Notes of "The Limit of A Function"

Jinxin Wang

1 Definition of the Limit of a Function

Definition 1 (The Limit of a Function (Basic Type)). *(ϵ-δ)*

Remark 1. *Based on the definition, we can see that limits are a kind of local characteristic of a function. With that, when using the definition to prove the limit of a function at $x = x_0$, we can discuss it within a certain neighborhood $O(x_0, \delta_0)$.*

2 Properties of the Limit of a Function

3 方法与技巧

3.1 证明与研究函数极限

- 定义法

- 变量代换

Remark 2. 在研究函数极限时使用变量代换是否总是成立？这个问题使用数学语言来描述如下：
Suppose that $\lim_{x \to a} g(x) = A$, $\lim_{x \to A} f(x) = B$, is it true that

$$\lim_{x \to a} f(g(x)) = \lim_{y \to A} f(y)$$

In fact it is not always true. Here is a counterexample: Let $g(x) \equiv 0$, and thus $\lim_{x \to 0} g(x) = 0$. Let $f(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}$. Then we have $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} f(g(x)) = 1$. 这里的数学直观是 $\lim_{x \to a} g(x) = A$ 决定了 $y = g(x) \to A$ 的方式。它与 $\lim_{x \to A} f(x) = B$ 中 $x \to A$ 的不同可能导致结果的不同。

Here are two propositions related to this problem:

Proposition 1. *Suppose that $\lim_{x \to a} g(x) = A$, $\lim_{x \to A} f(x) = B$. If any of the following conditions is true:*

- *$\exists \delta_0 > 0$ such that $\forall x \in O(a, \delta_0) \setminus \{a\}$: $g(x) \neq A$.*

- *$\lim_{x \to A} f(x) = f(A)$.*

- *$A = \infty$, and $\lim_{x \to \infty} f(x)$ is defined.*

then the following is true:

$$\lim_{x \to a} f(g(x)) = \lim_{y \to A} f(y)$$

证明. Hint:

- Notice the difference between the conclusion of the definition of $\lim_{x \to a} g(x) = A$, and the condition of the definition of $\lim_{x \to A} f(x) = B$.

- Notice what change the fact of continuity brings to the definition of $\lim_{x \to A} f(x) = B$.

$\square$

Proposition 2. *If* $\lim_{x \to a} g(x) = A$, $\lim_{x \to A} f(x) = B$, *then exact one of the following situation is true:*

- $\lim_{x \to a} f(g(x)) = B$

- $\lim_{x \to a} f(g(x)) = g(A)$

- $\lim_{x \to a} f(g(x))$ *is not defined*

证明. Hint: Using the first condition of the previous proposition to discuss different kinds of $g(x)$. $\square$