Lecture 12: Gradient

1 Gradient

1.1 Introduce Gradient

According to the chain rule, suppose that there is a function $w = w(x, y, z)$, where $x = x(t)$, $y = y(t)$, and $z = z(t)$, then

$$\begin{aligned}\frac{dw}{dt} &= w_x\frac{dx}{dt} + w_y\frac{dy}{dt} + w_z\frac{dz}{dt} \\ &= < w_x, w_y, w_z > \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > \\ &= \nabla w \cdot \frac{d\vec{r}}{dt} \\ \nabla w &= < w_x, w_y, w_z > \\ \frac{d\vec{r}}{dt} &= < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >\end{aligned}$$

The vector $< w_x, w_y, w_z >$ is called gradient, denoted as ∇w. Gradients on different positions can have different directions and magnitude.

1.2 A Property of Gradient

Theorem. *The gradient of a function is always perpendicular to the level surfaces of the function, where a level surface means the set of points on the function domain whose function values are equal to a constant.*

Notice that the level surfaces refer to points in the function domain, rather than on the function graph. It doesn't include the dimension of the function value.

Example 1. *Examine the relation between the gradient and the level surfaces for the function $w = a_1x + a_2y + a_3z$.*

$$\frac{\partial w}{\partial x} = a_1$$
$$\frac{\partial w}{\partial y} = a_2$$
$$\frac{\partial w}{\partial z} = a_3$$
$$\nabla w = < a_1, a_2, a_3 >$$

The level surfaces of the function $w = a_1x + a_2y + a_3z = c$, where c is any constant, are a series of planes parallel to each other. For any level surfaces of the function, the gradient ∇w is a normal vector to it. Therefore, the gradient is always perpendicular to the level surfaces for the function $w = a_1x + a_2y + a_3z$.

Example 2. *Examine the relation between the gradient and the level surfaces for the function $w = x^2 + y^2$.*

$$\frac{\partial w}{\partial x} = 2x$$
$$\frac{\partial w}{\partial y} = 2y$$
$$\nabla w = < 2x, 2y > = < x, y >$$

The level surfaces of the function $w = x^2 + y^2 = c$, where c is any constant, are a series of circles whose centers are the origin. At any point (x_0, y_0) in the function domain, the gradient vector is $< x_0, y_0 >$, which has the same direction as the radius through (x_0, y_0) on the level surface $x^2 + y^2 = x_0^2 + y_0^2$. A circle's radius is always perpendicular to the circle, hence the gradient vector is perpendicular to the level surface. Therefore, the gradient is always perpendicular to the level surfaces for the function $w = x^2 + y^2$.

Proof of the theorem:
Suppose there is a curve $\vec{r} = \vec{r}(t)$ that always stays on a level surface of a function f. The velocity vector $\frac{d\vec{r}}{dt}$ is tangent to the curve, and therefore tangent to the level surface on which the curve stays.
By chain rule, on any level surfaces it satisfies the following equation:

$$\frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

Therefore, at any points of a curve that stays on a level surface:

$$\nabla f \perp \frac{d\vec{r}}{dt}$$

The same reasoning applies to any curves on a level surface, so at any points on a level surface, the gradient ∇f is perpendicular to velocity vectors of every directiona, which are tangent to the level surface. Therefore, at any points on a level surface, the gradient ∇f is perpendicular to the tangent plane to the level surface.

1.3 Application of Gradient

We can use gradients to find the equation of the tangent line of a function graph at any points.

Example 3. *Find the equation of the tangent plane to the surface $x^2+y^2-z^2=4$ at the point (2, 1, 1).*
Solution:
The surface $x^2+y^2-z^2=4$ is a level surface of the function $w=x^2+y^2-z^2$.

$$\frac{\partial w}{\partial x}=2x$$
$$\frac{\partial w}{\partial y}=2y$$
$$\frac{\partial w}{\partial z}=-2z$$
$$\nabla w=<2x,2y,-2z>=<x,y,-z>$$

Hence the gradient of the function at the point (2, 1, 1) is $<2,1,-1>$. According to the property of gradients, the gradient is perpendicular to the tangent plane of the corresponding level surface at the point. Therefore, the equation of the tangent plane is

$$2x+y-z=c, \text{where c is a constant}$$

Then we can substitute the coordinate of the point (2, 1, 1) into the plane equation:

$$c=2\times 2+1\times 1-1\times 1=4$$

Therefore, the equation of the tangent plane to the surface $x^2+y^2-z^2=4$ at the point (2, 1, 1) is

$$2x+y-z=4$$

2 Directional Derivatives

How to calculate the partial derivative towards any direction $\vec{u}$, rather than only along the x axis and y axis?
Suppose there is a multivariable function $f(x,y)$, and an unit vector $\hat{\vec{u}}=<a,b>$. The partial derivative along a direction is the rate of change of the function value

$f(x, y)$ over the arclength, hence we need the differential of the arclength along the direction of $\hat{\vec{u}}$, which is denoted by ds.
By chain rule, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$