International Journal of Bifurcation and Chaos

Appendixes are included for review only, which will be removed if the paper can be accepted for publication.

Xiangxiang Jin
School of Sciences, Southwest Petroleum University
Chengdu, Sichuan 610500, China

Lingling Liu
School of Sciences, Southwest Petroleum University
Chengdu, Sichuan 610500, China
Data Recovery Key Laboratory of Sichuan Province
College of Mathematics and Information Science, Neijiang Normal University
Neijiang, Sichuan 641100, China
a600aa@163.com

Yong-Guo Shi*
Data Recovery Key Laboratory of Sichuan Province
College of Mathematics and Information Science, Neijiang Normal University
Neijiang, Sichuan 641100, China
scumat@163.com

*Corresponding author. E-mail addresses: scumat@163.com

Appendixes are included for review only, which will be removed if the paper can be accepted for publication.

Appendix A Sets and matlab code of $\mathcal{S}$ in Theorem 2.2

A.1 Sets of parameters

$$\mathcal{I}_3\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\frac{e_2}{h}<d_2<\frac{1}{1+\alpha_2}+\frac{e_2}{h},\frac{d_1}{1-\alpha_1d_1}<\kappa_1<\kappa_2<1\Big\},$$

$$\mathcal{I}_3\cap\mathcal{B}_{11}=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},$$
$$\frac{d_1}{1-\alpha_1d_1}<\kappa_1<\kappa_2<1\Big\},$$

$$\mathcal{I}_{21}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\frac{e_2}{h}<d_2<\frac{1}{1+\alpha_2},\kappa_2<1\Big\},$$

$$\mathcal{I}_{22}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho\geq\varpi,\frac{e_2}{h}<d_2<\frac{1}{1+\alpha_2},\frac{d_1}{1-\alpha_1d_1}<\kappa_2<1\Big\},$$

$$\mathcal{I}_{21}\cap\mathcal{B}_{11}=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},\kappa_2<1\Big\},$$

$$\mathcal{I}_{22}\cap\mathcal{B}_{11}=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho\geq\varpi,\max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},$$
$$\frac{d_1}{1-\alpha_1d_1}<\kappa_2<1\Big\},$$

$$\mathcal{I}_{11}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\frac{e_2}{h}<d_2<\min\{\frac{1}{1+\alpha_2},\kappa_2<1\Big\},$$

$$\mathcal{I}_{12}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho\leq\varpi,\frac{e_2}{h}<d_2<\min\{\frac{1}{1+\alpha_2},\kappa_1<\frac{d_1}{1-\alpha_1d_1},\kappa_2<1\Big\},$$

$$\mathcal{I}_{13}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\frac{e_2}{h}<d_2<\min\{\frac{1}{1+\alpha_2},\frac{d_1}{1-\alpha_1d_1}<\kappa_1<1<\kappa_2\Big\},$$

$$\mathcal{I}_{14}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\frac{e_2}{h}<d_2<\min\{\frac{1}{1+\alpha_2},\kappa_1<\frac{d_1}{1-\alpha_1d_1}<\kappa_2<1\Big\},$$

$$\mathcal{I}_{01}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho\geq\varpi,\frac{e_1+[(1+\alpha_1)d_1-1]e_2}{e_1(1+\alpha_2)}<d_2<\frac{1}{1+\alpha_2},\kappa_2<\frac{d_1}{1-\alpha_1d_1}\Big\},$$

$$\mathcal{I}_{02}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,0<d_2\leq\frac{e_1+[(1+\alpha_1)d_1-1]e_2}{e_1(1+\alpha_2)},\kappa_2<\frac{d_1}{1-\alpha_1d_1}\Big\},$$

$$\mathcal{I}_{03}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},0<d_2<\frac{1}{1+\alpha_2},\kappa_1>1\Big\},$$

$$\mathcal{I}_{04}\cap\mathcal{B}_2=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},0<d_2<\frac{1}{1+\alpha_2},e_2\varrho>\varpi\Big\},$$

$$\mathcal{I}_{11}\cap\mathcal{B}_{11}=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho<\varpi,\max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},,\kappa_2<1\Big\},$$

$$\mathcal{I}_{12}\cap\mathcal{B}_{11}=\Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1},e_2\varrho\leq\varpi,\max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},$$
$$\kappa_1<\frac{d_1}{1-\alpha_1d_1},\kappa_2<1\Big\},$$

$$\begin{aligned}
\mathcal{I}_{13}\cap\mathcal{B}_{11} &= \Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1}, e_2\varrho<\varpi, \max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},\\
&\qquad \frac{d_1}{1-\alpha_1 d_1}<\kappa_1<1<\kappa_2\Big\},\\
\mathcal{I}_{14}\cap\mathcal{B}_{11} &= \Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1}, e_2\varrho<\varpi, \max\{\frac{1}{1+\alpha_2},\frac{e_2}{h}\}<d_2<\min\{\frac{1}{1+\alpha_2}+\frac{e_2}{h},1\},\\
&\qquad \kappa_1<\frac{d_1}{1-\alpha_1 d_1}<\kappa_2<1\Big\},\\
\mathcal{I}_{01}\cap\mathcal{B}_{11} &= \Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1}, e_2\varrho\geq\varpi, \frac{1}{1+\alpha_2}\leq d_2<1, \kappa_2<\frac{d_1}{1-\alpha_1 d_1}\Big\},\\
\mathcal{I}_{03}\cap\mathcal{B}_{11} &= \Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1}, \frac{1}{1+\alpha_2}\leq d_2<1, \kappa_1>1\Big\},\\
\mathcal{I}_{04}\cap\mathcal{B}_{11} &= \Big\{\mu\in\mathcal{R}\Big|0<d_1<\frac{1}{1+\alpha_1}, \frac{1}{1+\alpha_2}\leq d_2<1, e_2\varrho>\varpi\Big\}.
\end{aligned}$$

