18.02 EXERCISES

Problem Set 3: Parametric Equations for Curves

Part I

Unit 1E Equations of Lines and Planes

4. Where does the line through $(0, 1, 2)$ and $(2, 0, 3)$ intersect the plane $x+4y+z=4$?
Solution:
A vector along the line is $< 2, -1, 1 >$. Hence, the parametric equation of the line is

$$\begin{cases} x = x(t) = 0 + 2t = 2t \\ y = y(t) = 1 - t \\ z = z(t) = 2 + t \end{cases}$$

For the intersection point of the line and the plane, it satisfies

$$\begin{aligned} x(t) + 4y(t) + z(t) &= 4 \\ 2t + 4(1 - t) + 2 + t &= 4 \\ -t &= -2 \\ t &= 2 \end{aligned}$$

Therefore, the coordinates of the intersection point is $(x(2), y(2), z(2))$, which is $(4, -1, 4)$.

7. Formulate a general method for finding the distance between two skew (i.e., non-intersecting) lines in space, and carry it out for two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).
Solution:
7-1. Formulate a general method for finding the distance between two parallel planes in space:
Suppose that the equations of two parallel planes are respectively

$$\begin{aligned} n_1x + n_2y + n_3z &= c_1 \\ n_1x + n_2y + n_3z &= c_2 \end{aligned}$$

Then the vector $\vec{n} =< n_1, n_2, n_3 >$ is one of the normal vectors of the two planes. Therefore, we can construct the parametric equation of a line perpendicular to the two planes as

$$\begin{cases} x(t) = n_1 t \\ y(t) = n_2 t \\ z(t) = n_3 t \end{cases}$$

Therefore, the two intersection points the line has with the two planes respectively satisfy

$$\begin{aligned} n_1 x(t_1) + n_2 y(t_1) + n_3 z(t_1) &= c_1 \\ n_1^2 t_1 + n_2^2 t_1 + n_3^2 t_1 &= c_1 \\ t_1 &= \frac{c_1}{n_1^2 + n_2^2 + n_3^2} \\ n_1 x(t_2) + n_2 y(t_2) + n_3 z(t_2) &= c_2 \\ n_1^2 t_2 + n_2^2 t_2 + n_3^2 t_2 &= c_1 \\ t_2 &= \frac{c_2}{n_1^2 + n_2^2 + n_3^2} \end{aligned}$$

The distance between two points in the line can be calculated as

$$\begin{aligned} d &= \sqrt{(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 + (z(t_1) - z(t_2))^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(t_1 - t_2)^2} \\ &= \sqrt{(n_1^2 + n_2^2 + n_3^2)(\frac{c_1 - c_2}{n_1^2 + n_2^2 + n_3^2})^2} \\ &= \frac{|c_1 - c_2|}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \\ &= \frac{|c_1 - c_2|}{|\vec{n}|} \end{aligned}$$

7-2. Formulate a general method for finding the distance between two non-parallel lines in space:
Suppose that there are two non-parallel lines in space named l_a and l_b. Their normal vectors are respectively $\vec{n_a} = \langle n_{ax}, n_{ay}, n_{az} \rangle$ and $\vec{n_b} = \langle n_{bx}, n_{by}, n_{bz} \rangle$,

and their parametric equations are respectively:

$$\begin{cases} x_a(t) = x_{a0} + n_{ax}t \\ y_a(t) = y_{a0} + n_{ay}t \\ z_a(t) = z_{a0} + n_{az}t \end{cases}$$

$$\begin{cases} x_b(t) = x_{b0} + n_{bx}t \\ y_b(t) = y_{b0} + n_{by}t \\ z_b(t) = z_{b0} + n_{bz}t \end{cases}$$

Then we can find the vector $\vec{n}$ which is perpendicular to both l_a and l_b by

$$\begin{aligned} \vec{n} &= \vec{n_a} \times \vec{n_b} \\ &= \langle n_x, n_y, n_z \rangle \end{aligned}$$

Then the distance between l_a and l_b is equal to the distance between the two planes P_a and P_b with $\vec{n}$ as their normal vectors where l_a and l_b lie within respectively.

The equation of P_a is

$$n_x x + n_y y + n_z z = n_x x_{a0} + n_y y_{a0} + n_z z_{a0}$$

The equation of P_b is

$$n_x x + n_y y + n_z z = n_x x_{b0} + n_y y_{b0} + n_z z_{b0}$$

According to the previous part, the distance between P_a and P_b is

$$\begin{aligned} d &= \frac{|(n_x x_{a0} + n_y y_{a0} + n_z z_{a0}) - (n_x x_{b0} + n_y y_{b0} + n_z z_{b0})|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot \langle (x_{a0} - x_{b0}), (y_{a0} - y_{b0}), (z_{a0} - z_{b0}) \rangle|}{|\vec{n}|} \\ &= \frac{|\vec{n} \cdot A_0\vec{B}_0|}{|\vec{n}|} \\ &= \frac{|(\vec{n_a} \times \vec{n_b}) \cdot A_0\vec{B}_0|}{|\vec{n_a} \times \vec{n_b}|} \end{aligned}$$

Therefore, the distance between l_a and l_b is

$$d = \frac{|(\vec{n_a} \times \vec{n_b}) \cdot A_0\vec{B}_0|}{|\vec{n_a} \times \vec{n_b}|}$$

7-3 Apply the above method to find the distance between two non-intersecting lines lying along the diagonals of two adjacent faces of the unit cube (place it in the first octant, with one vertex at the origin).

The two non-intersecting diagonals of two adjacent faces of the unit cube I choose are the diagonal between $(0,0,0)$ and $(1,0,1)$, and the diagonal between $(0,1,0)$ and $(0,0,1)$. Therefore, the parametric equations of these two lines along the described diagonals are respectively:

$$\begin{cases} x_a(t) = t \\ y_a(t) = 0 \\ z_a(t) = t \end{cases}$$

$$\begin{cases} x_b(t) = 0 \\ y_b(t) = 1 - t \\ z_b(t) = t \end{cases}$$

Therefore,

$$\begin{aligned} \vec{n_a} &= \langle 1,0,1 \rangle \\ \vec{n_b} &= \langle 0,-1,1 \rangle \\ A_0 &= (0,0,0) \\ B_0 &= (0,1,0) \\ \vec{A_0B_0} &= \langle 0,1,0 \rangle \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals is

$$\begin{aligned} d &= \frac{|(\vec{n_a} \times \vec{n_b}) \cdot \vec{A_0B_0}|}{|\vec{n_a} \times \vec{n_b}|} \\ &= \frac{|\langle 1,-1,-1 \rangle \cdot \langle 0,1,0 \rangle|}{|\langle 1,-1,-1 \rangle|} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Therefore, the distance between the two non-intersecting diagonals of two adjacent faces of the unit cube is $\frac{\sqrt{3}}{3}$.

Unit 1I Vector Functions and Parametric Equations

1. The point P moves with constant speed v in the direction of the constant vector $a\vec{i} + b\vec{j}$. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $\vec{r}(t)$?
Solution:
The direction of the constant vector $a\vec{i} + b\vec{j}$ is

$$dir(A) = \frac{a}{\sqrt{a^2+b^2}}\vec{i} + \frac{b}{\sqrt{a^2+b^2}}\vec{j}$$

Hence in a period of time t, the point P moves in a distance of vt in the direction $\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \rangle$. Hence along the x-axis it moves in a distance of $\frac{a}{\sqrt{a^2+b^2}}vt$, and along the y-axis it moves in a distance of $\frac{b}{\sqrt{a^2+b^2}}vt$.
Therefore, the position vector function of the point P is

$$\vec{r}(t) = \langle x_0 + \frac{a}{\sqrt{a^2+b^2}}vt, y_0 + \frac{b}{\sqrt{a^2+b^2}}vt \rangle$$

3. Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which P travels, and tell what part of the curve is actually traced out by P.
a) $\vec{r} = 2\cos^2 t\vec{i} + \sin^2 t\vec{j}$
b) $\vec{r} = \cos 2t\vec{i} + \cos t\vec{j}$
c) $\vec{r} = (t^2+1)\vec{i} + t^3\vec{j}$
d) $\vec{r} = \tan t\vec{i} + \sec t\vec{j}$
Solution:
a) According to the position vector function,

$$x(t) = 2\cos^2 t$$
$$y(t) = \sin^2 t$$

Therefore, we can find the relation between x and y:

$$\frac{x}{2} + y = 1$$
$$y = -\frac{1}{2}x + 1$$

which is a line in the xy-plane.
Since the ranges of x and y are respectively

$$x \in [0, 2]$$
$$y \in [0, 1]$$

the part of the curve that is actually traced out by P is the line segment from $(0,0)$ to $(2,1)$.
b) According to the position vector function,

$$x(t) = \cos 2t$$
$$y(t) = \cos t$$

Therefore, we can find the relation between x and y:

$$\cos 2t = 2\cos^2 t - 1$$
$$x = 2y^2 - 1$$

which is a parabola in the xy-plane.
Since the range of y is $y \in [-1, 1]$, and based on the characteristic of the parabola $x = 2y^2 - 1$, the part of the curve that is actually traced out by P is the part of the parabola between $(-1, 1)$ and $(1, 1)$.
c) According to the position vector function,

$$x(t) = t^2 + 1$$
$$y(t) = t^3$$

Therefore, we can find the relation between x and y:

$$y = (x-1)^{\frac{3}{2}}$$

which is a symmetric graph related to the x axis, in which the half above the x axis is the graph of a power function.
Since the range of x is $x \in [1, \infty)$, which is the domain of the function $y = (x-1)^{\frac{3}{2}}$, the full curve is actually traced out by P.
d) According to the position vector function,

$$x(t) = \tan t$$
$$y(t) = \sec t$$

Therefore, we can find the relation between x and y:

$$\sec^2 t - \tan^2 t = 1$$
$$y^2 - x^2 = 1$$

which is a hyperbola.
According to the graph of $x(t)$ and $y(t)$, the full hyperbola is actually traced out by P.

Question 1. *We know that the relation we find between x and y is correct, however, is it possible that x and y also satisfy other relation which is not equivalent to the one we find?*

5. A string is wound clockwise around the circle of radius a centered at the origin O; the initial position of the end P of the string is $(a, 0)$. Unwind the

string, always pulling it taut (so it stays tangent to the circle). Write parametric equations for the motion of P.
Solution:
Since the string is wound clockwise, when we unwind it, the direction is counterclockwise. Also, since the string is unwound around a circle, it is natural to use the parameter θ to describe the motion of P, where θ is the corresponding angle of the unwound part of the string. Hence the length of the unwound part of the string is $a\theta$.
During the unwinding, the string is always pulled taut, hence there is always a point P' on the circle which is the point of tangency between the circle and the unwound part of the string. The coordinates of P' can be described as $(a\cos\theta, a\sin\theta)$.
Next we need to explore the direction of the vector $\vec{P'P}$ in order to describe the coordinates of P. According to the tangency relationship between the circle and the unwound part of the string,

$$\vec{P'P} \perp \vec{OP'}$$
$$dir(\vec{OP'}) = \langle\cos\theta, \sin\theta\rangle$$

Therefore, we have the following system of equations

$$\begin{cases} dir(\vec{P'P}) \cdot \langle\cos\theta, \sin\theta\rangle = 0 \\ |dir(\vec{P'P})| = 1 \end{cases}$$

After solving the above system of equations, there are two possible solutions:

$$dir(\vec{P'P}) = \langle\sin\theta, -\cos\theta\rangle$$
$$dir(\vec{P'P}) = \langle-\sin\theta, \cos\theta\rangle$$

According to the geometric interpretation of unwinding the string counterclockwise, the correct direction of the vector $\vec{P'P}$ is $\langle\sin\theta, -\cos\theta\rangle$.
Therefore, the parametric equation for the motion of P is

$$\begin{cases} x(t) = a\cos\theta + a\theta\sin\theta \\ y(t) = a\sin\theta - a\theta\cos\theta \end{cases}$$

7. The cycloid is the curve traced out by a fix point P on a circle of radius a which rolls along the x-axis in the positive direction, starting when P is at the origin O. Find the vector function OP; use as variable the angle θ through which the circle has rolled.
Solution:
Let C denote the center of the rolling circle. At a given value of θ, suppose that the tangency point between the circle and the x axis is A.

Since the circle is rolling along the x axis without any slip, then

$$OA = a\theta$$

Also $AC = a$, hence the coordinates of C at the given value of θ is

$$C = (a\theta, a)$$

The direction of $\vec{CP}$ can be described by

$$dir(\vec{CP}) = \langle \cos(-\frac{\pi}{2} - \theta), \sin(-\frac{\pi}{2} - \theta) \rangle$$
$$dir(\vec{CP}) = \langle -\sin\theta, -\cos\theta \rangle$$

Therefore, the parametric equation for the motion of P is

$$\begin{cases} x(t) = a\theta - a\sin\theta \\ y(t) = a - a\cos\theta \end{cases}$$

