mathbb{E}[m_t^3]|\gamma-\widetilde{\gamma}|. \quad \text{(A.9)}\end{aligned}$$

This inequality (A.9) implies that $\{n^{-1/2}\mathbb{E}[f_t(\cdot)z_tz_t']\mathbb{E}[z_tz_t']^{-1}z_tu_t\}$ is also tight. Hence, it follows that for some $b<\infty$, $\mathbb{E}[\sup_{|\gamma-\widetilde{\gamma}|<\eta}|\widetilde{f}_t(\gamma) - \widetilde{f}_t(\widetilde{\gamma})|^{2+\omega}] \leq b\cdot\eta$. That is, $\{n^{-1/2}\sum_{t=1}^n\widetilde{f}_{n,t}(\cdot)\}$ is tight. From this and the fact that the finite-dimensional multivariate CLT holds, the weak convergence of $\{n^{-1/2}\sum_{t=1}^n\widetilde{f}_{n,t}(\cdot)\}$ is established.

(*b*) Next, we examine the limit behaviour of $n^{-1}Z'F(\cdot)F(\cdot)Z$. Note that $n^{-1}Z'F(\gamma)F(\gamma)Z = n^{-1}\sum_{t=1}^n f_t(\gamma)^2z_tz_t' - \left\{n^{-1}\sum_{t=1}^n f_t(\gamma)z_tz_t'\right\}\left\{n^{-1}\sum_{t=1}^n z_tz_t'\right\}^{-1}\left\{n^{-1}\sum_{t=1}^n f_t(\gamma)z_tz_t'\right\}$ and, given Assumptions 1, 2, 3, 4, and 6, $\sup_{\gamma\in\Gamma(\epsilon)}\|n^{-1}\sum_{t=1}^n f_t(\gamma)^2\;z_tz_t' - \mathbb{E}[f_t(\gamma)^2z_tz_t']\| \overset{\text{a.s.}}{\to} 0$ and $\sup_{\gamma\in\Gamma(\epsilon)}\|n^{-1}\sum_{t=1}^n f_t(\gamma)z_tz_t' - \mathbb{E}[f_t(\gamma)z_t\;z_t']\| \overset{\text{a.s.}}{\to} 0$ by Ranga Rao's (1962) uniform law of large numbers. Therefore, from the fact that $\|n^{-1}\sum_{t=1}^n z_tz_t' - \mathbb{E}[z_tz_t']\|_\infty = o_{\mathbb{P}}(1)$, it follows that $\sup_{\gamma\in\Gamma(\epsilon)}|n^{-1}Z'F(\gamma)MF(\gamma)Z - \{\mathbb{E}[f_t(\gamma)^2z_tz_t'] - \mathbb{E}[f_t(\gamma)z_tz_t']\mathbb{E}[z_tz_t']^{-1}\,\mathbb{E}[f_t(\gamma)z_tz_t']\}| = o_{\mathbb{P}}(1)$. Applying the converging-together lemma yields the desired result.

(*iii*) This result trivially follows from the fact that $\mathbb{E}[u_t^2|z_t] = \sigma_*^2$. ■

Proof of Lemma 2. Given Assumption 2, $\mathcal{H}_{02}$, and the definition of $H_j(\gamma)$, the j-th order derivative of $\mathcal{L}_n^{(2)}(\,\cdot\,,\theta)$ is obtained as

$$\begin{aligned}\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(\gamma,\theta) &= -\sum_{k=0}^{j}\binom{j}{k}\left\{\frac{\partial^k}{\partial\gamma^k}(y-F(\gamma)Z\theta)'\right\}M\left\{\frac{\partial^{j-k}}{\partial\gamma^{j-k}}(y-F(\gamma)Z\theta)\right\}\\ &= 2\theta'Z'H_j(\gamma)Mu-\sum_{k=1}^{j-1}\binom{j}{k}\theta'Z'H_j(\gamma)MH_{j-k}(\gamma)Z\theta \qquad \text{(A.10)}\end{aligned}$$

by iteratively applying the general Leibniz rule. We now evaluate this derivative at $\gamma=0$. Note that $H_j(0)=0$ if $j<\kappa$ by the definition of κ. This implies that $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\theta)=0$ for $j=1,2,\ldots,\kappa-1$. This also implies that $\binom{j}{k}\theta'Z'H_j(0)MH_{j-k}(0)Z\theta=0$ for $j=\kappa,\,\kappa+1,\,\ldots,\,2\kappa-1$. Therefore, $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0,\theta)=2\theta'Z'H_j(0)Mu$. Finally, we examine the case in which $j=2\kappa$. For each $j<2\kappa$, $H_j(0)=0$ and $H_\kappa(0)\neq 0$, so that the summand of the second term in the right side of (A.10) is different from zero only when $j=2\kappa$ and $k=\kappa$: $\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0,\theta)=2\theta'Z'H_{2\kappa}(\gamma)Mu-\binom{2\kappa}{\kappa}\theta'Z'H_\kappa(\gamma)MH_\kappa(\gamma)Z\theta$. This completes the proof. ■

Proof of Lemma 3. Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, we note that

$$\begin{aligned}QLR_n^{(2)} &:= \sup_\theta \overline{QLR}_n^{(2)}(\theta)\\ &= \sup_\theta\sup_\varsigma \frac{1}{\widehat{\sigma}_{n,0}^2}\left[\frac{2\{\theta'G_\kappa'u\}\varsigma^\kappa}{\kappa!\sqrt{n}}-\frac{1}{(2\kappa)!n}\left\{\binom{2\kappa}{\kappa}\theta'G_\kappa'G_\kappa\theta\right\}\varsigma^{2\kappa}\right]+o_{\mathbb{P}}(n). \qquad \text{(A.11)}\end{aligned}$$

Then, the FOC with respect to ς implies that $\widehat{\varsigma}_n^\kappa(\theta)=W_n(\theta)$, κ is odd; and $\widehat{\varsigma}_n^\kappa(\theta)=\max[0,W_n(\theta)]$, if κ is even by noting that $\widehat{\varsigma}_n^\kappa(\theta)$ cannot be negative. If we plug $\widehat{\varsigma}_n^\kappa(\theta)$ back into the right side of (A.11), the desired result follows. ■

Proof of Lemma 4. Before proving Lemma 4, we first show that for each j, $Z'H_j(0)Mu=O_{\mathbb{P}}(n^{1/2})$, so that $j=\kappa+1,\ldots,2\kappa-1$, $Z'H_j(0)Mu=o_{\mathbb{P}}(n^{j/2\kappa})$. Note that for $j=\kappa+1,\ldots,2\kappa$, $Z'H_jMu=\sum_{t=1}^n z_th_{t,j}(0)u_t-\sum_{t=1}^n z_th_{t,j}(0)z_t'\left(\sum_{t=1}^n z_tz_t'\right)^{-1}\sum_{t=1}^n z_tu_t$. First, we apply the ergodic theorem to $n^{-1}\sum_t z_th_{t,j}(0)z_t'$ and $n^{-1}\sum_t z_tz_t'$, respectively. Second, given Assumptions 1, 2, 3, 7, and 8, following the proof of Lemma 1, we have that $n^{-1/2}\sum_t z_tu_t$ is asymptotically normal. Furthermore, for all $j=\kappa+1,\ldots,2\kappa$, $n^{-1/2}\sum_t z_th_{t,j}(0)u_t$ is asymptotically normal. For this verification, note that $\{z_th_{t,j}(0)u_t,\mathcal{F}_t\}$ is a martingale difference sequence, so that for each j, $\mathbb{E}[z_th_{t,j}(0)u_t|\mathcal{F}_{t-1}]=0$. Next, we prove that for each j, $\mathbb{E}[z_{t,i}^2h_{t,j}^2(0)u_t^2]<\infty$. First note that using Assumption 7, $\mathbb{E}[|z_{t,i}^2h_{t,j}^2(0)u_t^2|]\leq\mathbb{E}[|u_t|^4]^{1/2}\mathbb{E}[|h_{t,j}^2(0)z_{t,i}^2|^2]^{1/2}\leq$

$\mathbb{E}[|u_t|^4]^{1/2}\mathbb{E}[|h_{t,j}(0)|^8]^{1/4}\, \mathbb{E}[|z_{t,i}|^8]^{1/4} < \infty$ by the Cauchy-Schwarz's inequality. For the same reason, $\mathbb{E}[|z_{t,i}^2 h_{t,j}^2(0)u_t^2|] \leq \mathbb{E}[|u_t\, h_{t,j}(0)|^4]^{1/2}\mathbb{E}[|z_{t,i}|^4]^{1/2} \leq \mathbb{E}[|u_t|^8]^{1/2}\mathbb{E}[|h_{t,j}(0)|^8]^{1/2}\mathbb{E}[|z_{t,i}|^4]^{1/2} < \infty$. By Assumption 8, $\mathbb{E}[u_t^2 z_t h_{t,j}(0)^2 z_t']$ is positive definite. It then follows by Theorem 5.25 of White (2001) that $n^{-1/2}\sum_t z_t h_{t,j}(0) u_t$ is asymptotically normal. Thus, $Z'H_j(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

We now consider the statements (*i*)–(*iii*).

