Notes of "Fundamental Theorem of Affine Transformations"

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1 Overview

2 The vector transformation induced by an affine transformation

Definition 1 (The vector transformation induced by an affine transformation). *Let $f : \pi \to \pi$ be an affine transformation on the plane π, then for all vectors $\alpha \parallel \pi$, define $f(\alpha) = \overrightarrow{f(A)f(B)}$ in which $A, B \in \pi$ and $\vec{AB} = \alpha$.*

Remark 1 (The image and preimage of the zero vector under an induced vector transformation). *From the definition, it is clear that $\alpha = 0 \Leftrightarrow f(\alpha) = 0$.*

Remark 2 (The fundamental reason behind an induced vector transformation). *From the proof of the uniqueness of $f(\alpha)$, we can see the fundamental reason behind the induced vector transformation is the property of an affine transformation that it transforms a line to a line, and keep the parallelism between two lines.*

3 Fundamental Theorem of Affine Transformations

Theorem 1 (Fundamental Theorem of Affine Transformations). *Let π be a plane.*

T1 *Suppose $f : \pi \to \pi$ is an affine transformation, $I = [O; \vec{e}_1, \vec{e}_2]$ is an affine coordinate system on π, then $I' = [f(O); f(\vec{e}_1), f(\vec{e}_2)]$ is also an affine coordinate system on π, and for all $P \in \pi$, the coordinates of P in I are the same as the one of $f(P)$ in I'.*

T2 *Let $I = [O; \vec{e}_1, \vec{e}_2]$ and $I' = [O'; \vec{e}_1', \vec{e}_2']$ be two affine coordinate systems on π. There exists an mapping $f : \pi \to \pi$ as following: for all $P \in \pi$ with the coordinates (x, y) in I, let $f(P)$ be the point with the same coordinates (x, y) in I', and $f : \pi \to \pi$ is an affine transformation.*

证明. (TODO) □

Remark 3. *A conclusion from the fundamental theorem of affine transformations is that for any two affine coordinate systems I and I' on a plane π, there exists a unique affine transformation $f : \pi \to \pi$ such that $f(I) = I'$.*

T2 *proves the existence of $f : \pi \to \pi$.*

T1 *proves the uniqueness of* $f : \pi \to \pi$. *If there are two affine transformations* $f_1 : \pi \to \pi$ *and* $f_2 : \pi \to \pi$ *with* $f_1(I) = I'$ *and* $f_2(I) = I'$, *then for all* $P \in \pi$, $f_1(P)$ *and* $f_2(P)$ *have the same coordinates in* I', *which is the coordinates of* P *in* I, *and thus* $f_1(P) = f_2(P)$. *Hence* $f_1 = f_2$.