

Notes of "The Limit of A Function"

Jinxin Wang

1 Definitions and Examples

Definition 1 (The Limit of a Function (Basic Type)). (ϵ - δ)

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in E (0 < |x - a| < \delta \Rightarrow |f(x) - A| < \epsilon) \quad (1)$$

Remark 1. Based on the definition, we can see that limits are a kind of local characteristic of a function. With that, when using the definition to prove the limit of a function at $x = x_0$, we can discuss it within a certain neighborhood $O(x_0, \delta_0)$.

Definition 2 (A deleted neighborhood of a point). A deleted neighborhood of a point is a neighborhood of the point from which the point itself has been removed.

Definition 3 (函数极限的邻域定义).

$$(\lim_{E \ni x \rightarrow a} f(x) = A) := \forall U_{\mathbb{R}}(A) \exists \dot{U}_E(a) (f(\dot{U}_E(a)) \subset U_{\mathbb{R}}(A)) \quad (2)$$

Remark 2. If f is convergent at a , a must be a limit point of its domain E .

Example 1 (符号函数 $\operatorname{sgn} x$).

Proposition 1 (Heine's Proposition).

Corollary 1 (Existence of the Limit of a Function by Limits of Sequences).

2 Properties of the Limit of a Function

Remark 3. In order to establish the properties of the limit of a function, we need only two properties of deleted neighborhoods of a limit point of a set:

- $\dot{U}_E(a) \neq \emptyset$
- $\forall \dot{U}_E^1(a) \dot{U}_E^2(a) \exists \dot{U}_E(a) (\dot{U}_E(a) \subset \dot{U}_E^1(a) \cap \dot{U}_E^2(a))$

This observation leads us to a general concept of a limit of a function and the possibility of using the theory of limits in the future not only for functions defined on sets of numbers.

2.1 General Properties

Definition 4 (最终常数函数).

Definition 5 (有界函数/最终有界函数, 上有界函数/最终上有界函数, 下有界函数/最终下有界函数).
A function $f : E \rightarrow \mathbb{R}$ is bounded, bounded above, or bounded below respectively if there is a number $C \in \mathbb{R}$ such that $|f(x)| < C$, $f(x) < C$, or $C < f(x)$ for all $x \in E$.

If one of these three relations holds only in some deleted neighborhood $\dot{U}_E(a)$, the function is said to be ultimately bounded, ultimately bounded above, or ultimately bounded below as $E \ni x \rightarrow a$ respectively.

Theorem 1. 1° (Ultimate Constant has the Limit) ($f : E \rightarrow \mathbb{R}$ is ultimately the constant A as $E \ni x \rightarrow a$) $\Rightarrow \lim_{E \ni x \rightarrow a} f(x) = A$

2° (Ultimately Boundness of the Limit) ($\exists \lim_{E \ni x \rightarrow a} f(x)$) $\Rightarrow (f : E \rightarrow \mathbb{R}$ is ultimately bounded as $E \ni x \rightarrow a$)

3° (Uniqueness of the Limit) ($\lim_{E \ni x \rightarrow a} f(x) = A_1$) \wedge ($\lim_{E \ni x \rightarrow a} f(x) = A_2$) $\Rightarrow A_1 = A_2$

证明. □

2.2 Properties Involving Arithmetic Operations

Definition 6 (两个函数的和、积与商).

Theorem 2 (四则运算中的函数极限). Let $f : E \rightarrow \mathbb{R}$ and $g : E \rightarrow \mathbb{R}$ be two functions with a common domain of definition. If $\lim_{E \ni x \rightarrow a} f(x) = A$ and $\lim_{E \ni x \rightarrow a} g(x) = B$, then

- $\lim_{E \ni x \rightarrow a} f(x) + g(x) = A + B$
- $\lim_{E \ni x \rightarrow a} f(x)g(x) = AB$
- $\lim_{E \ni x \rightarrow a} \frac{f(x)}{g(x)} = \frac{A}{B}$, if $B \neq 0$ and $g(x) \neq 0$ for $x \in E$.

Definition 7 (Infinitesimal). A function $f : E \rightarrow \mathbb{R}$ is said to be infinitesimal as $E \ni x \rightarrow a$ if $\lim_{E \ni x \rightarrow a} f(x) = 0$

Proposition 2 (无穷小函数的四则运算性质). Let $\alpha : E \rightarrow \mathbb{R}$ is an infinitesimal as $E \ni x \rightarrow a$, then

- If $\beta : E \rightarrow \mathbb{R}$ is also an infinitesimal as $E \ni x \rightarrow a$, then their sum $\alpha + \beta$ is also an infinitesimal as $E \ni x \rightarrow a$.
- If $\beta : E \rightarrow \mathbb{R}$ is also an infinitesimal as $E \ni x \rightarrow a$, then their product $\alpha\beta$ is also an infinitesimal as $E \ni x \rightarrow a$.
- If $\beta : E \rightarrow \mathbb{R}$ is ultimately bounded as $E \ni x \rightarrow a$, then their product $\alpha\beta$ is also an infinitesimal as $E \ni x \rightarrow a$.

证明. □

Remark 4.

$$\left(\lim_{E \ni x \rightarrow a} f(x) = A \right) \Leftrightarrow \left(\lim_{E \ni x \rightarrow a} (f(x) - A) = 0 \right) \quad (3)$$

2.3 Properties Involving Inequalities

Theorem 3. 1° (局部保序性) If the functions $f : E \rightarrow \mathbb{R}$ and $g : E \rightarrow \mathbb{R}$ are such that

2° (夹逼性)

Corollary 2 (局部保号性).

Corollary 3 (极限值的不等关系). Suppose $\lim_{E \ni x \rightarrow a} f(x) = A$ and $\lim_{E \ni x \rightarrow a} g(x) = B$. Let $\dot{U}_E(a)$ be a deleted neighborhood of $a \in E$.

(a) If $f(x) > g(x)$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.

(b) If $f(x) \geq g(x)$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.

(c) If $f(x) > B$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.

(d) If $f(x) \geq B$ for all $x \in \dot{U}_E(a)$, then $A \geq B$.

2.4 Two Important Examples

Example 2 (研究 $\frac{\sin x}{x}$ 在 $x = 0$ 处的极限).

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (4)$$

Example 3 (定义指数函数、对数函数和幂函数).

3 The General Definition of the Limit of a Function

3.1 Definition and Examples of a Base

Definition 8. A set \mathcal{B} of subsets $B \subset X$ of a set X is called a base in X if the following conditions hold:

- $\forall B \in \mathcal{B} \ (B \neq \emptyset)$
- $\forall B_1, B_2 \in \mathcal{B} \exists B \in \mathcal{B} \ (B \subset B_1 \cap B_2)$

Some useful bases in analysis

$x \rightarrow a$	$\dot{U}(a) := \{x \in \mathbb{R} \mid a - \delta_1 < x < a + \delta_2 \wedge x \neq a, \delta_1 > 0, \delta_2 > 0\}$
$x \rightarrow a + 0$	$E_a^+ := \{x \in \mathbb{R} \mid a < x\}, \dot{U}_{E_a^+}(a) := E_a^+ \cap \dot{U}(a) = \{x \in \mathbb{R} \mid a < x < a + \delta, \delta > 0\}$
$x \rightarrow a - 0$	$E_a^- := \{x \in \mathbb{R} \mid x < a\}, \dot{U}_{E_a^-}(a) := E_a^- \cap \dot{U}(a) = \{x \in \mathbb{R} \mid a - \delta < x < a, \delta > 0\}$
$x \rightarrow \infty$	$U(\infty) := \{x \in \mathbb{R} \mid x > \delta, \delta \in \mathbb{R}\}$
$x \rightarrow +\infty$	$E_\infty^+ := \{x \in \mathbb{R} \mid c < x, c \in \mathbb{R}\}, U_{E_\infty^+}(\infty) := E_\infty^+ \cap U(\infty) = \{x \in \mathbb{R} \mid c < x, c \in \mathbb{R}\}$
$x \rightarrow -\infty$	$E_\infty^- := \{x \in \mathbb{R} \mid x < c, c \in \mathbb{R}\}, U_{E_\infty^-}(\infty) := E_\infty^- \cap U(\infty) = \{x \in \mathbb{R} \mid x < c, c \in \mathbb{R}\}$
$E \ni x \rightarrow a$	$\dot{U}_E(a) := E \cap \dot{U}(a)$
$E \ni x \rightarrow a + 0$	$\dot{U}_E(a + 0) := E \cap E_a^+ \cap \dot{U}(a)$
$E \ni x \rightarrow a - 0$	$\dot{U}_E(a - 0) := E \cap E_a^- \cap \dot{U}(a)$
$E \ni x \rightarrow \infty$	$U_E(\infty) := E \cap U(\infty)$
$E \ni x \rightarrow +\infty$	$U_E(+\infty) := E \cap E_\infty^+ \cap U(\infty)$
$E \ni x \rightarrow -\infty$	$U_E(-\infty) := E \cap E_\infty^- \cap U(\infty)$

3.2 Limit over a Base

Definition 9 (The Limit of a Function over a Base).

$$(\lim_{\mathcal{B}} f(x) = A) := (\forall U(A) \exists B \in \mathcal{B} (f(B) \subset U(A))) \quad (5)$$

$$\lim_{x \rightarrow a} f(x) = A := (\forall \epsilon > 0, \exists \delta > 0, \forall x \in (a - \delta, a + \delta) : (|f(x) - A| < \epsilon))$$

$$\lim_{x \rightarrow a+0} f(x) = A$$

$$\lim_{x \rightarrow a-0} f(x) = A$$

$$\lim_{x \rightarrow \infty} f(x) = A$$

$$\lim_{x \rightarrow +\infty} f(x) = A$$

$$\lim_{x \rightarrow -\infty} f(x) = A$$

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a+0} f(x) = \infty$$

$$\lim_{x \rightarrow a-0} f(x) = \infty$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow +\infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = +\infty$$

$$\lim_{x \rightarrow a+0} f(x) = +\infty$$

$$\lim_{x \rightarrow a-0} f(x) = +\infty$$

$$\lim_{x \rightarrow \infty} f(x) = +\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a+0} f(x) = -\infty$$

$$\lim_{x \rightarrow a-0} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

4 The Existence of the Limit of a Function

4.1 Cauchy's Criterion

Definition 10 (Oscillation).

Theorem 4 (Cauchy's Criterion on the Limit of a Function).

4.2 The Limit of a Composite Function

Theorem 5 (Theorem of the Limit of a Composite Function).

Example 4.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad (6)$$

4.3 The Limit of a Monotonic Function

Definition 11 (Monotonic Functions). A function $f : E \rightarrow \mathbb{R}$ defined on a set $E \subset \mathbb{R}$ is said to be

- *increasing* if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$
- *nondecreasing* if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2))$
- *decreasing* if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$
- *nonincreasing* if $\forall x_1, x_2 \in E (x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2))$

Remark 5. Be mindful of that the monotonicity is defined on the complete domain of the definition, and thus the condition is $\forall x_1, x_2 \in E$ no matter what a set E is.

Theorem 6 (The Existence Criterion of the Limit of a Monotonic Function).

4.4 Comparison of the Limiting Behaviors of Functions

Proposition 3.

Remark 6. The above proposition can not be generalized to the limit of a sum of functions.

5 方法与技巧

5.1 证明与研究函数极限

- 定义法
- 变量代换

Remark 7. 在研究函数极限时使用变量代换是否总是成立？这个问题使用数学语言来描述如下：

Suppose that $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$, is it true that

$$\lim_{x \rightarrow a} f(g(x)) = \lim_{y \rightarrow A} f(y)$$

In fact it is not always true. Here is a counterexample: Let $g(x) \equiv 0$, and thus $\lim_{x \rightarrow 0} g(x) = 0$.

Let $f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$. Then we have $\lim_{x \rightarrow 0} f(x) = 0$, $\lim_{x \rightarrow 0} f(g(x)) = 1$. 这里的数学直观是

$\lim_{x \rightarrow a} g(x) = A$ 决定了 $y = g(x) \rightarrow A$ 的方式。它与 $\lim_{x \rightarrow A} f(x) = B$ 中 $x \rightarrow A$ 的不同可能导致结果的不同。

Here are two propositions related to this problem:

Proposition 4. Suppose that $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$. If any of the following conditions is true:

- $\exists \delta_0 > 0$ such that $\forall x \in O(a, \delta_0) \setminus \{a\}$: $g(x) \neq A$.
- $\lim_{x \rightarrow A} f(x) = f(A)$.
- $A = \infty$, and $\lim_{x \rightarrow \infty} f(x)$ is defined.

then the following is true:

$$\lim_{x \rightarrow a} f(g(x)) = \lim_{y \rightarrow A} f(y)$$

证明. Hint:

- Notice the difference between the conclusion of the definition of $\lim_{x \rightarrow a} g(x) = A$, and the condition of the definition of $\lim_{x \rightarrow A} f(x) = B$.
- Notice what change the fact of continuity brings to the definition of $\lim_{x \rightarrow A} f(x) = B$.

□

Proposition 5. If $\lim_{x \rightarrow a} g(x) = A$, $\lim_{x \rightarrow A} f(x) = B$, then exact one of the following situation is true:

- $\lim_{x \rightarrow a} f(g(x)) = B$
- $\lim_{x \rightarrow a} f(g(x)) = f(A)$
- $\lim_{x \rightarrow a} f(g(x))$ is not defined

证明. Hint: Using the first condition of the previous proposition to discuss different kinds of $g(x)$. □