

Notes of "The Limit of A Sequence"

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1 Definition of the Limit of a Sequence

Definition 1 (The Limit of a Sequence). (ϵ - N)
(neighborhood- N)

Remark 1. *The two definitions of the limit of a sequence is equivalent. The equivalence relation between them relies on the definition of a neighborhood.*

Definition 2 (Divergence).

2 Properties of the Limit of a Sequence

2.1 General Properties

- 有限点无关性: A finite number of terms of a sequence doesn't affect the convergence of the sequence. (Proof: definition)
- 唯一性: The limit of a convergent sequence is unique. (Proof: contradiction + definition)
- 有界性: A convergent sequence is bounded. (Proof: definition)

2.2 Properties Involving Arithmetic Operations

Theorem 1 (极限的四则运算).

2.3 Properties Involving Inequalities

Theorem 2 (保序性).

Theorem 3 (夹逼性).

3 Infinity

3.1 Definition of Infinity

Definition 3 (Infinity, Positive Infinity, Negative Infinity).

Corollary 1 (Relation between Infinity and Infinitesimal).

Definition 4 (Not an Infinity).

3.2 Operations Involving Infinity

3.3 待定型求极限: Stolz Theorem

4 Existence of the Limit of a Sequence

4.1 Cauchy's Convergence Criterion

4.2 Existence of the Limit of a Monotonic Sequence

4.2.1 e

4.2.2 pi

4.2.3 Euler Number

Theorem 4 (Weierstrass Theorem / 单调有界数列收敛定理).

Corollary 2. 当数列严格单调增加时,

4.3 Subsequences and the Partial Limits

4.4 闭区间套定理

5 The Limit of Transformed Sequences

5.1 Cauchy's Proposition

Proposition 1 (算术平均值形式). *If $\lim_{n \rightarrow \infty} a_n = a$, then*

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

Proposition 2 (算术平均值等价形式). *If $\lim_{n \rightarrow \infty} (a_n - a_{n-1}) = a$, then*

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = a$$

Proposition 3 (几何平均值形式). *If $\lim_{n \rightarrow \infty} a_n = a > 0$, then*

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a$$

5.2 Stolz's Theorem

Theorem 5 ($\frac{0}{0}$ type). *Suppose $\lim_{n \rightarrow \infty} a_n = 0$, $\lim_{n \rightarrow \infty} b_n = 0$, and $\{a_n\}$ is decreasing. If*

$$\lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = l$$

Theorem 6 ($\frac{*}{\infty}$ type). Suppose $\{a_n\}$ is increasing and $\lim_{n \rightarrow \infty} a_n = \infty$. If

$$\lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = l$$

证明. Method: By Toeplitz's Theorem

$$t_{nk} = \left\{ \frac{a_1}{a_n}, \frac{a_2 - a_1}{a_n}, \frac{a_3 - a_2}{a_n}, \dots, \frac{a_n - a_{n-1}}{a_n} \right\}$$

$$c_n = \left\{ \frac{b_1}{a_1}, \frac{b_2 - b_1}{a_2 - a_1}, \frac{b_3 - b_2}{a_3 - a_2}, \dots, \frac{b_n - b_{n-1}}{a_n - a_{n-1}} \right\}$$

□

5.3 Toeplitz's Theorem

Theorem 7. Suppose there exists a sequence $\{t_{nk}\}$ such that $\forall n, k \in \mathbb{N}^+$, $t_{nk} \geq 0$, $\sum_{k=1}^n t_{nk} = 1$, $\lim_{n \rightarrow \infty} t_{nk} = 0$. If $\lim_{n \rightarrow \infty} a_n = a$, then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n t_{nk} a_k = a$$

Remark 2. The condition in the Toeplitz' Theorem $\lim_{n \rightarrow \infty} t_{nk} = 0$ means that for any given k , in other words k is finite, t_{nk} tends to 0 when n tends to ∞ . This is supported by the proof, since in the proof we only need the first finite number of terms in the sequence $\{t_{nk}\}$ to converge to 0.

6 方法与技巧

6.1 求数列极限

- 定义法: 使用的前提是已知极限值。
- 夹逼法
- Cauchy Proposition, Stolz Theorem, Topelitz Theorem
- 单调有界数列收敛原理
- Cauchy's Convergence Criterion

6.2 判定数列发散

- 利用数列发散的定义 (数列收敛定义的反命题)
- 利用收敛数列性质的逆否命题
 - 无界数列一定发散
 - 有两个收敛到不同极限值的子列的数列一定发散
- 考察子列特性: 有一个发散子列的数列一定发散
- Cauchy's Convergence Criterion

6.3 考察数列单调性

6.4 数列放缩技巧

- 加一项减一项结合三角不等式