Notes of "The Limit of A Function"

Jinxin Wang

1 Definitions and Examples

Definition 1 (The Limit of a Function (Basic Type)). *(ϵ-δ)*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E (0 < |x - a| < \delta \Rightarrow |f(x) - A| < \epsilon) \tag{1}$$

Remark 1. *Based on the definition, we can see that limits are a kind of local characteristic of a function. With that, when using the definition to prove the limit of a function at $x = x_0$, we can discuss it within a certain neighborhood $O(x_0, \delta_0)$.*

Definition 2 (去心邻域). *A deleted neighborhood of a point is a neighborhood of the point from which the point itself has been removed.*

Definition 3 (函数极限的邻域定义).

$$(\lim_{E \ni x \to a} f(x) = A) := \forall U_{\mathbb{R}}(A) \exists \dot{U}_E(a) (f(\dot{U}_E(a)) \subset U_{\mathbb{R}}(A)) \tag{2}$$

Remark 2. *If f is convergent at a, a must be a limit point of its domain E.*

Example 1 (符号函数 $\operatorname{sgn} x$).

Proposition 1 (Heine's Proposition).

Corollary 1 (Existence of the Limit of a Function by Limits of Sequences).

2 Properties of the Limit of a Function

Remark 3. *In order to establish the properties of the limit of a function, we need only two properties of deleted neighborhoods of a limit point of a set:*

- $\dot{U}_E(a) \neq \emptyset$
- $\forall \dot{U}_E^1(a) \dot{U}_E^2(a) \exists \dot{U}_E(a) (\dot{U}_E(a) \subset \dot{U}_E^1(a) \cap \dot{U}_E^2(a))$

This observation leads us to a general concept of a limit of a function and the possibility of using the theory of limits in the future not only for functions defined on sets of numbers.

2.1 General Properties

Definition 4 (最终常数函数)**.**

Definition 5 (有界函数/最终有界函数，上有界函数/最终上有界函数，下有界函数/最终下有界函数)**.** *A function $f : E \to \mathbb{R}$ is bounded, bounded above, or bounded below respectively if there is a number $C \in \mathbb{R}$ such that $|f(x)| < C$, $f(x) < C$, or $C < f(x)$ for all $x \in E$.*

If one of these three relations holds only in some deleted neighborhood $\dot{U}_E(a)$, the function is said to be ultimately bounded, ultimately bounded above, or ultimatedly bounded below as $E \ni x \to a$ respectively.

Theorem 1.

1. *(Ultimate Constant has the Limit) ($f : E \to \mathbb{R}$ is ultimately the constant A as $E \ni x \to a$) $\Rightarrow \lim_{E\ni x\to a} f(x) = A$*
2. *(Ultimately Boundness of the Limit) ($\exists \lim_{E\ni x\to a} f(x)$) $\Rightarrow$ ($f : E \to \mathbb{R}$ is ultimately bounded as $E \ni x \to a$)*
3. *(Uniqueness of the Limit) $(\lim_{E\ni x\to a} f(x) = A_1) \wedge (\lim_{E\ni x\to a} f(x) = A_2) \Rightarrow A_1 = A_2$*

证明. □

2.2 Properties Involving Arithmetic Operations

Definition 6 (两个函数的和、积与商)**.**

Theorem 2 (四则运算中的函数极限)**.** *Let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two functions with a common domain of definition. If $\lim_{E\ni x\to a} f(x) = A$ and $\lim_{E\ni x\to a} g(x) = B$, then*

- $\lim_{E\ni x\to a} f(x) + g(x) = A + B$
- $\lim_{E\ni x\to a} f(x)g(x) = AB$
- $\lim_{E\ni x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}$*, if* $B \neq 0$ *and* $g(x) \neq 0$ *for* $x \in E$.

