Notes of "Algebra of Linear Operator"

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1 Overview

- Definition and basic properties of linear operators
  - Def: Linear operator over a vector space
  - Rmk: The matrix of a linear operator under a basis
  - Examples of linear operators
    * Eg: The zero operator
    * Eg: The reflection operator on $\mathbb{R}^2$
    * Eg: The rotation operator on $\mathbb{R}^2$
    * Eg: The stretch operation on any vector space
    * Eg: The differentiation operator $\mathcal{D} = \frac{d}{dt}$ on $K_n[t]$
  - Def: Inversible (degenerated, singular) operators, non-inversible (non-degenerated, non-singular) operators, and the inverse operator of a linear operator
  - Thm: The relationship between the dimension of the kernel and the dimension of the image of a linear operator
- The matrices of a linear operator under different bases
  - Rmk: The transformation between the matrices of a linear operator under two different bases
  - Def: Matrix similarity
  - Thm: The matrices of a linear operator under different bases are similar
  - Rmk: The matrices of a linear operator under different bases have the same determinant
  - Def: The determinant of a linear operator
  - Def: The trace of a square matrix
  - Rmk: The matrices of a linear operator under different bases have the same trace
  - Def: The trace of a linear operator
- Matrix similarity and its application
  - Prop: Matrix similarity is a equivalence relation
  - Rmk: Our interest in the equivalence classes of matrix similarity and its application
  - Eg: Find the terms of Fibonacci sequence through the exponentiation of matrices

  - Rmk: Explore the equivalence classes of matrix similarity by changing the basis of the space

- $\mathcal{L}(V)$ as an algebraic structure
  - Nta: $\mathcal{L}(V)$
  - Rmk: $\mathcal{L}(V)$ is a vector space with multiplication
  - Def: An algebra over a field
  - Examples of algebras over a field
    * $M_n(K)$
    * $K[x]$
    * An field is an algebra over itself
  - Rmk: $\sigma : \mathcal{L}(V) \to M_n(K)$ is an (algebra) isomorphism
  - Def: The determinant function and the trace function on $\mathcal{L}(V)$
  - Rmk: Some operational properties of the determinant function on $\mathcal{L}(V)$
  - Rmk: Some operational properties of the trace function on $\mathcal{L}(V)$

- The generated subalgebra of a linear operator
  - Def: The minimal polynomial of a linear operator
  - Thm: The existence of the minimal polynomial and its degree
  - Thm: A necessary and sufficient condition for a linear operator to be inversible in terms of the minimal polynomial
  - Thm: Every polynomial annihilating a linear operator is divisible by the minimal polynomial of it
  - Cor: The minimal polynomial of a linear operator is unique
  - Def: A linear operator is nilpotent and its index
  - Prop: A necessary and sufficient condition for a linear operator is nilpotent in terms of the minimal polynomial

2 Definition and basic properties of linear operators

Definition 1 (Linear operator over a vector space)**.**

Remark 1 (The matrix of a linear operator under a basis)**.**

Example 1 (The rotation operator on $\mathbb{R}^2$)**.**

Example 2 (The differentiation operator $\mathcal{D} = \frac{d}{dt}$ on $K_n[t]$)**.**

3 The matrices of a linear operator under different basis

4 Matrix similarity

Example 3 (Find the terms of Fibonacci sequence through the exponentiation of matrices)**.** *The idea is expressing the recursive relation of the Fibonacci sequence in matrix. Consider the pair* (a_n, a_{n+1}) *and let* $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$*, then*

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

5 $\mathcal{L}(V)$ as an algebraic structure

Remark 2 (Some operational properties of the trace function on $\mathcal{L}(V)$)**.** *Let V be a vector space over a field K. Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{L}(V)$ and $\lambda, \mu \in K$. The trace function on $\mathcal{L}(V)$ has the following operational properties:*

R1 $\operatorname{tr}(\lambda\mathcal{A} + \mu\mathcal{B}) = \lambda \operatorname{tr}(\mathcal{A}) + \mu \operatorname{tr}(\mathcal{B})$

R2 $\operatorname{tr}(\mathcal{A}\mathcal{B}) = \operatorname{tr}(\mathcal{B}\mathcal{A})$

6 The generated subalgebra of a linear operator