18.02 EXERCISES

Problem Set 2: Matrices and Systems of Equations

Part I

Unit 1E Equations of Lines and Planes

1. Find the equations of the following planes:
a) through (2, 0, -1) and perpendicular to $\vec{i} + 2\vec{j} - 2\vec{k}$.
b) through the origin, (1, 1, 0), and (2, -1, 3)
c) through (1, 0, 1), (2, -1, 2), (-1, 3, 2)
d) through the points on the x, y and z-axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
e) through (1, 0, 1) and (0, 1, 1) and parallel to $\vec{i} - \vec{j} + 2\vec{k}$
Solution:
a) According to the problem description, the vector $< 1, 2, -2 >$ is a normal vector to the plane. Therefore, the equation of the plane is

$$x + 2y - 2z = c, \text{where c is a constant}$$

Then we can put the point (2, 0, -1) into the equation:

$$c = 2 + 2 \times 0 - 2 \times (-1) = 4$$

Hence, the equation of the plane is

$$x + 2y - 2z = 4$$

b) According to the problem description, two vectors on the plane are

$$< 1, 1, 0 >, < 2, -1, 3 >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < 1, 1, 0 > \times < 2, -1, 3 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} \\ &= < 3, -3, -3 > \end{aligned}$$

Therefore, the equation of the plane is

$$3x - 3y - 3z = c, \text{where c is a constant}$$

Then we can put the point (0, 0, 0) into the equation:

$$c = 0$$

Hence the equation of the plane is

$$3x - 3y - 3z = 0$$

c) According to the problem description, two vectors on the plane are

$$< 1, -1, 1 >, < -2, 3, 1 >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < 1, -1, 1 > \times < -2, 3, 1 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 3 & 1 \end{vmatrix} \\ &=< -4, 1, 5 > \end{aligned}$$

Therefore, the equation of the plane is

$$-4x + y + 5z = c, \text{where c is a constant}$$

Then we can put the point (1, 0, 1) into the equation:

$$c = -4 \times 1 + 0 + 5 \times 1 = 1$$

Hence the equation of the plane is

$$-4x + y + 5z = 1$$

d) According to the problem description, two vectors on the plane are

$$< -a, b, 0 >, < -a, 0, c >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < -a, b, 0 > \times < -a, 0, c > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} \\ &=< bc, ac, ab > \end{aligned}$$

Therefore, the equation of the plane is

$$bcx + acy + abz = k, \text{where k is a constant}$$

Then we can put the point (a, 0, 0) into the equation:

$$k = bc \cdot a + ac \cdot 0 + ab \cdot 0 = abc$$

Hence the equation of the plane is

$$\begin{gathered} bcx + acy + abz = abc \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \end{gathered}$$

e) According to the problem description, one vector parallel to the plane can be derived from the two points (1, 0, 1) and (0, 1, 1), which is $< -1, 1, 0 >$. Therefore, a normal vector to the plane can be calculated as

$$
\begin{aligned}
< -1, 1, 0 > \times < 1, -1, 2 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} \\
&= < 2, 2, 0 >
\end{aligned}
$$

Therefore, the equation of the plane is

$$2x + 2y = c, \text{where c is a constant}$$

Then we can put the point (1, 0, 1) into the equation:

$$c = 2 \times 1 + 2 \times 0 = 2$$

Hence the equation of the plane is

$$
\begin{aligned}
2x + 2y &= 2 \\
x + y &= 1
\end{aligned}
$$

2. Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.
Solution:
Suppose that the intersection line between the planes $2x - y + z = 3$ and $x + y + 2z = 1$ is l. By the definition of dihedral angles, we can find a point O on the line l, and in the plane $2x - y + z = 3$ find a line $OA \perp l$, and in the plane $x + y + 2z = 1$ find a line $OB \perp l$. Then the angle $\angle AOB$ is the dihedral angle.
Notice that any normal vectors to the plane $2x - y + z = 3$ and the plane $x + y + 2z = 1$ are also perpendicular to the line l because l is on both planes. Therefore, the vector $\vec{n_1} =< 2, -1, 1 >$, which is a normal vector to the plane $2x - y + z = 3$, and the vector $\vec{n_2} =< 1, 1, 2 >$, which is a normal vector to the plane $x + y + 2z = 1$, are perpendicular to the line l.
Therefore, all four vectors $\vec{OA}$, $\vec{OB}$, $\vec{n_1}$, and $\vec{n_2}$ are all parallel to the same plane, whose normal vectors are along the line l, and they form a quadrilateral. Since $\vec{OA} \perp \vec{n_1}$, and $\vec{OB} \perp \vec{n_2}$, the angle $\angle AOB$ is either equal to or supplementary to the angle θ between two normal vectors $\vec{n_1}$ and $\vec{n_2}$.

$$
\begin{aligned}
\cos\theta &= \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\\
&= \frac{3}{\sqrt{6}\times\sqrt{6}}\\
&= \frac{1}{2}\\
\theta &= \frac{\pi}{3}
\end{aligned}
$$

Therefore, the dihedral angle between the two planes are $\frac{\pi}{3}$.
3. Find in parametric form the equations for
a) the line through (1, 0, -1) and parallel to $2\vec{i}-\vec{j}+3\vec{k}$
b) the line through (2, -1, -1) and perpendicular to the plane $x-y+2z=3$
c) all lines passing through (1, 1, 1) and lying in the plane $x+2y-z=2$
Solution:
a)

$$
\begin{aligned}
x(t) &= 1+2t\\
y(t) &= -t\\
z(t) &= -1+3t
\end{aligned}
$$

b) The line is perpendicular to the plane $x-y+2z=3 \iff$ The line is parallel to the normal vectors of the plane $x-y+2z=3 \iff$ The line is parallel to the vector $<1,-1,2>$.
Therefore, the parametric equation of the line is

$$
\begin{aligned}
x(t) &= 2+t\\
y(t) &= -1-t\\
z(t) &= -1+2t
\end{aligned}
$$

c) Since the line is lying in the plane $x+2y-z=2$, suppose the line vector is $<a,b,c>$, then it satisfies

$$
\begin{aligned}
<a,b,c>\cdot<1,2,-1> &= 0\\
a+2b-c &= 0\\
c &= a+2b
\end{aligned}
$$

Therefore, the parametric equations of all lines described above are

$$
\begin{aligned}
x(t) &= 1 + at \\
y(t) &= 1 + bt \\
z(t) &= 1 + (a + 2b)t
\end{aligned}
$$

where a and b are any constants.
5. The line passing through (1, 1, -1) and perpendicular to the plane $x+2y-z = 3$ intersects the plane $2x - y + z = 1$ at what point?
Solution:
The line is perpendicular to the plane $x + 2y - z = 3 \iff$ The line is parallel to the normal vectors of the plane $x + 2y - z = 3 \iff$ The line is parallel to the vector $< 1, 2, -1 >$.
Therefore, the parametric equation of the line is

$$
\begin{aligned}
x(t) &= 1 + t \\
y(t) &= 1 + 2t \\
z(t) &= -1 - t
\end{aligned}
$$

For the intersection point between the line and the plane, we have

$$
\begin{aligned}
2x(t) - y(t) + z(t) &= 1 \\
2(1 + t) - (1 + 2t) + (-1 - t) &= 1 \\
t &= -1
\end{aligned}
$$

Therefore, the intersection point is $(x(-1), y(-1), z(-1))$, which is $(0, -1, 0)$.
6. Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$.
Solution:
We need to find the parametric equation of the line through the origin and perpendicular to the plane $ax + by + cz = d$.
The line is perpendicular to the plane $ax+by+cz = d \iff$ The line is parallel to the normal vectors of the plane $ax + by + cz = d \iff$ The line is parallel to the vector $< a, b, c >$.
Therefore, the parametric equation of the line is

$$
\begin{aligned}
x(t) &= 0 + at = at \\
y(t) &= 0 + bt = bt \\
z(t) &= 0 + ct = ct
\end{aligned}
$$

For the intersection point between the line and the plane, we have

$$\begin{aligned}
ax(t) + by(t) + cz(t) &= d \\
a^2t + b^2t + c^2t &= d \\
t &= \frac{d}{a^2 + b^2 + c^2}
\end{aligned}$$

Therefore, the intersection point is $(\frac{ad}{a^2+b^2+c^2}, \frac{bd}{a^2+b^2+c^2}, \frac{cd}{a^2+b^2+c^2})$
Therefore, the distance D from the origin to the plane is equivalent to the distance between these two points:

$$\begin{aligned}
D &= \sqrt{(\frac{ad}{a^2 + b^2 + c^2})^2 + (\frac{bd}{a^2 + b^2 + c^2})^2 + (\frac{cd}{a^2 + b^2 + c^2})^2)} \\
&= \sqrt{\frac{(a^2 + b^2 + c^2)d^2}{(a^2 + b^2 + c^2)^2}} \\
&= \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

Unit 1F Matrix Algebra

5. a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3.

b) Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution:
a)

$$\begin{aligned}
A^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
A^3 &= A^2 \cdot A \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}
\end{aligned}$$

b)

$$
\begin{aligned}
A^2 &= A \cdot A \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
A^3 &= A^2 \cdot A \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

The first column in A determines that the first column in A^{k+1} will be the same as the first column in A^k where k can be any positive integers. The second column in A determines that the second column in A^{k+1} will be the sum of the first and second columns in A^k where k can be any positive integers. Therefore, we can induce that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

8. a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A?

b) If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is the 3×3 matrix A?

