Comprehensively Testing Linearity Hypothesis Using the Smooth Transition Autoregressive Model

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Abstract

This paper examines the null limit distribution of the quasi-likelihood ratio (QLR) statistic for testing linearity condition against the smooth transition autoregressive (STAR) model. We explicitly show that the QLR test statistic weakly converges to a functional of a multivariate Gaussian process under the null of linearity, which is done by resolving the issue of identification problem arises in two different ways under the null. In contrast with the Lagrange multiplier test that is widely employed for testing the linearity condition, the proposed QLR statistic has an omnibus power, and thus, it complements the existing testing procedure. We show the empirical relevance of our test by testing the neglected nonlinearity of the US fiscal multipliers and growth rates of US unemployment. These empirical examples demonstrate that the QLR test is useful for detecting the nonlinear structure among economic variables.

Key Words: QLR test statistic, STAR model, linearity test, multivariate Gaussian process

Subject Classification: C12, C18, C46, C52. H20, H62, H63.

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1 Introduction

The smooth transition autoregressive (STAR) model has been widely used in many areas including economics. It has a property which it shares with other nonlinear models such as the threshold autoregressive model or the hidden Markov or Markov-switching autoregressive model: it nests a linear autoregressive model. Even more importantly, these models are not identified when the data-generating process is a linear model. This lack of identification was first studied by Davies (1977, 1987); see also Watson and Engle (1982). If one estimates the STAR model before testing the linearity hypothesis, one may end up estimating an unidentified model. This typically causes numerical problems in estimation, and even when the estimation algorithm converges, the results are not reliable. It is therefore necessary to test linearity before estimating the STAR model. If the null of linearity is not rejected, the model builder can simply settle for a linear model and avoid the potential problems arising from fitting a more complicated nonlinear model.

The testing problem can be tackled head on by constructing an empirical null distribution by simulation or bootstrap, see, for example, Hansen (1996). Another popular approach consists of circumventing the identification problem by replacing the alternative by a Taylor series approximation around the null hypothesis and constructing a Lagrange multiplier (LM) test against this approximate alternative. For this solution, see Saikkonen and Luukkonen (1988) and Luukkonen, Saikkonen, and Teräsvirta (1988). Later, Granger and Teräsvirta (1993) and Teräsvirta (1994) made this test a part of their strategy for building STAR models.

However, the LM test statistic does not comprehensively test for the nonlinearity entailed by the STAR model. As will be detailed below, the STAR model violates the linearity condition in two different ways, and the LM statistic tests against only one of these two violations. The main goal of this study is to develop a testing procedure that complements the existing test by a test that has non-negligible power against arbitrary nonlinearity. Specifically, we resolve the foregoing identification issue by testing for nonlinearity in two different ways and combining the results into a single test.

An indication of the identification problem is that the model to be tested can be defined by more than one set of parameter restrictions on the alternative. When one such set is selected, some of the parameters of the alternative model remain unidentified under the null hypothesis. In this situation it is possible to choose a different set of restrictions such that the null model is defined using some or all of the parameters that were unidentified in the previous case. This implies that a different set of parameters, including the one or ones that defined the previous null model, are now unidentified when this null hypothesis holds. Following the previous literature we call this the twofold identification problem. Although the models under the null are observationally equivalent, testing procedures without any consideration of the twofold identification problem may not have omnibus power against the null hypothesis of linearity.

We resolve the associated issues by considering the quasi-likelihood ratio (QLR) test statistic, which is known to have omnibus power against arbitrary nonlinearity. As Stinchcombe and White (1998) pointed out, a linearity test acquires omnibus power if it is based on an analytic function, which is the case of the STAR model. As already mentioned, in the LM statistic this analytic function is approximated by a polynomial with the result that the omnibus power is not achieved. Nevertheless, the LM test is easy to compute and has an asymptotic χ^2 distribution under the null of linearity, which explains its popularity.

The QLR statistic in the context of linearity test is not novel. For instance, Cho, Ishida, and White (2011, 2014), Cho and Ishida (2012), White and Cho (2012), and Baek, Cho, and Phillips (2015), among others, study testing for neglected nonlinearity using analytic functions and note that the null of linearity can arise in two or three different ways, each of which carries its own identification problem. They propose a QLR test statistic to resolve the identification issues. We generalise the results in the previous literature to testing linearity against the STAR model and develop a testing procedure that is readily available for applications.

A similar testing procedure can be found in the literature on sub-vector inference. In particular, when the identification of some parameters depends on the identifiability of others, Cheng (2015) considers an inference method that remains valid uniformly on the parameter space by applying the methodology in Andrews and Cheng (2014). However, similarly to the LM test, only one side of the alternative hypothesis is concerned. The Wald test examined by Cheng (2015) does not comprehensively examine the null of linearity as for the QLR test examined by Andrews and Cheng (2014).

Once the QLR test has been constructed, it is appropriate to study its behaviour by applying it to economic data. In order to do this, we examine popular nonlinearity assumptions imposed in applied macroeconomic literature and revisit two published empirical studies. We first re-examine the macroeconomic data in Auerbach and Gorodnichenko (2012) who examined the government multiplier effect using the vector smooth transition autoregressive (VSTAR) model. They used nonstationary data in their analysis. Following Candelon and Lieb (2013), we transform Auerbach and Gorodnichenko's (2012) nonstationary VSTAR model into a stationary vector smooth transition error-correction (VSTEC) model. This makes it possible for us to apply the QLR (and the LM) test statistic. As it turns out, the QLR test statistic rejects linearity, which supports the use of the VSTEC model in studying nonlinear effects of fiscal policy in the US.

In addition, we provide evidence that the QLR and LM test statistics are complementary to each other. We extend the quarterly US unemployment rate series that has been previously studied by van Dijk, Teräsvirta, and Franses (2002). They tested linearity by the LM statistic, and in this study we illustrate the use of the QLR test statistic alongside the LM statistic and find nonlinear features in the series that could not have been found by the LM or the QLR statistic alone.

The plan of the paper is as follows. Testing linearity in the STAR framework is discussed in Section 2, where the null limit distribution of the QLR test statistic is derived. Section 3 contains applications of the QLR test statistic to the multiplier effect of US government spending and the US unemployment rate. Section 4 concludes.

The detailed proofs can be found in the Supplement. There our theory is applied to the ESTAR and LSTAR models. Results on Monte Carlo simulations are reported. In particular, we demonstrate the use of Hansen's (1996) weighted bootstrap in the context of the QLR statistic.

2 Testing Linearity against STAR

2.1 Preliminaries

In this subsection, we clarify the difference between the STAR model and the artificial neural network (ANN) model, in which the QLR test has hitherto been studied (*e.g.* Cho, Ishida, and White, 2011, 2014; White and Cho, 2012; Baek, Cho, and Phillips, 2015). This helps to explain how the current study contributes to the literature on the QLR test by tackling the twofold identification problem within the STAR family of models.

The standard single-hidden layer (univariate) ANN model is specified for stationary variables and has the following form:

$$y_t = \pi_0 + \widetilde{z}_t'\pi + \sum_{j=1}^{q} \theta_j f(z_t'\gamma_j) + \varepsilon_t \tag{1}$$

where $z_t := (1, \widetilde{z}_t')'$ with $\widetilde{z}_t := (y_{t-1}, ..., y_{t-p})'$, $f(0) =$ constant, and $\pi_0, \pi, \theta_j, \gamma_j, j = 1, ..., q$, are parameters. The ANN model (1) thus contains a linear combination of continuous and bounded functions (a hidden layer), typically logistic ones, although other bounded functions are possible. Nowadays, ANN models in applications often contain more than one hidden layer, but the single-hidden layer ANN model serves as a benchmark against which a STAR model may be compared. The twofold identification problem becomes obvious from (1). The model becomes linear by assuming either $\theta_j = 0$ or $\gamma_j = 0$ ($j = 1, ..., q$), so that if $\theta_j = 0$, γ_j disappears from the model; and if $\gamma_j = 0$, π_0 and θ_j are not separably estimable from it. Therefore, Davies' (1977, 1987) identification problem arises in two different ways. This makes the Wald test inapplicable, and so the previous studies focusing on the QLR test apply the likelihood-ratio principle.

In contrast, the following STAR model of order p is frequently specified as a prediction model of a time-series data y_t (*e.g.*, Teräsvirta, 1994; Granger and Teräsvirta, 1993): $\mathcal{M}_0 := \{h_0(\,\cdot\,, \pi, \theta, \gamma) : (\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma\}$, where

$$h_0(z_t, \pi, \theta, \gamma) := z_t'\pi + f(\widetilde{z}_t'\alpha, \gamma)(z_t'\theta), \tag{2}$$

$z_t := (1, \widetilde{z}_t')'$ is a $(p+1)\times 1$ vector of regressors with a transition variable $\widetilde{z}_t'\alpha$. Here, $\widetilde{z}_t := (y_{t-1}, y_{t-2}, \ldots, y_{t-p})'$, and $\alpha = (0, \ldots, 1, 0, \ldots, 0)'$ denotes a selection vector chosen by the researcher. The other parameter vectors $\pi := (\pi_0, \pi_1, \ldots, \pi_p)'$ and $\theta := (\theta_0, \theta_1, \ldots, \theta_p)'$ are the transition parameters, and γ is used to describe the smooth transition from one extreme regime to the other. Symbols Π, Θ, and Γ denote the parameter spaces of π, θ, and γ, respectively. The transition function $f(\cdot, \gamma)$ is a nonlinear, continuously differentiable, and uniformly bounded function. It is typically either exponential, $f_E(\widetilde{z}_t'\alpha, \gamma) := 1 - \exp(-\gamma(\widetilde{z}_t'\alpha)^2)$, or logistic, $f_L(\widetilde{z}_t'\alpha, \gamma) := \{1 + \exp(-\gamma\widetilde{z}_t'\alpha)\}^{-1}$. In both cases, $\gamma > 0$. This STAR model is a special case of the original STAR model in which the transition function $f(\widetilde{z}_t'\alpha - c, \gamma)$ with a constant c is substituted for $f(\widetilde{z}_t'\alpha, \gamma)$ in $\mathcal{M}_0$. We set $c = 0$ in $\mathcal{M}_0$ as in the regular exponential autoregressive model in Haggan and Ozaki (1981) and Auerbach and Gorodnichenko (2012) because the essential property in testing linearity is that $f(\widetilde{z}_t'\alpha, \cdot)$ is an analytic function. As we detail below, if c is estimated along with the other parameters π and θ, the inference becomes more complicated than ours, and this complexity limits its applicability.

The main difference between these two models is that the single hidden-layer ANN model contains a linear combination of several transitions that are themselves functions of linear combinations of elements of z_t, whereas in the standard STAR model a linear combination of these elements is multiplied by a transition function usually with a single argument.[1] Due to these differences, the analysis of the QLR test statistic needs to be generalised in order to make the QLR test statistic applicable in the STAR framework. Specifically, Cho, Ishida, and White (2011, 2014) characterised the null limit distribution of the QLR test statistic in the ANN context as a functional of a univariate Gaussian process. This limit distribution cannot, however, be applied to the STAR case because as it turns out, a multivariate Gaussian process is required for the null limit distribution when testing for the STAR model.

[1]This argument is most often an element of $\widetilde{z}_t$, although it can also be a weighted sum of several variables where the weights are assumed known. STAR models can also contain more than one additive transition, but this seems to be uncommon in applications.

The STAR model has a continuum number of regimes defined by transition functions obtaining values between zero to unity. This feature makes the model an appealing alternative in empirical studies because the behaviour of economic agents can often be best described by multiple regimes and smooth transitions between them. For more discussion on the STAR model the reader is referred to van Dijk, Teräsvirta, and Franses (2002), Teräsvirta (1994), Granger and Teräsvirta (1993), and Teräsvirta, Tjøstheim, and Granger (2010), among others.

The ESTAR and LSTAR models are specified by transforming the exponential function that is analytic, so it is generically comprehensively revealing for model misspecification as pointed out by Stinchcombe and White (1998). Therefore, the estimated parameters in the transition function become statistically significant such that the nonlinear component necessarily reduces the mean squared error of the model even when the assumed STAR model is misspecified. This implies that if the linear model is misspecified, the mean square error obtained from estimating the corresponding STAR model becomes smaller than that from the linear model. This in turn motivates testing linearity by comparing the estimated mean squared errors from the STAR and the linear model nested in the STAR. This process delivers an omnibus testing procedure for nonlinearity.

2.2 Brief Review of the LM Test

Before discussing the QLR test, we briefly review the model framework for the LM test statistics to make a comparison with the QLR test. The following auxiliary model is first estimated for the LM test statistics:

$$\mathcal{M}_{AUX} := \{h_{AUX}(\,\cdot\,, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) : (\alpha_0', \alpha_1', \alpha_1', \alpha_1', \alpha_1')' \in \Theta\},$$

where

$$h_{AUX}(z_t, \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) := \alpha_0' z_t + \alpha_1'(\widetilde{z}_t t_t) + \alpha_2'(\widetilde{z}_t t_t^2) + \alpha_3'(\widetilde{z}_t t_t^3) + \alpha_4'(\widetilde{z}_t t_t^4),$$

and t_t is the transition variable, viz., $\widetilde{z}_t'\alpha$. This model is obtained by applying a fourth-order Taylor expansion to the analytic function as an intermediate step to compute the LM test statistics conveniently which tests $\gamma_* = 0$. Luukkonen, Saikkonen, and Teräsvirta (1988) and Teräsvirta (1994) provided detailed rationales of carrying out testing linearity by examining the coefficients of nonlinear components of the approximate alternative.

To be specific, Teräsvirta (1994) and Escribano and Jordà (1999) specify the following four sets of hypotheses which are commonly considered in empirical studies:

$$\mathcal{H}_{0,1}:\alpha_{1*}=\alpha_{2*}=\alpha_{3*}=0|\alpha_{4*}=0 \quad \text{vs.} \quad \mathcal{H}_{1,1}:\alpha_{1*}\neq 0,\alpha_{2*}\neq 0, \text{ or } \alpha_{3*}\neq 0|\alpha_{4*}=0.$$

$$\mathcal{H}_{0,2}:\alpha_{1*}=\alpha_{2*}=\alpha_{3*}=\alpha_{4*}=0 \quad \text{vs.} \quad \mathcal{H}_{1,2}:\alpha_{1*}\neq 0,\alpha_{2*}\neq 0,\alpha_{3*}\neq 0, \text{ or } \alpha_{4*}\neq 0.$$

$$\mathcal{H}_{0,3}:\alpha_{1*}=\alpha_{3*}=0 \quad \text{vs.} \quad \mathcal{H}_{1,3}:\alpha_{1*}\neq 0 \text{ or } \alpha_{3*}\neq 0.$$

$$\mathcal{H}_{0,4}:\alpha_{2*}=\alpha_{4*}=0 \quad \text{vs.} \quad \mathcal{H}_{1,4}:\alpha_{2*}\neq 0 \text{ or } \alpha_{4*}\neq 0.$$

For later purpose, we denote the LM test statistics testing $\mathcal{H}_{0,i}$ as $LM_{i,n}$, $i=1,...,4$. Here, $LM_{1,n}$ and $LM_{2,n}$ are general tests against STAR, whereas $LM_{3,n}$ and $LM_{4,n}$ are tests against the LSTAR and ESTAR models, respectively.

2.3 Data generating process and the QLR Test Statistic

We consider a univariate STAR model and study the null limit behaviour of the QLR test statistic in this framework. In order to proceed, we make the following assumptions:

Assumption 1. $\{(y_t,\tilde{z}_t')'\in\mathbb{R}^{1+p}:t=1,2,\ldots\}$ $(p\in\mathbb{N})$ *is a strictly stationary and absolutely regular process defined on the complete probability space* $(\Omega,\mathcal{F},\mathbb{P})$*, with* $\mathbb{E}[|y_t|]<\infty$ *and mixing coefficient* β_τ *such that for some* $\rho>1$, $\sum_{\tau=1}^{\infty}\tau^{1/(\rho-1)}\beta_\tau<\infty$. □

Here, the mixing coefficient is defined as $\beta_\tau := \sup_{s\in\mathbb{N}}\mathbb{E}[\sup_{A\in\mathcal{F}_{s+\tau}^{\infty}}|\mathbb{P}(A|\mathcal{F}_{-\infty}^{s})-\mathbb{P}(A)|]$, where $\mathcal{F}_\tau^s$ is the σ-field generated by $(y_t,\ldots,y_{t+s})$. Many time series models satisfy this condition, and the autoregressive process is one of them. It is general enough to cover the stationary time series we are interested in. We impose the following regular STAR model condition:

Assumption 2. *Let* $f(\tilde{z}_t'\alpha,\cdot):\Gamma\mapsto[0,1]$ *be a non-polynomial analytic function with probability 1. Let* $\Pi\in\mathbb{R}^{p+1}$, $\Theta\in\mathbb{R}^{p+1}$*, and* $\Gamma\in\mathbb{R}$ *be non-empty convex and compact sets such that* $0\in\Gamma$*. Let* $h(z_t,\pi,\theta,\gamma):=z_t'\pi+\{f(\tilde{z}_t'\alpha,\gamma)-f(\tilde{z}_t'\alpha,0)\}(z_t'\theta)$*, and let* $\mathcal{M}:=\{h(\,\cdot\,,\pi,\theta,\gamma):(\pi,\theta,\gamma)\in\Pi\times\Theta\times\Gamma\}$ *be the model specified for* $\mathbb{E}[y_t|z_t]$. □

Note that $\mathcal{M}$ differs from $\mathcal{M}_0$. The transition function is centered at $f(\tilde{z}_t'\alpha,0)$ for analytical convenience. As $f(\tilde{z}_t'\alpha,0)$ is constant, the nonlinearity of the STAR model is not modified by this centering. For example, we have $f_E(\tilde{z}_t'\alpha,0)=0$ and $f_L(\tilde{z}_t'\alpha,0)=1/2$, so f_L will be centered to have value zero. Furthermore, centering reduces the dimension of the identification problem as we detail below. The parameters to estimate are π, θ, and γ. Here, the selection vector α is defined by the researcher.

Using Assumption 2, the linearity hypothesis and the alternative are specified as follows:

$$\mathcal{H}_0 : \exists \pi \in \mathbb{R}^{p+1} \quad \text{such that} \quad \mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) = 1; \quad \text{vs.} \quad \mathcal{H}_1 : \forall \pi \in \mathbb{R}^{p+1}, \, \mathbb{P}(\mathbb{E}[y_t|z_t] = z_t'\pi) < 1.$$

These hypotheses are the same as the ones in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). As in the previous literature, the focus is on developing an omnibus test statistic but now against STAR, and the QLR test statistic is a vehicle for reaching this goal. The QLR test statistic is formally defined as

$$QLR_n := n \left(1 - \frac{\widehat{\sigma}^2_{n,A}}{\widehat{\sigma}^2_{n,0}} \right),$$

where

$$\widehat{\sigma}^2_{n,0} := \min_{\pi} n^{-1} \sum_{t=1}^{n} (y_t - z_t'\pi)^2, \qquad \widehat{\sigma}^2_{n,A} := \min_{\pi,\theta,\gamma} n^{-1} \sum_{t=1}^{n} \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2,$$

and $f_t(\gamma) := f(\widetilde{z}_t'\alpha, \gamma) - f(\widetilde{z}_t'\alpha, 0)$. We let the nonlinear least squares (NLS) estimator $(\widehat{\pi}_n, \widehat{\theta}_n, \widehat{\gamma}_n)$ minimise the squared errors with respect to (π, θ, γ). Furthermore, $(\pi_*, \theta_*, \gamma_*)$ denotes the probability limit of the NLS estimator: $(\pi_*, \theta_*, \gamma_*) := \arg\min_{\pi,\theta,\gamma} \mathbb{E}[\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2]$ is the pseudo-true parameter. Note that this limit is not unique under the null.

