

18.02 EXERCISES

Problem Set 1: Vectors, Determinants and Planes

Part I

Unit 1 Vectors

1. Find the magnitude and direction of the vectors

a) $\vec{i} + \vec{j} + \vec{k}$

b) $2\vec{i} - \vec{j} + 2\vec{k}$

c) $3\vec{i} - 6\vec{j} - 2\vec{k}$

Solution:

a) Suppose $\vec{A} = \vec{i} + \vec{j} + \vec{k}$, then

$$\begin{aligned} |\vec{A}| &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{dir} \vec{A} &= \frac{\vec{A}}{|\vec{A}|} \\ &= \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\vec{i} + \frac{\sqrt{3}}{3}\vec{j} + \frac{\sqrt{3}}{3}\vec{k} \end{aligned}$$

b) Suppose $\vec{A} = 2\vec{i} - \vec{j} + 2\vec{k}$, then

$$\begin{aligned} |\vec{A}| &= \sqrt{2^2 + (-1)^2 + 2^2} \\ &= \sqrt{9} & |\vec{A}| &= \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{dir} \vec{A} &= \frac{\vec{A}}{|\vec{A}|} \\ &= \frac{2\vec{i} - \vec{j} + 2\vec{k}}{3} \\ &= \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k} \end{aligned}$$

c) Suppose $\vec{A} = 3\vec{i} - 6\vec{j} - 2\vec{k}$, then

$$\begin{aligned} |\vec{A}| &= \sqrt{3^2 + (-6)^2 + (-2)^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

$$\begin{aligned}
\text{dir } \vec{A} &= \frac{\vec{A}}{|\vec{A}|} \\
&= \frac{3\vec{i} - 6\vec{j} - 2\vec{k}}{7} \\
&= \frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k}
\end{aligned}$$

2. a) Let P and Q be two points in space, and X the midpoint of the line segment PQ . Let O be an arbitrary fixed point; show that as vectors, $OX = \frac{1}{2}(OP + OQ)$.

b) With the notation of part (a), assume that X divides the line segment PQ in the ratio $r : s$, where $r + s = 1$. Derive an expression for OX in terms of OP and OQ .

Solution:

a) Suppose $P = (a_1, b_1, c_1)$, $Q = (a_2, b_2, c_2)$, and $O = (a_0, b_0, c_0)$.

Since X is the midpoint of the line segment PQ , then

$$X = \left(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}, \frac{c_1 + c_2}{2} \right)$$

Therefore,

$$\begin{aligned}
\vec{OP} &= \langle a_1 - a_0, b_1 - b_0, c_1 - c_0 \rangle \\
\vec{OQ} &= \langle a_2 - a_0, b_2 - b_0, c_2 - c_0 \rangle \\
\vec{OX} &= \left\langle \frac{a_1 + a_2}{2} - a_0, \frac{b_1 + b_2}{2} - b_0, \frac{c_1 + c_2}{2} - c_0 \right\rangle \\
&= \frac{1}{2} \langle a_1 + a_2 - 2a_0, b_1 + b_2 - 2b_0, c_1 + c_2 - 2c_0 \rangle \\
&= \frac{1}{2}(\vec{OP} + \vec{OQ})
\end{aligned}$$

b) Since X divides the line segment PQ in the ratio $r : s$, where $r + s = 1$,

$$\vec{PX} = \frac{r}{r+s} \vec{PQ}$$

Then we also know

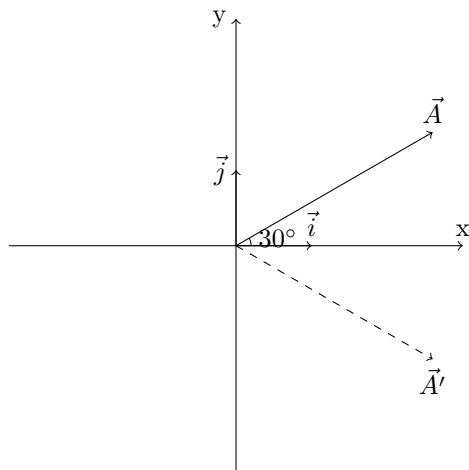
$$\begin{aligned}
\vec{PQ} &= \vec{OQ} - \vec{OP} \\
\vec{OX} &= \vec{OP} + \vec{PX}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{OX} &= \vec{OP} + \vec{PX} \\
&= \vec{OP} + \frac{r}{r+s} \vec{PQ} \\
&= \vec{OP} + \frac{r}{r+s} (\vec{OQ} - \vec{OP}) \\
&= \frac{s}{r+s} \vec{OP} + \frac{r}{r+s} \vec{OQ}
\end{aligned}$$

3. What are the $\vec{i}\vec{j}$ -components of a plane vector \vec{A} of length 3, if it makes an angle of 30° with \vec{i} and 60° with \vec{j} . Is the second condition redundant?

