18.02 EXERCISES

Problem Set 3: Parametric Equations for Curves

Part I

Unit 1E Equations of Lines and Planes

4. Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?
Solution:
A vector along the line is $< 2, -1, 1 >$. Hence, the parametric equation of the line is

$$\begin{cases} x = x(t) = 0 + 2t = 2t \\ y = y(t) = 1 - t \\ z = z(t) = 2 + t \end{cases}$$

For the intersection point of the line and the plane, it satisfies

$$\begin{aligned} x(t) + 4y(t) + z(t) &= 4 \\ 2t + 4(1 - t) + 2 + t &= 4 \\ -t &= -2 \\ t &= 2 \end{aligned}$$

Therefore, the coordinates of the intersection point is $(x(2), y(2), z(2))$, which is $(4, -1, 4)$.

7. Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).
Solution:
7-1. Formulate a general method for finding the distance between two parallel planes in space:
Suppose that the equations of two parallel planes are respectively

$$\begin{aligned} n_1x + n_2y + n_3z &= c_1 \\ n_1x + n_2y + n_3z &= c_2 \end{aligned}$$

Then the vector $\vec{n} =< n_1, n_2, n_3 >$ is one of the normal vectors of the two planes. Therefore, we can construct the parametric equation of a line perpendicular to the two planes as

$$\begin{cases} x(t) = n_1 t \\ y(t) = n_2 t \\ z(t) = n_3 t \end{cases}$$

Therefore, the two intersection points the line has with the two planes respectively satisfy

$$\begin{gathered} n_1 x(t_1) + n_2 y(t_1) + n_3 z(t_1) = c_1 \\ n_1^2 t_1 + n_2^2 t_1 + n_3^2 t_1 = c_1 \\ t_1 = \frac{c_1}{n_1^2 + n_2^2 + n_3^2} \\ n_1 x(t_2) + n_2 y(t_2) + n_3 z(t_2) = c_2 \\ n_1^2 t_2 + n_2^2 t_2 + n_3^2 t_2 = c_1 \\ t_2 = \frac{c_2}{n_1^2 + n_2^2 + n_3^2} \end{gathered}$$

The distance between two points in the line can be calculated as

$$\begin{aligned} d &= \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 + (z(t_1) - z(t_2))^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(t_1 - t_2)^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(\frac{c_1 - c_2}{n_1^2 + n_2^2 + n_3^2})^2} \\ &= \frac{|c_1 - c_2|}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \\ &= \frac{|c_1 - c_2|}{|\vec{n}|} \end{aligned}$$

7-2. Formulate a general method for finding the distance between two non-parallel lines in space:
Suppose that there are two non-parallel lines in space named l_a and l_b. Their normal vectors are respectively $\vec{n_a} = \langle n_{ax}, n_{ay}, n_{az} \rangle$ and $\vec{n_b} = \langle n_{bx}, n_{by}, n_{bz} \rangle$,

and their parametric equations are respectively:

$$\begin{cases} x_a(t) = x_{a0} + n_{ax}t \\ y_a(t) = y_{a0} + n_{ay}t \\ z_a(t) = z_{a0} + n_{az}t \end{cases}$$

$$\begin{cases} x_b(t) = x_{b0} + n_{bx}t \\ y_b(t) = y_{b0} + n_{by}t \\ z_b(t) = z_{b0} + n_{bz}t \end{cases}$$

Then we can find the vector $\vec{n}$ which is perpendicular to both l_a and l_b by

$$\begin{aligned} \vec{n} &= \vec{n_a} \times \vec{n_b} \\ &= \langle n_x, n_y, n_z \rangle \end{aligned}$$

Then the distance between l_a and l_b is equal to the distance between the two planes P_a and P_b with $\vec{n}$ as their normal vectors where l_a and l_b lie within respectively.
The equation of P_a is

$$n_x x + n_y y + n_z z = n_x x_{a0} + n_y y_{a0} + n_z z_{a0}$$

The equation of P_b is

$$n_x x + n_y y + n_z z = n_x x_{b0} + n_y y_{b0} + n_z z_{b0}$$

According to the previous part, the distance between P_a and P_b is

$$\begin{aligned} d &= \frac{|(n_x x_{a0} + n_y y_{a0} + n_z z_{a0}) - (n_x x_{b0} + n_y y_{b0} + n_z z_{b0})|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot \langle (x_{a0} - x_{b0}), (y_{a0} - y_{b0}), (z_{a0} - z_{b0}) \rangle|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot A_0\vec{B}_0|}{|\vec{n}|} \\ &= \frac{|(\vec{n_a} \times \vec{n_b}) \cdot A_0\vec{B}_0|}{|\vec{n_a} \times \vec{n_b}|} \end{aligned}$$

