Notes of "Closed Bounded Subsets of $\mathbb{E}^n$"

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1 Overview

- Def: Bounded subset of a Euclidean space
- Rmk: The closed and bounded subsets in $\mathbb{E}^n$ are of special interest to us because of their presence
- Def: Open cover and subcover of a topological space
- Examples of open covers and subcovers of topological spaces
  - Eg: An open cover of $\mathbb{E}^1$: $\{(k-1, k+1) \mid k \in \mathbb{Z}\}$ and its subcovers
  - Eg: An open cover of $\mathbb{E}^2$: $\left\{\mathring{B}((x,1)) \mid x = (a,b) \text{ in which } a, b \in \mathbb{Z}\right\}$ and its subcovers
  - Eg: An open cover of $[0,1]$: $[0, \frac{1}{5})$, $(\frac{4}{5}, 1]$, $\left\{(\frac{1}{n}, 1 - \frac{1}{n}) \mid n \in \mathbb{N} \wedge n > 2\right\}$ and its subcovers
  - Eg: An open cover of $(0,1]$: $\left\{(\frac{1}{n}, 1] \mid n \in \mathbb{N} \wedge n > 1\right\}$ and its subcovers
- Thm: A necessary and sufficient condition for a subset of $\mathbb{E}^n$ is closed and bounded in terms of open covers
- Def: Compact topological space
- Rmk: Compactness is a topological property of a space

2 Content

Definition 1 (Bounded subset of a Euclidean space)**.** *A subset X of $\mathbb{E}^n$ is bounded if there exists a ball $B(O,r)$ in which O is the origin and $r > 0$ such that $X \subset B(O,r)$.*

Definition 2 (Open cover and subcover of a topological space)**.** *Let X be a topological space. A family $\mathfrak{F}$ of open subsets of X is called an open cover of X if the union of the family is X. If $\mathfrak{F}'$ is a subfamily of $\mathfrak{F}$ and $\bigcup \mathfrak{F}' = X$, then $\mathfrak{F}'$ is called a subcover of $\mathfrak{F}$.*

Example 1 (An open cover of $\mathbb{E}^1$: $\{(k-1, k+1) \mid k \in \mathbb{Z}\}$ and its subcovers)**.** *The family of open subsets of $\mathbb{E}^1$ $\mathfrak{F} = \{(k-1, k+1) \mid k \in \mathbb{Z}\}$ is clearly an open cover of $\mathbb{E}^1$.*

Notice that $\mathbb{E}^1$ is unbounded, and $\mathfrak{F}$ does not have any subcover. Suppose we remove the open interval centred at $k (k \in \mathbb{Z})$ in $\mathfrak{F}$, then the point $x = k$ is not covered by any other open intervals in $\mathfrak{F}$ since the distance between it and other integer point in $\mathbb{E}^1$ is greater than or equal to 1.

Example 2 (An open cover of $\mathbb{E}^2$: $\left\{\mathring{B}((x,1)) \mid x=(a,b) \text{ in which } a,b\in\mathbb{Z}\right\}$ and its subcovers)**.** *Analogous to the previous example, the family of open subsets of* $\mathbb{E}^2$ $\mathfrak{F}=\left\{\mathring{B}((x,1)) \mid x=(a,b) \text{ in which } a,b\in\mathbb{Z}\right\}$ *is an open cover of* $\mathbb{E}^2$*, but it does not have any subcover since removing any open subset in* $\mathfrak{F}$ *will cause the centre of the removed open subset to be uncovered.*

Example 3 (An open cover of $[0,1]$: $[0,\frac{1}{5})$, $(\frac{4}{5},1]$, $\left\{(\frac{1}{n},1-\frac{1}{n}) \mid n\in\mathbb{N}\wedge n>2\right\}$ and its subcovers)**.** *Let* $[0,1]$ *have the subspace topology of* $\mathbb{E}^1$*. The family of open sets* $\mathfrak{F}=\left\{[0,\frac{1}{5}),(\frac{4}{5},1]\right\}\cup\left\{(\frac{1}{n},1-\frac{1}{n}) \mid n\in\mathbb{N}\wedge n>2\right\}$ *is an open cover of* $[0,1]$*.*

Notice that $[0,1]$ *is a closed and bounded subset of* $\mathbb{E}^1$*, and* $\mathfrak{F}$ *has a finite subcover* $\mathfrak{F}'=\{[0,\frac{1}{5}),(\frac{4}{5},1],(\frac{1}{3},\frac{2}{3}),(\frac{1}{4},\frac{3}{4}),(\frac{1}{5},\frac{4}{5}),(\frac{1}{6},\frac{5}{6})\}$*.*

Example 4 (An open cover of $(0,1]$: $\left\{(\frac{1}{n},1] \mid n\in\mathbb{N}\wedge n>1\right\}$ and its subcovers)**.** *Let* $(0,1]$ *have the subspace topology of* $\mathbb{E}^1$*. The family of open sets* $\mathfrak{F}=\left\{(\frac{1}{n},1] \mid n\in\mathbb{N}\wedge n>1\right\}$ *is an open cover of* $(0,1]$*.*

Notice that $(0,1]$ *is not a closed subset of* $\mathbb{E}^1$*, and* $\mathfrak{F}$ *does not have a finite subcover. Suppose that there is a finite subcover* $\mathfrak{F}'$ *of* $\mathfrak{F}$*, then* $\mathfrak{F}'$ *consists of finite open intervals with the form* $(\frac{1}{n},1]$*. Since they are finite we can pick the open interval with the largest* n*, denoted by* n_0*. Then* $(0,\frac{1}{n_0})$ *is not covered by* $\mathfrak{F}'$*, which contradicts that* $\mathfrak{F}$ *is a subcover. Hence* $\mathfrak{F}$ *does not have any finite subcover.*

Definition 3 (Compact topological space)**.** *A topological space* X *is compact if every open cover of* X *has a finite subcover.*