A.2 Matlab code for $\mathcal{S}$

```
alpha = zeros(1,2); d = zeros(1,2); e = zeros(1,2); A = zeros(1,4);
alpha(1)=input('Please enter alpha[1]:'); alpha(2)=input('Please enter alpha[2]:');
d(1)=input('Please enter d[1]:'); d(2)=input('Please enter d[2]:');
e(1)=input('Please enter e[1]:'); e(2)=input('Please enter e[2]:');
h = input('Please enter h:');A(1)=e(1)*d(2)*(1+h) - e(2)*(d(1)+e(1));
A(2)=e(2)(1-alpha(1)*d(1))e(1)((alpha(2)d(2)-1-h)+(1-alpha(1)-alpha(2))(e(2)-h*d(2))
A(3)=e(1)*alpha(1)*(e(2)-h*d(2)) + e(1)*(1-alpha(1))*(h + (e(2)-h*d(2))*alpha(2));
A(4)=e(1)*alpha(1)*(h + (e(2)-h*d(2))*alpha(2));
Delta=(A(2)*A(3)-9*A(1)*A(4))^2 - 4*(A(3)^2-3*A(2)*A(4))*(A(2)^2-3*A(1)*A(3));
symbol_Sequence = arrayfun(@(x) '+', A, 'UniformOutput', false);
symbol_Sequence(A < 0) = {'-'};
symbol_Sequence(A == 0) = {'0'};
if Delta > 0
    symbol_Sequence{5} = '+';
elseif Delta < 0
    symbol_Sequence{5} = '-';
else
    symbol_Sequence{5} = '0';
end
disp('symbolic Sequence S is:');
disp(['[A[0], A[1], A[2], A[3], Delta]=', ...
      '[', strjoin(symbol_Sequence, ','), ']']);
```

A.3 Expressions

The expressions in Theorem 4.2 is

$$\begin{aligned}
T_1\Psi_1+\Psi_2T_2 = \{&-\alpha_2\gamma_2(\alpha_2-1)^2[\gamma_1\gamma_2(\alpha_2+4)(1+\alpha_2)^3(\alpha_1\alpha_2-\alpha_1+2\alpha_2)^2e_1+4\left(\alpha_1^2+4\alpha_1+2\gamma_1+4\right)\alpha_2^6\\
&+4(\alpha_1\gamma_1\gamma_2+2\gamma_1\gamma_2+4\alpha_1+8\gamma_1+8)\alpha_2^5+\Big((-4\gamma_1\gamma_2-8\gamma_1-8)\alpha_1^2+(-4\gamma_1\gamma_2-32\gamma_1-16)\alpha_1+16+(8\gamma_2+24)\gamma_1\Big)\\
&\alpha_2^4+\Big(-4\alpha_1^2\gamma_1\gamma_2+(-52\gamma_1\gamma_2-32\gamma_1-16)\alpha_1-72\gamma_1\gamma_2\Big)\alpha_2^3-20\gamma_2(3\alpha_1+4)\gamma_1\alpha_1\alpha_2+((44\gamma_1\gamma_2+8\gamma_1+4)\alpha_1^2\\
&+132\alpha_1\gamma_1\gamma_2+56\gamma_1\gamma_2)\alpha_2^2+24\alpha_1^2\gamma_1\gamma_2]\};
\end{aligned}$$