(*i*) First, we show that $\theta' Z' H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$. By the definition of M,

$$Z'H_\kappa(0)Mu = \sum_{t=1}^{n} z_t h_{t,\kappa}(0)u_t - \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t' \left[\sum_{t=1}^{n} z_t z_t'\right]^{-1} \sum_{t=1}^{n} z_t u_t. \tag{A.12}$$

We examine all sums on the right-hand side of (A.12). First, $h_{t,\kappa}(0)$ is a function of z_t, which implies that, given the moment condition in Assumption 7, $n^{-1}\sum z_t h_{t,\kappa}(0)z_t'$ obeys the ergodic theorem. Second, similarly under Assumptions 1, 2, 3, 7, 8, and $\mathcal{H}_{02}$, $n^{-1}\sum z_t z_t'$ also obeys the ergodic theorem. Third, given the assumptions and the proof of Lemma 1, we have already proved that $n^{-1/2}\sum z_t u_t$ is asymptotically normally distributed. Finally, $n^{-1/2}\sum z_t h_{t,\kappa}(0) u_t$ is asymptotically normal, and the proof is similar to that of the asymptotic normality of $n^{-1/2}\sum_t z_t h_{t,j}(0)u_t$ ($j = \kappa+1, \ldots, 2\kappa$). All these facts imply that $Z'H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

(*ii*) $n^{-1}G_\kappa' G_\kappa \stackrel{\text{a.s.}}{\to} A_2$ by the ergodic theorem.

(*iii*) Note that

$$Z'H_\kappa(0)MH_\kappa(0)Z = \sum_{t=1}^{n} z_t h_{t,\kappa}(0)^2 z_t' - \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t' \left[\sum_{t=1}^{n} z_t z_t'\right]^{-1} \sum_{t=1}^{n} z_t h_{t,\kappa}(0) z_t'. \tag{A.13}$$

The limit of (A.13) is revealed by applying the ergodic theorem to each term on the right-hand side of this expression. Consequently, $n^{-1}Z'H_\kappa(0)MH_\kappa(0)Z \stackrel{\text{a.s.}}{\to} \mathbb{E}[g_{t,\kappa}g_{t,\kappa}']$, where $\mathbb{E}[g_{t,\kappa}g_{t,\kappa}'] := \mathbb{E}[z_t H_{2\kappa}(0)^2 z_t'] - \mathbb{E}[z_t H_{2\kappa}(0) z_t']\mathbb{E}[z_t z_t']^{-1}\, \mathbb{E}[z_t H_{2\kappa}(0)\, z_t']$. This completes the proof. ■

Proof of Lemma A.2. The distributional equivalence between $\dot{\mathcal{G}}(\cdot)$ and $\ddot{\mathcal{G}}(\cdot)$ can be established by showing that for all $\gamma, \widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$. We will proceed in three steps. First, we derive the functional form of $\ddot{\rho}(\gamma, \widetilde{\gamma})$. We show that if $\gamma, \widetilde{\gamma} > 0$, then $\ddot{k}_1(\gamma, \widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma) b_n(\widetilde{\gamma})$. This in turn implies that for γ, $\widetilde{\gamma} > 0$, $\ddot{\rho}(\gamma, \widetilde{\gamma}) = \ddot{k}_1(\gamma, \gamma)^{-1/2}\sum_{n=1}^{\infty} b_n(\gamma) b_n(\widetilde{\gamma}) \ddot{k}_1(\widetilde{\gamma}, \widetilde{\gamma})^{-1/2}$. It follows that the specific functional form of $\ddot{\rho}(\gamma, \widetilde{\gamma})$ can be obtained from this result and (A.4).

Second, similarly for all $\gamma, \widetilde{\gamma} \geq 0$, we derive the functional form of $\dot{\rho}(\gamma, \widetilde{\gamma})$ and compare it to $\ddot{\rho}(\gamma, \widetilde{\gamma})$. To do all this, we first note that for all $\gamma, \widetilde{\gamma} > 0$,

$$\ddot{k}_1(\gamma,\tilde{\gamma}) =\frac{1}{4}\mathbb{E}\left[y_t^2\tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right] - \frac{1}{4}\mathbb{E}\left[y_t^2\tanh\left(\frac{\gamma y_t}{2}\right)\right]\mathbb{E}\left[y_t^2\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right]\Big/\mathbb{E}[y_t^2]$$
$$=\frac{1}{4}\mathbb{E}\left[y_t^2\tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right]. \tag{A.14}$$

This follows from that fact that for any $x \in \mathbb{R}$, $\tanh(x) = -\tanh(-x)$ and that y_t follows the Laplace distribution with mean zero and variance 2, so that $\mathbb{E}\left[y_t^2\tanh\left(\gamma y_t/2\right)\right] = 0$. Given this, we can apply the Dirichlet series to $\tanh(\cdot)$ to obtain the functional form of $\ddot{k}_1(\cdot,\cdot)$. Thus, for any $x \in \mathbb{R}$, $\tanh(x) = \mathrm{sgn}(x)(1-2\sum_{k=0}^{\infty}(-1)^k\exp(-2|x|(k+1)))$ and, furthermore, that $\mathbb{E}\left[s_t^2\exp(-s_t\gamma k)\right] = 2/(1+\gamma k)^3$ and $\mathbb{E}[s_t^2] = 2$, where $s_t := |y_t|$ follows the exponential distribution with mean 1 and variance 2. Applying these to (A.14) yields

$$\begin{aligned}
\ddot{k}_1(\gamma,\widetilde{\gamma}) &= \mathbb{E}\left[\frac{y_t^2}{4}\tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\tilde{\gamma} y_t}{2}\right)\right]\\
&= \mathbb{E}\left[\frac{s_t^2}{4}\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\gamma k)\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\widetilde{\gamma} k)\right]\\
&\quad + \mathbb{E}\left[s_t^2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+j-2}\exp(-s_t(\gamma k+\widetilde{\gamma} j))\right]\\
&= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}\frac{2}{(1+\gamma k+\widetilde{\gamma} j)^3}.
\end{aligned}$$

Next, for $|x| < 1$ we have $(1-x)^{-3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}$, so that $(1+\gamma k+\widetilde{\gamma} j)^{-3} = (1+\gamma k)^{-3}(1+\widetilde{\gamma} j)^{-3}\left(1-\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^{-3}$, where we note that $(1-\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j})^{-3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left(\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^{n-1}$. Therefore, it follows that

$$\ddot{k}_1(\gamma,\widetilde{\gamma}) = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}\frac{n(n+1)(\gamma k)^{n-1}(\widetilde{\gamma} j)^{n-1}}{(1+\gamma k)^{n+2}(1+\widetilde{\gamma} j)^{n+2}}.$$

Furthermore,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}n(n+1)\frac{(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\frac{(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}}\\
&=2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(1+\widetilde{\gamma} j)^3} + \sum_{n=2}^{\infty}n(n+1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}},
\end{aligned}$$

which is equal to $2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$, where for $n = 2, 3, \ldots$, $a(\gamma) := \sum_{k=1}^{\infty} (-1)^{k-1}/(1+\gamma k)^3$ and $b_n(\gamma) := \sqrt{n(n+1)} \sum_{k=1}^{\infty} (-1)^{k-1}(\gamma k)^{n-1}/(1+\gamma k)^{n+2}$. In particular, $b_1(\gamma) := 2^{-1/2}(1 - 2a(\gamma))$, so that $\ddot{k}_1(\gamma, \widetilde{\gamma}) = \frac{1}{2} - a(\gamma) - a(\widetilde{\gamma}) + 2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma}) = \frac{1}{2}(1-2a(\gamma))(1-2a(\widetilde{\gamma})) + \sum_{n=2}^{\infty} b_n(\gamma)$ $b_n(\widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$. Then, for each γ and $\widetilde{\gamma} > 0$, $\ddot{\rho}_1(\gamma, \widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_1(\widetilde{\gamma})] = \ddot{k}_1(\gamma, \gamma)^{-1/2}$ $\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})\ddot{k}_1\ (\widetilde{\gamma}, \widetilde{\gamma})^{-1/2}$. In addition, for $\gamma > 0$, we examine $\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_2]$. Note that from (A.4), $\ddot{\rho}_3(\gamma) = \{\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)]\}/\{4\sqrt{6}\ddot{k}_1(\gamma,\gamma)^{1/2}\} = [48\gamma^4 + P_G(3, 1 + \frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]/$ $\{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}\}$ as affirmed by Mathematica. It follows that the specific functional form of $\ddot{\rho}(\gamma, \widetilde{\gamma})$ is given as

$$\ddot{\rho}(\gamma, \widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} > 0; \\ 1, & \text{if } \gamma = 0 \text{ and } \widetilde{\gamma} = 0; \\ \frac{48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} = 0, \end{cases} \cdot \tag{A.15}$$

Third, we examine the covariance kernel of $\dot{\mathcal{G}}(\cdot)$, viz., $\dot{\rho}(\cdot, \cdot)$. If we let $\gamma, \widetilde{\gamma} > 0$, $\dot{\rho}(\gamma, \widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma) \cdot \mathcal{G}(\widetilde{\gamma})] =$ $\ddot{k}_1(\gamma,\gamma)^{-1/2} \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2} = \ddot{k}_1(\gamma,\gamma)^{-1/2}\ddot{k}_1(\ \gamma,\widetilde{\gamma})\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{-1/2} = \ddot{\rho}_1(\gamma,\widetilde{\gamma})$. Furthermore, by some tedious algebra, $\mathrm{plim}_{\gamma\downarrow 0}\ddot{\mathcal{Z}}_1^2(\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{\mathcal{Z}}_1^2(\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{\mathcal{Z}}_1^2(\gamma) = \frac{1}{8}\{3\sqrt{2}Z_1 + \sqrt{6}Z_2\}^2$, $\mathrm{plim}_{\gamma\downarrow 0}\ddot{k}_1(\gamma,\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{k}_1(\gamma,\gamma) = 0$, and $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{k}_1(\gamma,\gamma) = 3$, so that $\mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma)^2 = (\frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2)^2$, which implies $\dot{\mathcal{G}}_2 := \mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2 \sim N(0,1)$. Consequently, if $\gamma > 0$,

