18.02 EXERCISES

Problem Set 2: Matrices and Systems of Equations

Part I

Unit 1E Equations of Lines and Planes

1. Find the equations of the following planes:
a) through (2, 0, -1) and perpendicular to $\vec{i} + 2\vec{j} - 2\vec{k}$.
b) through the origin, (1, 1, 0), and (2, -1, 3)
c) through (1, 0, 1), (2, -1, 2), (-1, 3, 2)
d) through the points on the x, y and z-axes where $x = a$, $y = b$, $z = c$ respectively (give the equation in the form $Ax + By + Cz = 1$ and remember it)
e) through (1, 0, 1) and (0, 1, 1) and parallel to $\vec{i} - \vec{j} + 2\vec{k}$
Solution:
a) According to the problem description, the vector $< 1, 2, -2 >$ is a normal vector to the plane. Therefore, the equation of the plane is

$$x + 2y - 2z = c, \text{where c is a constant}$$

Then we can put the point (2, 0, -1) into the equation:

$$c = 2 + 2 \times 0 - 2 \times (-1) = 4$$

Hence, the equation of the plane is

$$x + 2y - 2z = 4$$

b) According to the problem description, two vectors on the plane are

$$< 1, 1, 0 >, < 2, -1, 3 >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < 1, 1, 0 > \times < 2, -1, 3 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} \\ &= < 3, -3, -3 > \end{aligned}$$

Therefore, the equation of the plane is

$$3x - 3y - 3z = c, \text{where c is a constant}$$

Then we can put the point (0, 0, 0) into the equation:

$$c = 0$$

Hence the equation of the plane is

$$3x - 3y - 3z = 0$$

c) According to the problem description, two vectors on the plane are

$$< 1, -1, 1 >, < -2, 3, 1 >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < 1, -1, 1 > \times < -2, 3, 1 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 3 & 1 \end{vmatrix} \\ &= < -4, 1, 5 > \end{aligned}$$

Therefore, the equation of the plane is

$$-4x + y + 5z = c, \text{where c is a constant}$$

Then we can put the point (1, 0, 1) into the equation:

$$c = -4 \times 1 + 0 + 5 \times 1 = 1$$

Hence the equation of the plane is

$$-4x + y + 5z = 1$$

d) According to the problem description, two vectors on the plane are

$$< -a, b, 0 >, < -a, 0, c >$$

Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < -a, b, 0 > \times < -a, 0, c > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} \\ &= < bc, ac, ab > \end{aligned}$$

Therefore, the equation of the plane is

$$bcx + acy + abz = k, \text{where k is a constant}$$

Then we can put the point (a, 0, 0) into the equation:

$$k = bc \cdot a + ac \cdot 0 + ab \cdot 0 = abc$$

Hence the equation of the plane is

$$\begin{aligned} bcx + acy + abz &= abc \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 1 \end{aligned}$$

e) According to the problem description, one vector parallel to the plane can be derived from the two points (1, 0, 1) and (0, 1, 1), which is $< -1, 1, 0 >$. Therefore, a normal vector to the plane can be calculated as

$$\begin{aligned} < -1,1,0 > \times < 1,-1,2 > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} \\ &=< 2,2,0 > \end{aligned}$$

Therefore, the equation of the plane is

$$2x + 2y = c, \text{where c is a constant}$$

Then we can put the point (1, 0, 1) into the equation:

