Notes of "The Equation of a Line"

Jinxin Wang

1 Overview

- Two types of equations of a line
  - The point-direction equation of a line
  - The general equation of a line
  - Transformation between the two types of equations of a line
    - Rmk: Transform a point-direction equation to a general equation
    - Rmk: Transform a general equation to a point-direction equation
- Positional relationships between a line and a plane
- Positional relationships between two lines
  - Criterion for positional relationships with the point-direction form
  - Criterion for positional relationships with the general form

2 Two types of equations of a line

2.1 The point-direction equation of a line

2.2 The general equation of a line

2.3 Transformation between the two types of equations of a line

Remark 1 (Transform a point-direction equation to a general equation). *Given* $\frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z}$, *the equivalent general form is*

$$\begin{cases} \frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} \\ \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z} \end{cases}$$

$$\begin{cases} u_y x - u_x y + u_x y_0 - u_y x_0 = 0 \\ u_z y - u_y z + u_y z_0 - u_z y_0 = 0 \end{cases} \tag{1}$$

Remark 2 (Transform a general equation to a point-direction equation). *Given the general equation of a line l:*

$$\begin{cases} \pi_1 : A_1 x + B_1 y + C_1 z + D_1 = 0 \\ \pi_2 : A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$

we can find a solution (x_0, y_0, z_0) *to the linear system, which is the coordinate of a point on the line. Since a vector* $u(u_x, u_y, u_z) \parallel l$ *is equivalent to* $(u \parallel \pi_1) \wedge (u \parallel \pi_2)$*. According to the theorem of parallelism between a vector and a plane, the previous condition is equivalent to*

$$\begin{cases} A_1 u_x + B_1 u_y + C_1 u_z = 0 \\ A_2 u_x + B_2 u_y + C_2 u_z = 0 \end{cases}$$

then one solution of u *is*

$$(u_x, u_y, u_z) = \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right)$$

3 Positional relationships between a line and a plane

There are three kinds of positional relationships between a line and a plane:

- 平行不重合
- 重合
- 相交

3.1 The Point-Direction Form

3.2 The General Form

Suppose the equation in the general form of a line l is $\begin{cases} A_1 + B_1 + C_1 + D_1 = 0 \\ A_2 + B_2 + C_2 + D_2 = 0 \end{cases}$, and the equation of a plane π is $A_3 + B_3 + C_3 + D_3 = 0$, then to determine the positional relationship between them, our method is to transform the problem into studying the relationship between the three planes:

Proposition 1.

- $l \parallel \pi \wedge l \not\subseteq \pi \Leftrightarrow \begin{cases} A_1 + B_1 + C_1 + D_1 = 0 \\ A_2 + B_2 + C_2 + D_2 = 0 \\ A_3 + B_3 + C_3 + D_3 = 0 \end{cases}$ *has no solution.*
- l *intersected with* $\pi \Leftrightarrow \begin{cases} A_1 + B_1 + C_1 + D_1 = 0 \\ A_2 + B_2 + C_2 + D_2 = 0 \\ A_3 + B_3 + C_3 + D_3 = 0 \end{cases}$ *has a unique solution* $\Leftrightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0$
- $l \subset \pi \Leftrightarrow \begin{cases} A_1 + B_1 + C_1 + D_1 = 0 \\ A_2 + B_2 + C_2 + D_2 = 0 \\ A_3 + B_3 + C_3 + D_3 = 0 \end{cases}$ *has infinitely many solutions.*

4 共轴平面系

We can approach the relationship that a line belongs to a plane in another direction:

Proposition 2. *Suppose the equation in the general form of a line l is* $\begin{cases} A_1 + B_1 + C_1 + D_1 = 0 \\ A_2 + B_2 + C_2 + D_2 = 0 \end{cases}$*, and the equation of a plane π is $A_3 + B_3 + C_3 + D_3 = 0$, then $l \subset \pi \Leftrightarrow \exists \lambda, \mu$ which at least one of them are non-zero such that*

$$\lambda(A_1 + B_1 + C_1 + D_1) + \mu(A_2 + B_2 + C_2 + D_2 = 0) = A_3 + B_3 + C_3 + D_3 \tag{2}$$

Definition 1 (共轴平面系)**.**

5 Positional relationships between two lines

两直线间的位置关系总览，同时也是一种判断两条直线的位置关系的流程：

- 方向向量共线
  - 平行
  - 重合
- 方向向量不共线
  - 相交
  - 异面

5.1 Criterion for positional relationships with the point-direction form

Proposition 3. *Let l_1 be a line passing through M_1 and parallel to u_1, and l_2 be a line passing through M_2 and parallel to u_2.*

- l_1 与 l_2 重合 $\Leftrightarrow (u_1 \parallel u_2) \wedge \neg(\vec{M_1 M_2} \parallel u_1)$
- l_1 与 l_2 平行不重合 $\Leftrightarrow (u_1 \parallel u_2) \wedge (\vec{M_1 M_2} \parallel u_1)$
- l_1 与 l_2 相交 $\Leftrightarrow \neg(u_1 \parallel u_2) \wedge ((\vec{M_1 M_2}, u_1, u_2) = 0)$
- l_1 与 l_2 异面 $\Leftrightarrow \neg(u_1 \parallel u_2) \wedge ((\vec{M_1 M_2}, u_1, u_2) \neq 0)$

证明. □

5.2 Criterion for positional relationships with the general form

Proposition 4. *Let l_1 be a line with the general equation as* $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$*, and l_2 be a line with the general equation as* $\begin{cases} A_3x + B_3y + C_3z + D_3 = 0 \\ A_4x + B_4y + C_4z + D_4 = 0 \end{cases}$

- $l_1 \parallel l_2 \Leftrightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_4 & B_4 & C_4 \end{vmatrix} = 0$
- l_1 与 l_2 重合 $\Leftrightarrow$
- l_1 与 l_2 平行不重合 $\Leftrightarrow$
- l_1 与 l_2 相交 $\Leftrightarrow$
- l_1 与 l_2 异面 $\Leftrightarrow \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} \neq 0$

证明. □

6 Application: Find the Equation of Lines or Planes with Certain Positional Relationships

Example 1. 在一个仿射坐标系中，直线 l_1 有一般方程 $\begin{cases} x - y + z - 1 = 0 \\ y + z = 0 \end{cases}$，直线 l_2 过点 $M(0, 0, -1)$，平行于向量 $\vec{u}(2, 1, -2)$。平面 π 的方程为 $x + y + z = 0$。求由全体与 l_1, l_2 都相交，并且平行于 π 的直线所构成的曲面 S 的方程。

Solution:

- 平面参数法
- 直线参数法
- 双直线参数法
- 轨迹法