Notes of "Polynomials"

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1 一元多项式

Definition 1 (数域 P 上的一元多项式).

Definition 2 (一元多项式相等及零多项式).

Definition 3 (数域 P 上的一元多项式环).

2 整除的概念

Proposition 1 (带余除法的成立).

$$f(x) = q(x)g(x) + r(x) \tag{1}$$

$q(x)$ and $r(x)$ are uniquely determined. $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

Remark 1 (综合除法). 对于除式是次数为 *1* 的多项式的情况，我们有一个快速进行带余除法的技巧，叫做综合除法。它基于以下观察

$$f(x) = \Sigma_{i=0}^{n} a_i x^i, g(x) = x - c, q(x) = \Sigma_{i=0}^{n-1} b_i x^i, r(x) = r$$

$$q(x)g(x)+r(x) = b_{n-1}x^n+(b_{n-2}-cb_{n-1})x^{n-1}+(b_{n-3}-cb_{n-2})x^{n-2}+\cdots+(b_0-cb_1)x+r-cb_0 = \Sigma_{i=0}^{n} a_i x^i = f(x)$$

则系数间有如下关系:

$$\begin{cases} b_{n-1} = a_n \\ b_{n-2} = a_{n-1} + cb_{n-1} \\ b_{n-3} = a_{n-2} + cb_{n-2} \\ \dots \\ b_1 = a_2 + cb_2 \\ b_0 = a_1 + cb_1 \\ r = a_0 + cb_0 \end{cases}$$

根据上述关系，我们可以快速得到带余除法的结果。

注意上述关系只在除式 $g(x) = x + c$ 即一次项系数为 1 才成立，对于一次项系数不是 1 的除式，则有 $g(x) = c_1x + c_0 = c_1(x + \frac{c_0}{c_1}) = c_1 g'(x)$.

Definition 4 (整除).

Properties of Divisors:

- 对偶等价于相差常数倍
- 传递性
- 整除组合

3 最大公因式

Definition 5 (Common Divisor and Greatest Common Divisor)**.** *If $\phi(x)$ is a divisor of both $f(x)$ and $g(x)$, we say it is a common divisor of $f(x)$ and $g(x)$.*

Suppose $f(x) \in P[x]$ and $g(x) \in P[x]$. A polynomial $d(x) \in P[x]$ is the greatest common divisor of $f(x)$ and $g(x)$ if the following conditions are true:

- *$d(x)$ is a common divisor of $f(x)$ and $g(x)$.*
- *Every common divisor of $f(x)$ and $g(x)$ is a divisor of $d(x)$.*

Remark 2. *Since $\forall f(x) \in P[x]$ is a divisor of the zero polynomial, in other words $0 = 0 \cdot f(x)$, the greatest common divisor of $f(x)$ and 0 is 0. Especially, the greatest common divisor of 0 and 0 is 0, which is in accordance with the definition of GCD.*

Lemma 1 (Common Divisors in Euclidean Division)**.** *If it holds that $f(x) = q(x)g(x) + r(x)$ for $f(x) \in P[x]$ and $g(x) \in P[x]$, then the two pairs of polynomials $(f(x), g(x))$ and $(g(x), r(x))$ have the same common divisors.*

证明. □

Theorem 1 (Theorem of Polynomial Greatest Common Divisor)**.** *$\forall f(x) \in P[x]$ and $\forall g(x) \in P[x]$, there exists $d(x) \in P[x]$ that is the greatest common divisor of $f(x)$ and $g(x)$, and $d(x)$ can be expressed as a combination of $f(x)$ and $g(x)$, which is $\exists u(x) \in P[x], v(x) \in P[x]$ such that*

$$d(x) = u(x)f(x) + v(x)g(x)$$

证明. (TODO) □

Remark 3 (Euclid's Algorithm)**.**

Definition 6 (Coprime)**.** *(f(x), g(x)) = 1*

Theorem 2 (互素的等价条件)**.**

$$((f(x), g(x)) = 1) \Leftrightarrow u(x)f(x) + v(x)g(x) = 1$$

Remark 4. *This theorem contains a kind of symmetry because both $f(x)$ and $g(x)$ and $u(x)$ and $v(x)$ are coprime.*

Theorem 3. *If $(f(x), g(x)) = 1$, and $f(x)|g(x)h(x)$, then $f(x)|h(x)$.*

Corollary 1. *If $f_1(x)|g(x)$, $f_2(x)|g(x)$, and $(f_1(x), f_2(x)) = 1$, then $f_1(x)f_2(x)|g(x)$.*

上述结论均可推广到多个多项式。

4 因式分解定理

Definition 7 (不可约多项式)**.**

Theorem 4 (不可约多项式作因式)**.**

Theorem 5 (因式分解定理)**.**

5 重因式

6 多项式函数