Lecture 6: Kepler's Second Law

1 Describe Motion with Vectors

With parametric equations, we have a way not only to describe a motion, but also to analyze the motion with more details.

1.1 Position vectors

Position vectors are used to describe the position of a point in a motion.

$$\vec{r}(t) =< x(t), y(t), z(t) >$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\vec{r}(t) =< t - \sin t, 1 - \cos t >$$

1.2 Velocity vectors

Velocity vectors are used to describe how fast and in what direction a point moves in a motion.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &=< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > \end{aligned}$$

Speed is a scalar, and used to describe how fast a point moves in a motion.

$$speed = |\vec{v}|$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&=< \frac{d(t-\sin t)}{dt}, \frac{d(1-\cos t)}{dt} > \\
&=< 1-\cos t, \sin t > \\
|\vec{v}| &= \sqrt{(1-\cos t)^2 + \sin^2 t} \\
&= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\
&= \sqrt{2 - 2\cos t}
\end{aligned}$$

At $t = 0$, the velocity vector $\vec{v} =< 0, 0 >$, so the speed at that time is 0.

1.3 Acceleration vectors

Acceleration vectors are used to describe how velocity changes in a motion.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&=< \frac{d(1-\cos t)}{dt}, \frac{d(\sin t)}{dt} > \\
&=< \sin t, \cos t >
\end{aligned}$$

At $t = 0$, the acceleration vector $\vec{a} =< 0, 1 >$, which means the point at that time has an acceleration in the positive direction of the y axis. Combining with the previous conclusion that the velocity at this time is 0, it explains why the trajectory between two arches has a vertical tangent line.

1.4 Arc length

Arc length, usually denoted by s, is the distance traveled along the trajectory of a motion. According to this definition,

$$\frac{ds}{dt} = speed = |\vec{v}|$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example. The length of an arch of the cycloid is

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}dt$$

1.5 Unit tangent vector

Unit tangent vector, denoted by $\hat{T}$, is an unit vector in the direction of the tangent line at a position in a motion.
Since $\vec{v}$ is a tangent vector,

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

Another definition for the unit tangent vector:

$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} \\
&= \frac{d\vec{r}}{ds} \cdot |\vec{v}| \\
\frac{d\vec{r}}{ds} &= \frac{\vec{v}}{|\vec{v}|} \\
&= \hat{T} \\
\vec{v} &= \hat{T} \cdot \frac{ds}{dt}
\end{aligned}$$

Therefore, we can see the velocity vector can be interpreted as:

1. direction: the unit tangent vector $\hat{T}$.
2. magnitude: the speed $\frac{ds}{dt}$.

The geometric interpretation of $\frac{d\vec{r}}{ds} = \hat{T}$: during the time period of Δt,

$$\begin{aligned}
\Delta\vec{r} &= \vec{r}(t + \Delta t) - \vec{r}(t) \\
\Delta\vec{r} &\approx \hat{T} \cdot \Delta s \\
\frac{\Delta\vec{r}}{\Delta t} &\approx \hat{T} \cdot \frac{\Delta s}{\Delta t}
\end{aligned}$$

With Δt approaches 0, the above formula becomes

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \hat{T} \cdot \frac{ds}{dt} \\
\frac{d\vec{r}}{ds} &= \hat{T}
\end{aligned}$$

2 Example of Describing Motion with Vectors: Kepler's Second Law

2.1 Kepler's Second Law

Kepler's Second Law: Motion of planets is in a plane, and the area swept out by the line from sun to planet at a constant rate.
Newton later explain this law with gravitational attraction.

2.2 Translation with Vector Notations

The first part of Kepler's Second Law is: motion of planets is in a plane. The plane of planet motion is determined by two vectors: the position vector of the planet $\vec{r}$ and its velocity vector $\vec{v}$. The plane of planet motion remains unchanged $\iff$ the direction of the normal vector of the plane remains unchanged $\iff$ $dir(\vec{r} \times \vec{v})$ is constant.
The second part of Kepler's Second Law is: the area swept out by the line from sun to planet at a constant rate. The area swept out in unit time can be calculated as

$$\begin{aligned} dA &= \frac{1}{2}|\vec{r} \times d\vec{r}| \\ &= \frac{1}{2}|\vec{r} \times \vec{v}|dt \\ \frac{dA}{dt} &= \frac{1}{2}|\vec{r} \times \vec{v}| \end{aligned}$$

Therefore, the area swept out by the line from sun to planet at a constant rate $\iff$ $\frac{dA}{dt}$ is constant $\iff$ $|\vec{r} \times \vec{v}|$ is constant.
Therefore, Kepler's Second Law $\iff$ both $dir(\vec{r} \times \vec{v})$ and $|\vec{r} \times \vec{v}|$ are constant $\iff$ $\vec{r} \times \vec{v}$ is constant $\iff$ $\frac{d}{dt}\vec{r} \times \vec{v} = 0$.

$$\begin{aligned} \frac{d}{dt}\vec{r} \times \vec{v} &= 0 \\ \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} &= 0 \\ \vec{v} \times \vec{v} + \vec{r} \times \vec{a} &= 0 \\ 0 + \vec{r} \times \vec{a} &= 0 \\ \vec{r} \times \vec{a} &= 0 \\ \vec{r} &\parallel \vec{a} \end{aligned}$$

Therefore, Kepler's Second Law holds for any moving object if and only if its acceleration vector is always parallel to its position vector.