Notes of "Basic Lemmas Connected with the Completeness of the Real Numbers"

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1 Overview

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2 The Nested Interval Lemma (Cauchy-Cantor Principle)

Definition 1 (A Sequence of Elements of a Set)**.** *A function $f : \mathbb{N} \to X$ of a natural-number argument is called a sequence or, more fully, a sequence of elements of X.*

Definition 2 (A Sequence of Nested Intervals)**.** *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of sets. If $X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$, that is $X_n \supset X_{n+1}$ for all $n \in \mathbb{N}$, we say the sequence is nested.*

Lemma 1 (The Nested Interval Lemma, or Cauchy-Cantor Principle)**.** *For any nested sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of closed intervals, there exists a point $c \in \mathbb{R}$ belonging to all of these intervals.*

If in addition it is known that for any $\epsilon > 0$ there is an interval I_k whose length $|I_k|$ is less than ϵ, then c is the unique point common to all the intervals.

证明. Hint: By the least upper bound principle.

- Let $I_i = [a_i, b_i], i \in \mathbb{N}^+$. Since $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$, we have

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$$

$$b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$$

- Let $A = \{a_i \mid i \in \mathbb{N}^+\}$, and $B = \{b_i \mid i \in \mathbb{N}^+\}$. $\forall i \in \mathbb{N}^+$, a_i is a lower bound of B, and b_i is an upper bound of A. According to the least upper bound principle, there exist $s = \sup A$ and $i = \inf B$.
- We claim that $s \leq i$. Suppose that $s > i$, then there exists $j, k \in \mathbb{N}^+$ such that $a_j > b_k$, which is contradictory to our condition about the order between elements of A and B. Therefore, $C = \{c \mid \sup A \leq c \leq \inf B\}$ contains the only real numbers that belong to all intervals.
- If for any $\epsilon > 0$ there is an interval I_k whose length $|I_k|$ is less than ϵ, then we claim that $s = i$. If $s < i$, then let $d = i - s > 0$, and for all intervals their lengths should be greater than or equal to d, which is contradictory to that there exists I_k whose length $|I_k| < d$. Hence C contains only one element, which is $\sup A(\inf B)$.

□

Remark 1. *Notice the sequence of nested intervals are closed intervals. From the proof we can see, if the sequence of nested intervals are open intervals, the unique point* $\sup A(\inf B)$ *might be one of the endpoints of all intervals and thus is not included in the sequence of intervals. Example:* $\{I_n\}$ *where* $I_n = (0, \frac{1}{n})$.

3 The Finite Covering Lemma (Borel-Lebesgue Principle)

Definition 3 (A Cover of a Set)**.** *A system* $S = \{X\}$ *of sets* X *is said to cover a set* Y *if* $Y \subset \bigcup_{X \in S} X$*, that is, if every element* $y \in Y$ *belongs to at least one of the sets* X *in the system* S*.*

Lemma 2 (The Finite Covering Lemma, or Borel-Lebesgue Principle)**.** *Every system of open intervals covering a closed interval containing a finite subsystem that covers the closed interval.*

证明. (TODO) □

4 The Limit Point Lemma (Bolzano-Weierstrass Principle)

Definition 4 (A Limit Point of a Set)**.** *A point* $p \in \mathbb{R}$ *is a limit point of the set* $X \subset \mathbb{R}$ *if every neighborhood of the point contains an infinite subset of* X*.*

Remark 2. *An equivalent condition is that every neighborhood of* p *contains at least one point of* X *different from* p *itself.*

Remark 3. *The concept of a limit point is* 相对的*. It is clear from the definition that we must specify the set of a limit point. A limit point of a set* $X_1 \subset \mathbb{R}$ *might not be a limit point in terms of another set* $X_2 \subset \mathbb{R}$*. For example, a finite point set* $X \subset \mathbb{R}$ *can never have a limit point since it does not have any infinite subset.*

Lemma 3 (The Limit Point Principle, or Bolzano-Weierstrass Principle)**.** *Every bounded infinite set of real numbers has at least one limit point.*

证明. Hint: By the finite covering lemma and proof by contradiction.

- Let $X \subset \mathbb{R}$ be the bounded infinite set of real numbers. Since it is bounded, there exists a closed interval I such that $X \subset I$.
- Assume that $\forall p \in \mathbb{R}$ is not a limit point of X, and hence there exists a neighborhood $U(p)$ of p that contains only a finite subset of X.

□

证明. Hint: By the nested interval lemma. □