Notes of "Basic Definitions and Examples"

Jinxin Wang

1 Continuity of a Function at a Point

Definition 1 (Continuity of a Function at a Point with Domain in a Neighborhood of the Point)**.**

Definition 2 (Continuity of a Function at a Point with General Domain)**.**

$$f : E \to \mathbb{R} \text{ is continuous at } a \in E := (\forall V(f(a)))\exists U_E(a)(f(U_E(a)) \subset V(f(a)))$$

Remark 1. *Depending on the kind of point $x = a$ of the domain E:*

- *If a is an isolated point of E, then there exists $U_E(a) = \{a\}$, and $\forall V(f(a))$, $f(U_E(a)) = \{f(a)\} \subset V(f(a))$. Therefore, f is continuous at any isolated point of its domain E.*
- *If a is a limit point of E, then we have an equivalent definition of continuity at the point:*

$$(f : E \to \mathbb{R} \text{ is continuous at } a \in E, \text{where } a \text{ is a limit point of } E) \Leftrightarrow \lim_{E \ni x \to a} f(x) = f(a) \tag{1}$$

证明. (TODO) □

Remark 2. *Since we can rewrite*

$$\lim_{E \ni x \to a} f(x) = f(a) = f(\lim_{E \ni x \to a} x) \tag{2}$$

It leads to the conclusion that continuous functions and only the continuous ones can commute with the operation of passing to the limit at a point (只有连续函数可以与取极限交换运算顺序).

Remark 3. *By the Cauchy criterion we can give another equivalent definition of continuity at a point with the concept of the oscillation of a function at a point.*

Definition 3 (The Oscillation of a Function at a Point)**.** *The oscillation of $f : E \to \mathbb{R}$ at a, denoted as $\omega(f; a)$, is defined as*

$$\omega(f; a) = \lim_{\delta \to 0^+} \omega(f; U_E^\delta(a)) \tag{3}$$

Then we have the following statement:

$$(f : E \to \mathbb{R} \text{ is continuous at } a \in E) \Leftrightarrow (\omega(f; a) = 0)$$

Definition 4 (Continuity of a Function on a Set)**.**

2 Points of Discontinuity

Definition 5 (Point of Discontinuity)**.** *If the function $f : E \to \mathbb{R}$ is not continuous at a point of E, this point is called a point of discontinuity or simply a discontinuity of f.*

Remark 4. *A point of discontinuity of a function must belong to the domain of the definition of the function. Continuity or discontinuity of a function at point is not discussed outside of the domain of the function.*

Definition 6 (Removable Discontinuity)**.** *If a point of discontinuity $a \in E$ of the function $f : E \to \mathbb{R}$ is such that there exists a continuous function $\tilde{f} : E \to \mathbb{R}$ such that $f|_{E\setminus a} = \tilde{f}|_{E\setminus a}$, then a is called a removable discontinuity of the function f.*

Example 1.

$$f(x) = \begin{cases} \sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$$

Definition 7 (Discontinuity of First Kind)**.** *The point $a \in E$ is called a discontinuity of first kind for the function $f : E \to \mathbb{R}$ if the following limits exist*

$$f(a+0) := \lim_{E \ni x \to a+0} \text{ or } f(a-0) := \lim_{E \ni x \to a-0}$$

but at least one of them is not equal to the value $f(a)$ that the function assumes at a.

Remark 5. *A removable discontinuity is a discontinuity of first kind.*

Definition 8 (Discontinuity of Second Kind)**.** *If $a \in E$ is a point of discontinuity of the function $f : E \to \mathbb{R}$ and at least one of the two limits*

$$f(a+0) := \lim_{E \ni x \to a+0} \text{ or } f(a-0) := \lim_{E \ni x \to a-0}$$

does not exist, then a is called a discontinuity of second kind.

Example 2 (The Dirichlet Function)**.**

$$\mathcal{D}(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Example 3 (The Riemann Function)**.**

$$\mathcal{R}(x) = \begin{cases} \frac{1}{n}, \text{ if } x = \frac{m}{n} \in \mathbb{Q}, n \in \mathbb{N} \\ 0, \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$