BUSINESS MATHEMATICS for MBAs

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FIRST EDITION

Business Mathematics For MBAs

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Preface

This eBook has been developed from notes that formed the basis for the MBA Math *Bootcamp* class taught at Wharton. This class is now delivered through an online video format and students take it prior to their first Quarter of classes. Given that MBA students' most constrained resource is their time, this eBook is intentionally concise.

Though the original notes on which the eBook is based were developed to be used in conjunction with the video course, I believe that students will find the eBook useful in its own right as a stand-alone self-study business math course.

The topics covered in the eBook were identified through consultation with the faculty teaching the MBA core quantitative classes and these topics are considered an essential foundation for an MBA degree. The goal of this eBook is to provide a mathematical underpinning so students taking quantitative classes can focus on the substantive ideas in that class, rather than getting distracted by mathematical details.

The eBook includes links to questions that test students' grasp of the material, enabling them to get immediate feedback on their level of understanding. Some questions test concepts directly whilst others test ideas within the context of a business setting. There are solutions to every question and these solutions are linked directly from the eBook. Of the 120 practice questions more than 60% have video solutions with an accompanying PDF file of that video solution. Working through the eBook should take about 20 hours and completing the practice questions an additional 10 hours.

This eBook was created for the EPUB 3 standard. This format is supported by a variety of eBook readers. The mathematical content in the book is most reliably viewed with an eBook reader set to a *scrolling view* and orientated in the *landscape* direction. All images are in high resolution so will render clearly on a retina display quality device. My own experience suggests that the Apple iBooks reader is the most reliable eBook reader for mathematical content. The Firefox browser with the EPUBReader add-on installed also works well, but depending on which operating system and version of Firefox you have, you may need to download the MathML fonts for the mathematical symbols to be displayed correctly.

About the author

Richard Waterman holds a Ph.D. in Statistics from Penn State University. He is a Practice Professor of Statistics at the Wharton School and has been teaching there since 1993. He regularly taught the MBA Math "Bootcamp" class and teaches extensively in both the Executive and Regular MBA programs. Richard has regularly received teaching awards including the Helen Kardon Moss Anvil Award that is voted on by the MBA student body and presented to the Professor who has exemplified outstanding teaching quality during the previous year.

He has published work in research journals including *Statistical Science*, *Biometrika* and the *Journal of the Royal Statistical Society* on statistical modeling methodologies. He is also the co-author of two case-books: *Basic Business Statistics* and *Business Analysis Using Regression* published by Springer-Verlag.

In addition to teaching at Wharton, he runs a quantitative business consulting company and has consulted widely with clients ranging from Fortune 500 companies to start-ups. He currently focuses most of his consulting activity on quantitative issues related to legal matters and, in particular, to Intellectual Property and Copyright.

Introduction

MBA programs offer a variety of classes, some of them more quantitative than others. Within an MBA program, the ideas in this eBook are likely to appear in Finance, Marketing, Operations, Statistics and Managerial Economics classes.

The primary goal of this eBook is to provide students with a mathematical refresher to prepare them for these quantitative classes. This will allow them to concentrate on the new ideas presented in these courses rather than getting weighed down by the mathematical details.

The eBook focuses on the four key functions that are used the most often in business mathematics. These functions are the linear, power, exponential and log functions.

Module 1 introduces the concept of a mathematical function. The linear function is the focus of Module 2. Module 3 introduces the power, exponential, and log functions. Modules 4 and 5 illustrate the use of these functions in models for growth and decay. These models are important because they the basis for the calculation of the present and future value of an income stream.

The derivative is introduced in Module 6 and optimization is covered in both Modules 6 and 7. Module 8 reviews functions of more than one variable; these multivariate functions are required to create realistic models such as the multiple regression model frequently seen in an MBA statistics or business analytics class.

Modules 9 and 10 introduce probability and statistics.

There are five icons in the text and their actions are described in the table below.

The recommended approach to using this eBook is to read through the materials up until the point at which a link to a question appears. At this point, if you are already familiar and comfortable with the ideas, then just

Icon	Action
	Link to a question.
	Link to an answer.
	External video solution.
	PDF file of the video solution.
	Back to course content.

keep going. However, if you want some practice to reinforce the content, then work through the question. You can then follow another link to review the answer. If the answer still doesn't make sense, you can then follow the link to the video solution. This video will appear in its own window outside of the eBook. You can also download the PDF solution that matches the video.

Contents

Module 1

Relationships/Functions

1.1 Introduction to Module 1

1.1.1 Module 1 overview

Module 1 overview

Topics to be covered in this module:

- The key concept in this module is the *relationship* between variables, and in particular, mathematical functions of a single variable. We will see three ways of representing a function:

    1. The formula representation of a function.
    2. The tabular representation of a function.
    3. The graphical representation of a function.

- Graphing functions.
- The composition of functions: the output of one function becomes the input to another.
- Multivariate functions: relationships that depend on many input variables.
- Summary.

