Notes of ”Vector Space of Rows and Columns”

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1 Basic Definitions

Definition 1 (Row (Vector) Space and Row Vectors). *Given $n \in \mathbb{N}$,*

Properties of Vector Space

1. Associative property of addition
2. Commutative property of addition
3. Zero element
4. Inverse element of addition
5. Unit element in scalars
6. $\alpha(\beta X) = (\alpha\beta)X$
7. $(\alpha + \beta)X = \alpha X + \beta X$
8. $\alpha(X + Y) = \alpha X + \alpha Y$

There are also column vector space and column vectors.

2 Linear Combination and Linear Span

Definition 2 (Linear Combinations of Vectors).

Linear combinations have an interesting property:

Proposition 1. *A linear combination of linear combinations of vectors $X_1, X_2, \ldots, X_k \in \mathbb{R}^n$ is also a linear combination of vectors $X_1, X_2, \ldots, X_k \in \mathbb{R}^n$.*

证明. Hint: Express the linear combination by vectors $X_1, X_2, \ldots, X_k \in \mathbb{R}^n$ □

Therefore, if we consider the set V consisting of all linear combinations of vectors $X_1, X_2, \ldots, X_k \in \mathbb{R}^n$, it has the following property:

$$X, Y \in V \Rightarrow \alpha X + \beta Y \in V, \forall \alpha\beta \in \mathbb{R}$$

Definition 3 (Linear Span (Or Linear Hull)).

Definition 4 (Linear Span of a Subset in $\mathbb{R}^n$)**.** *Given a subset $S \subset \mathbb{R}^n$, the linear span of S, denoted by $\langle S \rangle$, is the set of all linear combinations of any finite numbers of vectors in S.*

Some interesting properties of linear spans:

Proposition 2. *Suppose V is a linear span in $\mathbb{R}^n$, then*

- $\langle V \rangle = V$.
- *If $S \subset V$, then $\langle S \rangle \subset V$.*

The second property leads to another definition of the linear span of a subset in $\mathbb{R}$:

Definition 5 (Linear Span (Or Linear Hull))**.** *The linear span of a subset $S \subset \mathbb{R}^{\ltimes}$ is the intersection of all linear spans in $\mathbb{R}^n$ that contains S:*

$$\langle S \rangle = \bigcap_{S \subset V} V$$

证明. Hint:

- $\langle S \rangle \subset \bigcap_{S \subset V} V$
- $\bigcap_{S \subset V} V \subset \langle S \rangle$
- $\bigcap_{S \subset V} V$ is a linear span

□

3 Linear Dependence

The concept of linear combination leads to a kind of relationship between a set of vectors:

Definition 6 (Linear Independent & Linear Dependent)**.**

Remark 1. *The order of the vectors doesn't affect linear independence because the addition operation in the vector space holds the commutative property.*

Theorem 1.

4 Basis and Dimension

Definition 7 (Basis)**.** *Suppose that V is a non-zero linear span in $\mathbb{R}^n$. A set of vectors $X_1, X_2, \ldots, X_r$ is said to be a basis of V, if they are linear independent, and their linear span is the same as V: $\langle X_1, X_2, \ldots, X_r \rangle = V$.*

Remark 2. *The basis of a vector space or a linear span is not unique.*

Remark 3. *As we proved before, the linear span of $\{E_{(1)}, E_{(2)}, \ldots, E_{(n)}\}$ is $\mathbb{R}^n$, hence $\{E_{(1)}, E_{(2)}, \ldots, E_{(n)}\}$ is a basis of $\mathbb{R}^n$, and is said to be the standard basis of $\mathbb{R}^n$.*

Proposition 3. *Given a basis of a vector space, the linear combination of a vector in the space with the basis is unique, and we call the coefficients of the linear combination the coordinate of the vector under the basis.*

Lemma 1. *Let V be a linear span in $\mathbb{R}$ with a basis of $X_1, X_2, \ldots, X_r$, and $Y_1, Y_2, \ldots, Y_s$ be a set of linear independent vectors in V, then $s \leq r$.*

证明. Hint: Proof by contradiction.

- Consider the definition of $Y_1, Y_2, \ldots, Y_s$ are linear independent.
- Expand the definition into a homogeneous linear system.
- Discuss the number of solutions of the linear system.

□

Theorem 2.

Definition 8 (Dimension of a Linear Span and Maximal Linearly Independent Subset)**.**

Definition 9 (Rank of a Set of Vectors)**.**