Notes of "Quotient Space"

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1 Overview

- Motivation and definition of quotient space
  - Rmk: The motivation of the quotient space is to define a subspace which is naturally isomorphic to any complemented subspace of a subspace
  - Def: The quotient space of a vector space by a subspace
    - Rmk: The operations defined on V/W is well-defined, i.e. do not depend on the choice of representatives
  - Examples of quotient spaces
    - Eg: The quotient space of $\mathbb{R}^2$ by the x-axis
- Properties of quotient space
  - Thm: The isomorphism between the quotient space of a vector space by a subspace and a complemented subspace
  - Cor: The relation between the dimensions of a vector space, a subspace, and the quotient space of the vector space mod by the subspace

2 Motivation and definition of quotient space

Definition 1 (The quotient space of a vector space by a subspace)**.** *Let V be a space over the field K and U be a subspace of V.*

We can define a binary relation in V as follows:

$$x \sim y \Leftrightarrow x - y \in U$$

It is clear that $\sim$ is a equivalence relation.

The set of all equivalent classes under the above equivalence relation is denoted as V/U, and the class where a vector x belongs is denoted as $\bar{x}$. It is clear that

$$\bar{x} = x + U = \{x + u \mid u \in U\}$$

We can define the addition and the scalar multiplication with K as follows:

$$\bar{x} + \bar{y} = \overline{x + y}$$

$$a\bar{x} = \overline{ax}$$

It is easy to verify that V/U is a vector space over K, and it is called the quotient space of V by U.

3 Properties of quotient space