$$c = 2 \times 1 + 2 \times 0 = 2$$

Hence the equation of the plane is

$$\begin{aligned} 2x + 2y &= 2 \\ x + y &= 1 \end{aligned}$$

2. Find the dihedral angle between the planes $2x - y + z = 3$ and $x + y + 2z = 1$.
Solution:
Suppose that the intersection line between the planes $2x - y + z = 3$ and $x + y + 2z = 1$ is l. By the definition of dihedral angles, we can find a point O on the line l, and in the plane $2x - y + z = 3$ find a line $OA \perp l$, and in the plane $x + y + 2z = 1$ find a line $OB \perp l$. Then the angle $\angle AOB$ is the dihedral angle.
Notice that any normal vectors to the plane $2x - y + z = 3$ and the plane $x + y + 2z = 1$ are also perpendicular to the line l because l is on both planes. Therefore, the vector $\vec{n_1} =< 2, -1, 1 >$, which is a normal vector to the plane $2x - y + z = 3$, and the vector $\vec{n_2} =< 1, 1, 2 >$, which is a normal vector to the plane $x + y + 2z = 1$, are perpendicular to the line l.
Therefore, all four vectors $\vec{OA}$, $\vec{OB}$, $\vec{n_1}$, and $\vec{n_2}$ are all parallel to the same plane, whose normal vectors are along the line l, and they form a quadrilateral. Since $\vec{OA} \perp \vec{n_1}$, and $\vec{OB} \perp \vec{n_2}$, the angle $\angle AOB$ is either equal to or supplementary to the angle θ between two normal vectors $\vec{n_1}$ and $\vec{n_2}$.

$$\begin{aligned}\cos\theta &= \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\\ &= \frac{3}{\sqrt{6}\times\sqrt{6}}\\ &= \frac{1}{2}\\ \theta &= \frac{\pi}{3}\end{aligned}$$

Therefore, the dihedral angle between the two planes are $\frac{\pi}{3}$.
3. Find in parametric form the equations for
a) the line through (1, 0, -1) and parallel to $2\vec{i}-\vec{j}+3\vec{k}$
b) the line through (2, -1, -1) and perpendicular to the plane $x-y+2z=3$
c) all lines passing through (1, 1, 1) and lying in the plane $x+2y-z=2$
Solution:
a)

$$\begin{aligned}x(t) &= 1+2t\\ y(t) &= -t\\ z(t) &= -1+3t\end{aligned}$$

b) The line is perpendicular to the plane $x-y+2z=3 \iff$ The line is parallel to the normal vectors of the plane $x-y+2z=3 \iff$ The line is parallel to the vector $<1,-1,2>$.
Therefore, the parametric equation of the line is

$$\begin{aligned}x(t) &= 2+t\\ y(t) &= -1-t\\ z(t) &= -1+2t\end{aligned}$$

c) Since the line is lying in the plane $x+2y-z=2$, suppose the line vector is $<a,b,c>$, then it satisfies

$$\begin{aligned}<a,b,c>\cdot<1,2,-1> &= 0\\ a+2b-c &= 0\\ c &= a+2b\end{aligned}$$

Therefore, the parametric equations of all lines described above are

$$x(t) = 1 + at$$
$$y(t) = 1 + bt$$
$$z(t) = 1 + (a + 2b)t$$

where a and b are any constants.
5. The line passing through (1, 1, -1) and perpendicular to the plane $x+2y-z = 3$ intersects the plane $2x - y + z = 1$ at what point?
Solution:
The line is perpendicular to the plane $x + 2y - z = 3 \iff$ The line is parallel to the normal vectors of the plane $x + 2y - z = 3 \iff$ The line is parallel to the vector $< 1, 2, -1 >$.
Therefore, the parametric equation of the line is

$$x(t) = 1 + t$$
$$y(t) = 1 + 2t$$
$$z(t) = -1 - t$$

For the intersection point between the line and the plane, we have

$$2x(t) - y(t) + z(t) = 1$$
$$2(1 + t) - (1 + 2t) + (-1 - t) = 1$$
$$t = -1$$

Therefore, the intersection point is $(x(-1), y(-1), z(-1))$, which is $(0, -1, 0)$.
6. Show that the distance D from the origin to the plane $ax + by + cz = d$ is given by the formula $D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$.
Solution:
We need to find the parametric equation of the line through the origin and perpendicular to the plane $ax + by + cz = d$.
The line is perpendicular to the plane $ax+by+cz = d \iff$ The line is parallel to the normal vectors of the plane $ax + by + cz = d \iff$ The line is parallel to the vector $< a, b, c >$.
Therefore, the parametric equation of the line is

$$x(t) = 0 + at = at$$
$$y(t) = 0 + bt = bt$$
$$z(t) = 0 + ct = ct$$

For the intersection point between the line and the plane, we have

$$\begin{aligned}
ax(t) + by(t) + cz(t) &= d \\
a^2t + b^2t + c^2t &= d \\
t &= \frac{d}{a^2+b^2+c^2}
\end{aligned}$$

