Notes of ”Invariant Subspace and Eigenvectors”

Jinxin Wang

1 Overview

- Invariant subspace
  - Def: An invariant subspace for a linear operator
    - Rmk: Two trivial invariant subspaces for a linear operator
  - Examples of invariant subspaces of linear operators
    - $\operatorname{Im}\mathcal{A}$ and $\ker\mathcal{A}$ are invariant subspaces of $\mathcal{A}$
    - $\mathcal{AB} = \mathcal{BA}$, then $\operatorname{Im}\mathcal{B}$ and $\ker\mathcal{B}$ are invariant subspaces of $\mathcal{A}$
  - Def: The restriction of a linear operator over an invariant space
  - Rmk: Connection between an invariant subspace for a linear operator and a matrix of the operator
  - Thm: A necessary and sufficient condition for the matrix of a linear operator under a basis has the form of block-wise diagonal matrix
  - Rmk: Generalization of the above theorem to multiple invariant subspaces for a linear operator
  - Rmk: Specialization of the above remark to invariant subspaces of dimension 1
  - Prop: Properties of invariant subspaces for a linear operator
- Eigenvectors and characteristic polynomial
  - Def: Eigenvector and eigenvalue of a linear operator
  - Rmk: Connection between eigenvectors and invariant subspaces
  - Rmk: The eigenvectors corresponding to the same eigenvalue and the zero vector form a subspace
  - Def: The eigenspace and the geometric multiplicity of an eigenvalue
  - Thm: A necessary and sufficient condition for a scalar to be an eigenvalue of a linear operator in terms of the roots of a polynomial
  - Def: The characteristic polynomial of a linear operator
  - Rmk: A method to find the roots of the characteristic polynomial of a linear operator
  - Def: The characteristic polynomial of a square matrix
  - Thm: The characteristic polynomials of similar square matrices are the same

  - Rmk: The existence of eigenvalues and eigenvectors of a linear operator on a general field and on an algebraically closed field
  - Study characteristic polynomials by invariant subspaces
    * Thm: The characteristic polynomial of a linear operator is divisible by the characteristic polynomial of the restriction of the operator over an invariant subspace
    * Def: The algebraic multiplicity of an eigenvalue
    * Cor: The geometric multiplicity of an eigenvalue is no greater than its algebraic multiplicity
    * Thm: The characteristic polynomial of a linear operator of a space that is a direct sum of invariant subspaces

- The criterion of diagonalizable linear operators
  - Def: Diagonalizable linear operator
  - Thm: A necessary and sufficient condition for a linear operator to be diagonalizable in terms of eigenvectors
  - Lma: The eigenspaces of different eigenvalues are linear independent
  - Def: The spectrum of a linear operator, the spectrum of a square matrix, 谱点 and 单谱点
  - Thm: A sufficient condition for a linear operator to be diagonalizable in terms of 单谱点
  - Thm: A necessary and sufficient condition for a linear operator to be diagonalizable in terms of 谱点
- The existence of invariant subspaces
  - Thm: A linear operator of a space over an algebraically closed field has one-dimensional invariant subspace(s)
  - Thm: A linear operator of a space over the real number field has one-dimensional or two-dimensional invariant subspace(s)

2 Invariant subspace

Definition 1 (An invariant subspace for a linear operator)**.** *Let $\mathcal{A}$ be a linear operator on a space V. A subspace U of V is called an invariant subspace for $\mathcal{A}$ if $\mathcal{A}U \subset U$, where $\mathcal{A}U = \{\mathcal{A}x \mid x \in U\}$.*

Definition 2 (The restriction of a linear operator over an invariant space)**.**

Remark 1 (Connection between an invariant subspace for a linear operator and a matrix of the operator)**.** *Let U be an invariant subspace for a linear operator $\mathcal{A}$ on a space V. Choose a basis of U and expand it into a basis of V as $\{e_1, e_2, \ldots, e_n\}$. Suppose among them $e_{i+1}, e_{i+2}, \ldots, e_j (i < j)$ are the basis of U and A_U is the matrix of the restriction $\mathcal{A}_U$ under the basis $e_{i+1}, e_{i+2}, \ldots, e_j$. Then the matrix of $\mathcal{A}$ under this basis is that*

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_U & A_{23} \\ A_{31} & 0 & A_{33} \end{pmatrix}$$

in which A_{11} is a square matrix of order i, and A_{33} is a square matrix of order $n-j$.

Proof: Since U is an invariant subspace of $\mathcal{A}$, for any of $e_{i+1}, e_{i+2}, \ldots, e_j$, $\mathcal{A}e_k \in U$ $(k = i+1, i+2, \ldots, j)$, and thus a linear combination of $e_{i+1}, e_{i+2}, \ldots, e_j$. Hence the nonzero elements in columns $i+1, i+2, \ldots, j$ can only appear in rows $i+1, i+2, \ldots, j$. More specifically, $\mathcal{A}(e_{i+1}, e_{i+2}, \cdots, e_j) = \mathcal{A}_U(e_{i+1}, e_{i+2}, \cdots, e_j) = (e_{i+1}, e_{i+2}, \cdots, e_j)A_U$, hence the block consisting of columns $i+1, i+2, \ldots, j$ in A is $\begin{pmatrix} 0 \\ A_U \\ 0 \end{pmatrix}$ which satisfies

$$\mathcal{A}(e_{i+1}, e_{i+2}, \cdots, e_j) = (e_1, e_2, \cdots, e_{i+1}, e_{i+2}, \cdots, e_j, e_{j+1}, \cdots, e_n) \begin{pmatrix} 0 \\ A_U \\ 0 \end{pmatrix}$$

Conversely, if the matrix of $\mathcal{A}$ under a basis $\{e_1, e_2, \ldots, e_n\}$ has the above form, i.e. , then the linear span $\langle e_{i+1}, e_{i+2}, \ldots, e_j \rangle$ is an invariant subspace for $\mathcal{A}$.

Remark 2 (Specialization of the above remark to invariant subspaces of dimension 1)**.** *If the space V is a direct sum of n invariant subspaces $V_1, V_2, \ldots, V_n$ of dimension 1, then with a basis that is compatible with these subspaces, the matrix of the linear operator under this basis has the form as*

$$\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

which is a diagonal matrix.

Conversely, if the matrix of the linear operator under a basis is a diagonal matrix, then the space can be expressed as a direct sum of n invariant subspaces of dimension 1 for the operator.

3 Eigenvectors and characteristic polynomial

Definition 3 (Eigenvector and eigenvalue of a linear operator)**.** *A non-zero vector v in a space V is called an eigenvector of a linear operator $\mathcal{A} : V \to V$ on V if there exists $\lambda \in K$ such that*

$$\mathcal{A}v = \lambda v$$

The scalar λ is called an eigenvalue of $\mathcal{A}$ that corresponds to the eigenvector v.

Remark 3 (Connection between eigenvectors and invariant subspaces)**.** *Every non-zero vector in an invariant subspace of dimension 1 for a linear operator is an eigenvector of the operator. Conversely, the linear span of an eigenvector of a linear operator is an invariant subspace for the operator.*

Proof: Let U be an invariant subspace of dimension 1 for a linear operator $\mathcal{A}$ of the space V.

P1 *For each non-zero vector $v \in U$, according to the definition of an invariant subspace, we have $\mathcal{A}v \in U$. Since $\dim V = 1$, v is also a basis of V, and hence $\mathcal{A}v = \lambda v$. Therefore v is an eigenvector of $\mathcal{A}$.*

P2 *Given an eigenvector v of $\mathcal{A}$ with corresponding eigenvalue λ, the linear span $\langle v \rangle = \{\mu v \mid \mu \in K\}$, then $\mathcal{A} \langle v \rangle = \{\mathcal{A}\mu v \mid \mu \in K\} = \{\mu \mathcal{A} v \mid \mu \in K\} = \{\mu\lambda v \mid \mu \in K\} \in \langle v \rangle$. Therefore, $\langle v \rangle$ is an invariant subspace for $\mathcal{A}$.*

Definition 4 (Eigenspace and geometric multiplicity of an eigenvalue)**.** *The eigenspace of an eigenvalue λ of a linear operator $\mathcal{A}$, denoted by V^λ, is defined as the set of vectors $V^\lambda = \{v \in V \mid \mathcal{A}v = \lambda v\}$. The dimension of the eigenspace is called the geometric multiplicity of the eigenvalue λ.*

4 The criterion of diagonalizable linear operators

Definition 5 (Diagonalizable linear operator)**.** *A linear operator is diagonalizable if the matrix of it under a basis is a diagonal matrix.*

Remark 4 (A equivalent condition for a linear operator to be diagonalizable in terms of eigenvectors)**.** *A linear operator $\mathcal{A}$ on a space V is diagonalizable if and only if V has a basis consists of eigenvectors of $\mathcal{A}$.*

Proof: (TODO)

5 The existence of invariant subspaces