Notes of ”Ruled Quadric Surfaces”

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1 Overview

Geometric interpretation of ruled surface: 由一簇直线构成的曲面叫做直纹面。

2 Quadric Cylindrical Surface 二次柱面

Based on the geometric interpretation of cylindrical surfaces, they are a kind of ruled surface.

3 Quadric Conical Surface 二次锥面

Based on the geometric interpretation of conical surfaces, they are a kind of ruled surface.

4 Hypobolic Paraboloid 双曲抛物面

4.1 Equation of Generatrix in Hypobolic Paraboloid

Given the standard equation of a hypobolic paraboloid:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z \tag{1}$$

$$(\frac{x}{a} + \frac{y}{b})(\frac{x}{a} - \frac{y}{b}) = 2z$$

Let $c = \frac{x}{a} + \frac{y}{b}$, then the line with the equation

$$l_c = \begin{cases} \frac{x}{a} + \frac{y}{b} = c \\ c(\frac{x}{a} - \frac{y}{b}) = 2z \end{cases}$$

belongs to the hypobolic paraboloid.

Let $c = \frac{x}{a} - \frac{y}{b}$, then the line with the equation

$$l'_c = \begin{cases} \frac{x}{a} - \frac{y}{b} = c \\ c(\frac{x}{a} + \frac{y}{b}) = 2z \end{cases}$$

belongs to the hypobolic paraboloid.

Therefore, we find two collection of straight generatrix in the hypoboloic paraboloid:

$$I = \{l_c | c \in \mathbb{R}\} I' = \{l'_c | c \in \mathbb{R}\} \tag{2}$$

Rewrite the equations of the two collections of straight generatrix in the point-direction form.
l_c passes through the point $M_c(ac, 0, \frac{c^2}{2})$, and is parallel to the vector $\vec{u_c}(-a, b, -c)$.
l'_c passes through the point $M'_c(ac, 0, \frac{c^2}{2})$, and is parallel to the vector $\vec{u_c}(a, b, c)$.

4.2 Properties of Generatrix in Hypobolic Paraboloid

- For every point $P \in S$, each collection of straight generatrix has exactly one generatrix passing through it.
- Generatrix from the same collection is parallel to a plane.
- Two generatrix from the same collection are skew with each other.
- Generatrix from different collections intersect.
- There is no straight generatrix belonging to both collections.
- All straight generatrix in S belong to either collection.

5 Hyperboloid of One Sheet 单叶双曲面

5.1 Equation of Generatrix in Hyperboloid of One Sheet

Given the standard equation of a hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

$$(\frac{x}{a} + \frac{z}{c})(\frac{x}{a} - \frac{z}{c}) = (1 + \frac{y}{b})(1 - \frac{y}{b})$$

$$l_{s:t} = \begin{cases} s(\frac{x}{a} + \frac{z}{c}) = t(1 + \frac{y}{b}) \\ t(\frac{x}{a} - \frac{z}{c}) = s(1 - \frac{y}{b}) \end{cases} \tag{4}$$

$$l'_{s:t} = \begin{cases} s(\frac{x}{a} + \frac{z}{c}) = t(1 - \frac{y}{b}) \\ t(\frac{x}{a} - \frac{z}{c}) = s(1 + \frac{y}{b}) \end{cases} \tag{5}$$

It is apparent that each straight generatrix one-to-one corresponds to a value of $s : t$ where

$$l_\theta = \begin{cases} \cos\theta(\frac{x}{a} + \frac{z}{c}) = \sin\theta(1 + \frac{y}{b}) \\ \sin\theta(\frac{x}{a} - \frac{z}{c}) = \cos\theta(1 - \frac{y}{b}) \end{cases} \tag{6}$$

$$l'_\theta = \begin{cases} \cos\theta(\frac{x}{a} + \frac{z}{c}) = \sin\theta(1 - \frac{y}{b}) \\ \sin\theta(\frac{x}{a} - \frac{z}{c}) = \cos\theta(1 + \frac{y}{b}) \end{cases} \tag{7}$$

Remark 1.

Rewrite the equations of the two collections of straight generatrix in the point-direction form. Take l_θ as an example:

$$l_\theta = \begin{cases} \frac{\cos\theta}{a}x - \frac{\sin\theta}{b}y + \frac{\cos\theta}{c}z = \sin\theta \\ \frac{\sin\theta}{a}x + \frac{\cos\theta}{b}y - \frac{\sin\theta}{c}z = \cos\theta \end{cases}$$

Let $z = 0$, then we solve the following linear system with two equations and two unknowns

$$\begin{cases} \frac{\cos\theta}{a}x - \frac{\sin\theta}{b}y = \sin\theta \\ \frac{\sin\theta}{a}x + \frac{\cos\theta}{b}y = \cos\theta \end{cases} \tag{8}$$

and find the only solution $x = a\sin 2\theta, y = b\cos 2\theta$. Therefore, the straight generatrix l_θ pass through the point $M_\theta(a\sin 2\theta, b\cos 2\theta, 0)$.

Besides, l_θ is parallel to the vector $\vec{u_\theta}(\begin{vmatrix} -\frac{\sin\theta}{b} & \frac{\cos\theta}{c} \\ \frac{\cos\theta}{b} & -\frac{\sin\theta}{c} \end{vmatrix}, \begin{vmatrix} \frac{\cos\theta}{c} & \frac{\cos\theta}{a} \\ -\frac{\sin\theta}{c} & \frac{\sin\theta}{a} \end{vmatrix}, \begin{vmatrix} \frac{\cos\theta}{a} & -\frac{\sin\theta}{b} \\ \frac{\sin\theta}{a} & \frac{\cos\theta}{b} \end{vmatrix})$, which is $\vec{u_\theta}(-\frac{\cos 2\theta}{bc}, \frac{\sin 2\theta}{ac}, \frac{1}{ab})$, or in better form $\vec{u_\theta}(-a\cos 2\theta, b\sin 2\theta, c)$.

Remark 2. *Previously I had this doubt that why l_θ passes through the point M_θ instead of any other points or more points in the xOy plane. Based on the equation (8), there is only one solution, which means l_θ has only one intersection point with the xOy plane. Geometrically it also makes sense since l_θ is not parallel to the xOy plane given its z-component is non-zero.*

Similarly, we can find that l'_θ passes through the point $M'_\theta(a\sin 2\theta, -b\cos 2\theta, 0)$, and is parallel to the vector $\vec{u'_\theta}(a\cos 2\theta, b\sin 2\theta, -c)$.

5.2 Properties of Generatrix in Hyperboloid of One Sheet

- For every point $P \in S$, each collection of straight generatrix has exactly one generatrix passing through it.
- Two generatrix from the same collection are skew with each other.
- Three generatrix from the same collection are not parallel to a plane.
- Generatrix from different collections are coplanar.
- There is no straight generatrix belonging to both collections.
- All straight generatrix in S belong to either collection.

6 Identification of Ruled Quadric Surface

There are only four kinds of ruled quadric surface except those within a plane:

- Quadric Cylindrical Surface
- Quadric Conical Surface
- Hypobolic Paraboloid

- Hyperboloid of One Sheet

We can use the characteristics of these four kinds of ruled quadric surface to differentiate them and identify a random ruled quadric surface:

- Parallelism of generatrix: There exist parallel generatrix in hyperboloid of one sheet, but any two generatrix in hypobolic paraboloid are not parallel.
- Parallelism between generatrix and planes: