Notes of "The Limit of A Sequence"

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1 Definition of the Limit of a Sequence

Definition 1 (The Limit of a Sequence)**.** *(ϵ-N)*

(neighborhood-N)

Remark 1. *The two definitions of the limit of a sequence is equivalent. The equivalence relation between them relies on the definition of a neighborhood.*

Definition 2 (Divergence)**.**

2 Properties of the Limit of a Sequence

2.1 General Properties

- 有限点无关性: A finite number of terms of a sequence doesn't affect the convergence of the sequence. (Proof: definition)
- 唯一性： The limit of a convergent sequence is unique. (Proof: contradiction + definition)
- 有界性： A convergent sequence is bounded. (Proof: definition)

2.2 Properties Involving Arithmetic Operations

Theorem 1 (极限的四则运算)**.**

2.3 Properties Involving Inequalities

Theorem 2 (保序性)**.**

Theorem 3 (夹逼性)**.**

3 Infinity

3.1 Definition of Infinity

Definition 3 (Infinity, Positive Infinity, Negative Infinity)**.**

Corollary 1 (Relation between Infinity and Infinitesimal)**.**

Definition 4 (Not an Infinity)**.**

3.2 Operations Involving Infinity

4 Existence of the Limit of a Sequence

4.1 Cauchy's Convergence Criterion

4.2 Existence of the Limit of a Monotonic Sequence

4.2.1 e

Proposition 1. *The sequences* $a_n = (1+\frac{1}{n})^n$ *and* $b_n = (1+\frac{1}{n})^{n+1}$ *are convergent, and they have the same limit values.*

证明. □

Definition 5.

Proposition 2. $e = \Sigma_{n=0}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$

证明. □

Proposition 3. $e = \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$

证明. □

4.2.2 pi

4.2.3 Euler Number

Theorem 4 (Weierstrass Theorem / 单调有界数列收敛定理)**.**

Corollary 2. 当数列严格单调增加时,

4.3 Subsequences and the Partial Limits

4.4 闭区间套定理

5 The Limit of Transformed Sequences

5.1 Undertermined Form

5.2 Toeplitz's Theorem

Theorem 5. *Suppose there exists a sequence* $\{t_{nk}\}$ *such that* $\forall n, k \in \mathbb{N}^+$, $t_{nk} \geq 0$, $\Sigma_{k=1}^{n} t_{nk} = 1$, $\lim_{n\to\infty} t_{nk} = 0$. *If* $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty}\Sigma_{k=1}^{n} t_{nk} a_k = a$$

Remark 2. *The condition in the Toeplitz' Theorem* $\lim_{n\to\infty} t_{nk} = 0$ *means that for any given* k, *in other words* k *is finite,* t_{nk} *tends to* 0 *when* n *tends to* ∞. *This is supported by the proof, since in the proof we only need the first finite number of terms in the sequence* $\{t_{nk}\}$ *to converge to* 0.

5.3 Stolz's Theorem

Theorem 6 ($\frac{0}{0}$ type)**.** *Suppose* $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, *and* $\{a_n\}$ *is decreasing. If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

Theorem 7 ($\frac{*}{\infty}$ type)**.** *Suppose* $\{a_n\}$ *is increasing and* $\lim_{n\to\infty} a_n = \infty$*. If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

证明. Method: By Toeplitz's Theorem

$$t_nk = \{\frac{a_1}{a_n}, \frac{a_2 - a_1}{a_n}, \frac{a_3 - a_2}{a_n}, \cdots, \frac{a_n - a_{n-1}}{a_n}\}$$

$$c_n = \{\frac{b_1}{a_1}, \frac{b_2 - b_1}{a_2 - a_1}, \frac{b_3 - b_2}{a_3 - a_2}, \cdots, \frac{b_n - b_{n-1}}{a_n - a_{n-1}}\}$$

□

5.4 Cauchy's Proposition

Proposition 4 (算术平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a$*, then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

Proposition 5 (算术平均值等价形式)**.** *If* $\lim_{n\to\infty}(a_n - a_{n-1}) = a$*, then*

$$\lim_{n\to\infty} \frac{a_n}{n} = a$$

Proposition 6 (几何平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a > 0$*, then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a$$

6 方法与技巧

6.1 求数列极限

- 定义法：使用的前提是已知极限值。
- 夹逼法
- Cauchy Proposition, Stolz Theorem, Topelitz Theorem
- 单调有界数列收敛原理
- Cauchy's Convergence Criterion

6.2 判定数列发散

- 利用数列发散的定义（数列收敛定义的否命题）
- 利用收敛数列性质的逆否命题
  - 无界数列一定发散
  - 有两个收敛到不同极限值的子列的数列一定发散
- 考察子列特性: 有一个发散子列的数列一定发散
- Cauchy's Convergence Criterion

6.3 考察数列单调性

6.4 数列放缩技巧

- 加一项减一项结合三角不等式