Lecture 1: Dot Product

1 Vectors

Vectors: quantities that have both direction and magnitude/length.
Scalars: quantities that only have magnitude.
Identities of vectors: as long as the directions and magnitudes of two vectors are the same, we say these two vectors are equal. The equality of vectors has nothing to do with their start points and end points.

Notations for Vectors

- $\vec{A}$: a vector with the name A.
- $\vec{AB}$: a vector from the point A to the point B.
- $|\vec{A}|$: the magnitude/length of the vector $\vec{A}$.
- $dir(\vec{A})$: the direction of the vector $\vec{A}$.

Definition. *The direction of $\vec{A}$ is defined by*

$$dir\vec{A} = \frac{\vec{A}}{|\vec{A}|}, \qquad (\vec{A} \neq 0);$$

it is the unit vector lying along $\vec{A}$ and pointed the same direction as $\vec{A}$.

1.1 Vectors From A Geometric Perspective

Geometrically, we typically express a vector as an arrow with a direction and a length.

Basic vector operations

- Addition of vectors: The result of $\vec{A} + \vec{B}$ is defined as the vector from the start of $\vec{A}$ to the end of $\vec{B}$ after moving the start of $\vec{B}$ to the end of $\vec{A}$ parallel.
- Negative vectors: The result of $-\vec{A}$ is defined as the vector with the same length and reverse direction of $\vec{A}$.

- Subtraction of vectors: The result of $\vec{A} - \vec{B}$ is equal to $\vec{A} + (-\vec{B})$, which is the vector from the end of $\vec{B}$ to the end of $\vec{A}$.
- Multiplication by scalars: The result of $c \times \vec{A}$ is defined as the vector with the same direction and c times of the length of $\vec{A}$.

1.2 Vectors From An Analytical Perspective

To compute things with vectors, we need to put them into coordinate systems. With the addition operation we defined for vectors, we can decompose a vector to a unique set of component vectors along coordinate axes.
For the convenience of expression, we define a unit vector for each coordinate axis, which has length 1 and same direction as the corresponding axis. By convention, the unit vector along the x axis is $\vec{i}$, the unit vector along the y axis is $\vec{j}$, and the unit vector along the z axis $\vec{k}$. In this way, every vector in the three dimension space can be expressed as

$$\vec{A} = a_1 * i + a_2 * j + a_3 * k$$

Or more generally

$$\vec{A} =< a_1, a_2, a_3 >$$

This kind of expression can be generalized into N dimensional space as:

$$\vec{A} =< a_1, a_2, a_3, ..., a_{n-1}, a_n >$$

Basic vector operations

- Addition of vectors: Since the addition of vectors satisfies the commutative and associative laws, $\vec{A} + \vec{B}$ is equal to the sum of the addition result of their corresponding component vectors. Hence

$$\vec{A} + \vec{B} =< a_1 + b_1, a_2 + b_2, ..., a_n + b_n >$$

- Negative vectors: It is obvious that

$$-\vec{A} =< -a_1, -a_2, ..., -a_n >$$

- Subtraction of vectors: The result of $\vec{A} - \vec{B}$ is equal to $\vec{A} + (-\vec{B})$. Hence

$$\vec{A} - \vec{B} =< a_1 - b_1, a_2 - b_2, ..., a_n - b_n >$$

- Multiplication by scalars: It is obvious that

$$c \times \vec{A} =< c \times a_1, c \times a_2, ..., c \times a_n$$

Example 1. *What is the length of the vector $\vec{A} =< 3, 2, 1 >$?*
We can define an intermediate vector $\vec{A_1} = 3\vec{i} + 2\vec{j}$. According to Pythagorean Theorem, $|\vec{A_1}| = \sqrt{3^2 + 2^2} = \sqrt{13}$. Then $\vec{A} = \vec{A_1} + \vec{k}$, and $\vec{A_1}$ is perpendicular to $\vec{k}$. Hence $|\vec{A}| = \sqrt{|\vec{A_1}|^2 + |\vec{k}|^2} = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$.

In general,

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$$

This formula holds because of the orthogonality of the axes in coordinate systems.

2 Dot Product

Definition. $\vec{A} \cdot \vec{B} = \sum a_i b_i = a_1 b_1 + a_2 b_2 + ... + a_n b_n$

Geometric Interpretation: Suppose the angle between $\vec{A}$ and $\vec{B}$ is θ, then $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$.
Proofs:
Firstly let's consider the dot product of a vector and itself. According to the definition of dot product:

$$\vec{A} \cdot \vec{A} = \sum_{i=1}^{n} a_i^2 = |\vec{A}|^2$$

Therefore the above statement holds in this case.
Then let's consider the dot product of two different vectors. One important insight about the definition of dot product is that it satisfies the commutative and distributive law. Let $\vec{C}$ denotes the result vector of $\vec{A} - \vec{B}$. Then

$$\vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B})$$

$$|\vec{C}|^2 = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2\vec{A} \cdot \vec{B}$$

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}$$

According to Law of Cosine, we have

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$$

Therefore

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$$

Property of the sign of dot product:

- $\vec{A} \cdot \vec{B} > 0$ if and only if $\theta < 90°$.

- $\vec{A} \cdot \vec{B} = 0$ if and only if $\theta = 90°$.
- $\vec{A} \cdot \vec{B} < 0$ if and only if $\theta > 90°$.

Application of Dot Product
1. Computing lengths and angles (especially angles).

Example 2. *Find the angle between PQ and PR.*
Solution:

$$\vec{PQ} =< -1, 1, 0 >$$
$$\vec{PR} =< -1, 0, 2 >$$

Then

$$\begin{aligned}\cos(\angle QPR) &= \frac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}| \cdot |\vec{PR}|} \\ &= \frac{-1 \cdot (-1) + 1 \cdot 0 + 0 \cdot 2}{\sqrt{(-1)^2 + 1^2 + 0^2} \cdot \sqrt{(-1)^2 + 0^2 + 2^2}} \\ &= \frac{1}{\sqrt{10}}\end{aligned}$$

2. Detect orthogonality.

Example 3. *The set of points where $x + 2y + 3z = 0$ describes*
A. The empty set
B. A single point
C. A line
D. A plane
E. A sphere
F. None of the above
Solution:
The set of points where $x + 2y + 3z = 0$ describes a plane. We can find three points which satisfy the equation and are not on a same line together, in order to exclude the option A, B, and C.

$$P_1 = (0, 0, 0)$$

$$P_2 = (1, 1, -1)$$
$$P_3 = (5, -1, -1)$$

If a set of points describes a plane, then we can find a vector which is perpendicular to any line formed by two points on the plane. That vector is called normal. The normal to the surface described by this equation is $< 1, 2, 3 >$. Suppose we have two random points on the surface described by the above equation: $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Then we have

$$x_1 + 2y_1 + 3z_1 = 0$$

$$x_2 + 2y_2 + 3z_2 = 0$$

Therefore

$$(x_1 - x_2) + 2(y_1 - y_2) + 3(z_1 - z_2) = 0$$

which proves that the vector $< 1, 2, 3 >$ *is perpendicular to the vector* $\vec{P_1P_2}$. *Since* P_1 *and* P_2 *are random points, we prove that the vector* $< 1, 2, 3 >$ *is perpendicular to any vectors formed by two points on that surface. Therefore, that surface is a plane.*