Notes of "Definition and Properties of the Limit of a Sequence"

Jinxin Wang

1 Overview

2 Definition of the Limit of a Sequence

Definition 1 (A Sequence). *A function $f : \mathbb{N} \to X$ is called a sequence.*

Definition 2 (The Limit of a Numerical Sequence). *A number $A \in \mathbb{R}$ is called the limit of the numerical sequence $\{x_n\}$ if for every neighborhood $V(A)$ of A there exists an index N (depending on $V(A)$) such that all terms of the sequence having index larger than N belong to the neighborhood $V(A)$.*

Remark 1 (An Equivalent Definition of the Limit of a Numerical Sequence with $\epsilon - N$). *Another equivalent definition of the limit of a numerical sequence is:*

A number $A \in \mathbb{R}$ is called the limit of the numerical sequence $\{x_n\}$ if for every $\epsilon > 0$ there exists an index N (depending on ϵ) such that $|x_n - A| < \epsilon$ for all $n > N$.

Proof of equivalence: (TODO)

Remark 2 (Formulation of definitions of the limit of a numerical sequence in symbolic logic).

$$(\lim_{n\to\infty} x_n = A) \coloneqq (\forall V(A) \exists N \in \mathbb{N} \forall n > N (x_n \in V(A)))$$

$$(\lim_{n\to\infty} x_n = A) \coloneqq (\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N (|x_n - A| < \epsilon))$$

Definition 3 (A convergent/divergent sequence). *If $\lim_{n\to\infty} x_n = A$, we say that the sequence $\{x_n\}$ converges to A or tends to A and write $x_n \to A$ as $n \to \infty$.*

A sequence having a limit is said to be convergent. A sequence that does not have a limit is said to be divergent.

3 Properties of the Limit of a Sequence

3.1 General Properties

The word "general" means that the properties in this section are possessed not only by numerical sequences, but by other kinds of sequences as well.

Definition 4 (An ultimately constant sequence). *If there exists a number A and an index N such that $x_n = A$ for all $n > N$, the sequence $\{x_n\}$ will be called ultimately constant.*

Definition 5 (A bounded sequence). *A sequence $\{x_n\}$ is bounded if there exists M such that $|x_n| < M$ for all $n \in \mathbb{N}$.*

Theorem 1. T1 *An ultimately constant sequence converges.*

T2 *Any neighborhood of the limit of a sequence contains all but a finite number of terms of the sequence.*

T3 *A convergent sequence cannot have two different limits.*

T4 *A convergent sequence is bounded.*

证明. **Pf of T1**

□

3.2 Properties Involving Arithmetic Operations

Definition 6 (The sum, product and quotient of two numerical sequences)**.** *If $\{x_n\}$ and $\{y_n\}$ are two numerical sequences, their sum, product, and quotient are the sequences*

$$\{(x_n + y_n)\}, \{(x_n \cdot y_n)\}, \{(\frac{x_n}{y_n})\}$$

The quotient is defined only when $y_n \neq 0$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $\{x_n\}$ and $\{y_n\}$ be numerical sequences. If $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$, then*

T1 $\lim_{n\to\infty}(x_n + y_n) = A + B$

T2 $\lim_{n\to\infty}(x_n \cdot y_n) = AB$

T3 $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{A}{B}$, *provided $y_n \neq 0 (n = 1, 2, \ldots)$ and $B \neq 0$*

证明. **Pf of T1**

Pf of T2

Pf of T3

□

3.3 Properties Involving Inequalities

Theorem 3. T1 *(保序性) Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences with $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$. If $A < B$, then there exists an index $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n > N$.*

T2 *(夹逼性) Suppose the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are such that $x_n \leq y_n \leq z_n$ for all $n > N \in \mathbb{N}$. If the sequences $\{x_n\}$ and $\{z_n\}$ both converge to the same limit, then the sequence $\{y_n\}$ also converges to that limit.*

证明. **Pf of T1**

Pf of T2

□

Corollary 1. *Suppose $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$. If there exists N such that for all $n > N$ we have*

C1 $x_n > y_n$, *then* $A \geq B$.

C2 $x_n \geq y_n$, *then* $A \geq B$.

C3 $x_n > B$, *then* $A \geq B$.

C4 $x_n \geq B$, *then* $A \geq B$.

Remark 3. *Notice that strict inequality in terms may become equality in the limit. Example:* $\frac{1}{n} > 0$ *for all* $n \in \mathbb{N}$ *but* $\lim_{n\to\infty} \frac{1}{n} = 0$.

4 Infinity

4.1 Definition of Infinity

Definition 7 (Infinity, Positive Infinity, Negative Infinity)**.**

Corollary 2 (Relation between Infinity and Infinitesimal)**.**

Definition 8 (Not an Infinity)**.**

4.2 Operations Involving Infinity