Therefore, the distance between l_a and l_b is

$$d = \frac{|(\vec{n_a} \times \vec{n_b}) \cdot A_0\vec{B}_0|}{|\vec{n_a} \times \vec{n_b}|}$$

7-3 Apply the above method to find the distance between two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

The two non-intersecting diagonals of two adjacent faces of the unit cube I choose are the diagonal between $(0,0,0)$ and $(1,0,1)$, and the diagonal between $(0,1,0)$ and $(0,0,1)$. Therefore, the parametric equations of these two lines along the described diagonals are respectively:

$$\begin{cases} x_a(t) = t \\ y_a(t) = 0 \\ z_a(t) = t \end{cases}$$

$$\begin{cases} x_b(t) = 0 \\ y_b(t) = 1 - t \\ z_b(t) = t \end{cases}$$

Therefore,

$$\begin{aligned} \vec{n_a} &= \langle 1, 0, 1 \rangle \\ \vec{n_b} &= \langle 0, -1, 1 \rangle \\ A_0 &= (0,0,0) \\ B_0 &= (0,1,0) \\ \vec{A_0B_0} &= \langle 0, 1, 0 \rangle \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals is

$$\begin{aligned} d &= \frac{|(\vec{n_a} \times \vec{n_b}) \cdot \vec{A_0B_0}|}{|\vec{n_a} \times \vec{n_b}|} \\ &= \frac{|\langle 1, -1, -1 \rangle \cdot \langle 0, 1, 0 \rangle|}{|\langle 1, -1, -1 \rangle|} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals of two adjacent faces of the unit cube is $\frac{\sqrt{3}}{3}$.

Unit 1I Vector Functions and Parametric Equations

1. The point P moves with constant speed v in the direction of the constant vector $a\vec{i} + b\vec{j}$. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $\vec{r}(t)$?
Solution:
The direction of the constant vector $a\vec{i} + b\vec{j}$ is

$$dir(A) = \frac{a}{\sqrt{a^2+b^2}}\vec{i} + \frac{b}{\sqrt{a^2+b^2}}\vec{j}$$

Hence in a period of time t, the point P moves in a distance of vt in the direction $\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \rangle$. Hence along the x-axis it moves in a distance of $\frac{a}{\sqrt{a^2+b^2}}vt$, and along the y-axis it moves in a distance of $\frac{b}{\sqrt{a^2+b^2}}vt$.
Therefore, the position vector function of the point P is

$$\vec{r}(t) = \langle x_0 + \frac{a}{\sqrt{a^2+b^2}}vt, y_0 + \frac{b}{\sqrt{a^2+b^2}}vt \rangle$$

3. Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.
a) $\vec{r} = 2\cos^2 t\vec{i} + \sin^2 t\vec{j}$
b) $\vec{r} = \cos 2t\vec{i} + \cos t\vec{j}$
c) $\vec{r} = (t^2+1)\vec{i} + t^3\vec{j}$
d) $\vec{r} = \tan t\vec{i} + \sec t\vec{j}$
Solution:
a) According to the position vector function,

$$x(t) = 2\cos^2 t$$
$$y(t) = \sin^2 t$$

Therefore, we can find the relation between x and y:

$$\frac{x}{2} + y = 1$$
$$y = -\frac{1}{2}x + 1$$

which is a line in the xy-plane.
Since the ranges of x and y are respectively

$$x \in [0, 2]$$
$$y \in [0, 1]$$

the part of the curve that is actually traced out by P is the line segment from $(0,0)$ to $(2,1)$.
b) According to the position vector function,

$$x(t) = \cos 2t$$
$$y(t) = \cos t$$

Therefore, we can find the relation between x and y:

