Notes of "Determinant: Construction and Basic Properties"

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1 Geometric Background

2 Combination-Analytic Method

Definition 1 (Determinant). *The determinant of a matrix A of order n is a number decided by the matrix that is*

$$\det A = \Sigma_{\sigma \in S_n} \epsilon_\sigma a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \tag{1}$$

Remark 1. *For a matrix of order n, its determinant is a sum with $n!$ terms, since S_n has $n!$ elements.*

3 Basic Properties of Determinants

For the convenience of description, some conventions of notations:

- The i-th row of a matrix A is denoted by $A_{(i)}$.
- The j-th column of a matrix A is denoted by $A^{(j)}$.
- A matrix of order n can be represented as a column of row vectors: $A = [A_{(1)}, A_{(2)}, \ldots, A_{(n)}]$
- A matrix of order n can be represented as a row of column vectors: $A = (A^{(1)}, A^{(2)}, \ldots, A^{(n)})$

According to the definition of a determinant, det is a mapping from a square matrix A to a number $|A|$. Our mission is to study how the mapping changes when we change rows or columns of a matrix.

Definition 2 (多重线性函数).

Definition 3 (斜对称函数).

Theorem 1 (Determinants of a Square Matrix and its Transpose).

$$\det A = \det A^T$$

证明. Suppose $A = (a_{ij})_{n\times n}$ and $A^T = (a'_{ij})_{n\times n}$, then $a_{ij} = a'_{ji}$.

$$\det A = \Sigma_{\sigma \in S_n} \epsilon_\sigma a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

If we rearrange the order of factors in each term of $\det A$ by its column index, then

$$\det A = \Sigma_{\sigma \in S_n} \epsilon_\sigma a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} \cdots a_{\sigma^{-1}(n)n}$$

Recall that $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$. Besides, since $S_n \to S_n : \sigma \mapsto \sigma^{-1}$ is a bijection,

$$\{\sigma \mid \sigma \in S_n\} = \{\sigma^{-1} \mid \sigma \in S_n\}$$

Therefore,

$$\begin{aligned}
\det A &= \Sigma_{\sigma \in S_n} \epsilon_{\sigma^{-1}} a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} \cdots a_{\sigma^{-1}(n)n} \\
&= \Sigma_{\sigma \in S_n} \epsilon_{\sigma^{-1}} a'_{1\sigma^{-1}(1)} a'_{2\sigma^{-1}(2)} \cdots a'_{n\sigma^{-1}(n)} \\
&= \Sigma_{\sigma^{-1} \in S_n} \epsilon_{\sigma^{-1}} a'_{1\sigma^{-1}(1)} a'_{2\sigma^{-1}(2)} \cdots a'_{n\sigma^{-1}(n)} \\
&= \det A^T
\end{aligned}$$

□

Remark 2. *The above property of determinants shows a kind of symmetry between the rows and columns of a determinant. More specifically, if there is a property of determinant with regards of rows, it also holds for columns, and vice versa.*

Remark 3. *The symmetry between rows and columns of a determinant is sometimes used to calculate the value of the determinant.*

Theorem 2 (行列式的多重线性和反对称性)**.** *The function defined on the set* $M_n(\mathbb{R})$ $\det : A \mapsto \det A$ *has the following properties:*

D1 det *is skew-symmetric with regards of the rows of a square matrix.*

D2 det *is multilinear with regards of the rows of a square matrix.*

D3 $\det E = 1$

证明. □

The function $\det : A \mapsto \det A$ has some more properties that can be derived from the above theorem, but we want to show that they holds for any function $D : M_n(\mathbb{R}) \to \mathbb{R}$ that is skew-symmetric and multilinear with regards of rows of a square matrix (in other words, satisfying the above property **D1** and **D2**).