The main reason for proceeding with the QLR statistic is that linearity leads to a twofold identification problem, and this statistic is able to handle both parts of it. If $\mathbb{E}[y_t|z_t]$ is linear with respect to z_t with coefficient π_*, we can generate a linear function from $h(\cdot, \pi_*, \theta_*, \gamma_*)$ in two different ways, either by letting $\theta_* = 0$ or by assuming $\gamma_* = 0$. Because of this, $(\pi_*, \theta_*, \gamma_*)$ is not uniquely determined. If $\theta_* = 0$, $h(\cdot, \pi_*, 0, \gamma_*) = z_t'\pi_*$, so that γ_* is not identified. We call this problem a type I identification problem, under which $(\pi_*, \theta_*, \gamma_*)$ becomes any element in $\{(\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma : \pi = \pi_*, \theta = 0\}$. If we employed $f(\widetilde{z}_t'\alpha - c, \gamma)$ instead of $f(\widetilde{z}_t'\alpha, \gamma)$ for $\mathcal{M}$ as in the original STAR model, neither γ_* nor the additional c_* is identified under $\theta_* = 0$, which leads to a more complicated identification problem. We fix our interest in the current derivative model $\mathcal{M}$ that excludes c_*. Alternatively, if $\gamma_* = 0$, $h(\cdot, \pi_*, \theta_*, 0) = z_t'\pi_*$, so that θ_* is not identified. This leads to a type II identification problem, in which $(\pi_*, \theta_*, \gamma_*)$ becomes any element in $\{(\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma : \pi = \pi_*, \gamma = 0\}$.

If the transition function is not centered at $f(\widetilde{z}_t'\alpha, 0)$, letting $\gamma_* = 0$ leads to $h_0(z_t, \pi_*, \theta_*, 0) = z_t'(\pi_* + f(\widetilde{z}_t'\alpha, 0)\theta_*)$. This makes the type II identification problem more complicated as π_* and θ_* are not separately identified. Centering thus transforms this complication into a relatively straightforward identification problem. Besides, mainly due to the invariance principle, the null limit distribution does not change by this centering. Note that π in $\mathcal{M}_0$ is reparameterised to $\pi - f(\tilde{z}_t'\alpha, 0)\theta$ in $\mathcal{M}$, so that the QLR test obtained by this reparameterisation becomes identical to that before the reparameterisation. Without it, the null model investigation has to be separately conducted by discerning the parameters with the type II identification problem. So, we avoid the involved complication by the centering and obtain the null limit distribution of the QLR test efficiently. This centering is also indirectly applied in the literature when the null limit distribution of the LM test statistic is being derived. If z_t contains a constant, this limit distribution is not affected by the centering, because the centered parameter is merged into the linear component in the Taylor expansion that forms the basis of the LM test statistic.

Now, the null holds for the following two sub-hypotheses: $\mathcal{H}_{01} : \theta_* = 0$ and $\mathcal{H}_{02} : \gamma_* = 0$. The limit distribution of the QLR test statistic can be derived under both $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$, leading to different null limit distributions even for the same statistic. We call these derivations type I and type II analyses, respectively. The null hypothesis of linearity against STAR is properly tested by tackling both $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ simultaneously, and we shall demonstrate that the QLR test is capable of doing this. For this purpose, we derive its null limit distribution from the separately obtained null weak limits in the spirit of likelihood-ratio principle. Specifically, we show how the two different weak limits are related to the null limit distribution of the QLR statistic.

Our view to testing linearity by accommodating type I and II analyses differs from the other tests in the literature. For example, the LM test statistic focuses on testing $\mathcal{H}_{02}$. The main argument for the LM test is that its asymptotic null distribution is chi-squared, which makes the test easily applicable. As another example, Cheng (2015) assumes the standard STAR model and analyses the standard Wald statistic for testing $\mathcal{H}_{01}$ in the vein of the type I identification problem. None of them accommodates the twofold identification problem.

2.4 The Null Limit Distribution of the QLR Test

We now derive the null limit distribution of the QLR test and highlight the difference between the STAR-based approach and the ANN-based one. We first study the limit distributions of the QLR test under $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ separately, combine them and, finally, obtain the limit distribution under $\mathcal{H}_0$. For this, we let our quasi-likelihood (QL) function be

$$\mathcal{L}_n(\pi, \theta, \gamma) := -\sum_{t=1}^{n}\{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2.$$

2.4.1 Type I Analysis: Testing $\mathcal{H}_{01} : \theta_* = 0$

In this subsection, we discuss the limit distribution of the QLR test under $\mathcal{H}_{01} : \theta_* = 0$. The problem is that γ_* is not identified under this hypothesis. We obtain the NLS estimator by maximising the QL function with respect to γ in the final stage for the purpose of testing $\mathcal{H}_{01}$: $\mathcal{L}_n^{(1)} := \max_\gamma \max_\theta \max_\pi -\sum_{t=1}^n \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$ and let $QLR_n^{(1)}$ denote the QLR statistic obtained by this optimisation process. That is,

$$\mathcal{L}_n^{(1)} := \max_{\gamma \in \Gamma} \{-u'Mu + u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu\},$$

where $u := [u_1, u_2, \ldots, u_n]'$, $u_t := y_t - \mathbb{E}[y_t|z_t]$, $Z := [Z_1, Z_2, \ldots, Z_n]'$, $M := I - Z(Z'Z)^{-1}Z'$, and $F(\gamma) := \mathrm{diag}[f_1(\gamma), f_2(\gamma), \ldots, f_n(\gamma)]$. Therefore, we found that

$$QLR_n^{(1)} := \max_{\gamma \in \Gamma} \widehat{\sigma}_{n,0}^{-2} u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu$$

under $\mathcal{H}_{01}$ using the fact that $y_t = \mathbb{E}[y_t|z_t] + u_t = z_t'\pi_* + u_t$. Note that the numerator of $QLR_n^{(1)}$ is identical to $n(\widehat{\sigma}_{n,0}^2 - \widehat{\sigma}_{n,A}^2)$ under $\mathcal{H}_{01} : \theta_* = 0$, so that the QLR test accords with $QLR_n^{(1)}$. Here, we cannot let $\gamma = 0$ when deriving $QLR_n^{(1)}$. If $\gamma = 0$, the alternative model reduces to the linear model, so that the QLR statistic cannot test the null model by letting $\gamma = 0$. We therefore examine its null limit distribution by supposing $\gamma \neq 0$.

We now derive the limit distribution of $QLR_n^{(1)}$ under $\mathcal{H}_{01}$. For this and to ensure a regular null limit distribution, we impose the following conditions:

Assumption 3. *(i)* $\mathbb{E}[u_t|z_t, u_{t-1}, z_{t-1}, \ldots] = 0$*; and (ii)* $\mathbb{E}[u_t^2|z_t, u_{t-1}, z_{t-1}, \ldots] = \sigma_*^2$. □

Assumption 4. $\sup_{\gamma \in \Gamma} |(\partial/\partial\gamma) f_t(\gamma)| \leq m_t$. □

Assumption 5. *There exists a sequence of stationary ergodic random variables* m_t *such that for* $i = 1, 2, \ldots, p$, $|\widetilde{z}_{t,i}| \leq m_t$, $|u_t| \leq m_t$, $|y_t| \leq m_t$, *and for some* $\omega \geq 2(\rho - 1)$, $\mathbb{E}[m_t^{6+3\omega}] < \infty$, *where* ρ *is in Assumption 1, and* $z_{t,i}$ *is the* i*-th row element of* z_t. □

Assumption 6. *For each* $\gamma \neq 0$, $V_1(\gamma)$ *and* $V_2(\gamma)$ *are positive definite, where for each* γ, $V_1(\gamma) := \mathbb{E}[u_t^2 \widetilde{r}_t(\gamma) \widetilde{r}_t(\gamma)']$ *and* $V_2(\gamma) := \mathbb{E}[\widetilde{r}_t(\gamma)\widetilde{r}_t(\gamma)']$ *with* $\widetilde{r}_t(\gamma) := (f_t(\gamma)z_t', z_t')'$. □

Assumption 3(*i*) implies that the model in Assumption 2 is not dynamically misspecified, and Assumption 3(*ii*) means that the errors are conditionally homoskedastic. Here, conditional homoskedasticity is not essential in achieving the main goal of this study, but this assumption will be assumed whenever it facilitates understanding the theoretical results intuitively. Assumption 4 plays an integral role in applying the tightness condition in Doukhan, Massart, and Rio (1995) to the QLR test statistic. Here it can be easily verified for the ESTAR and LSTAR models by noting that $|(\partial/\partial\gamma)f_E(\widetilde{z}_t'\alpha,\gamma)| = (1 - f_E(\widetilde{z}_t'\alpha,\gamma))(\widetilde{z}_t'\alpha)^2 \leq (\widetilde{z}_t'\alpha)^2$ and $|(\partial/\partial\gamma)f_L(\widetilde{z}_t'\alpha,\gamma)| = f_L(\widetilde{z}_t'\alpha,\gamma)(1 - f_L(\widetilde{z}_t'\alpha,\gamma))|(\widetilde{z}_t'\alpha)| \leq |(\widetilde{z}_t'\alpha)|$, so that we can let m_t in Assumption 4 be $(\widetilde{z}_t'\alpha)^2$ and $|(\widetilde{z}_t'\alpha)|$, respectively. The moment condition in Assumption 5 is stronger than those in Cho, Ishida, and White (2011, 2014), and it also implies that $\mathbb{E}[u_t^6]$ and $\mathbb{E}[y_t^6]$ are finite. The multiplicative component $f_t(\gamma)z_t'\theta$ in the STAR model makes the stronger moment condition necessary in the current study. Assumption 6 is imposed for the invertibility of the limit covariance matrix. This makes our test statistic non-degenerate. We have the following lemma:

Lemma 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and $\mathcal{H}_{01}$, (i) $\widehat{\sigma}_{n,0}^2 \overset{\text{a.s.}}{\to} \sigma_*^2 := \mathbb{E}[u_t^2]$; (ii) $\{n^{-1/2}Z'\,F(\cdot)Mu,\ \widehat{\sigma}_{n,0}^2 n^{-1}Z'F(\cdot)MF(\cdot)Z\} \Rightarrow \{\mathcal{Z}_1(\cdot),\ A_1(\cdot,\cdot)\}$ on $\Gamma(\epsilon)$ and $\Gamma(\epsilon)\times\Gamma(\epsilon)$, respectively, where $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$, $\mathcal{Z}_1(\cdot)$ is a continuous multivariate Gaussian process with $\mathbb{E}[\mathcal{Z}_1(\gamma)] = 0$, and for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{Z}_1(\gamma)\mathcal{Z}_1(\widetilde{\gamma})'] = B_1(\gamma,\widetilde{\gamma})$ such that $B_1(\gamma,\widetilde{\gamma}) := \mathbb{E}[u_t^2 f_t^*(\gamma) f_t^*(\widetilde{\gamma})']$ and $A_1(\gamma,\widetilde{\gamma}) := \sigma_*^2\mathbb{E}[f_t^*(\gamma)f_t^*(\widetilde{\gamma})']$ with $f_t^*(\gamma) = f_t(\gamma)z_t - \mathbb{E}[f_t(\gamma)z_t z_t']\,\mathbb{E}[z_t z_t']^{-1}z_t$; (iii) if, in addition, Assumption 3(ii) holds, $B_1(\gamma,\widetilde{\gamma}) = A_1(\gamma,\widetilde{\gamma})$.* □

There is a caveat to Lemma 1. It is clear from $QLR_n^{(1)}$ that its limit distribution is determined by the limit behaviour under $\mathcal{H}_{01}$ of both $n^{-1/2}Z'F(\cdot)Mu$ and $n^{-1}Z'F(\cdot)M\,F(\cdot)Z$. Furthermore, $\lim_{\gamma\to 0} Z'F(\gamma)Mu \overset{\text{a.s.}}{=} Z'F(0)Mu = 0$ and $\lim_{\gamma\to 0} Z'\,F(\gamma)MF(\gamma)Z \overset{\text{a.s.}}{=} Z'F(0)MF(0)Z = 0$. This implies that it is not straightforward to obtain the limit distribution of $QLR_n^{(1)}$ around $\gamma = 0$. We therefore assume for the moment that 0 is not included in Γ by considering $\Gamma(\epsilon)$ instead of Γ and accommodate this effect by restricting the QLR test statistic to

$$QLR_n^{(1)}(\epsilon) := \max_{\gamma\in\Gamma(\epsilon)} \frac{1}{\widehat{\sigma}_{n,0}^2} u'MF(\gamma)Z[Z'F(\gamma)MF(\gamma)Z]^{-1}Z'F(\gamma)Mu.$$

We relax this restriction when the limit distribution is examined under $\mathcal{H}_0$.

Lemma 1 is central in deriving the null limit distribution of $QLR_n^{(1)}(\epsilon)$ and corresponds to lemma 1 of Cho, Ishida, and White (2011). Despite being similar, the two lemmas are not identical; note that $\mathcal{Z}_1(\cdot)$ is mapped to $\mathbb{R}^{p+1}$, i.e., $(p+1)$-varaite Gaussian process, whereas their lemma obtains a univariate Gaussian process. This multivariate Gaussian process $\mathcal{Z}_1(\cdot)$ distinguishes the STAR model-based testing from the ANN-based approach. By this, the STAR model has a different null limit distribution, and the QLR test based upon the STAR model has power over alternatives in directions different from those of the ANN-based approach.

Theorem 1. *Given Assumptions 1, 2, 3(i), 4, 5, 6, and $\mathcal{H}_{01}$, for each $\epsilon > 0$, (i) $QLR_n^{(1)}(\epsilon) \Rightarrow \sup_{\gamma \in \Gamma(\epsilon)} \mathcal{G}_1(\gamma)'$ $\mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian process such that for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{G}_1(\gamma)] = 0$ and $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1^{-1/2}(\gamma, \gamma) B_1(\gamma, \widetilde{\gamma})\, A_1^{-1/2}(\widetilde{\gamma}, \widetilde{\gamma})$; (ii) if, in addition, Assumption 3 (ii) holds, then $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})'] = A_1^{-1/2}(\gamma, \gamma) A_1(\gamma, \widetilde{\gamma})\, A_1^{-1/2}(\widetilde{\gamma}, \widetilde{\gamma})$.* □

As continuous mapping makes proving Theorem 1 trivial, no proof is given.

Theorem 1 implies that $QLR_n^{(1)}(\epsilon)$ does not asymptotically follow a chi-squared distribution under $\mathcal{H}_{01}$ as does the LM statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993). The difficulty here is that the null limit distribution contains the unidentified nuisance parameter γ.

2.4.2 Type II Analysis: Testing $\mathcal{H}_{02} : \gamma_* = 0$

Here, the focus is on the limit distribution under $\mathcal{H}_{02} : \gamma_* = 0$. This hypothesis is tested using the LM statistic. As we know, θ_* is not identified under $\mathcal{H}_{02}$. We therefore maximise the QL function with respect to θ at the final stage: $\mathcal{L}_n^{(2)} := \sup_\theta \sup_\gamma \sup_\pi - \sum_{t=1}^n \{y_t - z_t'\pi - f_t(\gamma)(z_t'\theta)\}^2$, and denote the QLR test defined by this maximisation process by $QLR_n^{(2)}$.

Several remarks are in order. First, maximising the QL with respect to π is relatively simple due to linearity. We let the concentrated QL (CQL) function be $\mathcal{L}_n^{(2)}(\gamma, \theta) := \sup_\pi \mathcal{L}_n(\pi, \theta, \gamma) = -[y - F(\gamma)Z\theta]'M[y - F(\gamma)Z\theta]$, where $y := [y_1, y_2, \ldots, y_n]$. Here, we have to assume $\theta \neq 0$. If $\theta = 0$, the STAR model becomes linear, so the QLR test statistic cannot compare the null model with the alternative. Second, $\mathcal{L}_n^{(2)}(\cdot)$ is not linear with respect to γ, so that the next stage CQL function with respect to γ cannot be analytically derived. We approximate the CQL function with respect to γ around $\gamma_* = 0$ and capture its limit behaviour under $\mathcal{H}_{02}$. The first-order derivative of $\mathcal{L}_n^{(2)}(\gamma, \theta)$ with respect to γ is $(d/d\gamma)\, \mathcal{L}_n^{(2)}(\gamma, \theta) = 2[y - F(\gamma)Z\theta]'M(\partial F(\gamma)/\partial\gamma)Z\theta$, where $(\partial F(\gamma)/\partial\gamma) := (\partial/\partial\gamma)\,(f(\widetilde{z}_1'\alpha, \gamma), ..., f(\widetilde{z}_n'\alpha, \gamma))$. For the LSTAR model, $\partial f_L(\widetilde{z}_t'\alpha, \gamma)/\, \partial\gamma = f_L(\widetilde{z}_t'\alpha, \gamma)(1 - f_L(\widetilde{z}_t'\alpha, \gamma))\widetilde{z}_t'\alpha$ and $\partial F(0)/\partial\gamma = (1/4)(\widetilde{z}_1'\alpha, ..., \widetilde{z}_n'\alpha)'$, whereas for ESTAR, it follows that $\partial f_E(\widetilde{z}_t'\alpha, \gamma)/\, \partial\gamma = (\widetilde{z}_t'\alpha)^2(1 - f_E(\widetilde{z}_t'\alpha, \gamma))$, so $\partial F(0)/\partial\gamma = ((\widetilde{z}_1'\alpha)^2, ..., (\widetilde{z}_n'\alpha)^2)'$, implying that we can approximate the CQL function by a second-order approximation. Nevertheless, as Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011, 2014) pointed out, the first-order derivative of the CQL is often zero for many other models. For example, in $\mathcal{M}_A := \{\pi y_{t-1} + \theta\{1 + \exp(\gamma y_{t-1})\}^{-1} : (\pi, \theta, \gamma) \in \Pi \times \Theta \times \Gamma\}$, the first-order derivative of the CQL is zero when $\gamma_* = 0$. Due to this, we need a higher-order approximation. Cho, Ishida, and White (2014) adopt a sixth-order Taylor expansion, whereas

Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Cho, Ishida, and White (2011) use fourth-order Taylor expansions to obtain the null limit distributions of their tests. The order of expansion is determined by the functional form of $f(\widetilde{z}_t'\alpha, \cdot)$.

