Notes of ”The Algebra of Vectors”

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1 Vectors and Linear Operations

1.1 Decomposition of Vectors

Theorem 1 (Theorem of Decomposition of Vectors in Plane and Space).

1.2 Application in Geometry Problems

1.2.1 Vectors in a Plane

Proposition 1. *α, β, γ lie in a plane $\Leftrightarrow \exists \lambda, \mu, \nu$ that at least one of them is not equal to 0 such that*

$$\lambda\alpha + \mu\beta + \nu\gamma = 0 \tag{1}$$

1.2.2 Points in a Line

Proposition 2. *A, B, C lie in a line $\Leftrightarrow \exists s, t \in \mathbb{R}$ such that*

$$\vec{OC} = s\vec{OA} + t\vec{OB} \wedge s + t = 1 \tag{2}$$

Definition 1. $(A, B, C) = \frac{\vec{AC}}{\vec{CB}}$

Remark 1. $(A, B, C) \in (-\infty, -1) \cup (-1, \infty)$

1.2.3 Ceva's Theorem

1.2.4 Menelaus' Theorem

2 Affine Coordinate System

3 Dot Product of Vectors

3.1 Definition of Dot Product

3.2 Properties of Dot Product

3.3 Dot Product in Coordinate Systems

3.4 应用

3.4.1 余弦定理

3.4.2 三条高线交于一点

4 Cross Product of Vectors

4.1 三个不共面向量的定向

两个不共线向量 α, β 的定向：当从 α 转到 β 满足右手螺旋，则为右手系；满足左手螺旋，则为左手系。

三个不同面向量 α, β, γ 的定向：α, β 决定的平面将空间分成两个半空间。从 γ 指向的那一侧观察，若 α, β 组成右手系，则这三个向量组成右手系；若 α, β 组成左手系，则这三个向量组成左手系。

Properties of Orientation:

- When two vectors have a transposition, the orientation changes.
- When a vectors is replaced by its negative one, the orientation changes.

Remark 2. *It seems to be related to determinant.*

4.2 Definition of Cross Products of Vectors

4.3 Properties of Cross Products of Vectors

- Skew-symmetric

$$\alpha \times \beta = -\beta \times \alpha \tag{3}$$

- Double linearity

$$(\lambda\alpha) \times \beta = \lambda(\alpha \times \beta) = \alpha \times (\lambda\beta) \tag{4}$$

$$\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma, (\alpha + \beta) \times \gamma = \alpha \times \gamma + \beta \times \gamma \tag{5}$$

4.4 Cross Products of Vectors in Coordinate Systems

5 Mixed Product of Vectors

5.1 Double Cross Product

$$(\alpha \times \beta) \times \gamma = (\alpha \cdot \gamma)\beta - (\beta \cdot \gamma)\alpha \tag{6}$$

$$\alpha \times (\beta \times \gamma) = (\alpha \cdot \gamma)\beta - (\alpha \cdot \beta)\gamma \tag{7}$$

证明. Hint: 利用一个方便的坐标系来通过坐标进行验证。 □

Remark 3. 虽然证明过程依赖于坐标系的选取，但最终得到的等式并不依赖于坐标系的选取，甚至不依赖于坐标系，完全是向量之间的关系。

5.2 Mixed Product

几何意义：绝对值是以 α, β, γ 为边的平行六面体的体积。符号为三个向量组成的基的定向。

Properties of Mixed Product:

- α, β, γ in the same plane $\Leftrightarrow (\alpha, \beta, \gamma) = 0$
- $(\alpha, \beta, \gamma) = (\beta, \gamma, \alpha) = (\gamma, \alpha, \beta)$
- $\alpha \times \beta \cdot \gamma = (\alpha, \beta, \gamma) = \alpha \cdot \beta \times \gamma$
- Triple linearity

5.3 Mixed Product in Coordinate System

$$(\alpha, \beta, \gamma) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} (e_1, e_2, e_3) \tag{8}$$

Specially, in a Cartesian coordinate system, $(e_1, e_2, e_3) = 1$. Therefore,

$$(\alpha, \beta, \gamma) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{9}$$

5.4 Corollary and Application

5.4.1 Jacobi Identity

$$\alpha \times (\beta \times \gamma) + \beta \times (\gamma \times \alpha) + \gamma \times (\alpha \times \beta) = 0 \tag{10}$$

5.4.2 Lagrange Identity

$$\alpha \times \beta \cdot \gamma \times \delta = \begin{vmatrix} \alpha \cdot \gamma & \alpha \cdot \delta \\ \beta \cdot \gamma & \beta \cdot \delta \end{vmatrix} \tag{11}$$

证明. Hint: Use mixed product. □

5.4.3 Geometric Proof of Cramer Criterion in 3×3 Linear Systems