Nested-fixed point econometric models: Non-linear IV example

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August 16, 2017

Econometric model

- **Reference:** Berry et al. (ECMA, 1995)
- **Panel data-set:** Aggregate data on market shares s_{jt} and characteristics x_{jt}, and instrument vector w_{jt}. Where $j = 1, \ldots, J$ is a product identifier, and $t = 1, \ldots, T$ is a market identifier.
- Consider the following *mixed-logit* demand system:

$$\sigma_j(\delta_t, x_t; \theta) = \sum_i \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \exp(\delta_{jt} + \mu_{ijt})} \omega_i$$

  where ω_i is a **known** PDF weight for consumers of type i, and $\mu_{ijt} = \sum_k \theta_k \eta_{ik} x^{(2)}_{ijt,k}$ is the *idiosyncratic valuation* for product j, and $\delta_{jt} = x_{jt}\beta + \xi_{jt}$ is the *average valuation* for j.
- ξ_{jt} corresponds to the **residual** of the model: Unobserved quality of j.
- **Assumption:** $E(w_{jt} \cdot \xi_{jt}) = 0$ (i.e. moment restriction)

Econometric model

- To impose the moment restrictions, we need to define the *inverse demand* function:

$$\sigma_j^{-1}(s_{jt}, x_{jt}; \theta) = \delta_{jt} = x_{jt}\beta + \xi_{jt}$$

- The inverse function must be solved numerically using the following contraction mapping:

$$\delta_{jt}^{l+1} = \delta_{jt}^{l} + \ln s_{jt} - \ln \sigma_j(\delta_t^l, x_t; \theta)$$

Convergence: $|| \ln s_{jt} - \ln \sigma_j(\delta_t^l, x_t; \theta)|| < \varepsilon$ (e.g. 10^{-12})

- This inverse mapping implicitly defines the residual function (given parameters):

$$\rho_{jt}(s_t, x_t; \theta, \beta) = \sigma_j^{-1}(s_{jt}, x_{jt}; \theta) - x_{jt}\beta$$

- This leads to a non-linear IV problem (or GMM):

$$\min_{\theta,\beta} \rho(s_t, x_t; \theta, \beta)^T W A^{-1} W^T \rho(s_t, x_t; \theta, \beta)^T$$

Sketch of the code

1. Demand(δ_t,$x_t^{(2)}$,η,ω,θ):
   - Step 1: Calculate $\mu_{jt} = \sum_k \theta_k \eta_i$
   - Step 2: Calculate predicted market shares for all products in market t
   - Return $= \sigma_j(\delta, x_t; \theta)$
2. Inverse($x^{(2)}, \eta, \omega, \theta$)
   - For each market t, solve the fixed-point:

$$
\begin{aligned}
&\text{Starting values: } \delta_{jt}^0 \\
&\text{do}\{ \\
&\quad \hat{s}_{jt} = \text{Demand}(\delta_t^l, x_t^{(2)}, \eta, \omega, \theta) \\
&\quad \delta_{jt}^{l+1} = \delta_{jt}^l + \ln s_{jt} - \ln \hat{s}_{jt} \\
&\quad l = l + 1 \\
&\}\text{while}(||\delta_{jt}^l - \delta_{jt}^{l-1}|| < 10^{-12})
\end{aligned}
$$

   - **Note:** The fixed-point algorithm can easily be distributed across multiple processors.
   - Return $= \delta_{jt}$ for all j and t

Sketch of the code

- GMM(s, x, z, θ)
  - δ_{jt} =Inverse$(x^{(2)}, \eta, \omega, \theta)$
  - Find the value of β given θ (i.e. OLS):

$$\beta(\theta) = (x'x)^{-1}x'\delta$$

  - Compute residual: $\rho_{jt}(s_t, x_t; \theta, \beta) = \sigma_j^{-1}(s_{jt}, x_{jt}; \theta) - x_{jt}\beta$
  - Return $= \rho(s_t, x_t; \theta, \beta)^T WA^{-1}W^T \rho(s_t, x_t; \theta, \beta)^T$