As we do not assume a specific form for our STAR model, we simply let κ ($\kappa \in \mathbb{N}$) be the smallest order such that the κ-th order partial derivative with respect to γ is different from zero at $\gamma = 0$, so that for all $j < \kappa$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0, \cdot) \equiv 0$. For example, $\kappa = 3$ for $\mathcal{M}_A$. Then, the CQL function is expanded as

$$\mathcal{L}_n^{(2)}(\gamma, \theta) = \mathcal{L}_n^{(2)}(0, \theta) + \frac{1}{\kappa!}\frac{\partial^\kappa}{\partial\gamma^\kappa}\mathcal{L}_n^{(2)}(0, \theta)\gamma^\kappa + \ldots + \frac{1}{(2\kappa)!}\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0, \theta)\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{3}$$

Note that for $j = 1, 2, \ldots, \kappa - 1$, $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0, \theta) = 0$ by the definition of κ. If $\kappa = 1$, the first-order derivative differs from zero, so that none of the derivatives is zero, meaning that $j = 0$. The limit behaviours of the partial derivatives in (3) are given in the following lemma:

Lemma 2. *Given Assumption 2, the definition of κ, and $\mathcal{H}_{02}$, for each $\theta \neq 0$, $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0, \theta) = 2\theta' Z' H_j(0) M u$ for $\kappa \leq j < 2\kappa$; and $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0, \theta) = 2\theta' Z' H_{2\kappa}(0) M u - \binom{2\kappa}{\kappa}\theta' Z' H_\kappa(0) M\ H_\kappa(0) Z\theta$ for $j = 2\kappa$, where $H_j(\gamma) := (\partial^j/\partial\gamma^j)F(\gamma)$.* □

Using Lemma 2, we can specifically rewrite (3) as $\mathcal{L}_n^{(2)}(\gamma, \theta) - \mathcal{L}_n^{(2)}(0, \theta) = \sum_{j=\kappa}^{2\kappa} \frac{2}{j!}\{\theta' Z'\ H_j(0) M u\}\gamma^j - \frac{1}{(2\kappa)!}\binom{2\kappa}{\kappa}\theta' Z' H_\kappa(0) M H_\kappa(0) Z\theta\gamma^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa})$. To reduce notational clutter, we further let $G_j := [g_{j,1}, g_{j,2}, \ldots, g_{j,n}]' := M H_j(0) Z$, where $g_{j,t} := h_{j,t}(0) z_t - Z' H_j(0) Z\ (Z'Z)^{-1} Z' z_t$ and $\varsigma_n := n^{1/2\kappa}\gamma$ with $h_{j,t}(0)$ being the t-th diagonal element of $H_j(0)$. Then,

$$\mathcal{L}_n^{(2)}(\gamma, \theta) - \mathcal{L}_n^{(2)}(0, \theta) = \sum_{j=\kappa}^{2\kappa} \frac{2\{\theta' G_j' u\}}{j!\, n^{j/2\kappa}}\varsigma_n^j - \frac{1}{(2\kappa)!\, n}\binom{2\kappa}{\kappa}\{\theta' G_\kappa' G_\kappa \theta\}\varsigma_n^{2\kappa} + o_{\mathbb{P}}(\gamma^{2\kappa}). \tag{4}$$

We note that if $j = \kappa$, $n^{-j/2\kappa} G_j' u = O_{\mathbb{P}}(1)$ by applying the central limit theorem. Furthermore, for $j = \kappa + 1, \ldots, 2\kappa - 1$, $n^{-j/(2\kappa)}(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(\gamma, \theta) = o_{\mathbb{P}}(1)$ and $\theta' G_{2\kappa} u = o_{\mathbb{P}}(n)$ by the ergodic theorem, so that they become asymptotically negligible, implying that the smallest j-th component greater than κ and surviving at the limit becomes the second-final term in the right side of (4). Note that $n^{-1} G_\kappa' G_\kappa = O_{\mathbb{P}}(1)$, if the ergodic theorem applies, and the terms with $j > 2\kappa$ belong to $o_{\mathbb{P}}(\gamma^{2\kappa})$ by Taylor's theorem, so that they are asymptotically negligible under the null at any rate. Due to this fact, $\mathcal{L}_n^{(2)}(\cdot, \theta)$ is approximated by the 2κ-th degree polynomial function in (4), and we provide the following condition for the asymptotic analysis of the polynomial function:

Assumption 7. *For $j = \kappa, \kappa+1, \ldots, 2\kappa$ and $i = 0, 1, \ldots, p$, (i) $\mathbb{E}[|u_t|^8] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^4] < \infty$; or (ii) $\mathbb{E}[|u_t|^4] < \infty$, $\mathbb{E}[|h_{j,t}(0)|^8] < \infty$, and $\mathbb{E}[|z_{t,i}|^8] < \infty$.* $\square$

Using Assumption 7, we can apply the CLT to $n^{-1/2}G_j'u$ for $j = \kappa, \kappa+1, \ldots, 2\kappa$. Note that $G_j'u = \sum_{t=1}^{n}(u_t g_{j,t})$, and $\mathbb{E}[(u_t g_{j,t})^2] < \infty$ by the moment conditions in Assumption 7 and Cauchy-Schwarz inequality, so that for $j = \kappa+1, \ldots, 2\kappa-1$, $n^{-j/2\kappa}G_j'u = o_{\mathbb{P}}(1)$. Although the QLR test statistic is approximated by the 2κ-th degree polynomial function, the moment conditions in Assumption 7 are sufficient to apply the CLT to the first term in (4).

We establish the following lemma by collecting the asymptotically surviving terms:

Lemma 3. *Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, $QLR_n^{(2)} = \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(n)$, where for given $\theta \neq 0$,*

$$\overline{QLR}_n^{(2)}(\theta) := \sup_{\varsigma_n} \frac{1}{\widehat{\sigma}_{n,0}^2}\left\{\frac{2}{\kappa! n^{1/2}}\{\theta' G_\kappa' u\}\varsigma_n^\kappa - \frac{1}{(2\kappa)! n}\binom{2\kappa}{\kappa}\theta' G_\kappa' G_\kappa \theta \varsigma_n^{2\kappa}\right\}$$

and $\widehat{\varsigma}_n^\kappa(\theta)$ maximises the given objective function, so that $\widehat{\varsigma}_n^\kappa(\theta) = W_n(\theta)$, if κ is odd; and $\widehat{\varsigma}_n^\kappa(\theta) = \max[0, W_n(\theta)]$, if κ is even, where $W_n(\theta) := \kappa! n^{1/2}\{\theta' G_\kappa' u\}/\{\theta' G_\kappa' G_\kappa \theta\}$. $\square$

Lemma 3 implies that the functional form of $\overline{QLR}_n^{(2)}(\cdot)$ depends on κ: for each $\theta \neq 0$, $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}\frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}$, if κ is odd; and $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}\max\left[0, \frac{(\theta' G_\kappa' u)^2}{\theta' G_\kappa' G_\kappa \theta}\right]$, if κ is even. If θ is a scalar as in the previous literature, θ cancels out, so maximisation with respect to θ does not matter at the limit. This implies that $QLR_n^{(2)}$ and $\overline{QLR}_n^{(2)}(\cdot)$ are asymptotically equivalent under $\mathcal{H}_{02}$. On the other hand, if θ is a vector, the asymptotic null distribution of the test statistic has to be determined by further maximising $\overline{QLR}_n^{(2)}(\cdot)$ with respect to θ.

We now derive the regular limit distribution of QLR test statistic under $\mathcal{H}_{02}$. The following additional condition is sufficient for this:

Assumption 8. *$V_3(0)$ and $V_4(0)$ are positive definite, where for each γ, $V_3(\gamma) := \mathbb{E}[u_t^2 \bar{r}_t(\gamma)\ \bar{r}_t(\gamma)']$ and $V_4(\gamma) := \mathbb{E}[\bar{r}_t(\gamma)\bar{r}_t(\gamma)']$ with $\bar{r}_t(\gamma) := (h_{t,\kappa}(\gamma)z_t', z_t')'$.* $\square$

We note that the nuisance parameter γ does not play a significant role in Assumption 8 as it does in the type I analysis, because $\overline{QLR}_n(\cdot)$ has already concentrated the QL function with respect to γ. Given these regularity conditions, the key limit results of the components that constitute $\overline{QLR}_n^{(2)}(\cdot)$ appear in the following lemma:

Lemma 4. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i) $n^{-1/2}G'_{\kappa}u \Rightarrow \mathcal{Z}_2$, where $\mathbb{E}[\mathcal{Z}_2] = 0$ and $\mathbb{E}[\mathcal{Z}_2\mathcal{Z}_2'] = \mathbb{E}[u_t^2 g_{t,\kappa}g'_{t,\kappa}]$; (ii) $n^{-1}G'_{\kappa}G_{\kappa} \stackrel{\text{a.s.}}{\to} A_2$, where $A_2 := \mathbb{E}[g_{t,\kappa}g'_{t,\kappa}]$; and (iii) if, additionally, Assumption 3(iii) holds, $\mathbb{E}[u_t^2 g_{t,\kappa}g'_{t,\kappa}] = \sigma_*^2\mathbb{E}[g_{t,\kappa}g'_{t,\kappa}]$.* □

Using Lemma 4, Theorem 2 describes the limit distribution of $QLR_n^{(2)}$ under $\mathcal{H}_{02}$:

Theorem 2. *Given Assumptions 1, 2, 3(i), 4, 7, 8, and $\mathcal{H}_{02}$, (i) $QLR_n^{(2)} \Rightarrow \max_{\theta\in\Theta}\mathcal{G}_2^2(\theta)$ if κ is odd; and if κ is even, $QLR_n^{(2)} \Rightarrow \max_{\theta\in\Theta}\max[0,\mathcal{G}_2(\theta)]^2$, where $\mathcal{G}_2(\cdot)$ is a univariate Gaussian process such that for each θ, $\mathbb{E}[\mathcal{G}_2(\theta)] = 0$ and $\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = A_2^{-1/2}(\theta,\theta)B_2(\theta,\widetilde{\theta})A_2^{-1/2}(\widetilde{\theta},\widetilde{\theta})$, where $B_2(\theta,\widetilde{\theta}) := \theta'\mathbb{E}[u_t^2 g_{t,\kappa} g'_{t,\kappa}]\widetilde{\theta}$ and $A_2(\theta,\widetilde{\theta}) := \sigma_*^2\theta'\mathbb{E}[g_{t,\kappa}\ g'_{t,\kappa}]\widetilde{\theta}$; (ii) if, additionally, Assumption 3(iii) holds, $\mathbb{E}[\mathcal{G}_2(\theta)\mathcal{G}_2(\widetilde{\theta})] = A_2^{-1/2}(\theta,\theta)A_2(\theta,\widetilde{\theta})A_2^{-1/2}(\widetilde{\theta},\widetilde{\theta})$.* □

As Theorem 2 follows from Lemma 4 and continuous mapping, its proof is omitted.

Several remarks are in order. First, the covariance kernel of $\mathcal{G}_2(\cdot)$ is bilinear with respect to θ and $\widetilde{\theta}$. This implies that $\mathcal{G}_2(\theta)$ is a linear Gaussian process with respect to θ. Therefore, if $z \sim N(0,\mathbb{E}[u_t^2 g_{t,\kappa}g'_{t,\kappa}])$, $z'\theta$ as a function of θ is distributionally equivalent to $\mathcal{G}_2(\cdot)$. This fact relates the null limit distribution to the chi-squared distribution. Corollary 1 of Cho and White (2018) shows that $\max_{\theta\in\Theta}\mathcal{G}_2^2(\theta) \stackrel{\text{d}}{=} \mathcal{X}_{p+1}^2$ if $\mathcal{G}_2(\cdot)$ is a linear Gaussian process and $\mathbb{E}[u_t^2 g_{t,\kappa}g'_{t,\kappa}] = \sigma_*^2\mathbb{E}[g_{t,\kappa}g'_{t,\kappa}]$, where $\mathcal{X}_{p+1}^2$ is a chi-squared distribution with $p+1$ degrees of freedom. Second, the chi-squared null limit distributions of the LM test statistics in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994), and Granger and Teräsvirta (1993) follow from the fact that the LM test statistic is equivalent to the QLR test statistic under $\mathcal{H}_{02}$. Finally, if $\theta = 0$, $\mathcal{G}_2(\theta)$ is not well defined as the weak limit in Theorem 2 is obtained by assuming that $\theta \neq 0$. Nevertheless, the null limit distribution of the QLR test is well represented by Theorem 2 as it obtains the alternative model by letting $\theta \neq 0$.

2.4.3 Limit Distribution of the QLR Test Statistic under $\mathcal{H}_0$

In this subsection, we derive the limit distribution of the QLR test under $\mathcal{H}_0$ by examining the relationship between $QLR_n^{(1)}$ and $QLR_n^{(2)}$. Specifically, using the arguments similar to those of Cho, Ishida, and White (2011, 2014), we show that $QLR_n^{(1)} \geq QLR_n^{(2)}$, which means the limit distribution under $\mathcal{H}_0$ equals that of $QLR_n^{(1)}$.

The following lemma generalises the approach in Cho, Ishida, and White (2011, 2014):

Lemma 5. *Let $n(\gamma) := Z'F(\gamma)Mu$ and $D(\gamma) := Z'F(\gamma)MF(\gamma)Z'$ with $n^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)n(\gamma)$, and $D^{(j)}(\gamma) := (\partial^j/\partial\gamma^j)D(\gamma)$. Under Assumptions 1, 2 and 3, (i) for $j < \kappa$, $\lim_{\gamma\to 0} n^{(j)}(\gamma) \stackrel{\text{a.s.}}{=} 0$ and*

$\lim_{\gamma\to 0} D^{(j)}(\gamma) \overset{\text{a.s.}}{=} 0$; *(ii)* $\lim_{\gamma\to 0} n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G_\kappa' u$; *and (iii)* $\lim_{\gamma\to 0} D^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} G_\kappa' G_\kappa$. □

The limit obtained by letting $\gamma \to 0$ under $\mathcal{H}_{01}$ can be compared with that obtained under $\mathcal{H}_{02}$. More specifically, using Lemma 5 and L'Hôpital's rule, we obtain that $\lim_{\gamma\to 0} n(\gamma)' D(\gamma)^{-1} n(\gamma) \overset{\text{a.s.}}{=} \lim_{\gamma\to 0} n^{(\kappa)}(\gamma)' D^{(\kappa)}(\gamma)^{-1} n^{(\kappa)}(\gamma) \overset{\text{a.s.}}{=} u' G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' u$. From this, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta)$ as

$$QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} n(\gamma)' D(\gamma)^{-1} n(\gamma) \geq \lim_{\gamma\to 0} \frac{1}{\widehat{\sigma}_{n,0}^2} n(\gamma)' D(\gamma)^{-1} n(\gamma) \overset{\text{a.s.}}{=} \frac{1}{\widehat{\sigma}_{n,0}^2} u' G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' u.$$

Furthermore, $\overline{QLR}_n^{(2)}(\theta)$ is asymptotically equal to $\widehat{\sigma}_{n,0}^{-2} u' G_\kappa \theta \ (\theta' G_\kappa' G_\kappa \theta)^{-1} \ \theta' G_\kappa' u$. Thus, it follows that $QLR_n^{(1)} \geq \sup_\theta \overline{QLR}_n^{(2)}(\theta) + o_{\mathbb{P}}(1)$, if $G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa' - G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa'$ is positive semidefinite irrespective of θ. To show this, we first note that the two terms are idempotent and symmetric matrices, and make use of Exercise 8.58 in Abadir and Magnus (2005, p. 233). Then, $\{G_\kappa (G_\kappa' G_\kappa)^{-1} G_\kappa'\}\{G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa'\} = G_\kappa \theta (\theta' G_\kappa' G_\kappa \theta)^{-1} \theta' G_\kappa'$, so that it is positive semidefinite. This implies

$$QLR_n = \max[QLR_n^{(1)}, QLR_n^{(2)}] + o_{\mathbb{P}}(1) = \max[QLR_n^{(1)}, \sup_\theta \overline{QLR}_n^{(2)}(\theta)] + o_{\mathbb{P}}(1) = QLR_n^{(1)} + o_{\mathbb{P}}(1).$$

Given that $\Gamma(\epsilon)$ was considered in Theorem 1 to remove $\gamma = 0$ from Γ, if we select ϵ as small as possible to have $QLR_n = QLR_n(\epsilon) + o_{\mathbb{P}}(1)$ so that we can let $\gamma \to 0$ as posited in Lemma 5, it is now straightforward to show that the null limit distribution of the QLR test is characterised by the Gaussian process in Theorem 1. That is, under the conditions in Theorems 1 and 2, the null limit distribution of the QLR test statistic is obtained by combining Theorems 1 and 2. For this purpose, we first combine Assumptions 6 and 8 into a new assumption:

Assumption 9. *For each $\gamma \neq 0$, $V_5(\gamma)$ and $V_6(\gamma)$ are positive definite, where $V_5(\gamma) := \mathbb{E}[u_t^2 \ddot{r}_t(\gamma) \ddot{r}_t(\gamma)']$, $V_6(\gamma) := \mathbb{E}[\ddot{r}_t(\gamma) \ddot{r}_t(\gamma)']$, and $\ddot{r}_t(\gamma) := (h_{t,\kappa}(0) z_t', f_t(\gamma) z_t', z_t')'$.* □

Next, we provide the limit distribution of the QLR test under $\mathcal{H}_0$ in the following theorem:

Theorem 3. *Given Assumptions 1, 2, 3(i), 4, 5, 7, 9, and $\mathcal{H}_0$, (i) $QLR_n \Rightarrow \sup_{\gamma\in\Gamma} \mathcal{G}_1(\gamma)' \mathcal{G}_1(\gamma)$, where $\mathcal{G}_1(\cdot)$ is a Gaussian process such that for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\mathcal{G}_1(\gamma)] = 0$ with $\mathbb{E}[\mathcal{G}_1(\gamma) \mathcal{G}_1(\widetilde{\gamma})'] = A_1^{-1/2}(\gamma,\gamma) B_1(\gamma,\widetilde{\gamma}) A_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma})$; (ii) if Assumption 3(ii) additionally holds, then $\mathbb{E}[\mathcal{G}_1(\gamma) \mathcal{G}_1(\widetilde{\gamma})'] = A_1(\gamma,\gamma)^{-1/2} A_1(\gamma,\widetilde{\gamma}) A_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma})$.* □

Theorem 3 immediately follows from Theorems 1 and 2 and from our earlier argument that $QLR_n = QLR_n^{(1)} + o_{\mathbb{P}}(1)$, which is why we do not prove it in the Supplement. Note that the consequence of Theorem 3 is the same as that of Theorem 1, although the null hypothesis is extended from $\mathcal{H}_{01}$ to $\mathcal{H}_0$ by enlarging the parameter space from $\Gamma(\epsilon)$ to Γ with sufficiently small ϵ. Also note that the Gaussian process $\mathcal{G}_1(\cdot)$ is obtained by supposing that $\gamma \neq 0$. Otherwise, a meaningful QLR test statistic is not properly defined.

The null limit distribution in Theorem 3 is derived as in Cho, Ishida, and White (2011, 2014) and Baek, Cho, and Phillips (2015). Nevertheless, our proofs generalise theirs due to the complexity associated with the STAR model. Furthermore, from Section 2.2, it can be easily seen that the LM test statistics only test $\mathcal{H}_{02}$. In addition, it also extends the Wald test principle exploited by Cheng (2015) to test $\mathcal{H}_{01}$ but not $\mathcal{H}_{02}$. The QLR statistic tests the linear model hypothesis by combining the null hypotheses neglected by the LM and Wald tests separately.

Before closing this section, we remark on the simulations of the QLR test. The Supplement applies our theory to the ESTAR and LSTAR models and reports results on Monte Carlo simulations. Specifically, particular data conditions are assumed for the errors, and using this we derive the covariance kernel in Theorem 3 analytically, enabling us to represent the null limit distribution as an infinite sum of functions of γ multiplied by Gaussian random coefficients for both models. Thorough this, we compare the null limit distribution with the empirical null distributions of the QLR tests obtained under various model environments, to affirm that Theorem 3 is valid. In addition to this, we also demonstrate the use of Hansen's (1996) weighted bootstrap to the QLR test and show that it can be usefully exploited when the covariance kernel of the Gaussian kernel in Theorem 3 is not available for the empirical researcher.

3 Two Empirical Examples

In this section, we apply the QLR tests against the ESTAR and LSTAR models, denoted by QLR_n^E and QLR_n^L, to two empirical examples. These are compared with the LM tests against the four alternative hypotheses outlined in Section 2.2.

3.1 Testing linearity of the Fiscal Multiplier Effect

We first test the hypothesis of nonlinear government spending effect to other macro economic variables by revisiting the empirical work of Auerbach and Gorodnichenko (2012) who specified a VSTAR model for $y_t := (g_t, \tau_t, q_t)'$ with g_t, τ_t, and q_t being log real government spending, log real government net tax receipts, and log real GDP deflated by the 2012 GDP deflator, respectively.

