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Evolutionary game theory and multi-agent reinforcement learning

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Abstract

In this paper we survey the basics of reinforcement learning and (evolutionary) game theory, applied to the field of multi-agent systems. This paper contains three parts. We start with an overview on the fundamentals of reinforcement learning. Next we summarize the most important aspects of evolutionary game theory. Finally, we discuss the state-of-the-art of multi-agent reinforcement learning and the mathematical connection with evolutionary game theory.

1 Introduction

In this paper we describe the basics of reinforcement learning (RL) and evolutionary game theory (EGT), applied to the field of multi-agent systems (MASs). The uncertainty inherent to the multi-agent environment implies that an agent needs to learn from, and adapt to, this environment to be successful. Indeed, it is impossible to foresee all situations an agent can encounter beforehand. Therefore, learning and adaptiveness become crucial for the successful application of MASs to contemporary technological challenges, for instance, routing in telecom, e-commerce, Robocup, etc. RL is already an established and profound theoretical framework for learning in stand-alone or single-agent systems. Yet, extending RL to MASs does not guarantee the same theoretical grounding. As long as the environment an agent is experiencing is Markov,[1] and the agent can experiment enough, RL guarantees convergence to the optimal strategy. In a MAS, however, the reinforcement an agent receives may depend on the actions taken by the other agents present in the system. Hence, the Markov property no longer holds. As such, guarantees of convergence do no longer hold.

In the light of the above problem it is important to fully understand the dynamics of RL and the effect of exploration in MASs. For this aim we review EGT as a solid basis for understanding learning and constructing new learning algorithms. The replicator equations appear to be an interesting model to study learning in various settings. This model consists of a system of differential equations describing how a population (or a probability distribution) of strategies evolves over time, and plays a central role in biological and economical models.

In Section 2 we summarize the fundamentals of RL. More precisely, we discuss policy iteration (PI) and value iteration (VI) methods, RL as a stochastic approximation technique and some convergence issues. We also discuss distributed RL in this section. We discuss basic concepts of traditional theory and EGT in Section 3. We provide definitions and examples of the most basic concepts as Nash equilibrium, Pareto optimality, evolutionary stable strategies and the replicator

[1] The Markov property states that only the present state is relevant for the future behavior of the learning process. Knowledge of the history of the process does not add any new information.

equations. We also discuss the relationship between EGT and RL. Section 4 is dedicated to multi-agent RL (MARL). We discuss some possible approaches, their advantages and limitations. More precisely, we will describe the joint action space approach, independent learners, informed agents and an EGT approach. Finally, conclusions are given in Section 5.

2 Fundamentals of RL

RL finds its roots in animal learning. It is well known that we can teach an animal to respond in a desired way by rewarding and punishing it appropriately. For example we can train a dog to detect drugs in people's luggage at customs by rewarding it each time it responds correctly and punishing it otherwise. Based on this external feedback signal the dog adapts to the desired behavior. More generally, the objective of a reinforcement learner is to discover a policy, i.e. a mapping from situations to actions, so as to maximize the reinforcement it receives. The reinforcement is a scalar value that is usually negative to express a punishment and positive to indicate a reward. Unlike supervised learning techniques, reinforcement learning methods do not assume the presence of a teacher who is able to judge the action taken in a particular situation. Instead the learner finds out what the best actions are by trying them out and by evaluating the consequences of the actions by itself. For many problems the consequences of an action are not immediately apparent after performing the action, but only become apparent after a number of other actions have been taken. In other words the selected action may not only affect the immediate reward/punishment the learner receives, but also the reinforcement it might get in subsequent situations, i.e. the delayed rewards and punishments. Originally, RL was considered to be single-agent learning. All of the events the agent has no control over are considered to be part of the environment. In this section we consider the single-agent setting and in Section 4 we discuss different approaches to MARL.

2.1 RL and its relationship to dynamic programming

From a mathematical point of view RL is closely related to dynamic programming (DP). DP is a well known method to solve Markovian decision problems (MDP; see Bertsekas (1976)). A MDP is a multistage decision problem for which an optimal policy must be found, i.e. a policy that optimizes the expected long-term reward. Usually the expected discounted cumulative return is considered. However, other measures do exist (Bellman & Dreyfuss, 1962). The Markovian property assures that the learner can behave optimally observing its current state only, i.e. there is no need to keep track of the history, so the learner does not need to know how it got there. The DP techniques are usually classified into two approaches: the PI approach and the VI approach. The same classification is used in RL, and each RL approach can be viewed as an asynchronous, model-free approach of its DP counterpart. Before we present the two RL classes, we briefly introduce the DP counterparts. DP is a model-based approach, so it assumes that a model of the environment is available. In general the environment is stochastic, and its response is described by a transition matrix. This matrix gives the probability that the next state s_{t+1} will be reached and a reward r_{t+1} will be received, given the current state is s_t, and the action taken is a_t. This is represented by

$$P^a_{ss'} = P\{s_{t+1} = s' | s_t = s,\ a_t = a\}, \tag{1}$$

which are called the transition probabilities. Now, given any state and action s and a, together with any next state s', the expected value of the next reward is

$$R^a_{ss'} = E\{r_{t+1} | s_t = s,\ a_t = a,\ s_{t+1} = s'\}. \tag{2}$$

It is important here to note that we assume that the Markov property is valid. This allows us to determine the optimal action based on the observation of the current state only. Below we introduce the two approaches in DP, PI and VI, and introduce their RL counterparts.

2.1.1 PI in DP

The PI approach considers a current policy π, and tries to locally improve the policy based on the statevalues that correspond to the current policy π, More formally

$$V^{\pi}(s)=E_{\pi}\{r_{t+1}+\gamma r_{t+2}+\gamma^{2}r_{t+3}+\cdots|s_t=s\} \tag{3}$$

$$=E_{\pi}\{r_{t+1}+\gamma V^{\pi}(s_{t+1})|s_t=s\}. \tag{4}$$

It is well known that $V_{\pi}^{*}(s)$, with π^{*} the optimal policy, are the solutions of the Bellman optimality equation

$$V^{*}(s)=\max_{a}\sum_{s'}P_{ss'}^{a}\left[R_{ss'}^{a}+\gamma V^{*}(s')\right]. \tag{5}$$

$V_{\pi}(s)$ can be calculated using successive approximation as follows:

$$V_{k+1}(s)=E_{\pi}\{r_{t+1}+\gamma V_{k+1}(s_{t+1})|s_t=s\} \tag{6}$$

$$=\sum_{a}\pi\,(s,a)\sum_{s'}P_{ss'}^{a}\left[R_{ss'}^{a}+\gamma V_{k}(s')\right]. \tag{7}$$

to locally improve the policy in a given state s, the best action a is looked for based on the current state values $V_k(s)$. So π is improved in state s, by updating $\pi(s)$ into the action that maximizes the right-hand side of Equation (7), yielding an updated policy π. The PI algorithm is given in Algorithm 1.

Algorithm 1 PI

```
Choose a policy $\pi'$ arbitrarily
– loop
–    $\pi := \pi'$
–    Compute the value function $V$ of policy $\pi$:
–       solve the linear equations:
–       $V^{\pi}(s) = \sum_{s'} P_{ss'}^{\pi(s)}\left[R_{ss'}^{\pi(s)} + \gamma V^{\pi}(s')\right]$
–    Now improve the policy at each state:
(I)     $\pi'(s) = \mathrm{argmax}_a \sum_{s'} P_{ss'}^{a}\left[R_{ss'}^{a} + \gamma V^{\pi}(s')\right]$
– until $\pi = \pi'$
```

2.1.2 PI in RL

For a RL approach we usually assume that the model is not available and therefore we cannot improve the policy locally using Equation (I) from Algorithm 1. Instead, PI RL techniques build up their own internal evaluator or critic. Based on this internal critic the appropriateness of an action for a state is evaluated. An architecture realizing this type of learning is given in Figure 1.

This architecture was first proposed in Barto *et al.* (1983) and generalized in Anderson (1987). Since then it has been adopted by many others. The scheme in Figure 1 contains two units, the evaluation unit and the action unit. The former is the internal evaluator, while the latter is

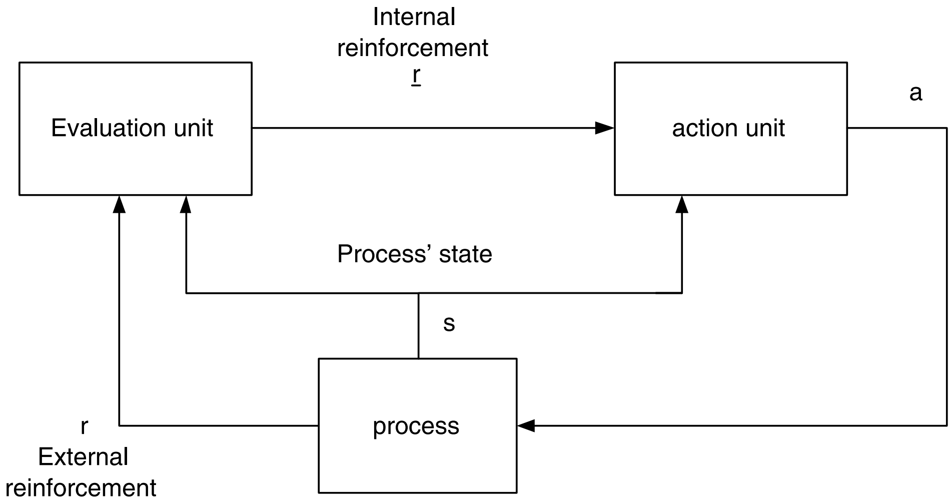


Figure 1 An architecture for PI RL

responsible for determining the actions that look most promising according to the internal evaluation. The evaluation unit yields an estimate of the current $V_{\pi}(s)$ values. In Barto *et al.* (1983) this component is called the adaptive critic element. Based on the external reinforcement signal r_t, and its own prediction, the evaluation unit adjusts its prediction on-line as follows:

$$V(s_t) = V(s_t) + \zeta(r_t + \gamma V(s_{t+1}) - V(s_t)) \tag{8}$$

where ζ is a positive constant determining the rate of change. This updating rule is the so-called temporal difference, TD(0), method of Sutton (1988). As stated above, the goal of the evaluation unit is to transform the environmental reinforcement signal r into a more informative internal signal $\underline{r}$. To generate the internal reinforcement, differences in the predictions between two successive states are used. If the process moves from a state with a prediction of lower reinforcement into a state with a prediction of higher reinforcement, the internal reinforcement signal will reward the action that caused the move. Barto *et al.* (1983) proposed using the following internal reinforcement signal:

$$\underline{r}(t) = r(t) + \gamma V(s_{t+1}) - V(s_t) \tag{9}$$

Given the current state, the action unit produces the action that will be applied next. Many different approaches exist for implementing this action unit. If the action unit contains a mapping from states to actions, the action that will be applied to the system can be generated by a two-step process. In the first step the most promising action is generated, this is the action to which the state is mapped. This action is then modified by means of a stochastic modifier S. This second step is necessary to allow exploration of alternative actions. Actions that are 'close' to the action that was generated in the first step are more likely to be the outcome of this second step. This approach is often used if the action set is a continuum. If an action that was applied to the system turned out to be better than expected, i.e. the internal reinforcement signal is positive, then the mapping will be 'shifted' towards this action. If the action set is discrete, a table representation can be used. Then the table maps states to probabilities of actions to be selected for a particular state, and the probabilities are updated directly. Below we discuss a simple mechanism to update these probabilities.

