Notes of "Linear Map between Vector Spaces"

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1 Overview

- Definition and related concepts
  - Def: A linear map from a space on K to the other on K
  - Examples of linear maps
    - Eg: A linear form of a vector space
    - Eg: Rotation on $\mathbb{R}^2$
    - Eg: The projection from $\mathbb{R}^3$ to its subspace $\{(a, b, 0) \mid a, b \in \mathbb{R}\}$
    - Eg: $C^k(\mathbb{R}) \to \mathbb{R}_{k+1}[x]$
    - Eg: $\mathcal{D} : K[x] \to K[x], f(x) \mapsto f'(x)$
  - Def: The kernel and image of a linear map
  - Rmk: The kernel and image of a linear map are subspaces of the domain and codomain of the map respectively
  - Thm: The relationship of the dimensions of the kernel, the image and the domain of a linear map
  - Cor: The dimension of the image of a linear map is no greater than the dimension of the domain of the map
- One-to-one correspondence between linear maps and matrices
  - Rmk: Generalize some properties of cases of linear maps
  - Rmk: A linear map from V to W is uniquely identified by the set of values on a basis
  - Rmk: A linear map from V to W is uniquely identified by a matrix under certain bases of V and W
  - Def: The rank of a linear map
  - Thm: Two bijections from $\mathcal{L}(V, W)$ to W^n and from $\mathcal{L}(V, W)$ to $M_{m,n}(K)$
- $\mathcal{L}(V, W)$ as an algebraic structure
  - Nta: $\mathcal{L}(V, W)$ and $Hom(V, W)$
  - Rmk: $\mathcal{L}(V, W)$ is a vector space and a subspace of W^V
  - Thm: The bijection $\sigma : \mathcal{L}(V, W) \to M_{m,n}(K)$ is a (linear) isomorphism

  - Cor: The dimension of $\mathcal{L}(V, W)$ is $\dim V \cdot \dim W$
  - Rmk: The bijection $\sigma : \mathcal{L}(V, W) \to M_{m,n}(K)$ 保持乘法
  - Thm: $\dim Im(fg) \le \dim Im(f)$ and $\dim Im(fg) \le \dim Im(g)$

- The matrix of a linear map represents its coordinate transformation
  - Rmk: The matrix of a linear map under certain bases of V and W represents the coordinate transformation under the bases of V and W

2 Definition and related concepts

3 One-to-one correspondence between linear maps and matrices

Remark 1 (A linear map from V to W is uniquely identified by the set of values on a basis)**.** *Let V and W be two vector spaces, and $e_1, e_2, \cdots, e_n$ be a basis of V. We use $\mathcal{L}(V, W)$ denote the set of linear operators from V to W, and W^n denote the set of n-tuple of vectors (e.g. $(\xi_1, \xi_2, \ldots, \xi_n)$) in W.*

We claim that given a basis $e_1, e_2, \cdots, e_n$ of V there is a bijection $\sigma : \mathcal{L}(V, W) \to W^n, f \mapsto (\xi_1, \xi_2, \cdots, \xi_n)$ by setting $f(e_i) = \xi_i, i = 1, 2, \ldots, n$.

First, a linear operator f specifies a set of values on $e_1, e_2, \cdots, e_n$, i.e. specifies a set of vectors as $f(e_1), f(e_2), \cdots, f(e_n)$. Different linear vectors have different sets of values on $e_1, e_2, \cdots, e_n$. If two linear operators f and g have the same sets of values on $e_1, e_2, \cdots, e_n$, i.e. $f(e_i) = g(e_i), i = 1, 2, \ldots, n$, then for every vector $\alpha = \Sigma_{i=1}^n a_i e_i$ in V, we have $f(\alpha) = f(\Sigma_{i=1}^n a_i e_i) = \Sigma_{i=1}^n a_i f(e_i) = \Sigma_{i=1}^n a_i g(e_i) = g(\Sigma_{i=1}^n a_i e_i) = g(\alpha)$. Hence $f = g$. We can conclude that sigma is a injection.

Second, given a set of vectors $\xi_1, \xi_2, \cdots, \xi_n$ in W, we can always define a linear map $f : V \to W$ by setting $f(e_i) = \xi_i, i = 1, 2, \ldots, n$. By the definition of linear map, for every vector $v = \Sigma_{i=1}^n a_i e_i$ in V, $f(v) = f(\Sigma_{i=1}^n a_i e_i) = \Sigma_{i=1}^n a_i f(e_i)$ is determined. Hence the linear map f is specified. Therefore, σ is a surjection.

Remark 2 (A linear map from V to W is uniquely identified by a matrix under certain bases of V and W)**.** *Let V and W be two vector spaces, $e_1, e_2, \cdots, e_n$ be a basis of V, and $w_1, w_2, \cdots, w_m$ be a basis of W. We use $\mathcal{L}(V, W)$ denote the set of linear operators from V to W, and $M_{m,n}(K)$ denote the set of matrices of order $m \times n$.*

We claim that given a basis $e_1, e_2, \cdots, e_n$ of V and a basis $w_1, w_2, \cdots, w_m$ of W there is a bijection $\sigma : \mathcal{L}(V, W) \to M_{m,n}(K), f \mapsto M_f$ in which the i-th ($i = 1, 2, \ldots, n$) column of M_f is the coordinate of $f(e_i)$ under the basis $w_1, w_2, \cdots, w_n$.

With the basis $e_1, e_2, \cdots, e_n$, we have a bijection $\sigma_1 : \mathcal{L}(V, W) \to W^n$. If we can find another bijection $\sigma_2 : W^n \to M_{m,n}(K)$, then the mapping $\sigma_2 \sigma_1$ is the one we are looking for in the claim.

With the basis $w_1, w_2, \cdots, w_m$, the bijection $\sigma_2 : W_n \to M_{m,n}(K), (\xi_1, \xi_2, \cdots, \xi_n) \mapsto M$ can be constructed by setting $M = (\xi_1, \xi_2, \cdots, \xi_n)$, i.e. the i-th ($i = 1, 2, \ldots, n$) column of M is the coordinate of ξ_i under the basis $w_1, w_2, \cdots, w_m$. It is clear that σ_2 is a bijection. Therefore, the claim above is proved to be correct.

Theorem 1 (Two bijections from $\mathcal{L}(V, W)$ to W^n and from $\mathcal{L}(V, W)$ to $M_{m,n}(K)$).

4 $\mathcal{L}(V, W)$ as an algebraic structure

5 The matrix of a linear map represents its coordinate transformation