Lecture 8: Partial Derivatives

1 Multivariable Functions

Multivariable functions are functions whose values depend on more than one independent variables.
Examples of Multivariable functions:

- $f(x,y) = x^2 + y^2$
- $f(x,y) = \frac{1}{x+y}$, $x + y \neq 0$
- $f(x,y) = \frac{1}{\sqrt{y}}$, $y > 0$
- The temperature of a certain point on earth at a given time is a multivariable function of latitude, longitude, and height.

For simplicity, in this course we will mainly study multivariable functions with two or three variables, but the concepts apply to any multivariable functions with any number of variables.

2 Visualization of Multivariable Functions with Two Variables

2.1 Function Graph

For a single-variable function f, we plot the function graph as $(x, f(x))$, which is very straightforward.
For a multivariable function with two variables f, we can similarly plot the function graph as $(x, y, f(x,y))$, which is a surface in 3D space.

Example 1. *Plot the graph of the function $f(x,y) = -y$.*
The equation of the function graph is $z = -y$, which is a plane.

Example 2. *Plot the graph of the function $f(x,y) = 1 - x^2 - y^2$.*
The equation of the function graph is $z = 1 - x^2 - y^2$, whose shape is not so easy to be identified.
We can first take a look at the intersection between the function graph and the $x - z$ plane, which means let $y = 0$. Hence, the equation of the intersection is $z = 1 - x^2$, which is a parabola.

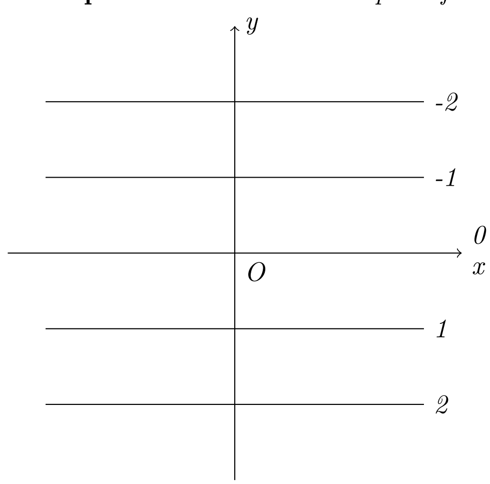
We can then take a look at the intersection between the function graph and the $y - z$ plane, which means let $x = 0$. Hence, the equation of the intersection is $z = 1 - y^2$, which is a parabola.
Finally, we can take a look at the intersection between the function graph and the $x - y$ plane, which means let $z = 0$. Hence the equation of the intersection is $x^2 + y^2 = 1$, which is a unit circle.
Based on the above information, we can guess the shape of the function graph is as follows:

From the above examples, we can see that the graph of multivariable functions with two variables are not only hard to plot, but also often hard to read.

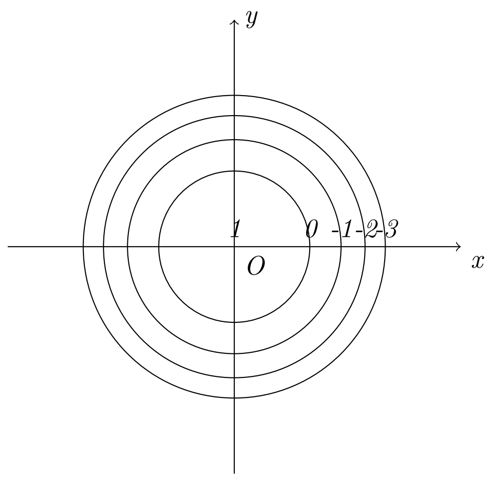
2.2 Contour Plot

Contour Plot is another way to visualize multivariable functions with two variables. It basically shows all the points where $f(x, y) =$ some fixed constants, which are chosen at regular intervals. Examples include contour map, isotherm map, etc.
From geometric point of view, contour plot is equivalent to using horizontal planes ($z =$ some fixed constants) to slice the function graph and combining the intersection points in the $x - y$ plane.

Example 3. *Find the contour plot of the function $f(x, y) = -y$.*



Example 4. *Find the contour plot of the function $f(x, y) = 1 - x^2 - y^2$.*



Notice that the distance between two adjacent contour lines becomes smaller going outwards, which means the slope becomes larger in that direction.