$P_2 \mid \tilde{d}_1 = -\gamma_2\vartheta_1 e_2^3 (\bar{x}\alpha_1+1)^2 (\bar{x}\alpha_2+1) [(\bar{x}\alpha_2+1)\vartheta_1\vartheta_2\gamma_1 + \bar{x}(\bar{x}-1)(\bar{x}\alpha_1+1)] + \vartheta_1\vartheta_2\{\gamma_2(\bar{x}\alpha_1+1)^2$
$\left[2h(\bar{x}\alpha_2+1)\vartheta_1\vartheta_2\gamma_1 + \bar{x}\left(h\bar{x}^2\alpha_1 - h\bar{x}\alpha_1 + h\bar{x} - h - 1\right)\right] + \bar{x}\gamma_1(\bar{x}\alpha_2+1)[(\bar{x}\alpha_1+1)^2(2\bar{x}\alpha_2-\alpha_2+1)\vartheta_1 h$
$+\bar{x}\alpha_2+1]\} - h\gamma_1 e_2\vartheta_1^2\vartheta_2^2\{h(\bar{x}\alpha_1+1)^2[\vartheta_2\gamma_2 + \bar{x}(2\bar{x}\alpha_2-\alpha_2+1)] + \bar{x}(\alpha_1-\alpha_2)\}$;
$P_1'P_2 \mid \tilde{d}_1 = \{-\gamma_2\vartheta_1 e_2^3(\bar{x}\alpha_1+1)^2(\alpha_2\bar{x}+1)[(\alpha_2\bar{x}+1)\vartheta_1\vartheta_2\gamma_1 + \bar{x}(\bar{x}-1)(\bar{x}\alpha_1+1)] + \vartheta_1\vartheta_2\{\gamma_2(\bar{x}\alpha_1+1)^2$
$\left[2h(\alpha_2\bar{x}+1)\vartheta_1\vartheta_2\gamma_1 + \bar{x}\left(h\bar{x}^2\alpha_1 - h\bar{x}\alpha_1 + h\bar{x} - h - 1\right)\right] + \bar{x}\gamma_1(\alpha_2\bar{x}+1)[(\bar{x}\alpha_1+1)^2(2\alpha_2\bar{x}-\alpha_2+1)\vartheta_1 h$
$+\alpha_2\bar{x}+1]\} - h\gamma_1 e_2\vartheta_1^2\vartheta_2^2\{h(\bar{x}\alpha_1+1)^2[\vartheta_2\gamma_2 + \bar{x}(2\alpha_2\bar{x}-\alpha_2+1)] + \bar{x}(\alpha_1-\alpha_2)\}\}[\gamma_1 e_1\vartheta_2^2 h^2 - \gamma_1 e_1(\bar{x}\alpha_2+1)e_2\vartheta_2 h$
$-\bar{x}\alpha_2 e_2^2]/[h\vartheta_2 - e_2(\bar{x}\alpha_2+1)](\bar{x}\alpha_2+1)e_1e_2$.

Appendix B Proof of Theorem 4.2.(ii-iii)

Proof. **(ii)** First, we compute the Jacobian matrix of the system at E_2 as follows

$$J_{E_2} = \begin{pmatrix} d_2(1+\alpha_2) - \frac{2d_2}{1-\alpha_2 d_2} & \frac{d_2}{1+(\alpha_1-\alpha_2)d_2} & -d_2 \\ 0 & a_{22} & 0 \\ \gamma_2[1-d_2(1+\alpha_2)] & \frac{\gamma_2 e_2[1-d_2(1+\alpha_2)]}{(1-\alpha_2 d_2)^2} & 0 \end{pmatrix}, \tag{B.1}$$

and then obtain the corresponding characteristic equation (B.2)

$$(\lambda-\theta_1)(\lambda^2+\theta_2\lambda+\theta_3 d_2) = 0, \tag{B.2}$$

where

$$\theta_1 = \frac{2d_2}{1-\alpha_2 d_2} - d_2(1+\alpha_2), \quad \theta_3 = \gamma_2[1-d_2(1+\alpha_2)],$$
$$\theta_2 = \gamma_1\left[\frac{d_2}{1+(\alpha_1-\alpha_2)d_2} - \frac{e_1(1-d_2(1+\alpha_2))}{(1-\alpha_2 d_2)^2} - d_1\right].$$

Assuming that equation (B.2) has a pair of conjugate complex roots $\lambda_{1,2} = u_2(d_2) \pm i v_2(d_2)$ and $u_2(d_2^0) = 0$, then by substituting into the equation (B.2) and taking the derivative for the bifurcation parameter $d_2^0 = \frac{\alpha_2-1}{\alpha_2(1+\alpha_2)}$, and separate the real and imaginary parts. Then we obtain that $v_2(d_2^0) = \frac{\sqrt{\gamma_2(\alpha_2^2-1)}}{\alpha_2(\alpha_2+1)} \neq 0$. The verification of the transversality condition for the Hopf bifurcation follows a process similar to the proof of Theorem 4.2, and thus is not detailed here for brevity. By calculation, we derive the following formula

$$\frac{\partial u_2}{\partial d_2}\Big|_{d_2=d_2^0} = \frac{\Psi_1(d_2)T_1(d_2) + \Psi_2(d_2)T_2(d_2)}{T_1^2(d_2)+T_2^2(d_2)}, \tag{B.3}$$

where

$$T_1(d_2) = \theta_1\theta_2 + 4d_2\theta_3, \quad \Psi_2(d_2) = -v_2(d_2)\left(\frac{\partial(\theta_1\theta_2)}{\partial d_2} + \frac{\partial(d_2\theta_3)}{\partial d_2}\right),$$
$$\Psi_1(d_2) = -\frac{\partial(\theta_1+\theta_2)}{\partial d_2} + \frac{\partial(d_2\theta_2\theta_3)}{\partial d_2}, \quad T_2(d_2) = 2v_2(d_2)(\theta_1+\theta_2).$$

Due to the extensive computation required, we used symbolic computation software. The transversality condition $\frac{\partial u_2}{\partial d_2}|_{d_2=d_2^0} \neq 0$ for a Hopf bifurcation is satisfied when the condition $T_1\Psi_1 + \Psi_2 T_2|_{d_2=d_2^0} \neq 0$ holds. Hence, system (2) will undergo a Hopf bifurcation at the boundary equilibrium E_2 when the parameter d_2 crosses the critical value d_2^0.
(iii) According to the characteristic equation (8) of the system at the interior equilibrium $\bar{E}$, the equation (8) can be written as

$$H(\lambda) = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3 = (\lambda^2+P_2)(\lambda+P_1) = 0. \tag{B.4}$$