Solution:



Apparently

$$\begin{aligned}\vec{A} &= \langle |\vec{A}| \cos(30^\circ), |\vec{A}| \sin(30^\circ) \rangle \\ &= \langle 3 \cdot \frac{\sqrt{3}}{2}, 3 \cdot \frac{1}{2} \rangle \\ &= \langle \frac{3\sqrt{3}}{2}, \frac{3}{2} \rangle \\ &= \frac{3\sqrt{3}}{2}\vec{i} + \frac{3}{2}\vec{j}\end{aligned}$$

The second condition that the vector \vec{A} makes an angle of 60° with \vec{j} is not redundant, otherwise the referred vector can be \vec{A} or \vec{A}' in the above diagram.

4. A small plane wishes to fly due north at 200 mph (as seen from the ground), in a wind blowing from the northeast at 50 mph. Tell with what vector velocity in the air it should travel (given the $\vec{i}\vec{j}$ -components).

Solution:

Suppose that \vec{i} represents east, then \vec{j} represents north, and $-\vec{i}$ represents west, and $-\vec{j}$ represents south. Let \vec{V}_w denote the velocity of the wind, \vec{V}_p denote the velocity of the plane in the air, and \vec{V} denote the velocity of plane seen from the ground.

\vec{V} should be the result of \vec{V}_p and \vec{V}_w applied together, therefore

$$\vec{V} = \vec{V}_p + \vec{V}_w$$

According to the problem description,

$$\begin{aligned}\vec{V}_w &= \langle 50 \cos 45^\circ, 50 \sin 45^\circ \rangle \\ &= \langle 25\sqrt{2}, 25\sqrt{2} \rangle\end{aligned}$$

$$\vec{V} = \langle 0, 200 \rangle$$

Therefore

$$\begin{aligned}\vec{V}_p &= \vec{V} - \vec{V}_w \\ &= \langle 0, 200 \rangle - \langle 25\sqrt{2}, 25\sqrt{2} \rangle \\ &= \langle -25\sqrt{2}, 200 - 25\sqrt{2} \rangle\end{aligned}$$

5. Let $\vec{A} = a\vec{i} + b\vec{j}$ be a plane vector; find in terms of a and b the vectors \vec{A}' and \vec{A}'' resulting from rotating \vec{A} by 90° a) clockwise b) counterclockwise.

c) Let $\vec{i}' = (3\vec{i} + 4\vec{j})/5$. Show that \vec{i}' is a unit vector, and use the first part of the exercise to find a vector \vec{j}' such that \vec{i}', \vec{j}' forms a right-handed coordinate system.

Solution:

a) According to the problem description, \vec{A}' is the result of rotating \vec{A} clockwise by 90° . Therefore, the $\vec{i}\vec{j}$ -components of \vec{A} should be rotated the same way to get the corresponding components of \vec{A}' . After being rotated in the mentioned way, \vec{i} becomes $-\vec{j}$, and \vec{j} becomes \vec{i} . Therefore,

$$\vec{A}' = b \cdot \vec{i} - a \cdot \vec{j}$$

b) Similar to a), after being rotated counterclockwise by 90° , \vec{i} becomes \vec{j} , and \vec{j} becomes $-\vec{i}$. Therefore,

$$\vec{A}'' = -b \cdot \vec{i} + a \cdot \vec{j}$$

c) To prove \vec{i}' is a unit vector,

$$\begin{aligned}|\vec{i}'| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} \\ &= 1\end{aligned}$$

To form a right-handed coordinate system, \vec{j}' should be the result of rotating \vec{i}' counterclockwise by 90° . According the part b), we can derive that

$$\vec{j}' = -\frac{4}{5} \cdot \vec{i} + \frac{3}{5} \cdot \vec{j}$$

6. The direction of a space vector is in engineering practice often given by its **direction cosines**. To describe these, let $\vec{A} = a\vec{i} + b\vec{j} + c\vec{k}$ be a space vector, represented as an origin vector, and let α , β , and γ be the three angles ($\leq \pi$) that \vec{A} makes respectively with \vec{i} , \vec{j} , and \vec{k} .