$$\begin{aligned} \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] &= \ddot{k}_1(\gamma,\gamma)^{-1/2}\mathbb{E}\left[\ddot{\mathcal{Z}}_1(\gamma)\left(\frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2\right)\right] = \ddot{k}_1(\gamma,\gamma)^{-1/2}\left[\frac{\sqrt{3}}{2}b_1(\gamma) + \frac{1}{2}b_2(\gamma)\right] \\ &= \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}\left[48\gamma^4 + P_G\left(3, 1 + \frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]. \end{aligned} \tag{A.16}$$

The last equality follows from the fact that $b_1(\gamma) = \frac{1}{8\sqrt{2}\gamma^3}[8\gamma^3 - P_G(2, 1 + \frac{1}{2\gamma}) + P_G(2, \frac{1+\gamma}{2\gamma})]$, $b_2(\gamma) = \frac{1}{16\sqrt{6}\gamma^4}[6\gamma P_G(2, \frac{1}{2\gamma}) - 6\gamma P_G(2, \frac{1+\gamma}{2\gamma}) + P_G(3, \frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]$, $P_G(2, \frac{1}{2\gamma}) - P_G(2, 1 + \frac{1}{2\gamma}) = -16\gamma^3$, and $P_G(3, \frac{1}{2\gamma}) - P_G(3, 1 + \frac{1}{2\gamma}) = 96\gamma^4$, as obtained by Mathematica. Equation (A.16) then leads to the following functional form for $\dot{\rho}(\gamma, \widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$:

$$\dot{\rho}(\gamma, \widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1(\gamma,\gamma)^{1/2}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} > 0; \\ 1, & \text{if } \gamma = 0 \text{ and } \widetilde{\gamma} = 0; \\ \frac{48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} = 0, \end{cases}$$

which is identical to the functional form of $\ddot{\rho}(\cdot, \cdot)$ in (A.15). This allows the conclusion that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$. ■

In the following, we provide additional supplementary claim in (A.5) that is given in the following lemma:

Lemma A.3. *Given the DGP and Model conditions in Section A.1.2,* $\lim_{\widetilde{\gamma}\downarrow 0} \ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 = [48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]^2/\{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}\}^2$. □

Lemma A.3 implies that $\text{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1(\gamma)^2 = \ddot{\mathcal{G}}_2^2$, so that $\sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1(\gamma)^2 \geq \ddot{\mathcal{G}}_2^2$ and $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1(\gamma)^2$.

Proof of Lemma A.3. From the definition of $\ddot{\rho}_1(\gamma,\widetilde{\gamma})$, note that $\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 := \ddot{k}_1(\gamma,\gamma)^{-1}\ddot{k}_1(\gamma,\widetilde{\gamma})^2\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma})^{-1}$. Furthermore, we have $\text{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = 0$, $\text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial}{\partial\widetilde{\gamma}}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = 0$, $\text{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 0$, $\text{plim}_{\widetilde{\gamma}\downarrow 0}(\partial/\partial\widetilde{\gamma})\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 0$, $\text{plim}_{\widetilde{\gamma}\downarrow 0}(\partial^2/\partial\widetilde{\gamma}^2)\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 3$, and $\text{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial^2}{\partial\widetilde{\gamma}^2}\ddot{k}_1(\gamma,\widetilde{\gamma})^2 = (\{48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})\}/\{32\sqrt{2}\gamma^4\})^2$ by some algebra using Mathematica. This property implies that $\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1(\gamma,\widetilde{\gamma})^2 = (\{48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})\}/\{32\sqrt{2}\gamma^4\})^2/\{3\ddot{k}_1(\gamma,\gamma)\} = [48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]^2/\{32\sqrt{6}\gamma^4\ddot{k}_1(\gamma,\gamma)^{1/2}\}^2$. This completes the proof. ■

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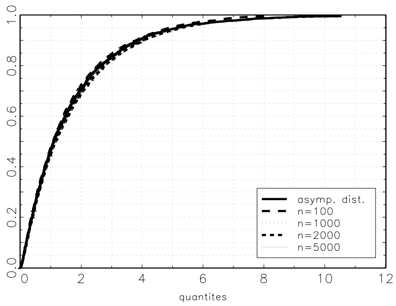

