Notes of "Rank of Matrices"

Jinxin Wang

1 Overview

- Retrospective of Linear System
- Rank of Matrices
  - Def: Column space and column rank, row space and row rank of a matrix.
  - Lemma: Invariance of the row rank and column rank of a matrix through elementary row operations.
  - Thm: Equality of row rank and column rank of a matrix.
  - Def: Rank of a matrix.
- Application: Sovlable Criterion of a Linear System
  - Corollary: Numbere of Major Unknowns in a Linear System with Rank.
  - Thm: Sovlability of a Linear System with Rank.

2 Retrospective of Linear System

With the notion of linear span, we have a new perspective of a linear system:
In a m-dimensional column vector space $\mathbb{R}^m$, consider a subset A with n vectors

$$A^{(j)} = [a_{1j}, a_{2j}, \ldots, a_{mj}], j = 1, 2, \ldots, n$$

and their linear span

$$V = \langle A \rangle = \langle A^{(1)}, A^{(2)}, \ldots, A^{(n)} \rangle$$

For a given vector $B \in \mathbb{R}^m$, two natural questions are as follows:

1. Whether it is true that $B \in V$?
2. If so, how to express B as a linear combination of vectors in A?

Remark 1. *If $\dim V = n$, then $A = \{A^{(1)}, A^{(2)}, \ldots, A^{(n)}\}$ is a base of V. Then the second question above is equivalent to find the coordinate of B under the base A.*

Formulate the above two questions in equations:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{1}$$

which is equivalent to a linear system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

Another perspective of a linear system that we already know is the matrix form:

$$AX = B \tag{3}$$

with the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{4}$$

and the augmented matrix

$$(A|B) \tag{5}$$

3 Rank of Matrices

Definition 1 (Column Space, Column Rank, Row Space, and Row Rank of a Matrix)**.** *Suppose A is a $m \times n$ matrix* $V_c(A) = \langle A^{(1)}, A^{(2)}, \ldots, A^{(n)} \rangle$

$r_c(A) = dimV_c(A)$

$V_r(A) = \langle A_{(1)}, A_{(2)}, \ldots, A_{(n)} \rangle$

$r_r(A) = dimV_r(A)$

Remark 2. *According to Theorem 2 in the Notes of "Vector Space of Rows and Columns", the dimension of a linear span is determined, then our notion of column rank and row rank of a matrix is well defined, because the definitions of column rank and row rank of a matrix rely on the linear span of column vectors and row vectors of the matrix. In other words, these properties of a matrix exists.*

From previous knowledge we know that applying two kinds of elementary row operations to a coefficient matrix can transform it to its row reduced echelon form. Notice an important fact about elementary row operations: both kinds are inversible. More specifically, the transformation of an elementary row operation to a matrix can reversed (undone) with the same kind of elementary row operation.

The above fact leads to the following important observation about the effect of elementary row operations on the column rank and row rank of a matrix.

Lemma 1. *If a matrix A' is transformed from a matrix A through finite number of elementary row operations, it holds that*

$$r_c(A') = r_c(A)$$
$$r_r(A') = r_r(A)$$

证明. Hint: 同解性 □

The above lemma helps us answer a natural and important question: whether the row rank and the column rank of a matrix is equal or not?

Theorem 1. *For a $m \times n$ matrix A, it always holds that*

$$r_c(A) = r_r(A)$$

证明. Hint: 化为阶梯型 □

Definition 2 (Rank of a Matrix)**.** *The rank of a matrix A, denoted by $\operatorname{rank} A$, is equal to its column rank, as well as its row rank.*

$$\operatorname{rank} A = r_c(A) = r_r(A)$$

Remark 3. 一个矩阵的秩是唯一确定的，是它的内在特征，不依赖于任何外界情况。

4 Application: Solvable Criterion of a Linear System

Through transforming a linear system to its row reduced echelon form, we can already answer the solvable question of a linear system by observing some characteristics of the row reduced echelon form, such as whether there is an equation with zero on the left side of the equal sign, and non-zero on the right side.

However, we are not satisfied with this approach because the process to transform a coefficient matrix to its row reduced echelon form is undertermined, in other words the process and the final form can vary from person to person. Instead, we want to get the answer of solvability from some unique property of the linear system, or equivalently the augmented matrix (including the coefficient matrix). That's where rank comes on the scene.

4.1 Invariant of a Linear System with Rank

Corollary 1 (Number of Major Unknowns in a Linear System with Rank)**.** *The number of the major unknowns in a homogenous linear system doesn't rely on the way how it reaches the row reduced echelon form. It is always equal to the rank of its coefficient matrix.*

证明. Hint: The number of non-zero rows in the row reduced echelon form. □

4.2 Sovlability of a Linear System with Rank

Theorem 2 (Rouché−Capelli theorem or Kronecker−Capelli theorem: Sovlability of a Linear System with Rank)**.** *A linear system is solvable if and only if the rank of its coefficient matrix is equal to the rank of its augmented matrix.*