Notes of ”Elementary Facts about Series”

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1 Overview

- The sum of a series and the Cauchy criterion for convergence of a series
  - Def: A (infinite) series
  - Def: The terms of a series and the n-th term
  - Def: The (n-th) partial sum of a series
  - Def: A convergent or divergent series
  - Def: The sum of a series
  - Thm: The Cauchy convergence criterion for a series
  - Cor: The equivalence of the convergence between two series with only finite terms different
  - Cor: A necessary condition for a series to be convergent in terms of the limit of its terms
- Absolute convergence, the comparison theorem and its consequences
  - Def: A series is absolutely convergent
  - Rmk: Absolute convergence implies convergence, but the opposite is not true
  - Thm: Criterion for convergence of series of nonnegative terms in terms of bounds
  - Thm: (Comparison theorem) Test for convergence of series of nonnegative terms in terms of comparison with another nonnegative series
  - Cor: (The Weierstrass M-test for absolute convergence)
  - Cor: (Cauchy's test)
  - Cor: (d'Alembert's test)
  - Prop: (Cauchy) A necessary and sufficient condition for a monotonic nonnegative series to be absolutely convergent in terms of the generated series $\Sigma_{k=0}^{\infty} 2^k a_{2^k}$
  - Cor: The convergence of the series $\zeta(p) = \Sigma_{n=1}^{\infty} \frac{1}{n^p}$
- The number e as the sum of a series

2 The sum of a series and the Cauchy criterion for convergence of a series

3 Absolute convergence, the comparison theorem and its consequences

Definition 1 (A series is absolutely convergent)**.** *The series* $\Sigma_{n=1}^{\infty} a_n$ *is absolutely convergent if the series* $\Sigma_{n=1}^{\infty} |a_n|$ *is convergent.*

Remark 1 (Absolute convergence implies convergence, but the opposite is not true)**.**

Theorem 1 (Criterion for convergence of series of nonnegative terms in terms of bounds)**.**

Theorem 2 ((Comparison theorem) Test for convergence of series of nonnegative terms in terms of comparison with another nonnegative series)**.**

Corollary 1 ((The Weierstrass M-test for absolute convergence))**.** *Let* $\Sigma_{n=1}^{\infty} a_n$ *and* $\Sigma_{n=1}^{\infty} b_n$ *be series. Suppose there exists an index* $N \in \mathbb{N}$ *such that* $|a_n| \leq b_n$ *for all* $n > N$*. Then a sufficient condition for absolute condition of the series* $\Sigma_{n=1}^{\infty} a_n$ *is that the series* $\Sigma_{n=1}^{\infty} b_n$ *converge.*

4 The number e as the sum of a series