Lecture 13: Lagrange Multipliers

1 Motivation of Lagrange Multipliers

The usage of Lagrange multipliers is to maximize/minimize the value of a function $f(x, y, z)$ where x, y, and z are not independent, in other words there is a constraint $g(x, y, z) = c$.
Real world example: In thermodynamics, we often deal with a system with parameters such as temperature T, pressure P, and volume V. These parameters are not independent, and they satisfy a relation $PV = nRT$.
Those kinds of problems cannot be solved by only checking the critical points of the function $f(x, y, z)$, because they probably don't satisfy the existing constraint $g(x, y, z) = c$.

Example 1. *Find the point closest to the origin on the hyperbola $xy = 3$.*
We need to minimize the function $f(x, y) = \sqrt{x^2 + y^2}$, or more conveniently, the function $f(x, y) = x^2 + y^2$, with the constraint $g(x, y) = xy = 3$.
From geometric perspective, we can plot the function graph of $g(x, y) = xy = 3$, and the contour plot of the function $f(x, y) = x^2 + y^2$. We can see that with a large constant c_1, the graphs of $g(x, y) = 3$ and $f(x, y) = c_1$ have four intersection points; with a small constant c_2, the two graphs have no intersection point. With the correct solution c_0, the two graphs have exactly two intersection points.

2 Solution with Lanrange Multipliers

One key observation to the above example: At the minimum, the level curve of the function $f(x, y)$ is tangent to the hyperbola $xy = 3$, which is another level curve of the function $g(x, y)$.
Then how to find the point (x, y) where the level curves of $f(x, y)$ and $g(x, y)$ are tangent to each other?
Notice: The level curves of $f(x, y)$ and $g(x, y)$ are tangent to each other
$\iff$ The level curves of $f(x, y)$ and $g(x, y)$ have the same tangent line
$\iff$ The gradients of $f(x, y)$ and $g(x, y)$ are parallel to each other
$\iff$ $\nabla f = \lambda \nabla g$, where $\lambda \neq 0$.
Therefore, we can derive a system of equations from the above statement:

From $\nabla f = \lambda \nabla g$, we can derive that

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

Also we have the constraint $g(x, y) = c$.
Therefore, for a optimization problem of a function with two variables, the derived system of equations is

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = c \end{cases}$$

It is sufficient to solve the point (x, y) as well as the factor λ. This factor λ is called the Lagrange multiplier.