2.1.3 Learning automata

Learning automata (LA) have their origins in the research labs of the former USSR. More precisely it started with the work of Tsetlin in the 1960s (Tsetlin, 1962; 1973).

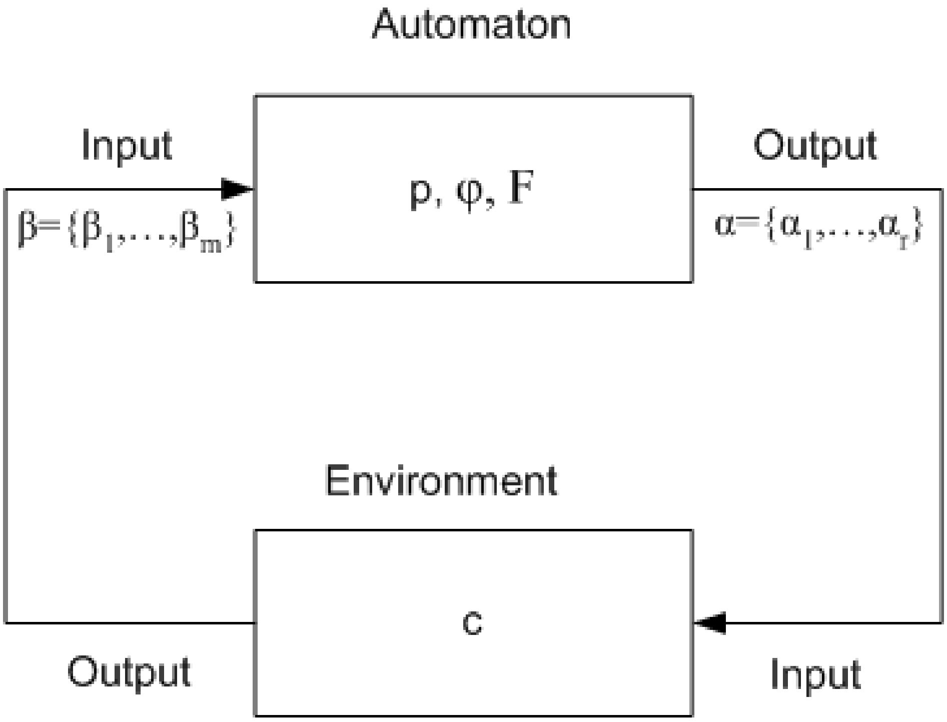

Automaton
Input
$\beta=\{\beta_1,\ldots,\beta_m\}$
p, φ, F
Output
$\alpha=\{\alpha_1,\ldots,\alpha_r\}$
Environment
c
Output
Input

Figure 2 The feedback connection of the LA–environment pair

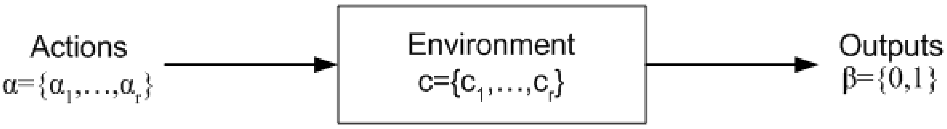


Figure 3 Zooming in on the environment of the LA

In those early days, LAs were deterministic and based on complete knowledge of the environment. Later developments came up with uncertainties in the system and the environment and lead to the stochastic automaton. More precisely, the stochastic automaton tries to provide a solution for the RL problem without having any information on the initial optimal action. It starts with equal probabilities on all actions and during the learning process these probabilities are updated, based on responses from the environment. We consider LAs to be a method for solving RL problems in a policy iterative fashion.

The term learning automaton (LA) was introduced for the first time in the work of Narendra & Thathacher (1974). Since then there has been a lot of development in the field and a number of survey papers and books on this topic have been published (Narendra & Thathacher, 1974; 1989; Thathacher, 2002).

In Figure 2 a LA is illustrated in its most general form. The automaton tries to determine an optimal action out of a set of possible actions to perform. Let us now first zoom in on the environment part of Figure 2. This part is illustrated in Figure 3. The environment responds to the input action α by producing an output β. The output also belongs to a set of possible outcomes, i.e. $\{0, 1\}$, which is probabilistically related to the set of inputs through the environment vector c.

The environment is represented by a triple $\{\alpha, c, \beta\}$, where α represents a finite action set, β represents the response set of the environment and c is a vector of penalty probabilities, where each component c_i corresponds to an action α_i.

The response β from the environment can take on 2 values β_1 or β_2. Often they are chosen to be 0 and 1, where 1 is associated with a penalty response (a failure) and 0 with a reward (a success).

Now, the penalty probabilities c_i can be defined as

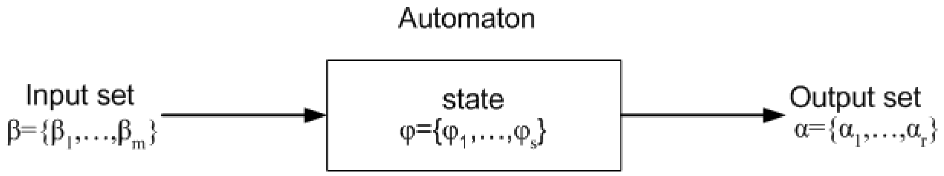


Figure 4 Zooming in on the automaton part of the LA

$$c_i = P(\beta(n) = 1 | \alpha(n) = \alpha_i). \tag{10}$$

Consequently, c_i is the probability that action α_i will result in a penalty response. If these probabilities are constant, the environment is called *stationary*.

Several models are recognized by the response set of the environment. Models in which the response β can only take two values are called P-models. Models that allow a finite number of values in the fixed interval [0, 1] are called Q-models. When β is a continuous random variable in the fixed interval [0, 1], the model is called an S-model.

Now we have considered the environment of the LA model of Figure 2, we now zoom in on the automaton itself. More precisely, Figure 4 illustrates this.

The automaton is represented by a set of states $\varphi = \{\varphi_1, \ldots, \varphi_s\}$. As opposed to the environment, β becomes the input and α the output. This implicitly defines a function $F{:}\varphi \times \beta \rightarrow \varphi$, mapping the current state and input into the next state, and a function $H{:}\varphi \times \beta \rightarrow \alpha$, mapping the current state and current input into the current output. In this article we use p as the probability vector over the possible actions of the automaton, which corresponds to the function H.

Summarizing, this brings us to the definition of a LA. More precisely, it is defined by a quintuple $\{\alpha, \beta, F, p, T\}$ for which α is the action or output set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of the automaton, β is a random variable in the interval [0, 1], F is the state transition function, p is the action probability vector of the automaton or agent and T denotes an update scheme. The output α of the automaton is actually the input to the environment. The input β of the automaton is the output of the environment, which is modeled through penalty probabilities c_i with $c_i = P[\beta|\alpha_i]$, $i = 1, \ldots, r$ over the actions.

The automaton can be either stochastic or deterministic, the former's output function H being composed of probabilities based on the environment's response, whilst the latter has a fixed mapping function between the internal state and the function to be performed.

Further sub-division of classification occurs when considering the transition or updating function F, which determines the next state of the automaton given its current state and the response from the environment. If this is fixed then the automaton is a *fixed structure deterministic* or a *fixed structure stochastic* automaton.

However, if the updating function is variable, allowing for the transition function to be modified so that choosing the operations or actions changes after each iteration, then the automaton is a *variable structure deterministic* or a *variable structure stochastic* automaton. In this paper we are mainly concerned with the variable structure stochastic automata, which have the potential of greater flexibility and therefore increased performance. Such an automaton A at timestep t is defined as

$$A(t) = \{\alpha, \beta, p, T(\alpha, \beta, p)\}$$

where we have an action set α with r actions, an environment response set β and a probability set p containing r probabilities, each being the probability of performing every action possible in the current internal automaton state. The function T is the reinforcement algorithm, which modifies the action probability vector p with respect to the performed action and the received response. The new probability vector can therefore be written as

$$p(t+1) = T\{\alpha, \beta, p(t)\}$$

with t the timestep.

Next we summarize the different update schemes.

The most important update schemes are *linear reward–penalty*, *linear reward–inaction* and *linear reward–ε-penalty*. The philosophy of those schemes is essentially to increase the probability of an action when it results in a success and to decrease it when the response is a failure. The general update algorithm is given by

$$p_i(t+1) \leftarrow p_i(t) + \alpha(1-\beta(t))(1-p_i(t)) - \beta(t)p_i(t)$$
$$\text{if } \alpha_i \text{ is the action taken at time } t \tag{11}$$
$$p_j(t+1) \leftarrow p_j(t) - \alpha(1-\beta(t))p_j(t) + b\beta(t)[(r-1)^{-1} - p_j(t)]$$
$$\text{if } \alpha_j \neq \alpha_i. \tag{12}$$

The constants a and b in]0, 1[are the reward and penalty parameters respectively. When $a=b$ the algorithm is referred to as linear reward–penalty ($L_{R\text{–}P}$), when $b=0$ it is referred to as linear reward–inaction ($L_{R\text{–}I}$) and when b is small compared to a it is called linear reward–ε-penalty ($L_{R\text{–}\varepsilon P}$).

