Notes of "Metric Characteristics of Conic Sections"

Jinxin Wang

1 Geometric Interpretation of Eigenvalues and Eigenvectors

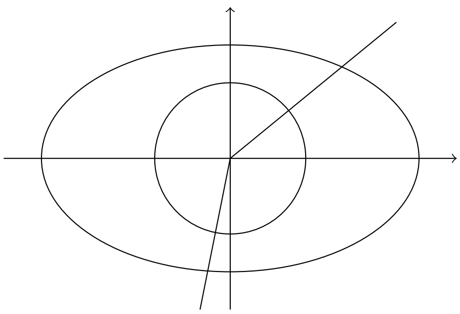
Notice that a characteristic of the major and minor axes of an ellipse is that the intersection between an ellipse and its major and minor axes, i.e. the vertex and covertex of an ellipse, have the maximum and minimum distance to the center respectively in the set of points on the ellipse.

In the language of analysis, let E be the set of points on an ellipse $\mathit{\Gamma}$, O be the center of $\mathit{\Gamma}$, and a function $f : E \to \mathbb{R}$ map a point $P \in E$ to its distance to O. Then the fact we stated above is that f assumes its global maximum and minimum values on the vertex and covertex of $\mathit{\Gamma}$ respectively, and the vectors from the center O to the vertex and covertex represent the directions of the major and minor axes of the ellipse $\mathit{\Gamma}$. Therefore, this observation might help us determine the symmetric axis of a quadratic curve.

Frame the problem: on the set E of points (x, y) such that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we need to find the points on which $f(x, y) = x^2 + y^2$ assumes global extreme values. To reduce the difficulty, we might consider an easier dual problem to the original one: on the set E of points (x, y) such that $x^2 + y^2 = 1$, we need to find the points on which $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ assumes global extreme values.

The dual relationship between the two problems can be understood in the following way:

- Ratio



- Change of basis

Recall that a global extremum is also a local extramum, and a necessary condition for an interior extremum of a differentiable function at the point is that the derivative is 0.

Proposition 1. *Given the equation of a quadratic curve $\mathit{\Gamma}$ in a plane:*

$$F(x, y) = (x, y, 1)A[x, y, 1] = 0$$

where A is the associated symmetric matrix, we define two values

$$\lambda_1 = \sup\{(x, y)A_0[x, y] \mid x^2 + y^2 = 1\} \tag{1}$$

$$\lambda_2 = \inf\{(x, y)A_0[x, y] \mid x^2 + y^2 = 1\} \tag{2}$$

then we have the following conclusions:

C1 *λ_1 and λ_2 are two eigenvalues of Γ.*

C2 *There exist unit vectors $\vec{u}_1(m_1, n_1)$ and $\vec{u}_2(m_2, n_2)$ such that*

$$\lambda_1 = (m_1, n_1)A_0[m_1, n_1],\ \lambda_2 = (m_2, n_2)A_0[m_2, n_2]$$

and they satisfy

$$A_0[m_1, n_1] = \lambda_1[m_1, n_1],\ A_0[m_2, n_2] = \lambda_2[m_2, n_2]$$

In other words, $\vec{u}_1$ and $\vec{u}_2$ are eigenvectors associated with λ_1 and λ_2 respectively.

C3 *If $\lambda_1 \neq \lambda_2$, then $\vec{u}_1 \perp \vec{u}_2$. If $\lambda_1 = \lambda_2$, then any vectors are eigenvectors of λ_1 and λ_2.*

证明. Hint:

Part 1 Prove the existence of λ_1 and λ_2, and the unit vectors $\vec{u}_1(m_1, n_1)$ and $\vec{u}_2(m_2, n_2)$.

Let $x = \cos\theta$ and $y = \sin\theta$ where $\theta \in [0, 2\pi]$ (the reason behind the closed interval is to apply the property of a continuous function on a compact set). In this way,

$$\begin{aligned} f(x, y) &= (x, y)A_0[x, y] \\ &= (\cos\theta, \sin\theta)A_0[\cos\theta, \sin\theta] \\ &= a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\cos\theta\sin\theta \end{aligned}$$

which is a continous function on $[0, 2\pi]$.

Part 2 Prove that they satisfy

$$A_0[m_1, n_1] = \lambda_1[m_1, n_1],\ A_0[m_2, n_2] = \lambda_2[m_2, n_2]$$

Since λ_1 and λ_2 are global extreme values of $f(\theta)$, then $f'(\theta_i) = 0, i = 1, 2$.

$$\begin{aligned} f'(\theta) &= (a_{11}\cos^2\theta + a_{22}\sin^2\theta + 2a_{12}\cos\theta\sin\theta)' \\ &= 2(-a_{11}\cos\theta\sin\theta + a_{12}(\cos^2\theta - \sin^2\theta) + a_{22}\cos\theta\sin\theta) \\ &= 2((-\sin\theta, \cos\theta)A_0[\cos\theta, \sin\theta]) \end{aligned}$$

Hence we have

$$(-\sin\theta_i, \cos\theta_i)A_0[\cos\theta_i, \sin\theta_i] = 0,\ i = 1, 2$$

i.e. $[-\sin\theta_i, \cos\theta_i] \perp A_0[\cos\theta_i, \sin\theta_i]$. Meanwhile, $[-\sin\theta_i, \cos\theta_i] \perp [\cos\theta_i, \sin\theta_i]$, then we have

$$A_0[\cos\theta_i, \sin\theta_i] = \mu[\cos\theta_i, \sin\theta_i]$$

in which if we multiply both sides with $(\cos\theta_i, \sin\theta_i)$ on the left side, we get

$$\begin{gathered} (\cos\theta_i, \sin\theta_i)\mu[\cos\theta_i, \sin\theta_i] = (\cos\theta_i, \sin\theta_i)A_0[\cos\theta_i, \sin\theta_i] \\ \mu(\cos\theta_i, \sin\theta_i)[\cos\theta_i, \sin\theta_i] = (\cos\theta_i, \sin\theta_i)A_0[\cos\theta_i, \sin\theta_i] \\ \mu = \lambda_i \end{gathered}$$

Part 3 Prove that if $\lambda_1 \neq \lambda_2$ (TODO)

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