

WEEK 5

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Eigenvalues and Eigenvectors: Application to Data Problems

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1 What are eigen-things

1.1 What are Eigenvalues and Eigenvectors

- linear transformations using matrices.(scalings, rotations, and shears)

In linear algebra, an **eigenvector** or **characteristic vector** of a linear transformation is a non-zero vector that changes by only a scalar factor when that linear transformation is applied to it. More formally, if T is a linear transformation from a vector space V over a field F into itself and v is a vector in V that is not the zero vector, then v is an eigenvector of T if $T(v)$ is a scalar multiple of v . This condition can be written as the equation

$$T(v) = \lambda v \quad (1)$$

where λ is a scalar in the field F , known as the **eigenvalue**, **characteristic value**, or **characteristic root** associated with the eigenvector v

- Eigenvalues and Eigenvectors

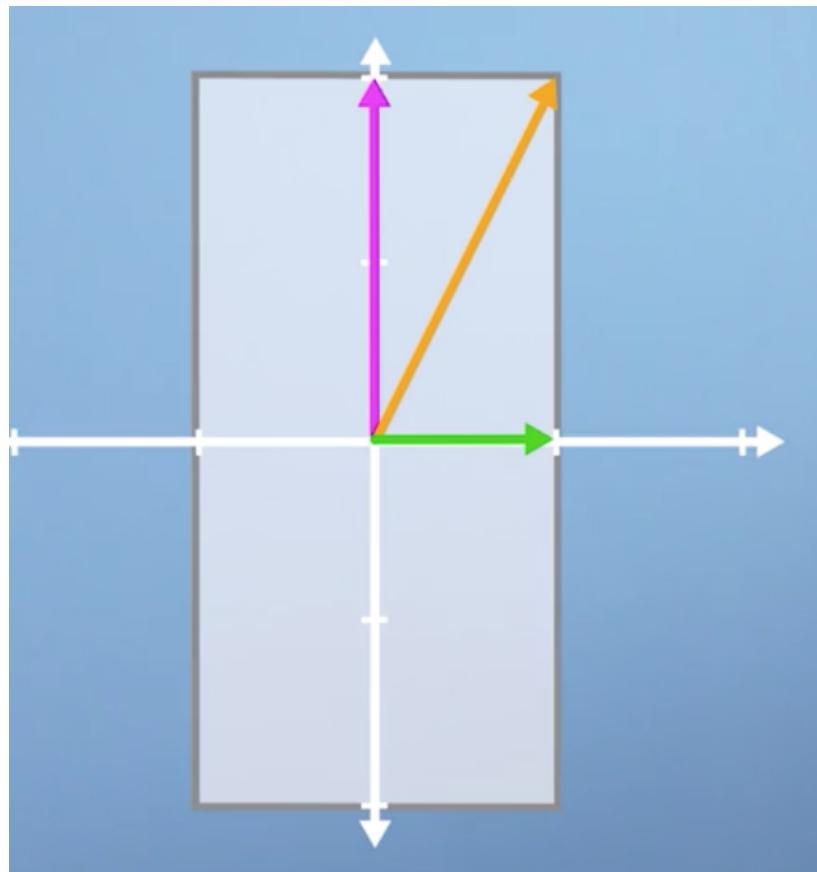


Figure 1: This is a vertical scale 2.

In the case above the purple and green vector only change in scale so they are Eigenvectors of this transformation.

2 Getting into the detail of eigenproblems

2.1 Special eigen-cases

recap: eigenvectors are those which lie along the same span both before and after applying a linear transform to a space. And then, eigenvalues are simply the amount that each of those vectors has been stretched in the process

- Uniform scaling: where we scale by the same amount in each direction. any vector would be an eigenvector
- Rotation: 180 degrees. all vectors for this transform are eigenvectors, and they all have eigenvalues of minus one.
- combination of a horizontal shear and a vertical scaling:

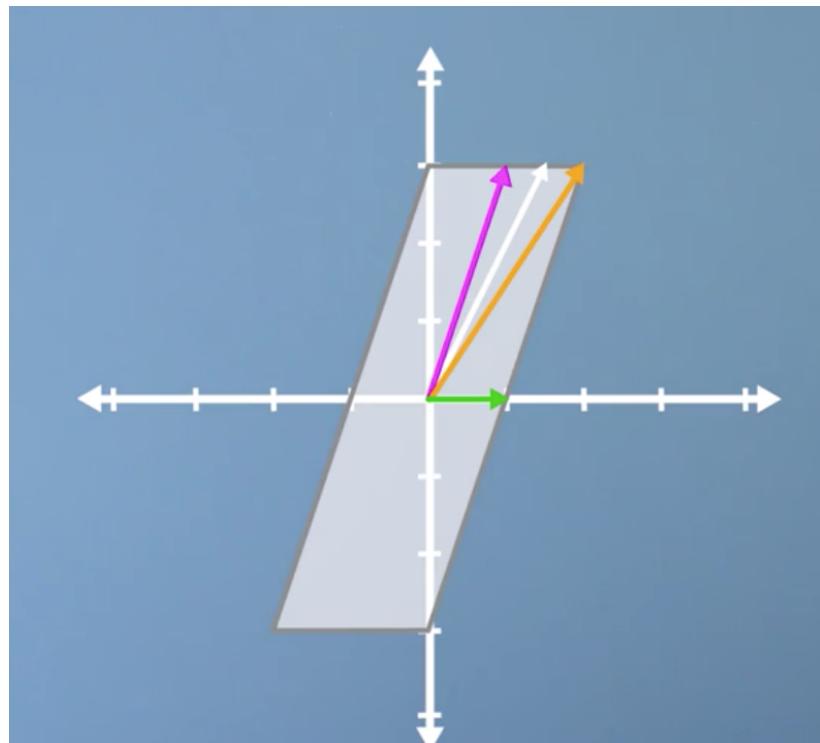


Figure 2: This is a combine horizontal shear and a vertical scaling

Green and White vector are eigenvectors. (not easy to spot)

- inverse transform:
 - tougher.
- 3D rotation

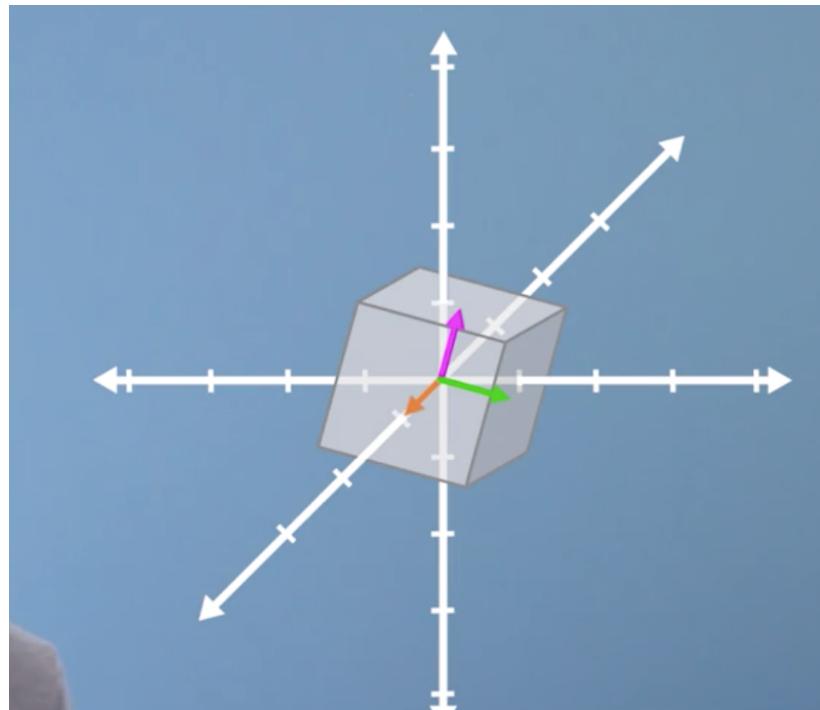


Figure 3: This is a 3D roation

If we find the eigenvector of a 3D rotation, it means we've also found the axis of rotation

2.2 Calculating eigenvectors

- Algrbra expression

$$A\mathbf{v} = \lambda\mathbf{v} \quad (2)$$

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \quad (3)$$

That is, the determinant of $(A - \lambda I)\mathbf{v} = \mathbf{0}$ must equal 0. We call $p(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** of A. The eigenvalues of A are simply the roots of the characteristic polynomial of A.

$$\begin{aligned}
 A\mathbf{x} &= \lambda\mathbf{x} \\
 (A - \lambda I)\mathbf{x} &= 0 \\
 \det(A - \lambda I) &= 0 \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
 \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) &= 0 \\
 \lambda^2 - (a+d)\lambda + ad - bc &= 0
 \end{aligned}$$

Figure 4: This is the equation for 2D transfer

- vertical scaling by a factor of two,

$$\begin{aligned}
 A &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \\
 (A - \lambda I)\mathbf{x} &= 0 & = (1-\lambda)(2-\lambda) = 0 \\
 @\lambda=1: \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0 \\
 @\lambda=2: \begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0 \\
 @\lambda=1: \mathbf{x} &= \begin{pmatrix} t \\ 0 \end{pmatrix} & @\lambda=2: \mathbf{x} = \begin{pmatrix} 0 \\ t \end{pmatrix}
 \end{aligned}$$

Figure 5: vertical scaling by a factor of two

So now we have two eigenvalues, and their two corresponding eigenvectors.

3. WHEN CHANGING TO THE EIGENBASIS IS REALLY USEFUL

- Rotation by 90-degrees anti-clockwise

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

Figure 6: vertical scaling by a factor of two

No eigenvalue and vector.

3 When changing to the eigenbasis is really useful

3.1 Changing to the eigenbasis

- diagonalisation vector times a transformation matrix is really costly. We want to change the T to diagonal matrix. Please look at the $AV = \lambda v$

$$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$T = C D C^{-1}$$

$$T^2 = C D C^{-1} C D C^{-1} = C D D C^{-1} = C D^2 C^{-1}$$

$$T^n = C D^n C^{-1}$$

Figure 7: eigenanalysis

This is the eigenanalysis refer to the formulae $Av = \lambda v$ A becomes T and λ become a diagonal matrix.

3.2 eigenbasis example

These are some examples to show the process above;

3. WHEN CHANGING TO THE EIGENBASIS IS REALLY USEFUL

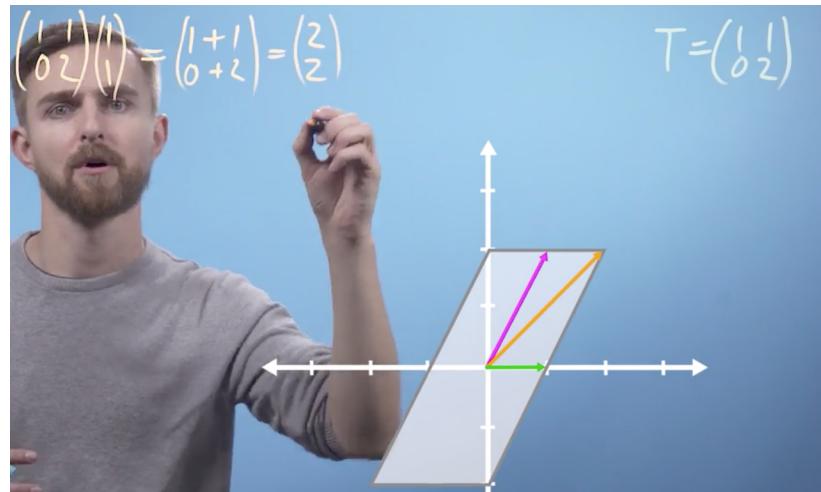


Figure 8: eigenanalysis example

$$\begin{aligned}
 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1+1 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} & T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\
 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1+1 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} & @\lambda=1: \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} &= \begin{pmatrix} 0+2 \\ 0+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} & @\lambda=2: \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1+3 \\ 0+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} & T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}
 \end{aligned}$$

Figure 9: eigenanalysis example

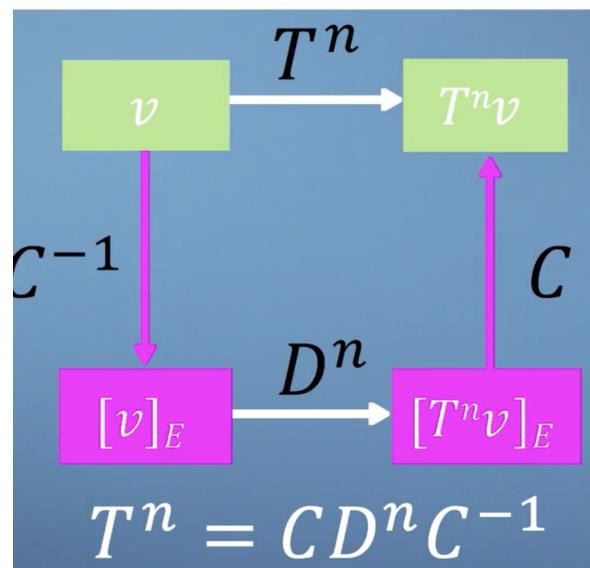
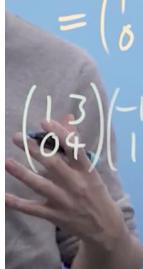


Figure 10: eigenanalysis example



$$\begin{aligned}
 T^2 &= CD^2C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} & @\lambda=1 : x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} & @\lambda=2 : x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
 \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1+3 \\ 0+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \checkmark & C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
 && C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

Figure 11: eigenanalysis example

4 Making the PageRank algorithm

5 Eigenvalues and Eigenvectors: Assessment