1.2 Relationships/Functions

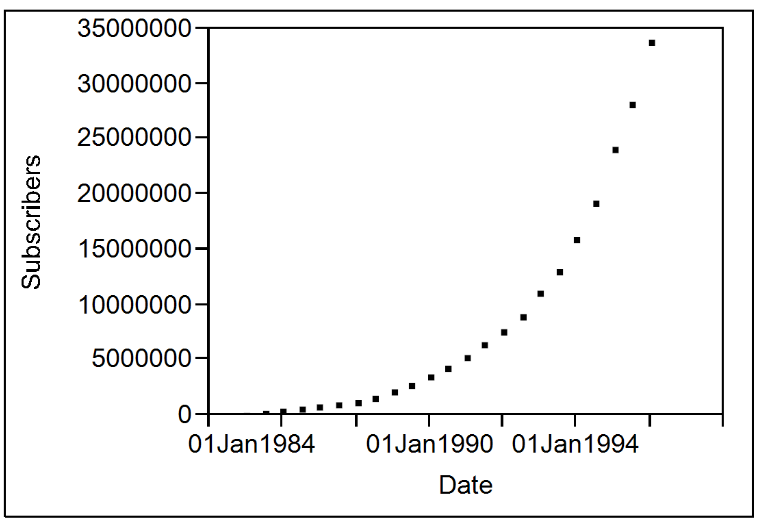
1.2.1 Introduction to functions

Introduction to functions

Interesting questions are usually about relationships between variables. Variables are denoted by symbols and can take on a range of values. In the business context, variables often correspond to metrics and values of processes of interest. The key question of interest is "how is a change in one variable associated with the change in another?" For example:

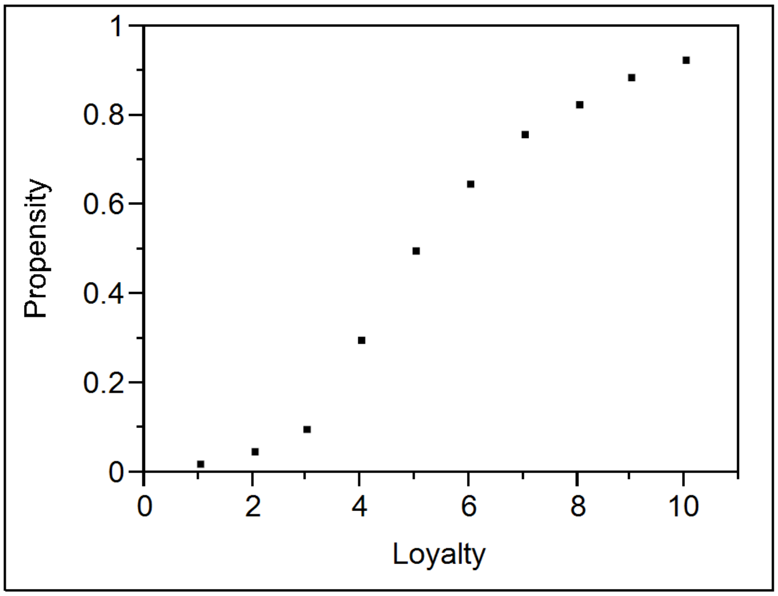
- How does the rate of inflation depend on the unemployment rate?
- How has the number of cellular phone subscribers in the US changed over time?

Figure 1.1: A time-series plot of the number of US cell phone subscribers between 1985 and 1996



- What is the relationship between a customer's current loyalty for a particular product and the chance (propensity) that they purchase this product again?

Figure 1.2: Propensity to purchase a product as a function of customer loyalty



- How does the number of production units transported relate to the **total cost** of transportation?
- These are all examples of the relationship between two variables.

1.2.2 What is a function?

What is a function?

- We can generalize these examples to the statement "how is the value of one variable associated with the value of another?"
- Mathematically, we describe relationships through the use of an object called a **function**.
- A function is simply a "rule". The rule takes a value as an input and provides a specific and unique output.

- Both the input and output are called **variables**, with the input often described as the **independent variable** and the output as the **dependent variable**.
- Another common terminology is to call the input the **x-variable** and the output the **y-variable**.
- The potential values that x can take are called the **domain** of the function.
- The potential values that y can take are called the **range** of the function.

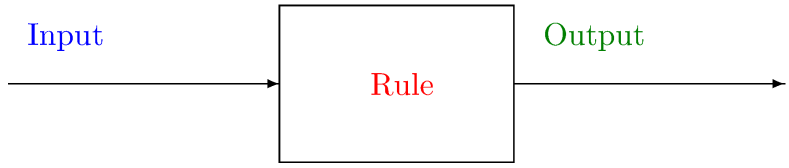
For the customer loyalty example:

- The x-variable is **customer loyalty**.
- The y-variable is **the chance that they repeat purchase**.
- The domain of the function that relates loyalty to chance of a repeat purchase is the **interval** [**1,10**] because that's how the loyalty metric was defined.
- The range of the function is [**0,1**] because it is a probability/propensity and probabilities must lie between 0 and 1 inclusive.