Definition 7 (Infinitesimal)**.** *A function $f : E \to \mathbb{R}$ is said to be infinitesimal as $E \ni x \to a$ if $\lim_{E\ni x\to a} f(x) = 0$*

Proposition 2 (无穷小函数的四则运算性质)**.** *Let $\alpha : E \to \mathbb{R}$ is an infinitesimal as $E \ni x \to a$, then*

- *If $\beta : E \to \mathbb{R}$ is also an infinitesimal as $E \ni x \to a$, then their sum $\alpha + \beta$ is also an infinitesimal as $E \ni x \to a$.*
- *If $\beta : E \to \mathbb{R}$ is also an infinitesimal as $E \ni x \to a$, then their product $\alpha\beta$ is also an infinitesimal as $E \ni x \to a$.*
- *If $\beta : E \to \mathbb{R}$ is ultimately bounded as $E \ni x \to a$, then their product $\alpha\beta$ is also an infinitesimal as $E \ni x \to a$.*

证明. □

Remark 4.

$$(\lim_{E\ni x\to a} f(x) = A) \Leftrightarrow (\lim_{E\ni x\to a} (f(x) - A) = 0) \tag{3}$$

2.3 Properties Involving Inequalities

Theorem 3. • *(局部保序性) If the functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are such that*

- *(夹逼性)*

Corollary 2 (局部保号性)**.**

Corollary 3 (极限值的不等关系)**.** *Suppose $\lim_{E \ni x \to a} f(x) = A$ and $\lim_{E \ni x \to a} g(x) = B$. Let $\dot{U}_E(a)$ be a deleted neighborhood of $a \in E$.*

(a) *If $f(x) > g(x)$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.*

(b) *If $f(x) \geq g(x)$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.*

(c) *If $f(x) > B$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.*

(d) *If $f(x) \geq B$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.*

2.4 Two Important Examples

Example 2 (研究 $\frac{\sin x}{x}$ 在 $x = 0$ 处的极限)**.**

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{4}$$

Example 3 (定义指数函数、对数函数和幂函数)**.**

3 The General Definition of the Limit of a Function

3.1 Definition and Examples of a Base

Definition 8. *A set $\mathcal{B}$ of subsets $B \subset X$ of a set X is called a base in X if the following conditions hold:*

- $\forall B \in \mathcal{B}\ (B \neq \emptyset)$
- $\forall B_1, B_2 \in \mathcal{B} \exists B \in \mathcal{B}\ (B \subset B_1 \cap B_2)$

Some useful bases in analysis

$x \to a$	$\dot{U}(a) := \{x \in \mathbb{R} \mid a - \delta_1 < x < a + \delta_2 \wedge x \neq a, \delta_1 > 0, \delta_2 > 0\}$
$x \to a + 0$	$E_a^+ := \{x \in \mathbb{R} \mid a < x\}, \dot{U}_{E_a^+}(a) := E_a^+ \cap \dot{U}(a) = \{x \in \mathbb{R} \mid a < x < a + \delta, \delta > 0\}$
$x \to a - 0$	$E_a^+ := \{x \in \mathbb{R} \mid x < a\}, \dot{U}_{E_a^-}(a) := E_a^- \cap \dot{U}(a) = \{x \in \mathbb{R} \mid a - \delta < x < a, \delta > 0\}$
$x \to \infty$	$U(\infty) := \{x \in \mathbb{R} \mid \lvert x \rvert > \delta, \delta \in \mathbb{R}\}$
$x \to +\infty$	$E_\infty^+ := \{x \in \mathbb{R} \mid c < x, c \in \mathbb{R}\}, U_{E_\infty^+}(\infty) := E_\infty^+ \cap U(\infty) = \{x \in \mathbb{R} \mid c < x, c \in \mathbb{R}\}$
$x \to -\infty$	$E_\infty^- := \{x \in \mathbb{R} \mid x < c, c \in \mathbb{R}\}, U_{E_\infty^-}(\infty) := E_\infty^- \cap U(\infty) = \{x \in \mathbb{R} \mid x < c, c \in \mathbb{R}\}$
$E \ni x \to a$	$\dot{U}_E(a) := E \cap \dot{U}(a)$
$E \ni x \to a + 0$	$\dot{U}_E(a + 0) := E \cap E_a^+ \cap \dot{U}(a)$
$E \ni x \to a - 0$	$\dot{U}_E(a - 0) := E \cap E_a^- \cap \dot{U}(a)$
$E \ni x \to \infty$	$U_E(\infty) := E \cap U(\infty)$
$E \ni x \to +\infty$	$U_E(+\infty) := E \cap E_\infty^+ \cap U(\infty)$
$E \ni x \to -\infty$	$U_E(-\infty) := E \cap E_\infty^- \cap U(\infty)$

3.2 Limit over a Base

Definition 9 (The Limit of a Function over a Base)**.**

$$(\lim_{\mathcal{B}} f(x) = A) := (\forall U(A) \exists B \in \mathcal{B}(f(B) \subset U(A))) \tag{5}$$

$$\lim_{x \to a} f(x) = A := (\forall \epsilon > 0, \exists \delta > 0, \forall x \in (a - \delta, a + \delta) : (|f(x) - A| < \epsilon))$$

$$\lim_{x \to a+0} f(x) = A$$

$$\lim_{x \to a-0} f(x) = A$$

$$\lim_{x \to \infty} f(x) = A$$

$$\lim_{x \to +\infty} f(x) = A$$

$$\lim_{x \to -\infty} f(x) = A$$

$$\lim_{x \to a} f(x) = \infty$$

$$\lim_{x \to a+0} f(x) = \infty$$

$$\lim_{x \to a-0} f(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to +\infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = \infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a+0} f(x) = +\infty$$

$$\lim_{x \to a-0} f(x) = +\infty$$

$$\lim_{x \to \infty} f(x) = +\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to a+0} f(x) = -\infty$$

$$\lim_{x \to a-0} f(x) = -\infty$$

$$\lim_{x \to \infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = -\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

4 The Existence of the Limit of a Function

4.1 Cauchy's Criterion

Definition 10 (Oscillation).

Theorem 4 (Cauchy's Criterion on the Limit of a Function).

4.2 The Limit of a Composite Function

Theorem 5 (Theorem of the Limit of a Composite Function).

Example 4.

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e \tag{6}$$

4.3 The Limit of a Monotonic Function

Definition 11 (Monotonic Functions). *A function $f : E \to \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is said to be*

- *increasing if* $\forall x_1, x_2 \in E(x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$
- *nondecreasing if* $\forall x_1, x_2 \in E(x_1 < x_2 \Rightarrow f(x_1) \le f(x_2))$
- *decreasing if* $\forall x_1, x_2 \in E(x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$
- *nonincreasing if* $\forall x_1, x_2 \in E(x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2))$

Remark 5. *Be mindful of that the monotonicity is defined on the complete domain of the definition, and thus the condition is $\forall x_1, x_2 \in E$ no matter what a set E is.*

Theorem 6 (The Existence Criterion of the Limit of a Monotonic Function).

4.4 Comparison of the Limiting Behaviors of Functions

Proposition 3.

Remark 6. *The above proposition can not be generalized to the limit of a sum of functions.*

5 方法与技巧

5.1 证明与研究函数极限

- 定义法
- 变量代换

Remark 7. *在研究函数极限时使用变量代换是否总是成立？这个问题使用数学语言来描述如下：
Suppose that $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$, is it true that*

$$\lim_{x\to a} f(g(x)) = \lim_{y\to A} f(y)$$

In fact it is not always true. Here is a counterexample: Let $g(x) \equiv 0$, *and thus* $\lim_{x\to 0} g(x) = 0$. *Let* $f(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}$. *Then we have* $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} f(g(x)) = 1$. 这里的数学直观是 $\lim_{x\to a} g(x) = A$ 决定了 $y = g(x) \to A$ 的方式。它与 $\lim_{x\to A} f(x) = B$ 中 $x \to A$ 的不同可能导致结果的不同。

Here are two propositions related to this problem:

Proposition 4. *Suppose that* $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$. *If any of the following conditions is true:*

- $\exists\delta_0 > 0$ *such that* $\forall x \in O(a, \delta_0) \setminus \{a\}$: $g(x) \neq A$.
- $\lim_{x\to A} f(x) = f(A)$.
- $A = \infty$, *and* $\lim_{x\to\infty} f(x)$ *is defined.*

then the following is true:

$$\lim_{x\to a} f(g(x)) = \lim_{y\to A} f(y)$$

证明. Hint:

- Notice the difference between the conclusion of the definition of $\lim_{x\to a} g(x) = A$, and the condition of the definition of $\lim_{x\to A} f(x) = B$.
- Notice what change the fact of continuity brings to the definition of $\lim_{x\to A} f(x) = B$.

□

Proposition 5. *If* $\lim_{x\to a} g(x) = A$, $\lim_{x\to A} f(x) = B$, *then exact one of the following situation is true:*

- $\lim_{x\to a} f(g(x)) = B$
- $\lim_{x\to a} f(g(x)) = g(A)$
- $\lim_{x\to a} f(g(x))$ *is not defined*

证明. Hint: Using the first condition of the previous proposition to discuss different kinds of $g(x)$. □