Notes of ”Subspace”

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1 Overview

- The sum of two subspaces of a vector space
  - Def: The sum of two (or finite) subspaces of a vector space
    - Rmk: The sum of two (or finite) subspaces is still a subspace
  - Prop: The existence of a basis of a subspace and the relation between the dimension of a space and the dimension of its subspace
  - Def: 向量空间的基与子空间相合
    - Rmk: A method to find a basis of a vector space 与其一个子空间相合
  - Thm: The existence of a basis 与两个子空间相合
    - Eg: An example that the theorem does not hold for three or more subspaces
  - Cor: The formula of the basis of the sum of two subspaces
  - TNta: k-dimensional plane, codimension (余维数) of a subspace, hyperplane (超平面), flag variety (旗簇)
- The direct sum of two subspaces of a vector space
  - Def: The linear independence and dependence of a class of subspaces
  - Prop: A necessary and sufficient condition for two subspaces to be linear independent
    - Rmk: This proposition cannot be generalized to three or more subspaces
  - Thm: A necessary and sufficient condition for a class of subspaces to be linear independent in terms of intersection
  - Thm: A necessary and sufficient condition for a class of subspaces to be linear independent in terms of basis
  - Thm: A necessary and sufficient condition for a class of subspaces to be linear independent in terms of dimension
  - Thm: A necessary and sufficient condition for a class of subspaces to be linear independent in terms of the decomposition of vectors in the sum of these subspaces
  - Def: The (internal) direct sum of a class of subspaces of a vector space
  - Thm: The existence of the complemented subspace of a subspace

    - * Rmk: Complemented subspaces of a subspace are not unique.
  - Def: The (external) direct sum of two vector spaces
  - Rmk: There is no difference in nature between internal direct sums and external direct sums

2 The sum of two subspaces of a vector space

Definition 1 (The sum of two (or finite) subspaces of a vector space)**.** *Let V be a vector space, and U and W be two subspaces of V, then the sum of U and W, denoted by $U+W$, is a set as follows:*

$$U + W = \{u + w \mid u \in U, w \in W\} \tag{1}$$

Similarly, the sum of finite subspaces $U_1, U_2, \ldots, U_n$ is defined as

$$U_1 + U_2 + \cdots + U_n = \{u_1 + u_2 + \cdots + u_n \mid u_i \in U_i, i = 1, 2, \ldots, n\} \tag{2}$$

3 The direct sum of two subspaces of a vector space

Definition 2 (The linear independence and dependence of finite subspaces)**.** *A finite class of subspaces $U_1, U_2, \ldots, U_n$ are linear independent if the equation $u_1 + u_2 + \cdots + u_n = 0$ where $u_i \in U_i, i = 1, 2, \ldots, n$ iff. $u_1 = u_2 = \cdots = u_n = 0$*

Theorem 1 (A necessary and sufficient condition for a class of subspaces to be linear independent in terms of basis)**.** *Let V be a vector space and $U_1, U_2, \ldots, U_n$ be subspaces of V. $U_1, U_2, \ldots, U_n$ are linear independent if and only if the union of the basis of all subspaces are linear independent.*

Theorem 2 (A necessary and sufficient condition for a class of subspaces to be linear independent in terms of the decomposition of vectors in the sum of these subspaces)**.** *Let V be a vector space and $U_1, U_2, \ldots, U_n$ be subspaces of V. $U_1, U_2, \ldots, U_n$ are linear independent if and only if there exists a unique way to express a vector in $U_1 + U_2 + \cdots + U_n$ as $u_1 + u_2 + \cdots + u_n = 0$ where $u_i \in U_i, i = 1, 2, \ldots, n$.*