Due:Nov 14

24 points

1. Indicate which of the following are True (provide a sentence explaining why your choices are correct)[8 points]:

   (a) $2n^3 - 8n^2 + 32n + 9 \in O(n^3)$ **SOLUTION:** True

   (b) $n^p \in O(e^n)$, where $p \in \mathbb{R}$ and $p \geq 0$ **SOLUTION:** True

   (c) $e^n \in O(n^p)$, where $p \in \mathbb{R}$ and $p \geq 0$ **SOLUTION:** False

   (d) $\sqrt{n} \in O(1)$ (side note: $O(1)$ is called 'constant time') **SOLUTION:** False

2. Show that for any real constants x and y, where $y > 0$,
   $(n + x)^y = \theta(n^y)$ [5 points].

   SOLUTION:

   To show that $(n + x)^y = \theta(n^y)$, we want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1 n^y \leq (n + x)^y \leq c_2 n^y$ for all $n \geq n_0$.

   Note that
   $n + x \leq n + |x| \leq 2n \qquad$ when $|x| \leq n$,
   and
   $n + x \geq n - |x| \geq \frac{1}{2}n \qquad$ when $|x| \leq \frac{1}{2}n$,

   Thus, when $n \geq 2|x|$, $0 \leq \frac{1}{2}n \leq n + x \leq 2n$.

   Since $y > 0$,the inequality still holds when all parts are raised to the power y:

   $0 \leq (\frac{1}{2}n)^y \leq (n + x)^y \leq (2n)^y$,

   $0 \leq (\frac{1}{2})^y n^y \leq (n + x)^y \leq 2^y n^y$,

   Thus, $c_1 = (\frac{1}{2})^y$, $c_2 = 2^y$, and $n_0 = 2|x|$ statisfy the definition.

3. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficiently small n. Make your bounds as tight as possible, and justify your answers[8 points].

   (a) $T(n) = 16T(\frac{n}{4}) + n^2$. **Solution**: By Master Theorem $\theta(n^2 lg(n))$

   (b) $T(n) = 2T(\frac{n}{4}) + \sqrt{n}$. **Solution**: By Master Theorem $\theta(\sqrt{n} lg(n))$

   (c) $T(n) = T(n-2) + n^2$.
   Solution: $T(n) = \theta(n^3)$
   $T(n) = T(n-2) + n^2 = T(n-4) + (n-2)^2 + n^2 = T(n-2j) + \sum_{i=0}^{j-1}(n-2i)^2$
   Assuming n is even, n mod $2 = 0$ (we could also solve it with odd). It won't change the asymptotic complexity.
   We have $j = n/2$
   $T(n) = T(0) + \sum_{i=0}^{n/2-1}(n-2i)^2 = \sum_{i=0}^{n/2-1}(n^2 - 4ni + 4i^2) + c$
   $T(n) = n^2(n/2-1)4n1/2n/2(n/2-1) + 41/6(n/2-1)n/2(n-1) + c$
   $T(n) = \theta(n^3)$

   (d) $T(n) = 3T(\frac{n}{3}) + n/lgn$.
   Solution: $T(n) = \theta(n)$

. We first show by substitution that $T(n) \leq n \lg(n)$.

$$T(n) = 3T(n/3)+n/\lg(n) \leq cn\lg(n)-cn\lg(3)+n/\lg(n) = cn\lg(n)+n(\frac{1}{\lg(n)}-c\lg(3)) \leq cn\lg(n)$$

now, we show that $T(n) \geq cn^{1-\epsilon}$ for every $\epsilon > 0$.

$$T(n) = 3T(n/3) + n/\lg(n) \geq 3c/3^{1-\epsilon}n^{1-\epsilon} + n/\lg(n) = 3^{\epsilon}cn^{1-\epsilon} + n/\lg(n)$$

showing that this is $\leq cn^{1-\epsilon}$ is the same as showing

$$3^{\epsilon} + n^{\epsilon}/(c\lg(n)) \geq 1$$

Since $\lg(n) \in o(n^{\epsilon})$ this inequality holds. So, we have that The function is soft Theta of n, see problem 3-5.