Lecture 16: Double Integrals

1 Definition of Double Integrals

For single-variable functions, we use integral to calculate the area under the function curve. For a multivariable function with two independent variables $z = f(x, y)$, we define double integrals to calculate the volume under the function graph over a region R in the xy-plane. The notation is $\iint_R f(x, y)dA$.

Definition of double integrals:
If we cut the region R into small pieces of area ΔA_i, then the volume can be calculated as

$$V \approx \sum_i f(x_i, y_i)\Delta A_i$$

If we take the limit of ΔA_i to 0, then it becomes the double integral

$$\iint_R f(x, y)dA = \lim_{\Delta A_i \to 0} \sum_i f(x_i, y_i)\Delta A_i$$

2 Calculation of Double Integrals

To compute a double integral $\iint_R f(x, y)dA$, we can take slices of the function graph along one dimension and use integral to get the area of slices. In this way, the area of all slices becomes a function of another dimension, then we can use the integral again to sum up those slices to get the volume.
More concretely, let $S(x)$ denote the area of slices by plane parallel to yz-plane. Then

$$\iint_R f(x, y)dA = \int_{x_min}^{x_max} S(x)dx$$

For a given x_0, the area of the slice

$$S(x_0) = \int_{y_min(x_0)}^{y_max(x_0)} f(x, y)dy$$

Therefore, the double integral

$$\begin{aligned} \iint_R f(x, y)dA &= \int_{x_min}^{x_max} S(x)dx \\ &= \int_{x_min}^{x_max} \int_{y_min(x_0)}^{y_max(x_0)} f(x, y)dydx \end{aligned}$$

This calculation method is called **iterated integral**, because we iterate twice through both dimensions to get the integral.
Notice that the bounds of the outer integral are numbers, and the bounds of the inner integral are numbers or functions depending on the variable of the outer integral. The bounds of both outer and inner integrals are determined by the region of the double integral.

3 Examples of Double Integral

Example 1. *Find the value of the double integral of the function* $z = 1-x^2-y^2$ *over the region* R *as* $0 \le x \le 1$, $0 \le y \le 1$.
Solution:
By using iterated integrals,

$$\iint_R f(x,y)dA = \int_0^1 \int_0^1 1 - x^2 - y^2 dydx$$

For the inner integral,

$$\begin{aligned}\int_0^1 1 - x^2 - y^2 dy &= (y - x^2 y - \frac{y^3}{3})|_0^1 \\ &= 1 - x^2 - \frac{1}{3} \\ &= \frac{2}{3} - x^2\end{aligned}$$

For the outer integral,

$$\begin{aligned}\int_0^1 \frac{2}{3} - x^2 dx &= (\frac{2x}{3} - \frac{x^3}{3})|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3}\end{aligned}$$

Therefore, the value of the double integral is $\frac{1}{3}$.

Example 2. *Find the value of the double integral of the function* $z = 1-x^2-y^2$ *over the region* R *as*

$$\begin{cases} x^2 + y^2 \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

By the equation of the region R*, we can get that for any given* x_0*, the range of* y *in the region* R *is* $[0, \sqrt{1-x^2}]$.
By using iterated integrals,

$$\iint_R f(x,y)dA = \int_0^1 \int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2 dydx$$

For the inner integral,

$$\begin{aligned}\int_0^{\sqrt{1-x^2}} 1-x^2-y^2dy &= (y-x^2y-\frac{y^3}{3})|_0^{\sqrt{1-x^2}}\\ &= \sqrt{1-x^2}-x^2\sqrt{1-x^2}-\frac{(1-x^2)^{\frac{3}{2}}}{3}\\ &= \frac{2}{3}(1-x^2)^{\frac{3}{2}}\end{aligned}$$

For the outer integral,

$$\begin{aligned}\int_0^1 \frac{2}{3}(1-x^2)^{\frac{3}{2}}dx &= \int_0^{\frac{\pi}{2}} \frac{2}{3}(1-\sin^2\theta)^{\frac{3}{2}}d(\sin\theta)\\ &= \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta\\ &= \frac{2}{3}\int_0^{\frac{\pi}{2}} (\frac{1+\cos 2\theta}{2})^2 d\theta\\ &= \frac{1}{6}\int_0^{\frac{\pi}{2}} (1+2\cos 2\theta+\cos^2 2\theta)d\theta\\ &= \frac{1}{6}\int_0^{\frac{\pi}{2}} (1+2\cos 2\theta+\frac{1+\cos 4\theta}{2})d\theta\\ &= \frac{1}{12}\int_0^{\frac{\pi}{2}} (3+4\cos 2\theta+\cos 4\theta)d\theta\\ &= \frac{1}{12}(3\theta+2\sin 2\theta+\frac{1}{4}\cos 4\theta)|_0^{\frac{\pi}{2}}\\ &= \frac{1}{12}\cdot\frac{3\pi}{2}\\ &= \frac{\pi}{8}\end{aligned}$$

Therefore, the value of the double integral is $\frac{\pi}{8}$.

4 Exchange the Order of Iterated Integrals

In theory, no matter in which order we calculate the iterated integral, the values always exist and are the same. However, in practice an order requires much more complex computation than the other; sometimes we might not even be able to compute the double integral with a specific order, but able to do it with the other order. Therefore, we need to be mindful about the order of iterated integral we use.
In some cases, we can exchange the order of an iterated integral, with some adaption to the bounds of integrals.

Example 3.

$$\int_0^1\int_0^2 dxdy = \int_0^2\int_0^1 dydx$$

For rectangle regions, the bounds of the inner integral don't depend on the variable of the outer integral, so we can exchange the inner integral and outer integral freely.
From the perspective of the definition of iterated integrals, for rectangle regions, slices of either direction don't change the set of small pieces of area dA_i to sum up.

Example 4. *Find the value of $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx$.*
It is too difficult to calculate $\int_x^{\sqrt{x}} \frac{e^y}{y} dy$, so we need to consider exchanging the order of integrals.
By looking at the region, we derive that

$$\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx = \int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dxdy$$

and the one on the right-hand side is doable.
For the inner integral,

$$\begin{aligned} \int_{y^2}^{y} \frac{e^y}{y} dx &= \frac{xe^y}{y} \Big|_{y^2}^{y} \\ &= e^y - ye^y \end{aligned}$$

For the outer integral,

$$\begin{aligned} \int_0^1 e^y - ye^y dy &= (2e^y - ye^y)\Big|_0^1 \\ &= 2e - e - 2 \\ &= e - 2 \end{aligned}$$

Therefore, the value of $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y} dydx$ is $e - 2$.