a) Show that $\text{dir}\vec{A} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$. (The three coefficients are called the *direction cosines* of \vec{A} .)

b) Express the direction cosines of \vec{A} in terms of a , b , c ; find the direction cosines of the vector $-\vec{i} + 2\vec{j} + 2\vec{k}$.

c) Prove that three numbers t , u , v are the direction cosines of a vector in space if and only if they satisfy $t^2 + u^2 + v^2 = 1$.

Solution:

a) According to the definition, $\vec{dir}\vec{A}$ is the unit vector with the same direction as \vec{A} , hence $\vec{dir}\vec{A}$ makes the same angles with \vec{i} , \vec{j} , \vec{k} as \vec{A} , i.e. α , β , and γ respectively. Then according to the definition of component vectors, i.e. the cast on a specific directions, we can derive that

$$\vec{dir}\vec{A} = \cos \alpha \cdot \vec{i} + \cos \beta \cdot \vec{j} + \cos \gamma \cdot \vec{k}$$

b) According to the definition of vector directions,

$$\begin{aligned} \vec{dir}\vec{A} &= \frac{\vec{A}}{|\vec{A}|} \\ &= \frac{a\vec{i} + b\vec{j} + c\vec{k}}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{a}{\sqrt{a^2 + b^2 + c^2}}\vec{i} + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\vec{j} + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\vec{k} \\ &= \cos \alpha \cdot \vec{i} + \cos \beta \cdot \vec{j} + \cos \gamma \cdot \vec{k} \end{aligned}$$

Therefore,

$$\begin{aligned} \cos \alpha &= \frac{a}{\sqrt{a^2 + b^2 + c^2}} \\ \cos \beta &= \frac{b}{\sqrt{a^2 + b^2 + c^2}} \\ \cos \gamma &= \frac{c}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Hence for the vector $-\vec{i} + 2\vec{j} + 2\vec{k}$,

$$\begin{aligned} \cos \alpha &= \frac{-1}{\sqrt{(-1)^2 + 2^2 + 2^2}} \\ &= -\frac{1}{3} \\ \cos \beta &= \frac{2}{\sqrt{(-1)^2 + 2^2 + 2^2}} \\ &= \frac{2}{3} \\ \cos \gamma &= \frac{2}{\sqrt{(-1)^2 + 2^2 + 2^2}} \\ &= \frac{2}{3} \end{aligned}$$

c) Proof:

If t, u, v are the direction cosines of a vector \vec{A} in space, then they are the components of its direction:

$$\text{dir} \vec{A} = t \cdot \vec{i} + u \cdot \vec{j} + v \cdot \vec{k}$$

Since $\text{dir} \vec{A}$ is a unit vector, then

$$|\text{dir} \vec{A}| = \sqrt{t^2 + u^2 + v^2} = 1$$

$$t^2 + u^2 + v^2 = 1$$

If t, u, v satisfy $t^2 + u^2 + v^2 = 1$, then we can construct a unit vector $\vec{D} = t \cdot \vec{i} + u \cdot \vec{j} + v \cdot \vec{k}$. Then t, u, v are the direction cosines of the constructed space vector.

7. Prove using vector methods (without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides \vec{A} and \vec{B} .)

Proof:

Suppose that the two sides are \vec{A} and \vec{B} , the third side is \vec{C} , and the line segment joining the midpoints of the two sides is \vec{C}' . Then

$$\vec{C} = \vec{A} - \vec{B}$$

or

$$\vec{C} = -(\vec{A} - \vec{B})$$

And

$$\begin{aligned} \vec{C}' &= \frac{1}{2}\vec{A} - \frac{1}{2}\vec{B} \\ &= \frac{1}{2}(\vec{A} - \vec{B}) \end{aligned}$$

or

$$\begin{aligned} \vec{C}' &= -(\frac{1}{2}\vec{A} - \frac{1}{2}\vec{B}) \\ &= -\frac{1}{2}(\vec{A} - \vec{B}) \end{aligned}$$

Therefore, either of the following equations must hold:

$$\vec{C}' = \frac{1}{2}\vec{C}$$

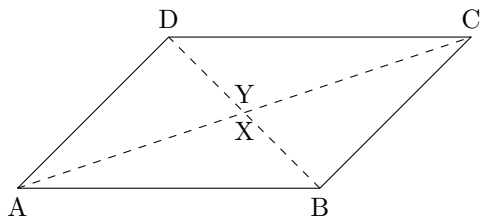
or

$$\vec{C}' = -\frac{1}{2}\vec{C}$$

Therefore, \vec{C}' is parallel to \vec{C} and the magnitude of \vec{C}' is the half of the one of \vec{C} .

