# Notes of "Differentiable Functions"

Jinxin Wang

## 1  Statement of the Problem and Introductory Considerations

Solving the Kepler problem of two bodies, which is to describe the motion of a planet relative to a star. With an appropriate coordinate system, we use two functions $x(t)$ and $y(t)$ to describe the motion of the planet.

$$\vec{F} = m\vec{a} \tag{1}$$

$$\vec{F} = G\frac{Mm}{|\vec{r}|^3}\vec{r} \tag{2}$$

Now our task is to define and compute the instaneous velocity of a motion governed by the law (1).

By the law of inertia, if no force is acted on a body, it moves in uniform motion. Hence if at time $t$ the external action on a body ceases, it will continue moving in a straight line with a certain velocity. Since forces are to change velocity instead of maintaining velocity, it is natural to consider the hypothetical velocity as the instaneous velocity at time $t$.

If the absolute magnitude of the rate of change of a quantity $f(t)$ is bounded over an time interval $[0, t]$, then as the time interval $\Delta t$ becomes smaller, the change of the quantity $\Delta f$ becomes smaller, which is equivalent to say that $f(t)$ is continuous.

Apparently the above analysis applies to the case of velocity where the rate of change of velocity is acceleration, whose absolute magnitude is bounded since the external force on a body is finite. Therefore, at all times $t$ close to some time $t_0$ the velocity $\vec{v}(t)$ of the body $m$ must be close to the value $\vec{v}(t_0)$ that we wish to determine.

**Remark 1.** *Notice that there seems to be a shift from a scalar $f(t)$ to a vector $\vec{v}(t)$, and a question may arise that whether the previous analysis about the bounded rate of change and continuity of a scalar function applies to a vector function. Consider the nature of change, it must be true that when a change of a quantity becomes smaller, the changed quantity is closer to the original quantity, no matter whether the quantity is a scalar or a vector. Therefore, the analysis about the bounded rate of change and continuity applies to both cases.*

# 2  Functions Differentiable at a Point

**Definition 1.** *A function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of $E$ if there exists a linear function $A(x - a)$ such that $f(x) - f(a)$ can be represented as*

$$f(x) - f(a) = A(x - a) + o(x - a) \text{ as } x \to a, x \in E \tag{3}$$

**Remark 2.** *If a function $f : E \to \mathbb{R}$ is differentiable at a point $a$, then $a$ is a limit point of $E$.*

**Corollary 1.** *If a function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of $E$, then it is also continuous at the point $a \in E$.*

**Remark 3.** *The conversion of the above corollary is not true. An counterexample is that $f(x) = x^{\frac{1}{3}}$ is continuous at $x = 0$, but it is not differentiable at $x = 0$.*

**Definition 2.** *The linear function $A(x - a)$ in the above definition is called the differential of the function $f$ at $a$.*

**Definition 3.** *The number*

$$f'(a) = \lim_{E \ni x \to a} \frac{f(x) - f(a)}{x - a} \tag{4}$$

*is called the derivative of the function $f$ at $a$.*

**Definition 4.** *A function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of $E$ if*

$$f(x + h) - f(h) = A(x)h + \alpha(x; h) \tag{5}$$

*where $h \mapsto A(x)h$ is a linear function in $h$ and $\alpha(x; h) = o(h)$ as $h \to 0, x + h \in E$.*

**Remark 4.** *The difference between this definition of differentiability at a point and the previous one is that this one takes into account that the quantity $A(x)$ and $\alpha(x; h)$ may change in terms of differentiability at different points $x$, so we write them as functions of $x$.*

**Definition 5.** *The function $h \mapsto A(x)h$ in the definition of differentiability at a point, which is linear in $h$, is called the differential of the function $f : E \to \mathbb{R}$ at the point $x \in E$ and is denoted $df(x)$ or $Df(x)$.*

**Remark 5** (The Principal Linear Part of the Increment of the Function)**.**

**Remark 6** (The Leibniz Notation of the Derivative)**.**

$$df(x)(h) = f'(x)h \tag{6}$$

$$dx(h) = 1 \cdot h = h$$

$$df(x)(h) = f'(x)dx(h) \tag{7}$$

$$df(x) = f'(x)dx \tag{8}$$

$$f'(x) = \frac{df(x)(h)}{dx(h)} \tag{9}$$

# 3   The Tangent Line; Geometric Meaning of the Derivative and Differential

$$f(x) = c_0 + c_1(x - x_0) + o(x - x_0) \text{ as } x \to x_0, x \in E \tag{10}$$

**Proposition 1.** *A function $f : E \to \mathbb{R}$ that is continuous at a point $x_0 \in E$ that is a limit point of $E \subset \mathbb{R}$ admits a linear approximation as*

$$f(x) = c_0 + c_1(x - x_0) + o(x - x_0) \text{ as } x \to x_0, x \in E \tag{11}$$

*if and only if it is differentiable at the point.*

**Remark 7.** *(TODO) What is the difference between this proposition and the definition of the differentiability at a point?*

**Definition 6** (Analytic Definition of a Tangent Line)**.** *If a function $f : E \to \mathbb{R}$ is defined on a set $E \subset \mathbb{R}$ and differentiable at a point $x_0 \in E$, the line defined by the equation*

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{12}$$

*is called the tangent to the graph of this function at the point $(x_0, f(x_0))$.*

**Definition 7** (Nth Order Contact at a Point between Two Mappings)**.** *If the mappings $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous at a point $x_0 \in E$ that is a limit point of $E$ and $f(x) - g(x) = o((x - x_0)^n)$ as $x \to x_0, x \in E$, we say that $f$ and $g$ have nth order contact at $x_0$ (more precisely, contact of order at least n).*

*For $n = 1$, we say that the mappings $f$ and $g$ are tangent to each other at $x_0$.*

**Example 1** (Polynomial Approximation)**.**

# 4   The Role of the Coordinate System

**Remark 8** (Geometric Definition of a Tangent Line)**.**