Lecture 19: Vector Fields

1 Definition of Vector Fields

A vector field can be described with such a formula:

$$\vec{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$$

where M and N are functions of coordinates.
In a vector field, at each point (x, y) there is a corresponding vector. In other words, vectors in a vector field are a function of the position.
Real world examples of a vector field:

- Velocity field in fluid $\vec{v}$.
- Force field $\vec{F}$.

2 Plot of Vector Fields

Plot of a vector field enables us to understand the vector field in a concrete way and probably provides some important insights.

Example 1. *Plot the vector field* $\vec{F} = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$.
TODO(jinxinwang): add the graph.

Example 2. *Plot the vector field* $\vec{F} = x\hat{\boldsymbol{\imath}}$.
TODO(jinxinwang): add the graph.
Notice that we usually use the length of arrows in the graph to relatively indicate the magnitudes of vectors in a vector field.

Example 3. *Plot the vector field* $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.
TODO(jinxinwang): add the graph.

Example 4. *Plot the vector field* $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$.
TODO(jinxinwang): add the graph.
This is the vector field of the uniform rotation at unit angular velocity.

Question. *Vectors in mathematics have no starting point. However, in vector fields each vector is associated with a starting point. Is it contradictory with the definition of vectors?*

3 Line Integrals

3.1 Definition of Line Integrals

The motivation of introducing line integrals is to calculate the work done along a trajectory in a force field.
Recall from physics:

$$W = \vec{F} \cdot \Delta\vec{r}$$

For a motion with changing force or curved trajectory, to calculate the total work, we need to apply the idea of integrals. We can divide the trajectory into many small pieces, with each piece as $\Delta\vec{r}$. For a piece of small trajectory $\Delta\vec{r}_i$, the work is

$$\Delta W = \vec{F} \cdot \Delta\vec{r}_i$$

With the number of divided pieces approaching infinity and each piece of trajectory approaching infinitesimal, we can add them up to get the total work:

$$\begin{aligned} W &= \lim_{\Delta\vec{r}_i \to \vec{0}} \sum_i \vec{F} \cdot \Delta\vec{r}_i \\ &= \int_C \vec{F} \cdot d\vec{r} \end{aligned}$$

which is the definition of a line integral.

3.2 Calculation of Line Integrals

To calculate a line integral,

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt \end{aligned}$$

Notice that in the definition of line integrals, $\vec{F}$ and the trajectory C are independent from each other.

Example 5. *Suppose that there is a force field as $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$, and a trajectory C as*

$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

where $0 \le t \le 1$. Calculate the work along the trajectory in the force field.
Solution:

We can use the line integral to calculate the total work.

$$
\begin{aligned}
W &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt
\end{aligned}
$$

According to the problem description,

$$
\begin{aligned}
\vec{F} &= \langle -y, x \rangle = \langle -t^2, t \rangle \\
\vec{r} &= \langle x, y \rangle = \langle t, t^2 \rangle \\
\frac{d\vec{r}}{dt} &= \langle 1, 2t \rangle
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W &= \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\
&= \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle dt \\
&= \int_0^1 t^2 dt \\
&= \frac{t^3}{3} |_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

3.3 Another Way of Calcuating Line Integrals

Another way to look at the calculation of line integrals. For a line integral $\int_C \vec{F} \cdot d\vec{r}$, we have

$$
\begin{aligned}
\vec{F} &= \langle M, N \rangle \\
d\vec{r} &= \langle dx, dy \rangle
\end{aligned}
$$

Question. *Why the following equation holds $d\vec{r} = \langle dx, dy \rangle$?*

Therefore,

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C \langle M, N \rangle \cdot \langle dx, dy \rangle \\
&= \int_C M dx + N dy
\end{aligned}
$$

To evaluate the expression $\int_C M dx + N dy$, we need to express x and y with a single variable, and do substitution in the integrand.

Question. *Why is it incorrect to evaluate* $\int_C Mdx + Ndy$ *by the following method:*

$$\begin{aligned}\int_C Mdx + Ndy &= \int_C Mdx + \int_C Ndy \\ &= \int_{x_1}^{x_2} Mdx + \int_{y_1}^{y_2} Ndy\end{aligned}$$

Answer:
I can think of a counterexample to prove the above method can produce incorrect result. Consider the following case:
There is a force field as $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$*, and a trajectory as*

$$\begin{cases} x = \sin t \\ y = \cos t \end{cases}$$

where $0 \le t \le 2\pi$*.*
Obviously the total along the described trajectory in the force field is not 0 *because at any points of the trajectory,* $\vec{F}$ *and* $d\vec{r}$ *have the same direction. However, using the above method*

$$\int_C Mdx + Ndy = \int_{x_1}^{x_2} Mdx + \int_{y_1}^{y_2} Ndy$$

the result would be 0 *because* $x_1 = x_2$ *and* $y_1 = y_2$*.*
In summary, the incorrect evaluation method cannot correctly evaluate line integrals with back-and-forth trajectory.

Example 6. *Suppose that there is a force field as* $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$*, and a trajectory* C *as*

$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

where $0 \le t \le 1$*. Calculate the work along the trajectory in the force field.*
Solution:
We can use line intergrals to calculate the total work:

$$\begin{aligned}W &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C \langle -y, x\rangle \cdot \langle dx, dy\rangle \\ &= \int_C -ydx + xdy\end{aligned}$$

According to the problem description,

$$\begin{aligned} x &= t \\ y &= t^2 \\ dx &= 1dt \\ dy &= 2tdt \end{aligned}$$

Therefore,

$$\begin{aligned} W &= \int_C -ydx + xdy \\ &= \int_0^1 -t^2 \cdot 1 \cdot dt + t \cdot 2t \cdot dt \\ &= \int_0^1 t^2 dt \\ &= \frac{1}{3} \end{aligned}$$

Note that the result of a line integral $\int_C \vec{F} \cdot d\vec{r}$ doesn't depend on the parameterization, in other words, the substitution of any parameters should produce the same results. For the above example, we can also apply the following parameterization, which would yield the same result.

$$\begin{aligned} x &= \sin\theta \\ y &= \sin^2\theta \\ 0 \leq \theta &\leq \frac{\pi}{2} \end{aligned}$$

Therefore, we should choose the parameterization which is the easiest to calculate.

3.4 Geometric Approach

$$\begin{aligned} \vec{v} = \frac{d\vec{r}}{dt} = \vec{T}|\frac{ds}{dt}| = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \\ d\vec{r} = \vec{T}ds = \langle dx, dy \rangle \\ \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \cdot ds \end{aligned}$$