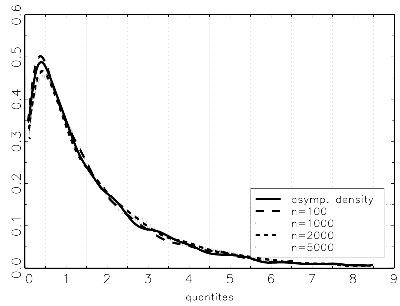

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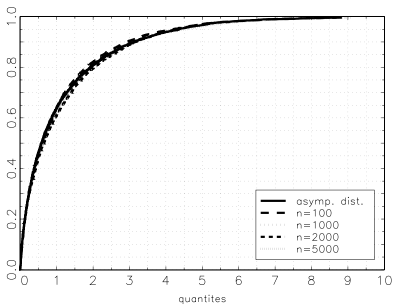

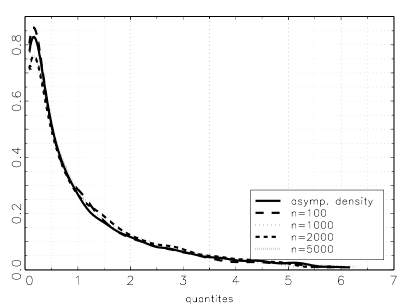

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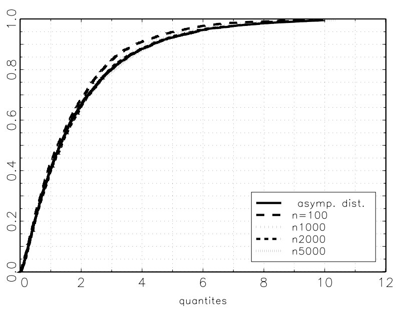

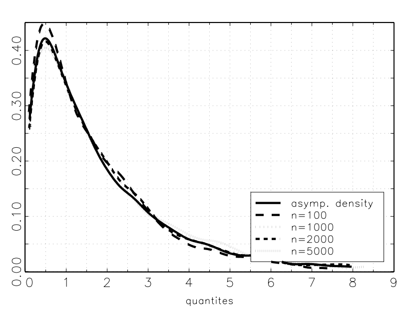

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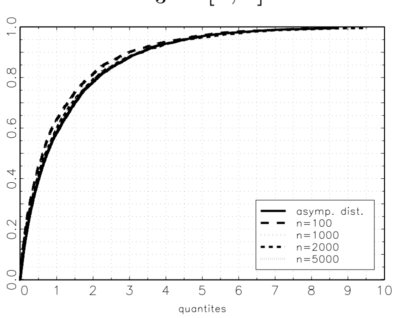
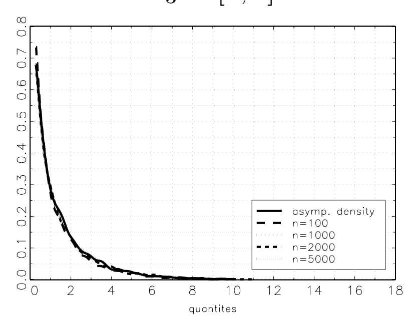
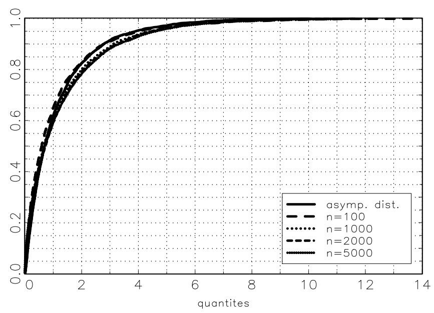




$\Gamma_4 = [0, 5]$ $\Gamma_4 = [0, 5]$

Figure A.1: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\} + u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



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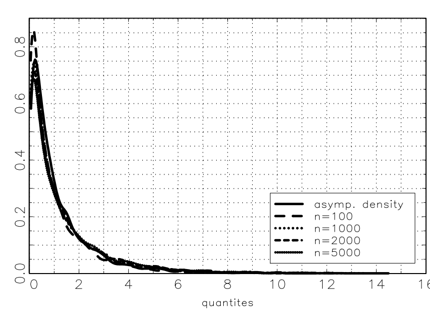

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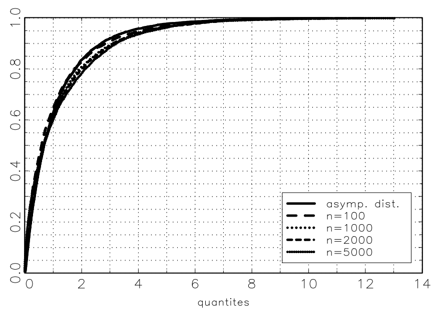