8. Prove using vector methods (without components) that the diagonals of a parallelogram bisect each other. (One way: let X and Y be the midpoints of the two diagonals; show $X = Y$.)

Proof:



As shown in the diagram, suppose that the four endpoints of a parallelogram are A, B, C, D , and the midpoint of the diagonal AC is X , and the midpoint of the diagonal BD is Y . Then

$$\begin{aligned}\vec{AX} &= \frac{1}{2}\vec{AC} \\ &= \frac{1}{2}(\vec{AB} + \vec{AD}) \\ &= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AD}\end{aligned}$$

$$\begin{aligned}\vec{AY} &= \vec{AB} + \vec{BY} \\ &= \vec{AB} + \frac{1}{2}\vec{BD} \\ &= \vec{AB} + \frac{1}{2}(\vec{AD} - \vec{AB}) \\ &= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AD}\end{aligned}$$

Therefore, $\vec{AX} = \vec{AY}$, which means the midpoints of the two diagonals are the same point. Hence, the two diagonals of a parallelogram bisect each other.

Unit 2 Dot Product

1. Tell for what values of c the vectors $c\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + 2\vec{k}$ will
a) be orthogonal b) form an acute angle

Solution:

- a) The vectors $c\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + 2\vec{k}$ are orthogonal if and only if

$$(c\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} + 2\vec{k}) = 0$$

$$c - 2 - 2 = 0$$

$$c = 4$$

- b) The vectors $c\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + 2\vec{k}$ form an acute angle if and only if

$$(c\vec{i} + 2\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} + 2\vec{k}) > 0$$

$$c - 2 - 2 > 0$$

$$c > 4$$

2. Using vectors, find the angle between a longest diagonal PQ of a cube, and
a) a diagonal PR of one of its faces; b) an edge PS of the cube.
(Choose a size and position for the cube that makes calculation easiest.)

Solution:

a) Suppose that $P = (0, 0, 0)$, $Q = (1, 1, 1)$, and $R = (1, 1, 0)$. Therefore, $\vec{PQ} = \langle 1, 1, 1 \rangle$, and $\vec{PR} = \langle 1, 1, 0 \rangle$. Then, the angle θ between PQ and PR satisfies

$$\begin{aligned} \cos \theta &= \frac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}| \cdot |\vec{PR}|} \\ &= \frac{1 \times 1 + 1 \times 1 + 1 \times 0}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2 + 1^2 + 0^2}} \\ &= \frac{2}{\sqrt{3} \times \sqrt{2}} \\ &= \frac{\sqrt{6}}{3} \end{aligned}$$

b) Suppose that $P = (0, 0, 0)$, $Q = (1, 1, 1)$, and $S = (1, 0, 0)$. Therefore, $\vec{PQ} = \langle 1, 1, 1 \rangle$, and $\vec{PS} = \langle 1, 0, 0 \rangle$. Then, the angle θ between PQ and PS satisfies

$$\begin{aligned} \cos \theta &= \frac{\vec{PQ} \cdot \vec{PS}}{|\vec{PQ}| \cdot |\vec{PS}|} \\ &= \frac{1 \times 1 + 1 \times 0 + 1 \times 0}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2 + 0^2 + 0^2}} \\ &= \frac{1}{\sqrt{3} \times 1} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

