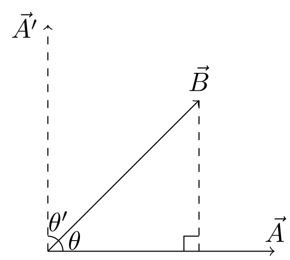
Lecture 2: Determinants, Cross Product

1 Determinants

1.1 Determinants in 2D plane

With two vectors, how do we calculate the area of the triangle enclosed by these two vectors?

1.1.1 Calculate the area of the triangle formed by two vectors with an acute angle



Apparently

$$Area = \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$$

But it is hard to calculate $\sin\theta$, whereas it is easy to calculate $\cos\theta$ with dot products. Therefore, we can rotate $\vec{A}$ towards $\vec{B}$ by 90° to get $\vec{A}'$ and hence

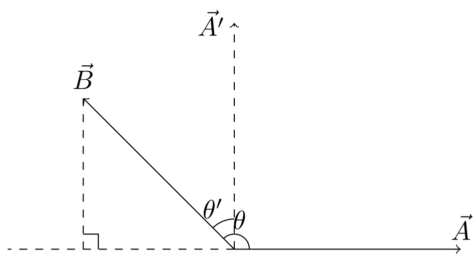
$$\sin\theta = \cos(90^\circ - \theta) = \cos(\theta')$$
$$\begin{aligned} Area &= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta \\ &= \frac{1}{2}|\vec{A}'||\vec{B}|\cos\theta' \\ &= \frac{1}{2}\vec{A}' \cdot \vec{B} \end{aligned}$$

Suppose that $\vec{A} =< a_1, a_2 >$, and $\vec{B} =< b_1, b_2 >$, then $\vec{A}' =< -a_2, a_1 >$. Therefore,

$$
\begin{aligned}
\vec{A'} \cdot \vec{B} &=< -a_2, a_1 > \cdot < b_1, b_2 > \\
&= a_1 b_2 - a_2 b_1 \\
&= det(\vec{A}, \vec{B}) \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}
$$

Notice that it is also possible that $\vec{A'} =< a_2, -a_1 >$, depending on the relative positions of $\vec{A}$ and $\vec{B}$, so the result could be the negative value of the area.

1.1.2 Calculate the area of the triangle formed by two vectors with an obtuse angle



$$
\begin{aligned}
Area &= \frac{1}{2}|\vec{A}||\vec{B}|\sin(\pi - \theta) \\
&= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta \\
&= \frac{1}{2}|\vec{A}||\vec{B}|\sin(\pi + \theta') \\
&= \frac{1}{2}|\vec{A'}||\vec{B}|\cos\theta' \\
&= \frac{1}{2}\vec{A'} \cdot \vec{B}
\end{aligned}
$$

Similarly, suppose that $\vec{A} =< a_1, a_2 >$, and $\vec{B} =< b_1, b_2 >$, then $\vec{A'} =< -a_2, a_1 >$. Therefore,

$$
\begin{aligned}
\vec{A'} \cdot \vec{B} &=< -a_2, a_1 > \cdot < b_1, b_2 > \\
&= a_1 b_2 - a_2 b_1 \\
&= det(\vec{A}, \vec{B}) \\
&= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}
\end{aligned}
$$

1.1.3 Geometric interpretation of determinants in 2D plane

$det(\vec{A}, \vec{B})$ measures the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$, but the value can be positive or negative of it.

1.2 Determinants in 3D space

Suppose that $\vec{A} =< a_1, a_2, a_3 >$, $\vec{B} =< b_1, b_2, b_3 >$, and $\vec{C} =< c_1, c_2, c_3 >$, then

$$\begin{aligned} det(\vec{A}, \vec{B}, \vec{C}) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \end{aligned}$$

We can expand the 3×3 determinants to expressions of 2×2 determinants according to the first row to calculate their values. The rules are as follows:

1.2.1 Geometric interpretation of determinants in 3D space

Theorem. *Geometrically,* $det(\vec{A}, \vec{B}, \vec{C}) = \pm$ *volume of the parallelepiped formed by* $\vec{A}$, $\vec{B}$, *and* $\vec{C}$.

2 Cross Product

Definition. *The cross product of two vectors* $\vec{A} =< a_1, a_2, a_3 >$ *and* $\vec{B} =< b_1, b_2, b_3 >$ *is a vector defined as:*

$$\begin{aligned} \vec{A} \times \vec{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \end{aligned}$$

Notice that the 3×3 determinants in the above definition is not a legit determinant expression. It is just a symbolic notation helping memorize the formula.

Theorem. $|\vec{A} \times \vec{B}| =$ *the area of the parallelogram formed by* $\vec{A}$ *and* $\vec{B}$ *in the 3D space.*

Theorem. *The direction of* $\vec{A} \times \vec{B}$, *or* $dir(\vec{A} \times \vec{B})$ *is perpendicular to the parallelogram formed by* $\vec{A}$ *and* $\vec{B}$ *in the 3D space.*

There are two directions which is perpendicular to the parallelogram formed by $\vec{A}$ and $\vec{B}$, which one is the direction of $\vec{A} \times \vec{B}$?
Right-hand rule: Use your right hand to rotate from $\vec{A}$ to $\vec{B}$ then your right thumb is pointing to the direction of $\vec{A} \times \vec{B}$.

Example 1.

2.1 Properties of cross product

2.1.1 Commutative property

Cross product does not satisfy the commutative property.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

In particular:

$$\vec{A} \times \vec{A} = \vec{0}$$

2.1.2 Associative property

Cross product does not satisfy the associative property.
Counter example: Suppose $\vec{A} =< 1,0,1 >$, $\vec{B} =< 1,1,0 >$, $\vec{C} =< 0,1,1 >$, then we can get

$$(\vec{A} \times \vec{B}) \times \vec{C} =< 0,1,-1 >$$
$$\vec{A} \times (\vec{B} \times \vec{C}) =< 1,0,-1 >$$

2.2 Application of cross product

Example 2. *Given the coordinates of three points that are not in the same line in 3D space, find the equation of the plane determined by these three points.*
Solution:
Finding the equation of a plane is equivalent to finding a method of determining whether a random point is in the plane or not.
Suppose the three known points are P_1, P_2, and P_3, and a random point is P.
Method 1: P is in the plane if and only if the parallelepiped formed by $\vec{P_1P_2}$, $\vec{P_1P_3}$, and $\vec{P_1P}$ has a volume 0. Therefore, according to the geometric interpretation of determinants in 3D space, we can derive that P is in the plane if and only if

$$det(\vec{P_1P_2}, \vec{P_1P_3}, \vec{P_1P}) = 0$$

Method 2: Using the normal vector. The normal vector of a plane refers to the vector that are perpendicular to the plane. A plane can have infinite normal vectors. We usually use $\vec{N}$ to represent a normal vector.
The result of a cross product of two vectos is a normal vector of the parallelogram formed by the two vectos. With that being said, P is in the plane if and only if $\vec{P_1P} \perp \vec{N}$, which is equivalent to

$$\vec{P_1P} \cdot \vec{N} = 0$$
$$\vec{P_1P} \cdot (\vec{P_1P_2} \times \vec{P_1P_3}) = 0$$

Therefore, the two equations are actually equivalent:

$$det(\vec{P_1P_2}, \vec{P_1P_3}, \vec{P_1P}) = \vec{P_1P} \cdot (\vec{P_1P_2} \times \vec{P_1P_3}) = 0$$