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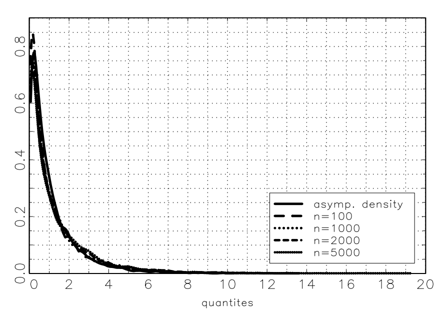

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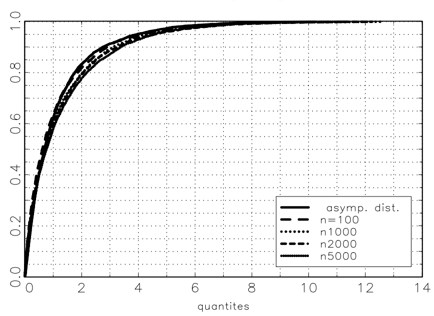

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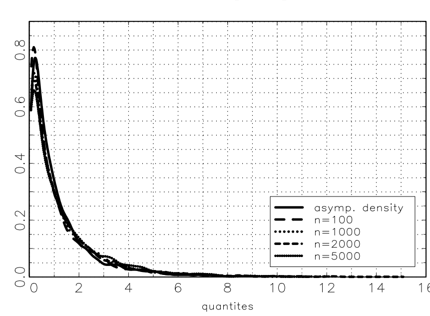

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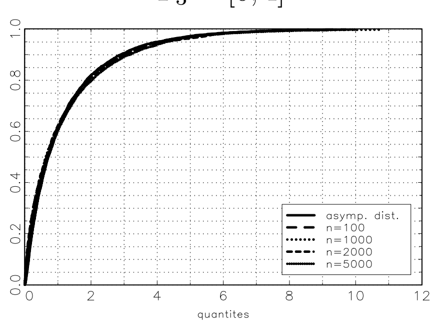

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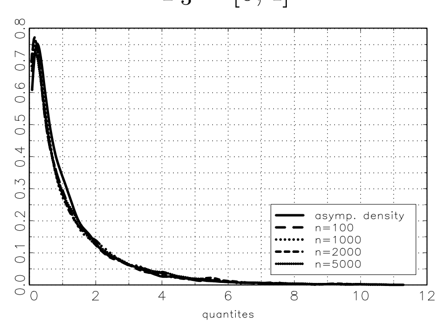
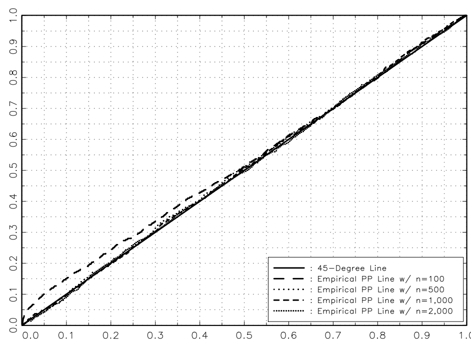


$\Gamma_4 = [0, 5]$

Figure A.2: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim \text{Laplace}(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1 + \exp(-\gamma y_{t-1}))^{-1}\} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim \text{Laplace}(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



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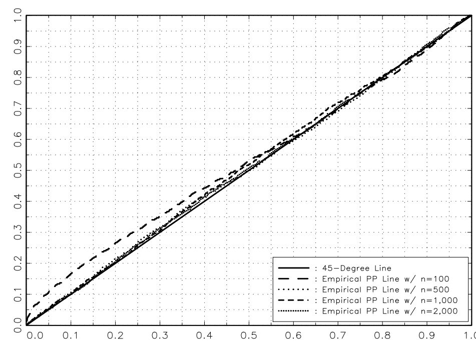

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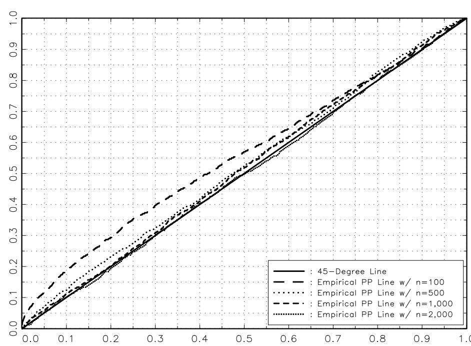

