

Notes of "The Limit of A Function"

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1 Definitions and Examples

Definition 1 (The Limit of a Function (Basic Type)). (ϵ - δ)

Remark 1. *Based on the definition, we can see that limits are a kind of local characteristic of a function. With that, when using the definition to prove the limit of a function at $x = x_0$, we can discuss it within a certain neighborhood $O(x_0, \delta_0)$.*

Definition 2 (去心邻域).

Definition 3 (函数极限的邻域定义).

Example 1 (符号函数 $\operatorname{sgn} x$).

Proposition 1 (Heine's Proposition).

Corollary 1 (Existence of the Limit of a Function by Limits of Sequences).

2 Properties of the Limit of a Function

2.1 General Properties

Definition 4 (有界函数/局部有界函数, 上有界函数/局部上有界函数, 下有界函数/局部下有界函数).

Theorem 1. 1. (局部常值函数收敛)

2. (局部有界)

3. (在某一点处极限值唯一)

2.2 Properties Involving Arithmetic

Definition 5 (两个函数的和、积与商).

Theorem 2 (四则运算中的函数极限).

Definition 6 (无穷小函数).

Proposition 2 (无穷小函数的四则运算性质).

2.3 Properties Involving Inequalities

Theorem 3. • (局部保序性)

- (夹逼性)

Corollary 2 (局部保号性).

Corollary 3 (极限值的不等关系).

Example 2 (研究 $\frac{\sin x}{x}$ 在 $x = 0$ 处的极限).

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (1)$$

Example 3 (定义指数函数、对数函数和幂函数).

3 The General Definition of the Limit of a Function

3.1 Definition and Examples of a Base

3.2 Limit over a Base

4 The Existence of the Limit of a Function

4.1 Cauchy's Criterion

Definition 7 (Oscillation).

Theorem 4 (Cauchy's Criterion on the Limit of a Function).

4.2 The Limit of a Composite Function

Theorem 5 (Theorem of the Limit of a Composite Function).

Example 4.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad (2)$$

4.3 The Limit of a Monotonic Function

Definition 8 (Monotonic Functions). A function $f : E \rightarrow \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is said to be

- increasing if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$
- nondecreasing if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2))$
- decreasing if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$
- nonincreasing if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2))$

Remark 2. Be mindful of that the monotonicity is defined on the complete domain of the definition, and thus the condition is $\forall x_1, x_2 \in E$ no matter what a set E is.

Theorem 6 (The Existence Criterion of the Limit of a Monotonic Function).

4.4 Comparison of the Limiting Behaviors of Functions

Proposition 3.

Remark 3. The above proposition can not be generalized to the limit of a sum of functions.

5 方法与技巧

5.1 证明与研究函数极限

- 定义法
- 变量代换

Remark 4. 在研究函数极限时使用变量代换是否总是成立？这个问题使用数学语言来描述如下：

Suppose that $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$, is it true that

$$\lim_{x \rightarrow a} f(g(x)) = \lim_{y \rightarrow A} f(y)$$

In fact it is not always true. Here is a counterexample: Let $g(x) \equiv 0$, and thus $\lim_{x \rightarrow 0} g(x) = 0$.

Let $f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$. Then we have $\lim_{x \rightarrow 0} f(x) = 0$, $\lim_{x \rightarrow 0} f(g(x)) = 1$. 这里的数学直观是

$\lim_{x \rightarrow a} g(x) = A$ 决定了 $y = g(x) \rightarrow A$ 的方式。它与 $\lim_{x \rightarrow A} f(x) = B$ 中 $x \rightarrow A$ 的不同可能导致结果的不同。

Here are two propositions related to this problem:

Proposition 4. Suppose that $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$. If any of the following conditions is true:

- $\exists \delta_0 > 0$ such that $\forall x \in O(a, \delta_0) \setminus \{a\}$: $g(x) \neq A$.
- $\lim_{x \rightarrow A} f(x) = f(A)$.
- $A = \infty$, and $\lim_{x \rightarrow \infty} f(x)$ is defined.

then the following is true:

$$\lim_{x \rightarrow a} f(g(x)) = \lim_{y \rightarrow A} f(y)$$

证明. Hint:

- Notice the difference between the conclusion of the definition of $\lim_{x \rightarrow a} g(x) = A$, and the condition of the definition of $\lim_{x \rightarrow A} f(x) = B$.
- Notice what change the fact of continuity brings to the definition of $\lim_{x \rightarrow A} f(x) = B$.

□

Proposition 5. If $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$, then exact one of the following situation is true:

- $\lim_{x \rightarrow a} f(g(x)) = B$

- $\lim_{x \rightarrow a} f(g(x)) = g(A)$
- $\lim_{x \rightarrow a} f(g(x))$ *is not defined*

证明. Hint: Using the first condition of the previous proposition to discuss different kinds of $g(x)$. \square