Notes of "Existence of the Limit of a Sequence"

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1 Overview

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  - Def: The inferior limit and superior limit of a sequence
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  - Prop: The relationship between the inferior (superior) limits and the

2 The Cauchy Criterion

Definition 1 (A fundamental or Cauchy sequence). *A sequence $\{x_n\}$ is called a fundamental or Cauchy sequence if for any $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ whenever $n > N$ and $m > N$.*

Theorem 1 (Cauchy's convergence criterion). *A numerical sequence converges if and only if it is a Cauchy sequence.*

Example 1.

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$\begin{aligned} x_{2n} - x_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &> n \cdot \frac{1}{2n} \\ &= \frac{1}{2} \end{aligned}$$

3 A Criterion for the Existence of the Limit of a Monotonic Sequence

Definition 2 (An increasing/nondecreasing/nonincreasing/decreasing sequence and monotonic sequences)**.** *A sequence* $\{x_n\}$ *is*

- *increasing if* $x_n < x_{n+1}$ *for all* $n \in \mathbb{N}$
- *nondecreasing if* $x_n \leq x_{n+1}$ *for all* $n \in \mathbb{N}$
- *nonincreasing if* $x_n \geq x_{n+1}$ *for all* $n \in \mathbb{N}$
- *decreasing if* $x_n > x_{n+1}$ *for all* $n \in \mathbb{N}$

Sequences of these four types are called monotonic sequences.

Definition 3 (A bounded-above sequence)**.**

Theorem 2. *In order for a nondecreasing sequence to have a limit it is necessary and sufficient that it be bounded above.*

证明. □

Remark 1. *An analogous theorem exists that it is sufficient and necessary for a nonincreasing sequence to have a limit that it be bounded below.*

Example 2. $\lim_{n\to\infty} \frac{n}{q^n} = 0$ *if* $q > 1$.

Corollary 1.

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

Corollary 2.

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 \text{ for any } a > 0$$

Example 3. $\lim_{n\to\infty} \frac{q^n}{n!} = 0$ *where* $q \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

4 The Number e

Proposition 1. *The sequences $a_n = (1 + \frac{1}{n})^n$ and $b_n = (1 + \frac{1}{n})^{n+1}$ are convergent, and they have the same limit values.*

证明. □

Definition 4.

$$e \coloneqq \lim_{n\to\infty}(1 + \frac{1}{n})^n$$

Proposition 2. $e = \Sigma_{n=0}^{\infty}\frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$

证明. □

Proposition 3. $e = \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}$

证明. □

5 Subsequences and the Partial Limits

Definition 5 (A subsequence of a sequence)**.** *If $x_1, x_2, \ldots, x_n, \ldots$ is a sequence and $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, then the sequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ is called a subsequence of the sequence $\{x_n\}$.*

Lemma 1 (Bolzano-Weierstrass theorem)**.** *Every bounded sequence of real numbers contains a convergent subsequence.*

证明. Hint:

- Let E be the set of values of x_n. Hence E is bounded.
- If the number of elements of E is finite, then there exists $c \in E$ such that $x_{n_1} = x_{n_2} = \cdots = x_{n_k} = \cdots = c$. Hence the subsequence $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots\}$ converges to c.
- If the number of elements of E is infinite, then by Bolzano-Weierstrass principle it has a limit point c.

□

Definition 6 (A sequence tends to positive infinity)**.** *We shall write $x_n \to +\infty$ and say that the sequence $\{x_n\}$ tends to positive infinity if for each number c there exists $N \in \mathbb{N}$ such that $x_n > c$ for all $n > N$.*

Remark 2 (Definitions of a sequence tends to negative infinity and tends to infinity)**.** *There are two analogous definitions of a sequence tends to negative infinity and tends to infinity:*

- *The sequence $\{x_n\}$ tends to negative infinity if for each number c there exists $N \in \mathbb{N}$ such that $x_n < c$ for all $n > N$.*
- *The sequence $\{x_n\}$ tends to infinity if for each number c there exists $N \in \mathbb{N}$ such that $|x_n| > c$ for all $n > N$.*

Remark 3 (Definitions of a sequence tends to positive infinity, negative infinity, and infinity in symbolic logic)**.**

$$(x_n \to +\infty) \coloneqq (\forall c \in \mathbb{R} \exists N \in \mathbb{N} \forall n > N (c < x_n))$$
$$(x_n \to -\infty) \coloneqq (\forall c \in \mathbb{R} \exists N \in \mathbb{N} \forall n > N (x_n < c))$$
$$(x_n \to \infty) \coloneqq (\forall c \in \mathbb{R} \exists N \in \mathbb{N} \forall n > N (c < |x_n|))$$

Lemma 2. *From each sequence of real numbers one can extract either a convergent subsequence or a subsequence that tends to infinity.*

证明. Hint:

- If the sequence is bounded, then by Bolzano-Weierstrass theorem we can extract a convergent subsequence.
- If the sequence is unbounded, then for any $c \in \mathbb{R}$, there exists at least one term $|x_k| > c$. So we can construct such a subsequence where the k-th term x_{n_k} holds that $|x_{n_k}| > k$ for all $k \in \mathbb{N}$. It is clear that such a subsequence exists and it tends to infinity.

□

Definition 7 (The inferior limit of a sequence)**.** *The number $l = \lim_{n\to\infty} \inf_{k\geq n} x_k$ is called the inferior limit of the sequence $\{x_k\}$ and denoted $\underline{\lim}_{k\to\infty} x_k$ or $\liminf_{k\to\infty} x_k$. If $i_n \to +\infty$, it is said that the inferior limit of the sequence equals positive infinity, and we write $\underline{\lim}_{k\to\infty} x_k = +\infty$ or $\liminf_{k\to\infty} x_k = +\infty$. If the original sequence $\{x_k\}$ is not bounded below, then we shall have $i_n = \inf_{k\geq n} x_k = -\infty$ for all n. In that case we say that the inferior limit of the sequence equals negative infinity and write $\underline{\lim}_{k\to\infty} x_k = -\infty$ or $\liminf_{k\to\infty} x_k = -\infty$.*

$$\underline{\lim}_{k\to\infty} x_k \coloneqq \lim_{n\to\infty} \inf_{k\geq n} x_k$$

Definition 8 (The superior limit of a sequence)**.**

$$\overline{\lim}_{k\to\infty} x_k \coloneqq \lim_{n\to\infty} \sup_{k\geq n} x_k$$

Definition 9 (A partial limit of a sequence)**.** *A number (or the symbol $+\infty$ or $-\infty$) is called a partial limit of a sequence, if the sequence contains a subsequence converging to that number.*

Proposition 4. *The inferior and superior limit of a bounded sequence are respectively the smallest and largest partial limits of the sequence.*

证明. □

Remark 4. *The Bolzano-Weierstrass Lemma in its restricted formulation follows from the above proposition.*

Proposition 5. *For any sequence, the inferior limit is the smallest of its partial limits and the superior limit is the largest of its partial limits.*

Remark 5. *The Bolzano-Weierstrass Lemma in its wider formulation follows from the above proposition.*

Corollary 3. *A sequence has a limit or tends to negative or positive infinity if and only if its inferior and superior limits are the same.*

Corollary 4. *A sequence converges if and only if every subsequence of it converges.*

6 The Limit of a Transformed Sequence

6.1 Toeplitz's Theorem

Theorem 3. *Suppose there exists a sequence* $\{t_{nk}\}$ *such that* $\forall n, k \in \mathbb{N}^+$, $t_{nk} \geq 0$, $\Sigma_{k=1}^n t_{nk} = 1$, $\lim_{n\to\infty} t_{nk} = 0$. *If* $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} \Sigma_{k=1}^n t_{nk} a_k = a$$

Remark 6. *The condition in the Toeplitz' Theorem* $\lim_{n\to\infty} t_{nk} = 0$ *means that for any given* k, *in other words* k *is finite,* t_{nk} *tends to* 0 *when* n *tends to* ∞. *This is supported by the proof, since in the proof we only need the first finite number of terms in the sequence* $\{t_{nk}\}$ *to converge to* 0.

6.2 Stolz's Theorem

Theorem 4 ($\frac{0}{0}$ type)**.** *Suppose* $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, *and* $\{a_n\}$ *is decreasing. If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

Theorem 5 ($\frac{*}{\infty}$ type)**.** *Suppose* $\{a_n\}$ *is increasing and* $\lim_{n\to\infty} a_n = \infty$. *If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

证明. Method: By Toeplitz's Theorem

$$t_n k = \{\frac{a_1}{a_n}, \frac{a_2 - a_1}{a_n}, \frac{a_3 - a_2}{a_n}, \cdots, \frac{a_n - a_{n-1}}{a_n}\}$$

$$c_n = \{\frac{b_1}{a_1}, \frac{b_2 - b_1}{a_2 - a_1}, \frac{b_3 - b_2}{a_3 - a_2}, \cdots, \frac{b_n - b_{n-1}}{a_n - a_{n-1}}\}$$

□

6.3 Cauchy's Proposition

Proposition 6 (算术平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

Proposition 7 (算术平均值等价形式)**.** *If* $\lim_{n\to\infty}(a_n - a_{n-1}) = a$, *then*

$$\lim_{n\to\infty} \frac{a_n}{n} = a$$

Proposition 8 (几何平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a > 0$, *then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a$$