18.02 EXERCISES

Problem Set 3: Parametric Equations for Curves

Part I

Unit 1E Equations of Lines and Planes

4. Where does the line through $(0,1,2)$ and $(2,0,3)$ intersect the plane $x+4y+z=4$?
Solution:
A vector along the line is $< 2,-1,1 >$. Hence, the parametric equation of the line is

$$\begin{cases} x = x(t) = 0 + 2t = 2t \\ y = y(t) = 1 - t \\ z = z(t) = 2 + t \end{cases}$$

For the intersection point of the line and the plane, it satisfies

$$\begin{aligned} x(t) + 4y(t) + z(t) &= 4 \\ 2t + 4(1-t) + 2 + t &= 4 \\ -t &= -2 \\ t &= 2 \end{aligned}$$

Therefore, the coordinates of the intersection point is $(x(2), y(2), z(2))$, which is $(4,-1,4)$.

7. Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).
Solution:
7-1. Formulate a general method for finding the distance between two parallel planes in space:
Suppose that the equations of two parallel planes are respectively

$$\begin{aligned} n_1x + n_2y + n_3z &= c_1 \\ n_1x + n_2y + n_3z &= c_2 \end{aligned}$$

Then the vector $< n_1, n_2, n_3 >$ is one of the normal vectors of the two planes. Therefore, we can construct the parametric equation of a line perpendicular to the two planes as

$$\begin{cases} x(t) = n_1 t \\ y(t) = n_2 t \\ z(t) = n_3 t \end{cases}$$

Therefore, the two intersection points the line has with the two planes respectively satisfy

$$\begin{aligned} n_1 x(t_1) + n_2 y(t_1) + n_3 z(t_1) &= c_1 \\ n_1^2 t_1 + n_2^2 t_1 + n_3^2 t_1 &= c_1 \\ t_1 &= \frac{c_1}{n_1^2 + n_2^2 + n_3^2} \\ n_1 x(t_2) + n_2 y(t_2) + n_3 z(t_2) &= c_2 \\ n_1^2 t_2 + n_2^2 t_2 + n_3^2 t_2 &= c_1 \\ t_2 &= \frac{c_2}{n_1^2 + n_2^2 + n_3^2} \end{aligned}$$

The distance between two points in the line can be calculated as

$$\begin{aligned} d &= \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 + (z(t_1) - z(t_2))^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(t_1 - t_2)^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(\frac{c_1 - c_2}{n_1^2 + n_2^2 + n_3^2})^2} \\ &= \frac{|c_1 - c_2|}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \end{aligned}$$

7-2. Formulate a general method for finding the distance between two parallel lines in space:
Suppose that the equations of two parallel lines are respectively

$$\begin{cases} x(t) = x_1 + n_1 t \\ y(t) = y_1 + n_2 t \\ z(t) = z_1 + n_3 t \end{cases}$$

$$\begin{cases} x(t) = x_2 + n_1 t \\ y(t) = y_2 + n_2 t \\ z(t) = z_2 + n_3 t \end{cases}$$

The two parallel lines determine a plane in space. The normal vector of the plane can be calculated as the cross product of the two vectors:

$$< x_1 - x_2, y_1 - y_2, z_1 - z_2 > \times < n_1, n_2, n_3 >$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ n_1 & n_2 & n_3 \\ (x_1 - x_2) & (y_1 - y_2) & (z_1 - z_2) \end{vmatrix}$$
$$=< n_2(z_1 - z_2) - n_3(y_1 - y_2), n_3(x_1 - x_2) - n_1(z_1 - z_2), n_1(y_1 - y_2) - n_2(x_1 - x_2) >$$

Then we can find the vector perpendicular to the two parallel lines and lying in the plane determined by the two parallel lines by calculating the cross product of the normal vector and the line vector $< n_1, n_2, n_3 >$:

Part II

1. A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P. Suppose that the disk is tangent to the x-axis with the center initially at $(0, 2)$ and P initially at $(0, 1)$ and that it starts to roll to the right on the x-axis at unit speed. Let C be the curve traced out by the point P.
a) Make a sketch of what you think the curve C will look like.
b) Use vectors to find the parametric equations for $\vec{OP}$ as a function of time t.
c) Open the 'Mathlet' Wheel (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C. If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph.