Notes of "Differentiable Functions"

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1 Statement of the Problem and Introductory Considerations

Solving the Kepler problem of two bodies, which is to describe the motion of a planet relative to a star. With an appropriate coordinate system, we use two functions $x(t)$ and $y(t)$ to describe the motion of the planet.

$$\vec{F} = m\vec{a} \tag{1}$$

$$\vec{F} = G\frac{Mm}{|\vec{r}|^3}\vec{r} \tag{2}$$

Now our task is to define and compute the instaneous velocity of a motion governed by the law (1).

By the law of inertia, if no force is acted on a body, it moves in uniform motion. Hence if at time t the external action on a body ceases, it will continue moving in a straight line with a certain velocity. Since forces are to change velocity instead of maintaining velocity, it is natural to consider the hypothetical velocity as the instaneous velocity at time t.

If the absolute magnitude of the rate of change of a quantity $f(t)$ is bounded over an time interval $[0, t]$, then as the time interval Δt becomes smaller, the change of the quantity Δf becomes smaller, which is equivalent to say that $f(t)$ is continuous.

Apparently the above analysis applies to the case of velocity where the rate of change of velocity is acceleration, whose absolute magnitude is bounded since the external force on a body is finite. Therefore, at all times t close to some time t_0 the velocity $\vec{v}(t)$ of the body m must be close to the value $\vec{v}(t_0)$ that we wish to determine.

Remark 1. *Notice that there seems to be a shift from a scalar $f(t)$ to a vector $\vec{v}(t)$, and a question may arise that whether the previous analysis about the bounded rate of change and continuity of a scalar function applies to a vector function. Consider the nature of change, it must be true that when a change of a quantity becomes smaller, the changed quantity is closer to the original quantity, no matter whether the quantity is a scalar or a vector. Therefore, the analysis about the bounded rate of change and continuity applies to both cases.*

2 Functions Differentiable at a Point

Definition 1. *A function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of E if there exists a linear function $A(x-a)$ such that $f(x) - f(a)$ can be represented as*

$$f(x) - f(a) = A(x-a) + o(x-a) \text{ as } x \to a, x \in E \tag{3}$$

Corollary 1. *If a function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of E, then it is also continuous at the point $a \in E$.*

Remark 2. *The conversion of the above corollary is not true. An counterexample is that $f(x) = x^{\frac{1}{3}}$ is continuous at $x = 0$, but it is not differentiable at $x = 0$.*

Definition 2. *The linear function $A(x-a)$ in the above definition is called the differential of the function f at a.*

Definition 3. *The number*

$$f'(a) = \lim_{E \ni x \to a} \frac{f(x) - f(a)}{x - a} \tag{4}$$

is called the derivative of the function f at a.

Definition 4. *A function $f : E \to \mathbb{R}$ is differentiable at a point $a \in E$ that is a limit point of E if*

$$f(x+h) - f(h) = A(x)h + \alpha(x;h) \tag{5}$$

where $h \mapsto A(x)h$ is a linear function in h and $\alpha(x;h) = o(h)$ as $h \to 0, x+h \in E$.

Remark 3. *The difference between this definition of differentiability at a point and the previous one is that this one takes into account that the quantity $A(x)$ and $\alpha(x;h)$ may change in terms of differentiability at different points x, so we write them as functions of x.*

Definition 5. *The function $h \mapsto A(x)h$ in the definition of differentiability at a point, which is linear in h, is called the differential of the function $f : E \to \mathbb{R}$ at the point $x \in E$ and is denoted $df(x)$ or $Df(x)$.*

Remark 4 (The Principal Linear Part of the Increment of the Function)**.**

Remark 5 (The Leibniz Notation of the Derivative)**.**

$$df(x)(h) = f'(x)h \tag{6}$$

$$dx(h) = 1 \cdot h = h$$

$$df(x)(h) = f'(x)dx(h) \tag{7}$$

$$df(x) = f'(x)dx \tag{8}$$

$$f'(x) = \frac{df(x)(h)}{dx(h)} \tag{9}$$

3 The Tangent Line; Geometric Meaning of the Derivative and Differential

$$f(x) = c_0 + c_1(x - x_0) + o(x - x_0) \text{ as } x \to x_0, x \in E \tag{10}$$

Proposition 1. *A function $f : E \to \mathbb{R}$ that is continuous at a point $x_0 \in E$ that is a limit point of $E \subset \mathbb{R}$ admits a linear approximation as*

$$f(x) = c_0 + c_1(x - x_0) + o(x - x_0) \text{ as } x \to x_0, x \in E \tag{11}$$

if and only if it is differentiable at the point.

Remark 6. *(TODO) What is the difference between this proposition and the definition of the differentiability at a point?*

Definition 6 (Analytic Definition of a Tangent Line)**.** *If a function $f : E \to \mathbb{R}$ is defined on a set $E \subset \mathbb{R}$ and differentiable at a point $x_0 \in E$, the line defined by the equation*

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{12}$$

is called the tangent to the graph of this function at the point $(x_0, f(x_0))$.

Definition 7 (Nth Order Contact at a Point between Two Mappings)**.** *If the mappings $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are continuous at a point $x_0 \in E$ that is a limit point of E and $f(x) - g(x) = o((x - x_0)^n)$ as $x \to x_0, x \in E$, we say that f and g have nth order contact at x_0 (more precisely, contact of order at least n).*

For $n = 1$, we say that the mappings f and g are tangent to each other at x_0.

Example 1 (Polynomial Approximation)**.**

4 The Role of the Coordinate System

Remark 7 (Geometric Definition of a Tangent Line)**.**