Notes of "Properties of Continuous Functions"

Jinxin Wang

1 Local Properties

2 Global Properties

Theorem 1 (The Bolzano-Cauchy Intermediate-Value Theorem)**.**

Corollary 1. *If the function ϕ is continuous on an open interval and assumes values $\phi(a) = A$ and $\phi(b) = B$ at points a and b, then for any number C between A and B, there is a point c between a and b at which $\phi(c) = C$.*

证明. □

Theorem 2 (The Weierstrass Maximum-Value Theorem)**.**

Definition 1 (Uniform Continuity)**.**

Theorem 3 (The Contor-Heine Theorem on Uniform Continuity)**.**

Proposition 1. *A continuous mapping $f : E \to \mathbb{R}$ of a closed interval $E = [a, b]$ into $\mathbb{R}$ is injective if and only if the function f is strictly monotonic on $[a, b]$.*

Proposition 2. *Each strictly monotonic function $f : X \to \mathbb{R}$ defined on a numerical set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f, and has the same kind of monotonicity on Y that f has on X.*

Proposition 3. *The discontinuities of a function $f : E \to \mathbb{R}$ that is monotonic on the set $E \subset \mathbb{R}$ can be only discontinuities of first kind.*

Corollary 2. *If a is a point of discontinuity of a monotonic function $f : E \to \mathbb{R}$, then at least one of the limits $\lim_{E \ni x \to a^+} = f(a^+)$ or $\lim_{E \ni x \to a^-} = f(a^-)$ exists, and strict inequality holds in at least one of the inequalities $f(a^-) \leq f(a) \leq f(a^+)$ when f is nondecreasing and $f(a^-) \geq f(a) \geq f(a^+)$ when f is nonincreasing. The function assumes no values in the open interval defined by the strict inequality. Open intervals of this kind determined by different points of discontinuity have no points in common.*

Corollary 3. *The set of points of discontinuity of a monotonic function is at most countable.*

Proposition 4 (A Criterion for Continuity of a Monotonic Function)**.** *A monotonic function $f : E \to \mathbb{R}$ defined on a closed interval $E = [a, b]$ is continuous if and only if its set of values $f(E)$ is the closed interval with endpoints $f(a)$ and $f(b)$.*

Theorem 4 (The Inverse Function Theorem)**.** *A function $f : X \to \mathbb{R}$ that is strictly monotonic on a set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f, and has the same kind of monotonicity on Y that f has on X.*

If in addition X is a closed interval $[a, b]$ and f is continuous on X, then the set $Y = f(X)$ is the closed interval with endpoints $f(a)$ and $f(b)$ and the function $f^{-1} : Y \to \mathbb{R}$ is continuous on it.