

Notes of "Properties of Continuous Functions"

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1 Local Properties

2 Global Properties

Theorem 1 (The Bolzano-Cauchy Intermediate-Value Theorem).

Corollary 1.

Theorem 2 (The Weierstrass Maximum-Value Theorem).

Definition 1 (Uniform Continuity).

Theorem 3 (The Contor-Heine Theorem on Uniform Continuity).

Proposition 1. *A continuous mapping $f : E \rightarrow \mathbb{R}$ of a closed interval $E = [a, b]$ into \mathbb{R} is injective if and only if the function f is strictly monotonic on $[a, b]$.*

Proposition 2. *Each strictly monotonic function $f : X \rightarrow \mathbb{R}$ defined on a numerical set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \rightarrow \mathbb{R}$ defined on the set $Y = f(X)$ of values of f , and has the same kind of monotonicity on Y that f has on X .*

Proposition 3. *The discontinuities of a function $f : E \rightarrow \mathbb{R}$ that is monotonic on the set $E \subset \mathbb{R}$ can be only discontinuities of first kind.*

Corollary 2. *If a is a point of discontinuity of a monotonic function $f : E \rightarrow \mathbb{R}$, then at least one of the limits $\lim_{E \ni x \rightarrow a^+} f(x) = f(a^+)$ or $\lim_{E \ni x \rightarrow a^-} f(x) = f(a^-)$ exists, and strict inequality holds in at least one of the inequalities $f(a^-) \leq f(a) \leq f(a^+)$ when f is nondecreasing and $f(a^-) \geq f(a) \geq f(a^+)$ when f is nonincreasing. The function assumes no values in the open interval defined by the strict inequality. Open intervals of this kind determined by different points of discontinuity have no points in common.*

Corollary 3. *The set of points of discontinuity of a monotonic function is at most countable.*

Proposition 4 (A Criterion for Continuity of a Monotonic Function). *A monotonic function $f : E \rightarrow \mathbb{R}$ defined on a closed interval $E = [a, b]$ is continuous if and only if its set of values $f(E)$ is the closed interval with endpoints $f(a)$ and $f(b)$.*

Theorem 4 (The Inverse Function Theorem). *A function $f : X \rightarrow \mathbb{R}$ that is strictly monotonic*