3. Three points in space are $P : (a, 1, -1)$, $Q : (0, 1, 1)$, $R : (a, -1, 3)$. For what value(s) of a will PQR be

a) a right angle b) an acute angle

Solution:

a) PQR will be a right angle $\iff \vec{PQ} \cdot \vec{QR} = 0$

$$\vec{PQ} = \langle -a, 0, 2 \rangle$$

$$\vec{QR} = \langle a, -2, 2 \rangle$$

$$\vec{PQ} \cdot \vec{QR} = 0$$

$$\langle -a, 0, 2 \rangle \cdot \langle a, -2, 2 \rangle = 0$$

$$-a^2 + 4 = 0$$

$$a = \pm 2$$

b) PQR will be an acute angle $\iff \vec{PQ} \cdot \vec{QR} > 0$

$$\vec{PQ} \cdot \vec{QR} > 0$$

$$\langle -a, 0, 2 \rangle \cdot \langle a, -2, 2 \rangle > 0$$

$$-a^2 + 4 > 0$$

$$a^2 < 4$$

$$-2 < a < 2$$

4. Find the component of the force $\vec{F} = 2\vec{i} - 2\vec{j} + \vec{k}$ in

a) the direction $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ b) the direction of the vector $3\vec{i} + 2\vec{j} - 6\vec{k}$.

Solution:

a) To get the component of a vector along a certain direction, we can calculate the dot product of the vector and the direction. Since dot product satisfies the distributive property, the dot product of a vector and a direction is equivalent to the dot product of the direction and the component vector of the vector along the direction, plus the dot product of the direction and the other component vector, which is perpendicular to the direction. Therefore, the result is only the dot product of the direction and the component vector of the vector along the direction, which is the component value.

$$\begin{aligned} c &= (2\vec{i} - 2\vec{j} + \vec{k}) \cdot \left(\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}} \right) \\ &= \langle 2, -2, 1 \rangle \cdot \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right\rangle \\ &= -\frac{\sqrt{3}}{3} \end{aligned}$$

b) Similar to a),

$$\begin{aligned}
 \text{dir}(3\vec{i} + 2\vec{j} - 6\vec{k}) &= \frac{3\vec{i} + 2\vec{j} - 6\vec{k}}{|3\vec{i} + 2\vec{j} - 6\vec{k}|} \\
 &= \frac{3\vec{i} + 2\vec{j} - 6\vec{k}}{7} \\
 &= \left\langle \frac{3}{7}, \frac{2}{7}, -\frac{6}{7} \right\rangle \\
 c &= (2\vec{i} - 2\vec{j} + \vec{k}) \cdot \text{dir}(3\vec{i} + 2\vec{j} - 6\vec{k}) \\
 &= \left\langle 2, -2, 1 \right\rangle \cdot \left\langle \frac{3}{7}, \frac{2}{7}, -\frac{6}{7} \right\rangle \\
 &= -\frac{4}{7}
 \end{aligned}$$

5. Prove using vector methods (without components) that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

Proof:

Let \vec{A} and \vec{B} denote the vectors of two adjacent sides of a parallelogram, and \vec{D}_1 and \vec{D}_2 denote the two diagonals of the parallelogram.

$$\begin{aligned}
 \vec{D}_1 &= \vec{A} + \vec{B} \\
 \vec{D}_2 &= \vec{A} - \vec{B}
 \end{aligned}$$

The two diagonals have equal lengths is equivalent to

$$\begin{aligned}
 |\vec{D}_1| &= |\vec{D}_2| \\
 \iff |\vec{A} + \vec{B}| &= |\vec{A} - \vec{B}| \\
 \iff |\vec{A} + \vec{B}|^2 &= |\vec{A} - \vec{B}|^2 \\
 \iff (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) &= (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) \\
 \iff |\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A} \cdot \vec{B} &= |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B} \\
 \iff 4\vec{A} \cdot \vec{B} &= 0 \\
 \iff \vec{A} \cdot \vec{B} &= 0 \\
 \iff \vec{A} \perp \vec{B}
 \end{aligned}$$

which is equivalent to the parallelogram is a rectangle.

Therefore, it is proved that the diagonals of a parallelogram have equal lengths if and only if it is a rectangle.

6. Prove using vector methods (without components) that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

Proof:

Let \vec{A} and \vec{B} denote the vectors of two adjacent sides of a parallelogram, and \vec{D}_1 and \vec{D}_2 denote the two diagonals of the parallelogram.

$$\begin{aligned}\vec{D}_1 &= \vec{A} + \vec{B} \\ \vec{D}_2 &= \vec{A} - \vec{B}\end{aligned}$$

The two diagonals are perpendicular is equivalent to

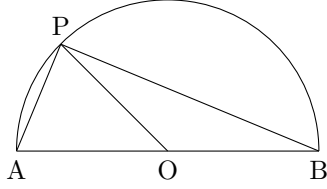
$$\begin{aligned}\vec{D}_1 &\perp \vec{D}_2 \\ \iff \vec{D}_1 \cdot \vec{D}_2 &= 0 \\ \iff (\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) &= 0 \\ \iff |\vec{A}|^2 - |\vec{B}|^2 &= 0 \\ \iff |\vec{A}|^2 &= |\vec{B}|^2 \\ \iff |\vec{A}| &= |\vec{B}|\end{aligned}$$

which is equivalent to the four sides of the parallelogram are equal.

Therefore, it is proved that the diagonals of a parallelogram are perpendicular if and only if it is a rhombus, i.e., its four sides are equal.

7. Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle.

Proof:



Suppose P is a random point on the semicircle, then APB is an angle inscribed in a semicircle.

$$\begin{aligned}\vec{PA} \cdot \vec{PB} &= (\vec{OA} - \vec{OP}) \cdot (\vec{OB} - \vec{OP}) \\ &= \vec{OA} \cdot \vec{OB} + |\vec{OP}|^2 - \vec{OA} \cdot \vec{OP} - \vec{OB} \cdot \vec{OP} \\ &= -|\vec{OA}|^2 + |\vec{OP}|^2 - (\vec{OA} + \vec{OB}) \cdot \vec{OP}\end{aligned}$$

Given that APB is inscribed in a semicircle, $|\vec{OA}| = |\vec{OB}| = |\vec{OP}|$, and $\vec{OA} = -\vec{OB}$.

$$\begin{aligned}\vec{PA} \cdot \vec{PB} &= -|\vec{OA}|^2 + |\vec{OP}|^2 - (\vec{OA} + \vec{OB}) \cdot \vec{OP} \\ &= -|\vec{OP}|^2 + |\vec{OP}|^2 - \vec{0} \cdot \vec{OP} \\ &= 0\end{aligned}$$

$$\vec{PA} \perp \vec{PB}$$

Therefore, it is proved that an angle inscribed in a semicircle is a right angle.

8. Prove the trigonometric formula: $\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$.

Proof:

Suppose the angle between the unit vector \vec{u}_1 and \vec{i} is θ_1 , and the angle between the unit vector \vec{u}_2 and \vec{i} is θ_2 .

$$\vec{u}_1 = \langle \cos \theta_1, \sin \theta_1 \rangle$$

$$\vec{u}_2 = \langle \cos \theta_2, \sin \theta_2 \rangle$$

The angle between \vec{u}_1 and \vec{u}_2 is $|\theta_1 - \theta_2|$. Hence,

$$\begin{aligned}\vec{u}_1 \cdot \vec{u}_2 &= |\vec{u}_1| \cdot |\vec{u}_2| \cdot \cos(|\theta_1 - \theta_2|) \\ &= |\vec{u}_1| \cdot |\vec{u}_2| \cdot \cos(\theta_1 - \theta_2) \\ &= \cos(\theta_1 - \theta_2)\end{aligned}$$

$$\begin{aligned}\vec{u}_1 \cdot \vec{u}_2 &= \langle \cos \theta_1, \sin \theta_1 \rangle \cdot \langle \cos \theta_2, \sin \theta_2 \rangle \\ &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \\ \cos(\theta_1 - \theta_2) &= \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2\end{aligned}$$

Therefore, it is proved that $\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$.

Unit 3 Determinants

1. Calculate the value of the determinants

a) $\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}$

b) $\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix}$

Solution:

a)

$$\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} = (1 \times (-1)) - (4 \times 2) = -9$$

b)

$$\begin{vmatrix} 3 & -4 \\ -1 & -2 \end{vmatrix} = (3 \times (-2)) - ((-4) \times (-1)) = -10$$

2. Calculate $\begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix}$ using the Laplace expansion by the cofactors of:

a) the first row b) the first column

Solution:

a)

$$\begin{aligned} \begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} &= (-1) \cdot \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} + 4 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} \\ &= (-1) \times 2 - 0 \times (-7) + 4 \times (-8) \\ &= -34 \end{aligned}$$

b)

$$\begin{aligned} \begin{vmatrix} -1 & 0 & 4 \\ 1 & 2 & 2 \\ 3 & -2 & -1 \end{vmatrix} &= (-1) \cdot \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 4 \\ -2 & -1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 4 \\ 2 & 2 \end{vmatrix} \\ &= (-1) \times 2 - 1 \times (8) + 3 \times (-8) \\ &= -34 \end{aligned}$$

3. Find the area of the plane triangle whose vertices lie at

a) $(0, 0), (1, 2), (1, -1)$ b) $(1, 2), (1, -1), (2, 3)$

Solution:

a) The vectors of the two edges of the plane triangle are $\langle 1, 2 \rangle$ and $\langle 1, -1 \rangle$.

