Notes of "Basis and Dimension"

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1 Overview

- Basis and dimension
  - Def: A basis and its basis vectors of a vector space
    - Rmk: A basis of a vector contains only finite vectors
  - Examples of a basis and its basis vectors of a vector space
    - Eg: The standard basis of the coordinate space
  - Thm: The uniqueness of the linear representation (the coordinate) of every vector in a space with a basis
  - Thm: The existence of a (linear) isomorphism between a space of n dimension and K^n
  - Def: A (linear) homomorphism between two vector spaces, injective homomorphism and surjective homomorphism
  - Examples of (linear) homomorphisms between two vector spaces
    - Eg: $\phi : \mathbb{R}^n \to \mathbb{R}^{\mathbb{N}}$
    - Eg: $\phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$
  - Def: A (linear) isomorphism between two vector spaces
    - Rmk: Two vector spaces can form an isomorphism only if they are on the same field.
    - Rmk: An isomorphic map between two vector spaces maps the zero vector in one space to the one in the other, and it maps a basis of one space to a basis of the other.
    - Rmk: If $\phi : V \to U$, $\psi : U \to W$ are both isomorphic maps, then $\phi^{-1} : U \to V$ and $\phi\psi : V \to W$ are both isomorphic maps.
  - Examples of (linear) isomorphism between two vector spaces
    - Eg: $\mathbb{C} \simeq \mathbb{R}^2$
    - Eg: 实斐波那契数列空间 $\simeq \mathbb{R}^2$
    - Eg: $M_n(K) \simeq K^{n^2}$
- Properties of a basis of a vector space
  - Prop: The uniqueness of the number of vectors in every basis of a vector space
  - Def: The dimension of a vector space

- Prop: A necessary and sufficient condition of two vector spaces on the same field to be isomorphic
- Thm: The linear independence of $n+1$ vectors of a space of dimension n $(n < \infty)$
- Thm: Two necessary and sufficent conditions of a set of vectors to be a basis of a space of dimension n $(n < \infty)$
- Thm: The possibility of a set of linear independent vectors to become a basis of a space of dimension n $(n < \infty)$
- Def: Finite-dimensional spaces and infinite-dimensional spaces
  * Examples of finite-dimensional spaces
  * Examples of infinite-dimensional spaces
- Def: A maximal linearly independent subset of a set of vectors
- Thm: Existence, uniqueness of the number of vectors, and the linear span of maximal linearly independent subsets of a set in a finite-dimensional space
- Def: The rank of a set of vectors

• Transition matrix (Change-of-basis matrix)
  - Def: A transition matrix from a basis to another
    * Rmk: The uniqueness of the transition matrix from a basis to another
  - Rmk: Transform the coordinate of a vector under a basis of its space to the coordinate under another basis with the transition matrix between the two basis
  - Thm: The inversibility of the transition matrix between two basis of a vector space and its meaning
  - Thm: The composition of two transition matrices between three basis of a vector space and its meaning

• The order of the vectors in a basis
  - Rmk: By default, a basis of a vector space is an ordered set of vectors, and its order is fixed

2 Basis and dimension

Definition 1 (A basis and its basis vectors of a vector space). *A set of vectors $v_1, v_2, \ldots, v_n$ is said to be a basis of a space V if they are linear independent and their linear span is V.*

Remark 1 (A basis of a vector contains only finite vectors). *A basis of a vector contains only finite vectors.*

Definition 2 (A (linear) homomorphism between two vector spaces, injective homomorphism and surjective homomorphism). *Two vector spaces U and V over a field K are homomorphic if there exists a bijective $\phi : V \to U$ such that $\forall a, b \in K$ and $\forall u, v \in V$ we have*

$$\phi(au + bv) = a\phi(u) + b\phi(v) \tag{1}$$

3 Properties of a basis of a vector space

Theorem 1 (Two necessary and sufficent conditions of a set of vectors to be a basis of a space of dimension n ($n < \infty$))**.** *Let V be a vector space of dimension n, then:*

T1 *If n vectors $v_1, v_2, \ldots, v_n$ in V are linear independent, then they form a basis of V.*

T2 *If the linear span of n vectors $v_1, v_2, \ldots, v_n$ in V is V itself, then $v_1, v_2, \ldots, v_n$ are linear independent, and thus are a basis of V.*

Definition 3 (Finite-dimensional and infinite-dimensional spaces)**.** *If any finite set of vectors in a space cannot span the space, then the space is called a infinite-dimensional space. Otherwise, it is a finite-dimensional space.*

Definition 4 (A maximal linearly independent subset of a set of vectors)**.** *A subset of set of vectors is called a maximal linearly independent subset if the vectors in the subset are linear independent and every other vector in the set is a linear combination of the vectors in the subset.*

Theorem 2 (Existence, uniqueness of the number of vectors, and the linear span of maximal linearly independent subsets of a set in a finite-dimensional space)**.** *Let S be a set of vectors in a finite-dimensional space, and it contains non-zero vector, then:*

T1 *S contains maximal linearly independent subsets.*

T2 *Any two maximal linearly independent subsets of S consists of the same number of vectors.*

T3 *A maximal linearly independent subset of S is a basis of $\langle S \rangle$.*

Definition 5 (The rank of a set of vectors)**.** *The rank of a set of vectors is defined as the dimension of the linear span of the set of vectors.*

4 Transition matrix (Change-of-basis matrix)

5 The order of the vectors in a basis