Notes of "Existence of the Limit of a Sequence"

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1 Overview

2 The Cauchy Criterion

3 A Criterion for the Existence of the Limit of a Monotonic Sequence

4 The Number e

Proposition 1. *The sequences $a_n = (1 + \frac{1}{n})^n$ and $b_n = (1 + \frac{1}{n})^{n+1}$ are convergent, and they have the same limit values.*

证明. □

Definition 1.

Proposition 2. $e = \Sigma_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$

证明. □

Proposition 3. $e = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}}$

证明. □

4.1 pi

4.2 Euler Number

Theorem 1 (Weierstrass Theorem / 单调有界数列收敛定理)**.**

Corollary 1. 当数列严格单调增加时,

5 Subsequences and the Partial Limits

6 The Limit of a Transformed Sequence

6.1 Toeplitz's Theorem

Theorem 2. *Suppose there exists a sequence* $\{t_{nk}\}$ *such that* $\forall n, k \in \mathbb{N}^+$, $t_{nk} \geq 0$, $\Sigma_{k=1}^{n} t_{nk} = 1$, $\lim_{n\to\infty} t_{nk} = 0$. *If* $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} \Sigma_{k=1}^{n} t_{nk} a_k = a$$

Remark 1. *The condition in the Toeplitz' Theorem* $\lim_{n\to\infty} t_{nk} = 0$ *means that for any given* k, *in other words* k *is finite,* t_{nk} *tends to* 0 *when* n *tends to* ∞. *This is supported by the proof, since in the proof we only need the first finite number of terms in the sequence* $\{t_{nk}\}$ *to converge to* 0.

6.2 Stolz's Theorem

Theorem 3 ($\frac{0}{0}$ type)**.** *Suppose* $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, *and* $\{a_n\}$ *is decreasing. If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

Theorem 4 ($\frac{*}{\infty}$ type)**.** *Suppose* $\{a_n\}$ *is increasing and* $\lim_{n\to\infty} a_n = \infty$. *If*

$$\lim_{n\to\infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = l$$

then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = l$$

证明. Method: By Toeplitz's Theorem

$$t_nk = \{\frac{a_1}{a_n}, \frac{a_2 - a_1}{a_n}, \frac{a_3 - a_2}{a_n}, \cdots, \frac{a_n - a_{n-1}}{a_n}\}$$
$$c_n = \{\frac{b_1}{a_1}, \frac{b_2 - b_1}{a_2 - a_1}, \frac{b_3 - b_2}{a_3 - a_2}, \cdots, \frac{b_n - b_{n-1}}{a_n - a_{n-1}}\}$$

□

6.3 Cauchy's Proposition

Proposition 4 (算术平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$$

Proposition 5 (算术平均值等价形式)**.** *If* $\lim_{n\to\infty}(a_n - a_{n-1}) = a$, *then*

$$\lim_{n\to\infty} \frac{a_n}{n} = a$$

Proposition 6 (几何平均值形式)**.** *If* $\lim_{n\to\infty} a_n = a > 0$, *then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a$$