If the penalty probabilities c_i of the environment are constant, the probability $p(n+1)$ is fully determined by $p(n)$ and hence $p(n)_{n>0}$ is a discrete-time homogeneous Markov process. Convergence results for the different schemes are obtained under the assumptions of constant penalty probabilities, see Narendra & Thathacher (1989). LAs belong to the PI approach, because action probabilities are updated directly. In the VI approach the quality of an action is determined and the action with the highest quality is part of the optimal policy.

2.1.4 VI in DP

The VI approach turns the Bellman equation (5) into an update rule. As the $V^*(s)$ are the unknowns, an estimate of these values is used at the right-hand side, these estimates $V_k(s)$ are iteratively updated using the update rule:

$$V_{k+1}(s) = \max_a E\{r_{t+1} + \gamma V_k(s_{t+1}) | s_t = s, a_t = a\} \tag{13}$$

$$= \max_a \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V_k(s') \right]. \tag{14}$$

As for all states $k \rightarrow V_k(s)$ is a contraction mapping, the $V_k(s)$ values converge in the limit to the optimal values $V^*(s)$. In practice the updating is stopped when the changes become very small and the corresponding optimal policy does not change. As VI in DP assumes a transition model of the system is available, the optimal policy π^* can be obtained using the following equation:

$$\pi^*(s) = \text{argmax}_a \sum_{s'} P^a_{ss'} \left[R^a_{ss'} + \gamma V_k(s') \right]. \tag{15}$$

2.1.5 VI in RL

The best known counterpart of VI of DP in RL is Q-learning (Watkins & Dayan, 1992). As RL is model free the optimal policy cannot be retrieved from Equation (15). Therefore, Q-learning stores explicitly the Quality of each action for each state. These values are called Q-values, denoted by $Q^*(s, a)$. The relationship between the $V^*(s)$ values and the $Q^*(s, a)$ values is

$$V^*(s) = \max_a Q^*(s, a). \tag{16}$$

The $Q^*(s, a)$ are equal to the expected return of taking action a in state s and from then on behaving according to the optimal policy π^*, i.e.

$$Q^* (s, a) = \sum_{s'} P^a_{ss'} E\left[R^a_{ss'} + \gamma V^*(s')\right] \tag{17}$$

and therefore we also have that

$$Q^* (s, a) = \sum_{s'} P^a_{ss'} E\left[R^a_{ss'} + \gamma \max_a Q^*(s', a)\right]. \tag{18}$$

The same way as in VI of DP, the Q-values are iteratively updated. However, because RL does not include a model of the environment, we do not know the $P^a_{ss'}$, nor the $E[R^a_{ss'}]$, therefore stochastic approximation is used, yielding the well-known Q-learning updating rule:

$$Q (s, a) \leftarrow Q (s, a) + \alpha (r + \gamma \max_a Q (s', a') - Q (s, a)) \tag{19}$$

where α is a learning factor.

2.2 *Some convergence issues*

Not all RL techniques come with a proof of convergence. In particular, the PI approaches often lack such a proof. The LA, introduced above as an example of one stage PI, does have a proof of convergence. A single LA that uses the reward–inaction updating scheme is guaranteed to converge (Narendra & Thathacher, 1989), the same is true for a set of independent LAs (see Section 4).

In the VI approach, Q-learning is also proved to converge if applied in a Markovian environment, provided that some very reasonable assumptions apply such as appropriate settings for α (see Section 2.2.2). The Markovian property is really crucial: as soon as this no longer holds, the guarantee for convergence is lost. However, this does not mean that RL cannot be applied in non-Markovian environments, but care has to be taken.

2.2.1 *Partially observable Markov decision processes*

RL has been successfully applied to partially observable Markov decision processes (POMDPs). Here the states are only partially observable such that the learner cannot distinguish sufficiently between the states to let the Markovian property hold. For example, this may be due to a continuous state problem has been translated into a discrete problem, and the discretization is too coarse. A denser discretization can restore the Markovian property, but leads to an increased size of the state space. Another reason to not have the Markovian property is an agent might miss a particular component of the state description, due to the lack of a sensor that can measure that component. For example, if temperature is a critical component to have a Markovian view on the environment, and the agent cannot measure this, the environment looks non-Markovian to the agent. Or if the agent only has a local view of the environment, like a mobile robot, then it is possible that two different locations result in the same sensor input to the agent. A technique that is often used to tackle the non-Markovianism is to guide the exploration. If the agent can actively take part, the exploration can be steered such that two indistinguishable states can be differentiated via their reactions from certain actions and in so doing produce a limited history. As we do not believe that this kind of guided exploration is the way to solve the non-Markovian environments, where MASs operates, we do not go into more detail in this issue. For more details see Cassandra *et al.* (1994), Kaelbling *et al.* (1996) and Perkins & Pendrith (2002).

2.2.2 *The convergence of Q-learning*

Amongst others, Tsitsiklis (1993) has proved that under very reasonable assumptions, Q-learning is guaranteed to converge to the optimal policy. The proof of Tsitsiklis is very interesting because it considers Q-learning as a stochastic approximation technique and can be adapted to prove all kinds of variants to Q-learning. See, for example, the distributed version in the next section. We will not discuss the proof in detail here, but it is worthwhile to have a look at the assumptions that have an impact on the setting of the learning process. A first assumption says that the learning parameter in the Q-learning updating rule (i.e. α in Section 2.1.5) must be decreased in time such that $\sum_{t=0}^{\infty} \alpha_{(s,a)}(t)=\infty$ and $\sum_{t=0}^{\infty} \alpha_{(s,a)}^2(t)<\infty$. This allows Q-values to be updated in an asynchronous way.

Another assumption states that it is no problem that past experience is used to guide the exploration, so the agent can actively take part in the exploration.

A last interesting assumption is that the agent can use outdated information as long as old information is eventually discarded, i.e. in the updating rule of Q-learning, the update of the $Q_{k+1}(s, a)$ values can be performed based on $Q_{k-i}(s, a)$ values, with i between 0 and k. It becomes clear that Q-learning can be considered as a stochastic approximation method by rewriting the Q-learning update rule as follows:

$$Q_{k+1}(s, a)=Q_k(s, a)+\alpha_{(s,a)}(k)[Q_k(s, a)(Q_k)-Q_k(s, a)+W_k(s, a)]. \tag{20}$$

With $Q_k(s, a)(Q_k)=E[r(s, a)+\gamma\sum_{s'}P_{ss'}^{a} max_{a'}Q_k(s', a')]$ and (Q_k) is the vector containing all $Q_k(s, a)$ values. For $\alpha_{(s,a)}(k)=0$ if $Q_{k+1}(s, a)$ is not updated at time step $k+1$, otherwise $\alpha_{(s,a)}(k) \in\ ]0, 1]$ obeying the restrictions stated above.

Also

$$W_k\,(s, a) = (r_k(s, a)+\gamma \max_{a} Q_k\,(s', a')) - E\left[r(s, a)+\gamma \sum_{s'} P_{ss'}^{a} \max_{a} Q_k\,(s', a')\right]. \tag{21}$$

Q is a contraction mapping, meaning that it is monotonically converging, and W is proved to behave as white noise (Tsitsiklis, 1993).

In the next section we briefly introduce a first MAS setting of RL, however we prefer to refer to it as a simple distributed approach.

2.3 Distributed RL

As the proof of Tsitsiklis allows that Q-values are updated asynchronously and based on outdated information, it is rather straightforward to come up with a parallel or distributed version of Q-learning.

Assume we subdivide the state space in different regions. In each region an agent gets the responsibility of updating the corresponding Q-values by exploring its region. Agents can explore their own region, and make updates in their copy of the table of the Q-values to the Q-values that are their own region. As long as they make transitions in their own region, they can apply the usual updating rule. However, if they make a transition to another region, with Q-values that are the responsibility of another agent, they should not directly communicate to that other agent and ask for the particular Q-value but use the information they have in their own copy table, i.e. use out-dated information and steer the exploration so as to get back to their own region. As out-dated information needs to be updated from time to time, the agents should communicate from time to time and distribute the Q-values for which they have the responsibility. As we do not put the Markovian property in danger by this approach, the proof of Tsitsiklis can be applied and convergence is still assured. Although this approach can be considered as a MAS, we prefer to refer to it as a distributed or parallel version of Q-learning. The approach has been successfully applied to the problem of Call Admission Control in telecommunications (Steenhaut *et al.*, 1997).[2]

[2] In this paper we focus on genuine model-free RL. However, many RL techniques incorporate domain knowledge. If this knowledge is available then it is a good idea to use it one way or another. A useful feature of RL is that if the domain knowledge seems imperfect or even totally incorrect, the RL techniques will still converge, at the end, to the optimal policy. The incorporation of domain knowledge can, for example, be obtained by initialing the Q-values based on the knowledge, or by hypothetical moves, so updates are based on the model.

3 EGT

3.1 Introduction

Originally, game theory (GT) was launched by von Neumann & Morgenstern (1944) in their book *Theory of Games and Economic Behavior*.[3]

GT is an economical theory that models interactions between rational agents as games of two or more players that can choose from a set of strategies and the corresponding preferences. It is the mathematical study of interactive decision making in the sense that the agents involved in the decisions take into account their own choices and those of others. Choices are determined by: (1) stable preferences concerning the outcomes of their possible decisions; and (2) that agents act strategically, in other words, they take into account the relation between their own choices and the decisions of other agents. Different economical situations lead to different rational strategies for the players involved.

When John Nash discovered the theory of games at Princeton, in the late 1940s and early 1950s, the impact was enormous. The impact of the developments in GT were particularly relevant in the field of economics, where its concepts played an important role in, for instance, the study of international trade, bargaining, the economics of information and the organization of corporations. However, the importance of GT also became clear in other disciplines such as social and natural sciences, and examples include studies of legislative institutions, voting behavior, warfare, international conflicts and evolutionary biology.

However, von Neumann and Morgenstern had only managed to define an equilibrium concept for two-person zero-sum games. Zero-sum games correspond to situations of pure competition, whatever one player wins, must be lost by another. Nash addressed the case of competition with mutual gain by defining best-reply functions and using Kakutani's fixed-point theorem.[4] The main results of his work were the development of the *Nash equilibrium* and the *Nash bargaining solution* concept.

Despite the great usefulness of the Nash equilibrium, the assumptions traditional GT makes, such as hyper-rational players that correctly anticipate the other players in an equilibrium, made GT stagnate for quite some time (Weibull, 1996; Samuelson, 1997; Gintis, 2000). A lot of refinements of Nash equilibria came along (for instance, *trembling hand perfection*), which made it hard to choose the appropriate equilibrium in a particular situation. Almost any Nash equilibrium could be justified in terms of some particular refinement. This made clear that the static Nash concept did not reflect the (dynamic) real world where people do not make decisions under hyper-rationality assumptions.

This is where EGT originated. More precisely, Maynard-Smith adopted the idea of evolution from biology (Maynard-Smith & Price, 1973; Maynard-Smith, 1982). He applied GT to biology, which made him relax some of the premises of GT. Under these biological circumstances, it becomes impossible to judge what choices are the most rational. The question now becomes how a player can learn to optimize its behavior and maximize its return. This learning process is the core of evolution in biology.

These new ideas led Smith and Price to the concept of evolutionary stable strategies (ESSs), a special case of the Nash condition. In contrast to GT, EGT is descriptive and starts from more realistic views of the game and its players. Here the game is no longer played once by rational players who know all the details of the game, such as each others preferences over outcomes. Instead, EGT assumes that the game is played repeatedly by players randomly drawn from large populations, uninformed of the preferences of the opponent players.

[3] We do not intend to ignore previous game theoretic results, for instance the theorem of Zermelo, which asserts that chess is strictly determined. We only state that this is the first book under the name GT assembling many different results.

[4] Kakutani's fixed-point theorem goes as follows. Consider X a non-empty set and F a point-to-set map from X to subsets of X. Now, if F is continuous, X is compact and convex, and for each x in X, $F(x)$ is non-empty and convex, F has a fixed point. Applying this theorem (and thus checking its conditions) to the best response function proves the existence of a Nash equilibrium.

Table 1 Matrix (A) defines the payoff for the row player for the Prisoner's dilemma. Strategy D is defect and strategy C is cooperate

$A=$

D	1	5
C	0	3

Table 2 Matrix (B) defines the payoff for the column player for the prisoner's dilemma. Strategy D is defect and strategy C is cooperate

$B=$

D	C
1	0
5	3

EGT offers a solid basis for rational decision making in an uncertain world by how individuals make decisions and interact in complex environments in the real world. Modeling learning agents in the context of MASs requires insight into the type and form of interactions with the environment and other agents in the system. Usually, these agents are modeled in a similar to the different players in a standard game theoretical model. In other words, these agents assume complete knowledge of the environment, have the ability to correctly anticipate the opposing player (hyper-rationality) and know that the optimal strategy in the environment is always the same (static Nash equilibrium). The intuition that in the real world people are not completely knowledgeable hyper-rational players and that an equilibrium can change dynamically led to the development of EGT.

Before introducing the most elementary concepts from EGT we summarize some well-known examples of strategic interaction in the next section.

3.2 *Examples of strategic interaction*

3.2.1 *The prisoner's dilemma*

As the first example of a strategic game we consider the prisoner's dilemma game (Weibull, 1996; Gintis, 2000).

In this game, two prisoners, who committed a crime together, have a choice to either collaborate with the police (to defect) or work together and deny everything (to cooperate). If the first criminal (row player) defects and the second cooperates, the first gets off the hook (expressed by a maximum reward of 5) and the second gets the most severe punishment (reward 0). If they both defect, they get the second most severe punishment (expressed by a payoff of 1). If both cooperate, they both get a small punishment (reward 3).

The rewards are summarized in payoff Tables 1 and 2. The first table has to be read from a row perspective and the second from a column perspective. For instance if the row player chooses to Defect (D), the payoff has to be read from the first row. It then depends on the strategy of the column player as to what payoff the row player will receive.

3.2.2 *The battle of the sexes*

The second example of a strategic game we consider is the battle of the sexes game (Weibull, 1996; Gintis, 2000).

In this game, a married couple loves each other so much they want to do everything together. One evening the husband wants to see a football game and the wife wants to go to the opera. If they both choose their own preference and do their activities separately they receive the lowest payoff. This situation is described by the payoff matrices of Tables 3 and 4.

Table 3 Matrix (A) defines the payoff for the row player for the battle of the sexes. Strategy F is choosing football and strategy O is choosing the opera

$A=$

F	2	0
O	0	1

Table 4 Matrix (B) defines the payoff for the column player for battle of the sexes. Strategy F is choosing football and strategy O is choosing the opera

$B=$

F	O
1	0
0	2

Table 5 Matrix (A) defines the payoff for the row player for the matching pennies game. Strategy H is playing heads and strategy T is playing tails

$A=$

H	1	−1
T	−1	1

Table 6 Matrix (B) defines the payoff for the column player for the matching pennies game. Strategy H is playing heads and strategy T is playing tails

$B=$

H	T
−1	1
1	−1

3.2.3 Matching pennies

The third example of a strategic game we consider is the matching pennies game (Weibull, 1996; Gintis, 2000).

In this game two children hold both a penny and independently choose which side of the coin to show (heads or tails). The first child wins if both coins show the same side, otherwise the second child wins. This is an example of a zero-sum game as can be seen from the payoff Tables 5 and 6. Whatever is lost by one player, must be won by the other.

3.3 Elementary concepts

In this section we review the key concepts of GT and EGT and their mutual relationships. We start by defining strategic games and concepts such as Nash equilibrium, Pareto optimality and ESSs. Then we discuss the relationships between these concepts and provide some examples.

3.3.1 Strategic games

An n-player normal form game models a conflict situation involving gains and losses between n players. In such a game n players interact with each other by all choosing an action (or strategy) to play. All players choose their strategy at the same time without being informed about the choices of the others. For reasons of simplicity, we limit the pure strategy set of the players to two strategies.

Table 7 The left matrix (A) defines the payoff for the row-player, the right matrix (B) defines the payoff for the column-player

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

A strategy is defined as a probability distribution over all possible actions. In the pure two-strategy case, we have $s_1=(1, 0)$ and $s_2=(0, 1)$. A mixed strategy s_{m} is then defined by $s_{\mathrm{m}}=(x_1, x_2)$ with $x_1, x_2 \neq 0$ and $x_1+x_1=1$.

Defining a game more formally, we restrict ourselves to the two-player two-action game. Nevertheless, an extension to n-player n-actions games is straightforward, but examples in the n-player case do not show the same illustrative strength as in the two-player case. A game $G=(S_1, S_2, P_1, P_2)$ is defined by the payoff functions P_1, P_2 and their strategy sets S_1 for the first player and S_2 for the second player. In the two-player two-strategies case, the payoff functions $P_1{:}S_1 \times S_2 \rightarrow \Re$ and $P_2{:}S_1 \times S_2 \rightarrow \Re$ are defined by the payoff matrices, A for the first player and B for the second player, see Table 7. The payoff tables A, B define the instantaneous rewards. Element a_{ij} is the reward the row-player (player 1) receives for choosing pure strategy s_i from set S_1 when the column-player (player 2) chooses the pure strategy s_j from set S_2. Element b_{ij} is the reward for the column-player for choosing the pure strategy s_j from set S_2 when the row-player chooses pure strategy s_i from set S_1.

The family of 2×2 games is usually classified in the following three subclasses, following Redondo (2001).

Subclass 1: if $(a_{11}-a_{21})(a_{12}-a_{22})>0$ or $(b_{11}-b_{12})(b_{21}-b_{22})>0$, at least one of the two players has a dominant strategy, therefore there is just one strict equilibrium.

Subclass 2: if $(a_{11}-a_{21})(a_{12}-a_{22})<0$, $(b_{11}-b_{12})(b_{21}-b_{22})<0$ and $(a_{11}-a_{21})(b_{11}-b_{12})>0$, there are two pure equilibria and one mixed equilibrium.

Subclass 3: if $(a_{11}-a_{21})(a_{12}-a_{22})<0$, $(b_{11}-b_{12})(b_{21}-b_{22})<0$ and $(a_{11}-a_{21})(b_{11}-b_{12})<0$, there is just one mixed equilibrium.

The first subclass includes those types of game where each player has a dominant strategy,[5] as for instance the prisoner's dilemma. However, it includes a larger collection of games as only one of the players needs to have a dominant strategy. In the second subclass none of the players has a dominant strategy (e.g. battle of the sexes). However, both players receive the highest payoff by both playing their first or second strategy. This is expressed in the condition $(a_{11}-a_{21})(b_{11}-b_{12})>0$. The third subclass only differs from the second in the fact that the players do not receive their highest payoff by both playing the first or the second strategy (e.g. matching pennies game). This is expressed by the condition $(a_{11}-a_{21})(b_{11}-b_{12})<0$.

3.3.2 Nash equilibrium

In traditional GT it is assumed that the players are hyper-rational, meaning that every player will choose the action that is best for themselves, given their beliefs about the other players' actions. A basic definition of a Nash equilibrium is stated as follows. If there is a set of strategies for a game with the property that no player can increase its payoff by changing his strategy while the other players keep their strategies unchanged, then that set of strategies and the corresponding payoffs constitute a Nash equilibrium.

Formally, a Nash equilibrium is defined as follows. When two players play the strategy profile $s=(s_i, s_j)$ belonging to the product set $S_1 \times S_2$ then s is a Nash equilibrium if $P_1(s_i, s_j) \geq P_1(s_x, s_j)$ for all $x \in \{1, \ldots, n\}$ and $P_2(s_i, s_j) \geq P_2(s_i, s_x)$ for all $x \in \{1, \ldots, m\}$.[6]

[5] A strategy is dominant if it is always better than any other strategy, regardless of what the opponent may do.

[6] For a definition in terms of best reply or best response functions we refer the reader to Weibull (1996).

3.3.3 Pareto optimality

Intuitively a Pareto-optimal solution of a game can be defined as follows: a combination of actions of agents in a game is Pareto optimal if there is no other solution for which all players do at least as well and at least one agent is strictly better off.

More formally we have that a strategy combination $s=(s_1, \ldots, s_n)$ for n agents in a game is Pareto optimal if there does not exist another strategy combination $s'=(s_1, \ldots, s_n)$ for which each player receives at least the same payoff P_i and at least one player j receives a strictly higher payoff than P_j, i.e. $P_i(s) \leq P_j(s)$ for all I and there exists $j{:}P_i(s) < P_j(s)$.

Another related concept is that of Pareto dominance: an outcome of a game is Pareto dominated if some other outcome would make at least one player better off without hurting any other player. That is, some other outcome is weakly preferred by all players and strictly preferred by at least one player. If an outcome is not Pareto dominated by any other, then it is Pareto optimal.

3.3.4 ESSs

The core equilibrium concept of EGT is that of an ESS. The idea of an ESS was introduced by Maynard- Smith & Price (1973). Imagine a population of agents playing the same strategy. Assume that this population is invaded by a different strategy, which is initially played by a small number of the total population. If the reproductive success of the new strategy is smaller than the original, it will not overrule the original strategy and will eventually disappear. In this case we say that the strategy is evolutionary stable against this new appearing strategy. More generally, we say a strategy is an ESS if it is robust against evolutionary pressure from any appearing mutant strategy.

Formally an ESS is defined as follows. Suppose that a large population of agents is programmed to play the (mixed) strategy s and suppose that this population is invaded by a small number of agents playing strategy s'. The population share of agents playing this mutant strategy is $\varepsilon \in \,]0, 1[$. When an individual is playing the game against a random chosen agent, chances that he is playing against a mutant are ε and against a non-mutant are $1-\varepsilon$. The expected payoff for the first player being a non-mutant is

$$P(s, (1-\varepsilon)s+\varepsilon s')=(1-\varepsilon)P(s, s)+\varepsilon p(s, s')$$

and being a mutant is

$$P(s', (1-\varepsilon)s+\varepsilon s').$$

Now we can state that a strategy s is an ESS if for all $s' \neq s$ there exists some $\delta \in \,]0, 1[$ such that for all $\varepsilon{:}0<\varepsilon<\delta$,

$$P(s, (1-\varepsilon)s+\varepsilon s')>P(s', (1-\varepsilon)s+\varepsilon s')$$

holds. The condition for all $\varepsilon{:}0<\varepsilon<\delta$ expresses that the share of mutants needs to be sufficiently small.

3.3.5 The relation between Nash equilibria and ESSs

This section explains how the core equilibria concepts from classical and EGT relate to one another. The set of ESSs for a particular game are contained in the set of Nash equilibria for that same game,

$$\{ESS\} \subset \{NE\}.$$

The condition for an ESS is more strict than the Nash condition. Intuitively this can be understood as follows: as defined above, a Nash equilibrium is a best reply against the strategies of the other players. Now if a strategy s_1 is an ESS then it is also a best reply against itself, and as such optimal.

Table 8 Prisoner's dilemma: the left matrix (A) defines the payoff for the row player, the right matrix (B) defines the payoff for the column player

$$A = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}$$

If it was not optimal against itself there would have been a strategy s_2 that would lead to a higher payoff against s_1 than s_1 itself:

$$P(s_2 s_1) > P(s_1, s_1).$$

So, if the population share ε of mutant strategies s_2 is small enough then s_1 is not evolutionary stable, because

$$P(s_2, (1-\varepsilon)s_1 + \varepsilon s_2) > P(s_1, (1-\varepsilon)s_1 + \varepsilon s_2).$$

Another important property for an ESS is the following. If s_1 is an ESS and s_2 is an alternative best reply to s_1, then s_1 has to be a better reply to s_2 than s_2 to itself. This can easily be seen as follows, because s_1 is an ESS, we have for all s_2

$$P(s_1, (1-\varepsilon)s_1 + \varepsilon s_2) > P(s_2, (1-\varepsilon)s_1 + \varepsilon s_2),$$

i.e.

$$P(s_2, s_2) = P(s_1, s_2).$$

If s_2 does as well against itself as s_1 does, then s_2 earns at least as much against $(1-\varepsilon)s_1 + \varepsilon s_2$ as s_1 and then s_1 is no longer evolutionary stable. To summarize we now have the following two properties for an ESS s_1:

1. $P(s_2, s_1) \leq P(s_1, s_1)$ for all s_2;
2. $P(s_2, s_1) = P(s_1, s_1) \Rightarrow P(s_2, s_2) < P(s_1, s_2)$ for all $s_2 \neq s_1$.

3.3.6 Examples

In this section we provide an example for each class of game described in Section 3.3.1 and illustrate the Nash equilibrium concept and ESS concept as well as Pareto optimality.

For the first subclass we consider the prisoner's dilemma game. The strategic setup of this game has been explained in Section 3.2. The payoffs of the game are repeated in Table 8. As one can see, both players have one dominant strategy, more precisely *defect*.

For both players, defecting is the dominant strategy and therefore always the best reply toward any strategy of the opponent. So the Nash equilibrium in this game is for both players to defect. Let us now determine whether this equilibrium is also an ESS. Suppose $\varepsilon \in [0, 1]$ is the number of cooperators in the population. The expected payoff of a cooperator is $3\varepsilon + (1-0\varepsilon)$ and that of a defector is $5\varepsilon + (1-1\varepsilon)$. As for all ε,

$$5\varepsilon + 1(1-\varepsilon) > 3\varepsilon + 0(1-\varepsilon),$$

defect is an ESS. So the number of defectors will always increase and the population will eventually only consist of defectors. In Section 3.4 this dynamical process will be illustrated by the replicator equations.

This equilibrium, which is both Nash and ESS, is not a Pareto-optimal solution. This can be easily seen if we look at the payoff tables. The combination (*defect,defect*) yields a payoff of (1, 1),

Table 9 Battle of the sexes: the left matrix (A) defines the payoff for the row player, the right matrix (B) defines the payoff for the column player

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Table 10 The left matrix (A) defines the payoff for the row player, the right matrix (B) defines the payoff for the column player

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

which is a smaller payoff for both players than the combination (*cooperate,cooperate*), which yields a payoff of (3, 3). Moreover, the combination (*cooperate,cooperate*) is a Pareto optimal solution. However, if we apply the definition of Pareto optimality, then also (*defect,cooperate*) and (*cooperate,defect*) are Pareto optimal. However, both of these Pareto-optimal solutions do not Pareto dominate the Nash equilibrium and therefore are not of interest to us. The combination (*cooperate,cooperate*) is a Pareto-optimal solution for which the Pareto dominates the Nash equilibrium.

For the second subclass we considered the battle of the sexes game (Weibull, 1996; Gintis, 2000). In this game there are two pure strategy Nash equilibria, i.e. (*football,football*) and (*opera,opera*), which both are also evolutionary stable (as demonstrated in Section 3.4.4). There is also one mixed Nash equilibrium, i.e. where the row player (the husband) plays *football* with 2/3 probability and *opera* with 1/3 probability and the column player (the wife) plays *opera* with 2/3 probability and *football* with 1/3 probability. However, this equilibrium is not an evolutionary stable one.

The third class consists of the games with a unique mixed equilibrium ((1/2, 1/2), (1/2, 1/2)). For this category we used the game defined by the matrices in Table 10, i.e. matching pennies. This equilibrium is not evolutionary stable. Typical for this class of games is that the interior trajectories define closed orbits around the equilibrium point.

3.4 Population dynamics

In this section we discuss the replicator dynamics (RDs) in a single- and a multi-population setting. We discuss the relation with concepts seen as Nash equilibrium and ESSs and illustrate the described ideas with some examples.

3.4.1 Single-population RDs

The basic concepts and techniques developed in EGT were initially formulated in the context of evolutionary biology (see Maynard-Smith, 1982; Weibull, 1996; Samuelson, 1997). In this context, the strategies of all of the players are genetically encoded (called genotype). Each genotype refers to a particular behavior, which is used to calculate the payoff of the player. The payoff of each player's genotype is determined by the frequency of other player types in the environment.

One way in which EGT proceeds is by constructing a dynamic process in which the proportions of various strategies in a population evolve. Examining the expected value of this process gives an approximation, which is called the RD. An abstraction of an evolutionary process usually combines two basic elements: *selection* and *mutation*. Selection favors some varieties over others, while mutation provides variety in the population. The RDs highlight the role of selection: it describes how systems consisting of different strategies change over time. They are formalized as a system of differential equations. Each replicator (or genotype) represents one (pure) strategy s_i. This strategy is inherited by all of the offspring of the replicator. The general form of RD is the following:

$$\frac{dx_i}{dt} = [(A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x}]x_i. \tag{22}$$

In Equation (22), x_i represents the density of strategy s_i in the population, A is the payoff matrix that describes the different payoff values each individual replicator receives when interacting with other replicators in the population. The state of the population ($\mathbf{x}$) can be described as a probability vector $\mathbf{x}=(x_1, x_2, \ldots, x_J)$, which expresses the different densities of all the different types of replicators in the population. Hence $(A\mathbf{x})_i$ is the payoff that replicator s_i receives in a population with state x and $\mathbf{x}\cdot A\mathbf{x}$ describes the average payoff in the population. The growth rate $(dx_i/dt)/x_i$ of the population share using strategy s_i equals the difference between the strategy's current payoff and the average payoff in the population. For further information we refer the reader to Weibull (1996) and Hofbauer & Sigmund (1998).

3.4.2 Multi-population RDs

So far the study of population dynamics was limited to a single population. However, in many situations interaction takes place between two or more individuals from different populations. In this section we study this situation in the two-player multi-population case for reasons of simplicity. Games played by individuals of different populations are commonly called *evolutionary asymmetric games*. Here we consider a game to be played between the members of two different populations. As a result, we need two systems of differential equations: one for the row player (R) and one for the column player (C). This setup corresponds to a RD for asymmetric games. If $A=B^{\mathrm{t}}$ (the transpose of B), Equation (22) would result again. Player R has a probability vector p over its possible strategies and player C a probability vector q over its strategies.

This translates into the following replicator equations for the two populations:

$$\frac{dp_i}{dt} = [(A\mathbf{q})_i - \mathbf{p} \cdot A\mathbf{q}]p_i \tag{23}$$

$$\frac{dq_i}{dt} = [(B\mathbf{p})_i - \mathbf{q} \cdot B\mathbf{p}]q_i. \tag{24}$$

As can be seen in Equations (23) and (24), the growth rate of the types in each population is now determined by the composition of the other population. Note that, when calculating the rate of change using these systems of differential equations, two different payoff matrices (A and B) are used for the two different players.

