Notes of "More Properties of Determinants"

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1 行列式按一行或一列的元素展开

Example 1.

$$\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = (a+(n-1)b)(a-b)^{n-1}$$

Example 2 (奇数阶反对称（斜对称）行列式)**.**

Example 3 (Verdemond Determinant)**.**

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \Pi_{1 \le i < j \le n}(a_j - a_i)$$

Pf: By mathematical induction

$$D_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 - a_1 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ a_1^2 - a_1 a_1 & a_2^2 - a_2 a_1 & a_3^2 - a_3 a_1 & \cdots & a_n^2 - a_n a_1 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-2} - a_1^{n-3} a_1 & a_2^{n-2} - a_2^{n-3} a_1 & a_3^{n-2} - a_3^{n-3} a_1 & \cdots & a_n^{n-2} - a_n^{n-3} a_1 \\ a_1^{n-1} - a_1^{n-2} a_1 & a_2^{n-1} - a_2^{n-2} a_1 & a_3^{n-1} - a_3^{n-2} a_1 & \cdots & a_n^{n-1} - a_n^{n-2} a_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_2^{n-3}(a_2 - a_1) & a_3^{n-3}(a_3 - a_1) & \cdots & a_n^{n-3}(a_n - a_1) \\ 0 & a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{pmatrix}$$

$$= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_2 & a_3 & a_4 & \cdots & a_n \\ a_2^2 & a_3^2 & a_4^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_2^{n-2} & a_3^{n-2} & a_4^{n-2} & \cdots & a_n^{n-2} \end{pmatrix}$$

$$= \Pi_{j=2}^{n}(a_j - a_1) D_{n-1}$$

Example 4 (三对角行列式)**.**

$$A = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}$$

If $a^2 - 4bc = 0$*, then let* α *be the only solution of the equation* $x^2 - ax + bc = 0$*, and thus* $\det A = \frac{(n+1)\alpha^n}{2^n}$.

If $a^2 - 4bc \neq 0$*, then let* α, β *be the two solutions of the equation* $x^2 - ax + bc = 0$*, and thus* $\det A = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$.

Example 5 (循环行列式)**.**

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_0 \end{vmatrix} = f(1) f(\omega) f(\omega^2) \cdots f(\omega^{n-1})$$

in which

$$f(x) = \Sigma_{i=0}^{n-1} a_i x^i, \omega = \exp(\frac{2\pi i}{n})$$

2 特殊矩阵的行列式