Lecture 11: Chain Rule

Now with partial derivatives, we can learn more tools to study multivariable functions.

1 Total Differential

Total differential describes the rate of change of the value of a function in terms of all variables.
Suppose there is a function $f(x, y, z)$:

$$\begin{aligned} df &= f_x dx + f_y dy + f_z dz \\ &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \end{aligned}$$

Notice that df is not a real number, nor is it Δf. It is an abstract notion. So are dx, dy, and dz.

Question 1. *What is the intuitive way to understand the total differential formula? It is hard to reason about whether it is valid or not.*

How do we use the total differential?

- The total differential encodes how changes in independent variables affect the function value.
- We can replace dx, dy, and dz with small variations Δx, Δy, and Δz to get approximated change of the function value, which is the approximation formula:
$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$
- We can divide both sides of the total differential by the differential of a common parameter, such as dt, to get the rate of change in terms of this common parameter.

2 Chain Rule

Actually, the third usage of the total differential described above derives the formula of chain rule in multivariable functions.

Suppose there is a function $f(x, y, z)$, and there exists $x = x(t)$, $y = y(t)$, and $z = z(t)$, then

$$\begin{aligned}\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\end{aligned}$$

Why is the formula of chain rule valid?
We can start with the approximation formula. We know that

$$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$$

Then we can divide both sides of the equation by a change of the parameter t, which is Δt:

$$\frac{\Delta f}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + f_z \frac{\Delta z}{\Delta t}$$

With $\Delta t \to 0$, according to the definition of derivatives:

$$\begin{aligned}\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} &= \frac{df}{dt} \\ \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} &= \frac{dx}{dt} \\ \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} &= \frac{dy}{dt} \\ \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} &= \frac{dz}{dt}\end{aligned}$$

and the approximation becomes more and more accurate. Eventually

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

Example 1. *Given that $w = f(x, y, z) = x^2 y + z$, $x = t$, $y = e^t$, and $z = \sin t$, find $\frac{dw}{dt}$.*
Solution:

$$
\begin{aligned}
\frac{dw}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\
f_x &= 2xy = 2te^t \\
f_y &= x^2 = t^2 \\
f_z &= 1 \\
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= e^t \\
\frac{dz}{dt} &= \cos t \\
\frac{dw}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\
&= 2t \cdot e^t \cdot 1 + t^2 \cdot e^t + 1 \cdot \cos t \\
&= 2te^t + t^2 e^t + \cos t
\end{aligned}
$$

We can verify the result by first find the formula of w with respect to t, and then calculate its derivative:

$$
\begin{aligned}
w &= x^2 y + z \\
&= t^2 e^t + \sin t \\
\frac{dw}{dt} &= \frac{d(t^2 e^t + \sin t)}{dt} \\
&= 2te^t + t^2 e^t + \cos t
\end{aligned}
$$

The results from the two methods are the same.

We can use chain rule of multivariable functions to derive the product and quotient rule of derivatives of single-variable functions:

- Product rule:

$$
\begin{aligned}
f &= u \cdot v \\
\frac{df}{dt} &= \frac{d(u \cdot v)}{dt} \\
&= f_u \frac{du}{dt} + f_v \frac{dv}{dt} \\
&= \frac{du}{dt} v + u \frac{dv}{dt}
\end{aligned}
$$

- Quotient rule:

$$\begin{aligned}
f &= \frac{u}{v} \\
\frac{df}{dt} &= \frac{d(\frac{u}{v})}{dt} \\
&= f_u \frac{du}{dt} + f_v \frac{dv}{dt} \\
&= \frac{1}{v} \cdot \frac{du}{dt} + \frac{-u}{v^2} \cdot \frac{dv}{dt} \\
&= \frac{\frac{du}{dt} v - u \frac{dv}{dt}}{v^2}
\end{aligned}$$

3 Chain Rule with More Variables

Suppose there is a function $w = f(x, y)$, where $x = x(u, v)$, and $y = y(u, v)$, how to find the total differential of w with respect to u and v?
According to the total differential formula,

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$