Notes of "General Theory of Affine Coordinate Transformation"

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1 Transformation Matrix and Transformation Formulas of Vectors and Points in Two Coordinate Systems

Remark 1. 由于坐标变换前后的方程描述的是同一个几何对象，因此许多几何性质在坐标变换前后保持不变，比如直线在变换后还是直线。但某些性质也会改变，如由于坐标轴单位的不同，不同方向上的比例关系发生了改变。

1.1 Transformation Formula of Points in Two Coordinate Systems

Given the coordinate (x', y', z') in I' of a point M, how to find its corresponding coordinate (x, y, z) in I?

- $M(x', y', z')$ in $I' \Leftrightarrow O'M(x', y', z')$ in I'
- $O'M$ in I has the coordinate $C[x', y', z']$
- Suppose OO' in I, in other words, the coordinate of the point O' in I, is d_1, d_2, d_3
- $OM = OO' + O'M$ has the coordinate $[x, y, z] = C[x', y', z'] + [d_1, d_2, d_3]$
- $OM(x, y, z)$ in $I \Leftrightarrow M(x, y, z)$ in I

$$\begin{cases} x = c_{11}x' + c_{12}y' + c_{13}z' + d_1 \\ y = c_{21}x' + c_{22}y' + c_{23}z' + d_2 \\ z = c_{31}x' + c_{32}y' + c_{33}z' + d_3 \end{cases} \tag{1}$$

2 Transformation Formula of a Graph in Two Coordinate Systems

2.1 Surface

2.2 Curves

Example 1 (Change of Basis of a Line)**.** *Suppose the transition matrix from I to I' is* $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$*. The standard equation of a line in I is $\frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z}$. To find the equation* *of the line in I', we have two methods:*

- *To utilize the method of change of basis of a plane, we transform the standard equation of the line to the general form:*

$$\begin{cases} \frac{x-x_0}{u_x} = \frac{y-y_0}{u_y} \\ \frac{y-y_0}{u_y} = \frac{z-z_0}{u_z} \end{cases}$$

3 Properties of Transformation Matrices

Proposition 1. *All transformation matrix between two affine coordinate systems are inversible matrices.*

Proposition 2. *If the transformation matrix from I to I' is C_1, and the transformation matrix from I' to I'' is C_2, then the transformation matrix from I to I'' is C_1C_2.*

Corollary 1. *If the transformation matrix from I to I' is C, then the transformation matrix from I' to I is C^{-1}.*

Proposition 3. *Suppose C is the transformation matrix from I to I'. I and I' has the same orientation if and only if $|C| > 0$. I and I' has the opposite orientation if and only if $|C| < 0$*

证明.

$$\begin{pmatrix} \vec{e'}_1 \\ \vec{e'}_2 \\ \vec{e'}_3 \end{pmatrix} = C^T \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

$$\begin{vmatrix} \vec{e'}_1 \\ \vec{e'}_2 \\ \vec{e'}_3 \end{vmatrix} = |C^T| \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}$$

$$(\vec{e'}_1, \vec{e'}_2, \vec{e'}_3) = |C|(\vec{e}_1, \vec{e}_2, \vec{e}_3)$$

□

4 Transformation between Cartesian Coordinate Systems

$I[O; e_1, e_2, e_3]$ $I'[O'; e'_1, e'_2, e'_3]$. The transformation matrix from I to I' is

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

which is equivalent to

$$\begin{cases} \vec{e'}_1 = c_{11}\vec{e}_1 + c_{21}\vec{e}_2 + c_{31}\vec{e}_3 \\ \vec{e'}_2 = c_{12}e_1 + c_{22}e_2 + c_{32}e_3 \\ \vec{e'}_3 = c_{13}e_1 + c_{23}e_2 + c_{33}e_3 \end{cases}$$

$$\vec{e'}_i \cdot \vec{e'}_j = c_{1i}c_{1j} + c_{2i}c_{2j} + c_{3i}c_{3j}$$

Since I' is a Cartesian coordinate system,

$$\vec{e'}_i \cdot \vec{e'}_j = \delta_{ij}$$

$$c_{1i}c_{1j} + c_{2i}c_{2j} + c_{3i}c_{3j} = \delta_{ij}$$

The above equation means that the inner product of two column vectors of C is 0, and the inner product between a column vector of C and itself is 1. Using matrix multiplication to express it:

$$C^T C = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} \vec{e'}_1 \cdot \vec{e'}_1 & \vec{e'}_2 \cdot \vec{e'}_1 & \vec{e'}_3 \cdot \vec{e'}_1 \\ \vec{e'}_1 \cdot \vec{e'}_2 & \vec{e'}_2 \cdot \vec{e'}_2 & \vec{e'}_3 \cdot \vec{e'}_2 \\ \vec{e'}_1 \cdot \vec{e'}_3 & \vec{e'}_2 \cdot \vec{e'}_3 & \vec{e'}_3 \cdot \vec{e'}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

which shows that C is an orthogonal matrix.

Remark 2. *It is clear that the properties of the basis of the two Cartesian coordinate systems, namely the vectors in a basis are unit vectors and they are orthogonal to each other, decide that the transformation matrix between them is an orthogonal matrix.*

Therefore, we have the following proposition:

Proposition 4. *The transformation matrix between two Cartesian coordinate systems is an orthogonal matrix.*

By the property of an inverse matrix, it also holds that

$$CC^T = C^T C = E$$

which means that the inner product of two row vectors of C is 0, and the inner product between a row vector of C and itself is 1.

4.1 Transformation between Two Cartesian Coordinate Systems in a Plane

Suppose C is the transformation matrix between two Cartesian coordinate systems in a plane:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

According to the above proposition, C is an orthogonal matrix, thus it satisfies two conditions:

1. $c_{11}^2 + c_{12}^2 = c_{11}^2 + c_{21}^2 = c_{12}^2 + c_{22}^2 = c_{21}^2 + c_{22}^2 = 1$
2. $c_{11}c_{12} + c_{21}c_{22} = c_{11}c_{21} + c_{12}c_{22} = 0$

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$C = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

in which $\theta \in [0, 2\pi)$