Solution:

a) According to the definition of matrix product, the result of $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the first column of A, the result of $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is the second column of A, and the result of

$A\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the third column of A. Therefore,

$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$

a) (Method 2) If we combine the three matrix equations, we can get

$$A \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$
$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}^{-1}$$
$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$
$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$

b) (Method 1) Suppose $A = \begin{pmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{pmatrix}$. Then

$$A\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$
$$\begin{pmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{pmatrix} \cdot \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$
$$\begin{pmatrix}2a_1\\2b_1\\2c_1\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$
$$a_1 = -1, b_1 = 0, c_1 = 2$$

Also

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$
$$a_1 + a_2 + a_3 = 3$$
$$b_1 + b_2 + b_3 = 0$$
$$c_1 + c_2 + c_3 = 3$$

And

$$A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 2a_2 + a_3 \\ 2b_2 + b_3 \\ 2c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$
$$2a_2 + a_3 = 7$$
$$2b_2 + b_3 = 1$$
$$2c_2 + c_3 = 1$$

Hence we derive the following system of equations:

$$\begin{cases} a_1 = -1 \\ b_1 = 0 \\ c_1 = 2 \\ a_1 + a_2 + a_3 = 3 \\ b_1 + b_2 + b_3 = 0 \\ c_1 + c_2 + c_3 = 3 \\ 2a_2 + a_3 = 7 \\ 2b_2 + b_3 = 1 \\ 2c_2 + c_3 = 1 \end{cases}$$

We can solve this system of equations and get

$$A = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

b) (Method 2) If we combine the three matrix equations, we can get

$$A \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 7 \\ 0 & 0 & 1 \\ 4 & 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 3 & 7 \\ 0 & 0 & 1 \\ 4 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

9. A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that
a) each row of A is a row vector of length 1.
b) two different rows are orthogonal vectors.
Solution:
Suppose $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, then $A^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$.

$$A \cdot A^T = I_n$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expand the matrix equation, we can derive the following equations:

$$
\begin{aligned}
a_1^2 + a_2^2 + a_3^3 = 1 \\
b_1^2 + b_2^2 + b_3^3 = 1 \\
c_1^2 + c_2^2 + c_3^3 = 1 \\
a_1b_1 + a_2b_2 + a_3b_3 = 0 \\
b_1c_1 + b_2c_2 + b_3c_3 = 0 \\
a_1c_1 + a_2c_2 + a_3c_3 = 0
\end{aligned}
$$

which is equivalent to that each row of A is a row vector of length 1, and two different rows in A are orthogonal.
Therefore, the statement in the problem description is proved.

Unit 1G Solving Square Systems; Inverse Matrices

3. $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $Ax = b$ by finding A^{-1}.
Solution:

$$
\begin{aligned}
Ax = b \\
x = A^{-1}b \\
x = \frac{1}{det(A)} adj(A) \cdot b \\
x = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \\
x = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \\
x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

4. Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, x_2 + x_3 = y_2, -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i.
Solution:

$$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_2 + x_3 = y_2 \\ -x_1 - x_2 + 2x_3 = y_3 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{cases} x_1 = \frac{3}{5}y_1 + \frac{1}{5}y_2 - \frac{2}{5}y_3 \\ x_2 = -\frac{1}{5}y_1 + \frac{3}{5}y_2 - \frac{1}{5}y_3 \\ x_3 = \frac{1}{5}y_1 + \frac{2}{5}y_2 + \frac{1}{5}y_3 \end{cases}$$

5. Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.
Solution:
By the definition of inverse matrix,

$$AB \cdot (AB)^{-1} = I_n$$
$$A^{-1}AB \cdot (AB)^{-1} = A^{-1}I_n$$
$$B \cdot (AB)^{-1} = A^{-1}I_n$$
$$B^{-1}B \cdot (AB)^{-1} = B^{-1}A^{-1}I_n$$
$$(AB)^{-1} = B^{-1}A^{-1}I_n$$
$$(AB)^{-1} = B^{-1}A^{-1}$$

Part II