Thus, equation (B.4) has a pair of pure imaginary characteristic roots and a negative root at $d_1 = \tilde{d}_1$. Specifically, the roots are: $\lambda_1 = \sqrt{P_2}i, \lambda_2 = -\sqrt{P_2}i, \lambda_3 = -P_1$. Next, we will verify the transversality condition for the Hopf bifurcation in system (2). First, we compute the partial derivative of the function (B.4) with respect to d_1 as follows,

$$\frac{\partial H(\lambda)}{\partial d_1} = (3\lambda^2 + 2P_1\lambda + P_2)\lambda' + P_1'\lambda^2 + P_2'\lambda + P_3' = 0, \tag{B.5}$$

further, we have

$$\lambda'|_{\lambda=\lambda_{1,2}=\pm\sqrt{P_2}i} = \frac{\left(P_2P_1' \pm iP_2'\sqrt{P_2} + P_3'\right)\left(P_2 \pm iP_1\sqrt{P_2}\right)}{2P_2\left(P_1^2 + P_2\right)},$$

where $'$ denotes the partial derivative concerning d_1. Hence, we get

$$Re\left(\frac{\partial \lambda}{\partial d_1}|_{\lambda_{1,2}=\pm\sqrt{P_2}i}\right) = \frac{P_1'P_2}{P_2 + P_1^2}|_{d_1=\tilde{d}_1}. \tag{B.6}$$

From the condition $P_1'P_2 \neq 0$, then we have $Re(\frac{\partial\lambda}{\partial d_1})|_{d_1=\tilde{d}_1} \neq 0$, which confirms that the transversality condition for the Hopf bifurcation is satisfied. Therefore, a Hopf bifurcation will occur at the interior equilibrium $\bar{E}$ of system (2) near the critical value $d_1 = \tilde{d}_1$. ■

Appendix C Stability and Direction of Hopf bifurcation

Proof. We first perform a translation transformation on system (2), moving the boundary equilibrium $E_1(d_1^0) = (\frac{\alpha_1-1}{2\alpha_1}, \frac{(\alpha_1+1)^2}{4\alpha_1}, 0)$ to the origin. i.e.,

$$\begin{cases} x_1 = x - \frac{\alpha_1-1}{2\alpha_1} \\ x_2 = y - \frac{(\alpha_1+1)^2}{4\alpha_1} \\ x_3 = z. \end{cases} \tag{C.1}$$

Then, system (2) becomes system (C.2) as follow:

$$\dot{X} = J_{E_1}(d_1^0)X + F(X), X = (x_1, x_2, x_3)^T, F(X) = (F_1, F_2, F_3)^T, \tag{C.2}$$

where $F_i(X)(i = 1, 2, 3)$ are the non-linear terms of the right hand functions of (C.2), detailed as follows:

$$\begin{aligned} F_1 = & -\frac{x_1^2(\alpha_1-1)}{\alpha_1+1} - \frac{4x_1x_2}{(\alpha_1+1)^2} - \frac{4\alpha_1^2x_1x_3}{((\alpha_2+2)\alpha_1-\alpha_2)^2} - \frac{4\alpha_1x_1^3}{(\alpha_1+1)^2} + \frac{8\alpha_1x_1^2x_2}{(\alpha_1+1)^3} \\ & + \frac{8\alpha_2\alpha_1^3x_1^2x_3}{((\alpha_2+2)\alpha_1-\alpha_2)^3} + O\left(X^4\right), \\ F_2 = & -\frac{2\gamma_1x_1^2}{\alpha_1+1} + \frac{4\gamma_1x_1x_2}{(\alpha_1+1)^2} - \frac{16\gamma_1e_1\alpha_1^2x_2x_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^2} + \frac{4\alpha_1\gamma_1x_1^3}{(\alpha_1+1)^2} - \frac{8\gamma_1\alpha_1x_1^2x_2}{(\alpha_1+1)^3} \\ & + \frac{64\gamma_1e_1h\alpha_1^3x_2^2x_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^3} + O\left(X^4\right), \\ F_3 = & \frac{4\gamma_2\alpha_1^2x_1x_3}{((\alpha_2+2)\alpha_1-\alpha_2)^2} + \frac{16e_2\gamma_2\alpha_1^2x_2x_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^2} - \frac{8\gamma_2\alpha_2\alpha_1^3x_1^2x_3}{((\alpha_2+2)\alpha_1-\alpha_2)^3} \\ & - \frac{64e_2\gamma_2\alpha_1^3hx_2^2x_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^3} + O\left(X^4\right). \end{aligned}$$

