Notes of "Rank of Matrices"

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1 Retrospective of Linear System

With the notion of linear span, we have a new perspective of a linear system:

In a m-dimensional column vector space $\mathbb{R}^m$, consider a subset A with n vectors

$$A^{(j)} = [a_{1j}, a_{2j}, \dots, a_{mj}], j = 1, 2, \dots, n$$

and their linear span

$$V = \langle A \rangle = \langle A^{(1)}, A^{(2)}, \dots, A^{(n)} \rangle$$

For a given vector $B \in \mathbb{R}^m$, two natural questions are as follows:

1. Whether it is true that $B \in V$?
2. If so, how to express B as a linear combination of vectors in A?

Remark 1. *If $\dim V = n$, then $A = \{A^{(1)}, A^{(2)}, \dots, A^{(n)}\}$ is a base of V. Then the second question above is equivalent to find the coordinate of B under the base A.*

Formulate the above two questions in equations:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{1}$$

which is equivalent to a linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

Another perspective of a linear system that we already know is the matrix form:

$$AX = B \tag{3}$$

with the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{4}$$

and the augmented matrix

$$(A|B) \tag{5}$$

2 Rank of Matrices

Definition 1 (Column Space, Column Rank, Row Space, and Row Rank of a Matrix)**.** *Suppose A is a $m \times n$ matrix* $V_c(A) = \langle A^{(1)}, A^{(2)}, \ldots, A^{(n)} \rangle$

$r_c(A) = dim V_c(A)$

$V_r(A) = \langle A_{(1)}, A_{(2)}, \ldots, A_{(n)} \rangle$

$r_r(A) = dim V_r(A)$

Remark 2. *According to Theorem 2 in the Notes of "Vector Space of Rows and Columns", the dimension of a linear span is determined, then our notion of column rank and row rank of a matrix is well defined, because the definitions of column rank and row rank of a matrix rely on the linear span of column vectors and row vectors of the matrix. In other words, these properties of a matrix exists.*

前面学到过，我们可以对系数矩阵进行两种初等变换，将其化为阶梯型矩阵。注意到关于初等变化的一个重要事实：两类初等变换都是 inversible. More specifically, the matrix A' transformed from a matrix A with elementary row operations can be transformed back to A with the same type of elementary row operations.

The above fact leads to the following important observation about the effect of elementary row operations on the column rank and row rank of a matrix.

Lemma 1. *If a matrix A' is transformed from a matrix A through finite number of elementary row operations, it holds that*

$$r_c(A') = r_c(A)$$

$$r_r(A') = r_r(A)$$

证明. Hint: 同解性 □

The above lemma helps us answer a natural and important question: whether the row rank and the column rank of a matrix is equal or not?

Theorem 1. *For a $m \times n$ matrix A, it always holds that*

$$r_c(A) = r_r(A)$$

证明. Hint: 化为阶梯型 □

Definition 2 (Rank of a Matrix)**.** *The rank of a matrix A, denoted by* $\operatorname{rank} A$*, is equal to its column rank, as well as its row rank.*

$$\operatorname{rank} A = r_c(A) = r_r(A)$$

Remark 3. 一个矩阵的秩是唯一确定的，是它的内在特征，不依赖于任何外界情况。

3 Application: Solvable Criterion of a Linear System

Through transforming a linear system to its row reduce echelon form, we can already answer the solvable question of a linear system.

Corollary 1. *The number of the major unknowns in a homogenous linear system doesn't rely on the way how it reaches the row reduced echelon form. It is always equal to the rank of its coefficient matrix.*

证明. Hint: The number of non-zero rows in the row reduced echelon form. □

Theorem 2 (Rouché–Capelli theorem or Kronecker–Capelli theorem)**.** *A linear system is solvable if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix.*