ECON 200: Spring 2018
Assignment 1, due on Friday, March 23.

1. Consider the following function:

$$f(x) = \begin{cases} x^2(\sin \frac{1}{x} - 1), x \neq 0 \\ \\ 0, x = 0 \end{cases}.$$

Can you show that this function is everywhere differentiable but is not continuously differentiable at $x = 0$?

Answer: Since $d \sin x/dx = \cos x$, we can apply the chain and the product rules to show that for all $x \neq 0$,

$$f'(x) = 2x \left[\sin\left(x^{-1}\right) - 1\right] - \cos\left(x^{-1}\right).$$

As $-2x^2 \leq f(x) \leq 0$ for all x and $f(0) = 0$,

$$\frac{-2x^2 - 0}{x} \leq \frac{f(x) - f(0)}{x} \leq \frac{0 - 0}{x},$$

it is clear that $f'(0) = 0$. Thus, $f(x)$ is everywhere differentiable. But as

$$\lim_{x \to 0} 2x \left[\sin\left(x^{-1}\right) - 1\right] - \cos\left(x^{-1}\right) \neq 0,$$

we see that $f'(x)$ is not continuously at $x = 0$.

2. Examine the definiteness and semi-definiteness of the following quadratic forms:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 0 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ A_4 = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

Answer: $[Z_1, Z_2, Z_3] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_1 \\ Z_3 \end{bmatrix} = 2Z_1Z_3 + Z_2^2 \Rightarrow A_1$ is indefinite;

$[Z_1, Z_2, Z_3] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_1 \\ Z_3 \end{bmatrix} = (Z_1 + 2Z_2)(Z_1 + 2Z_2 + 6Z_3) \Rightarrow A_2$ is indefinite;

$[Z_1, Z_2, Z_3] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_1 \\ Z_3 \end{bmatrix} = (Z_1 + Z_3)^2 + Z_2^2 \geq 0 \Rightarrow A_3$ is positive semi-definite;

$[Z_1, Z_2, Z_3] \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_1 \\ Z_3 \end{bmatrix} = -(Z_1 - 2Z_2 + Z_3)^2 \leq 0 \Rightarrow A_4$ is negative semi-definite.

3. Specify the Hessian $D^2 f$ of each of the following functions. Evaluate the Hessians at the specified points, and examine if the Hessian is positive definite, negative definite, positive semi-definite, negative semi-definite, or indefinite:

a. $f : \mathbb{R}^2 \to \mathbb{R},\ f(x) = x_1^2 + \sqrt{x_2}$, at $x = (1, 1)$.

Answer:

$$D^2 f(x_1, x_2) = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{4}x_2^{-\frac{3}{2}} \end{bmatrix} \Rightarrow D^2 f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \text{ which is indefinite.}$$

b. $f : \mathbb{R}^2 \to \mathbb{R},\ f(x) = \sqrt{x_1 x_2}$, at an arbitrary point $x \in \mathbb{R}^2_{++}$.

Answer:

$$D^2 f(x_1, x_2) = \begin{bmatrix} -\frac{1}{4}x_1^{-\frac{3}{2}}x_2^{\frac{1}{2}} & \frac{1}{4}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}} \\ \frac{1}{4}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}} & -\frac{1}{4}x_1^{\frac{1}{2}}x_2^{-\frac{3}{2}} \end{bmatrix} \Rightarrow \triangle_1 < 0, \triangle_2 = 0$$

$$\Rightarrow \quad D^2 f(x_1, x_2) \text{ is negative semi-definite at an arbitrary point } x \in \mathbb{R}^2_{++}.$$

c. $f : \mathbb{R}^2 \to \mathbb{R},\ f(x) = (x_1 x_2)^2$, at an arbitrary point $x \in \mathbb{R}^2_{++}$.

Answer:

$$\begin{aligned} D^2 f(x_1, x_2) &= \begin{bmatrix} 2x_2^2 & 4x_1x_2 \\ 4x_1x_2 & 2x_1^2 \end{bmatrix} \Rightarrow [Z_1, Z_2]\, D^2 f \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \\ &= 2x_2^2 Z_1^2 + 8x_1x_2 Z_1 Z_2 + 2Z_2^2 x_1^2 \Rightarrow D^2 f(x_1, x_2) \text{ is indefinite.} \end{aligned}$$

d. $f : \mathbb{R}^3_+ \to \mathbb{R},\ f(x) = \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3}$, at $x = (2, 2, 2)$.