Unit 1J Differentiation of Vector Functions

1. For each of the following vector functions of time, calculate the velocity, speed $|ds/dt|$, unit tangent vector (in the direction of velocity), and acceleration.
a) $e^t\vec{i} + e^{-t}\vec{j}$
b) $t^2\vec{i} + t^3\vec{j}$
c) $(1-2t^2)\vec{i} + t^2\vec{j} + (-2+2t^2)\vec{k}$
Solution:

a)

$$
\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d(e^t\vec{i} + e^{-t}\vec{j})}{dt} \\
&= e^t\vec{i} - e^{-t}\vec{j} \\
|ds/dt| &= |\vec{v}| \\
&= |e^t\vec{i} - e^{-t}\vec{j}| \\
&= \sqrt{(e^t)^2 + (e^{-t})^2} \\
&= \frac{\sqrt{e^{4t}+1}}{e^t} \\
\vec{T} &= dir(\vec{v}) \\
&= \frac{\vec{v}}{|\vec{v}|} \\
&= \frac{e^{2t}}{\sqrt{e^{4t}+1}}\vec{i} + \frac{1}{\sqrt{e^{4t}+1}}\vec{j} \\
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(e^t\vec{i} - e^{-t}\vec{j})}{dt} \\
&= e^t\vec{i} + e^{-t}\vec{j}
\end{aligned}
$$

b)

$$
\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d(t^2\vec{i} + t^3\vec{j})}{dt} \\
&= 2t\vec{i} + 3t^2\vec{j}
\end{aligned}
$$

$$
\begin{aligned}
|ds/dt| &= |\vec{v}| \\
&= |2t\vec{i} + 3t^2\vec{j}| \\
&= \sqrt{(2t)^2 + (3t^2)^2} \\
&= \sqrt{9t^4 + 4t^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{T} &= dir(\vec{v}) \\
&= \frac{\vec{v}}{|\vec{v}|} \\
&= \frac{2t}{\sqrt{9t^4 + 4t^2}}\vec{i} + \frac{3t^2}{\sqrt{9t^4 + 4t^2}}\vec{j}
\end{aligned}
$$

$$
\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(2t\vec{i} + 3t^2\vec{j})}{dt} \\
&= 2\vec{i} + 6t\vec{j}
\end{aligned}
$$

c)

$$
\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d((1-2t^2)\vec{i} + t^2\vec{j} + (-2+2t^2)\vec{k})}{dt} \\
&= -4t\vec{i} + 2t\vec{j} + 4t\vec{k} \\
|ds/dt| &= |\vec{v}| \\
&= |-4t\vec{i} + 2t\vec{j} + 4t\vec{k}| \\
&= \sqrt{(-4t)^2 + (2t)^2 + (4t)^2} \\
&= 6|t| \\
\vec{T} &= dir(\vec{v}) \\
&= \frac{\vec{v}}{|\vec{v}|} \\
&= \begin{cases} -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}, \ t \geq 0 \\ \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}, \ t < 0 \end{cases} \\
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d(-4t\vec{i} + 2t\vec{j} + 4t\vec{k})}{dt} \\
&= -4\vec{i} + 2\vec{j} + 4\vec{k}
\end{aligned}
$$

2. Let $OP = \frac{1}{1+t^2}\vec{i} + \frac{t}{1+t^2}\vec{j}$ be the position vector for a motion.
a) Calculate $\vec{v}$, $|ds/dt|$, and $\vec{T}$.
b) At what point in the speed greatest? smallest?
c) Find the xy-equation of the curve along which the point P is moving, and describe it geometrically.
Solution:

a)

$$
\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d(\frac{1}{1+t^2}\vec{i} + \frac{t}{1+t^2}\vec{j})}{dt} \\
&= -\frac{2t}{(t^2+1)^2}\vec{i} + \frac{1-t^2}{(t^2+1)^2}\vec{j} \\
|ds/dt| &= |\vec{v}| \\
&= |-\frac{2t}{(t^2+1)^2}\vec{i} + \frac{1-t^2}{(t^2+1)^2}\vec{j}| \\
&= \sqrt{(-\frac{2t}{(t^2+1)^2})^2 + (\frac{1-t^2}{(t^2+1)^2})^2} \\
&= \frac{1}{t^2+1} \\
\vec{T} &= dir(\vec{v}) \\
&= \frac{\vec{v}}{|\vec{v}|} \\
&= -\frac{2t}{t^2+1}\vec{i} + \frac{1-t^2}{t^2+1}\vec{j}
\end{aligned}
$$

b) According to the expression of the speed $|ds/dt|$, it is obvious that when $t = 0$, the speed has its greatest value which is 1, and when $t = \infty$ or $t = -\infty$, the speed has its smallest value which is 0.
c) According to the position vector function,

$$
\begin{aligned}
x(t) &= \frac{1}{1+t^2} \\
y(t) &= \frac{t}{1+t^2}
\end{aligned}
$$

Therefore, we can find the relation between x and y as

$$
\begin{gathered}
(\frac{1}{1+t^2})^2 + (\frac{t}{1+t^2})^2 = \frac{1+t^2}{(1+t^2)^2} = \frac{1}{1+t^2} \\
x^2 + y^2 = x \\
x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \\
(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}
\end{gathered}
$$

which is a circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$.
According to the graph of x and y, the full circle is actually traced out by the motion of P.

3. Prove the rule for differentiating the scalar product of two plane vector functions:

$$\frac{d}{dt}\vec{r}\cdot\vec{s} = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}$$

by calculating with components, letting $\vec{r} = x_1\vec{i} + y_1\vec{j}$, and $\vec{s} = x_2\vec{i} + y_2\vec{j}$.
Solution:

$$\vec{r}\cdot\vec{s} = x_1x_2 + y_1y_2$$

$$\begin{aligned}
\frac{d}{dt}\vec{r}\cdot\vec{s} &= \frac{d(x_1x_2 + y_1y_2)}{dt} \\
&= \frac{d(x_1x_2)}{dt} + \frac{d(y_1y_2)}{dt} \\
&= \frac{dx_1}{dt}x_2 + x_1\frac{dx_2}{dt} + \frac{dy_1}{dt}y_2 + y_1\frac{dy_2}{dt} \\
&= (\frac{dx_1}{dt}x_2 + \frac{dy_1}{dt}y_2) + (x_1\frac{dx_2}{dt} + y_1\frac{dy_2}{dt}) \\
&= (\langle\frac{dx_1}{dt}, \frac{dy_1}{dt}\rangle\cdot\langle x_2, y_2\rangle) + (\langle x_1, y_1\rangle\cdot\langle\frac{dx_2}{dt}, \frac{dy_2}{dt}\rangle) \\
&= \frac{d\vec{r}}{dt}\cdot d\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}
\end{aligned}$$

Q.E.D.

4. Suppose a point P moves on the surface of a sphere with center at the origin; let

$$OP = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

Show that the velocity vector v is always perpendicular to $\vec{r}$ in two different ways:
a) using the x, y, z-coordinates
b) without coordinates (use the formula in **1J-3**, which is also valid in space).
Also Prove the converse: if $\vec{r}$ and $\vec{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin.
Solution:
To prove that the velocity vector v is always perpendicular to $\vec{r}$ for the described motion of P:

a)

$$
\begin{aligned}
\vec{v}(t) &= \frac{d(\vec{r}(t))}{dt} \\
&= \frac{d(x(t))}{dt}\vec{i} + \frac{d(y(t))}{dt}\vec{j} + \frac{d(z(t))}{dt}\vec{k} \\
\vec{r}\cdot\vec{v} &= (x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k})\cdot(\frac{d(x(t))}{dt}\vec{i} + \frac{d(y(t))}{dt}\vec{j} + \frac{d(z(t))}{dt}\vec{k}) \\
&= x(t)\frac{d(x(t))}{dt} + y(t)\frac{d(y(t))}{dt} + z(t)\frac{d(z(t))}{dt}
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{d((x(t))^2)}{dt} &= \frac{d(x(t))}{dt}x(t) + x(t)\frac{d(x(t))}{dt} \\
\frac{d((x(t))^2)}{dt} &= 2x(t)\frac{d(x(t))}{dt} \\
x(t)\frac{d(x(t))}{dt} &= \frac{1}{2}\frac{d((x(t))^2)}{dt}
\end{aligned}
$$

then

$$
\begin{aligned}
\vec{r}\cdot\vec{v} &= x(t)\frac{d(x(t))}{dt} + y(t)\frac{d(y(t))}{dt} + z(t)\frac{d(z(t))}{dt} \\
&= \frac{1}{2}\frac{d((x(t))^2)}{dt} + \frac{1}{2}\frac{d((y(t))^2)}{dt} + \frac{1}{2}\frac{d((z(t))^2)}{dt} \\
&= \frac{1}{2}\frac{d}{dt}((x(t))^2 + (y(t))^2 + (z(t))^2)
\end{aligned}
$$

Since the point P moves on the surface of a sphere with center at the origin, then

$$(x(t))^2 + (y(t))^2 + (z(t))^2 = R^2,\ \text{R is the radius of the sphere}$$

Therefore,

$$
\begin{aligned}
\vec{r}\cdot\vec{v} &= \frac{1}{2}\frac{d}{dt}((x(t))^2 + (y(t))^2 + (z(t))^2) \\
&= \frac{1}{2}\frac{d(R^2)}{dt} \\
&= 0
\end{aligned}
$$