Therefore, the intersection point is $(\frac{ad}{a^2+b^2+c^2}, \frac{bd}{a^2+b^2+c^2}, \frac{cd}{a^2+b^2+c^2})$
Therefore, the distance D from the origin to the plane is equivalent to the distance between these two points:

$$\begin{aligned}
D &= \sqrt{(\frac{ad}{a^2+b^2+c^2})^2 + (\frac{bd}{a^2+b^2+c^2})^2 + (\frac{cd}{a^2+b^2+c^2})^2)} \\
&= \sqrt{\frac{(a^2+b^2+c^2)d^2}{(a^2+b^2+c^2)^2}} \\
&= \frac{|d|}{\sqrt{a^2+b^2+c^2}}
\end{aligned}$$

Unit 1F Matrix Algebra

5. a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Compute A^2, A^3.

b) Find A^2, A^3, A^n if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Solution:
a)

$$\begin{aligned}
A^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
A^3 &= A^2 \cdot A \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}
\end{aligned}$$

b)

$$
\begin{aligned}
A^2 &= A \cdot A \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
A^3 &= A^2 \cdot A \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

The first column in A determines that the first column in A^{k+1} will be the same as the first column in A^k where k can be any positive integers. The second column in A determines that the second column in A^{k+1} will be the sum of the first and second columns in A^k where k can be any positive integers. Therefore, we can induce that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

8. a) If $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, what is the 3×3 matrix A?

b) If $A \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$, $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$, what is the 3×3 matrix A?

Solution:

a) According to the definition of matrix product, the result of $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the first column of A, the result of $A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is the second column of A, and the result of

$A\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the third column of A. Therefore,

$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$

a) (Method 2) If we combine the three matrix equations, we can get

$$A \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$

$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}^{-1}$$

$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$A = \begin{pmatrix}2 & -1 & 1\\3 & 0 & 1\\1 & 4 & -1\end{pmatrix}$$

b) (Method 1) Suppose $A = \begin{pmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{pmatrix}$. Then

$$A\begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$$\begin{pmatrix}a_1 & a_2 & a_3\\b_1 & b_2 & b_3\\c_1 & c_2 & c_3\end{pmatrix} \cdot \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$$\begin{pmatrix}2a_1\\2b_1\\2c_1\end{pmatrix} = \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$$a_1 = -1, b_1 = 0, c_1 = 2$$

Also

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$$

$$a_1 + a_2 + a_3 = 3$$

$$b_1 + b_2 + b_3 = 0$$

$$c_1 + c_2 + c_3 = 3$$

And

$$A \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a_2 + a_3 \\ 2b_2 + b_3 \\ 2c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$2a_2 + a_3 = 7$$

$$2b_2 + b_3 = 1$$

$$2c_2 + c_3 = 1$$

Hence we derive the following system of equations:

$$\begin{cases} a_1 = -1 \\ b_1 = 0 \\ c_1 = 2 \\ a_1 + a_2 + a_3 = 3 \\ b_1 + b_2 + b_3 = 0 \\ c_1 + c_2 + c_3 = 3 \\ 2a_2 + a_3 = 7 \\ 2b_2 + b_3 = 1 \\ 2c_2 + c_3 = 1 \end{cases}$$

We can solve this system of equations and get

$$A = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

b) (Method 2) If we combine the three matrix equations, we can get

$$A \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 7 \\ 0 & 0 & 1 \\ 4 & 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 3 & 7 \\ 0 & 0 & 1 \\ 4 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$$

9. A square $n \times n$ matrix is called **orthogonal** if $A \cdot A^T = I_n$. Show that this condition is equivalent to saying that
a) each row of A is a row vector of length 1.
b) two different rows are orthogonal vectors.
Solution:
Suppose $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, then $A^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$.

$$A \cdot A^T = I_n$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expand the matrix equation, we can derive the following equations:

$$
\begin{aligned}
a_1^2 + a_2^2 + a_3^3 = 1 \\
b_1^2 + b_2^2 + b_3^3 = 1 \\
c_1^2 + c_2^2 + c_3^3 = 1 \\
a_1 b_1 + a_2 b_2 + a_3 b_3 = 0 \\
b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 \\
a_1 c_1 + a_2 c_2 + a_3 c_3 = 0
\end{aligned}
$$

which is equivalent to that each row of A is a row vector of length 1, and two different rows in A are orthogonal.
Therefore, the statement in the problem description is proved.

Unit 1G Solving Square Systems; Inverse Matrices

3. $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. Solve $Ax = b$ by finding A^{-1}.
Solution:

$$
\begin{aligned}
Ax &= b \\
x &= A^{-1} b \\
x &= \frac{1}{\det(A)} adj(A) \cdot b \\
x &= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \\
x &= \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \\
x &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

4. Referring to Exercise 3 above, solve the system

$$x_1 - x_2 + x_3 = y_1, x_2 + x_3 = y_2, -x_1 - x_2 + 2x_3 = y_3$$

for the x_i as functions of the y_i.
Solution:

$$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_2 + x_3 = y_2 \\ -x_1 - x_2 + 2x_3 = y_3 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{cases} x_1 = \frac{3}{5}y_1 + \frac{1}{5}y_2 - \frac{2}{5}y_3 \\ x_2 = -\frac{1}{5}y_1 + \frac{3}{5}y_2 - \frac{1}{5}y_3 \\ x_3 = \frac{1}{5}y_1 + \frac{2}{5}y_2 + \frac{1}{5}y_3 \end{cases}$$

5. Show that $(AB)^{-1} = B^{-1}A^{-1}$, by using the definition of inverse matrix.
Solution:
By the definition of inverse matrix,

$$\begin{gathered} AB \cdot (AB)^{-1} = I_n \\ A^{-1}AB \cdot (AB)^{-1} = A^{-1}I_n \\ B \cdot (AB)^{-1} = A^{-1}I_n \\ B^{-1}B \cdot (AB)^{-1} = B^{-1}A^{-1}I_n \\ (AB)^{-1} = B^{-1}A^{-1}I_n \\ (AB)^{-1} = B^{-1}A^{-1} \end{gathered}$$

Unit 1H Theorems about Square Systems

3. a) For what c-value(s) will

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ -x_1 + cx_2 + 2x_3 = 0 \end{cases}$$

have a non-trivial solution?
b) For what c-value(s) will

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$$

have a non-trivial solution? (Write it as a system of homogeneous equations.)
c) For each value of c in part (a), find a non-trivial solution to the corresponding system.
d) For each value of c in part (b), find a non-trivial solution to the corresponding system.

Solution:
a)

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ -x_1 + cx_2 + 2x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This is a homogeneous linear system. It always has a trivial solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The linear system has a non-trivial solution
$\iff$ The linear system has more than one solution
$\iff$ The transformation matrix is not inversible
$\iff$ The determinant of the transformation matrix is 0

$$\det(A) = 0$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{vmatrix} = 0$$

$$1 \times (2 - c) + 1 \times (4 + 1) + 1 \times (2c + 1) = 0$$

$$c + 8 = 0$$

$$c = -8$$

Therefore when $c = -8$ the linear system have a non-trivial solution.

b)

$$\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 2x + y = cx \\ -y = cy \end{cases}$$

$$\begin{cases} (2-c)x + y = 0 \\ (c+1)y = 0 \end{cases}$$

$$\begin{pmatrix} 2-c & 1 \\ 0 & c+1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Similar to part (a), it is a homogeneous linear system and it has a trivial solution 0. The system has a non-trivial solution $\iff$ the determinant of the transformation matrix is 0.

$$\det(A) = 0$$

$$\begin{vmatrix} 2-c & 1 \\ 0 & c+1 \end{vmatrix} = 0$$

$$(c+1)(2-c) = 0$$

$$c = 2, c = -1$$

Therefore, when $c = 2$ or $c = -1$, the linear system has a non-trivial solution.

c) When $c = -8$, the system of equations becomes

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ -x_1 - 8x_2 + 2x_3 = 0 \end{cases}$$

We can try to directly solve it. Though we cannot get a unique solution by solving it, we can get some relations between x_1, x_2, and x_3 helping us find a valid solution.

If we add up the first and second equations, and add up the first and third equations, we can get

$$3x_1 + 2x_3 = 0$$

$$-3x_2 + x_3 = 0$$

Therefore, we can construct a solution as $(x_1, x_2, x_3) = (-2, 1, 3)$ and we can verify that it is a valid solution.

d) When $c = 2$, the system of equations becomes

$$\begin{cases} y = 0 \\ 3y = 0 \end{cases}$$

Therefore, it is obvious that a non-trivial solution can be $(x, y) = (1, 0)$.
When $c = -1$, the system of equations becomes

$$3x + y = 00 = 0$$

Therefore, it is obvious that a non-trivial solution can be $(x, y) = (-1, 3)$.
7. Suppose we want to find a pure oscillation (sine wave) of frequency 1 passing through two given points. In other words, we want to choose constants a and b so that the function

$$f(x) = a\cos x + b\sin x$$

has prescribed values at two given x-values: $f(x_1) = y_1$, $f(x_2) = y_2$:
a) Show this is possible in one and only one way, if we assume that $x_2 \neq x_1 + n\pi$, for every integer n.
b) If $x_2 = x_1 + n\pi$ for some integer n, when can a and b be found?
Solution:
a) According to the problem description, the values of a and b satisfy the following system of equations:

$$\begin{cases} f(x_1) = y_1 \\ f(x_2) = y_2 \end{cases}$$

$$\begin{cases} \cos x_1 a + \sin x_1 b = y_1 \\ \cos x_2 a + \sin x_2 b = y_2 \end{cases}$$

$$\begin{pmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Therefore, the values of a and b satisfy a linear system. According to the property of linear systems, we know that:
The linear system has a unique solution
$\iff$ the determinant of the transformation matrix doesn't equal to 0.

$$\begin{aligned} \det(A) &= \begin{vmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{vmatrix} \\ &= \sin x_2 \cos x_1 - \sin x_1 \cos x_2 \\ &= \sin(x_2 - x_1) \end{aligned}$$

We also know that

$$\begin{gathered}
x_2 \neq x_1 + n\pi \\
x_2 - x_1 \neq n\pi \\
\sin(x_2 - x_1) \neq 0 \\
\det(A) \neq 0
\end{gathered}$$

Therefore, the linear system of a and b has a unique solution.
b) First of all, if $(y_1, y_2) = (0, 0)$, the linear system of a and b is homogeneous, hence it always has a trivial solution which is $(a, b) = (0, 0)$.
For non-homogeneous cases, if $x_2 = x_1 + n\pi$ holds for an odd integer n, then the linear system becomes

$$\begin{cases} \cos x_1 a + \sin x_1 b = y_1 \\ -\cos x_1 a - \sin x_1 b = y_2 \end{cases}$$

It is obvious that when $y_2 = -y_1$, we can find solutions for a and b, otherwise the linear system has no solution.
For non-homogeneous cases where $x_2 = x_1 + n\pi$ holds for an even integer n, the linear system becomes

$$\begin{cases} \cos x_1 a + \sin x_1 b = y_1 \\ \cos x_1 a + \sin x_1 b = y_2 \end{cases}$$

It is obvious that when $y_1 = y_2$, we can find solutions for a and b, otherwise the linear system has no solution.

Part II

1. Suppose we know that when the three planes P_1, P_2 and P_3 in $\mathbb{R}^3$ intersect in pairs, we get three lines L_1, L_2, and L_3 which are distinct and parallel.
a) Sketch a picture of this situation.
b) Show that the three normals to P_1, P_2, and P_3 all lie in one plane, using a geometric argument.
c) Show that the three normals to P_1, P_2, and P_3 all lie in one plane, using an algebraic argument.
Solution:
a) TODO(jinxinwang)
b) Suppose that the intersection of P_1 and P_2 is L_1, the intersection of P_2 and P_3 is L_2, and the intersection of P_3 and P_1 is L_3. Let $\vec{n_1}$ be the normal to P_1, $\vec{n_2}$ be the normal to P_2, and $\vec{n_3}$ be the normal to P_3.

According to the intersection relationship, we can derive

$$\begin{gathered}\vec{n_1} \perp L_1 \\ \vec{n_2} \perp L_1 \\ \vec{n_2} \perp L_2 \\ \vec{n_3} \perp L_2 \\ \vec{n_3} \perp L_3 \\ \vec{n_1} \perp L_3\end{gathered}$$

Since L_1, L_2, L_3 are parallel to each other, and if $l_1 \perp l_2$, then l_1 is also perpendicular to all lines parallel to l_2, we can also derive that

$$\begin{gathered}\vec{n_1} \perp L_2 \\ \vec{n_2} \perp L_3 \\ \vec{n_3} \perp L_1\end{gathered}$$

Most importantly, we derive that

$$\begin{gathered}\vec{n_1} \perp L_1 \\ \vec{n_2} \perp L_1 \\ \vec{n_3} \perp L_1\end{gathered}$$

Then I claim that if a vector is perpendicular to another vector $\vec{n}$, then it lies in the plane whose normal vector is $\vec{n}$. From the geometric perspective, this claim can be proved by contradiction:
Suppose that a vector $\vec{a}$ is perpendicular to another vector $\vec{n}$, and it doesn't lie in the plane P whose normal vector is $\vec{n}$. From a random point O in the plane P, we construct a line l_a in the direction of $\vec{a}$. From a random point A on the line l_a other than the point O, we construct a line towards the plane in the same or reverse direction of $\vec{n}$, and the intersection point between the line and the plane is A'. AOA' forms a triangle. Since $\vec{a} \perp \vec{n}$, $AA' \perp OA$ and $\angle OAA' = \frac{\pi}{2}$. Since $\vec{n}$ is the normal vector of the plane P and OA' is on the plane, $AA' \perp OA'$ and $\angle AA'O = \frac{\pi}{2}$. Since l_a is not on the plane P, $\angle AOA' > 0$. Therefore, the sum of the three angles of the triangle AOA' is greater than π, which is a contradiction. Therefore, any vector which is perpendicular to a vector $\vec{n}$ must lie in the plane whose normal vector is $\vec{n}$.
Therefore, $\vec{n_1}$, $\vec{n_2}$, and $\vec{n_3}$ all lie in the plane whose normal vector is along the line L_1.

c) Using the same notation and conclusion as part (b), we derive that

$$\vec{n_1} \perp L_1$$
$$\vec{n_2} \perp L_1$$
$$\vec{n_3} \perp L_1$$

Then I claim that if a vector is perpendicular to another vector $\vec{n}$, then it lies in the plane whose normal vector is $\vec{n}$, and I will prove it from the algebraic perspective:
Suppose that the equation of the plane is $n_1x + n_2y + n_3z = c$, and one of its normal vectors is $\vec{n} =< n_1, n_2, n_3 >$. Also suppose that the vector that is perpendicular to $\vec{n}$ is $\vec{a}$. Let P_0 denote a random point on the plane $n_1x+n_2y+n_3z = c$ and $P_0 = (x_0, y_0, z_0)$. Then we have

$$n_1x_0 + n_2y_0 + n_3z_0 = c$$

For any points $P = (x, y, z)$ that $\vec{P_0P} \parallel \vec{a}$, since $\vec{a} \perp \vec{n}$, then

$$< x - x_0, y - y_0, z - z_0 > \cdot < n_1, n_2, n_3 >= 0$$
$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$
$$n_1x + n_2y + n_3z = n_1x_0 + n_2y_0 + n_3z_0$$
$$n_1x + n_2y + n_3z = c$$

Therefore, P is also on the plane, which means the vector $\vec{a}$ lies in the plane. Hence it is proved that a vector is perpendicular to another vector $\vec{n}$, then it lies in the plane whose normal vector is $\vec{n}$.
Therefore, $\vec{n_1}$, $\vec{n_2}$, and $\vec{n_3}$ all lie in the plane whose normal vector is along the line L_1.

2. A manufacturing process mixes three raw materials M_1, M_2, and M_3 to produce three products P_1, P_2, and P_3. The ratios of the amounts of the raw materials (in the order M_1, M_2, M_3) which are used to make up each of the three products are as follows: For P_1 the ratio is $1 : 2 : 3$; for P_2 the ratio is $1 : 3 : 5$; and for P_3 the ratio is $3 : 5 : 8$. In a certain production run, 137 units of M_1, 279 units of M_2, and 448 units of M_3 were used. The problem is to determine how many units of each of the products P_1, P_2, and P_3 were produced in that run.
a) Set this problem up in matrix form. Use the letter A for the matrix, and write down the (one-line) formula for the solution in matrix form.
b) Compute the inverse matrix of A and use it to solve for the production vector P.
c) Find a choice for the ratios for the third product (in lowest form), different from the other ratios, and for which the resulting system has non-unique solutions.

Solution:
a) Suppose that in the mentioned run, x_1 units of P_1, x_2 units of P_2, and x_3 units of P_3 were produced. According to the problem description, we can set the following linear system:

$$AX = B$$
$$\begin{pmatrix}1 & 1 & 3\\ 2 & 3 & 5\\ 3 & 5 & 8\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}137\\ 279\\ 448\end{pmatrix}$$
$$X = A^{-1}B$$
$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 3\\ 2 & 3 & 5\\ 3 & 5 & 8\end{pmatrix}^{-1}\begin{pmatrix}137\\ 279\\ 448\end{pmatrix}$$

b)

$$\begin{aligned}A^{-1} &= \frac{1}{\det(A)}adj(A)\\ &= \frac{1}{1}\begin{pmatrix}-1 & -1 & 1\\ 7 & -1 & -2\\ -4 & 1 & 1\end{pmatrix}^T\\ &= \begin{pmatrix}-1 & 7 & -4\\ -1 & -1 & 1\\ 1 & -2 & 1\end{pmatrix}\end{aligned}$$
$$X = A^{-1}B$$
$$\begin{aligned}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} &= \begin{pmatrix}-1 & 7 & -4\\ -1 & -1 & 1\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}137\\ 279\\ 448\end{pmatrix}\\ &= \begin{pmatrix}24\\ 32\\ 27\end{pmatrix}\end{aligned}$$

c) Suppose that the consumption ratio of the product P_3 is $a_1 : a_2 : a_3$. Then the transformation matrix A in the linear system becomes

$$A = \begin{pmatrix}1 & 1 & a_1\\ 2 & 3 & a_2\\ 3 & 5 & a_3\end{pmatrix}$$

For a non-homogeneous linear system like the one in this problem,
The linear system has non-unique system

$\iff$ The transformation matrix A is not inversible
$\iff$ The corresponding determinant of the transformation matrix A equals 0.

$$\det(A) = 0$$
$$1 \cdot (3a_3 - 5a_2) - 1 \cdot (2a_3 - 3a_2) + a_1 \cdot 1 = 0$$
$$a_1 - 2a_2 + a_3 = 0$$

Hence, a possible choice of the consumption ratio of the product P_3 which makes the linear system has non-unique solutions is $1 : 1 : 1$.

3. For any plane P which is not parallel to the x-y plane, define the steepest direction on P to be the direction of any vector which lies in P and which makes the largest (acute) angle with the x-y plane.
a) Let P be the plane through the origin with the normal vector $\vec{n}$. Derive a formula, in terms of $\vec{n}$, for a vector $\vec{w}$ which points in the steepest direction on P.
b) Now Let P be the plane through the origin which contains two non-parallel vectors $\vec{u}$ and $\vec{v}$, where $\vec{u}$ and $\vec{v}$ do not both lie in the x-y plane. Derive a formula, in terms of $\vec{u}$ and $\vec{v}$, for a vector $\vec{w}$ which points in the steepest direction on P.
Solution: (TODO)
a) Suppose $\vec{n} =< n_1, n_2, n_3 >$, and $\vec{w} =< w_1, w_2, w_3 >$. Since $\vec{n}$ is a normal vector to P, and $\vec{w}$ lies in P,

$$\vec{n} \cdot \vec{w} = 0$$
$$n_1 w_1 + n_2 w_2 + n_3 w_3 = 0$$

Also since $\vec{w}$ makes the largest (acute) angle with the x-y plane, from the geometric perspective, it means the angle between $\vec{w}$ and the unit vector $\vec{k}$ is the smallest, and hence $\vec{w} \cdot \vec{k}$ has the largest result.
b)

$$\vec{u} \times \vec{v} = \vec{n}$$