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ECE 590: Theory and Practice of Algorithms
September 2, 2024

Homework 1

Question 1. Prove that

$$\forall P, Q \in \mathbb{B}.(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

Note that B is "booleans", so P and Q are propositions. Also note that another way to phrase this questions is to show that the two are equivalent. While you may typically just use the logical equivalences in your logic handout, you may not just directly use this equivalence, as we are asking you to prove it.[10 points]

To prove that

$$\forall P, Q \in \mathbb{B},\ (P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q),$$

We just need to divide into 2 steps:

1) $(P \Rightarrow Q) \implies (\neg P \vee Q)$.
2) $(\neg P \vee Q) \implies (P \Rightarrow Q)$.

1. $P \Rightarrow Q$ IMPLIES $\neg P \vee Q$

Recall this equation is defined as:

$$P \Rightarrow Q \equiv \neg P \vee Q.$$

On the left-hand side: There are 2 conditions that $P \Rightarrow Q$ is true in Boolean terms:

- If P is false(i.e., $\neg P$ is true), $\neg P \vee Q$ is true regardless of Q.
- If P is true, then Q must be true for the implication to hold, and $\neg P$ is false. Thus, Q must be true, making $\neg P \vee Q$ true.

$\therefore P \Rightarrow Q$ implies $\neg P \vee Q$.

2. $\neg P \vee Q$ IMPLIES $P \Rightarrow Q$

Similarly, on the right hand Consider the disjunction $\neg P \vee Q$:

1) If $\neg P$ is true (i.e., P is false), then $P \Rightarrow Q$ holds regardless of Q because an implication with a false premise is always true.
2) If Q is true, then $P \Rightarrow Q$ holds regardless of P because an implication with a true conclusion is always true.

$\therefore \neg P \vee Q$ implies $P \Rightarrow Q$.

CONCLUSION

$\because P \Rightarrow Q$ implies $\neg P \vee Q$ and $\neg P \vee Q$ implies $P \Rightarrow Q$,
$\therefore (P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$. This completes the proof.

Question 2. Prove that

$$\forall n \in \mathbb{N}. \sum_{i=0}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$$

Hint: use induction on n. [20 points]

For $n = 0$:

$$\sum_{i=0}^{0} i^3 = 0^3 = 0,$$

$$\frac{1}{4} \cdot 0^2 \cdot (0+1)^2 = \frac{1}{4} \cdot 0 \cdot 1 = 0.$$

$\therefore$the statement is correct $n = 0$.

Assume the formula holds for $n = k$.

$$\sum_{i=0}^{k} i^3 = \frac{1}{4} k^2 (k+1)^2.$$

For $n = k + 1$:

$$\sum_{i=0}^{k+1} i^3 = \sum_{i=0}^{k} i^3 + (k+1)^3.$$

Now we can use the formula above to substitute $\sum_{i=0}^{k+1} i^3$

$$\sum_{i=0}^{k+1} i^3 = \frac{1}{4} k^2 (k+1)^2 + (k+1)^3.$$

Factor out $(k+1)^2$:

$$\sum_{i=0}^{k+1} i^3 = \frac{1}{4} (k+1)^2 \left(k^2 + 4(k+1) \right).$$

$\because$

$$k^2 + 4(k+1) = (k+2)^2.$$

$\therefore$

$$\sum_{i=0}^{k+1} i^3 = \frac{1}{4} (k+1)^2 (k+2)^2.$$

Conclusion: By induction, the formula

$$\sum_{i=0}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$$

is true for all $n \in \mathbb{N}$.