The area of the plane triangle A can be calculated as

$$\begin{aligned} \det(\langle 1, 2 \rangle, \langle 1, -1 \rangle) &= \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \\ &= -3 \\ A &= |\det(\langle 1, 2 \rangle, \langle 1, -1 \rangle)| = 3 \end{aligned}$$

b) The vectors of the two edges of the plane triangle are $\langle 0, -3 \rangle$ and $\langle 1, 1 \rangle$.

The area of the plane triangle A can be calculated as

$$\begin{aligned} \det(\langle 0, -3 \rangle, \langle 1, 1 \rangle) &= \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} \\ &= 3 \\ A &= |\det(\langle 0, -3 \rangle, \langle 1, 1 \rangle)| = 3 \end{aligned}$$

4. a) Show that the value of a 2×2 determinants is unchanged if you add to the second row a scalar multiple of the first row.

b) Show that the value of a 2×2 determinants is unchanged if you add to the second column a scalar multiple of the first column.

Solution:

a)

$$\begin{aligned}
 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} &= a_1 b_2 - a_2 b_1 \\
 \begin{vmatrix} a_1 & a_2 \\ b_1 + c \cdot a_1 & b_2 + c \cdot a_2 \end{vmatrix} &= a_1 \cdot (b_2 + c \cdot a_2) - a_2 \cdot (b_1 + c \cdot a_1) \\
 &= a_1 b_2 + c \cdot a_1 a_2 - a_2 b_1 - c \cdot a_1 a_2 \\
 &= a_1 b_2 - a_2 b_1
 \end{aligned}$$

b)

$$\begin{aligned}
 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} &= a_1 b_2 - a_2 b_1 \\
 \begin{vmatrix} a_1 & c \cdot a_1 + a_2 \\ b_1 & c \cdot b_1 + b_2 \end{vmatrix} &= a_1 \cdot (c \cdot b_1 + b_2) - b_1 \cdot (c \cdot a_1 + a_2) \\
 &= c \cdot a_1 b_1 + a_1 b_2 - c \cdot a_1 b_1 - a_2 b_1 \\
 &= a_1 b_2 - a_2 b_1
 \end{aligned}$$

5. Use a Laplace expansion and Exercise 4a to show the value of a 3×3 determinants is unchanged if you add to the second row a scalar multiple of the third row.

Solution:

$$\begin{aligned}
 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + k \cdot c_1 & b_2 + k \cdot c_2 & b_3 + k \cdot c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \cdot \begin{vmatrix} b_2 + k \cdot c_2 & b_3 + k \cdot c_3 \\ c_2 & c_3 \end{vmatrix} - \\
 &\quad a_2 \cdot \begin{vmatrix} b_1 + k \cdot c_1 & b_3 + k \cdot c_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 + k \cdot c_1 & b_2 + k \cdot c_2 \\ c_1 & c_2 \end{vmatrix}
 \end{aligned}$$

According to the result of Exercise 4a,

$$\begin{aligned}
 \begin{vmatrix} b_2 + k \cdot c_2 & b_3 + k \cdot c_3 \\ c_2 & c_3 \end{vmatrix} &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \\
 \begin{vmatrix} b_1 + k \cdot c_1 & b_3 + k \cdot c_3 \\ c_1 & c_3 \end{vmatrix} &= \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \\
 \begin{vmatrix} b_1 + k \cdot c_1 & b_2 + k \cdot c_2 \\ c_1 & c_2 \end{vmatrix} &= \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + k \cdot c_1 & b_2 + k \cdot c_2 & b_3 + k \cdot c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \cdot \begin{vmatrix} b_2 + k \cdot c_2 & b_3 + k \cdot c_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 + k \cdot c_1 & b_3 + k \cdot c_3 \\ c_1 & c_3 \end{vmatrix} + \\
 &\quad a_3 \cdot \begin{vmatrix} b_1 + k \cdot c_1 & b_2 + k \cdot c_2 \\ c_1 & c_2 \end{vmatrix} \\
 &= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
 &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
 \end{aligned}$$

6. Let (x_1, y_1) and (x_2, y_2) both range over all unit vectors. Find the maximum value of the function $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$.

