Notes of "Basic Lemmas Connected with the Completeness of the Real Numbers"

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1 Overview

- The Nested Interval Lemma
  - Definition: A sequence of elements of a set
  - Definition: A sequence of nested intervals
  - Lemma: The Nested Interval Lemma
- The Finite Covering Lemma
  - Definition: A cover of a set
  - Lemma: The Finite Covering Lemma
- The Limit Point Lemma
  - Definition: A limit point of a set
  - Lemma: The Limit Point Lemma

2 The Nested Interval Lemma (Cauchy-Cantor Principle)

Definition 1 (A Sequence of Elements of a Set)**.** *A function $f : \mathbb{N} \to X$ of a natural-number argument is called a sequence or, more fully, a sequence of elements of X.*

Definition 2 (A Sequence of Nested Intervals)**.** *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of sets. If $X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$, that is $X_n \supset X_{n+1}$ for all $n \in \mathbb{N}$, we say the sequence is nested.*

Lemma 1 (The Nested Interval Lemma, or Cauchy-Cantor Principle)**.** *For any nested sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of closed intervals, there exists a point $c \in \mathbb{R}$ belonging to all of these intervals.*

If in addition it is known that for any $\epsilon > 0$ there is an interval I_k whose length $|I_k|$ is less than ϵ, then c is the unique point common to all the intervals.

证明. (TODO) □

Remark 1. *Notice the sequence of nested intervals are closed intervals. If they are open intervals (TODO)*

3 The Finite Covering Lemma (Borel-Lebesgue Principle)

Definition 3 (A Cover of a Set)**.** *A system $S = \{X\}$ of sets X is said to cover a set Y if $Y \subset \bigcup_{X \in S} X$, that is, if every element $y \in Y$ belongs to at least one of the sets X in the system S.*

Lemma 2 (The Finite Covering Lemma, or Borel-Lebesgue Principle)**.** *Every system of open intervals covering a closed interval containing a finite subsystem that covers the closed interval.*

证明. (TODO) □

4 The Limit Point Lemma (Bolzano-Weierstrass Principle)

Definition 4 (A Limit Point of a Set)**.** *A point $p \in \mathbb{R}$ is a limit point of the set $X \subset \mathbb{R}$ if every neighborhood of the point contains an infinite subset of X.*

Remark 2. *An equivalent condition is that every neighborhood of p contains at least one point of X different from p itself.*

Remark 3. *The concept of a limit point is 相对的. It is clear from the definition that we must specify the set of a limit point. A limit point of a set $X_1 \subset \mathbb{R}$ might not be a limit point in terms of another set $X_2 \subset \mathbb{R}$. For example, a finite point set $X \subset \mathbb{R}$ can never have a limit point since it does not have any infinite subset.*

Lemma 3 (The Limit Point Principle, or Bolzano-Weierstrass Principle)**.** *Every bounded infinite set of real numbers has at least one limit point.*

证明. (TODO) □