Lecture 9: Max-Min and Least Squares

1 Application of Partial Derivatives: Optimization Problem

Optimization problems refer to the tasks of finding the maximum or minimum point of a function. Here we focus on finding the maximum or minimum point of multivariable functions, such as $f(x, y)$.
Another concept is local max/min point, which means the function value of this point is greater/less than the function values of adjacent points, but it is not necessarily the greatest or leaast function value in the function domain.

Theorem. *If a point in a function f is a local max/min point, then all of the partial derivatives at this point are equal to 0.*
Proof:
We will prove it by contradiction. Suppose there is a local max/min point in a function $f(x_1, x_2, ..., x_n)$, and one of its partial derivative $\frac{\partial f}{\partial x_i}$ is not 0. According to the definition of partial derivatives, in that direction the function values of the two adjacent point are

$$f(x_1, x_2, ..., x_i + \Delta x_i, ..., x_n) = f(x_1, x_2, ..., x_n) + \frac{\partial f}{\partial x_i}\Delta x_i$$
$$f(x_1, x_2, ..., x_i - \Delta x_i, ..., x_n) = f(x_1, x_2, ..., x_n) - \frac{\partial f}{\partial x_i}\Delta x_i$$

Therefore, one of them is less than the local max/min point and the other is greater than the local max/min point. Hence, it is not a local max/min point. There is a contradiction.
Therefore, it is proved that if a point in a function f is a local max/min point, then all of the partial derivatives at this point are equal to 0.

Therefore, A point is a local max/min point
$\Rightarrow$ All partial derivatives at that point are 0
$\iff$ The tangent plane at that point is horizontal
$\iff$ The point is a critical point.

2 Example of Optimization Problem: Least Squares