3.4.3 Relating Nash, ESSs and the RDs

Being a system of differential equations, the RDs have some rest points or equilibria. An interesting question is how these RDs equilibria relate to the concepts of Nash equilibria and ESSs. We briefly summarize some known results from the EGT literature (Osborne & Rubinstein, 1994; Weibull, 1996; Hofbauer & Sigmund, 1998; Gintis, 2000; Redondo, 2001). An important result is that every Nash equilibrium is an equilibrium of the RDs. However, the opposite is not true. This can be easily understood as follows. Let us consider the vector space or simplex of mixed strategies determined by all pure strategies. Formally, the unit simplex is defined by

$$\Delta = \left\{ x \in \mathcal{R}^{\mathrm{m}}_{+} : \sum_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{x_i} = 1 \right\}$$

where x is a mixed strategy in m-dimensional space (there are m pure strategies) and x_i is the probability with which strategy s_i is played. Calculating the RDs for the unit vectors of this space (putting all the weight on a particular pure strategy), yields zero. This is simply due to the properties of the simplex Δ, where the sum of all population shares remains equal to one and no population share can ever turn negative. So, if all pure strategies are present in the population at any time, then they always have been and always will be present, and if a pure strategy is absent from the population at any time, then it always has been and always will be absent.[7] So, this means that the pure strategies are rest points of the RDs, but depending on the structure of the game that is played, these pure strategies do not need to be a Nash equilibrium. Hence, not every rest point of the RDs is a Nash equilibrium. So the concept of dynamic equilibrium or stationarity alone is not enough to have a better understanding of the RDs.

For this reason the criterion of asymptotic stability came along, where you have some kind of local test of dynamic robustness (local in the sense of minimal perturbations). For a formal definition of asymptotic stability, we refer the reader to Hirsch & Smale (1974). Here we give an intuitive definition. An equilibrium is asymptotic stable if the following two conditions hold.

- Any solution path of the RDs that starts sufficiently close to the equilibrium remains arbitrarily close to it. This condition is called the *Liapunov stability*.
- Any solution path that starts close enough to the equilibrium, converges to the equilibrium.

Now, if an equilibrium of the RDs is asymptotically stable (i.e. being robust to local perturbations) then it is a Nash equilibrium. For a proof, the reader is referred to Redondo (2001). An interesting result due to Hofbauer & Sigmund (1998) is the following: if s is an ESS, then the population state $x=s$ is asymptotically stable in the sense of the RDs (for a proof see Hofbauer & Sigmund (1998) and Redondo (2001)). So, by this result we have some kind of refinement of the asymptotic stable rest points of the RDs and it provides a way of selecting equilibria from the RDs that show dynamic robustness.

3.4.4 Examples

In this section we continue with the examples of Section 3.2 and the classification of games of Section 3.3.1. We start over with the prisoner's dilemma game. In Figure 5 we plotted the direction field of the replicator equations applied to the prisoner's dilemma. A direction field is a very elegant and excellent tool to understand and illustrate a system of differential equations. The direction fields presented here consist of a grid of arrows tangential to the solution curves of the system. Its a graphical illustration of the vector field indicating the direction of the movement at every point of the grid in the state space. Filling in the parameters for each game in Equations (23) and (24) allowed us to plot this field.

The x-axis represents the probability with which the first player will play defect and the y-axis represents the probability with which the second player will play defect. So the Nash equilibrium and the ESS lie at coordinates (1, 1). As you can see from the field plot, all the movement goes towards this equilibrium.

Figure 6 illustrates the direction field diagram for the battle of the sexes game. As you may recall from Section 3.3.6 this game has two pure Nash equilibria and one mixed Nash equilibrium. These equilibria can be seen in the figure at coordinates (0, 0), (1, 1), (2/3, 1/3). The two pure equilibria are ESSs as well. This is also easy to verify from the plot; more precisely, any small perturbation away from the equilibrium is led back to the equilibrium by the dynamics.

The mixed equilibrium, which is a Nash equilibrium, is not an asymptotic stable strategy, which is obvious from the plot. From Section 3.3.6, we can now also conclude that this equilibrium is not evolutionary stable either.

[7] Of course, a solution orbit can evolve towards the boundary of the simplex as time goes to infinity and thus in the limit, when the distance to the boundary goes to zero, a pure strategy can disappear from the population of strategies. For a more formal explanation, we refer the reader to Weibull (1996).

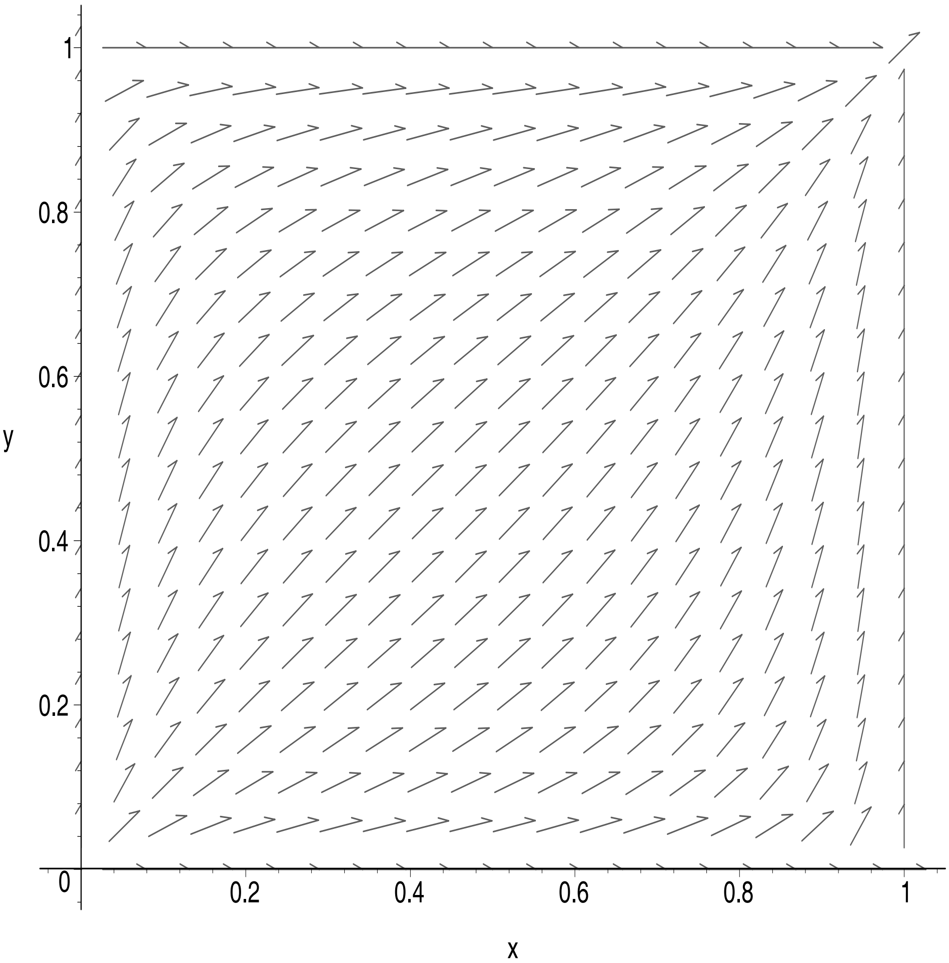


Figure 5 The direction field of the RD of the prisoner's dilemma using payoff Table 8

3.5 The role of EGT in MASs

In this section we discuss the most interesting properties that link the fields of EGT and MASs. These properties make clear that there exists an important role for EGT in MASs.

Traditional GT is an economical theory that models interactions between rational agents as games of two or more players that can choose from a set of strategies and the corresponding preferences. It is the mathematical study of interactive decision making in the sense that the agents involved in the decisions take into account their own choices and those of others. Choices are determined by:

1. stable preferences concerning the outcomes of their possible decisions;
2. agents act strategically, in other words, they take into account the relation between their own choices and the decisions of other agents.

Typical for the traditional game-theoretic approach is to assume perfectly rational players (or hyperrationality) who try to find the most rational strategy to play. These players have a perfect knowledge of the environment and the payoff tables and they try to maximize their individual payoff. These assumptions made by classical GT just do not apply to the real world and multi-agent settings in particular.

In contrast, EGT is descriptive and starts from a more realistic views of the game and its players. A game is not played only once, but repeatedly with changing opponents. Moreover, the players are

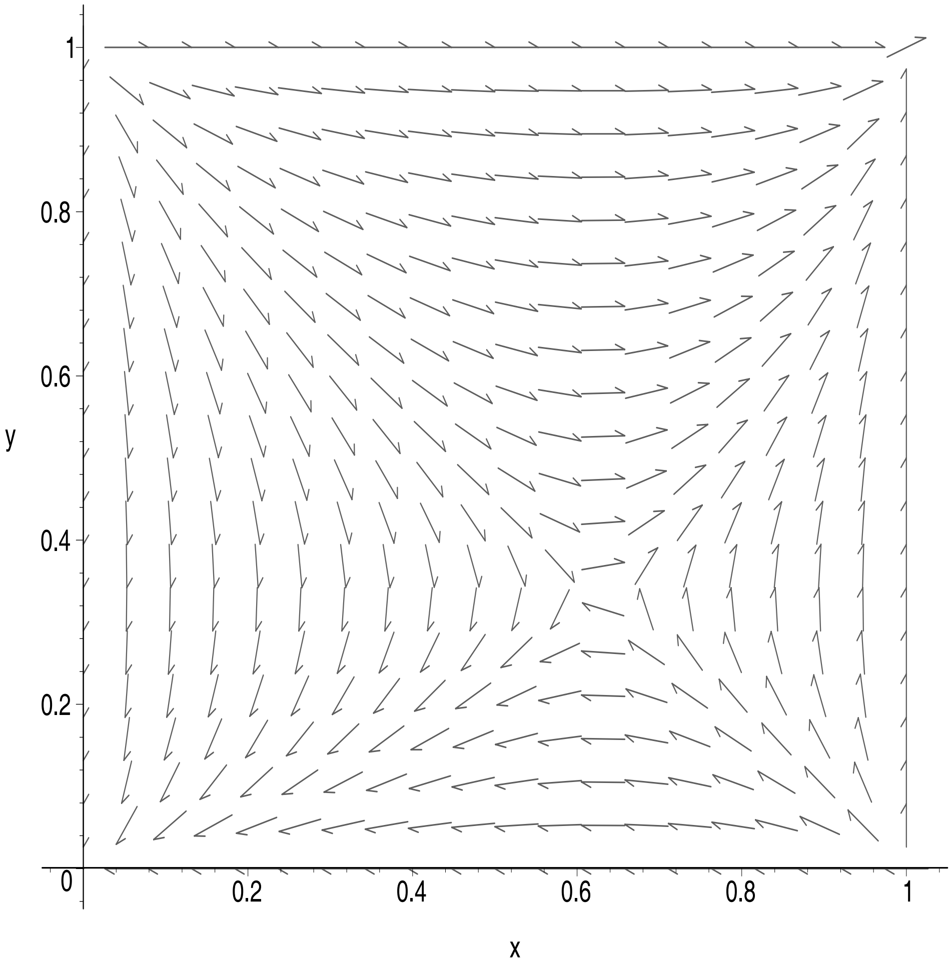


Figure 6 The direction field of the RDs of the battle of the sexes game using payoff Table 9

not completely informed, they sometimes misinterpret each others' actions, and are not completely rational but also biologically and sociologically conditioned. Under these circumstances, it becomes impossible to judge what choices are the most rational. The question now becomes how a player can learn to optimize its behavior and maximize its return. Mathematical models are developed for this learning process, e.g. replicator equations.