Depicting a function

We can consider a function as depicted by the following diagram: an input goes into the box, the rule is in the box, and the output comes out. Pictorially it would look like this:

Figure 1.3: Pictorial depiction of a function



Notation for functions

One of the benefits of thinking mathematically is that it provides a concise shorthand notation. In particular, we represent a function in the form $y = f(x)$.

Here, y is the output variable, x is the input variable and the rule is denoted simply by the letter f. In English, the expression $y = f(x)$, can be translated as

> The function f, takes an input variable, x, and produces an output variable, y.

For the customer loyalty example described previously, the input is loyalty, the output is propensity to repeat purchase, and the function f describes how propensity depends on loyalty.

1.2.3 Why do we care about functions?

Why do we care about functions?

Ask yourself an interesting practical question and see if it's not about the relationship between variables.

To accomplish tasks you have to understand the world in which you work. To understand, we need to describe. There are many ways to describe the world and a mathematical description (through a function) offers one view. This is an extremely powerful representation because:

- A function forces you to define terms.
- A function makes you think through assumptions.
- A function allows you to explore scenarios.
- A function provides for a concise description of a problem.

In summary, we can understand functions as descriptions and summaries of the world about us.

1.2.4 Examples of functions

The constant function

The most simple function of all is an extremely straightforward one. It's called the **constant** function.

- An example of the constant function is the fixed-price all-you-can-eat buffet.
- Recall that a function is a rule that describes how an output depends on an input. Here the input is the quantity you eat and the output of the "cost function" is the price you pay for it ($8.99 in this instance). Of course, by definition there is **no** relationship between quantity eaten (x-variable) and the price (y-variable), that is, the amount you pay does not depend on how much you eat.

Figure 1.4: The constant function

Examples of the constant function

The constant function sometimes shows up as a component of more complicated functions. Two examples of the constant function that have interpretations in a business context are:

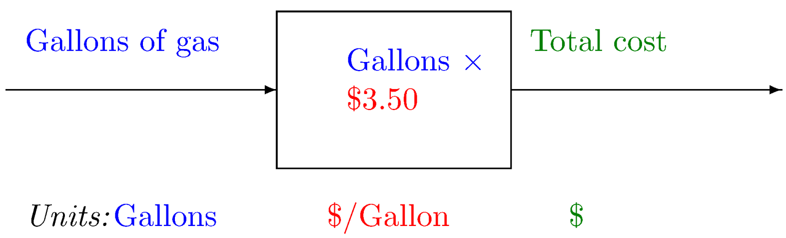
- **Fixed** costs in a cost model. Fixed costs are that part of total cost that do not vary with volume. So by definition the fixed costs do not depend on the output volume and are therefore constant
- **Start-up time** in a productivity function. Start-up time is the component of the time required to perform a job that does not depend on the volume of output and, again, by definition is constant.

The multiplicative function

The next function we will discuss is called the **multiplicative** function. It takes an input and multiplies it by a fixed value.

A familiar example of this function is the *multiplicative* function that relates the quantity of gas purchased to the total cost of filling up a vehicle. For example, if gas is priced at $3.50/gallon, then we have the following rule for determining the total cost:

Figure 1.5: The multiplicative function



Balancing the units of measurement

- Continuing the previous example, written mathematically, we have $y = f(x)$, where f is the function that multiplies the number of gallons (x) by 3.50 to produce the total cost (y).
- In a business school setting we stress the practical interpretation of concepts covered. In the above example it is very helpful for interpretation to recognize that all the quantities in the equation have units attached.
- For example, x is the number of gallons and y is the cost in dollars. It follows that in order to make the equation balance (the units on the right hand side must be the same as the units on the left hand side), the multiplicative factor must be measured in dollars per gallon.
- Balancing the units of measurement is a helpful idea when you are given a function and asked to interpret it.

The function as a table

Another representation of a function is as a table, where the two columns denote the values of x and y respectively, and each row maps the value of x into y. For example, the following table indicates that 20 gallons of gas will cost \$70.00.

Table 1.1: Tabular representation of the function relating gallons of gas purchased to total cost.

x = # of Gallons	y = Total cost
1	3.50
2	7.00
5	17.50
10	35.00
20	70.00
30	105.00

The linear function

- A straightforward way to understand the **linear function** is to start with the example of converting temperatures between measurement scales.
- To convert from Celsius to Fahrenheit you use the rule (function) "multiply by 9/5 and then add 32". If a function only involves multiplication and addition of the input, then it is termed a **linear** function.
- You can see that this example is a combination of the previous two functions. It involves a multiplication step (multiply by 9/5) and then the addition of a constant (32).

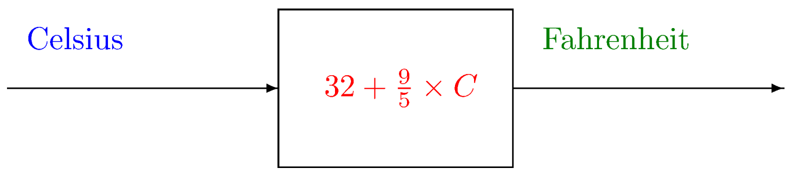
Module 1. Q7.

