

# Lecture 4: Square Systems

## 1 Plane Equations

An equation of the form  $ax + by + cz = d$  ( $a, b, c, d$  are constants) defines a plane.

**Example 1.** Find the equation of the plane through the origin with a normal vector  $\vec{N} = \langle 1, 5, 10 \rangle$ .

By thinking geometrically, a point  $P$  is in the plane  $\iff OP \perp \vec{N} \iff \vec{OP} \cdot \vec{N} = 0$ . Therefore, the equation of the described plane is

$$\langle x, y, z \rangle \cdot \langle 1, 5, 10 \rangle = 0$$

$$x + 5y + 10z = 0$$

**Example 2.** Find the equation of the plane through the point  $P_0 (2, 1, -1)$  with a normal vector  $\vec{N} = \langle 1, 5, 10 \rangle$ .

Similarly by thinking geometrically, a point  $P (x, y, z)$  is in the plane  $\iff \vec{P_0P} \perp \vec{N} \iff \vec{P_0P} \cdot \vec{N} = 0$ . Therefore, the equation of the described plane is

$$\langle x - 2, y - 1, z + 1 \rangle \cdot \langle 1, 5, 10 \rangle = 0$$

$$(x - 2) + 5(y - 1) + 10(z + 1) = 0$$

$$x + 5y + 10z = -3$$

From the above example we can see that the coefficients of the equation of a plane are actually the components of one of its normal vectors.

The right-hand side constant in the equation of a plane is an indicator of the distance to its parallel plane through the origin. For example, we can derive that  $x + 5y + 10z = 3$  and  $x + 5y + 10z = -3$  are in the two sides of the plane  $x + 5y + 10z = 0$  respectively, and  $x + 5y + 10z = -1$  has a short distance to  $x + 5y + 10z = 0$  than  $x + 5y + 10z = -3$ .

**Example 3.** The vector  $\vec{v} = \langle 1, 2, -1 \rangle$  and the plane  $x + y + 3z = 5$  are A.

A. parallel

B. perpendicular

C. neither

*Reasons:*

*A normal vector of the plane  $x + y + 3z = 5$  is  $\vec{N} = \langle 1, 1, 3 \rangle$ . Then*

$$\begin{aligned}\vec{v} \cdot \vec{N} &= \langle 1, 2, -1 \rangle \cdot \langle 1, 1, 3 \rangle \\ &= 1 + 2 - 3 \\ &= 0\end{aligned}$$

*Hence  $\vec{v} \perp \vec{N}$ , from which we can derive that  $\vec{v}$  is perpendicular to the plane.*

## 2 Geometric Interpretation of Linear Systems

### 2.1 Linear systems in geometry

A linear system describes the intersection of some objects or point sets in geometry.

Take a  $3 \times 3$  linear system as an example:

$$\begin{cases} x + z = 1 \\ x + y = 2 \\ x + 2y + 3z = 3 \end{cases}$$

The solution to this linear system is the set of points  $(x, y, z)$  satisfying all three linear equations, where each of them defines a plane. Therefore, the points in the solution set are in all of three planes. Hence, the solution to the linear system describes the intersection of these three planes in geometry.

To find the solution to a linear system, algebraic methods are easier than geometric methods. Recall that a linear system can be expressed with matrix products as:

$$AX = B$$

$$X = A^{-1}B$$

$A^{-1}B$  is the unique solution to the linear system, which is the intersection point in geometry.

### 2.2 Exceptions of unique solution in geometry

However, there are exceptions to this algebraic method.

**Example 4.** *If the solution set to a  $3 \times 3$  linear system is not a single point, it could be A C E.*

*A. no solution*

*B. two points*

*C. a line*

*D. a tetrahedron*

*E. a plane*

*F. I don't know*

*Reasons: For A, the situation would be at least two of the three planes are parallel to each other and not the same plane.*

*For C, the situation would be the intersection of two planes, which is a line, is contained in the third plane.*

*For E, the situation would be that the three planes are the same.*

The exceptions to a single solution to a linear system are described in the above example.

### 2.3 Exceptions of unique solutions in algebra

Recall that the formula of the unique solution to a linear system is

$$X = A^{-1}B$$

It turns out this formula doesn't always hold. In those exceptions,  $A^{-1}$  doesn't exist, i.e.  $A$  is not invertible.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Since  $\operatorname{adj}(A)$  always exists,  $A^{-1}$  exists  $\iff A$  is invertible  $\iff \det(A) \neq 0$ .