Summarizing the above we can say that EGT treats agents' objectives as a matter of fact, not logic, with a presumption that these objectives must be compatible with appropriate evolutionary dynamics (Gintis, 2000). Evolutionary models do not predict self-interested behavior. It describes how agents can make decisions in complex environments where they interact with other agents. In such complex environments, software agents must be able to learn from their environment and adapt to its non-stationarity.

The basic properties of a MAS correspond exactly with that of EGT. First of all, a MAS is made up of interactions between two or more agents, who each try to accomplish a certain, possibly conflicting, goal. No agent has the guarantee to be completely informed about the other agents intentions or goals, nor has it the guarantee to be completely informed about the complete state of the environment. It is of great importance that EGT offers us a solid basis to understand dynamic iterative situations in the context of strategic games. A MAS has a typical dynamical character, which makes it hard to model and brings along a lot of uncertainty. At this stage EGT seems to offer us a helping hand in understanding this typical dynamical processes in a MAS and modeling them in simple settings as iterative games of two or more players.

4 MARL

Coordination is an important problem in MARL. For cooperative multi-agent environments several algorithms and techniques exist. However, they are usually defined for one-state problems or games: only few of them are suited for the multi-stage case, moreover restrictions on the structure of the multi-stage case are imposed. Below we give an overview of the most important one-stage techniques, and the results of a comparative study where we tested the techniques on a variety of settings, even for situations for which some of the algorithms were originally not developed.

We shed light on some useful characteristics and strengths of each algorithm studied. For learning in MAS, two extreme approaches can be recognized. On the one hand, the presence of other agents, who are possibly influencing the effects a single agent experiences, can be completely ignored. Thus a single agent is learning as if the other agents are not around.

On the other hand, the presence of other agents can be modeled explicitly. This results in a joint action space approach, which recently received quite a lot of attention (Littman, 1994; Claus & Boutilier, 1998; Hu & Wellman, 1999).

4.1 The joint action space approach

In the joint action space technique, learning happens in the product space of the set of states S, and the collections of action sets $A_1, \ldots, A_n$ (one set for every agent). The state transition function $T{:}S\times A_1\times\ldots\times A_n\to P(S)$ maps a state and an action from each agent onto a probability distribution on S and each agent receives an associated reward, defined by the reward function $R_i{:}S\times A_1\times\ldots\times A_n\to P(\mathfrak{R})$. This is the underlying model for the stocastic games, also referred to as Markov games in Littman (1994) and Hu & Wellman (1999).

The joint action space approach is a safe technique in the sense that the influence of an agent on every other agent can be modeled in such a manner that the Markov property still holds. This, combined with a unique solution concept such as the Stackelberg equilibrium bootstrapping as is usually done in RL techniques (Konenen, 2004). However, the joint action space approach violates the basic principles of MASs: distributed control, asynchronous actions, incomplete information, cost of communication, etc.

4.2 Independent RL

Independent RL agents try to optimize their behavior without any form of communication with the other agents, they only use the feedback they receive from the environment. These independent RL agents might use traditional RL algorithms, created for stationary, one-agent settings. As the feedback coming from the environment is, in general, dependent on the combination of action taken by the agents and not just the action of a single agent, the Markov property no longer holds and the problem becomes non-stationary state dependent (Navendra & Thathacher, 1989). It is shown in Navendra & Thathacher (1989) that if the agents use a reward–inaction updating scheme they are able to converge to a pure Nash equilibrium state. In case no pure Nash equilibrium exists, we need to extend the RL algorithm as discussed in the last section of this paper.

In our study, we also consider optimistic independent agents, introduced by Lauer & Riedmiller (2000). Optimistic independent agents will only perform their update when the performance they receive from the new experience is better than was expected until then.

4.3 Informed RL

Informed agents use communication through *revelation schemes*, in order to coordinate on the optimal joint action (Mukhurjee & Sen, 2001). Revelation comes in different two-player schemes, ranging from not allowing agents to communicate actions to each other (no revelation), to allowing agents in turn to reveal their actions (alternate revelation) and to allowing agents to reveal their actions simultaneously (simultaneous revelation). The agents, who are allowed to communicate, decide for themselves whether they reveal their action or not.

4.4 Exploration–exploitation schemes for independent reinforcement learning

The last group of techniques used for the comparison of Section 4.5 tries extended exploration–exploitation schemes to push independent agents toward their part of the optimal joint-action. Two algorithms have been considered in this study. The frequency maximum$Q$ (FMQ) technique described in Kapetanakis & Kudenko (2002) adds a heuristic value to the Q-value of an action in the Boltzmann exploration strategy. This FMQ heuristic takes into account how frequently an action produces its maximum corresponding reward. In Verbeeck *et al.* (2002) a new exploration technique is used for coordination games. It is based on exploring, selfish RL (ESRL) agents, playing selfish for a period of time and then excluding actions from their private action space, so that the joint action space gets considerably smaller and the agents are able to converge to a Nash equilibrium of the remaining subgame. By repeatedly excluding actions, the agents are able to figure out the Pareto front and decide on which combination of actions is preferable.

4.5 Comparison of techniques

The two-player games we used as a test-bed were, respectively, two revelation games: the penalty game and the climbing game. The revelation games were extracted from Mukherjee & Sen (2001). The first was constructed in such a way that each agent has a preferred action no matter what the other agent does. However, the joint combination of these choices is not optimal. The second revelation game has a Pareto optimal joint action; however, this is not a Nash equilibrium. The other two games were originally used by Claus & Boutulier (1998). The challenge in the climbing game is to reach the optimal joint action when it is surrounded by heavy penalties. The penalty game has the additional difficulty of having two Pareto-optimal solutions on which the agents should agree. A complete overview of the results can be found in Peeters (2003).We summarize the most important conclusions, flaws and strong points here.

The independent learners produced the most remarkable result in one of the revelation games. They were able to perform better than the revelation agents. This shows that providing more information to an agent may result in a worse performance. However, the other games showed that acting independently is not always enough to guarantee convergence to the Pareto-optimal Nash equilibrium. Independent learners are also very dependent on the settings of the learning rate and the exploration temperature. The optimistic assumption is useful in driving the agents to the Pareto-optimal solution; however, this technique will not work in stochastic environments.

The revelation learners were originally not created for the climbing and penalty game, and the penalties in these games turned out to be too difficult to overcome. The revelation learners would probably behave better in an auction environment. The modeling of other agents might give them an advantage and the concept of revelation might lead to interesting results (perhaps extending the agents to give them the ability to lie).

The FMQ learners have a convergence rate of 100% in identical payoff games (i.e. the games the algorithm is designed for). For the other games, convergence is not always assured, but if it does converge the convergence time is very good. When we play games where the optimal group utility differs from the agents' personal utility, FMQ learners fail to reach the optimal solution. Also at this stage, FMQ learners have problems with genuine stochastic games.

The ESRL learners reach the Pareto-optimal solution in every game we played under the assumption that they were allowed enough time to converge to a Nash equilibrium. A downside is that this time has to be set in advance. The time they need to converge is usually longer than for FMQ learners, because playing in periods and visiting all equilibria is a slow process. However, due to the agents playing in periods of time and therefore allowing them to sample enough information on joint actions, exclusion learners are able to work in stochastic games.

4.6 An EGT perspective on multi-agent learning

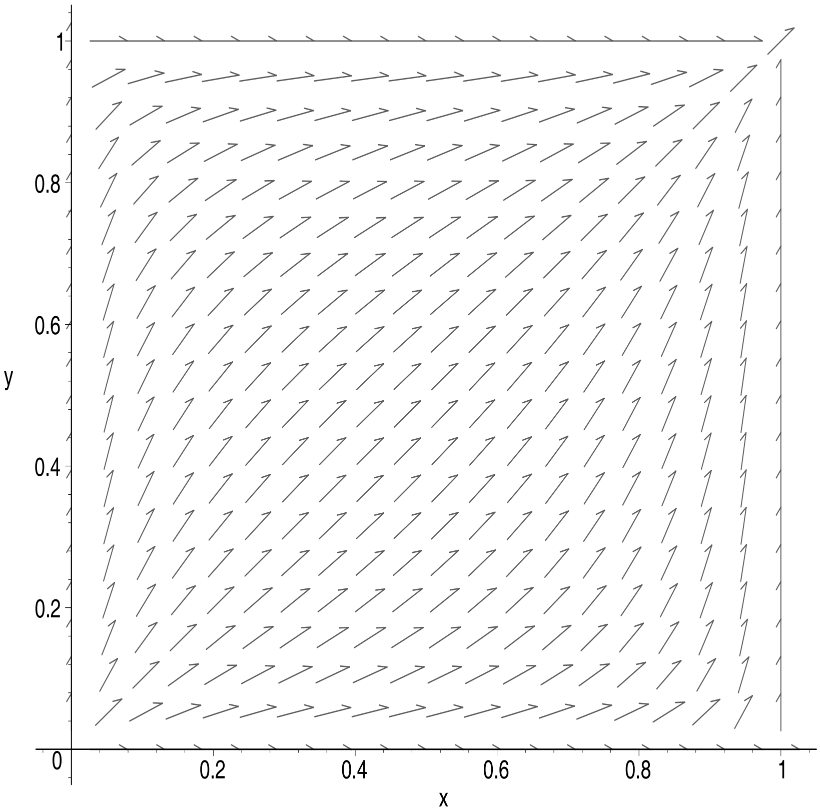
Relating RL and EGT is not new. Börgers and Sarin (1997) were the first[8] to mathematically connect RL with EGT. Their work is located in the field of economics, where other researchers are also very active on the link between RL and EGT, such as, for instance, Redondo (2001). However, this relation has received far less attention in the field of artificial intelligence and more specifically in the field of MASs. Börgers and Sarin (1997) have made the formal connection between the replicator equations and cross learning (a simple RL model) explicit in their work. The evolutionary approach to GT increasingly attracts attention of researchers from different fields, such as economics, computer science and artificial intelligence (Weibull, 1996; Bazzan, 1997; Börgers and Sarin, 1997; Samuelson, 1997; Nowé *et al*., 1999a; Gintis, 2000). The possible successful application of EGT concepts and models in these different fields becomes increasingly apparent.

