Notes of ”Properties of Continuous Functions”

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1 Local Properties

Theorem 1. T1 *(局部有界性)*

T2 *(局部保号性)*

T3 *(四则运算保持连续性)*

T4 *(复合函数保持连续性)*

2 Global Properties

Theorem 2 (The Bolzano-Cauchy Intermediate-Value Theorem)**.** *If a function is continuous on a closed interval, and assumes values with opposite signs on the endpoints of the interval, then the function has at least one zero point in the interval.*

In logic expression:

$$(f \in C[a, b]) \wedge (f(a) \cdot f(b) < 0) \Rightarrow \exists c \in (a, b)(f(c) = 0)$$

证明. Hint: Bisection method. □

证明. Hint: The Lebesgue method.

- Let $A = \{t \mid (a \leq t) \wedge (\forall x \in [a, t](f(x) < 0))\}$
- The set A is not empty.
- b is an upper bound of A.
- According to the least upper bound principle, there exists $s = \sup A \in [a, b]$.
- $f(s) < 0$ is impossible othewise s is not an upper bound of A. $f(s) > 0$ is impossible otherwise s is not the least upper bound of A. Therefore $f(s) = 0$.

□

Remark 1 (Connectivity of the domain)**.**

Corollary 1. *If the function ϕ is continuous on an open interval and assumes values $\phi(a) = A$ and $\phi(b) = B$ at points a and b, then for any number C between A and B, there is a point c between a and b at which $\phi(c) = C$.*

证明. □

Theorem 3 (The Weierstrass Maximum-Value Theorem). *If a function is continuous on a closed interval, then it is bounded on the interval. The function assumes the maximum value and minimum value on the interval.*

证明. Hint: The finite covering lemma □

Remark 2 (Compactness of the domain).

2.1 Uniform continuity

Definition 1 (Uniform continuity). *A function $f : E \to \mathbb{R}$ is uniformly continuous on a set $E \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ for all points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$.*

In logic expression:

$$
\begin{aligned}
&(f : E \to \mathbb{R} \text{ is uniformly continuous }) :=\\
&(\forall \epsilon > 0 \exists \delta > 0 \forall x_1 \in E \forall x_2 \in E(|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon))
\end{aligned}
$$

Remark 3. *If a function is uniformly continuous on a set, then it is continuous on any points in the set.*

Remark 4. *In general, a function is continuous on a set cannot derive that the function is uniformly continuous on the set.*

Remark 5 (The definition of the negation of uniform continuity for a function in logic expression).

$$
\begin{aligned}
&(f : E \to \mathbb{R} \text{ is not uniformly continuous }) :=\\
&(\exists \epsilon > 0 \forall \delta > 0 \exists x_1 \in E \exists x_2 \in E(|x_1 - x_2| < \delta \wedge |f(x_1) - f(x_2)| \geq \epsilon))
\end{aligned}
$$

Example 1. $f(x) = \sin \frac{1}{x}$

Remark 6 (The difference between continuity on a set and uniform continuity on a set). *In the definition of a function $f : E \to \mathbb{R}$ being continuous on E, the number δ depends on the number ϵ and the point a. Hence the number δ may vary on different points in E.*

In the case of uniform continuity, we choose the number δ depending on only the number ϵ, and it should work for every point in E. In other words, for every $\epsilon > 0$, there exists an greatest lower bound of the set of δ for a function $f : E \to \mathbb{R}$ to be continuous on each point $a \in E$.

Remark 7 (A sufficient condition of the negation of uniform continuity). *If the function $f : E \to \mathbb{R}$ is unbounded in every neighborhood of a fixed point $x_0 \in E$, then it is not uniformly continuous on E. If E consists of open intervals, then x_0 can be an endpoint of E since x_0 is a limit point of E and every neighborhood $U(x_0)$ contains infinite points of E.*

Remark 8 (Another sufficient and necessary condition of the negation of uniform continuity). *For a function $f : E \to \mathbb{R}$, if there exist a sequence $\{x'_n\}$ and $\{x''_n\}$ such that $\lim_{n\to+\infty}(x'_n - x''_n) = 0$ but $(f(x'_n) - f(x''_n))$ 不收敛于 0.*

Theorem 4 (The Contor-Heine theorem on uniform continuity)**.** *A function that is continuous on a closed interval is uniformly continuous on that interval.*

证明. Hint □

2.2 Monotonic continuous functions

Proposition 1. *A continuous mapping $f : E \to \mathbb{R}$ of a closed interval $E = [a, b]$ into $\mathbb{R}$ is injective if and only if the function f is strictly monotonic on $[a, b]$.*

Proposition 2. *Each strictly monotonic function $f : X \to \mathbb{R}$ defined on a numerical set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f, and has the same kind of monotonicity on Y that f has on X.*

Proposition 3. *The discontinuities of a function $f : E \to \mathbb{R}$ that is monotonic on the set $E \subset \mathbb{R}$ can be only discontinuities of first kind.*

Corollary 2. *If a is a point of discontinuity of a monotonic function $f : E \to \mathbb{R}$, then at least one of the limits $\lim_{E \ni x \to a^+} = f(a^+)$ or $\lim_{E \ni x \to a^-} = f(a^-)$ exists, and strict inequality holds in at least one of the inequalities $f(a^-) \leq f(a) \leq f(a^+)$ when f is nondecreasing and $f(a^-) \geq f(a) \geq f(a^+)$ when f is nonincreasing. The function assumes no values in the open interval defined by the strict inequality. Open intervals of this kind determined by different points of discontinuity have no points in common.*

Corollary 3. *The set of points of discontinuity of a monotonic function is at most countable.*

Proposition 4 (A Criterion for Continuity of a Monotonic Function)**.** *A monotonic function $f : E \to \mathbb{R}$ defined on a closed interval $E = [a, b]$ is continuous if and only if its set of values $f(E)$ is the closed interval with endpoints $f(a)$ and $f(b)$.*

Theorem 5 (The Inverse Function Theorem)**.** *A function $f : X \to \mathbb{R}$ that is strictly monotonic on a set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f, and has the same kind of monotonicity on Y that f has on X.*

If in addition X is a closed interval $[a, b]$ and f is continuous on X, then the set $Y = f(X)$ is the closed interval with endpoints $f(a)$ and $f(b)$ and the function $f^{-1} : Y \to \mathbb{R}$ is continuous on it.