1.2.5 Ways of expressing a function

Formula representation of a linear function

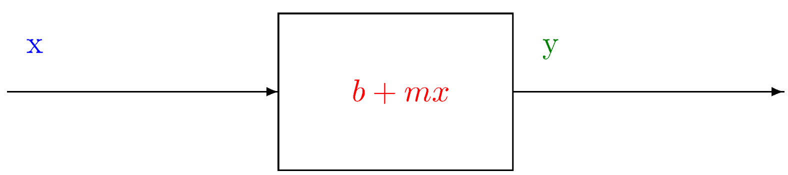
This temperature conversion function can be represented as:

Figure 1.6: Transforming Celsius to Fahrenheit



This is a specific case of the general linear function that is represented as:

Figure 1.7: The linear function

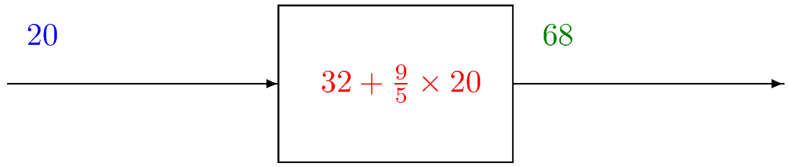


For the temperature conversion $b = 32$ and $m = \frac{9}{5}$.

Converting 20 degrees Celsius to Fahrenheit

For example, we can see how the rule takes as input 20 degrees Celsius and converts it to Fahrenheit:

Figure 1.8: An example of the conversion of Celsius to Fahrenheit



Tabular representation of the temperature conversion function

The tabular representation of the function that describes the conversion between the two temperature scales is presented below:

Table 1.2: The tabular representation of the Celsius to Fahrenheit conversion

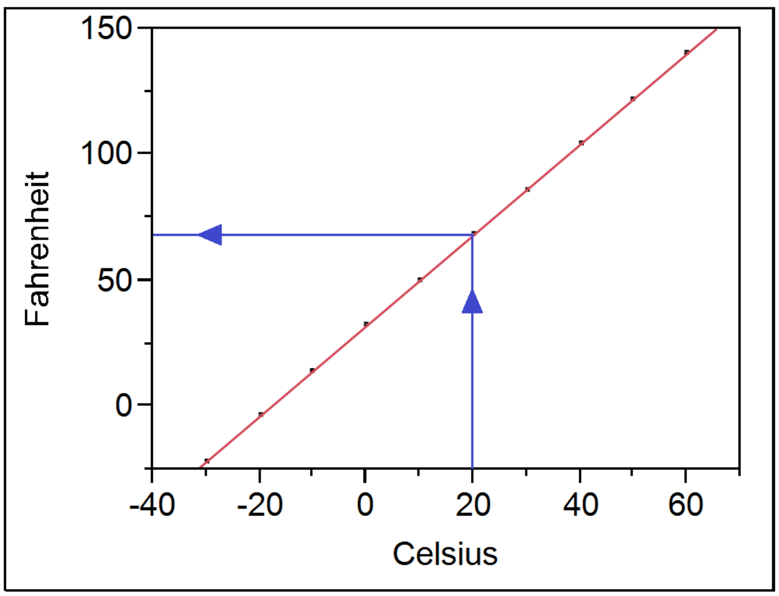
Celsius	Fahrenheit
-50	-58
-40	-40
-30	-22
-20	-4
-10	14
0	32
10	50
20	68
30	86
40	104
50	122

Module 1. Q1. Module 1. Q6.

Graphical representation of the function that converts Celsius to Fahrenheit

The temperature conversion function can also be represented graphically. For any particular value of temperature in degrees Celsius, draw a vertical line from the value until it reaches the red line and then read off the value in degrees Fahrenheit from the scale on the vertical axis. This is illustrated below for 20 degrees Celsius.

Figure 1.9: The graphical representation of the Celsius to Fahrenheit conversion, with the case of 20 degrees Celsius identified



1.2.6 Scratches on paper

Scratches on paper

It is important to realize that when a function is written $y = f(x)$, the letters used are just a convention. The input variable does not have to be written as x, nor the output as y. After all, these are nothing but scratch marks on paper.

The Celsius to Fahrenheit conversion function could just as well be written $f = g(c)$, where c stands for the temperature in degrees Celsius, f for the temperature in degrees Fahrenheit, and now g represents the rule "multiply by 9/5 then add 32". That is:

$$f = g(c) = 32 + \frac{9}{5} \times c.$$

Again, c is the input, g represents the rule and f now stands for the output.

What all functions have in common is that they take an input, apply a rule to it and produce a unique output. How we label the input, the rule, and the output is arbitrary. However, it is extremely important that we tell everyone else what our labels mean.

Module 1. Q11.

Recap

To recap, we have seen that functions are simply rules. There are three ways to represent a function:

- As a mathematical formula.
- As a table.
- Graphically.