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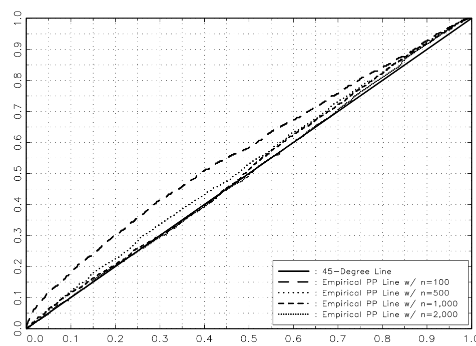
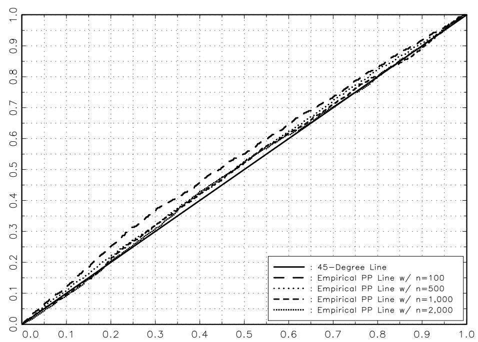


$\Gamma_4 = [0, 5]$

Figure A.3: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\}u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



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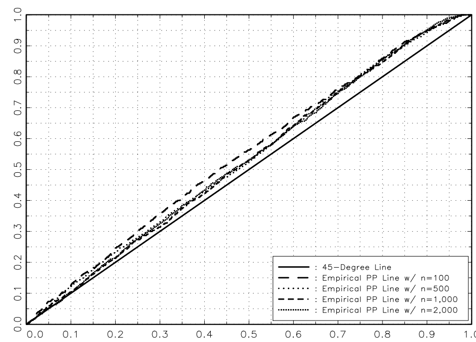

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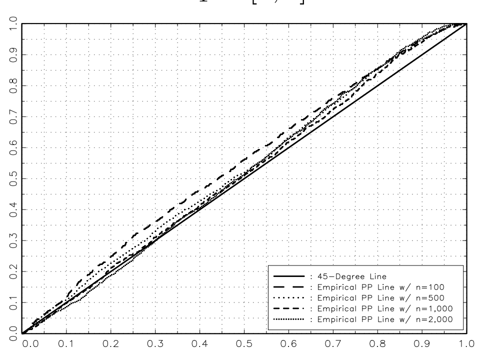

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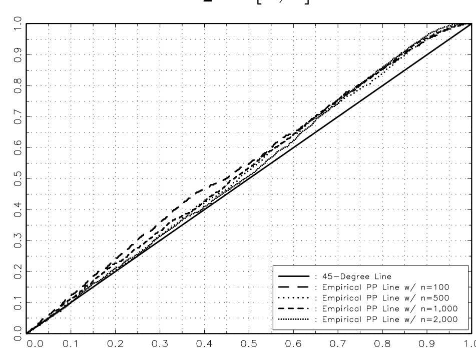

$\Gamma_4 = [0, 5]$

Figure A.4: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.25$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1 + \exp(-\gamma y_{t-1}))^{-1} - 1/2\} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.

Periods	Transition Variable	$LM_{1,n}$	$LM_{2,n}$	$LM_{3,n}$	$LM_{4,n}$	QLR_n^L	QLR_n^E
1968.06∼1999.12	$\Delta_{12}y_{t-1}$	0.150	0.532	0.412	0.895	**0.000**	**0.045**
	$\Delta_{12}y_{t-2}$	**0.037**	0.093	0.057	0.195	**0.000**	**0.028**
	$\Delta_{12}y_{t-3}$	0.162	0.326	0.163	0.555	**0.012**	0.054
	$\Delta_{12}y_{t-4}$	0.665	0.745	0.546	0.619	**0.014**	0.098
	$\Delta_{12}y_{t-5}$	0.662	0.886	0.954	0.830	**0.003**	0.099
	$\Delta_{12}y_{t-6}$	0.588	0.306	0.121	0.234	**0.003**	0.157
1968.06∼2015.08	$\Delta_{12}y_{t-1}$	**0.000**	**0.000**	**0.000**	0.098	**0.000**	**0.000**
	$\Delta_{12}y_{t-2}$	**0.000**	**0.000**	**0.000**	**0.016**	**0.000**	**0.000**
	$\Delta_{12}y_{t-3}$	**0.001**	**0.000**	**0.008**	**0.045**	**0.000**	**0.014**
	$\Delta_{12}y_{t-4}$	**0.008**	**0.012**	0.070	0.111	**0.000**	**0.009**
	$\Delta_{12}y_{t-5}$	**0.038**	0.237	0.274	0.861	**0.000**	**0.049**
	$\Delta_{12}y_{t-6}$	**0.003**	0.068	**0.017**	0.582	**0.000**	0.350

Table A.1: LINEARITY TESTS FOR THE MONTHLY US UNEMPLOYMENT RATE. Notes: The p-values of the linearity tests for the first differenced monthly US unemployment rate are provided. The p-values in the top panel are obtained using observations from 1968.06 to 1999.12, and the p-values of the bottom panel are obtained using observations from 1968.06 to 2015.08. The null linear model is given as AR(15) by AIC, and the twelve-month differences are considered as a transition variable. Boldface p-values indicate significance levels less than or equal to 0.05.