Notes of ”The Limit of A Function”

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1 Definitions and Examples

Definition 1 (The Limit of a Function (Basic Type))**.** *(ϵ-δ)*

Remark 1. *Based on the definition, we can see that limits are a kind of local characteristic of a function. With that, when using the definition to prove the limit of a function at $x = x_0$, we can discuss it within a certain neighborhood $O(x_0, \delta_0)$.*

Definition 2 (去心邻域)**.**

Definition 3 (函数极限的邻域定义)**.**

Example 1 (符号函数 $\operatorname{sgn} x$)**.**

Proposition 1 (Heine's Proposition)**.**

Corollary 1 (Existence of the Limit of a Function by Limits of Sequences)**.**

2 Properties of the Limit of a Function

2.1 General Properties

Definition 4 (有界函数/局部有界函数，上有界函数/局部上有界函数，下有界函数/局部下有界函数)**.**

Theorem 1.

1. *(局部常值函数收敛)*
2. *(局部有界)*
3. *(在某一点处极限值唯一)*

2.2 Properties Involving Arithmetic

Definition 5 (两个函数的和、积与商)**.**

Theorem 2 (四则运算中的函数极限)**.**

Definition 6 (无穷小函数)**.**

Proposition 2 (无穷小函数的四则运算性质)**.**

2.3 Properties Involving Inequalities

Theorem 3.
- *(局部保序性)*
- *(夹逼性)*

Corollary 2 (局部保号性)**.**

Corollary 3 (极限值的不等关系)**.**

Example 2 (研究 $\frac{\sin x}{x}$ 在 $x = 0$ 处的极限)**.**

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{1}$$

Example 3 (定义指数函数、对数函数和幂函数)**.**

3 The General Definition of the Limit of a Function

3.1 Definition and Examples of a Base

3.2 Limit over a Base

4 The Existence of the Limit of a Function

4.1 Cauchy's Criterion

Definition 7 (Oscillation)**.**

Theorem 4 (Cauchy's Criterion on the Limit of a Function)**.**

4.2 The Limit of a Composite Function

Theorem 5 (Theorem of the Limit of a Composite Function)**.**

Example 4.

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e \tag{2}$$

4.3 The Limit of a Monotonic Function

Definition 8 (Monotonic Functions)**.**

Theorem 6 (The Existence Criterion of the Limit of a Monotonic Function)**.**

4.4 Comparison of the Limiting Behaviors of Functions

Proposition 3.

Remark 2. *The above proposition can not be generalized to the limit of a sum of functions.*

5 方法与技巧

5.1 证明与研究函数极限

- 定义法
- 变量代换

Remark 3. 在研究函数极限时使用变量代换是否总是成立？这个问题使用数学语言来描述如下：*Suppose that* $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$, *is it true that*

$$\lim_{x\to a} f(g(x)) = \lim_{y\to A} f(y)$$

In fact it is not always true. Here is a counterexample: Let $g(x) \equiv 0$, *and thus* $\lim_{x\to 0} g(x) = 0$. *Let* $f(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}$. *Then we have* $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} f(g(x)) = 1$. 这里的数学直观是 $\lim_{x\to a} g(x) = A$ 决定了 $y = g(x) \to A$ 的方式。它与 $\lim_{x\to A} f(x) = B$ 中 $x \to A$ 的不同可能导致结果的不同。

Here are two propositions related to this problem:

Proposition 4. *Suppose that* $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$. *If any of the following conditions is true:*

- $\exists \delta_0 > 0$ *such that* $\forall x \in O(a, \delta_0) \setminus \{a\}$: $g(x) \neq A$.
- $\lim_{x\to A} f(x) = f(A)$.
- $A = \infty$, *and* $\lim_{x\to\infty} f(x)$ *is defined.*

then the following is true:

$$\lim_{x\to a} f(g(x)) = \lim_{y\to A} f(y)$$

证明. Hint:

- Notice the difference between the conclusion of the definition of $\lim_{x\to a} g(x) = A$, and the condition of the definition of $\lim_{x\to A} f(x) = B$.
- Notice what change the fact of continuity brings to the definition of $\lim_{x\to A} f(x) = B$.

□

Proposition 5. *If* $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$, *then exact one of the following situation is true:*

- $\lim_{x\to a} f(g(x)) = B$
- $\lim_{x\to a} f(g(x)) = g(A)$
- $\lim_{x\to a} f(g(x))$ *is not defined*

证明. Hint: Using the first condition of the previous proposition to discuss different kinds of $g(x)$. □