Each has its own advantages:

- Formula: good for mathematical manipulation to gain further insight.
- Table: everything is pre-calculated, no new calculations are necessary (e.g. fast computer look-ups, tax tables).
- Graphically: good for presentation and qualitative descriptions.

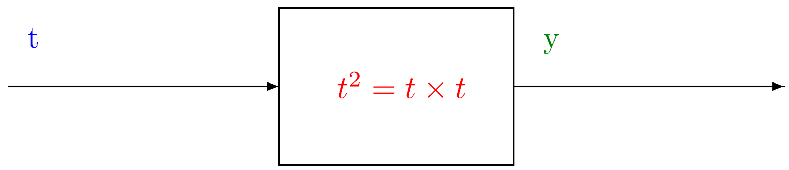
1.2.7 Non-linear functions

Non-linear functions

- So far, the three functions we have looked at have all been linear functions.
- The constant and multiplicative functions are special cases of a linear function.
  - The constant function is a linear function where the multiplicative factor is zero.
  - The multiplicative function is a linear function in which the constant is zero.
- Linear functions are fundamental to all applied mathematics. They are the main subject of Module 2 in this series.
- You will see them everywhere, particularly in Managerial Economics, Operations, Finance, and Statistics.
- However, they are not the only functions. Functions that are not linear, are, not surprisingly, called **non-linear**.
- A function is non-linear when the magnitude of the change in the output for a given absolute change in the input depends on the value of the input itself.
- For example, if an increase in age from 25 to 30 years is associated with an expected increase in salary of $10,000, but an increase of age from 40 to 45 is associated with an expected rise in salary of $15,000 then this is a non-linear function. It is non-linear because age is changing by the same amount, 5 years in each case, but the expected increase in salary is different, $10,000 and $15,000 respectively.

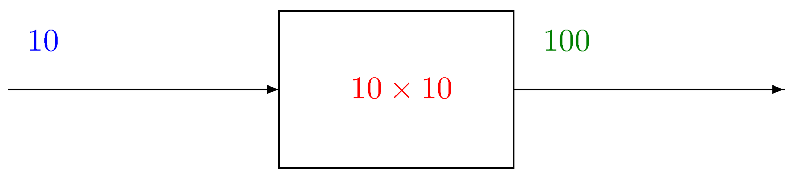
An example of a non-linear function is the **square function**. The rule for the square function is to multiply the input by itself. It can be represented as:

Figure 1.10: The square function



In the particular case of x = 10 we would have:

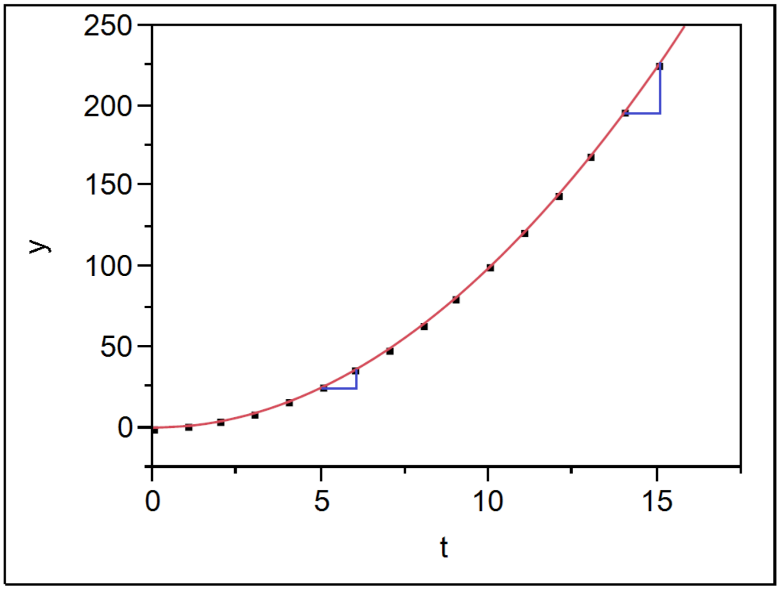
Figure 1.11: A particular instance of the square function



A graph of the square function

Notice that the same one-unit absolute change in the value of x, illustrated in the graph as x changing from 5 to 6 and x changing from 14 to 15, is associated with a different absolute change in y. When x goes from 5 to 6, the value of the squared function changes by 11 (that is 36 - 25), but when x moves from 14 to 15 it changes by 29 (that is 225 - 196).

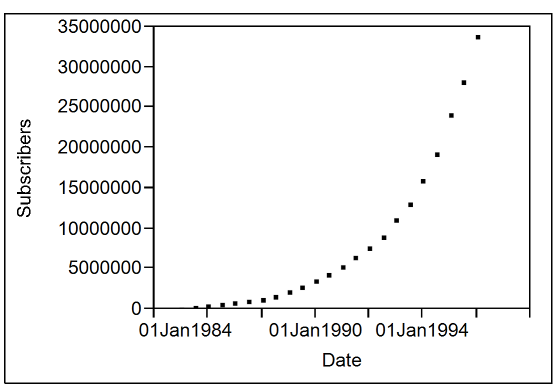
Figure 1.12: Graph of the square function

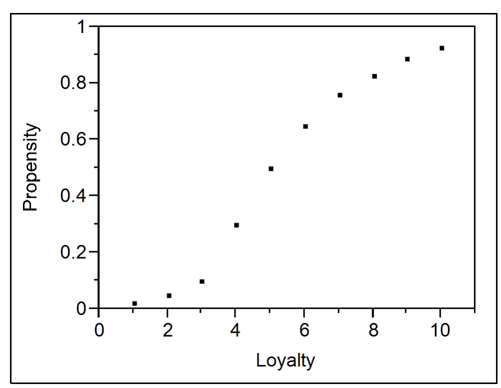


This is the essential feature of a non-linear function: that the amout that y changes as x changes, depends on the value of x itself.

The two example relationships

Both of the introductory examples on cell phones and customer loyalty show *non-linear* behavior.





Module 1. Q2. Module 1. Q5.

1.3 Graphing functions

1.3.1 Graphing functions

Graphing functions

- Now that we have discussed linear and non-linear functions, we will move on to sketching and graphing them. It is not uncommon to be given a function in the form of an equation. However, we gain additional insight by graphing the function.

- Since a function, by definition, relates inputs to outputs, it is natural to think of these inputs and outputs as **pairs** of numbers.

- For example the Celsius to Fahrenheit table can be represented by the pairs: (-50,-58), (-40,-40), (-30,-22), (-20,-4), (-10,14), (0,32), (10,50), (20,68), (30,86), (40,104), (50,122).

- These pairs can be graphically represented as points. The first element of the pair is the x-coordinate and the second element is the y-coordinate of the point.

- The values in the point pairs indicate how far the point is away from a set of **axes**. The horizontal axis is called the **x-axis**, and the vertical is the **y axis**. For example, the first point is (-50,-58), which means that the x-value is -50 and the y-value is -58.

- The graph provides a visual description of the relationship and is often the most direct way of understanding the behavior of a function.

- In practice, computer software is typically used to do the graphing.

The questions below will give you practice in graphing functions.

Module 1. Q3. Module 1. Q9.

1.4 Functions working on functions

1.4.1 Functions working on functions

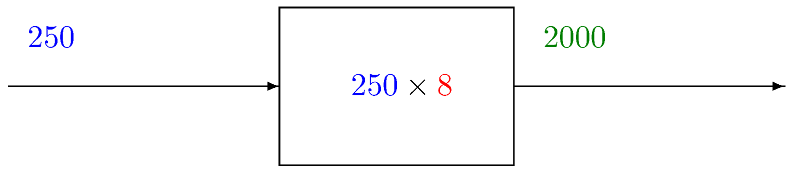
Functions working on functions

- Many processes are more complicated than a single input with a single output. In particular, the output of one process often leads into another.
- Put another way, a key problem-solving skill is to be able to take a complex problem and break it down into a sequence of simpler ones.
- As an example of the output of one process becoming the input of another, assume that an organization has 250 employees and each employee works 8 hours a day. The hourly wage rate is \$20/hour.
- In order to calculate the total labor costs per day, a two-step procedure is needed. First, calculate the total employee hours per day. This is represented by the function $y = f(x)$, where y is the total number of employee hours per day, x is the number of employees, and f is the rule "multiply by 8". That is, $y = 8 \times x$ or $y = 8x$ (as a convention we often drop the multiplication sign between a number and a variable).

Example

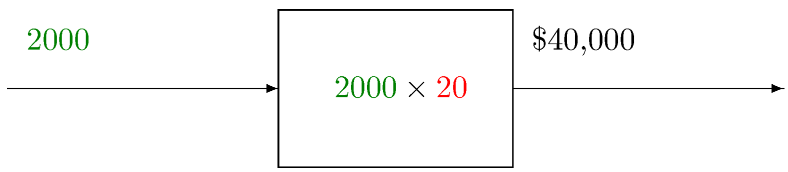
In particular:

Figure 1.13: The labor hours function



As the wage rate is \$20/hour, if we take the employee hours per day and multiply by 20, then we get the total labor cost per day. This can be represented as $z = g(y)$, where z is the total labor cost per day, y is the total employee hours per day, and g is the rule "multiply by 20".

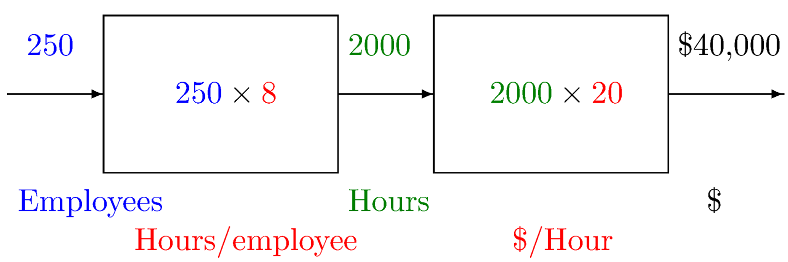
Figure 1.14: The labor cost function



The composition of functions

We could combine the above two rules into a single representation known as the **composition** of functions:

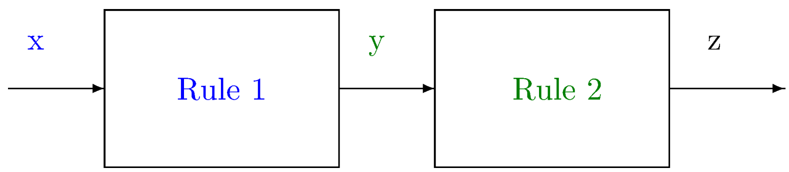
Figure 1.15: The composition of functions



The text beneath the boxes show how the units change with the application of the function. *Employees* goes into the first box where it is multiplied by *Hours/employee* to get *Hours*, which then goes into the second box and is multiplied by *dollars/hour* to give a final output in dollars.

Using the functional representation for the inputs and outputs, we have:

Figure 1.16: The general composition of functions



This is written symbolically as $z = g(y) = g(f(x))$. It is called a **composition** of functions.

The value of this representation, is, as stated earlier, that good problem solving often involves breaking down complex problems into a chain of simpler ones. Later in this eBook, we will study particular techniques for analyzing these compositions (or chains) of functions.

Module 1. Q8.

1.5 Functions with more than one input

1.5.1 Introduction

Multivariate functions, functions with more than one input

- So far we have only looked at functions that depend on a single input. In order to have a richer set of descriptors for a variety of processes, it is useful to recognize that relationships often depend on more than a single input.

- For example, a model for a person's salary is likely to have at least two inputs: education **and** experience.

- Likewise, the amount of money realized by an investment depends on both the interest rate and the length of time that the investment is held.
- This leads to the idea of **multivariate** functions. These functions depend on more than a single input.
- In particular, a function of two variables is a rule that takes two inputs and applies the rule to them to produce a unique output.

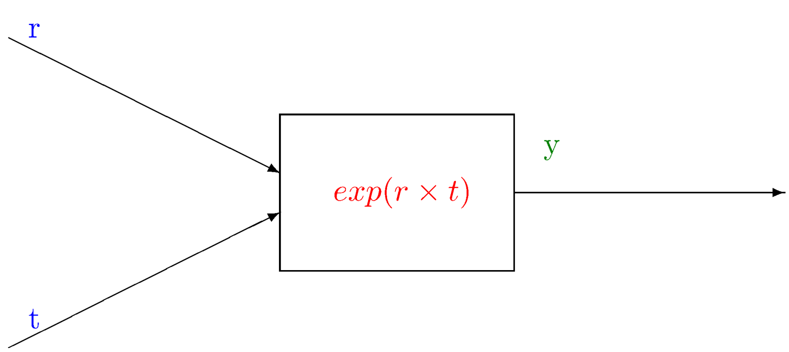
1.5.2 Example: compounding investments

Example: compounding investments

Later in this eBook we will study continuous compounding, which among other things, can be used to describe how money grows over time. If one dollar is initially invested at an annual nominal interest rate of $r\%$, then t years in the future the amount of money available is equal to $exp(r \times t)$. In this formula, the term **exp** stands for the *exponential* function which will be studied in detail in Module 3.

This relationship can be represented as:

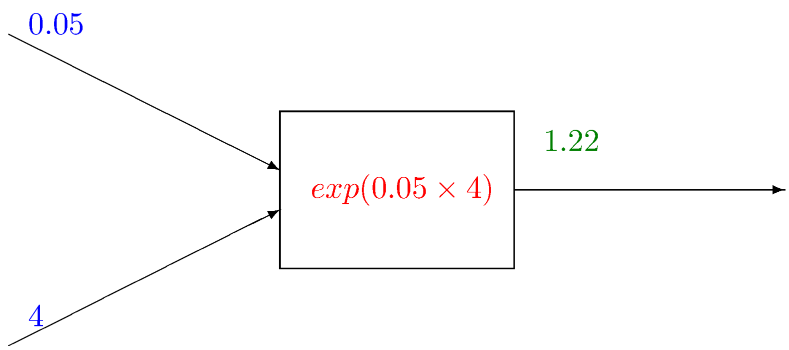
Figure 1.17: A function with two inputs



An example calculation using a multivariate function

Suppose the interest rate is 5%, and that the dollar is invested for 4 years, then $r = 0.05$ and $t = 4$. This can be represented in the diagram as:

Figure 1.18: A particular case of the multivariate function with two inputs



Symbolically we write:

$$\text{Value} = \text{y} = \text{f(r,t)},$$

which displays the fact that the value of the investment, y, depends on two quantities, r, the interest rate, and t, the length of time that the investment is held. Don't worry about how the 1.22 is calculated. We will get to that in Module 3.

Module 1. Q10.

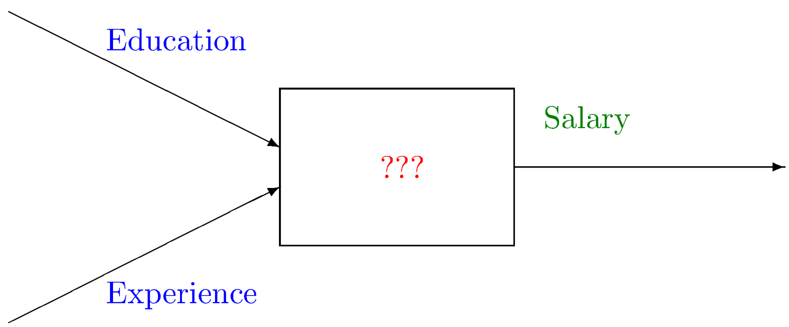
1.5.3 Turning the question around: What's in the box?

Reverse engineering what is in the box

One of the tasks that an analyst faces comes from the fact that often the exact form of a function is unknown. The job of the analyst is then to identify a reasonable mathematical representation of the relationship being studied. This is exactly what we mean by the term *modeling*.

For example, we could represent the relationship between education, experience and salary as:

Figure 1.19: Thinking about what form a function might take



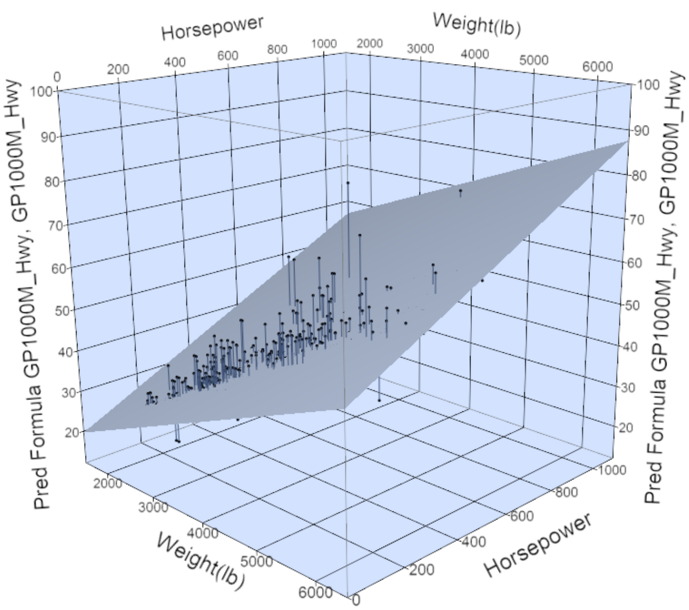
Of course, it's not obvious how the relationship works. The role of the analyst is to empirically determine the form of the function, that is, find what goes in the box. Mathematical and statistical modeling are the disciplines that address this fundamental question and this eBook provides the foundation to start this modeling process.

1.5.4 Visual presentation of functions of more than one variable

Visual presentation of functions of more than one variable

Just as functions of one variable (one input) can be represented graphically, so can functions of two variables. However, we now need a 3-dimensional picture.

Figure 1.20: A graphical representation of the relationship between weight, horsepower and fuel economy



This graphic shows the relationship between the weight of a car, the horsepower of its engine, and its fuel economy. Here, the fuel economy is the dependent variable and weight and horsepower are the independent variables. Each point in the picture corresponds to an actual car and the plane is used as the mathematical model that captures the nature of the relationship between the variables.

- A **plane** is drawn through the points and captures the form of the relationship. A plane is the extension to 3-dimensions of the 2-dimensional construct of the straight line. We will investigate planes in more detail in Module 2.
- The formula for a plane is:

  $$z = f(x, y) = a + bx + cy,$$

  where a, b and c are fixed constants.

- More complicated relationships may, in fact, depend on many independent variables; there is no reason why we have to stop at 2. For example, a function of 3 variables is represented as z = f(s,t,u), indicating that there are 3 inputs.
- In statistical modeling classes you may well find yourself creating functions that depend on a very large number of input variables – possibly hundreds!

Module 1. Q4.

Tips for dealing with functions

- The best way to get a feel for a function that has been presented as a formula is to plug in some numbers and graph it. This way you can identify important qualitative features of the function. Questions to be asked that address these qualitative features include:
- Is the function increasing or decreasing as x increases?
- If it is increasing, then is the rate of increase itself a constant or is it increasing/decreasing too (accelerating or decelerating)?
- Are there points where the function attains a maximum value or a minimum value?
- Module 6 will explore qualitative features of functions. These features are very important because when we model relationships in the business world, it is necessary that our mathematical model is able to mimic the qualitative characteristics we observe or expect to see in the relationship.

1.6 Summary

Module summary

Having reached the end of this module you should understand the following topics:

- The concept of a function.

- The three ways of representing a function and their relative strengths:
  - The tabular representation.
  - The graphical representation.
  - The formula representation.
- The composition of functions.
- Functions of more than one variable.