Notes of ”Basic Topology”

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1 Metric Spaces

Definition 1 (Euclidean Space)

Theorem 1 (Properties of Elements in Euclidean Space)

Definition 2 (Distance Function and Metric Space)

Definition 3 (Segment, Interval, and k-Cell)

2 Point Set

Definition 4 (Neighborhood, Limit Point, Isolated Point, and Interior Point)

Remark 1
A point p is a limit point of a set E doesn’t require $p \in E$.

Definition 5 (Closed, Open, Perfect, Bounded, and Dense Set)

Theorem 2
Every neighborhood is an open set.

证明. Hint: Recall the definition of a neighborhood. □

Theorem 3 (Property of Neighborhoods of a Limit Point)
If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

证明. Hint: Proof by contradiction. □

The converse is easy to prove by the definition of a limit point. Then it leads to another definition of a limit point.

Definition 6 (The Second Definition of a Limit Point)
A point p is a limit point of a set E if every neighborhood of p contains infinitely many points of E.

Corollary 1
A finite point set has no limit points.

Theorem 4
Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c) \tag{1}$$

证明. Hint: To prove the subset relation of both directions, consider a random element in the corresponding side. □

Theorem 5 (Open and Close for a Set and its Complement)
A set E is open if and only if its complement is closed.

Corollary 2
A set E is closed if and only if its complement is open.

Theorem 6 (Open and Close for Unions and Intersections)
Suppose $\{G_\alpha\}$ is a collection of open sets and $\{F_\alpha\}$ is a collection of closed sets.

- $\bigcup_\alpha G_\alpha$ *is open.*
- $\bigcap_\alpha F_\alpha$ *is closed.*
- *If $\{G_\alpha\}$ has finite number of elements, then $\bigcap_{i=1}^n G_i$ is open.*
- *If $\{F_\alpha\}$ has finite number of elements, then $\bigcup_{i=1}^n F_i$ is closed.*

证明. Hint:(TODO) □

Definition 7 (Closure)
If X is a metric space, $E \subset X$, and E' denotes the set of all limit points of E in X, then the closure of E is the set $\bar{E} = E \cup E'$.

Theorem 7 (Closure and Closed Set)
If X is a metric space and $E \subset X$, then

- $\bar{E}$ *is closed.*
- $E = \bar{E}$ *if and only if E is closed.*
- $\bar{E} \subset F$ *for every closed set $F \subset X$ and $E \subset F$.*
- $\bar{E}$ *is the smallest closed set that contains E.*

证明. Hint:(TODO) □

Theorem 8 (Least Upper Bound and Limit Point)
Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Definition 8 (Relative Open Subset)
Suppose $E \subset Y \subset X$, where X is a metric space. We say that E is open relative to Y if for each $p \in E$ there is an associated $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

Remark 2
A set E may be open relative to Y without being an open subset in X. For example, a segment (a, b) is not an open subset in $\mathbb{R}^2$ but it is open relative to segments Y containing it, such as $(a-1, b+1)$. In other words, relative open subset is a weaker property than open subset.

Theorem 9 (Relation Between Relative Open Subset and Open Subset)
Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

证明. Hint:(TODO) □

3 Compact Set