Hence the velocity vector v is always perpendicular to $\vec{r}$. Q.E.D.

b)

$$\begin{aligned}
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= \frac{d\vec{r}}{dt}\cdot\vec{r}+\vec{r}\cdot\frac{d\vec{r}}{dt}\\
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= 2\frac{d\vec{r}}{dt}\cdot\vec{r}\\
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= 2\vec{v}\cdot\vec{r}\\
\vec{v}\cdot\vec{r} &= \frac{1}{2}\frac{d}{dt}\vec{r}\cdot\vec{r}\\
&= \frac{1}{2}\frac{d}{dt}|\vec{r}|^2
\end{aligned}$$

Since the point P moves on the surface of a sphere with center at the origin, then

$$|\vec{r}|^2 = R^2,\ \text{R is the radius of the sphere}$$

Therefore,

$$\begin{aligned}
\vec{r}\cdot\vec{v} &= \frac{1}{2}\frac{d}{dt}|\vec{r}|^2\\
&= \frac{1}{2}\frac{d(R^2)}{dt}\\
&= 0
\end{aligned}$$

Hence the velocity vector v is always perpendicular to $\vec{r}$. Q.E.D.
To prove that if $\vec{r}$ and $\vec{v}$ are perpendicular, then the motion of P is on the surface of a sphere centered at the origin:

$$\begin{aligned}
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= \frac{d\vec{r}}{dt}\cdot\vec{r}+\vec{r}\cdot\frac{d\vec{r}}{dt}\\
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= 2\frac{d\vec{r}}{dt}\cdot\vec{r}\\
\frac{d}{dt}(\vec{r}\cdot\vec{r}) &= 2\vec{v}\cdot\vec{r}\\
\frac{d}{dt}(|\vec{r}|^2) &= 2\vec{v}\cdot\vec{r}
\end{aligned}$$

Since $\vec{r}$ and $\vec{v}$ are perpendicular, then

$$\vec{r}\cdot\vec{v} = 0$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}(|\vec{r}|^2) &= 2\vec{v}\cdot\vec{r}\\
&= 0\\
|\vec{r}|^2 &= c,\ \text{c is a constant}
\end{aligned}$$

Therefore, the motion of the point P is on a sphere of radius $\sqrt{c}$ with center at the origin. Q.E.D.

5. a) Suppose a point moves with constant speed. Show that its velocity vector and acceleration vector are perpendicular. (Use the formula in **1J-3**)
b) Show the converse: if the velocity and acceleration vectors are perpendicular, the point P moves with constant speed.
Solution:
a)

$$\begin{aligned}
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= \frac{d\vec{v}}{dt}\cdot\vec{v}+\vec{v}\cdot\frac{d\vec{v}}{dt} \\
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= 2\frac{d\vec{v}}{dt}\cdot\vec{v} \\
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= 2\vec{a}\cdot\vec{v} \\
\vec{a}\cdot\vec{v} &= \frac{1}{2}\frac{d}{dt}\vec{v}\cdot\vec{v} \\
&= \frac{1}{2}\frac{d}{dt}|\vec{v}|^2
\end{aligned}$$

Since the described point moves with constant, then

$$\begin{aligned}
|\vec{v}|^2 &= c, \text{ c is a constant} \\
\vec{a}\cdot\vec{v} &= \frac{1}{2}\frac{d}{dt}|\vec{v}|^2 \\
&= 0
\end{aligned}$$

Hence the velocity vector v is always perpendicular to the acceleration vector $\vec{a}$. Q.E.D.
b)

$$\begin{aligned}
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= \frac{d\vec{v}}{dt}\cdot\vec{v}+\vec{v}\cdot\frac{d\vec{v}}{dt} \\
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= 2\frac{d\vec{v}}{dt}\cdot\vec{v} \\
\frac{d}{dt}(\vec{v}\cdot\vec{v}) &= 2\vec{a}\cdot\vec{v} \\
\frac{d}{dt}(|\vec{v}|^2) &= 2\vec{a}\cdot\vec{v}
\end{aligned}$$

Since $\vec{a}$ and $\vec{v}$ are perpendicular, then

$$\begin{aligned}\vec{a}\cdot\vec{v} &= 0\\ \frac{d}{dt}(|\vec{v}|^2) &= 2\vec{a}\cdot\vec{v}\\ &= 0\\ |\vec{v}|^2 &= c,\ \text{c is a constant}\end{aligned}$$

Therefore, the described point P moves with constant speed. Q.E.D.