Secondly, introduce an invertible transformation as follows:

$$(x_1, x_2, x_3)^T = P(y_1, y_2, y_3)^T,$$

where

$$P=\begin{pmatrix}\xi_1 & -\frac{\sqrt{\alpha_1^2-1}}{\sqrt{\gamma_1}(1+\alpha_1)} & \frac{\sqrt{\alpha_1^2-1}}{\sqrt{\gamma_1}(1+\alpha_1)}\\ \xi_2 & 1 & 1\\ 1 & 0 & 0\end{pmatrix},$$

$$\begin{aligned}
\xi_1 = {} & \{(\alpha_1-1)(\alpha_1+1)\left(h\alpha_1^2+2h\alpha_1+h+4\alpha_1\right)[e_1(\alpha_1+1)(\alpha_1\alpha_2+2\alpha_1-\alpha_2)^2(h\alpha_1^2+2h\alpha_1\\
& +h+4\alpha_1)\gamma_1+4\alpha_1^2\gamma_2(h+1)(h\alpha_1^3\alpha_2d_2+2h\alpha_1^3d_2+h\alpha_1^2\alpha_2d_2-\alpha_1^3\alpha_2e_2-h\alpha_1^3+4h\alpha_1^2d_2\\
& -h\alpha_1\alpha_2d_2-2\alpha_1^3e_2+4\alpha_1^2\alpha_2d_2-\alpha_1^2\alpha_2e_2-h\alpha_1^2+2h\alpha_1d_2-h\alpha_2d_2+8\alpha_1^2d_2-4\alpha_1^2e_2\\
& -4\alpha_1\alpha_2d_2+\alpha_1\alpha_2e_2+h\alpha_1-4\alpha_1^2-2\alpha_1e_2+\alpha_2e_2+h+4\alpha_1)]\}/\{4(h+1)[(\alpha_1-1)\\
& \left(h\alpha_1^2+2h\alpha_1+h+4\alpha_1\right)^2(\alpha_1\alpha_2+2\alpha_1-\alpha_2)^2\gamma_1+\alpha_1^2\gamma_2^2(\alpha_1+1)\left(h\alpha_1^3\alpha_2d_2+2h\alpha_1^3d_2-h\alpha_1^2\right.\\
& +h\alpha_1^2\alpha_2d_2-\alpha_1^3\alpha_2e_2-h\alpha_1^3+4h\alpha_1^2d_2-h\alpha_1\alpha_2d_2-2\alpha_1^3e_2+4\alpha_1^2\alpha_2d_2-\alpha_1^2\alpha_2e_2+2h\alpha_1d_2\\
& \left.-h\alpha_2d_2+8\alpha_1^2d_2-4\alpha_1^2e_2-4\alpha_1\alpha_2d_2+\alpha_1\alpha_2e_2+h\alpha_1-4\alpha_1^2-2\alpha_1e_2+\alpha_2e_2+h+4\alpha_1\right)^2]\},\\
\xi_2 = {} & \{\alpha_1\gamma_1(\alpha_1+1)(\alpha_1\alpha_2+2\alpha_1-\alpha_2)\left(h\alpha_1^2+2h\alpha_1+h+4\alpha_1\right)[h\alpha_1^5\alpha_2d_2e_1\gamma_2+2h\alpha_1^5d_2e_1\gamma_2\\
& +3h\alpha_1^4\alpha_2d_2e_1\gamma_2-\alpha_1^5\alpha_2e_1e_2\gamma_2-h\alpha_1^5e_1\gamma_2+8h\alpha_1^4d_2e_1\gamma_2+2h\alpha_1^3\alpha_2d_2e_1\gamma_2-2\alpha_1^5e_1e_2\gamma_2\\
& +4\alpha_1^4\alpha_2d_2e_1\gamma_2-3\alpha_1^4\alpha_2e_1e_2\gamma_2-3h\alpha_1^4e_1\gamma_2+12h\alpha_1^3d_2e_1\gamma_2-2h\alpha_1^2\alpha_2d_2e_1\gamma_2+8\alpha_1^4d_2e_1\gamma_2\\
& -8\alpha_1^4e_1e_2\gamma_2+4\alpha_1^3\alpha_2d_2e_1\gamma_2-2\alpha_1^3\alpha_2e_1e_2\gamma_2-2h\alpha_1^3e_1\gamma_2+8h\alpha_1^2d_2e_1\gamma_2-3h\alpha_1\alpha_2d_2e_1\gamma_2\\
& -4\alpha_1^4e_1\gamma_2+16\alpha_1^3d_2e_1\gamma_2-12\alpha_1^3e_1e_2\gamma_2-4\alpha_1^2\alpha_2d_2e_1\gamma_2+2\alpha_1^2\alpha_2e_1e_2\gamma_2-4h^2\alpha_1^3+2h\alpha_1^2e_1\gamma_2\\
& +2h\alpha_1d_2e_1\gamma_2-h\alpha_2d_2e_1\gamma_2-4\alpha_1^3e_1\gamma_2+8\alpha_1^2d_2e_1\gamma_2-8\alpha_1^2e_1e_2\gamma_2-4\alpha_1\alpha_2d_2e_1\gamma_2+3\alpha_1\alpha_2e_1e_2\gamma_2\\
& -4h^2\alpha_1^2-4h\alpha_1^3+3h\alpha_1e_1\gamma_2+4\alpha_1^2e_1\gamma_2-2\alpha_1e_1e_2\gamma_2+\alpha_2e_1e_2\gamma_2+4h^2\alpha_1-20h\alpha_1^2+he_1\gamma_2\\
& +4\alpha_1e_1\gamma_2+4h^2+20h\alpha_1-16\alpha_1^2+4h+16\alpha_1]\}/\{4(h+1)[(\alpha_1-1)\left(h\alpha_1^2+2h\alpha_1+h+4\alpha_1\right)^2\\
& (\alpha_1\alpha_2+2\alpha_1-\alpha_2)^2\gamma_1+\alpha_1^2\gamma_2^2(\alpha_1+1)(h\alpha_1^3\alpha_2d_2+2h\alpha_1^3d_2+h\alpha_1^2\alpha_2d_2-\alpha_1^3\alpha_2e_2-h\alpha_1^3+4h\alpha_1^2d_2\\
& -h\alpha_1\alpha_2d_2-2\alpha_1^3e_2+4\alpha_1^2\alpha_2d_2-\alpha_1^2\alpha_2e_2-h\alpha_1^2+2h\alpha_1d_2-h\alpha_2d_2+8\alpha_1^2d_2-4\alpha_1^2e_2-4\alpha_1\alpha_2d_2\\
& +\alpha_1\alpha_2e_2+h\alpha_1-4\alpha_1^2-2\alpha_1e_2+\alpha_2e_2+h+4\alpha_1)^2]\}.
\end{aligned}$$

Then, system (C.2) becomes

$$\dot{Y}=P^{-1}J_{E_1}(d_1^0)PY+Q, Y=(y_1,y_2,y_3)^T, Q=(Q^1,Q^2,Q^3)^T, \tag{C.3}$$

where

$$P^{-1}J_{E_1}(d_1^0)P=\begin{pmatrix}\beta & 0 & 0\\ 0 & -\frac{\sqrt{\gamma_1(\alpha_1^2-1)}}{\alpha_1(\alpha_1+1)} & 0\\ 0 & 0 & \frac{\sqrt{\gamma_1(\alpha_1^2-1)}}{\alpha_1(\alpha_1+1)}\end{pmatrix},$$

$$\begin{aligned}
\beta &= \gamma_2\left(\frac{\alpha_1-1}{(\alpha_2+2)\alpha_1-\alpha_2}+\frac{e_2(\alpha_1+1)^2}{h\alpha_1^2+(2h+4)\alpha_1+h}-d_2\right),\\
Q^1 &= \frac{4\alpha_1^2\gamma_2y_1y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}+\frac{16e_2\gamma_2\alpha_1^2y_2y_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^2}+\frac{8\alpha_1^3\alpha_2\gamma_2y_1^2y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^3}\\
&-\frac{64e_2\gamma_2\alpha_1^3hy_2^2y_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^3}+O\left(Y^4\right),\\
Q^2 &= -\frac{\gamma_1y_1^2}{\alpha_1+1}+\frac{2\gamma_1y_1y_2}{(\alpha_1+1)^2}-\frac{8\gamma_1e_1\alpha_1^2y_2y_3}{\left(\alpha_1^2h+(2h+4)\alpha_1+h\right)^2}-\frac{4\gamma_1\alpha_1y_1^2y_2}{(\alpha_1+1)^3}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha_1\gamma_1 y_1^3}{(\alpha_1+1)^2}+\frac{1}{\sqrt{\alpha_1^2-1}}\left\{(\frac{\xi_1\sqrt{\gamma_1}}{2}-\frac{\xi_2\sqrt{\alpha_1^2-1}}{2}+\frac{\alpha_1\xi_1\sqrt{\gamma_1}}{2})\left[\frac{4\alpha_1^2\gamma_2 y_1 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}\right.\right.\\
&\left.\left.+\frac{8\alpha_1^3\alpha_2\gamma_2 y_1^2 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^3}+\frac{16e_2\gamma_2\alpha_1^2 y_2 y_3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2}-\frac{64e_2\gamma_2\alpha_1^3 h y_2^2 y_3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^3}\right]\right\}\\
&+\frac{1}{2\sqrt{\alpha_1^2-1}}\left\{\sqrt{\gamma_1}(\alpha_1+1)\left[\frac{(\alpha_1-1)y_1^2}{\alpha_1+1}+\frac{4y_1y_2}{(\alpha_1+1)^2}+\frac{4\alpha_1 y_1^3}{(\alpha_1+1)^2}-\frac{8\alpha_1 y_1^2 y_2}{(\alpha_1+1)^3}\right.\right.\\
&\left.\left.+\frac{4\alpha_1^2 y_1 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}+\frac{8\alpha_1^3\alpha_2 y_1^2 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^3}\right]\right\}+\frac{32\gamma_1 e_1 h\alpha_1^3 y_2^2 y_3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^3}+O\left(Y^4\right),\\
Q^3=&-\frac{1}{\sqrt{\alpha_1^2-1}}\left\{(\frac{\xi_1\sqrt{\gamma_1}}{2}+\frac{\xi_2\sqrt{\alpha_1^2-1}}{2}+\frac{\alpha_1\xi_1\sqrt{\gamma_1}}{2})\left[\frac{4\alpha_1^2\gamma_2 y_1 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}\right.\right.\\
&\left.\left.+\frac{16e_2\gamma_2\alpha_1^2 y_2 y_3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2}+\frac{8\alpha_1^3\alpha_2\gamma_2 y_1^2 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^3}-\frac{64e_2\gamma_2\alpha_1^3 h y_2^2 y_3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^3}\right]\right\}\\
&-\frac{1}{2\sqrt{\alpha_1^2-1}}\left\{\sqrt{\gamma_1}(\alpha_1+1)\left[\frac{(\alpha_1-1)y_1^2}{\alpha_1+1}+\frac{4y_1y_2}{(\alpha_1+1)^2}+\frac{4\alpha_1 y_1^3}{(\alpha_1+1)^2}-\frac{8\alpha_1 y_1^2 y_2}{(\alpha_1+1)^3}\right.\right.\\
&\left.\left.+\frac{4\alpha_1^2 y_1 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}+\frac{8\alpha_1^3\alpha_2 y_1^2 y_3}{(\alpha_2-(\alpha_2+2)\alpha_1)^3}\right]\right\}-\frac{\gamma_1 y_1^2}{\alpha_1+1}+\frac{2\alpha_1\gamma_1 y_1^3}{(\alpha_1+1)^2}+\frac{2\gamma_1 y_1 y_2}{(\alpha_1+1)^2}\\
&-\frac{4\gamma_1 y_1^2 y_2\alpha_1}{(\alpha_1+1)^3}-\frac{8\gamma_1 y_2 y_3 e_1\alpha_1^2}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2}+\frac{32\gamma_1 y_2^2 y_3 e_1 h\alpha_1^3}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^3}+O\left(Y^4\right).
\end{aligned}
$$

Next, we calculate the following of the nonlinear terms

$$
\begin{aligned}
g_{20}(0,0,0)&=\frac{\sqrt{\gamma_1\left(\alpha_1^2-1\right)}}{(\alpha_1+1)\left(\alpha_1^2-1\right)}+\frac{2\gamma_1-i\gamma_1(\alpha_1+3)}{2(\alpha_1+1)^2}+\frac{i\left[\left(\alpha_1^2-5\right)\sqrt{\gamma_1}\sqrt{\alpha_1^2-1}(\alpha_1-1)\right]}{4(\alpha_1+1)^2(\alpha_1-1)},\\
g_{11}(0,0,0)&=\frac{i}{4}\left(-\frac{2\gamma_1}{\alpha_1+1}+\frac{\sqrt{\gamma_1\left(\alpha_1^2-1\right)}}{\alpha_1+1}\right),\quad W_{11}(0,0,0)=\frac{2\gamma_1+\sqrt{\gamma_1\left(\alpha_1^2-1\right)}}{4\beta_1(\alpha_1+1)},\\
g_{02}(0,0,0)&=-\frac{\sqrt{\gamma_1\left(\alpha_1^2-1\right)}}{(\alpha_1+1)\left(\alpha_1^2-1\right)}-\frac{\gamma_1}{(\alpha_1+1)^2}+\frac{i\left(\left(\alpha_1^2+3\right)\sqrt{\gamma_1}\sqrt{\alpha_1^2-1}-2\gamma_1(\alpha_1-1)^2\right)}{4(\alpha_1+1)^2(\alpha_1-1)},\\
g_{21}(0,0,0)&=G_{101}W_{20}+2G_{110}W_{11}+G_{21},\quad G_{21}=\frac{i}{8},\\
G_{110}(0,0,0)&=\frac{2\alpha_1^2\gamma_2}{(\alpha_2-(\alpha_2+2)\alpha_1)^2}+\frac{4\left(-\xi_2\sqrt{\alpha_1^2-1}+\xi_1\sqrt{\gamma_1}(\alpha_1+1)\right)e_2\gamma_2\alpha_1^2}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2\sqrt{\alpha_1^2-1}},\\
&-\frac{4\gamma_1 e_1\alpha_1^2}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2}+i\left[\frac{(\xi_1\sqrt{\gamma_1}(\alpha_1-1)-\xi_2\sqrt{\alpha_1^2-1})\alpha_1^2\gamma_2}{(\alpha_2-(\alpha_2+2)\alpha_1)^2\sqrt{\alpha_1^2-1}}\right.\\
&\left.+\frac{\sqrt{\gamma_1}(\alpha_1+1)\alpha_1^2}{(\alpha_2-(\alpha_2+2)\alpha_1)^2\sqrt{\alpha_1^2-1}}-\frac{8e_2\gamma_2\alpha_1^2}{\left(\alpha_1^2 h+(2h+4)\alpha_1+h\right)^2}\right],\\
W_{20}(0,0,0)&=-\frac{(2i\sqrt{\gamma_1}\sqrt{\alpha_1^2-1}+\beta\alpha_1^2+\beta\alpha_1)\alpha_1}{\{(\alpha_1+1)\left(\beta^2\alpha_1^3+\beta^2\alpha_1^2+4\alpha_1\gamma_1-4\gamma_1\right)\}}\left\{\left[-\sqrt{\gamma_1\left(\alpha_1^2-1\right)}\alpha_1^2\right.\right.\\
&\left.\left.+4i\alpha_1\gamma_1-2\alpha_1^2\gamma_1+4i\sqrt{\gamma_1\left(\alpha_1^2-1\right)}-4i\gamma_1+\sqrt{\gamma_1\left(\alpha_1^2-1\right)}+2\gamma_1\right]\right\},
\end{aligned}
$$

