Lecture 5: Parametric Equations

1 Equations of Lines in Space

From previous lecture, we see that a line can be represented as the intersection of two planes. However, as equations it is not so easy to use because we need to solve it first.
Another representation is to treat a line as the trajectory of a moving point, and include a parameter in the equation to describe different positions in the line. Such representation is called **parametric equations**.

Example 1. *Find the parametric equation of the line through* $Q_0 = (-1, 2, 2)$ *and* $Q_1 = (1, 3, -1)$.
Given that the line can be the trajectory of a moving point, the parameter in the equation could be the time t. *Suppose that at* $t = 0$, *the moving point is in* Q_0, *and at* $t = 1$, *the moving point is in* Q_1, *and suppose the point moves at a constant speed.*
With parametric equations, the coordinates of the point Q *in the line are functions of the parameter. Therefore,* $Q = (x(t), y(t), z(t))$. *The point* Q *in the line also satisfies the following equation:*

$$\vec{Q_0Q} = t \cdot \vec{Q_0Q_1}$$

$$< x + 1, y - 2, z - 2 >= t \cdot < 2, 1, -3 >$$

$$\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 2 \\ z(t) = -3t + 2 \end{cases}$$

That is the parametric equation of the described line.

Notice that the coefficients of the parameter are components of a vector with the same direction as the line (either one of the two directions a line has). The constant terms in the parametric equation are coordinates of the point when the parameter is equal to 0.
A line can have infinite different parametric equations. With different time assumption and speed, the coefficients can be different, but they all express the same line.
The reason why the parametric equation of a line only has one parameter is that the points on a line only have one dimension (degree of freedom). It doesn't matter whether it is a line or a curve.

Example 2. *Consider the plane $x+2y+4z=7$, and two points $Q_0=(-1,2,2)$ and $Q_1=(1,3,-1)$.*
1) Whether Q_0 and Q_1 are $\underline{B}$.
A. on the same side.
B. on the opposite sides.
C. on the plane.
D. cannot be decided.
Reasons:
For $Q_0=(-1,2,2)$, $x+2y+4z=11$. Therefore, Q_0 is in the plane P_0 whose equation is $x+2y+4z=11$.
For $Q_1=(1,3,-1)$, $x+2y+4z=3$. Therefore, Q_1 is in the plane P_1 whose equation is $x+2y+4z=3$.
Assume the plane with the equation $x+2y+4z=7$ is called P.
The intersection point of these three planes with the z axis are $(0,0,\frac{11}{4})$, $(0,0,\frac{3}{4})$, and $(0,0,\frac{7}{4})$ respectively, among which the intersection point of the plane P are in between the ones of other two planes.
Since the planes P, P_0, and P_1 are all parallel to each other, P is in between other two planes. Hence, P is in between Q_0 and Q_1. Therefore, Q_0 and Q_1 are on the opposite sides of the plane $x+2y+4z=7$.

2) Does the line through Q_0 and Q_1 pass through the plane? If so, what is the intersection point?
According to the previous example, we know that the parametric equation of the line is:

$$\begin{cases} x(t)=2t-1 \\ y(t)=t+2 \\ z(t)=-3t+2 \end{cases}$$

For the intersection point, the following equation holds:

$$x(t)+2y(t)+4z(t)=7$$

$$(2t-1)+2(t+2)+4(-3t+2)=7$$

$$-8t+11=7$$

$$t=\frac{1}{2}$$

Therefore, the intersection point $Q=(0,\frac{5}{2},\frac{1}{2})$.
Notice that the coefficient of the parameter for the intersection point is the dot product of the direction of the line and the normal vector of the plane. If the coefficient is 0, from geometric point of view it means the direction of the line is perpendicular to the normal vector of the plane, which means the line is parallel to the plane; from algebraic point of view, the equation of the intersection point either always holds or has no solution, which also means the line is parallel to the plane.

2 General Usage of Parametric Equations

More generally, we can use parametric equations for arbitrary trajectory in the plane or space.

Example 3. *Find the parametric equation of a cycloid, which is the curve traced by a fixed point on a circle as it rolls along a straight line without slipping.*
Solution:
In this case, there is a natural parameter which is the angle θ the circle has rolled, because the position of the fixed point is solely decided by the angle. Suppose the start point of the circle is when the intersection between the circle and the x axis is the origin O, which is also the tracing point. After rolling an angle θ, the fix point is at position B, and the current intersection point is A. The center of the circle is C, and its radius is a.
To find the parametric equation of the cycloid, we need to find the parametric equation of $\vec{OB}$.

$$\vec{OB} = \vec{OA} + \vec{AC} + \vec{CB}$$

Since the circle rolls along the x axis without slipping,

$$\vec{OA} =< a\theta, 0 >$$

Since $\vec{AC}$ is always perpendicular to the x axis and the magnitude is the radius of the circle,

$$\vec{AC} =< 0, a >$$

We can decompose $\vec{CB}$ into horizontal and vertical directions,

$$\vec{CB} =< -a\sin\theta, -a\cos\theta >$$

Notice that we can verify that the equations of the three vectors hold for any θ, not only in $[0, 2\pi]$. Therefore,

$$\begin{aligned}\vec{OB} &= \vec{OA} + \vec{AC} + \vec{CB} \\ &=< a\theta - a\sin\theta, a - a\cos\theta >\end{aligned}$$

Therefore, the parametric equation of a cycloid is

$$\begin{cases} x(t) = a\theta - a\sin\theta \\ y(t) = a - a\cos\theta \end{cases}$$

Following: What does the curve look like between two humps, i.e. around $\theta = 2n\pi$?

To figure it out, we need to look at the derivative $\frac{dy}{dx}$ *around* $\theta = 2n\pi$*.*

$$\begin{aligned}
\frac{dy}{dx}|_{\theta=2n\pi} &= (\frac{dy}{d\theta}/\frac{dx}{d\theta})|_{\theta=2n\pi} \\
&= (\frac{d(a - a\cos\theta)}{d\theta}/\frac{d(a\theta - a\sin\theta)}{d\theta})|_{\theta=2n\pi} \\
&= \frac{a\sin\theta}{a - a\cos\theta}|_{\theta=2n\pi} \\
&= \frac{0}{0}
\end{aligned}$$

According to L'Hospital rule,

$$\begin{aligned}
\frac{dy}{dx}|_{\theta=2n\pi} &= \frac{a\sin\theta}{a - a\cos\theta}|_{\theta=2n\pi} \\
&= (\frac{d(a\sin\theta)}{d\theta}/\frac{d(a - a\cos\theta)}{d\theta})|_{\theta=2n\pi} \\
&= \frac{a\cos\theta}{a\sin\theta}|_{\theta=2n\pi} \\
&= \frac{a}{0} \\
&= \infty
\end{aligned}$$

Therefore, the slope of a cycloid between two humps, i.e. around $\theta = 2n\pi$*, is infinity, which means the tangent line in the intersection of two humps is vertical, perpendicular to the* x *axis.*