Notes of "Definitions and Examples"

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1 Overview

- Definitions related to vector spaces
  - Def: (Left) group action
  - Def: Vector space (linear space)
  - Def: 实向量空间和复向量空间
- Properties of operations in a vector space
  - Prop: Properties of operations in a vector space
- Examples of vector spaces
  - Eg: Zero vector space
  - Eg: Vector spaces of numbers, vectors, or matrices
    * Eg: 父域向量空间
    * Eg: Coordinate space
    * Eg: $M_{m,n}(K)$ 和 $M_n(K)$
  - Eg: Vector spaces of mappings
    * Eg: $\{(a_i)_{i\in\mathbb{N}} \mid a_i \in \mathbb{R}\}$
    * Eg: $\{(a_i)_{i\in\mathbb{N}} \mid a_i \in \mathbb{R}, a_i = a_{i-1} + a_{i-2}, i \geq 2\}$
    * Eg: 集合 X 到向量空间 U 的映射空间 U^X
    * Eg: The set of real-valued functions defined on $\mathbb{R}$
    * Eg: The set of real-valued continuous functions $C(X)$ defined on a subset $X \subset \mathbb{R}$
  - Eg: Vector spaces of polynomials
    * Eg: $K[x]$ the polynomial ring in x over K
    * Eg: $K[x_1, x_2, \ldots, x_m]$ the polynomial ring in m indeterminates over K
    * Eg: m 元多项式环 $K[x_1, x_2, \ldots, x_m]$ 中次数为 n 的齐次多项式全体加上零多项式
    * Eg: m 元多项式环 $K[x_1, x_2, \ldots, x_m]$ 中次数不超过 n 的多项式全体
- Linear subspaces
  - Def: A linear (or vector) subspace of a vector space

    * Rmk: Two trivial subspaces of every vector space
  - Examples of linear subspaces
    * Eg: Any line passing through the origin in $\mathbb{E}^2$
    * Eg: All vectors parallel to a vector in $\mathbb{E}^3$ forms a subspace of $\mathbb{E}^3$
    * Eg: The set of solution of the linear system $Ax = 0$ where $A \in M_{m,n}(K)$ forms a subspace of K^n
    * Eg: $\mathbb{R}$ 的子集 X 上的全体可微函数 $C^1(X)$ 是 $C(X)$ 的一个子空间
    * Eg: 实数域上定义的偶函数全体，周期函数全体，形如 $ae^x + be^- x, a, b \in \mathbb{R}$ 的函数全体都是实数域上的实函数空间的子空间
  - Prop: The intersection of any collection of subspaces of a space is a subspace of it

2 Definitions related to vector spaces

3 Properties of operations in a vector space

Proposition 1. *Let $-v$ denote the inverse element of v.*

P1 $0v = 0$

P2 $a0 = 0$

P3 $(-1)v = -v$

P4 $a(-v) = -av$

P5 $a(u - v) = au - av$

P6 $(a - b)v = av - bv$

P7 $av = 0 \Rightarrow a = 0$ or $v = 0$

证明. Just prove a few of them:

P1

P3

P7

□

4 Examples of vector spaces

Example 1 (Coordinate space)**.** *All row vectors $(a_1, a_2, \dots, a_n)$ of length n whose elements are in the field K.*

Same as all column vectors $[a_1, a_2, \dots, a_n]$ of length n whose elements are in the field K.

Example 2 (Zero vector space)**.** *The trivial abel group:* $\{0\}$*, with the scalar multiplication with the field* K *defined as:* $a0 = 0$ *for each* $a \in K$*.*

It is clear that the zero vector space is a vector space on any field.

Example 3 (The vector space of real-valued functions)**.** *Let* E *be a set. The set of all real-valued functions defined on* E *is a vector space on* $\mathbb{R}$ *with the following addition and scalar multiplication operations:*

$$(f + g)(x) = f(x) + g(x)$$

$$(af)(x) = a(f(x))$$

for each $x \in E$ *and* $a \in \mathbb{R}$*.*

Example 4 (The vector space of mappings to a vector space)**.** *Let* U *be a vector space on the field* K*. The set of all mappings from a set* X *to* U *is a vector space on* K*, denoted* U^X*. The addition and scalar multiplication are defined as:*

$$(f + g)(x) = f(x) + g(x)$$

$$(af)(x) = a(f(x))$$

Since $f(x), g(x) \in U$ *and* $a \in K$*, it holds that* $f(x) + g(x) \in U$ *and* $a(f(x)) \in U$ *given that* U *is a vector space on* K*. Therefore,* U^X *is a vector space on* K *is guaranteed by that* U *is a vector space on* K*.*

Especially, K^X *is a vector space on* K*. The above example, which is the vector space of all real-valued functions defined on a set* E*, is actually* $\mathbb{R}^E$*. Hence,* U^X *is a generalization of the vector space of real-valued functions.*

Example 5.

5 Linear subspaces

Definition 1 (A linear (or vector) subspace of a vector space)**.** *Let* U *be a subset of a vector space* V *on a field* K*.* U *is a linear subspace of* V *if it is an addition subgroup of* V*, and* $\forall k \in K$ *and* $\forall u \in U$ *it holds that* $ku \in U$*.*