3 Partial Derivatives

For single-variable functions, we have derivatives to describe the rate of change of the function value with the independent variable.
How do we do the same for multivariable functions, which have more than one independent variables? We can assume that only one independent variable in a multivariable function can change, i.e. treating other independent variables as constants, in this way we can adapt the definition of derivatives from single-variable functions to multivariable functions. Such derivatives are called **Partial Derivatives**. The notation of partial derivatives is $\frac{\partial f}{\partial x}$, where the symbol ∂ is read as *partial*, and the independent variable in the denominator is called the *partial variable*, which is the variable that can change, or is not treated as a constant. In applied mathematics, the notation of partial derivatives can also be f_x.

3.1 Definition of Partial Derivatives

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$
$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

3.2 Geometric Interpretation of Partial Derivatives

A partial derivative of a multivariable function can be viewed as extracting a 2D slice out of the plot of the function, and using the partial derivative to describe the slope of the tangent line in that slice.

3.3 Computation of Partial Derivatives

Since the definition of partial derivatives is the same as the derivative of single-variable functions, so the formulas with the derivative of single-variable functions also apply to partial derivatives. In the computation, we just need to treat other variables as constants and apply the proper derivative formulas.

Reason why formulas of the derivative of single-variable functions also apply to partial derivatives:
According to the definition of partial derivatives,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Since y_0 doesn't change in the computation, we can ignore it in the formula just as ignoring other constant terms and factors in the function. Therefore,

$$\begin{aligned}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\end{aligned}$$

which is the definition of the derivative of single-variable functions. Therefore, we can apply formulas of the derivative of single-variable functions to compute partial derivatives with treating other independent variables as constants.

Example 5. *Find the partial derivatives regarding x and y of the function $f(x, y) = x^3 y + y^2$.*
Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3yx^2 \\ \frac{\partial f}{\partial y} &= x^3 + 2y\end{aligned}$$

3.4 Application of Partial Derivatives: Approximation Formula

Suppose $z = f(x, y)$, If we change both $x \to x + \Delta x$, and $y \to y + \Delta y$, the change of the function value Δz can be approximated as

$$\Delta z \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

The geometric interpretation of the approximation formula is that the function graph of a multivariable function is close to its tangent plane near a given point $(x_0, y_0, z_0 = f(x_0, y_0))$, where the tangent line is determined by the two tangent lines in the plane $x = x_0$ and $y = y_0$.
In the plane $y = y_0$, the slope of the tangent line is

$$\frac{\partial f}{\partial x}(x_0, y_0) = a$$

Therefore, the equation of the tangent line in space is

$$\begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$$

By finding two points on the line (x_0, y_0, z_0) and $(2x_0, y_0, z_0 + ax_0)$, we can get a vector of the tangent line as $< x_0, 0, ax_0 >$, or $< 1, 0, a >$.
In the plane $x = x_0$, the slope of the tangent line is

$$\frac{\partial f}{\partial y}(x_0, y_0) = b$$

Therefore, the equation of the tangent line in space is

$$\begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$$

By finding two points on the line (x_0, y_0, z_0) and $(x_0, 2y_0, z_0 + by_0)$, we can get a vector of the tangent line as $< 0, y_0, by_0 >$, or $< 0, 1, b >$.
Therefore, a normal vector of the tangent plane would be

$$\begin{aligned} < 1, 0, a > \times < 0, 1, b > &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} \\ &= < -a, -b, 1 > \end{aligned} \tag{1}$$

Therefore, the equation of the tangent plane would be

$$-ax - by + z = c$$

Since we know the point (x_0, y_0, z_0) is on the tangent plane, we can replace (x, y, z) in the plane equation with (x_0, y_0, z_0) and we get

$$c = -ax_0 - by_0 + z_0$$

Therefore, the equation of the tangent plane is

$$-ax - by + z = -ax_0 - by_0 + z_0$$
$$z = z_0 + a(x - x_0) + b(y - y_0)$$

Question. *How do we know that the function graph at a given point is close to its tangent plane at that point?*
Probably continuity.

Trick. *In a 2D plane, one of the line vectors of the line with slope as a would be* $< 1, a >$.