Notes of "Solution of Linear Systems"

Jinxin Wang

1 Solution Space of a Homogenous Linear System

2 Solution Set of a Non-Homogenous Linear System

3 同解与秩

Proposition 1. *Suppose* $A = [A_{(1)}, A_{(2)}, \dots, A_{(s)}]$, $B = [B_{(1)}, B_{(2)}, \dots, B_{(t)}]$, *in which* $A_{(i)}$ *and* $B_{(j)}$ *are row vectors with* n *elements. If every solution of* $AX = 0$ *is also a solution of* $BX = 0$, *then every row vector of* B *is a linear combination of the row vectors of* A.

Corollary 1. *Suppose* $A = (a_{ij})_{s\times n}$, $B = (b_{ij})_{t\times n}$. $AX = 0$ *and* $BX = 0$ *has the same solution space if and only if the row vectors of* A *and* B *are in the row space of each other.*

Corollary 2. *Suppose* $A = (a_{ij})_{s\times n}$, $B = (b_{ij})_{t\times n}$. *If* $AX = 0$ *and* $BX = 0$ *has the same solution space, then* $\operatorname{rank} A = \operatorname{rank} B$.

Remark 1. *The converse is not true. Consider* $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, *which have the same rank as* 2, *but different solution space.*

The above remark shows that given any two matrices $A_{s\times n}$ and $B_{t\times n}$ with the same number of columns, from the condition that $\operatorname{rank} A = \operatorname{rank} B$ we cannot conclude anything about the solution spaces of their corresponding homogenous linear systems $AX = 0$ and $BX = 0$. However, if the two matrices are connected by matrix multiplication, then we have

Proposition 2. *Suppose* $A = (a_{ij})_{s\times t}$, $B = (b_{ij})_{t\times n}$. $(\operatorname{rank} AB = \operatorname{rank} B) \Leftrightarrow ABX = 0$ *and* $BX = 0$ *has the same solution space* $\Leftrightarrow$ *Every solution of* $ABX = 0$ *is also a solution of* $BX = 0$.

4 带有特殊秩的矩阵

4.1 秩 1 矩阵

4.2 秩 0 矩阵

Proposition 3. *Let* A *be a square matrix of order* 2, *and* k *be an interger greater than* 2. $A^k = 0$ *if and only if* $A^2 = 0$