Notes of ”The Axiom System and Some General Properties of the Set of Real Numbers”

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1 Overview

- Definition of the Set of Real Numbers
- Some General Algebraic Properties of Real Numbers
- The Completeness Axiom and the Existence of a Least Upper (or Greatest Lower) Bound of a Set of Numbers
  - Definition: A bounded-above/bounded below set
  - Definition: A bounded set
  - Definition: The maximal/minimal element of a set
    - Remark: The formal expression of the definition
    - Remark: The uniqueness of the maximal/minimal element of a set
    - Remark: The existence of the maximal/minimal element of a set
  - Definition: The least upper bound and the greatest lower bound
    - Remark: The formal express of the definition
  - Lemma: The least upper bound principle

2 Definition of the Set of Real Numbers

3 Some General Algebraic Properties of Real Numbers

4 The Completeness Axiom and the Existence of a Least Upper (or Greatest Lower) Bound of a Set of Numbers

Definition 1 (A Bounded-Above/Bounded-Below Set). *A set $X \subset \mathbb{R}$ is said to be bounded above (resp. bounded below) if there exists a number $c \in \mathbb{R}$ such that $x \leq c$ (resp. $c \leq x$) for all $x \in X$.*

Definition 2 (A Bounded Set). *A set that is bounded both above and below is called bounded.*

Definition 3 (The Maximal/Minimal Element of a Set). *An element $a \in X$ is called the largest or maximal (resp. the smallest or minimal) element of X if $x \leq a$ (resp. $a \leq x$) for all $x \in X$*

Remark 1 (The Formal Expression of the Definition)**.** *We can write the above definition in a formal expression:*

$$(a = \max X) \coloneqq (a \in X \wedge \forall x \in X (x \leq a))$$

$$(a = \min X) \coloneqq (a \in X \wedge \forall x \in X (a \leq x))$$

Remark 2 (The Uniqueness of the Maximal/Minimal Element of a Set)**.**

Remark 3 (The Existence of the Maximal/Minimal Element of a Set)**.** *Not every set, or not even every bounded set, has a maximal or minimal element. Example:* $I = \{x \mid 0 \leq x < 1\}$ *does not have a maximal element. In my opinion it is due to the* 稠密性 *of real numbers.*

Definition 4 (The Least Upper Bound)**.** *The minimal number that bounds a set* $X \subset \mathbb{R}$ *above is called the least upper bound of* X *and denoted* $\sup X$ *(the supremum of* X*) or* $\sup_{x \in X} x$*.*

Remark 4.

$$(s = \sup X) \coloneqq ((\forall x \in X (x \leq s)) \wedge (\forall s' < s \exists x' \in X (s' < x')))$$

Definition 5 (The Greatest Lower Bound)**.** *Similarly, the greatest lower bound of* X*, denoted* $\inf X$ *(the infimum of* X*) or* $\inf_{x \in X} x$*, is defined as*

$$(i = \inf X) \coloneqq ((\forall x \in X (i \leq x)) \wedge (\forall i' > i \exists x' \in X (x' < i')))$$

Lemma 1 (The Least Upper Bound Principle)**.** *Every nonempty set of real numbers that is bounded from above has a unique least upper bound.*

证明. (TODO) □

Lemma 2 (The Greatest Lower Bound Principle)**.**

$$(\text{X is nonempty and bounded below}) \Leftarrow (\exists! \inf X)$$