

18.02 EXERCISES

Problem Set 3: Parametric Equations for Curves

Part I

Unit 1E Equations of Lines and Planes

4. Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x + 4y + z = 4$?

Solution:

A vector along the line is $\langle 2, -1, 1 \rangle$. Hence, the parametric equation of the line is

$$\begin{cases} x = x(t) = 0 + 2t = 2t \\ y = y(t) = 1 - t \\ z = z(t) = 2 + t \end{cases}$$

For the intersection point of the line and the plane, it satisfies

$$\begin{aligned} x(t) + 4y(t) + z(t) &= 4 \\ 2t + 4(1 - t) + 2 + t &= 4 \\ -t &= -2 \\ t &= 2 \end{aligned}$$

Therefore, the coordinates of the intersection point is $(x(2), y(2), z(2))$, which is $(4, -1, 4)$.

7. Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

Solution:

7-1. Formulate a general method for finding the distance between two parallel planes in space:

Suppose that the equations of two parallel planes are respectively

$$\begin{aligned} n_1x + n_2y + n_3z &= c_1 \\ n_1x + n_2y + n_3z &= c_2 \end{aligned}$$

Then the vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is one of the normal vectors of the two planes. Therefore, we can construct the parametric equation of a line perpendicular to the two planes as

$$\begin{cases} x(t) = n_1 t \\ y(t) = n_2 t \\ z(t) = n_3 t \end{cases}$$

Therefore, the two intersection points the line has with the two planes respectively satisfy

$$\begin{aligned} n_1 x(t_1) + n_2 y(t_1) + n_3 z(t_1) &= c_1 \\ n_1^2 t_1 + n_2^2 t_1 + n_3^2 t_1 &= c_1 \\ t_1 &= \frac{c_1}{n_1^2 + n_2^2 + n_3^2} \\ n_1 x(t_2) + n_2 y(t_2) + n_3 z(t_2) &= c_2 \\ n_1^2 t_2 + n_2^2 t_2 + n_3^2 t_2 &= c_2 \\ t_2 &= \frac{c_2}{n_1^2 + n_2^2 + n_3^2} \end{aligned}$$

The distance between two points in the line can be calculated as

$$\begin{aligned} d &= \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 + (z(t_1) - z(t_2))^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(t_1 - t_2)^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2) \left(\frac{c_1 - c_2}{n_1^2 + n_2^2 + n_3^2} \right)^2} \\ &= \frac{|c_1 - c_2|}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \\ &= \frac{|c_1 - c_2|}{|\vec{n}|} \end{aligned}$$

7-2. Formulate a general method for finding the distance between two non-parallel lines in space:

Suppose that there are two non-parallel lines in space named l_a and l_b . Their normal vectors are respectively $\vec{n}_a = \langle n_{ax}, n_{ay}, n_{az} \rangle$ and $\vec{n}_b = \langle n_{bx}, n_{by}, n_{bz} \rangle$,

and their parametric equations are respectively:

$$\begin{cases} x_a(t) = x_{a0} + n_{ax}t \\ y_a(t) = y_{a0} + n_{ay}t \\ z_a(t) = z_{a0} + n_{az}t \end{cases} \quad \begin{cases} x_b(t) = x_{b0} + n_{bx}t \\ y_b(t) = y_{b0} + n_{by}t \\ z_b(t) = z_{b0} + n_{bz}t \end{cases}$$

Then we can find the vector \vec{n} which is perpendicular to both l_a and l_b by

$$\begin{aligned} \vec{n} &= \vec{n}_a \times \vec{n}_b \\ &= \langle n_x, n_y, n_z \rangle \end{aligned}$$

Then the distance between l_a and l_b is equal to the distance between the two planes P_a and P_b with \vec{n} as their normal vectors where l_a and l_b lie within respectively.

The equation of P_a is

$$n_x x + n_y y + n_z z = n_x x_{a0} + n_y y_{a0} + n_z z_{a0}$$

The equation of P_b is

$$n_x x + n_y y + n_z z = n_x x_{b0} + n_y y_{b0} + n_z z_{b0}$$

According to the previous part, the distance between P_a and P_b is

$$\begin{aligned} d &= \frac{|(n_x x_{a0} + n_y y_{a0} + n_z z_{a0}) - (n_x x_{b0} + n_y y_{b0} + n_z z_{b0})|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot \langle (x_{a0} - x_{b0}), (y_{a0} - y_{b0}), (z_{a0} - z_{b0}) \rangle|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot A_0 \vec{B}_0|}{|\vec{n}|} \\ &= \frac{|(\vec{n}_a \times \vec{n}_b) \cdot A_0 \vec{B}_0|}{|\vec{n}_a \times \vec{n}_b|} \end{aligned}$$

Therefore, the distance between l_a and l_b is

$$d = \frac{|(\vec{n}_a \times \vec{n}_b) \cdot A_0 \vec{B}_0|}{|\vec{n}_a \times \vec{n}_b|}$$

7-3 Apply the above method to find the distance between two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

The two non-intersecting diagonals of two adjacent faces of the unit cube I choose are the diagonal between $(0, 0, 0)$ and $(1, 0, 1)$, and the diagonal between $(0, 1, 0)$ and $(0, 0, 1)$. Therefore, the parametric equations of these two lines along the described diagonals are respectively:

$$\begin{cases} x_a(t) = t \\ y_a(t) = 0 \\ z_a(t) = t \end{cases}$$

$$\begin{cases} x_b(t) = 0 \\ y_b(t) = 1 - t \\ z_b(t) = t \end{cases}$$

Therefore,

$$\begin{aligned} \vec{n}_a &= \langle 1, 0, 1 \rangle \\ \vec{n}_b &= \langle 0, -1, 1 \rangle \\ A_0 &= (0, 0, 0) \\ B_0 &= (0, 1, 0) \\ A_0\vec{B}_0 &= \langle 0, 1, 0 \rangle \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals is

$$\begin{aligned} d &= \frac{|(\vec{n}_a \times \vec{n}_b) \cdot A_0\vec{B}_0|}{|\vec{n}_a \times \vec{n}_b|} \\ &= \frac{|\langle 1, -1, -1 \rangle \cdot \langle 0, 1, 0 \rangle|}{|\langle 1, -1, -1 \rangle|} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals of two adjacent faces of the unit cube is $\frac{\sqrt{3}}{3}$.

Unit 1I Vector Functions and Parametric Equations

1. The point P moves with constant speed v in the direction of the constant vector $a\vec{i} + b\vec{j}$. If at time $t = 0$ it is at (x_0, y_0) , what is its position vector function $\vec{r}(t)$?

Solution:

The direction of the constant vector $a\vec{i} + b\vec{j}$ is

$$\text{dir}(A) = \frac{a}{\sqrt{a^2 + b^2}}\vec{i} + \frac{b}{\sqrt{a^2 + b^2}}\vec{j}$$

Hence in a period of time t , the point P moves in a distance of vt in the direction $\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \rangle$. Hence along the x -axis it moves in a distance of $\frac{a}{\sqrt{a^2+b^2}}vt$, and along the y -axis it moves in a distance of $\frac{b}{\sqrt{a^2+b^2}}vt$. Therefore, the position vector function of the point P is

$$\vec{r}(t) = \langle x_0 + \frac{a}{\sqrt{a^2+b^2}}vt, y_0 + \frac{b}{\sqrt{a^2+b^2}}vt \rangle$$

3. Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞ . In each case, give the xy -equation of the curve along which P travels, and tell what part of the curve is actually traced out by P .

- a) $\vec{r} = 2 \cos^2 t \vec{i} + \sin^2 t \vec{j}$
- b) $\vec{r} = \cos 2t \vec{i} + \cos t \vec{j}$
- c) $\vec{r} = (t^2 + 1) \vec{i} + t^3 \vec{j}$
- d) $\vec{r} = \tan t \vec{i} + \sec t \vec{j}$

Solution:

a)

5. A string is wound clockwise around the circle of radius a centered at the origin O ; the initial position of the end P of the string is $(a, 0)$. Unwind the string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P .

7. The cycloid is the curve traced out by a fix point P on a circle of radius a which rolls along the x -axis in the positive direction, starting when P is at the origin O . Find the vector function \vec{OP} ; use as variable the angle θ through which the circle has rolled.

Part II

1. A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P . Suppose that the disk is tangent to the x -axis with the center initially at $(0, 2)$ and P initially at $(0, 1)$ and that it starts to roll to the right on the x -axis at unit speed. Let C be the curve traced out by the point P .

- a) Make a sketch of what you think the curve C will look like.
- b) Use vectors to find the parametric equations for \vec{OP} as a function of time t .
- c) Open the 'Mathlet' Wheel (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C . If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph.