Notes of "General Theory of Affine Coordinate Transformation"

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1 Transformation Matrix and Transformation Formulas of Vectors and Points in Two Coordinate Systems

1.1 Transformation Formula of Points in Two Coordinate Systems

Given the coordinate (x', y', z') in I' of a point M, how to find its corresponding coordinate (x, y, z) in I?

- $M(x', y', z')$ in $I' \Leftrightarrow O'M(x', y', z')$ in I'
- $O'M$ in I has the coordinate $C[x', y', z']$
- Suppose OO' in I, in other words, the coordinate of the point O' in I, is d_1, d_2, d_3
- $OM = OO' + O'M$ has the coordinate $[x, y, z] = C[x', y', z'] + [d_1, d_2, d_3]$
- $OM(x, y, z)$ in $I \Leftrightarrow M(x, y, z)$ in I

$$\begin{cases} x = c_{11}x' + c_{12}y' + c_{13}z' + d_1 \\ y = c_{21}x' + c_{22}y' + c_{23}z' + d_2 \\ z = c_{31}x' + c_{32}y' + c_{33}z' + d_3 \end{cases} \tag{1}$$

2 Properties of Transformation Matrices

Proposition 1. *All transformation matrix between two affine coordinate systems are inversible matrices.*

Proposition 2. *If the transformation matrix from I to I' is C_1, and the transformation matrix from I' to I'' is C_2, then the transformation matrix from I to I'' is C_1C_2.*

Corollary 1. *If the transformation matrix from I to I' is C, then the transformation matrix from I' to I is C^{-1}.*

Proposition 3. *Suppose C is the transformation matrix from I to I'. I and I' has the same orientation if and only if $|C| > 0$. I and I' has the opposite orientation if and only if $|C| < 0$*

证明.

$$\begin{pmatrix} \vec{e'}_1 \\ \vec{e'}_2 \\ \vec{e'}_3 \end{pmatrix} = C^T \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}$$

$$\begin{vmatrix} \vec{e'}_1 \\ \vec{e'}_2 \\ \vec{e'}_3 \end{vmatrix} = |C^T| \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}$$

$$(\vec{e'}_1, \vec{e'}_2, \vec{e'}_3) = |C|(\vec{e}_1, \vec{e}_2, \vec{e}_3)$$

□

3 Transformation between Cartesian Coordinate Systems

$I[O; e_1, e_2, e_3]$ $I'[O'; e'_1, e'_2, e'_3]$. The transformation matrix from I to I' is

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

which is equivalent to

$$\begin{cases} \vec{e'}_1 = c_{11}\vec{e}_1 + c_{21}\vec{e}_2 + c_{31}\vec{e}_3 \\ \vec{e'}_2 = c_{12}e_1 + c_{22}e_2 + c_{32}e_3 \\ \vec{e'}_3 = c_{13}e_1 + c_{23}e_2 + c_{33}e_3 \end{cases}$$

$$\vec{e'}_i \cdot \vec{e'}_j = c_{1i}c_{1j} + c_{2i}c_{2j} + c_{3i}c_{3j}$$

Since I' is a Cartesian coordinate system,

$$\vec{e'}_i \cdot \vec{e'}_j = \delta_{ij}$$

$$c_{1i}c_{1j} + c_{2i}c_{2j} + c_{3i}c_{3j} = \delta_{ij}$$

The above equation means that the inner product of two column vectors of C is 0, and the inner product between a column vector of C and itself is 1. Using matrix multiplication to express it:

$$C^TC = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} \vec{e'}_1 \cdot \vec{e'}_1 & \vec{e'}_2 \cdot \vec{e'}_1 & \vec{e'}_3 \cdot \vec{e'}_1 \\ \vec{e'}_1 \cdot \vec{e'}_2 & \vec{e'}_2 \cdot \vec{e'}_2 & \vec{e'}_3 \cdot \vec{e'}_2 \\ \vec{e'}_1 \cdot \vec{e'}_3 & \vec{e'}_2 \cdot \vec{e'}_3 & \vec{e'}_3 \cdot \vec{e'}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

which shows that C is an orthogonal matrix.

Remark 1. *It is clear that the properties of the basis of the two Cartesian coordinate systems, namely the vectors in a basis are unit vectors and they are orthogonal to each other, decide that the transformation matrix between them is an orthogonal matrix.*

Therefore, we have the following proposition:

Proposition 4. *The transformation matrix between two Cartesian coordinate systems is an orthogonal matrix.*

By the property of an inverse matrix, it also holds that

$$CC^T = C^TC = E$$

which means that the inner product of two row vectors of C is 0, and the inner product between a row vector of C and itself is 1.

3.1 Transformation between Two Cartesian Coordinate Systems in a Plane

Suppose C is the transformation matrix between two Cartesian coordinate systems in a plane:

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

According to the above proposition, C is an orthogonal matrix, thus it satisfies two conditions:

1. $c_{11}^2 + c_{12}^2 = c_{11}^2 + c_{21}^2 = c_{12}^2 + c_{22}^2 = c_{21}^2 + c_{22}^2 = 1$
2. $c_{11}c_{12} + c_{21}c_{22} = c_{11}c_{21} + c_{12}c_{22} = 0$

$$C = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$C = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

in which $\theta \in [0, 2\pi)$