Because the time series used by Auerbach and Gorodnichenko (2012) are not stationary, we cannot apply our tests to their VSTAR model (they do not test linearity). Following Candelon and Lieb (2013), we bypass this difficulty by first converting this model into the following VSTEC form:

$$\Delta y_t = \Psi_{1*}(L)w_{t-1} + f_t(\gamma_*)\Psi_{2*}(L)w_{t-1} + u_t, \tag{5}$$

where $u_t = (u_{1t}, u_{2t}, u_{3t})'$, $w_{t-1} := [y^{*\prime}_{t-1}, \Delta y'_{t-1}]'$, $\Psi_{1*}(L) := [\alpha_{R*}, \widetilde{\Pi}_{R*}(L)]$, and $\Psi_{2*}(L) := [\alpha_{D*}, \widetilde{\Pi}_{D*}(L)]$ with $\alpha_{D*} := \alpha_{R*} - \alpha_{E*}$ and $\widetilde{\Pi}_{D*}(L) := \widetilde{\Pi}_{R*}(L) - \widetilde{\Pi}_{E*}(L)$. Here, $\widetilde{\Pi}_{R*}(L)$ and $\widetilde{\Pi}_{E*}(L)$ are the VSTEC coefficients associated with the recession and expansion periods, respectively; β_* is the $m-$dimensional cointegrating vector, which is invariance to the economic state; and $y^*_{t-1} = \beta'_* y_{t-1}$ (*e.g.*, Rothman, van Dijk, and Franses, 2001; Hubrich and Teräsvirta, 2013). In addition, α_* denotes the adjustment coefficient, which is a $3 \times m$ matrix, and it is assumed that $\alpha_* = \alpha_* = (1 - f_t(\gamma_*))\alpha_{E*} + f_t(\gamma_*)\alpha_{R*}$, where α_{E*} is not necessarily equal to α_{R*}.[2]

We now test the nonlinear effect of government spending in the following order. First, we marginalise the model (5) under the normality condition of u_t as assumed by Auerbach and Gorodnichenko (2012). That is,

$$\Delta y_{jt} = \theta'_{j*}\Delta y_{-jt} + \xi_{j1*}(L)'w_{t-1} + f_t(\gamma_*)\xi_{j2*}(L)'w_{t-1} + \epsilon_{jt}, \tag{6}$$

for $j = 1, 2, 3$, where $\theta'_{j*} := \mathbb{E}[u_{jt}u_{-jt}]\mathbb{E}[u_{-jt}u'_{-jt}]^{-1}$, and, further, $\xi_{j1*}(L)' := \psi_{j1*}(L)' - \theta'_{j*}\psi_{-j1*}(L)'$ and $\xi_{j2*}(L)' := \psi_{j2*}(L)' - \theta'_{j*}\psi_{-j2*}(L)'$. Here, u_{jt} and u_{-jt} denote the j-th row element of u_t and the 2×1 vector obtained by removing u_{jt} from u_t, respectively. Furthermore, for each $i = 1$ and 2, $\psi_{ji*}(L)'$ and $\psi_{-ji*}(L)'$ are the j-th row vector of $\Psi_{i*}(L)$ and $2 \times (m+3)$ matrix obtained by removing the j-th row from $\Psi_{i*}(L)$, respectively. Second, we estimate the model. For this, we first let $\widehat{\beta}_n$ denote the maximum likelihood estimator for β_* estimated from (5), which is super-consistent (see Johansen, 1995), making it possible to estimate the other parameters by NLS by replacing y^*_{t-1} with $\widehat{y}_{t-1} := \widehat{\beta}'_n y_{t-1}$. Finally, we use the marginal model of (6) as our baseline model for testing for nonlinearity, where w_{t-1} is replaced with $\widehat{w}_{t-1} := [\widehat{y}'_{t-1}, \Delta y'_{t-1}]'$. We apply the QLR and LM tests for each $j = 1, 2, 3$. Note that rejecting the linearity hypothesis in at least one individual equation is sufficient for rejecting the linearity hypothesis of the whole VSTEC equation.

We now report our empirical findings using the data of Auerbach and Gorodnichenko (2012), which is

[2] The model above is similar to the one considered in Candelon and Lieb (2013), but our model is different from theirs as our model assumes a continuum of states and is not restricted to have $\alpha_{E*} = \alpha_{R*}$. In addition, we test the linearity hypothesis under the assumption of conditional heteroscedasticity, without imposing a restriction on the error covariance matrix.

comprised US quarterly macroeconomic variables. Their sample ranges from 1947Q1 to 2008Q4. In order to estimate the cointegration rank m, we apply Johansen's (1988, 1991) trace testing procedure with lag equal to 3 selected by AIC and BIC[3] and cannot reject the hypothesis that $m = 2$ at the 5% significance level. Using this rank, we estimate the cointegration coefficient β_* to obtain $\widehat{y}_{t-1}$. Next, we apply the diagnostic testing procedure to validate the assumption on the nonlinearity. As explained above, we replace w_{t-1} in the marginal model with $\widehat{w}_{t-1} := [\widehat{y}'_{t-1}, \Delta y'_{t-1}]'$ and test the linearity hypothesis by the QLR tests based upon ESTAR and LSTAR models. We report the p-values of the QLR and LM tests in Table 1, where the p-values for our tests are obtained by applying Hansen's (1996) weighted bootstrap with 20,000 replications. In general, both the QLR and LM tests strongly reject the linearity hypothesis. However, when the dependent variable is given by g_t, the estimated p-value of $LM_{3,n}$ is far above the significance level. A similar result is found when we apply $LM_{4,n}$ to the model whose dependent variable is given by τ_t. Although the latter could be caused by specifying the exponential transition function under the alternative, this is in contrast with the testing results of QLR statistic that reports substantially small p-values irrespective of the dependent variables. This suggests, at least in this particular case, the QLR test would be more robust, and both tests could be used complementary to each other. In general, the linearity testing results imply that the linear error-correction model is not adequate and the VSTEC model can better capture the dynamic interrelationship among the variables.

3.2 Application to US Unemployment Rates

We now examine the performance of the tests when the QLR test is applied to the monthly US unemployment rate. van Dijk, Teräsvirta, and Franses (2002) tested linearity of the series running from June 1968 to December 1999. We perform the tests both with their time series and the same series extended to August 2015.[4]

van Dijk, Teräsvirta, and Franses (2002) point out that the US unemployment rate is a persistent series with an asymmetric adjustment process and strong seasonality. They specify a STAR model with monthly dummy variables mainly because first differences of the seasonally unadjusted unemployment rate of males aged 20 and over is used for Δy_t. They test linearity against STAR assuming that the transition variable is a lagged twelve-month difference of the unemployment rate. The alternative (STAR) model has the following form (the

[3] The lag order is also identical to that selected by Auerbach and Gorodnichenko (2012). To test serial correlation in the errors of the specified VSTEC model, we applied the multivariate Ljung-Box test and failed to reject the null of no serial correlation at 5% level of significance. Testing results are available from the authors upon request.

[4] The data set used by van Dijk, Teräsvirta, and Franses (2002) is available at `<http://swopec.hhs.se/hastef/abs/hastef0380.htm>` that was originally retrieved from the Bureau of Labor Statistics.

lag length has been determined by AIC):

$$\Delta y_t = \pi_0 + \pi_1 y_{t-1} + \sum_{p=1}^{15} \pi_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \pi_{17+k} d_{t,k}$$
$$+ \left[\theta_0 + \theta_1 y_{t-1} + \sum_{p=1}^{15} \theta_{p+2} \Delta y_{t-p} + \sum_{k=1}^{11} \theta_{17+k} d_{t,k} \right] f(\Delta_{12} y_{t-d}, \gamma) + u_t,$$

where y_t is the monthly US unemployment rate in question; Δy_t is the first difference of y_t; $f(\cdot, \cdot)$ is a nonlinear transition function; $\Delta_{12} y_t$ is the twelve-month difference of y_t; $d_{t,k}$ is the dummy for month k; and $u_t \sim$ IID $(0, \sigma^2)$. The twelve-month difference $\Delta_{12} y_{t-d}$ is not included as an explanatory variable in the null (linear) model. The theory in Section 2 can nonetheless be used without modification as a null model including $\Delta_{12} y_{t-d}$ can be thought of having a zero coefficient for this variable. Following van Dijk, Teräsvirta, and Franses (2002), we test linearity by letting $\Delta_{12} y_{t-d}$, $d = 1, 2, \ldots, 6$, be the transition variable.

Our test results using the same series as van Dijk, Teräsvirta, and Franses (2002) are reported in the top panel of Table 2. Both the LM tests and QLR_n^L reject linearity when $d = 2$, and, besides, $LM_{3,n}$ that has power against LSTAR yields $p = 0.037$ for $d = 2$. The p-values of QLR_n^L, however, lie at or below 0.05 for all six lags, suggesting that at least in this particular case this QLR test is more powerful than the LM tests. The smallest p-value is even here attained for $d = 2$. The results from QLR_n^E are quite different in that they reject the null only for $d = 1, 2$, but not for other lags. This makes sense as this statistic is designed for ESTAR , and asymmetry in the unemployment rate is best described by an LSTAR model.

The bottom panel of Table 2 contains the results from the series extended to August 2015.[5] Now there seems to be plenty of evidence of asymmetry: all p-values of $LM_{1,n}$ are rather small. $LM_{3,n}$ also has small values for the first three lags, as has $LM_{2,n}$. The p-values from QLR_n^L are smallest of all, which is in line with the results in the top panel. Even QLR_n^E rejects the null of linearity at the 5% level for $d = 1, 2, 3, 4, 5$. This outcome may be expected as the QLR statistics are omnibus tests and as such respond to any deviation from the null hypothesis as the sample size increases. Note, however, that even $LM_{4,n}$ now yields two p-values $(d = 2, 3)$ that lie below 0.05, although the test does not have the omnibus property. The behaviour of the unemployment rate during and after the financial crisis (a quick upswing and slow decrease) has probably contributed to these results.

[5]The recent observations of the monthly US unemployment rate are available at <`http://beta.bls.gov/dataViewer/view/timeseries/LNU04000025`>.

4 Conclusion

The current study examines the null limit distribution of the QLR test statistic for neglected nonlinearity using the STAR model. The QLR test statistic contains a twofold identification problem under the null, and we explicitly examine how the twofold identification problem affects the null limit distribution of the QLR test statistic. We show that the QLR test statistic is shown to converge to a functional of a multivariate Gaussian process under the null of linearity by extending the testing scope of the LM test statistic in Luukkonen, Saikkonen, and Teräsvirta (1988), Teräsvirta (1994) and Granger and Teräsvirta (1993).

Finally, two empirical examples are revisited to demonstrate use of the QLR test statistic. We test for neglected nonlinearity in the multiplier effect of US government spending and the growth rates of US unemployment using the QLR test statistic by revisiting the empirical data examined by Auerbach and Gordonichenko (2012) and van Dijk, Teräsvirta, and Franses (2002), respectively. Through these examinations, the QLR test statistic turns out useful for detecting the nonlinear structure among the economic variables and complements the Lagrange multiplier test statistic in Teräsvirta (1994).

References

ABADIR, K. AND MAGNUS, J.R. (2005): *Matrix Algebra*. NY: Cambridge University Press.

AUERBACH, A. J. AND GORODNICHENKO, Y. (2012): "Measuring the Output Responses to Fiscal Policy," *American Economic Journal: Economic Policy*, 4, 1–27.

ANDREWS, D.W. AND CHENG, X. (2014): "GMM Estimation and Uniform Subvector Inference with Possible Identification Failure," *Econometric Theory*, 287–333.

BAEK, Y.I., CHO, J.S., AND PHILLIPS, P.C.B. (2015): "Testing Linearity Using Power Transforms of Regressors," *Journal of Econometrics*, 187, 376–384.

CANDELON, B. AND LIEB, L. (2013): "Fiscal Policy in Good and Bad Times," *Journal of Economic Dynamics and Control*, 37, 2679–2694.

CHENG, X. (2015): "Robust Inference in Nonlinear Models with Mixed Identification Strength," *Journal of Econometrics*, 189, 207–228.

CHO, J.S. AND ISHIDA, I. (2012): "Testing for the Effects of Omitted Power Transformations," *Economics Letters*, 117, 287–290.

CHO, J.S. ISHIDA, I., AND WHITE, H. (2011): "Revisiting Tests for Neglected Nonlinearity Using Artificial Neural Network," *Neural Computation*, 23, 1133–1186.

CHO, J.S. ISHIDA, I., AND WHITE, H. (2014): "Testing for Neglected Nonlinearity Using Twofold Unidentified Models under the Null and Hexic Expansions," *Essays in Nonlinear Time Series Econometrics*. Eds. Niels Haldrup, Mika Meitz, and Pentti Saikkonen, 3–27. Oxford: Oxford University Press.

CHO, J.S. AND PHILLIPS, P.C.B. (2018): "Sequentially Testing Polynomial Model Hypotheses Using Power Transforms of Regressors," *Journal of Applied Econometrics*, 33, 141–159.

CHO, J.S. AND WHITE, H. (2018): "Directionally Differentiable Econometric Models," *Econometric Theory*, 34, 1101–1131.

DAVIES, R. (1977): "Hypothesis Testing When a Nuisance Parameter is Present only under the Alternative," *Biometrika*, 64, 247–254.

DAVIES, R. (1987): "Hypothesis Testing When a Nuisance Parameter is Present only under the Alternative," *Biometrika*, 74, 33–43.

DOUKHAN, P., MASSART, P., AND RIO, E. (1995): "Invariance Principles for Absolutely Regular Empirical Processes," *Annales de l'Institut Henri Poincaré, Probabilites et Statistiques*, 31 393–427.

GRANGER, C. W. J., AND TERÄSVIRTA, T. (1993): *Modelling Non-Linear Economic Relationships* . Oxford: Oxford University Press.

HAGGAN, V. AND OZAKI, T. (1981): "Modelling Non-Linear Random Vibrations Using an Amplitude-Dependent Time Series Model," *Biometrika*, 68, 189–196.

HANSEN, B. (1996): "Inference When a Nuisance Parameter is Not Identified under the Null Hypothesis," *Econometrica*, 64, 413–430.

HUBRICH, K. AND TERÄ SVIRTA, T. (2013): "Thresholds and Smooth Transitions in Vector Autoregressive Models," *VAR Models in Macroeconomics–New Developments and Applications: Essays in Honor of*

Christopher A. Sims. Eds. Thomas B. Fomby, Lutz Kilian, and Anthony Murphy. *Advances in Econometrics*. Vol. 32, 273–326. Bingley, U.K.: Emerald.

JOHANSEN, S. (1988): "Statistical Analysis of Cointegration Vectors," *Journal of Economic Dynamics and Control*, 12, 231–254.

JOHANSEN, S. (1991): "Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models," *Econometrica*, 59, 1551–1580.

JOHANSEN, S. (1995): *Likelihood-based Inference in Cointegrated Vector Autoregressive Models* . Oxford, U.K.: Oxford University Press.

LUUKKONEN, R., SAIKKONEN, P., AND TERÄSVIRTA, T. (1988): "Testing Linearity Against Transition Autoregressive Models," *Biometrika*, 75, 491–499.

ROTHMAN, P., VAN DIJIK, D., AND FRANSES, P.H. (2001): "Multivariate STAR Analysis of Money-Output Relationship," *Macroeconomic Dynamics*, 5, 506–532.

STINCHCOMBE, M. AND WHITE, H. (1998): "Consistent Specification Testing with Nuisance Parameters Present Only under the Alternative," *Econometric Theory*, 14, 295–324.

TERÄSVIRTA, T. (1994): "Specification, Estimation, and Evaluation of Smooth Transition Autoregressive Models," *Journal of the American Statistical Association*, 89, 208–218.

TERÄSVIRTA, T., TJØSTHEIM, D., AND GRANGER, C.W.J. (2010): *Modelling Nonlinear Economic Time Series*. Oxford: Oxford University Press.

WATSON, M. AND ENGLE, E. (1982): "Tesintg for a Regression Coefficient Stability with a Stationary AR(1) Alternative," *Review of Economics and Statistics*, 67, 341–346.

WHITE, H. AND CHO, J.S. (2012): "Higher-Order Approximations for Testing Neglected Nonlinearity," *Neural Computation*, 24, 273–287.

Linearity tests \ variables	Δg_t	$\Delta \tau_t$	Δq_t
QLR_n^L	**0.000**	**0.004**	**0.000**
QLR_n^E	**0.000**	**0.000**	**0.009**
$LM_{1,n}$	**0.000**	**0.000**	**0.000**
$LM_{2,n}$	**0.000**	**0.000**	**0.000**
$LM_{3,n}$	0.530	**0.000**	**0.000**
$LM_{4,n}$	**0.035**	**0.000**	0.122

Table 1: p-values of the Diagnostic Test Statistics. Notes: The p-values of the linearity tests for the VSTEC model is reported. The figures in QLR_n row show the p-values of the QLR test statistics for linearity based upon the LSTAR VSTEC model, and they are obtained using 20,000 bootstrap replications. The variables in the first row denote the dependent variables in the marginal models. Boldface p-values indicate significance levels less than or equal to 0.05.

Periods	Transition Variable	$LM_{1,n}$	$LM_{2,n}$	$LM_{3,n}$	$LM_{4,n}$	QLR_n^L	QLR_n^E
1968.06∼1999.12	$\Delta_{12} y_{t-1}$	0.150	0.532	0.412	0.895	**0.000**	**0.045**
	$\Delta_{12} y_{t-2}$	**0.037**	0.093	0.057	0.195	**0.000**	**0.028**
	$\Delta_{12} y_{t-3}$	0.162	0.326	0.163	0.555	**0.012**	0.054
	$\Delta_{12} y_{t-4}$	0.665	0.745	0.546	0.619	**0.014**	0.098
	$\Delta_{12} y_{t-5}$	0.662	0.886	0.954	0.830	**0.003**	0.099
	$\Delta_{12} y_{t-6}$	0.588	0.306	0.121	0.234	**0.003**	0.157
1968.06∼2015.08	$\Delta_{12} y_{t-1}$	**0.000**	**0.000**	**0.000**	0.098	**0.000**	**0.000**
	$\Delta_{12} y_{t-2}$	**0.000**	**0.000**	**0.000**	**0.016**	**0.000**	**0.000**
	$\Delta_{12} y_{t-3}$	**0.001**	**0.000**	**0.008**	**0.045**	**0.000**	**0.014**
	$\Delta_{12} y_{t-4}$	**0.008**	**0.012**	0.070	0.111	**0.000**	**0.009**
	$\Delta_{12} y_{t-5}$	**0.038**	0.237	0.274	0.861	**0.000**	**0.049**
	$\Delta_{12} y_{t-6}$	**0.003**	0.068	**0.017**	0.582	**0.000**	0.350

Table 2: Linearity Tests for the Monthly US Unemployment Rate. Notes: The p-values of the linearity tests for the first differenced monthly US unemployment rate are provided. The p-values in the top panel are obtained using observations from 1968.06 to 1999.12, and the p-values of the bottom panel are obtained using observations from 1968.06 to 2015.08. The null linear model is given as AR(15) by AIC, and the twelve-month differences are considered as a transition variable. Boldface p-values indicate significance levels less than or equal to 0.05.

A Supplement

In this Supplement we examine testing linearity against commonly applied STAR models and also provides simulation evidence of our methodology. We also demonstrate how Hansen's (1996) weighted bootstrap is applied to enhance the applicability of our methodology. Finally, we provide the proofs of the theoretical results in the paper

A.1 Monte Carlo Experiments and Application of the Weighted Bootstrap

A.1.1 Monte Carlo Experiments Using the ESTAR Model

To simplify our illustration, we assume that for all $t = 1, 2, \ldots$, $u_t \sim$ IID $N(0, \sigma_*^2)$ and $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$. Under this DGP, we specify the following first-order ESTAR model: $\mathcal{M}_{ESTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 - \exp[-\gamma y_{t-1}^2]\} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma\}$. The model does not contain an intercept, and the transition variable is y_{t-1}. The nonlinear function $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ is defined on Γ which is compact and convex, and the exponential function is analytic. This means that the QLR test statistic is generically comprehensively revealing. To identify the model it is assumed that $\gamma_* > 0$. In our model set-up, we allow 0 to be included in Γ. The nonlinear function $f_t(\cdot)$ satisfies $f_t(0) = 0$. Given this model, the following hypotheses are of interest: $\mathcal{H}_0' : \exists \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) = 1$; vs. $\mathcal{H}_1' : \forall \pi \in \mathbb{R}, \mathbb{P}(\mathbb{E}[y_t|y_{t-1}] = \pi y_{t-1}) < 1$. Two parameter restrictions make $\mathcal{H}_0'$ valid: either $\theta_* = 0$ or $\gamma_* = 0$. The sub-hypotheses are thus $\mathcal{H}_{01}' : \theta_* = 0$ and $\mathcal{H}_{02}' : \gamma_* = 0$.

