Notes of "Basic Rules of Differentiation"

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1 Differentiation and the Arithmetic Operations

Theorem 1 (Rules of Differentiation of Results of Arithmetic Operations)**.**

Corollary 1. *The derivative of a linear combination of differentiable functions equals the same linear combination of the derivatives of these functions.*

Corollary 2. *If the functions $f_1, f_2, \dots, f_n$ is differentiable at x, then*

$$(f_1 f_2 \cdots f_n)'(x) = f_1'(x)f_2(x)\cdots f_n(x) + f_1(x)f_2'(x)\cdots f_n(x) + \cdots + f_1(x)f_2(x)\cdots f_n'(x)$$

Corollary 3. *The rules of differentiation of a result of arithmetic operations can also be written in terms of differentials:*

- $d(f+g)(x)$
- $d(f \cdot g)(x)$
- $d(\frac{f}{g})(x)$

2 Differentiation of a Composite Function (Chain Rule)

Theorem 2 (Differentiation of a Composite Function)**.** *If the function $f : X \to Y \subset \mathbb{R}$ is differentiable at a point $x \in X$ and the function $g : Y \to \mathbb{R}$ is differentiable at the point $y = f(x) \in Y$, then the composite function $g \circ f : X \to \mathbb{R}$ is differentiable at x, and the differential $d(g \circ f)(x) : T\mathbb{R}(x) \to T\mathbb{R}(g(f(x)))$ of their composition equals the composition $dg(y) \circ df(x)$ of their differentials*

$$df(x) : T\mathbb{R}(x) \to T\mathbb{R}(y = f(x)) \text{ and } dg(y = f(x)) : T\mathbb{R}(y) \to T\mathbb{R}(g(y))$$

证明. Hint: □

Remark 1. *Let $z = g(y), y = f(x)$, one may consider using the following relation to prove the rule of the differentiation of a composite function:*

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x}$$

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x}$$

The difficulty that appears in this method is that as $\Delta x \to 0$, Δy may be 0, which makes the above equation invalid.

In the above proof, we use the following techniques to handle this difficulty:

1. *First, we use differentials rather than derivatives to avoid division.*

2. *Second, we make the differential of $g(y)$ valid at $\Delta y = 0$ by defining $o(t) = 0$ when $t = 0$.*

Corollary 4. *The derivative $(g \circ f)'(x)$ of the composition of differentiable real-valued functions equals the product $g'(f(x)) \cdot f'(x)$ of the derivatives of these functions computed at the corresponding points.*

Corollary 5. *If the composition $(f_n \circ \cdots \circ f_2 \circ f_1)(x)$ of differentiable functions $y_1 = f_1(x), y_2 = f_2(y_1), \ldots, y_n = f_n(y_{n-1})$ exists, then*

$$(f_n \circ \cdots \circ f_2 \circ f_1)'(x) = f_n'(y_{n-1})f_{n-1}'(y_{n-2}) \cdots f_2'(y_1)f_1'(x)$$

3 Differentiation of an Inverse Function

Theorem 3 (The Derivative of an Inverse Function)**.** *Let the functions $f : X \to Y$ and $f^{-1} : Y \to X$ be mutually inverse and continuous at points $x_0 \in X$ and $f(x_0) = y_0 \in Y$ respectively. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is also differentiable at the points y_0, and*

$$(f^{-1})'(y_0) = f'(x_0)^{-1}$$

证明. Hint:

- f and f^{-1} are mutually inverse $\Rightarrow ((x = f^{-1}(y) \neq x_0 = f^{-1}(y_0)) \Leftrightarrow (y = f(x) \neq y_0 = f(x_0)))$

- f and f^{-1} are continuous at x_0 and $y_0 = f(x_0)$ respectively $\Rightarrow (X \ni x \to x_0) \Leftrightarrow (Y \ni y \to y_0)$

□

4 Table of Derivatives of the Basic Elementary Functions

5 Differentiation of a Very Simple Implicit Function

Example 1 (The Law of Addition of Velocities)**.**

6 Higher-Order Derivatives

Definition 1 (Higher-Order Derivatives)**.** *By induction, if the derivative $f^{(n-1)}(x)$ of order $n-1$ of f has been defined, then the derivative of order n is defined by the formula*

$$f^{(n)}(x) := (f^{(n-1)})'(x)$$

In convention, the derivative of order n is denoted by $f^{(n)}(x)$ or $\frac{d^n f(x)}{dx^n}$

Example 2 (Leibniz's Formula)**.** *Let $u(x)$ and $v(x)$ be functions having derivatives up to order n inclusive on a common set E.*

$$(uv)^{(n)} = \Sigma_{m=0}^{n} \binom{n}{m} u^{(n-m)} v^{(m)} \tag{1}$$

证明. Hint:

- By principle of induction.
- Use the formula of binomial coefficient: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

□

Example 3 (Finding the Coefficients of the Talor's Formula)**.** *If* $P_n(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, *then*

$$c_0 =,$$

$$c_1 =,$$

$$c_2 =,$$

$$c_n$$

Thus,

$$P_n(x) =$$

Example 4 (Acceleration)**.**

Example 5 (The Second Derivative of a Simple Implicit Function)**.** *Let* $y = y(t)$ *and* $x = x(t)$

$$\begin{aligned} y^{(n)}_{x^n} &= \frac{d}{dx}(y^{(n-1)}_{x^{n-1}}) \\ &= \frac{d}{dt}(y^{(n-1)}_{x^{n-1}})\frac{dt}{dx} \\ &= \frac{(y^{(n-1)}_{x^{n-1}})'_t}{x'_t} \end{aligned}$$