Notes of "Linear Form"

Jinxin Wang

1 Overview

- Linear form, dual space and dual basis of a vector space
  - Def: A linear form (or linear functional or covector) of a vector space
  - Examples of linear forms of vector spaces
    - Eg: $f(x_1, x_2, \cdots, x_n) = \Sigma_{i=1}^{n} a_i x_i$ is a linear map from K^n to K
    - Eg: $\int_a^b f(x)dx$ is a linear map from the space of all continuous functions on $[a, b]$ to $\mathbb{R}$
    - Eg: $Tr : M_n(\mathbb{R}) \to \mathbb{R}, M \mapsto Tr(M)$ is a linear map from $M_n(\mathbb{R})$ to $\mathbb{R}$
  - Rmk: A necessary and sufficient condition to determine a linear form of a space
  - Rmk: The set of linear forms from a vector space V to its field K is a subspace of K^V
  - Def: The dual space of a vector space
  - Rmk: A basis of the dual space of a finite-dimensional space
  - Thm: The dimension of the dual space of a finite-dimensional space and the dual basis of a basis of the space
  - Examples of dual space and dual basis
    - Eg: The dual space of $P_n = \langle 1, t, t^2, \ldots, t^{n-1} \rangle$ and the dual basis of the basis $1, t, t^2, \ldots, t^{n-1}$ of the space P_n
    - Eg: The dual space of $V = \langle \sin t, \sin 2t, \sin 3t, \ldots, \sin nt \rangle$ and the dual basis of the basis $\sin t, \sin 2t, \sin 3t, \ldots, \sin nt$ of the space V
- A natural isomorphism from a space to the dual space of its dual space
  - Rmk: The motivation of examining V^{**}
  - Thm: The map $f : x \mapsto f_x$ where $f_x : V^* \to K, \alpha \mapsto \alpha(x)$ is a (linear) isomorphism from V to V^{**}
    - Rmk: The (linear) isomorphism $f : x \mapsto f_x$ is natural
    - Rmk: The symmetry between V and V^* through the isomorphism from V to V^{**}
  - Cor: The one-to-one correspondence between basis of V and basis of V^*
  - Rmk: The conclusion about the equality of the dimension of a finite-dimensional space and the dimension of its dual space doesn't hold for infinite-dimensional spaces and the counterexample of $K\,|x|$

- Application of linear form: determine the rank of a set of vector in a space
  - Lma: The determinant of n linear forms on n linear dependent vectors in a space is 0
  - Lma: A necessary and sufficient condition for a set of vectors to be linear independent in terms of the determinant of a basis of V^* on those vectors
  - Thm: The rank of a set of vectors in a space V is the same as the rank of the matrix consists of the basis of V^* on those vectors
- A natural correspondence from subspaces of a space to subspaces of its dual space
  - Def: The annihilator of a subspace of a space
    * Rmk: The annihilator of a subspace of a space is a subspace of its dual space
  - Thm: The dimension of the annihilator in terms of the dimension of the space and the subspace
  - Thm: The annihilator of the annihilator of a subspace give rise to the subspace itself under the isomorphism
  - Cor: The correspondence between subspaces of a space and subspaces of its dual space in terms of annihilators
- A geometric interpretation of the solution space of a homogeneous linear system
  - Rmk: The generalization of homogeneous linear systems on a vector space
  - Thm: The dimension of the solution space can be determined by the dimension of the space and the rank of the set of linear forms
  - Thm: Every subspace of a vector space is the solution space of a homogeneous linear system on the space

2 Linear form, dual space and dual basis of a vector space

3 A natural isomorphism from a space to the dual space of its dual space

Theorem 1 (The map $f : x \mapsto f_x$ where $f_x : V^* \to K, \alpha \mapsto \alpha(x)$ is a (linear) isomorphism from V to V^{**})**.** *The map $f : x \mapsto f_x$ where $f_x : V^* \to K, \alpha \mapsto \alpha(x)$ is a (linear) isomorphism from V to V^{**}*

证明*.* First, we need to prove that $f_x : V^* \to K, \alpha \mapsto \alpha(x)$ is a linear form of V^*.

$$\begin{aligned} f_x(a\alpha + b\beta) &= (a\alpha + b\beta)(x) \\ &= a(\alpha(x)) + b(\beta(x)) \\ &= a(f_x(\alpha)) + b(f_x(\beta)) \end{aligned}$$

From this we can confirm that $f : x \mapsto f_x$ is a map from V to V^{**}.

Second, we need to prove that f is a linear map from V to V^{**}.

Third, we need to prove that f is bijective. To prove this, we can prove that f maps a basis of V to a basis of V^{**}. □

4 Application of linear form: determine the rank of a set of vector in a space

5 A natural correspondence from subspaces of a space to subspaces of its dual space

6 A geometric interpretation of the solution space of a homogeneous linear system