6. For the helical motion $\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k}$,
a) calculate $\vec{v}$, $\vec{a}$, $\vec{T}$, $|ds/dt|$
b) show that $\vec{v}$ and $\vec{a}$ are perpendicular; explain using **1J-5**
Solution:
a)

$$\begin{aligned}\vec{v} &= \frac{d(\vec{r}(t))}{dt}\\ &= \frac{d(a\cos t\vec{i} + a\sin t\vec{j} + bt\vec{k})}{dt}\\ &= -a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k}\\ \vec{a} &= \frac{d(\vec{v}(t))}{dt}\\ &= \frac{d(-a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k})}{dt}\\ &= -a\cos t\vec{i} - a\sin t\vec{j}\\ |ds/dt| &= |\vec{v}(t)|\\ &= |-a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k}|\\ &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\\ &= \sqrt{a^2+b^2}\end{aligned}$$

$$\begin{aligned}\vec{T} &= dir(\vec{v})\\ &= \frac{\vec{v}}{|\vec{v}|}\\ &= \frac{-a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k}}{\sqrt{a^2+b^2}}\\ &= -\frac{a}{\sqrt{a^2+b^2}}\sin t\vec{i} + \frac{a}{\sqrt{a^2+b^2}}\cos t\vec{j} + \frac{b}{\sqrt{a^2+b^2}}\vec{k}\end{aligned}$$

b) Since

$$\begin{aligned}\vec{v}\cdot\vec{a} &= (-a\sin t\vec{i} + a\cos t\vec{j} + b\vec{k})\cdot(-a\cos t\vec{i} - a\sin t\vec{j})\\ &= a^2\sin t\cos t - a^2\sin t\cos t + 0\cdot b\\ &= 0\end{aligned}$$

$\vec{v}$ and $\vec{a}$ are perpendicular.
According to the conclusion in **1J-5**, since we know that the speed of the movement is constant, i.e.

$$|ds/dt| = \sqrt{a^2 + b^2}$$

then the velocity $\vec{v}$ and the accceleration $\vec{a}$ are perpendicular.

9. A point P is moving in space, with position vector

$$\vec{r} = OP = 3\cos t\vec{i} + 5\sin t\vec{j} + 4\cos t\vec{k}$$

a) Show it moves on the surface of a sphere.
b) Show its speed is constant.
c) Show the acceleration is directed toward the origin.
d) Show it moves in a plane through the origin.
e) Describe the path of the point.
Solution:
a) According to the position vector of the point P, the parametric equation of the movement of the point P is

$$\begin{cases} x(t) = 3\cos t \\ y(t) = 5\sin t \\ z(t) = 4\cos t \end{cases}$$

Then the coordinates of the point P satisfies

$$\begin{aligned}(x(t))^2 + (y(t))^2 + (z(t))^2 &= (3\cos t)^2 + (5\sin t)^2 + (4\cos t)^2 \\ &= 9\cos^2 t + 25\sin^2 t + 16\cos^2 t \\ &= 25\end{aligned}$$

Therefore, the movement of the point P is always on the surface of $x^2+y^2+z^2 = 25$, which is a sphere.
b)

$$\begin{aligned}\vec{v} &= \frac{d(\vec{r}(t))}{dt} \\ &= \frac{d(3\cos t\vec{i} + 5\sin t\vec{j} + 4\cos t\vec{k})}{dt} \\ &= -3\sin t\vec{i} + 5\cos t\vec{j} - 4\sin t\vec{k} \\ |ds/dt| &= |\vec{v}| \\ &= |-3\sin t\vec{i} + 5\cos t\vec{j} - 4\sin t\vec{k}| \\ &= \sqrt{(-3\sin t)^2 + (5\cos t)^2 + (-4\sin t)^2} \\ &= 5\end{aligned}$$

c)

$$\begin{aligned}\vec{a} &= \frac{d(\vec{v}(t))}{dt} \\ &= \frac{d(-3\sin t\vec{i} + 5\cos t\vec{j} - 4\sin t\vec{k})}{dt} \\ &= -3\cos t\vec{i} - 5\sin t\vec{j} - 4\cos t\vec{k}\end{aligned}$$

Therefore,

$$\vec{a} = -\vec{r} = -OP = PO$$

Since $\vec{r}$ is the vector from the origin to the point P, then at the point P, $\vec{a}$, which has the reverse direction of $\vec{r}$, points back to the origin.

d) According to the parametric equation of the point P in part (a), the coordinates of the point P satisfy

$$4x(t) + 0y(t) - 3z(t) = 4 \cdot 3\cos t + 0 \cdot 5\sin t - 3 \cdot 4\cos t = 0$$

Therefore, the point P moves in a plane through the origin whose normal vector is $\langle 4, 0, -3\rangle$.

e) Since the movement of the point P is on the surface of a sphere whose center is the origin, and also in a plane through the origin, then the path of the point P is the intersection of the two geometric objects, which is a circle of radius 5 whose center is the origin.

The equation of the circle can be given as

$$\begin{cases} x^2 + y^2 + z^2 = 25 \\ 4x - 3z = 0 \end{cases}$$

Unit 1K. Kepler's Second Law

2. Let $\vec{s}(t)$ be a vector function. Prove by using components that

$$\frac{d\vec{s}}{dt} = \vec{0} \;\Rightarrow\; \vec{s}(t) = \vec{K}, \text{ where } \vec{K} \text{ is a constant vector.}$$

Solution:

Suppose that $\vec{s}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then

$$\frac{d\vec{s}}{dt} = \frac{d(x(t))}{dt}\vec{i} + \frac{d(y(t))}{dt}\vec{j} + \frac{d(z(t))}{dt}\vec{k} = \vec{0}$$

which is equivalent to

$$\begin{cases} \frac{d(x(t))}{dt} = 0 \\ \frac{d(y(t))}{dt} = 0 \\ \frac{d(z(t))}{dt} = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} x(t) = x_0, \ x_0 \text{ is a constant} \\ y(t) = y_0, \ y_0 \text{ is a constant} \\ z(t) = z_0, \ z_0 \text{ is a constant} \end{cases}$$

Therefore,

$$\begin{aligned} \vec{s}(t) &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \\ &= x_0\vec{i} + y_0\vec{j} + z_0\vec{k} \end{aligned}$$

where x_0, y_0, and z_0 are constants. Hence, $\vec{s}(t)$ is a constant vector. Moreover, this conclusion is valid no matter how many dimensions the vector has.

Question 2. *I believe the converse is also true. Prove it.*

3. In our proof that Kepler's second law is equivalent to the force being central, used the following steps to show the second law implies a central force. Kepler's second law says the motion is in a plane and

$$2\frac{dA}{dt} = |\vec{r} \times \vec{v}| \text{ is constant.}$$

This implies $\vec{r} \times \vec{v}$ is constant. So,

$$0 = \frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}$$

This implies $\vec{a}$ and $\vec{r}$ are parallel, i.e. the force is central.
Reverse the two steps to prove the converse: for motion under any type of central force, the path of motion will lie in a plane and area will be swept out by the radius vector at a constant rate.
Solution:
The motion is only under a central force
$\iff \vec{r} \parallel \vec{a}$
$\iff \vec{r} \times \vec{a} = 0$
Therefore,

$$\begin{gathered} \vec{r} \times \vec{a} = 0 \\ \vec{r} \times \vec{a} + \vec{v} \times \vec{v} = 0 \\ \frac{d}{dt}(\vec{r} \times \vec{v}) = 0 \\ \vec{r} \times \vec{v} = \vec{K}, \text{ where } \vec{K} \text{ is a constant vector} \\ \begin{cases} |\vec{r} \times \vec{v}| \text{ is constant.} \\ dir(\vec{r} \times \vec{v}) \text{ is constant.} \end{cases} \end{gathered}$$

Since

$$\begin{aligned}
|\vec{r} \times \vec{v}| &= |\vec{r} \times \frac{d\vec{r}}{dt}| \\
&= \frac{|\vec{r} \times d\vec{r}|}{dt} \\
&= 2\frac{dA}{dt}
\end{aligned}$$

Then

$$|\vec{r} \times \vec{v}| \text{ is constant.} \iff \frac{dA}{dt} \text{ is constant.}$$

Since

$$dir(\vec{r} \times \vec{v}) \text{ is constant.}$$

Then the normal vector of the plane determined by the position vector and the velocity vector is constant. Hence the movement is always in the plane.
Q.E.D.

Part II

1. A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P. Suppose that the disk is tangent to the x-axis with the center initially at $(0, 2)$ and P initially at $(0, 1)$ and that it starts to roll to the right on the x-axis at unit speed. Let C be the curve traced out by the point P.
a) Make a sketch of what you think the curve C will look like.
b) Use vectors to find the parametric equations for $\vec{OP}$ as a function of time t.
c) Open the 'Mathlet' Wheel (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C. If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph.