# Lecture 18: Change of Variables

## 1 Introduction to Change of Variables in Double Integrals

**Example 1.** *Calculate the area of the ellipse with semiaxes a and b.*
*We can use double integrals to calculate the area. By definition of double integrals, the area of the described ellipse*

$$A = \iint_R 1 dA$$
$$= \iint_{(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1} dxdy$$

*To simplify the equation, we can set*

$$u = \frac{x}{a}$$
$$v = \frac{y}{b}$$

*Then*

$$du = \frac{dx}{a}$$
$$dv = \frac{dy}{b}$$
$$dudv = \frac{dxdy}{ab}$$
$$dxdy = abdudv$$

*Therefore, the equation of the double integral is equivalent to*

$$A = \iint_{(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1} dxdy$$

$$= \iint_{u^2 + v^2 = 1} abdudv$$

$$= ab \iint_{u^2 + v^2 = 1} dudv$$

$$= \pi ab$$

*Therefore, the area of the ellipse with semiaxes $a$ and $b$ is $\pi ab$.*

From the above example we can see, in general when we change the variables in a double integral, we need to find the scaling factor between $dA = dxdy$ and $dA' = dudv$. In order to ensure the correctness of the double integral after changing the variables, we need to multiply the scaling factor with differentials of changed variables, to ensure the differential of area is still $dA$ rather than $dA'$. Also note that the scaling factor between $dA$ and $dA'$ might not be constant in the region $R$.

**Question.** *Why when we make sure $dA$ is correctly expressed by $dudv$ the double integral with change of variables yields the correct result?*
*Answer:*
*Probably think about it from the definition of double integral, and the geometric interpretation of change of variables.*

# 2 Change of Variables with Linear Transformation

**Example 2.** *Find the scaling factor between $dA = dxdy$ and $dA' = dudv$ with the following linear transformation:*

$$\begin{cases} u = 3x - 2y \\ v = x + y \end{cases}$$

*In the xy-coordinates, $dA = dxdy$ represents a small rectangle. Regarding the geometric interpretation of the linear transformation, we have the following two observations:*

- *A rectangle in the xy-coordinates corresponds to a parallelogram in the uv-coordinates. The ratio of the area of the parallelogram in the uv-coordinates to the area of the rectangle in the xy-coordinates is the area scaling factor.*

- *The area scaling factor of the linear transformation doesn't depend on the choice of rectangles, either the position or the area.*

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*Based on the above two observations, we can study a special case to find the scaling factor of the linear transformation. The special case we choose is a unit rectangle with the left bottom corner on the origin.*
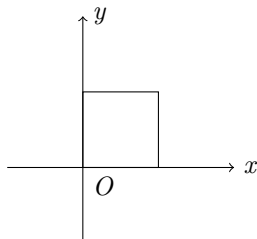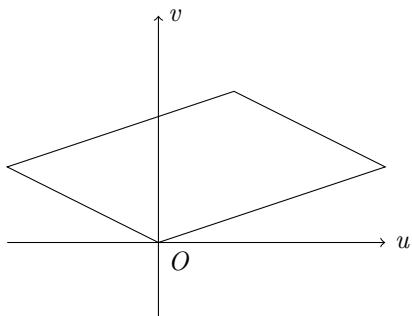


*After the linear transformation,*

$$(0,0) \to (0,0)$$
$$(1,0) \to (3,1)$$
$$(1,1) \to (1,2)$$
$$(0,1) \to (-2,1)$$



*We can use the determinant to calculate the area of a parallelogram in a plane*

$$A = \det(<3,1>,<-2,1>)$$
$$= \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}$$
$$= 5$$

*Therefore, the scaling factor of the linear transformation is* $5$:

$$dA' = 5dA$$
$$dA = \frac{1}{5}dA'$$
$$dxdy = \frac{1}{5}dudv$$

**Question.** *Why the scaling factor in the above example is* $5$ *rather than* $\frac{1}{5}$ *?*
*Answer:*

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# 3 Change of Variables in General