$$G_{101}(0,0,0) = \frac{2\alpha_1^2\gamma_2}{\left(\alpha_2-(\alpha_2+2)\,\alpha_1\right)^2} - \frac{4\left(-\xi_2\sqrt{\alpha_1^2-1}+\xi_1\sqrt{\gamma_1}\,(\alpha_1+1)\right)e_2\gamma_2\alpha_1^2}{\left(\alpha_1^2h+(2h+4)\,\alpha_1+h\right)^2\sqrt{\alpha_1^2-1}}$$
$$+\frac{4\gamma_1e_1\alpha_1^2}{\left(\alpha_1^2h+(2h+4)\,\alpha_1+h\right)^2} + i\left[-\frac{\left(-\xi_2\sqrt{\alpha_1^2-1}+\xi_1\sqrt{\gamma_1}\,(\alpha_1+1)\right)\alpha_1^2\gamma_2}{\left(\alpha_2-(\alpha_2+2)\,\alpha_1\right)^2\sqrt{\alpha_1^2-1}}\right.$$
$$\left.-\frac{\sqrt{\gamma_1}\,(\alpha_1+1)\,\alpha_1^2}{\left(\alpha_2-(\alpha_2+2)\,\alpha_1\right)^2\sqrt{\alpha_1^2-1}} + \frac{8e_2\gamma_2\alpha_1^2}{\left(\alpha_1^2h+(2h+4)\,\alpha_1+h\right)^2}\right].$$

Finally, according to [Singh et al. 2013], we have

$$l_1 = \frac{\alpha_1(\alpha_1+1)i}{2\sqrt{\gamma_1(\alpha_1^2-1)}}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{g_{21}}{2},\ l_1^0 = Re\{l_1\},\ \sigma = -\frac{4l_1^0}{1-\alpha_1^2},$$

where

$$\begin{aligned}
l_1^0 = {} & \{64\alpha_1^4[(h^2\alpha_1^4\gamma_2+4\gamma_2h\,(h+2)\,\alpha_1^3+(\left(2\alpha_2^2e_2\xi_2+8e_2\alpha_2\xi_2+6h^2+8e_2\xi_2+16h+16\right)\gamma_2 \\
& +2e_1\gamma_1\,(\alpha_2+2)^2)\alpha_1^2+\left(\left(-4\alpha_2^2e_2\xi_2-8e_2\alpha_2\xi_2+4h^2+8h\right)\gamma_2-4e_1\alpha_2\gamma_1\,(\alpha_2+2)\right)\alpha_1 \\
& +\left(2\alpha_2^2e_2\xi_2+h^2\right)\gamma_2+2\alpha_2^2e_1\gamma_1)(\alpha_1^2-1)^{1/2}-2\xi_1e_2\gamma_2\left((\alpha_2+2)\,\alpha_1-\alpha_2\right)^2(\alpha_1+1)\,\gamma_1^{1/2}] \\
& \{\{\beta\,(\alpha_1+1)^2\,\gamma_1^{1/2}+32\gamma_1^3\,(\alpha_1-1))\,(\alpha_1-1)\,(\alpha_1^2-1)^{1/2}\}/64+(\gamma_1^{5/2}+\beta\gamma_1/32)\alpha_1^3 \\
& +(-\gamma_1^{5/2}+\beta\gamma_1/32)\alpha_1^2+(-\gamma_1^{5/2}-\beta\gamma_1/32)\alpha_1+\gamma_1^{5/2}+\left(\alpha_1^2-1\right)^{3/2}\gamma_1^2/2-\beta\gamma_1/32\}\} \\
& /\{(\alpha_1+1)\,(\alpha_1^2-1)^{1/2}(\alpha_1^3\beta^2+\alpha_1^2\beta^2+4\gamma_1(\alpha_1-1))(\alpha_1\alpha_2+2\alpha_1-\alpha_2)^2(\alpha_1^2h+2h\alpha_1+h \\
& +4\alpha_1)^2\}-\{[12\gamma_1\left(\alpha_1^2-2\alpha_1-3\right)(\alpha_1^2-1)\gamma_1^{1/2}+12\,(\alpha_1-2\gamma_1+1)\,\gamma_1\,(\alpha_1-1)\,(\alpha_1+1)]\alpha_1\} \\
& /\{384\,(\alpha_1+1)^3\,(\alpha_1-1)\,[\gamma_1(\alpha_1^2-1)]^{1/2}\}+\{2\gamma_1+[\gamma_1(\alpha_1^2-1)]^{1/2}\}/[4\beta\,(\alpha_1+1)] \\
& \{2\alpha_1^2\gamma_2/[(\alpha_2-(\alpha_2+2)\,\alpha_1)^2]+[4(-\xi_2(\alpha_1^2-1)^{1/2}+\xi_1\gamma_1^{1/2}\,(\alpha_1+1))e_2\gamma_2\alpha_1^2] \\
& /[(\alpha_1^2h+(2h+4)\,\alpha_1+h)^2(\alpha_1^2-1)^{1/2}]-4\gamma_1e_1\alpha_1^2/[\left(\alpha_1^2h+(2h+4)\,\alpha_1+h\right)^2]\}.
\end{aligned}$$

Thus, according to [Singh et al. 2013], the limit cycle of system is unstable and the hopf bifurcation is subcritical if $\sigma < 0$, and the limit cycle of system is stable and the hopf bifurcation is supercritical if $\sigma > 0$. For the equilibrium $E_2, \bar{E}$ can be derived similarly and will not be proved here. ■