Notes of "Linear Mapping and Matrix Operations"

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1 Linear Mapping and Matrix

2 Matrix Multiplication

2.1 Block Matrix Multiplication

3 Matrix Transposition

4 Rank of a Product of Matrices

5 Square Matrix

5.1 Commuting Matrices

Definition 1 (Diagonal Matrix and Scalar Matrix)**.** *A diagonal matrix is a a matrix in which all entries outside the main diagonal are all zero.*

Definition 2 (Identity Matrix)**.**

$$E = (\delta_{ij}), \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \tag{1}$$

Definition 3 (Scalar Matrix)**.** *A scalar matrix is the result of the identity matrix multiplied with a scalar.*

Definition 4 (Matrix Unit)**.** *A matrix unit is a matrix with only one nonzero entry with value 1. The matrix unit with the nonzero entry in the i-th row and j-th column is denoted as E_{ij}*

Remark 1. *A matrix unit is not necessarily a square matrix.*

Definition 5 (Commuting Matrices)**.** *Given two matrices $A, B \in M_n(\mathbb{R})$, A and B are said to commute if $AB = BA$, or equivalently their commutator $[A, B] = AB - BA = 0$.*

Remark 2. *Notice that commuting matrices must be square matrix, because if $A \in M_{s\times n}, B \in M_{n\times r}, s \neq r$, then $AB \in M_{s\times r}, BA \in M_{r\times s}$, hence it is invalid to compare them.*

Theorem 1. *If a matrix $A \in M_n(\mathbb{R})$ commutes with $\forall B \in M_n(\mathbb{R})$, then A is a scalar matrix.*

证明. Hint:

- If $AE_{12} = E_{12}A$, then $\forall k \neq 1, a_{k1} = 0$, and $\forall k \neq 2, a_{2k} = 0$, and $a_{11} = a_{22}$.
- If $AE_{ij} = E_{ij}A$, then $\forall k \neq i, a_{ki} = 0$, and $\forall k \neq j, a_{jk} = 0$, and $a_{ii} = a_{jj}$.
- Consider $\forall B \in M_n\mathbb{R}, AB = BA$.

□

Remark 3. *When proving a property applies to any matrix in $M_{m\times n}$, one method is to consider all matrix units in $M_{m\times n}$, since the set of matrix units is a basis of $M_{m\times n}$.*

5.2 Inverse Matrix

Lemma 1 (Uniqueness of Inverse Matrix)**.**

$$A' = A'E = A'(AA'') = (A'A)A'' = A''$$

Definition 6 (Inverse Matrix)**.**

Definition 7 (Non-degenerate Matrix and Degenerate Matrix)**.** *A matrix $A \in M_n(\mathbb{R})$ is non-degenerate if* $\operatorname{rank} A = n$. *$A$ is degenerate if* $\operatorname{rank} A < n$.

Remark 4. *We only talk about non-degenerate matrices and degenerate matrices when it comes to square matrices.*

Theorem 2. *$A \in M_n(\mathbb{R})$ is non-degenerate if and only if A is inversible.*

证明. Hint:
$\Leftarrow$: Use the rank of the product of two matrices.
$\Rightarrow$: Use the unqiueness of the solution of $AX = 0$, and the transpose of the product of two matrices. □

Corollary 1. *If $A \in M_n\mathbb{R}$ is inversible, then A^T is inversible, and $(A^T)^{-1} = (A^{-1})^T$.*

Corollary 2. *If $B \in M_m(\mathbb{R})$ and $C \in M_n(\mathbb{R})$ are non-degenerate, $\forall A \in M_{m\times n}(\mathbb{R})$, it holds that*

$$\operatorname{rank} BAC = \operatorname{rank} A$$

Corollary 3. *If $A, B \in M_n(\mathbb{R})$, and $AB = E \vee BA = E$, then $B = A^{-1}$.*

Corollary 4. *If $A, B, \ldots, C, D \in M_n(\mathbb{R})$ are non-degenerate, then $AB\cdots CD$ is non-degenerate, and its inverse matrix is*

$$(AB\cdots CD)^{-1} = D^{-1}C^{-1}\cdots B^{-1}A^{-1}$$

5.3 Calculation of Powers of a Matrix

Example 1 (Powers of a Scalar Matrix)**.**

Example 2.

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$$

Example 3.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

6 Equivalence Class of Matrices

Definition 8 (Elementary Matrix)**.**

Theorem 3. *$M_{m\times n}(\mathbb{R})$ has a partition of $\min(m,n)+1$ equivalence classes. All matrices with its rank as r including the representative element is in a equivalence class.*

Corollary 5. *Every non-degenerate matrix $A \in M_n(\mathbb{R})$ can be expressed as the product of elementary matrices.*

7 Calculation of Inverse Matrix

8 Solution Space