We first examine the null distribution of the QLR test under $\mathcal{H}_{01}'$. By Theorem 1, the null limit distribution of this test statistic is given as $QLR_n^{(1)} = \sup_{\gamma \in \Gamma} \frac{1}{\widehat{\sigma}_{n,0}^2} (u'MF(\gamma)Z)^2 / \ Z'F(\gamma)MF(\gamma)Z \Rightarrow \sup_{\gamma \in \Gamma} \widetilde{\mathcal{G}}_1^2(\gamma)$, where $\widetilde{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure $\widetilde{\rho}_1(\gamma, \widetilde{\gamma}) = c_1^{-1/2}(\gamma, \gamma)\widetilde{k}_1(\gamma, \widetilde{\gamma})$ $\widetilde{c}_1^{-1/2}(\widetilde{\gamma}, \widetilde{\gamma})$ with $\widetilde{k}_1(\gamma, \widetilde{\gamma}) = \widetilde{c}_1(\gamma, \widetilde{\gamma}) = \sigma_*^2\{\mathbb{E}[y_t^2 \exp(-(\gamma + \widetilde{\gamma})y_t^2)] - \mathbb{E}[y_t^2 \exp(-\gamma y_t^2)]\mathbb{E}[y_t^2 \exp(-\widetilde{\gamma} y_t^2)] / \mathbb{E}[y_t^2]\}$. Furthermore, under $\mathcal{H}_{01}'$, y_t is normally distributed with $\mathbb{E}[y_t] = 0$ and $\text{var}[y_t] = \sigma_y^2 := \sigma_*^2/(1 - \pi_*^2)$, so that y_t^2 follows the gamma distribution with shape parameter $1/2$ and scale parameter $2\sigma_*^2/(1 - \pi_*^2)$. Define $\widetilde{m}(\gamma) := (1 + 2\sigma_*^2/1 - \pi_*^2\gamma)^{-\frac{1}{2}}$, and $\widetilde{h}(\gamma, \widetilde{\gamma}) := \frac{1}{\sigma_y^2}([(1 + 2\sigma_y^2\gamma)(1 + 2\sigma_y^2\widetilde{\gamma})/\{1 + 2\sigma_y^2(\gamma + \widetilde{\gamma})\}]^{3/2} - 1)$. Note that $\widetilde{m}(\gamma) = \mathbb{E}[\exp(-\gamma y_t^2)]$, so that $\mathbb{E}[y_t^2 \exp(-\gamma y_t^2)] = -\widetilde{m}'(\gamma)$. As a result, $\widetilde{\rho}_1(\gamma, \widetilde{\gamma})$ is further simplified to $\widetilde{k}_1(\gamma, \widetilde{\gamma}) = \sigma_*^2 \widetilde{m}'(\gamma)\widetilde{m}'(\widetilde{\gamma})\widetilde{h}(\gamma, \widetilde{\gamma})$, and $\widetilde{\rho}_1(\gamma, \widetilde{\gamma}) = \widetilde{c}_1^{-1/2}(\gamma, \gamma)\widetilde{k}_1(\gamma, \widetilde{\gamma})\widetilde{c}_1^{-1/2}(\widetilde{\gamma}, \widetilde{\gamma}) = \widetilde{h}^{-1/2}(\gamma, \gamma)\widetilde{h}(\gamma, \widetilde{\gamma})\widetilde{h}^{-1/2}(\widetilde{\gamma}, \widetilde{\gamma})$.

We next examine the limit distribution of the QLR test statistic under $\mathcal{H}_{02}'$: $\gamma_* = 0$. The first-order derivative $(\partial/\partial\gamma) f_t(\gamma) = y_{t-1}^2 \exp(-\gamma y_{t-1}^2)$, which is different from zero even when $\gamma = 0$, so that in this case $\kappa = 1$. Thus, we can apply the second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$. As a result, $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}_{n,0}^2}(\theta' G'_\kappa u)^2/\theta' G'_\kappa G_\kappa \theta$, where $\theta' G'_\kappa u = \theta[\sum y_{t-1}^3 u_t - \sum y_{t-1}^4 \sum y_{t-1} u_t / \sum y_{t-1}^2]$ and $\theta' G'_\kappa G_\kappa \theta = \theta^2[\sum y_{t-1}^6 - (\sum y_{t-1}^4)^2/\sum y_{t-1}^2]$. Here, θ is a scalar, so that cancels out, and it follows that $QLR_n^{(2)} \Rightarrow \widetilde{\mathcal{G}}_2^2$, where $\widetilde{\mathcal{G}}_2 \sim N(0,1)$.

These two separate results can be combined, which means that we can examine the limit distribution of the QLR test under $\mathcal{H}'_0$. We have $QLR_n \Rightarrow \sup_{\gamma\in\Gamma} \widetilde{\mathcal{G}}^2(\gamma)$, where $\widetilde{\mathcal{G}}(\gamma) = \widetilde{\mathcal{G}}_1(\gamma)$, if $\gamma \neq 0$; and $\widetilde{\mathcal{G}}(\gamma) = \widetilde{\mathcal{G}}_2$, otherwise, and $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_1(\gamma,\widetilde{\gamma})$, if $\gamma \neq 0, \widetilde{\gamma} \neq 0$; $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = 1$, if $\gamma = 0, \widetilde{\gamma} = 0$; and $\mathbb{E}[\widetilde{\mathcal{G}}(\gamma)\widetilde{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_3(\gamma)$, if $\gamma \neq 0, \widetilde{\gamma} = 0$ with $\widetilde{\rho}_3(\gamma) := \mathbb{E}[\widetilde{\mathcal{G}}_1(\gamma)\widetilde{\mathcal{G}}_2] = \sqrt{6}\sigma_y^2\gamma/\{\widetilde{h}^{1/2}(\gamma,\gamma)(1+2\sigma_y^2\gamma)\}$ such that $\widetilde{\rho}_3^2(\gamma) = \lim_{\widetilde{\gamma}\to 0} \widetilde{\rho}_1^2(\gamma,\widetilde{\gamma}) = (\sqrt{6}\sigma_y^2\gamma/\{\widetilde{h}^{1/2}(\gamma,\gamma)(1+2\sigma_y^2\gamma)\})^2$. Thus, we conclude that $QLR_n \Rightarrow \sup_\gamma \widetilde{\mathcal{G}}^2(\gamma)$, which agrees with Theorem 3.

The null limit distribution can be approximated numerically by simulating a distributionally equivalent Gaussian process. To do this we present the following lemma:

Lemma A. 1. *If $\{z_k : k = 0,1,2,\ldots\}$ is an IID sequence of standard normal random variables, $\widetilde{\mathcal{G}}(\cdot) \overset{\mathrm{d}}{=} \overline{\mathcal{G}}(\cdot)$, where for each $\gamma \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$, $\overline{\mathcal{G}}(\gamma) := \sum_{k=1}^\infty c(\gamma) a^k(\gamma)\ [(-1)^k \binom{-3/2}{k}]^{1/2} z_k$, $c(\gamma) := \{\sum_{k=1}^\infty (-1)^k a^{2k}(\gamma)\binom{-3/2}{k}\ \}^{-1/2}$, and $a(\gamma) := 2\sigma_y^2\gamma/(1+2\sigma_y^2\ \gamma)$.* □

Note that the term $(-1)^k\binom{-3/2}{k}$ in Lemma A. 1 is always positive irrespective of k, and for any γ,

$$\lim_{k\to\infty} \mathrm{var}\left[a^k(\gamma)\left((-1)^k\binom{-3/2}{k}\right)^{1/2} z_k\right] = \lim_{k\to\infty} a^{2k}(\gamma)(-1)^k\binom{-3/2}{k} = 0 \tag{A.1}$$

and $\widetilde{h}(\gamma,\gamma) = \sum_{k=1}^\infty a^{2k}(\gamma)(-1)^k\binom{-3/2}{k}$. Using these facts Lemma A. 1 shows that for any γ and $\widetilde{\gamma} \neq 0$, $\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}(\widetilde{\gamma})] = \widetilde{\rho}_1(\gamma,\widetilde{\gamma})$. Here, the non-negative parameter space condition for Γ is necessary for $\overline{\mathcal{G}}(\cdot)$ to be properly defined on Γ. Without this condition, $\overline{\mathcal{G}}(\gamma)$ cannot be properly generated. We note that $\lim_{\gamma\downarrow 0} \overline{\mathcal{G}}(\gamma) \overset{\text{a.s.}}{=} z_1$, so that if we let $z_1 = \overline{\mathcal{G}}_2$, $\mathbb{E}[\overline{\mathcal{G}}(\gamma)\overline{\mathcal{G}}_2] = \sqrt{6}\sigma_y^2\gamma\widetilde{h}^{-1/2}(\gamma,\gamma)(1+2\sigma_y^2\gamma)^{-1} = \widetilde{\rho}_3(\gamma)$. It follows that the distribution of $\widetilde{\mathcal{G}}(\cdot)$ can be simulated by iteratively generating $\overline{\mathcal{G}}(\cdot)$. In practice,

$$\overline{\mathcal{G}}(\gamma;K) := \sum_{k=1}^K a^k(\gamma)\left[(-1)^k\binom{-3/2}{k}\right]^{1/2} z_k \Bigg/ \sqrt{\sum_{k=1}^K a^{2k}(\gamma)(-1)^k\binom{-3/2}{k}}$$

is generated by choosing K to be sufficiently large . By (A.1), if this is the case, the difference between the distributions of $\overline{\mathcal{G}}(\cdot)$ and $\overline{\mathcal{G}}(\cdot;K)$ becomes negligible.

We now conduct Monte Carlo experiment and examine the empirical distributions of the QLR statistic under several different environments. First, we consider four different parameter spaces: $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, and $\Gamma_4 = [0, 5]$. They are selected to examine how the null limit distribution of the QLR test is influenced by the choice of Γ. We obtain the limit distribution by simulating $\sup_{\gamma\in\Gamma} \bar{\mathcal{G}}^2(\gamma; K)$ 5,000 times with $K = 2,000$, where Γ is in turn Γ_1, Γ_2, Γ_3, and Γ_4. Second, we study how the empirical distribution of the QLR test statistic changes with the sample size. We consider the sample sizes $n = 100$, $1,000$, $2,000$, and $5,000$.

Figure A.1 summarises the simulation results and shows that the empirical distribution approaches the null limit distribution under different parameter space conditions. We also provide the estimates of the probability density functions next to the empirical distributions. For every parameter space considered, the empirical rejection rates of the QLR test statistics are most accurate when $n = 2,000$. The empirical rejection rates are closer to the nominal levels when the parameter space is small. This result is significant when $n = 100$: the empirical rejection rates for $\Gamma = \Gamma_1$ are closer to the nominal ones than when $\Gamma = \Gamma_4$. Nonetheless, this difference becomes negligible as the sample size increases. The empirical rejection rates obtained using $n = 2,000$ are already satisfactorily close to the nominal levels, and this result is more or less similar to that from 5,000 observations. This suggests that the theory in Section 2 is effective for the ESTAR model. Considering even larger parameter spaces for γ yielded similar results, so they are not reported here.

A.1.2 Illustration Using the LSTAR Model

As another illustration, we consider testing against the first-order LSTAR model. We assume that the data-generating process is $y_t = \pi_* y_{t-1} + u_t$ with $\pi_* = 0.5$ and $u_t = \ell_t$ with probability $1 - \pi_*^2$; and $u_t = 0$ with probability π_*^2, where $\{\ell_t\}_{t=1}^n$ follows the Laplace distribution with mean 0 and variance 2. Under this assumption, y_t follows the same distribution as ℓ_t that makes the algebra associated with the LSTAR model straightforward. For example, the covariance kernel of the Gaussian process associated with the null limit distribution of the QLR test statistic is analytically obtained thanks to this distributional assumption. This data-generating process is a variation of the exponential autoregressive model in Lawrence and Lewis (1980). Their exponential distribution is replaced by the Laplace distribution to allow y_t to obtain negative values.

Given this DGP, the first-order LSTAR model for $\mathbb{E}[y_t|y_{t-1}, y_{t-2}, \ldots]$ is defined as follows: $\mathcal{M}^0_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{1 + \exp(-\gamma y_{t-1})\}^{-1} : \pi \in \Pi,\ \theta \in \Theta, \text{ and } \gamma \in \Gamma := [0, \bar{\gamma}]\}$. The nonlinear logistic function $\{1 + \exp(-\gamma y_{t-1})\}^{-1}$ contains an exponential function. It is therefore analytic, and this fact delivers a consistent power for the QLR test statistic. Note, however, that for $\gamma = 0$ the value of the logistic function equals 1/2. This difficulty is avoided by subtracting 1/2 from the logistic function when carrying out the test, viz., $\mathcal{M}_{LSTAR} := \{\pi y_{t-1} + \theta y_{t-1}\{[1+\exp(-\gamma y_{t-1})]^{-1} - 1/2\} : \pi \in \Pi,\ \theta \in \Theta,$ and $\gamma \in \Gamma := [0, \bar{\gamma}]\}$. By the invariance principle, this shift does not affect the null limit distribution of the QLR test statistic. We here let $\gamma \geq 0$ so that transition function is bounded, which modifies the limit space of ς_n into $\mathbb{R}^+$. The null and the alternative hypotheses are identical to those in the ESTAR case.

Before proceeding, note that $\{1+\exp(-\gamma y_{t-1})\}^{-1} - \frac{1}{2} = \frac{1}{2}\tanh\left(\frac{\gamma y_{t-1}}{2}\right)$. Using the hyperbolic tangent function as in Bacon and Watts (1971) makes it easy to find a Gaussian process that is distributionally equivalent to the Gaussian process obtained under the null.

Using this fact, the limit distribution of QLR test statistic under $\mathcal{H}'_{01}$ is derived as in before. By Theorem 1, $QLR_n^{(1)} = \sup_{\gamma\in\Gamma} \frac{1}{\widehat{\sigma}^2_{n,0}}(u'MF^2(\gamma)Z)/Z'F(\gamma)MF(\gamma)Z \Rightarrow \sup_{\gamma\in\Gamma} \ddot{\mathcal{G}}_1^2(\gamma)$, where $\ddot{\mathcal{G}}_1(\cdot)$ is a mean-zero Gaussian process with the covariance structure $\ddot{\rho}_1(\gamma, \widetilde{\gamma}) := \ddot{c}_1^{-1/2}(\gamma,\gamma)\ddot{k}_1(\gamma,\ \widetilde{\gamma})\ddot{c}_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma})$. The function $\ddot{k}_1(\gamma,\tilde{\gamma})$ is equivalent to $\ddot{c}_1(\gamma,\tilde{\gamma})$ by the conditional homoskedasticity condition, and for each nonzero γ and $\widetilde{\gamma}$, we now obtain that $\ddot{k}_1(\gamma,\tilde{\gamma}) = \frac{1}{4}\mathbb{E}[y_t^2 \tanh(\frac{\gamma y_t}{2})\tanh(\frac{\widetilde{\gamma} y_t}{2})] - \frac{1}{4}\mathbb{E}[y_t^2\tanh(\frac{\gamma y_t}{2})]\mathbb{E}[y_t^2\tanh(\frac{\widetilde{\gamma} y_t}{2})]/\,\mathbb{E}[y_t^2]$. In the proof of Lemma A. 2 given below, we further show that $\ddot{k}_1(\gamma,\widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n\,(\widetilde{\gamma})$, where $b_1(\gamma) := \frac{1}{\sqrt{2}}(1-2a(\gamma))$ with $a(\gamma) := \sum_{k=1}^{\infty}(-1)^{k-1}/(1+\gamma k)^3$ and for $n = 2, 3, \ldots$, $b_n(\gamma) := \sqrt{n(n+1)}\sum_{k=1}^{\infty}(-1)^{k-1}(\gamma k)^{n-1}/(1+\gamma k)^{n+2}$.

Next, we derive the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$. Note that for $\gamma = 0$, $(\partial/\partial\gamma)f_t(\gamma) = y_{t-1}\exp(\gamma y_{t-1})/[1+\exp(-\gamma y_{t-1})]^2 \neq 0$, implying that κ is unity as for the ESTAR case, so that we can apply a second-order Taylor expansion to obtain the limit distribution of the QLR test statistic under $\mathcal{H}'_{02}$: $\overline{QLR}_n^{(2)}(\theta) = \frac{1}{\widehat{\sigma}^2_{n,0}}(\theta' G'_\kappa u)^2/(\theta' G'_\kappa G_\kappa \theta)$, where, similarly to the ESTAR case, $\theta' G'_\kappa u = \frac{\theta}{4}[\sum y_{t-1}^2 u_t - \sum y_{t-1}^3 \sum y_{t-1}u_t/\sum y_{t-1}^2]$ and $\theta' G'_\kappa G_\kappa\theta = \frac{\theta^2}{16}[\sum y_{t-1}^4 - (\sum y_{t-1}^3)^2/\sum y_{t-1}^2]$. From this, it follows that $QLR_n^{(2)} \Rightarrow \ddot{\mathcal{G}}_2^2$, where $\ddot{\mathcal{G}}_2 \sim N(0,1)$.

Therefore, we conclude that $QLR_n \Rightarrow \sup_\gamma \ddot{\mathcal{G}}^2(\gamma)$, where $\ddot{\mathcal{G}}(\gamma) := \ddot{\mathcal{G}}_1(\gamma)$, if $\gamma \neq 0$; and $\ddot{\mathcal{G}}(\gamma) := \mathcal{G}_2$, otherwise. The limit variance of $\ddot{\mathcal{G}}(\gamma)$ is given as $\ddot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})]$ such that $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \ddot{\rho}_1(\gamma,\widetilde{\gamma})$ if $\gamma \neq 0$ and $\widetilde{\gamma} \neq 0$; $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = 1$, if $\gamma = 0$ and $\widetilde{\gamma} = 0$; and $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \ddot{\rho}_3(\gamma)$, if $\gamma \neq 0$ and $\widetilde{\gamma} = 0$, where $\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}_2] = \ddot{k}_1^{-1/2}(\gamma,\gamma)\ddot{r}_1(\gamma)\ddot{q}^{-1/2}$ with $\ddot{r}_1(\gamma) := \frac{1}{2}\mathbb{E}\left[y_{t-1}^3\tanh\left(\frac{\gamma y_{t-1}}{2}\right)\right]$ and $\ddot{q} := \mathbb{E}[y_t^4] - \mathbb{E}[y_t^3]^2/\mathbb{E}[y_t^2]$. From this it follows that $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}^2(\gamma)$. Furthermore, $\mathbb{E}[y_t^3] = 0$ and $\mathbb{E}[y_t^4] = 24$ given our DGP, so that

$$\ddot{\rho}_3(\gamma) = \frac{\mathbb{E}[y_t^3\tanh(\gamma y_t/2)]}{4\sqrt{6}\ddot{k}_1^{1/2}(\gamma,\gamma)}. \tag{A.2}$$

Here, we note that

$$\mathbb{E}[y_t^3 \tanh(\gamma y_t/2)] = \frac{1}{8\gamma^4}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right] \tag{A.3}$$

by some tedious algebra assisted by Mathematica, where $P_G(n,x)$ is the polygamma function: $P_G(n,x) := d^{n+1}/d\,x^{n+1}\log(\Gamma(x))$. Inserting (A.3) into (A.2) yields

$$\ddot{\rho}_3(\gamma) = \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]. \tag{A.4}$$

In addition, we show in Lemma A. 3 given below that applying L'Hôpital's rule iteratively yields that

$$\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1^2(\gamma,\widetilde{\gamma}) = \left[\frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3, \frac{1+\gamma}{2\gamma}\right)\right]\right]^2. \tag{A.5}$$

This fact implies that $\mathrm{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1^2(\gamma) = \ddot{\mathcal{G}}_2^2$. That is, the weak limit of the QLR test statistic under $\mathcal{H}_{02}'$ can be obtained from $\ddot{\mathcal{G}}_1^2(\cdot)$ by letting γ converging to zero, so that $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1^2(\gamma)$ under $\mathcal{H}_0'$.

Next, we derive another Gaussian process that is distributionally equivalent to $\ddot{\mathcal{G}}(\cdot)$ and conduct Monte Carlo simulations using it. The process is presented in the following lemma.

Lemma A. 2. *If $\{z_k\}_{k=1}^{\infty}$ is an IID sequences of standard normal random variables, then for each γ and $\widetilde{\gamma} \in \Gamma := \{\gamma \in \mathbb{R} : \gamma \geq 0\}$, $\ddot{\mathcal{G}}(\cdot) \overset{\mathrm{d}}{=} \dot{\mathcal{G}}(\cdot)$, where $\ddot{\mathcal{Z}}_1(\gamma) := \sum_{n=1}^{\infty} b_n(\gamma)z_n$ and $\dot{\mathcal{G}}(\gamma) := \left(\sum_{n=1}^{\infty} b_n^2(\gamma)\right)^{-1/2}\ddot{\mathcal{Z}}_1(\gamma)$.*

□

We prove Lemma A. 2 by showing that the Gaussian process $\dot{\mathcal{G}}(\cdot)$ given in Lemma A. 2 has the same covariance structure as $\ddot{\mathcal{G}}(\cdot)$, and for this purpose, we focus on proving that for all $\gamma, \widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$. If $\gamma, \widetilde{\gamma} > 0$, the desired equality trivially follows from the definition of $\dot{\mathcal{G}}(\cdot)$. On the other hand, applying L'Hôpital's rule iterative shows that $\mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2 \sim N(0,1)$, so that if we let $\dot{\mathcal{G}}_2 := \lim_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma)$, then for $\gamma \neq 0$, $\mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] = \left[\sqrt{3}b_1(\gamma) + b_2(\gamma)\right]/\{2\ddot{k}_1^{1/2}(\gamma,\gamma)\}$. We show in the proof of Lemma A. 2 that the term on the right side is identical to $\ddot{\rho}_3(\gamma)$ in (A.4), so that the covariance kernel of $\dot{\mathcal{G}}(\cdot)$ is identical to $\ddot{\rho}(\cdot,\cdot)$. This fact implies that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$, and $\dot{\mathcal{G}}_2$ can be regarded as the weak limit obtained under $\mathcal{H}_{02}'$.

Lemma A. 2 can be used to obtain the approximate null limit distribution of the QLR test statistic. We cannot generate $\dot{\mathcal{G}}(\cdot)$ using the infinite number of $b_n(\cdot)$, but we can simulate the following process to approximate the distribution of $\dot{\mathcal{G}}(\cdot)$: $\dot{\mathcal{G}}(\gamma; K) := (\sum_{n=1}^{K} b_{K,n}^2(\gamma))^{-1/2} \sum_{n=1}^{K} b_{K,n}(\gamma) z_k$, where for $n = 2, 3, \ldots$, $b_{K,1}(\gamma) := (1-2a_K(\gamma))/\sqrt{2}$, $a_K(\gamma) := \sum_{k=1}^{K} (-1)^{k-1}/(1+\gamma k)^3$ and $b_{K,n}(\gamma) := \sqrt{n(n+1)} \sum_{k=1}^{K} (-1)^{k-1}(\gamma k)^{n-1}/(1+\gamma k)^3$. If K is sufficiently large, the distribution of $\dot{\mathcal{G}}(\cdot; K)$ is close to that of $\dot{\mathcal{G}}(\cdot)$ as can be easily affirmed by simulations.

We conduct Monte Carlo Simulations for the LSTAR case as in the ESTAR case. The only aspect different from the ESTAR case is that the DGP is the one defined in the beginning of this section. Simulation results are summarised into Figure A.2. We use the same parameter spaces $\Gamma = \Gamma_i$, $i = 1, ..., 4$, as before, and we can see that the empirical distribution and PDF estimate of the QLR test approach the null limit distribution and its PDF that are obtained using $\dot{\mathcal{G}}(\cdot; K)$ with $K = 2,500$. This shows that the theory in Section 2 is also valid for the LSTAR model. When the parameter space Γ for γ becomes even larger, we obtain similar results. To save space, they are not reported.

A.1.3 Application of the Weighted Bootstrap

The standard approach to obtaining the null limit distribution of the QLR test is not applicable for empirical analysis because it requires knowledge of the error distribution. Without this information it is not possible in practice to obtain a distributionally equivalent Gaussian process. Hansen's (1996) weighted bootstrap is useful for this case. We apply it to our models as in Cho and White (2010), Cho, Ishida, and White (2011, 2014), and Cho, Cheong, and White (2011).

Although the relevant weighted bootstrap is available in Cho, Cheong, and White (2011), we provide here a version adapted to the structure of the STAR model. We consider the previously studied ESTAR and LSTAR models and proceed as follows. First, we compute the following score for each grid point of $\gamma \in \Gamma$: $\widetilde{W}_n(\gamma) := n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t^2(\gamma) z_t z_t' - n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' [n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t']^{-1} n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t'$ and $\widetilde{d}_{n,t}(\gamma) := z_t f_t(\gamma)\, \widetilde{u}_{n,t} - n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 f_t(\gamma) z_t z_t' [n^{-1} \sum_{t=1}^{n} \widetilde{u}_{n,t}^2 z_t z_t']^{-1} z_t \widetilde{u}_{n,t}$, where $\widetilde{u}_{n,t} := y_t - y_{t-1}\widetilde{\theta}_n$ and $\widetilde{\theta}_n$ is the least squares estimator of θ from the null model. Here, $f_t(\gamma) = 1 - \exp(-\gamma y_{t-1}^2)$ for ESTAR and $f_t(\gamma) = \{1 + \exp(\gamma y_{t-1})\}^{-1} - 1/2$ for the LSTAR model. Second, given these functions, we construct the following score function and pseudo-QLR test statistic: $\overline{QLR}_{b,n} := \sup_{\gamma \in \Gamma} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma)' \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \widetilde{s}_{n,t}(\gamma)$ and $\widetilde{s}_{n,t}(\gamma) := \{\widetilde{W}_n(\gamma)\}^{-1/2}\, \widetilde{d}_{n,t}(\gamma) z_{b,t}$, where $z_{b,t} \sim$ IID $(0, 1)$ with respect to b and t, $b = 1, 2, \ldots, B$, and B is the number of bootstrap replications. For example, we can resample $z_{b,t}$ from the standard normal distribution. For possible two-point distributions, see Davidson *et al.* (2007). Third, we estimate the empirical p-value by $\widehat{p}_n := B^{-1} \sum_{b=1}^{B} \mathbb{I}[QLR_n < \overline{QLR}_{b,n}]$, where $\mathbb{I}[\cdot]$ is the indicator function. We set $B = 300$ to obtain

$\widehat{p}_n^{(i)}$ with $i = 1, 2, \ldots, 2,000$. Finally, for a specified nominal value of α, we compute $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$. When the null hypothesis holds, this proportion converges to α.

The intuition of the weighted bootstrap is straightforward. Note that if the null hypothesis is valid, the QLR test statistic is bounded in probability, and its null limit distribution can be revealed by the covariance structure of $\widetilde{s}_{n,t}(\cdot)$ asymptotically. That is, for each γ and $\widetilde{\gamma}$, $\mathbb{E}[\widetilde{s}_{n,t}(\gamma)\widetilde{s}_{n,t}(\widetilde{\gamma})']$ converges to $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})']$ from the fact that $z_{b,t}$ is independent of $\widetilde{d}_{n,t}(\cdot)$ such that its population mean is zero and variance is unity. This means that $\mathbb{E}[\widetilde{s}_{n,t}(\gamma)\widetilde{s}_{n,t}(\widetilde{\gamma})']$ is asymptotically equivalent to $\widetilde{W}_n^{-1/2}(\gamma)\mathbb{E}[\widetilde{d}_{n,t}(\gamma)\widetilde{d}_{n,t}(\widetilde{\gamma})']\widetilde{W}_n^{-1/2}(\widetilde{\gamma})$, which is asymptotically equivalent to $\mathbb{E}[\mathcal{G}_1(\gamma)\mathcal{G}_1(\widetilde{\gamma})']$ as given in Theorem 3. Therefore, the null limit distribution can be asymptotically revealed by the resampling distribution of $\overline{QLR}_{b,n}$. On the contrary, if the alternative hypothesis is valid, the QLR test statistic is not bounded in probability, but $\overline{QLR}_{b,n}$ is bounded in probability from the fact that $z_{b,t}$ is distributed around zero, so that the chance for the QLR test statistic to be bounded by the critical value obtained by the resampling distribution of $\overline{QLR}_{b,n}$ gets smaller, as n increases. This aspect implies that the weighted bootstrap is asymptotically consistent.

The simulation results are displayed in the percentile-percentile (PP) plots for the ESTAR and LSTAR models in Figures A.3 (ESTAR) and A.4 (LSTAR). The horizonal unit interval stands for α, and the vertical unit interval is the space of p-values. As a function of α, the aforementioned proportion should converge to the 45-degree line under the null hypothesis. As before, the four parameter spaces are considered: $\Gamma = \Gamma_i$, $i = 1, ..., 4$. The results are summarised as follows. First, as a function of α, the proportion $\frac{1}{2000}\sum_{i=1}^{2000}\mathbb{I}[\widehat{p}_n^{(i)} < \alpha]$ does converge to the 45-degree line. Second, the empirical rejection rates estimated by the weighted bootstrap are closest to the nominal levels when the parameter space is small. Although the overall finite sample level distortions are smaller for the ESTAR model than the LSTAR model, the empirical rejection rate is close to the nominal significance level if α is close to zero. Finally, as the size of the parameter space increases, more observations are needed to better approximate the 45-degree line in the PP plots. We have conducted simulations using even larger parameter spaces and obtained similar results. We omit reporting them for brevity.

A.2 Proofs

Proof of Lemma 1. (*i*) Given Assumptions 1, 2, 3, and 5, it is trivial to show that $\widehat{\sigma}^2_{n,0} \stackrel{\text{a.s.}}{\to} \sigma^2_*$ by the ergodic theorem.

(*ii*) The null limit distribution of $QLR_n^{(1)}$ is determined by the two terms in $QLR_n^{(1)}$: $Z'F(\cdot)Mu$ and $Z'F(\cdot)M$ $F(\cdot)Z$. We examine their null limit behaviour one by one and combine the limit results using the converging-together lemma in Billingsley (1999, p. 39).

(a) We show the weak convergence part of $n^{-1/2}Z'F(\cdot)Mu$. Using the definition of $M := I - Z(Z'Z)^{-1}Z'$ we have $Z'F(\gamma)Mu = Z'F(\gamma)u - Z'F(\gamma)Z(Z'Z)^{-1}Z'u$, and we now examine the components on the right-hand side of this equation separately. For each $\gamma \in \Gamma$, we define $\widehat{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - (\sum_{t=1}^n f_t(\gamma)z_t z_t')(\sum_{t=1}^n z_t z_t')^{-1}\sum_{t=1}^n z_t u_t$, $\widetilde{f}_{n,t}(\gamma) := f_t(\gamma)u_t z_t - \mathbb{E}[f_t(\gamma)z_t z_t']\mathbb{E}[z_t z_t']^{-1}\sum_{t=1}^n z_t u_t$ and show that

$$\sup_{\gamma\in\Gamma(\epsilon)} \left\| n^{-1/2}\sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right]\right\|_\infty = o_{\mathbb{P}(1)}, \tag{A.6}$$

where $\Gamma(\epsilon) := \{\gamma \in \Gamma : |\gamma| \geq \epsilon\}$ and $\|\cdot\|_\infty$ is the uniform matrix norm. We have

$$\begin{aligned}
&\sup_{\gamma\in\Gamma(\epsilon)} \left\| \frac{1}{\sqrt{n}}\sum_{t=1}^n \left[\widehat{f}_{n,t}(\gamma) - \widetilde{f}_{n,t}(\gamma)\right]\right\|_\infty \\
&\quad \leq \sup_{\gamma\in\Gamma(\epsilon)} \left\| \left(\frac{1}{n}\sum_{t=1}^n f_t(\gamma)z_t z_t'\right)\left\{\left(\frac{1}{n}\sum_{t=1}^n z_t z_t'\right)^{-1} - \mathbb{E}[z_t z_t']^{-1}\right\} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty \\
&\quad + \sup_{\gamma\in\Gamma(\epsilon)} \left\| \left\{\left(n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t'\right) - \mathbb{E}[f_t(\gamma)z_t z_t']\right\}\mathbb{E}[z_t z_t']^{-1} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty .
\end{aligned} \tag{A.7}$$

We show that each term on the right-hand side of (A.7) is $o_{\mathbb{P}}(1)$. Now, $\{z_t u_t, \mathcal{F}_t\}$ is a martingale difference sequence, where $\mathcal{F}_t$ is the smallest sigma-field generated by $\{z_t u_t, z_{t-1}u_{t-1}, \ldots\}$. Therefore, $\mathbb{E}[z_t u_t|\mathcal{F}_{t-1}] = 0$, $\mathbb{E}[|Z_{t,j}u_t|^2] = \mathbb{E}[u_t^4]^{1/2}\mathbb{E}[|Z_{t,j}|^4]^{1/2} \leq \mathbb{E}[m_t^4]^{1/2}\mathbb{E}[Z_{t,j}^4]^{1/2} < \infty$, and $\mathbb{E}[u_t^2 z_t z_t']$ is positive definite. Thus, $n^{-1/2}\sum_{t=1}^n z_t u_t$ is asymptotically normal. Next, we note that $n^{-1/2}\sum_{t=1}^n f_t(\gamma)u_t z_t$ is also asymptotically normal. This follows from the fact that $\{f_t(\gamma)u_t z_t, \mathcal{F}_t\}$ is a martingale difference sequence, and for each j, $|f_t(\gamma)u_t z_{t,j}|^2 \leq m_t^6$, and $\mathbb{E}[m_t^6] < \infty$ by Assumptions 4 and 5. Furthermore, $\sup_{\gamma\in\Gamma}\|n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t' - \mathbb{E}[f_t(\gamma)z_t z_t']\|_\infty = o_{\mathbb{P}}(1)$ by Ranga Rao's (1962) uniform law of large numbers. Thus,

$$\sup_{\gamma\in\Gamma(\epsilon)} \left\| \left\{n^{-1}\sum_{t=1}^n f_t(\gamma)z_t z_t' - \mathbb{E}[f_t(\gamma)z_t z_t']\right\}\mathbb{E}[z_t z_t']^{-1} n^{-1/2}\sum_{t=1}^n z_t u_t\right\|_\infty = o_{\mathbb{P}}(1). \tag{A.8}$$

This shows that the second term of (A.7) is $o_{\mathbb{P}}(1)$. We now demonstrate that the first term of (A.7) is also $o_{\mathbb{P}}(1)$. By Assumption 4 and the ergodic theorem, we note that $\|n^{-1}\sum_{t=1}^n z_t z_t' - \mathbb{E}[z_t z_t']\|_\infty = o_{\mathbb{P}}(1)$, and $|\sum_{t=1}^n f_t(\gamma)z_{t,j}z_{t,i}| \leq \sum_{t=1}^n m_t^3 = O_{\mathbb{P}}(n)$, so that (A.8) follows, leading to (A.6). Therefore, $n^{-1/2}Z'F(\gamma)Mu \overset{\text{A}}{\sim} N[0, B_1(\gamma,\gamma)]$ by noting that $\mathbb{E}[\widetilde{f}_{n,t}(\gamma)\widetilde{f}_{n,t}(\gamma)'] = B_1(\gamma,\gamma)$. Using the same methodology, we can show that for each $\gamma, \widetilde{\gamma} \in \Gamma(\epsilon)$,

$$\frac{1}{\sqrt{n}} \left[\begin{array}{c} Z'F(\gamma)Mu \\ Z'F(\widetilde{\gamma})Mu \end{array} \right] \overset{\text{A}}{\sim} N \left[\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left[\begin{array}{cc} B_1(\gamma, \gamma) & B_1(\gamma, \widetilde{\gamma}) \\ B_1(\widetilde{\gamma}, \gamma) & B_1(\widetilde{\gamma}, \widetilde{\gamma}) \end{array} \right] \right].$$

Finally, we have to show that $\{\widetilde{f}_{n,t}(\cdot)\}$ is tight. First note that by Assumptions 1, 2, and 4, it follows that $|f_t(\gamma) z_{t,j} u_t - f_t(\widetilde{\gamma}) z_{t,j} u_t| \le m_t\, |z_{t,j} u_t|\ \ |\gamma - \widetilde{\gamma}|$ for each j. From this we obtain that $\sup_{|\gamma - \widetilde{\gamma}| < \eta} |f_t(\gamma) z_{t,j} u_t - f_t(\widetilde{\gamma}) z_{t,j} u_t|^{2+\omega} \le m_t^{2+\omega} |z_{t,j} u_t|^{2+\omega} \eta^{2+\omega} \le m_t^{6+3\omega} \eta^{2+\omega}$, so that it follows that $\mathbb{E}[\sup_{|\gamma - \widetilde{\gamma}| < \eta} |f_t(\gamma) z_{t,j} u_t - f_t(\widetilde{\gamma}) z_{t,j} u_t|^{2+\omega}]^{\frac{1}{2+\omega}} \le \mathbb{E}[m_t^{6+3\omega}]^{\frac{1}{2+\omega}} \eta$ for each j. This implies that $\{n^{-1/2} f_t(\cdot) z_{t,j} u_t\}$ is tight because Ossiander's $L^{2+\omega}$ entropy is finite.

Next, for some $c > 0$, it holds that $\|\mathbb{E}[f_t(\gamma) z_t z_t'] \mathbb{E}[z_t z_t']^{-1} z_t u_t - \mathbb{E}[f_t(\widetilde{\gamma}) z_t z_t'] \mathbb{E}[z_t z_t']^{-1} z_t u_t\|_\infty = \|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\} z_t z_t'] \mathbb{E}[z_t\ z_t']^{-1} z_t u_t\|_\infty \le c m_t^2 \left\|\mathbb{E}[z_t z_t']^{-1}\right\|_\infty \|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\} z_t z_t']\|_\infty$ by the property of the uniform norm and Assumption 5. Also note that $\|\mathbb{E}[f_t(\gamma) z_t z_t' - f_t(\widetilde{\gamma}) z_t z_t']\|_\infty \le \|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\} z_t z_t']\|_1$ and by Assumption 4, for each $i, j = 1, 2, \ldots, m+1$, $|z_{t,j} z_{t,i} [f_t(\gamma) - f_t(\widetilde{\gamma})]| \le m_t^3 |\gamma - \widetilde{\gamma}|$, where $\|[g_{i,j}]\|_1 := \sum_i \sum_j |g_{i,j}|$. Therefore,

$$\begin{aligned} &\left\|\mathbb{E}[f_t(\gamma) z_t z_t'] \mathbb{E}[z_t z_t']^{-1} z_t u_t - \mathbb{E}[f_t(\widetilde{\gamma}) z_t z_t'] \mathbb{E}[z_t z_t']^{-1} z_t u_t\right\|_\infty \\ &\quad \le c m_t^2 \left\|\mathbb{E}[z_t z_t']^{-1}\right\|_\infty \left\|\mathbb{E}[\{f_t(\gamma) - f_t(\widetilde{\gamma})\} z_t z_t']\right\|_\infty \le c^2 (m+1) m_t^2 \left\|\mathbb{E}[z_t z_t']^{-1}\right\|_\infty \mathbb{E}[m_t^3] |\gamma - \widetilde{\gamma}|. \end{aligned} \tag{A.9}$$

This inequality (A.9) implies that $\{n^{-1/2} \mathbb{E}[f_t(\cdot) z_t z_t'] \mathbb{E}[z_t z_t']^{-1} z_t u_t\}$ is also tight. Hence, it follows that for some $b < \infty$, $\mathbb{E}[\sup_{|\gamma - \widetilde{\gamma}| < \eta} |\widetilde{f}_t(\gamma) - \widetilde{f}_t(\widetilde{\gamma})|^{2+\omega}] \le b \cdot \eta$. That is, $\{n^{-1/2} \sum_{t=1}^n \widetilde{f}_{n,t}(\cdot)\}$ is tight. From this and the fact that the finite-dimensional multivariate CLT holds, the weak convergence of $\{n^{-1/2} \sum_{t=1}^n \widetilde{f}_{n,t}(\cdot)\}$ is established.

(b) Next, we examine the limit behaviour of $n^{-1} Z'F(\cdot)F(\cdot)Z$. Note that $n^{-1} Z'F(\gamma)F(\gamma)Z = n^{-1} \sum_{t=1}^n f_t^2(\gamma) z_t z_t' - \left\{n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t'\right\} \left\{n^{-1} \sum_{t=1}^n z_t z_t'\right\}^{-1} \left\{n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t'\right\}$ and, given Assumptions 1, 2, 3, 4, and 6, $\sup_{\gamma \in \Gamma(\epsilon)} \|n^{-1} \sum_{t=1}^n f_t^2(\gamma)\ z_t z_t' - \mathbb{E}[f_t^2(\gamma) z_t z_t']\| \overset{\text{a.s.}}{\to} 0$ and $\sup_{\gamma \in \Gamma(\epsilon)} \|n^{-1} \sum_{t=1}^n f_t(\gamma) z_t z_t' - \mathbb{E}[f_t(\gamma) z_t\ z_t']\| \overset{\text{a.s.}}{\to} 0$ by Ranga Rao's (1962) uniform law of large numbers. Therefore, from the fact that $\|n^{-1} \sum_{t=1}^n z_t z_t' - \mathbb{E}[z_t z_t']\|_\infty = o_{\mathbb{P}}(1)$, it follows that $\sup_{\gamma \in \Gamma(\epsilon)} |n^{-1} Z'F(\gamma)MF(\gamma)Z - \{\mathbb{E}[f_t^2(\gamma) z_t z_t'] - \mathbb{E}[f_t(\gamma) z_t z_t'] \mathbb{E}[z_t z_t']^{-1}\ \mathbb{E}[f_t(\gamma) z_t z_t']\}| = o_{\mathbb{P}}(1)$. Applying the converging-together lemma yields the desired result.

(iii) This result trivially follows from the fact that $\mathbb{E}[u_t^2 | z_t] = \sigma_*^2$. ■

Proof of Lemma 2. Given Assumption 2, $\mathcal{H}_{02}$, and the definition of $H_j(\gamma)$, the j-th order derivative of $\mathcal{L}_n^{(2)}(\,\cdot\,,\theta)$ is obtained as

$$\begin{aligned}\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(\gamma,\theta) &= -\sum_{k=0}^{j}\binom{j}{k}\left\{\frac{\partial^k}{\partial\gamma^k}(y-F(\gamma)Z\theta)'\right\}M\left\{\frac{\partial^{j-k}}{\partial\gamma^{j-k}}(y-F(\gamma)Z\theta)\right\}\\ &= 2\theta'Z'H_j(\gamma)Mu-\sum_{k=1}^{j-1}\binom{j}{k}\theta'Z'H_j(\gamma)MH_{j-k}(\gamma)Z\theta \qquad \text{(A.10)}\end{aligned}$$

by iteratively applying the general Leibniz rule. We now evaluate this derivative at $\gamma=0$. Note that $H_j(0)=0$ if $j<\kappa$ by the definition of κ. This implies that $(\partial^j/\partial\gamma^j)\mathcal{L}_n^{(2)}(0,\theta)=0$ for $j=1,2,\ldots,\kappa-1$. This also implies that $\binom{j}{k}\theta'Z'H_j(0)MH_{j-k}(0)Z\theta=0$ for $j=\kappa,\ \kappa+1,\ \ldots,\ 2\kappa-1$. Therefore, $\frac{\partial^j}{\partial\gamma^j}\mathcal{L}_n^{(2)}(0,\theta)=2\theta'Z'H_j(0)Mu$. Finally, we examine the case in which $j=2\kappa$. For each $j<2\kappa$, $H_j(0)=0$ and $H_\kappa(0)\neq 0$, so that the summand of the second term in the right side of (A.10) is different from zero only when $j=2\kappa$ and $k=\kappa$: $\frac{\partial^{2\kappa}}{\partial\gamma^{2\kappa}}\mathcal{L}_n^{(2)}(0,\theta)=2\theta'Z'H_{2\kappa}(\gamma)Mu-\binom{2\kappa}{\kappa}\theta'Z'H_\kappa(\gamma)MH_\kappa(\gamma)Z\theta$. This completes the proof. ■

Proof of Lemma 3. Given Assumptions 1, 2, 7, and $\mathcal{H}_{02}$, we note that

$$\begin{aligned}QLR_n^{(2)} &:= \sup_\theta \overline{QLR}_n^{(2)}(\theta)\\ &= \sup_\theta\sup_\varsigma \frac{1}{\widehat{\sigma}_{n,0}^2}\left[\frac{2\{\theta'G_\kappa'u\}\varsigma^\kappa}{\kappa!\sqrt{n}}-\frac{1}{(2\kappa)!n}\left\{\binom{2\kappa}{\kappa}\theta'G_\kappa'G_\kappa\theta\right\}\varsigma^{2\kappa}\right]+o_{\mathbb{P}}(n). \qquad \text{(A.11)}\end{aligned}$$

Then, the FOC with respect to ς implies that $\widehat{\varsigma}_n^\kappa(\theta)=W_n(\theta)$, κ is odd; and $\widehat{\varsigma}_n^\kappa(\theta)=\max[0,W_n(\theta)]$, if κ is even by noting that $\widehat{\varsigma}_n^\kappa(\theta)$ cannot be negative. If we plug $\widehat{\varsigma}_n^\kappa(\theta)$ back into the right side of (A.11), the desired result follows. ■

Proof of Lemma 4. Before proving Lemma 4, we first show that for each j, $Z'H_j(0)Mu=O_{\mathbb{P}}(n^{1/2})$, so that $j=\kappa+1,\ldots,2\kappa-1$, $Z'H_j(0)Mu=o_{\mathbb{P}}(n^{j/2\kappa})$. Note that for $j=\kappa+1,\ldots,2\kappa$, $Z'H_jMu=\sum_{t=1}^n z_th_{t,j}(0)u_t-\sum_{t=1}^n z_th_{t,j}(0)z_t'\left(\sum_{t=1}^n z_tz_t'\right)^{-1}\sum_{t=1}^n z_tu_t$. First, we apply the ergodic theorem to $n^{-1}\sum_t z_th_{t,j}(0)z_t'$ and $n^{-1}\sum_t z_tz_t'$, respectively. Second, given Assumptions 1, 2, 3, 7, and 8, following the proof of Lemma 1, we have that $n^{-1/2}\sum_t z_tu_t$ is asymptotically normal. Furthermore, for all $j=\kappa+1,\ldots,2\kappa$, $n^{-1/2}\sum_t z_th_{t,j}(0)u_t$ is asymptotically normal. For this verification, note that $\{z_th_{t,j}(0)u_t,\mathcal{F}_t\}$ is a martingale difference sequence, so that for each j, $\mathbb{E}[z_th_{t,j}(0)u_t|\mathcal{F}_{t-1}]=0$. Next, we prove that for each j, $\mathbb{E}[z_{t,i}^2h_{t,j}^2(0)u_t^2]<\infty$. First note that using Assumption 7, $\mathbb{E}[|z_{t,i}^2h_{t,j}^2(0)u_t^2|]\le\mathbb{E}[|u_t|^4]^{1/2}\mathbb{E}[|h_{t,j}^2(0)z_{t,i}^2|^2]^{1/2}\le$

$\mathbb{E}[|u_t|^4]^{1/2}\mathbb{E}[|h_{t,j}(0)|^8]^{1/4}\ \mathbb{E}[|z_{t,i}|^8]^{1/4} < \infty$ by the Cauchy-Schwarz's inequality. For the same reason, $\mathbb{E}[|z_{t,i}^2 h_{t,j}^2(0)u_t^2|] \leq \mathbb{E}[|u_t\ h_{t,j}(0)|^4]^{1/2}\mathbb{E}[|z_{t,i}|^4]^{1/2} \leq \mathbb{E}[|u_t|^8]^{1/2}\mathbb{E}[|h_{t,j}(0)|^8]^{1/2}\ \mathbb{E}[|z_{t,i}|^4]^{1/2} < \infty$. By Assumption 8, $\mathbb{E}[u_t^2 z_t h_{t,j}^2(0)z_t']$ is positive definite. It then follows by Theorem 5.25 of White (2001) that $n^{-1/2}\sum_t z_t h_{t,j}(0)\ u_t$ is asymptotically normal. Thus, $Z'H_j(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

We now consider the statements (*i*)–(*iii*).

(*i*) First, we show that $\theta' Z' H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$. By the definition of M,

$$Z'H_\kappa(0)Mu = \sum_{t=1}^{n} z_t h_{t,\kappa}(0)u_t - \sum_{t=1}^{n} z_t h_{t,\kappa}(0)z_t' \left[\sum_{t=1}^{n} z_t z_t'\right]^{-1} \sum_{t=1}^{n} z_t u_t. \tag{A.12}$$

We examine all sums on the right-hand side of (A.12). First, $h_{t,\kappa}(0)$ is a function of z_t, which implies that, given the moment condition in Assumption 7, $n^{-1}\sum z_t h_{t,\kappa}(0)z_t'$ obeys the ergodic theorem. Second, similarly under Assumptions 1, 2, 3, 7, 8, and $\mathcal{H}_{02}$, $n^{-1}\sum z_t z_t'$ also obeys the ergodic theorem. Third, given the assumptions and the proof of Lemma 1, we have already proved that $n^{-1/2}\sum z_t u_t$ is asymptotically normally distributed. Finally, $n^{-1/2}\sum z_t h_{t,\kappa}(0)u_t$ is asymptotically normal, and the proof is similar to that of the asymptotic normality of $n^{-1/2}\sum_t z_t h_{t,j}(0)u_t$ ($j = \kappa+1,\ldots,2\kappa$). All these facts imply that $Z'H_\kappa(0)Mu = O_{\mathbb{P}}(n^{1/2})$.

(*ii*) $n^{-1}G_\kappa' G_\kappa \overset{\text{a.s.}}{\to} A_2$ by the ergodic theorem.

(*iii*) Note that

$$Z'H_\kappa(0)MH_\kappa(0)Z = \sum_{t=1}^{n} z_t h_{t,\kappa}^2(0)z_t' - \sum_{t=1}^{n} z_t h_{t,\kappa}(0)z_t' \left[\sum_{t=1}^{n} z_t z_t'\right]^{-1} \sum_{t=1}^{n} z_t h_{t,\kappa}(0)z_t'. \tag{A.13}$$

The limit of (A.13) is revealed by applying the ergodic theorem to each term on the right-hand side of this expression. Consequently, $n^{-1}Z'H_\kappa(0)MH_\kappa(0)Z \overset{\text{a.s.}}{\to} \mathbb{E}[g_{t,\kappa}g_{t,\kappa}']$, where $\mathbb{E}[g_{t,\kappa}g_{t,\kappa}'] := \mathbb{E}[z_t H_{2\kappa}^2(0)z_t'] - \mathbb{E}[z_t H_{2\kappa}(0)z_t']\mathbb{E}[z_t z_t']^{-1}\ \mathbb{E}[z_t H_{2\kappa}(0)\ z_t']$. This completes the proof. ■

Proof of Lemma A. 2. The distributional equivalence between $\dot{\mathcal{G}}(\cdot)$ and $\ddot{\mathcal{G}}(\cdot)$ can be established by showing that for all $\gamma, \widetilde{\gamma} \geq 0$, $\mathbb{E}[\ddot{\mathcal{G}}(\gamma)\ddot{\mathcal{G}}(\widetilde{\gamma})] = \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$. We will proceed in three steps. First, we derive the functional form of $\ddot{\rho}(\gamma,\widetilde{\gamma})$. We show that if $\gamma, \widetilde{\gamma} > 0$, then $\ddot{k}_1(\gamma,\widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$. This in turn implies that for $\gamma, \widetilde{\gamma} > 0$, $\ddot{\rho}(\gamma,\widetilde{\gamma}) = \ddot{k}_1^{-1/2}(\gamma,\gamma)\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})\ddot{k}_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma})$. It follows that the specific functional form of $\ddot{\rho}(\gamma,\widetilde{\gamma})$ can be obtained from this result and (A.4).

Second, similarly for all $\gamma, \widetilde{\gamma} \geq 0$, we derive the functional form of $\dot{\rho}(\gamma,\widetilde{\gamma})$ and compare it to $\ddot{\rho}(\gamma,\widetilde{\gamma})$. To do all this, we first note that for all $\gamma, \widetilde{\gamma} > 0$,

$$\begin{aligned}\ddot{k}_1(\gamma, \tilde{\gamma}) =& \frac{1}{4}\mathbb{E}\left[y_t^2 \tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right] - \frac{1}{4}\mathbb{E}\left[y_t^2 \tanh\left(\frac{\gamma y_t}{2}\right)\right]\mathbb{E}\left[y_t^2 \tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right] \Big/ \mathbb{E}[y_t^2] \\ =& \frac{1}{4}\mathbb{E}\left[y_t^2 \tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\widetilde{\gamma} y_t}{2}\right)\right]. \end{aligned} \tag{A.14}$$

This follows from that fact that for any $x \in \mathbb{R}$, $\tanh(x) = -\tanh(-x)$ and that y_t follows the Laplace distribution with mean zero and variance 2, so that $\mathbb{E}\left[y_t^2 \tanh\left(\gamma y_t/2\right)\right] = 0$. Given this, we can apply the Dirichlet series to $\tanh(\cdot)$ to obtain the functional form of $\ddot{k}_1(\cdot,\cdot)$. Thus, for any $x \in \mathbb{R}$, $\tanh(x) = \mathrm{sgn}(x)(1 - 2\sum_{k=0}^{\infty}(-1)^k \exp(-2|x|(k+1)))$ and, furthermore, that $\mathbb{E}\left[s_t^2 \exp(-s_t\gamma k)\right] = 2/\left(1+\gamma k\right)^3$ and $\mathbb{E}[s_t^2] = 2$, where $s_t := |y_t|$ follows the exponential distribution with mean 1 and variance 2. Applying these to (A.14) yields

$$\begin{aligned}\ddot{k}_1(\gamma, \widetilde{\gamma}) &= \mathbb{E}\left[\frac{y_t^2}{4}\tanh\left(\frac{\gamma y_t}{2}\right)\tanh\left(\frac{\tilde{\gamma} y_t}{2}\right)\right] \\ &= \mathbb{E}\left[\frac{s_t^2}{4}\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\gamma k)\right] - \mathbb{E}\left[\frac{s_t^2}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\exp(-s\widetilde{\gamma} k)\right] \\ &\qquad + \mathbb{E}\left[s_t^2 \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+j-2}\exp(-s_t(\gamma k + \widetilde{\gamma} j))\right] \\ &= \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}\frac{2}{(1+\gamma k + \widetilde{\gamma} j)^3}. \end{aligned}$$

Next, for $|x| < 1$ we have $(1-x)^{-3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}$, so that $(1+\gamma k + \widetilde{\gamma} j)^{-3} = (1+\gamma k)^{-3}(1+\widetilde{\gamma} j)^{-3}\left(1 - \frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^{-3}$, where we note that $(1 - \frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j})^{-3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left(\frac{\gamma k}{1+\gamma k}\frac{\widetilde{\gamma} j}{1+\widetilde{\gamma} j}\right)^{n-1}$. Therefore, it follows that

$$\ddot{k}_1(\gamma, \widetilde{\gamma}) = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\widetilde{\gamma} k)^3} + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}\frac{n(n+1)(\gamma k)^{n-1}(\widetilde{\gamma} j)^{n-1}}{(1+\gamma k)^{n+2}(1+\widetilde{\gamma} j)^{n+2}}.$$

Furthermore,

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k-2}n(n+1)\frac{(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\frac{(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}} \\ &= 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(1+\gamma k)^3}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(1+\widetilde{\gamma} j)^3} + \sum_{n=2}^{\infty}n(n+1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\gamma k)^{n-1}}{(1+\gamma k)^{n+2}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(\widetilde{\gamma} j)^{n-1}}{(1+\widetilde{\gamma} j)^{n+2}}, \end{aligned}$$

which is equal to $2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$, where for $n = 2, 3, \ldots$, $a(\gamma) := \sum_{k=1}^{\infty}(-1)^{k-1}/(1+\gamma k)^3$ and $b_n(\gamma) := \sqrt{n(n+1)}\sum_{k=1}^{\infty}(-1)^{k-1}(\gamma k)^{n-1}/(1+\gamma k)^{n+2}$. In particular, $b_1(\gamma) := 2^{-1/2}(1-2a(\gamma))$, so that $\ddot{k}_1(\gamma,\widetilde{\gamma}) = \frac{1}{2} - a(\gamma) - a(\widetilde{\gamma}) + 2a(\gamma)a(\widetilde{\gamma}) + \sum_{n=2}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma}) = \frac{1}{2}(1-2a(\gamma))(1-2a(\widetilde{\gamma})) + \sum_{n=2}^{\infty} b_n(\gamma)\ b_n(\widetilde{\gamma}) = \sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})$. Then, for each γ and $\widetilde{\gamma} > 0$, $\ddot{\rho}_1(\gamma,\widetilde{\gamma}) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_1(\widetilde{\gamma})] = \ddot{k}_1^{-1/2}(\gamma,\gamma)\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})\ddot{k}_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma})$. In addition, for $\gamma > 0$, we examine $\ddot{\rho}_3(\gamma) := \mathbb{E}[\ddot{\mathcal{G}}_1(\gamma)\ddot{\mathcal{G}}_2]$. Note that from (A.4), $\ddot{\rho}_3(\gamma) = \{\mathbb{E}[y_t^3\tanh(\gamma y_t/2)]\}/\{4\sqrt{6}\ddot{k}_1^{1/2}(\gamma,\gamma)\} = [48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3,\frac{1+\gamma}{2\gamma})]/\{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)\}$ as affirmed by Mathematica. It follows that the specific functional form of $\ddot{\rho}(\gamma,\widetilde{\gamma})$ is given as

$$\ddot{\rho}(\gamma,\widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1^{1/2}(\gamma,\gamma)\ddot{k}_1^{1/2}(\widetilde{\gamma},\widetilde{\gamma})}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} > 0; \\ 1, & \text{if } \gamma = 0 \text{ and } \widetilde{\gamma} = 0; \\ \frac{48\gamma^4 + P_G(3,1+\frac{1}{2\gamma}) - P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} = 0, \end{cases} \cdot \tag{A.15}$$

Third, we examine the covariance kernel of $\dot{\mathcal{G}}(\cdot)$, viz., $\dot{\rho}(\cdot,\cdot)$. If we let $\gamma, \widetilde{\gamma} > 0$, $\dot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma)\cdot\mathcal{G}(\widetilde{\gamma})] = \ddot{k}_1^{-1/2}(\gamma,\gamma)\sum_{n=1}^{\infty} b_n(\gamma)b_n(\widetilde{\gamma})\ddot{k}_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma}) = \ddot{k}_1^{-1/2}(\gamma,\gamma)\ddot{k}_1(\gamma,\widetilde{\gamma})\ddot{k}_1^{-1/2}(\widetilde{\gamma},\widetilde{\gamma}) = \ddot{\rho}_1(\gamma,\widetilde{\gamma})$. Furthermore, by some tedious algebra, $\mathrm{plim}_{\gamma\downarrow 0}\ddot{\mathcal{Z}}_1^2(\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{\mathcal{Z}}_1^2(\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{\mathcal{Z}}_1^2(\gamma) = \frac{1}{8}\{3\sqrt{2}Z_1 + \sqrt{6}Z_2\}^2$, $\mathrm{plim}_{\gamma\downarrow 0}\ddot{k}_1(\gamma,\gamma) = 0$, $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial}{\partial\gamma}\ddot{k}_1(\gamma,\gamma) = 0$, and $\mathrm{plim}_{\gamma\downarrow 0}\frac{\partial^2}{\partial\gamma^2}\ddot{k}_1(\gamma,\gamma) = 3$, so that $\mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}^2(\gamma) = (\frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2)^2$, which implies $\dot{\mathcal{G}}_2 := \mathrm{plim}_{\gamma\downarrow 0}\dot{\mathcal{G}}(\gamma) = \frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2 \sim N(0,1)$. Consequently, if $\gamma > 0$,

$$\begin{aligned}\mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}_2] &= \ddot{k}_1^{-1/2}(\gamma,\gamma)\mathbb{E}\left[\ddot{\mathcal{Z}}_1(\gamma)\left(\frac{\sqrt{3}}{2}z_1 + \frac{1}{2}z_2\right)\right] = \ddot{k}_1^{-1/2}(\gamma,\gamma)\left[\frac{\sqrt{3}}{2}b_1(\gamma) + \frac{1}{2}b_2(\gamma)\right] \\ &= \frac{1}{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)}\left[48\gamma^4 + P_G\left(3, 1+\frac{1}{2\gamma}\right) - P_G\left(3,\frac{1+\gamma}{2\gamma}\right)\right]. \end{aligned} \tag{A.16}$$

The last equality follows from the fact that $b_1(\gamma) = \frac{1}{8\sqrt{2}\gamma^3}[8\gamma^3 - P_G(2, 1+\frac{1}{2\gamma}) + P_G(2,\frac{1+\gamma}{2\gamma})]$, $b_2(\gamma) = \frac{1}{16\sqrt{6}\gamma^4}[6\gamma P_G(2,\frac{1}{2\gamma}) - 6\gamma P_G(2,\frac{1+\gamma}{2\gamma}) + P_G(3,\frac{1}{2\gamma}) - P_G(3,\frac{1+\gamma}{2\gamma})]$, $P_G(2,\frac{1}{2\gamma}) - P_G(2, 1+\frac{1}{2\gamma}) = -16\gamma^3$, and $P_G(3,\frac{1}{2\gamma}) - P_G(3, 1+\frac{1}{2\gamma}) = 96\gamma^4$, as obtained by Mathematica. Equation (A.16) then leads to the following functional form for $\dot{\rho}(\gamma,\widetilde{\gamma}) := \mathbb{E}[\dot{\mathcal{G}}(\gamma)\dot{\mathcal{G}}(\widetilde{\gamma})]$:

$$\dot{\rho}(\gamma,\widetilde{\gamma}) = \begin{cases} \frac{\ddot{k}_1(\gamma,\widetilde{\gamma})}{\ddot{k}_1^{1/2}(\gamma,\gamma)\ddot{k}_1^{1/2}(\widetilde{\gamma},\widetilde{\gamma})}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} > 0; \\ 1, & \text{if } \gamma = 0 \text{ and } \widetilde{\gamma} = 0; \\ \frac{48\gamma^4 + P_G(3,1+\frac{1}{2\gamma}) - P_G(3,\frac{1+\gamma}{2\gamma})}{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)}, & \text{if } \gamma > 0 \text{ and } \widetilde{\gamma} = 0, \end{cases}$$

which is identical to the functional form of $\ddot{\rho}(\cdot,\cdot)$ in (A.15). This allows the conclusion that $\ddot{\mathcal{G}}(\cdot)$ has the same distribution as $\dot{\mathcal{G}}(\cdot)$. ■

In the following, we provide additional supplementary claim in (A.5) that is given in the following lemma:

Lemma A. 3. *Given the DGP and Model conditions in Section A.1.2,* $\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1^2(\gamma,\widetilde{\gamma}) = [48\gamma^4 + P_G(3, 1 + \frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]^2/\{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)\}^2$. □

Lemma A. 3 implies that $\mathrm{plim}_{\gamma\downarrow 0}\ddot{\mathcal{G}}_1^2(\gamma) = \ddot{\mathcal{G}}_2^2$, so that $\sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1^2(\gamma) \geq \ddot{\mathcal{G}}_2^2$ and $QLR_n \Rightarrow \sup_{\gamma\in\Gamma}\ddot{\mathcal{G}}_1^2(\gamma)$.

Proof of Lemma A. 3. From the definition of $\ddot{\rho}_1(\gamma,\widetilde{\gamma})$, note that $\ddot{\rho}_1^2(\gamma,\widetilde{\gamma}) := \ddot{k}_1^{-1}(\gamma,\gamma)\,\ddot{k}_1^2(\gamma,\widetilde{\gamma})\ddot{k}_1^{-1}(\widetilde{\gamma},\widetilde{\gamma})$. Furthermore, we have $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1^2(\gamma,\widetilde{\gamma}) = 0$, $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial}{\partial\widetilde{\gamma}}\ddot{k}_1^2(\gamma,\widetilde{\gamma}) = 0$, $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 0$, $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}(\partial/\partial\widetilde{\gamma})$ $\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 0$, $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}(\partial^2/\partial\widetilde{\gamma}^2)\ddot{k}_1(\widetilde{\gamma},\widetilde{\gamma}) = 3$, and $\mathrm{plim}_{\widetilde{\gamma}\downarrow 0}\frac{\partial^2}{\partial\widetilde{\gamma}^2}\ddot{k}_1^2(\gamma,\widetilde{\gamma}) = (\{48\gamma^4 + P_G(3, 1+\frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})\}$ $/\{32\sqrt{2}\gamma^4\})^2$ by some algebra using Mathematica. This property implies that $\lim_{\widetilde{\gamma}\downarrow 0}\ddot{\rho}_1^2(\gamma,\widetilde{\gamma}) = (\{48\gamma^4 + P_G(3, 1 + \frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})\}/\ \{32\sqrt{2}\gamma^4\})^2/\{3\ddot{k}_1(\gamma,\gamma)\} = [48\gamma^4 + P_G(3, 1 + \frac{1}{2\gamma}) - P_G(3, \frac{1+\gamma}{2\gamma})]^2/\{32\sqrt{6}\gamma^4\ddot{k}_1^{1/2}(\gamma,\gamma)\}^2$. This completes the proof. ■

References

BACON, D. W. AND WATTS, D.G. (1971): "Estimating the Transition between Two Intersecting Straight Lines," *Biometrika*, 58, 525–534.

BILLINGSLEY, P. (1999): *Convergence of Probability Measures*. New York: Wiley.

CHO, J. S., CHEONG, T. U., AND WHITE, H. (2011): "Experience with the Weighted Bootstrap in Testing for Unobserved Heterogeneity in Exponential and Weibull Duration Models," *Journal of Economic Theory and Econometrics*, 22:2, 60–91.

CHO, J.S. AND WHITE, H. (2010): "Testing for Unobserved Heterogeneity in Exponential and Weibull Duration Models," *Journal of Econometrics*, 157, 458–480.

DAVIDSON, J., MONTICINI, A., AND PEEL, D. (2007): "Implementing the Wild Bootstrap Using a Two-Point Distribution," *Economics Letters*, 96, 309–315.

ESCRIBANO, Á AND JORDÀ, Ò. (1999): "Testing Nonlinearity: Decision Rules for Selecting between Logistic and Expnential STAR Models," *Spanish Economic Review*, 3, 193–209.

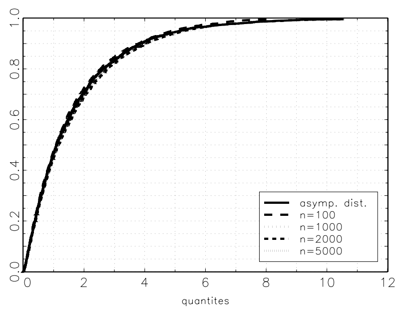
KOOP, G., PESARAN, M.H. AND POTTER, S.M. (1996): “Impulse response analysis in nonlinear multivariate models,” *Journal of econometrics*, 74, 119–147.

LAWRANCE, A. J., AND LEWIS, P. A. W. (1980): “The Exponential Autoregressive-Moving Average EARMA (p, q) Process,” *Journal of the Royal Statistical Society. Series B*, 47, 150–161.

RANGA RAO, R. (1962): “Relations Between Weak and Uniform Convergence of Measures with Applications,” *Annals of Mathematical Statistics*, 33, 659–680.

VAN DIJK, D., TERÄSVIRTA, T., AND FRANSES, P. H. (2002): “Smooth Transition Autoregressive Models–a Survey of Recent Developments,” *Econometric Reviews*, 21, 1–47.

WHITE, H. (2001): *Asymptotic Theory for Econometricians*. Orlando, FL: Academic Press.



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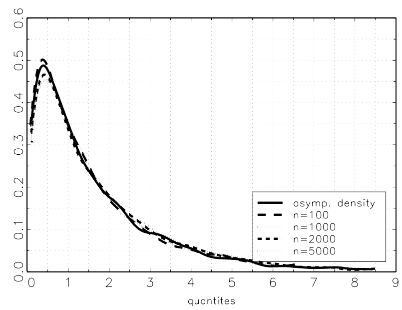

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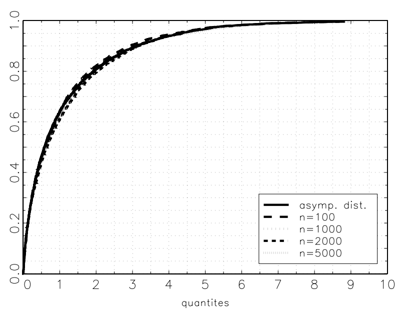

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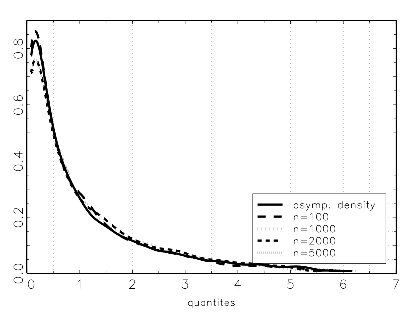

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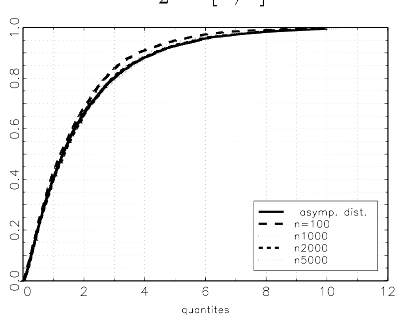

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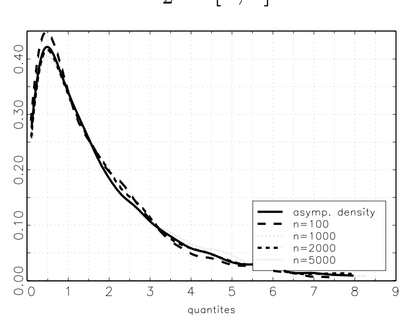

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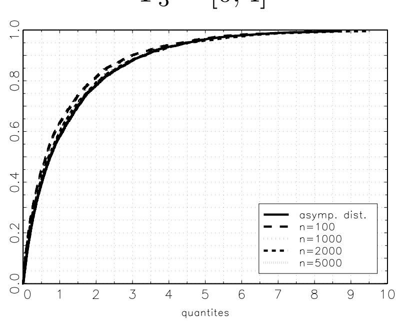

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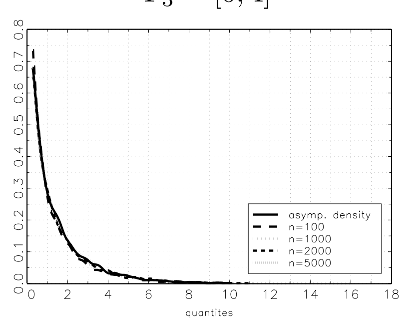
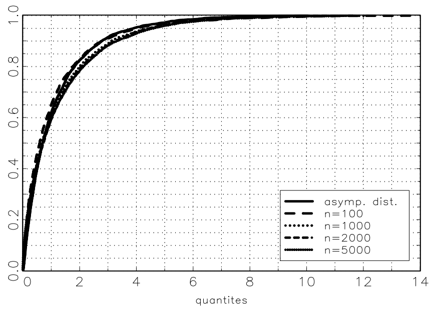


$\Gamma_4 = [0, 5]$

Figure A.1: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\} + u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



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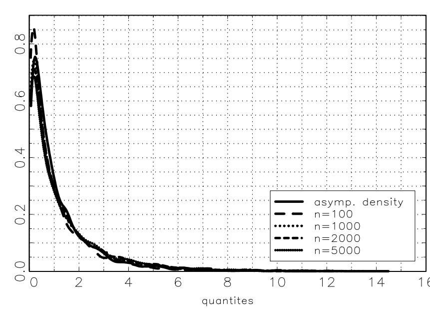

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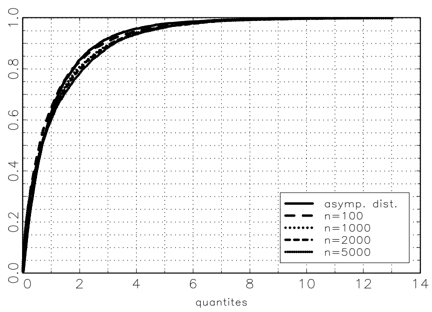

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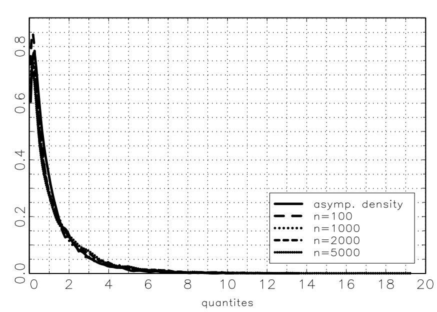

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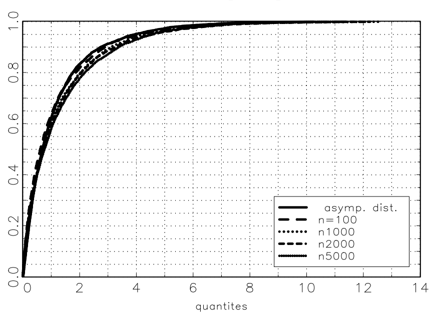

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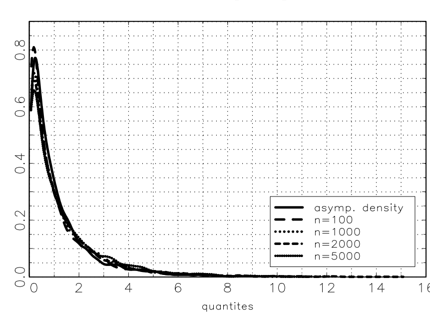

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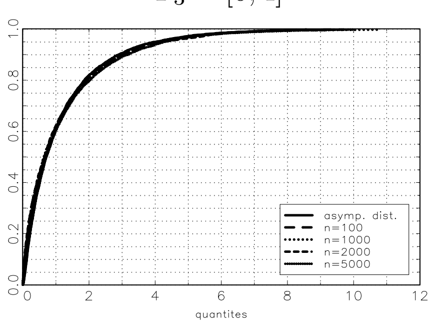

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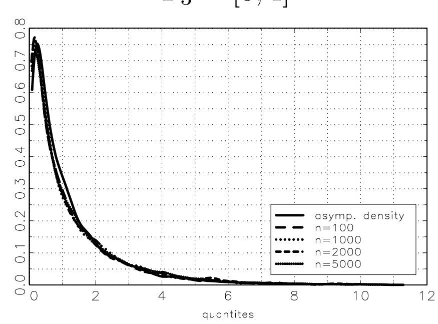
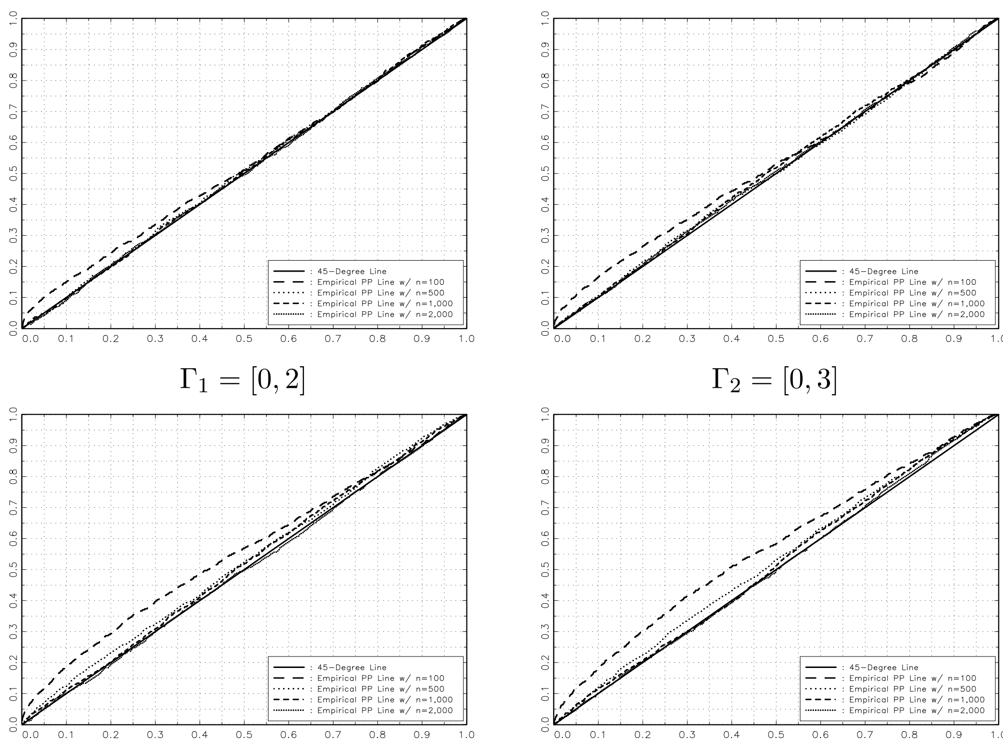


$\Gamma_4 = [0, 5]$

Figure A.2: EMPIRICAL NULL DISTRIBUTIONS OF THE QLR STATISTIC AND ITS NULL LIMIT DISTRIBUTION (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 5,000; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t \ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim \text{Laplace}(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1 + \exp(-\gamma y_{t-1}))^{-1}\} + u_t$ and $u_t = i_t \ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim \text{Laplace}(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.



$\Gamma_1 = [0, 2]$ $\Gamma_2 = [0, 3]$

$\Gamma_3 = [0, 4]$ $\Gamma_4 = [0, 5]$

Figure A.3: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (ESTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t \sim$ IID $N(0, 1)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{1 - \exp(-\gamma y_{t-1}^2)\}u_t$ and $u_t \sim$ IID $N(0, 1)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.

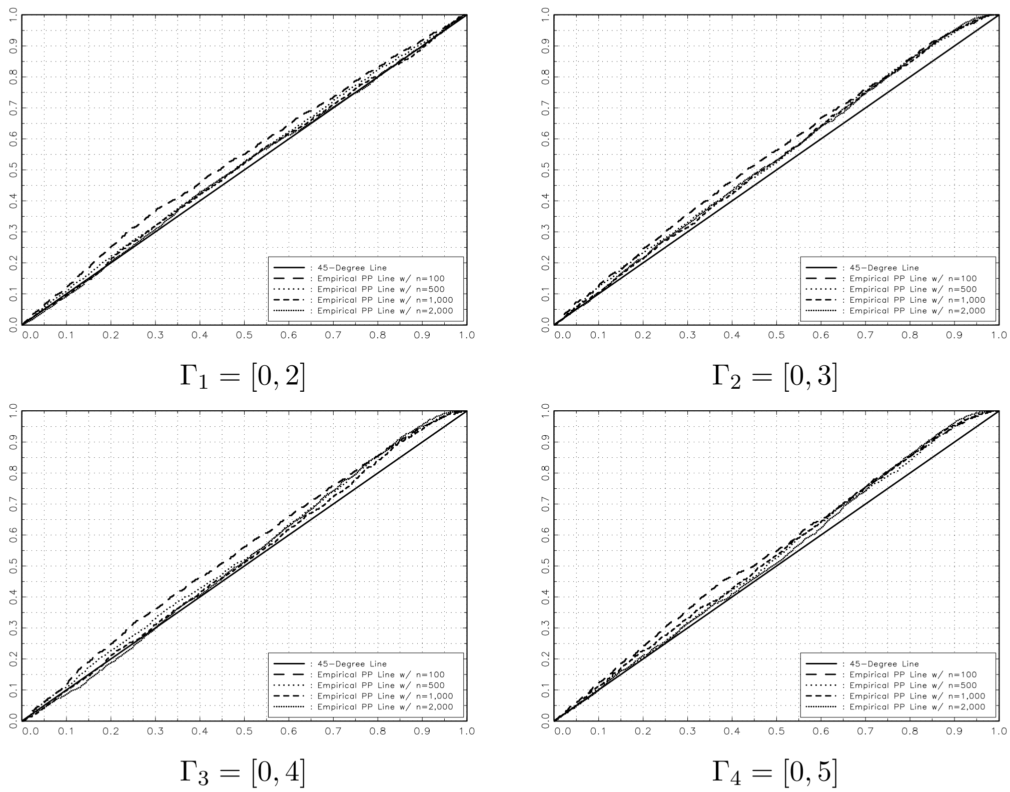


Figure A.4: PP PLOTS OF THE QLR STATISTIC USING THE WEIGHTED BOOTSTRAP (LSTAR MODEL CASE). Notes: (*i*) Number of Iterations: 2,000, Bootstrap Iterations: 300; (*ii*) DGP: $y_t = 0.5y_{t-1} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.25$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; (*iii*) Model: $y_t = \pi y_{t-1} + \theta y_{t-1}\{(1 + \exp(-\gamma y_{t-1}))^{-1} - 1/2\} + u_t$ and $u_t = i_t\ell_t$, where $\{i_t\}$ is an IID sequence in which $\mathbf{P}\{i_t = 1\} = 1 - 0.5^2$ and $\{\ell_t\} \sim$ Laplace$(0, 2)$; and (*iv*) $\Gamma_1 = [0, 2]$, $\Gamma_2 = [0, 3]$, $\Gamma_3 = [0, 4]$, $\Gamma_4 = [0, 5]$.