

# Lecture 6: Kepler's Second Law

## 1 Describe Motion with Vectors

With parametric equations, we have a way not only to describe a motion, but also to analyze the motion with more details.

### 1.1 Position vectors

Position vectors are used to describe the position of a point in a motion.

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$$

### 1.2 Velocity vectors

Velocity vectors are used to describe how fast and in what direction a point moves in a motion.

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} \\ &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \end{aligned}$$

Speed is a scalar, and used to describe how fast a point moves in a motion.

$$speed = |\vec{v}|$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \left\langle \frac{d(t - \sin t)}{dt}, \frac{d(1 - \cos t)}{dt} \right\rangle \\
&= \langle 1 - \cos t, \sin t \rangle \\
|\vec{v}| &= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\
&= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\
&= \sqrt{2 - 2\cos t}
\end{aligned}$$

At  $t = 0$ , the velocity vector  $\vec{v} = \langle 0, 0 \rangle$ , so the speed at that time is 0.

### 1.3 Acceleration vectors

Acceleration vectors are used to describe how velocity changes in a motion.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example:

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \left\langle \frac{d(1 - \cos t)}{dt}, \frac{d(\sin t)}{dt} \right\rangle \\
&= \langle \sin t, \cos t \rangle
\end{aligned}$$

At  $t = 0$ , the acceleration vector  $\vec{a} = \langle 0, 1 \rangle$ , which means the point at that time has an acceleration in the positive direction of the  $y$  axis. Combining with the previous conclusion that the velocity at this time is 0, it explains why the trajectory between two arches has a vertical tangent line.

### 1.4 Arc length

Arc length, usually denoted by  $s$ , is the distance traveled along the trajectory of a motion. According to this definition,

$$\frac{ds}{dt} = \text{speed} = |\vec{v}|$$

Take the cycloid of the wheel of radius 1 rolling at unit speed as an example. The length of an arch of the cycloid is

$$\int_0^{2\pi} \sqrt{2 - 2\cos t} dt$$

## 1.5 Unit tangent vector

Unit tangent vector, denoted by  $\hat{T}$ , is an unit vector in the direction of the tangent line at a position in a motion.

Since  $\vec{v}$  is a tangent vector,

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

Another definition for the unit tangent vector:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt} \\ &= \frac{d\vec{r}}{ds} \cdot |\vec{v}| \\ \frac{d\vec{r}}{ds} &= \frac{\vec{v}}{|\vec{v}|} \\ &= \hat{T} \\ \vec{v} &= \hat{T} \cdot \frac{ds}{dt}\end{aligned}$$

Therefore, we can see the velocity vector can be interpreted as:

1. direction: the unit tangent vector  $\hat{T}$ .
2. magnitude: the speed  $\frac{ds}{dt}$ .

The geometric interpretation of  $\frac{d\vec{r}}{ds} = \hat{T}$ : during the time period of  $\Delta t$ ,

$$\begin{aligned}\Delta\vec{r} &= \vec{r}(t + \Delta t) - \vec{r}(t) \\ \Delta\vec{r} &\approx \hat{T} \cdot \Delta s \\ \frac{\Delta\vec{r}}{\Delta t} &\approx \hat{T} \cdot \frac{\Delta s}{\Delta t}\end{aligned}$$

With  $\Delta t$  approaches 0, the above formula becomes

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \hat{T} \cdot \frac{ds}{dt} \\ \frac{d\vec{r}}{ds} &= \hat{T}\end{aligned}$$

## 2 Example of Describing Motion with Vectors: Kepler's Second Law

### 2.1 Kepler's Second Law

Kepler's Second Law: Motion of planets is in a plane, and the area swept out by the line from sun to planet at a constant rate.  
Newton later explain this law with gravitational attraction.

### 2.2 Translation with Vector Notations

The first part of Kepler's Second Law is: motion of planets is in a plane. The plane of planet motion is determined by two vectors: the position vector of the planet  $\vec{r}$  and its velocity vector  $\vec{v}$ . The plane of planet motion remains unchanged  $\iff$  the direction of the normal vector of the plane remains unchanged  $\iff \text{dir}(\vec{r} \times \vec{v})$  is constant.

The second part of Kepler's Second Law is: the area swept out by the line from sun to planet at a constant rate. The area swept out in unit time can be calculated as

$$\begin{aligned} dA &= \frac{1}{2} |\vec{r} \times d\vec{r}| \\ &= \frac{1}{2} |\vec{r} \times \vec{v}| dt \\ \frac{dA}{dt} &= \frac{1}{2} |\vec{r} \times \vec{v}| \end{aligned}$$

Therefore, the area swept out by the line from sun to planet at a constant rate  $\iff \frac{dA}{dt}$  is constant  $\iff |\vec{r} \times \vec{v}|$  is constant.  
Therefore, Kepler's Second Law  $\iff$  both  $\text{dir}(\vec{r} \times \vec{v})$  and  $|\vec{r} \times \vec{v}|$  are constant  $\iff \vec{r} \times \vec{v}$  is constant  $\iff \frac{d}{dt} \vec{r} \times \vec{v} = 0$ .

$$\begin{aligned} \frac{d}{dt} \vec{r} \times \vec{v} &= 0 \\ \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} &= 0 \\ \vec{v} \times \vec{v} + \vec{r} \times \vec{a} &= 0 \\ 0 + \vec{r} \times \vec{a} &= 0 \\ \vec{r} \times \vec{a} &= 0 \\ \vec{r} &\parallel \vec{a} \end{aligned}$$

Therefore, Kepler's Second Law holds for any moving object if and only if its acceleration vector is always parallel to its position vector.