Answer:

$$\begin{aligned} D^2 f(x_1, x_2, x_3) &= \begin{bmatrix} -\frac{1}{4}x_1^{-\frac{3}{2}} & 0 & 0 \\ 0 & -\frac{1}{4}x_2^{-\frac{3}{2}} & 0 \\ 0 & 0 & -\frac{1}{4}x_3^{-\frac{3}{2}} \end{bmatrix}, \\ D^2 f(2,2,2) &= \begin{bmatrix} -2^{-\frac{7}{2}} & 0 & 0 \\ 0 & -2^{-\frac{7}{2}} & 0 \\ 0 & 0 & -2^{-\frac{7}{2}} \end{bmatrix} \\ &\Rightarrow \quad \triangle_1 < 0, \triangle_2 > 0, \triangle_3 < 0 \Rightarrow D^2 f(2,2,2) \text{ is negative definite.} \end{aligned}$$

e. $f : \mathbb{R}^3_+ \to \mathbb{R},\ f(x) = x_1x_2 + x_2x_3 + x_3x_1$, at $x = (1, 1, 1)$.

Answer:

$$\begin{aligned} D^2 f(x_1, x_2, x_3) &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow [Z_1, Z_2, Z_3] D^2 f \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \\ &= 2(Z_1Z_2 + Z_2Z_3 + Z_1Z_3) \Rightarrow D^2 f(1,1,1) \text{ is indefinite.} \end{aligned}$$

f. $f : \mathbb{R}^3_+ \to \mathbb{R}$, $f(x) = ax_1 + bx_2 + cx_3$ for some constants a, b, $c \in \mathbb{R}$, at $x = (2, 2, 2)$.

Answer:

$$D^2 f(x_1, x_2, x_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow D^2 f(2, 2, 2) \text{ is both positive and negative semi-definite.}$$

4. Find and classify the critical points (local maximum, local minimum, neither) of each of the following functions. Are any of the local optima also global optima?

a. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$, $(x, y) \in \mathbb{R}^2$.

Answer:

$$\text{F.O.C.:} \begin{cases} \dfrac{\partial f}{\partial x} = 6x^2 + y^2 + 10x = 0 \\ \\ \dfrac{\partial f}{\partial y} = 2xy + 2y = 0 \end{cases} \Rightarrow \text{critical points: } (0, 0), (-5/3, 0), (-1, 2), (-1, -2).$$

$$D^2 f(x, y) = \begin{bmatrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{bmatrix}.$$

$D^2 f(0, 0) = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow D^2 f(0, 0)$ is positive definite

$\Rightarrow (0, 0)$ is a strict local minimizer. It is not a global minimizer since $f(-\infty, 0) = -\infty$.

$D^2 f(-5/3, 0) = \begin{bmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix} \Rightarrow D^2 f(-5/3, 0)$ is negative definite

$\Rightarrow (-5/3, 0)$ is a strict local maximizer. It is not a global maximizer since$f(+\infty, 0) = +\infty$

$D^2 f(-1, 2) = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$, $D^2 f(-1, -2) = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}$, both are indefinite,

$\Rightarrow$ Neither local minimizer nor local maximizer.

b. $f(x, y) = x^4 + x^2 y^2 - y$, $(x, y) \in \mathbb{R}^2$.

Answer:

$$\text{F.O.C.:} \begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 + 2xy^2 = 0 \\ \\ \dfrac{\partial f}{\partial y} = 2x^2 y - 1 = 0 \end{cases}$$

$\Rightarrow$ No critical points $\Rightarrow$ No local minimizer or local maximizer.

c. $f(x, y) = x^4 + y^4 - x^3$, $(x, y) \in \mathbb{R}^2$.

Answer:

$$\text{F.O.C.:} \begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 3x^2 = 0 \\ \\ \dfrac{\partial f}{\partial y} = 4y^3 = 0 \end{cases} \Rightarrow \text{critical points } (0, 0), (3/4, 0).$$

$$D^2 f(x,y) = \begin{bmatrix} 12x^2 - 6x & 0 \\ 0 & 12y^2 \end{bmatrix}.$$

$D^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow D^2 f(0,0)$ is both positive and negative semi-definite. But $(0,0)$ is neither a local maximizer nor a local minimizer, since $f(0,\varepsilon_y) = \varepsilon_y^4 > 0$ when $|\varepsilon_y|$ is very small and $f(\varepsilon_x, 0) = \varepsilon_x^3(\varepsilon_x - 1) < 0$ when ε_x is a very small positive number.

$D^2 f(3/4,0) = \begin{bmatrix} \frac{9}{4} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow D^2 f(3/4,0)$ is postive semi-definite. $(3/4,0)$ is both a local minimizer and a global minimizer. (Note that the objective function is seperable with respect to x and y.)