$$\cos 2t = 2\cos^2 t - 1$$
$$x = 2y^2 - 1$$

which is a parabola in the xy-plane.
Since the range of y is $y \in [-1, 1]$, and based on the characteristic of the parabola $x = 2y^2 - 1$, the part of the curve that is actually traced out by P is the part of the parabola between $(-1, 1)$ and $(1, 1)$.
c) According to the position vector function,

$$x(t) = t^2 + 1$$
$$y(t) = t^3$$

Therefore, we can find the relation between x and y:

$$y = (x - 1)^{\frac{3}{2}}$$

which is a symmetric graph related to the x axis, in which the half above the x axis is the graph of a power function.
Since the range of x is $x \in [1, \infty)$, which is the domain of the function $y = (x - 1)^{\frac{3}{2}}$, the full curve is actually traced out by P.
d) According to the position vector function,

$$x(t) = \tan t$$
$$y(t) = \sec t$$

Therefore, we can find the relation between x and y:

$$\sec^2 t - \tan^2 t = 1$$
$$y^2 - x^2 = 1$$

which is a hyperbola.
According to the graph of $x(t)$ and $y(t)$, the full hyperbola is actually traced out by P.

5. A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.
Solution:

Since the string is wound clockwise, when we unwind it, the direction is counter-clockwise. Also, since the string is unwound around a circle, it is natural to use the parameter θ to describe the motion of P, where θ is the corresponding angle of the unwound part of the string. Hence the length of the unwound part of the string is $a\theta$.
During the unwinding, the string is always pulled taut, hence there is always a point P' on the circle which is the point of tangency between the circle and the unwound part of the string. The coordinates of P' can be described as $(a\cos\theta, a\sin\theta)$.
Next we need to explore the direction of the vector $\vec{P'P}$ in order to describe the coordinates of P. According to the tangency relationship between the circle and the unwound part of the string,

$$\vec{P'P} \perp \vec{OP'}$$
$$dir(\vec{OP'}) = \langle \cos\theta, \sin\theta \rangle$$

Therefore, we have the following system of equations

$$\begin{cases} dir(\vec{P'P}) \cdot \langle \cos\theta, \sin\theta \rangle = 0 \\ |dir(\vec{P'P})| = 1 \end{cases}$$

After solving the above system of equations, there are two possible solutions:

$$dir(\vec{P'P}) = \langle \sin\theta, -\cos\theta \rangle$$
$$dir(\vec{P'P}) = \langle -\sin\theta, \cos\theta \rangle$$

According to the geometric interpretation of unwinding the string counter-clockwise, the correct direction of the vector $\vec{P'P}$ is $\langle \sin\theta, -\cos\theta \rangle$.
Therefore, the parametric equation for the motion of P is

$$\begin{cases} x(t) = a\cos\theta + a\theta\sin\theta \\ y(t) = a\sin\theta - a\theta\cos\theta \end{cases}$$

7. The cycloid is the curve traced out by a fix point P on a circle of radius a which rolls along the x-axis in the positive direction, starting when P is at the origin O. Find the vector function OP; use as variable the angle θ through which the circle has rolled.
Solution:
Let C denote the center of the rolling circle. At a given value of θ, suppose that the tangency point between the circle and the x axis is A.
Since the circle is rolling along the x axis without any slip, then

$$OA = a\theta$$

Also $AC = a$, hence the coordinates of C at the given value of θ is

$$C = (a\theta, a)$$

The direction of $\vec{CP}$ can be described by

$$dir(\vec{CP}) = \langle \cos(-\frac{\pi}{2} - \theta), \sin(-\frac{\pi}{2} - \theta) \rangle$$
$$dir(\vec{CP}) = \langle -\sin\theta, -\cos\theta \rangle$$

Therefore, the parametric equation for the motion of P is

$$\begin{cases} x(t) = a\theta - a\sin\theta \\ y(t) = a - a\cos\theta \end{cases}$$

Part II

1. A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P. Suppose that the disk is tangent to the x-axis with the center initially at $(0, 2)$ and P initially at $(0, 1)$ and that it starts to roll to the right on the x-axis at unit speed. Let C be the curve traced out by the point P.
a) Make a sketch of what you think the curve C will look like.
b) Use vectors to find the parametric equations for $\vec{OP}$ as a function of time t.
c) Open the 'Mathlet' Wheel (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C. If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph.