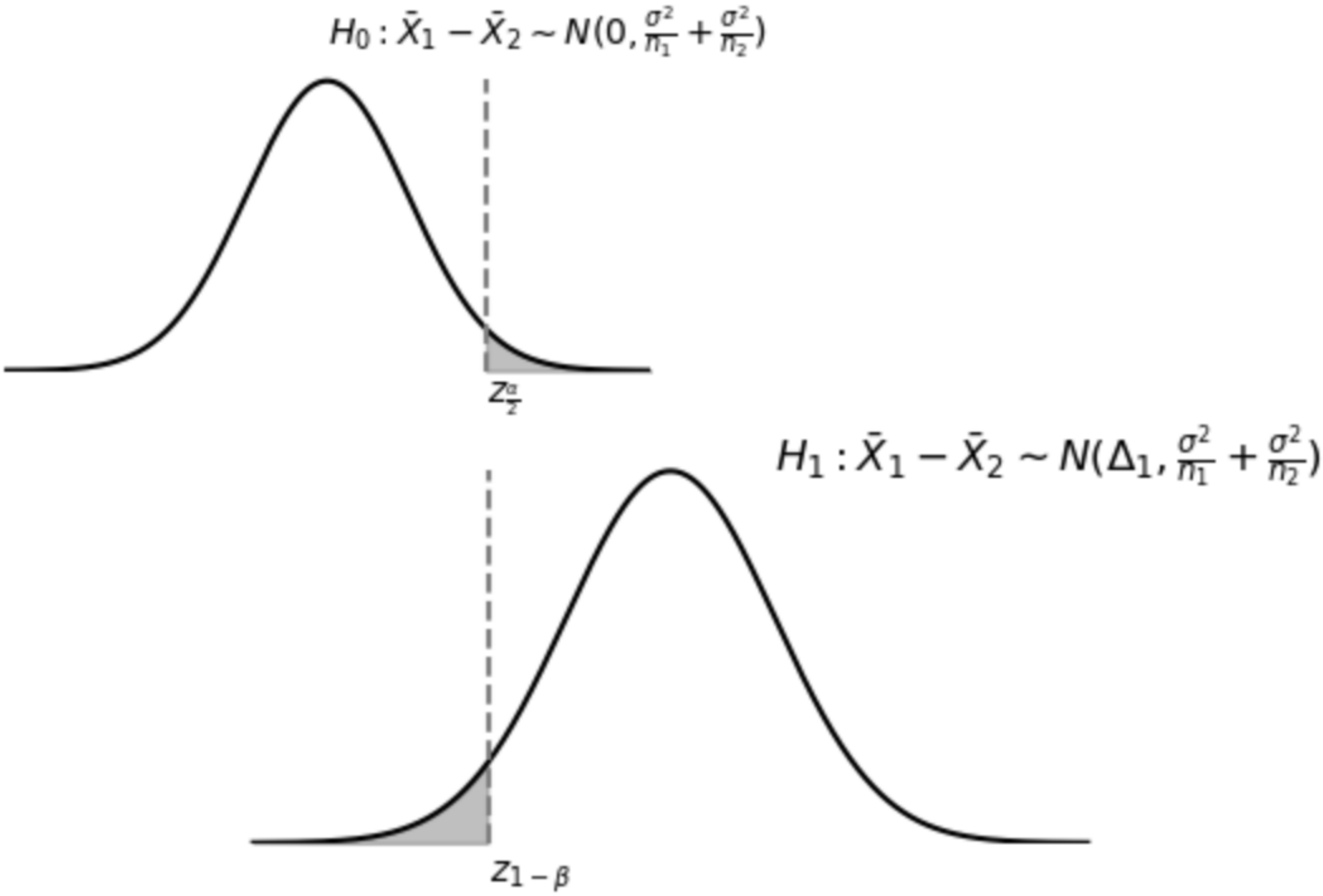
方差缩减技术



根据统计功效定义有$P($拒绝原假设$H_0|$原假设H_0错误$) = 1 - \beta$

以上公式暗含两个问题：

在原假设下，拒绝原假设需要样本均值的差值满足什么条件。

以双边检验为例，我们知道，当样本差值满足$P(\frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_1^2}{n_2}}} >= Z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$时，我们会拒绝原假设。其中$Z_{\frac{\alpha}{2}}$为标准正态分布上$\frac{\alpha}{2}$分位数。解上式有$\bar{x}_1 - \bar{x}_2 >= Z_{\frac{\alpha}{2}} \times \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

在备择假设正确的条件下，且统计功效等于$1 - \beta$时, 我们有

$$P(\bar{x}_1-\bar{x}_2>=Z_{\frac{\alpha}{2}}\times\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}})=1-\beta \tag{1}$$

$$\text{基于}H_1\text{假设，标准化后有}P(\frac{\bar{x}_1-\bar{x}_2-\Delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}>=Z_{\frac{\alpha}{2}}-\frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}})=1-\beta \tag{2}$$

$$\text{基于标准正态分布的定义有，}Z_{\frac{\alpha}{2}}-\frac{\Delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}=Z_{1-\beta} \tag{3}$$

$$\text{移项有，}\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}=\frac{\Delta}{Z_{\frac{\alpha}{2}}-Z_{1-\beta}} \tag{4}$$

$$\text{等式两边同时平方有,}\frac{\sigma^2}{n_1}+\frac{\sigma^2}{n_2}=\frac{\Delta^2}{(Z_{\frac{\alpha}{2}}-Z_{1-\beta})^2} \tag{5}$$

$$\text{当}n_1=n_2=\frac{n}{2}\text{时有，}\frac{\sigma_1^2+\sigma_2^2}{n_1}=\frac{\Delta^2}{(Z_{\frac{\alpha}{2}}-Z_{1-\beta})^2} \tag{6}$$

$$\text{移项有，}n_1=\frac{(Z_{\frac{\alpha}{2}}-Z_{1-\beta})^2(\sigma_1^2+\sigma_2^2)}{\Delta^2} \tag{7}$$

$$\text{根据正态分布的对称性有，}Z_{\frac{\alpha}{2}}=-Z_{1-\frac{\alpha}{2}}\text{,带入有，}n_1=\frac{(Z_{1-\frac{\alpha}{2}}+Z_{1-\beta})^2(\sigma_1^2+\sigma_2^2)}{\Delta^2} \tag{8}$$

$$\text{当}\sigma_1^2=\sigma_2^2=\sigma^2\text{时，有}n_1=\frac{(Z_{1-\frac{\alpha}{2}}+Z_{1-\beta})^2\times 2\sigma^2}{\Delta^2} \tag{9}$$