If the word evolution is used in the biological sense, then this means that we are concerned with environments in which behavior is genetically determined. Strategy selection then depends on the reproductive success of their carriers, i.e. genes. Often, evolution is not intended to be understood in a biological sense, but rather as a learning process that we call cultural evolution (Bjornstedt, 1995). Of course it is implicit and intuitive that there is an analogy between biological evolution and learning. We can now look at this analogy at two different levels. First there is the individual level. An individual decision maker usually has many ideas or strategies in his mind according to which he can behave. Which one of these ideas dominates and which are given less attention depends on the experiences of the individual. We can regard such a set of ideas as a population of possible behaviors. The changes that such a population undergoes in the individual's mind can be very similar to biological evolution. Second, it is possible that individual learning behavior is different from biological evolution. An example is best response learning where individuals adjust too rapidly to be similar to evolution. However, then it might be the case that at the population level, consisting of different individuals, a process is operating analogously to biological evolution. In this paper we describe the similarity between biological evolution and learning at the individual level in a formal and experimental manner.

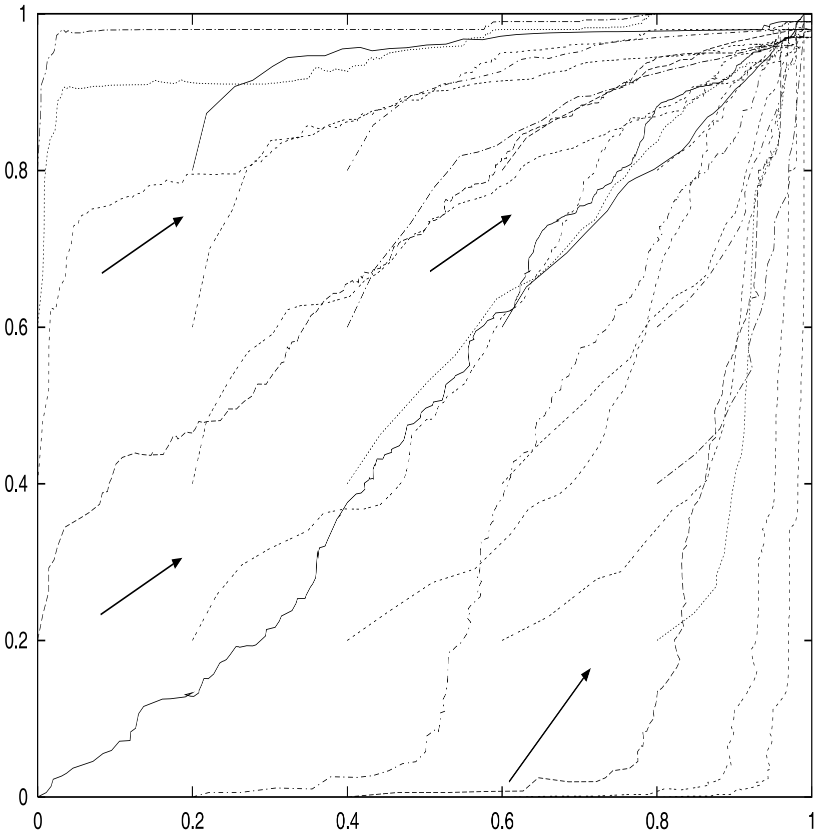
In this section we discuss or merely point out the results in making this relation between MARL and EGT explicit. Börgers and Sarin have shown how the two fields are related in terms of dynamic behavior, i.e. the relation between cross learning and the RDs. The RDs postulate gradual movement from worse to better strategies. This is in contrast to classical GT, which is a static theory and does not prescribe the dynamics of adjustment to equilibrium. The main result of Börgers and Sarin is that in an appropriately constructed continuous time limit, the cross learning model converges to the asymmetric, continuous time version of the RDs. The continuous time limit is constructed in such a manner that each time interval sees many iterations of the game, and that the adjustments that the players (or cross learners) make between two iterations of the game are very small. If the limit is constructed in this manner, the (stochastic) learning process becomes in the limit deterministic. This limit process satisfies the system of differential equations, which characterizes the RDs. For more details see Bürgers & Sarin (1997). We illustrate this result with the prisoner's dilemma game. In Figure 7 we plotted the direction field of the replicator equations for the prisoner's dilemma game and we also plotted the cross-learning process for this same game.

For both players we plotted the probability of choosing their first strategy (in this case defect). The x axis represents the probability with which the row-player plays *defect* and the y-axis represents the probability with which the column-player plays this same strategy. As you can see the sample paths of the cross-learning process approximates the paths of the RDs.

[8] As far as we know.



(a)



(b)

In previous work the authors have extended the results of Börgers and Sarin to popular RL models such as LA and Q-learning. In Tuyls *et al.* (2002) the authors have shown that the cross-learning model is a LA with a linear-reward–inaction updating scheme. All details and experiments are available in Tuyls *et al.* (2002).

Next, we continue with the mathematical relation between Q-learning and the RDs. In Tuyls *et al.* (2003b) we derived mathematically the dynamics of Boltzmann Q-learning. We investigated here whether there is a possible relation with the evolutionary dynamics of EGT. More precisely we constructed a continuous time limit of the Q-learning model, where Q-values are interpreted as Boltzmann probabilities for the action selection, in an analogous manner of Börgers and Sarin for cross learning. We briefly summarize the findings here. All details can be consulted in Tuyls *et al.* (2003b). The derivation has been restricted to a two-player situation for reasons of simplicity. Each agent (or player) has a probability vector over his action set, more precisely $x_1, \ldots, x_n$ over action set $a_1, \ldots, a_n$ for the first player and $y_1, \ldots, y_m$ over $b_1, \ldots, b_m$ for the second player. Formally the Boltzmann distribution is described by

$$x_i\,(k) = \frac{e^{\tau Q_{a_i}(k)}}{\sum_{j=1}^{n} e^{\tau Q_{a_j}(k)}}$$

where $x_i(k)$ is the probability of playing strategy i at time step k and τ is the temperature. The temperature determines the degree of exploring different strategies. As the trade-off between exploration and exploitation is very important in RL, it is important to set this parameter correctly.

Now suppose that we have payoff matrices A and B for the two players. Calculating the time limit results in

$$\frac{dx_i}{dt} = x_i\alpha\tau((A\mathbf{y})_i - \mathbf{x} \cdot A\mathbf{y}) + x_i\alpha \sum_j x_j \, 1\text{n}\left(\frac{x_j}{x_i}\right) \tag{25}$$

for the first player and, analogously, in

$$\frac{dy_i}{dt} = y_i\alpha\tau((B\mathbf{x})_i - \mathbf{y} \cdot B\mathbf{x}) + y_i\alpha \sum_j y_j \, 1\text{n}\left(\frac{y_j}{y_i}\right) \tag{26}$$

for the second player.

Comparing (25) or (26) with the RDs in (22), we see that the first term of (25) or (26) is exactly the RD and thus takes care of the selection mechanism (Weibull, 1996). The mutation mechanism for Q-learning is therefore left in the second term, and can be rewritten as

$$x_i\alpha \sum_j x_j \, 1\text{n}\,(x_j) - 1\text{n}\,(x_i). \tag{27}$$

In Equation (27) we recognize two entropy terms, one over the entire probability distribution x and one over strategy x_i. Relating entropy and mutation is not new. It is a well-known fact (Stauffer, 1999; Schneider, 2000) that mutation increases entropy. In Stauffer (1999), it is stated that the

Figure 7 (a) The direction field plot of the RDs of the prisoner's dilemma game. The x-axis represents the probability with which the first player (or row-player) plays *defect*, and the y-axis represents the probability with which the second player (or column-player) plays *defect*. The strong attractor and Nash equilibrium of the game lies at coordinates (1, 1) as one can see in the plot. (b) The paths induced by the cross-learning process of the prisoner's dilemma game. The arrows point out the direction of the learning process. These probabilities are now learned by the cross-learning algorithm concepts are familiar with thermodynamics in the following sense: the selection mechanism is analogous to *energy* and mutation to *entropy*. So, generally speaking, mutations tend to increase entropy. Exploration can be considered as the mutation concept, as both concepts take care of providing variety.

Equations (25) and (26) now express the dynamics of both Q-learners in terms of Boltzmann probabilities, from which the RDs emerge.

In Tuyls (2003c) we answered the question of whether it is possible to first define the dynamic behavior in terms of EGT and then develop the appropriate RL algorithm. For these reasons we extended the RDs of EGT. We call it the extended RDs. Details on this work can be found in Tuyls (2003c). The main result is that the extended dynamics guarantee an evolutionary stable outcome in all types of one-stage game.

Finally, in Hoen & Tuyls (2004) the authors have shown how the EGT approach can be used for dispersion games (Grenager, 2002). In this cooperative game, n agents must learn to choose from k tasks using local rewards and full utility is only achieved if each agent chooses a distinct task. We visualized the learning process of the MAS and showed typical phenomena of distributed learning in a MAS. Moreover, we showed how the derived fine tuning of parameter settings from the RDs can support application of the collective intelligence (COIN) framework of Wolpert *et al.* (1998, 1999) using dispersion games. COIN is a proven engineering approach for learning cooperative tasks in MASs. Broadly speaking, COIN defines the conditions that the private utility function of an agent has to meet to increase the probability that learning to optimize this function leads to increased performance of the collective of agents. Thus, the challenge is to define suitable private utility functions for the individual agents, given the performance of the collective. We showed that the derived link between RDs and RL predicts performance of the COIN framework and visualizes the incentives provided in COIN towards a cooperative behavior.

5 Final remarks

In this survey article we investigated RL and EGT in a multi-agent setting. We provided most of the fundamentals of RL and EGT and, moreover, showed their remarkable similarities. We also discussed some of the excellent existing MARL algorithms and gave a more detailed description of the EGT approach of the authors. However, a lot of work remains and some problems are still unresolved. In particular, overcoming problems of incomplete information and large state spaces in MASs (for instance, sensor webs) are still difficult. More precisely, under these conditions it becomes impossible to learn models over other agents, storing information on them and using a lot of communication.

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