Solution:

According to the geometric interpretation of 2×2 determinants, the value of the function $f(x_1, x_2, y_1, y_2) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ is the positive or negative value of the parallelogram formed by the two unit vectors (x_1, y_1) and (x_2, y_2) . The area of the parallelogram formed by these two vectors can be calculated as

$$\begin{aligned}
 A &= | \langle x_1, y_1 \rangle \times \langle x_2, y_2 \rangle | \cdot \sin \theta \\
 &= \sin \theta
 \end{aligned}$$

where θ is the angle between these two vectors.

Apparently, the maximum value of the area of the parallelogram formed by these two vectors is 1, which is achieved when $\theta = \frac{\pi}{2}$, i.e. the two unit vectors are perpendicular to each other.

Unit 4 Cross Product

1. Find $\vec{A} \times \vec{B}$ if

a) $\vec{A} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{B} = 2\vec{i} - \vec{j} - \vec{k}$

b) $\vec{A} = 2\vec{i} - 3\vec{k}$, $\vec{B} = \vec{i} + \vec{j} - \vec{k}$

Solution:

a)

$$\begin{aligned}
 \vec{A} \times \vec{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & -1 & -1 \end{vmatrix} \\
 &= \vec{i} \cdot \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} \\
 &= 3\vec{i} + 3\vec{j} + 3\vec{k}
 \end{aligned}$$

b)

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -3 \\ 2 & 1 & -1 \end{vmatrix} \\ &= \vec{i} \cdot \begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 3\vec{i} - 4\vec{j} + 2\vec{k}\end{aligned}$$

2. Find the area of the triangle in space having its vertices at the points

$$P : (2, 0, 1), Q : (3, 1, 0), R : (-1, 1, -1).$$

Solution:

$$\begin{aligned}\vec{PQ} &= \langle 1, 1, -1 \rangle \\ \vec{PR} &= \langle -3, 1, -2 \rangle \\ \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -3 & 1 & -2 \end{vmatrix} \\ &= \langle -1, 5, 4 \rangle \\ A &= \frac{|\vec{PQ} \times \vec{PR}|}{2} \\ &= \frac{\sqrt{(-1)^2 + 5^2 + 4^2}}{2} \\ &= \frac{\sqrt{42}}{2}\end{aligned}$$

Therefore, the area of the described triangle is $\frac{\sqrt{42}}{2}$.

3. Two vectors \vec{i}' and \vec{j}' of a right-handed coordinate system are to have the directions respectively of the vectors $\vec{A} = 2\vec{i} - \vec{j}$ and $\vec{B} = \vec{i} + 2\vec{j} + \vec{k}$. Find all three vectors \vec{i}' , \vec{j}' , \vec{k}' .

Solution:

According to the problem description, \vec{i}' is the direction of the vector $\vec{A} = 2\vec{i} - \vec{j}$, hence

$$\begin{aligned}\vec{i}' &= \frac{\vec{A}}{|\vec{A}|} \\ &= \frac{\langle 2, -1, 0 \rangle}{\sqrt{5}} \\ &= \langle \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}, 0 \rangle\end{aligned}$$

Similarly, since \vec{j}' is the direction of the vector $\vec{B} = \vec{i} + 2\vec{j} + \vec{k}$,

$$\begin{aligned}\vec{j}' &= \frac{\vec{B}}{|\vec{B}|} \\ &= \frac{\langle 1, 2, 1 \rangle}{\sqrt{6}} \\ &= \langle \frac{\sqrt{6}}{6}, \frac{2\sqrt{6}}{6}, \frac{\sqrt{6}}{6} \rangle\end{aligned}$$

According to the geometric interpretation of cross product,

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} \\ &= \langle -1, -2, 5 \rangle \\ \vec{k}' &= \text{dir}(\vec{A} \times \vec{B}) \\ &= \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \\ &= \frac{\langle -1, -2, 5 \rangle}{\sqrt{30}} \\ &= \langle -\frac{\sqrt{30}}{30}, -\frac{\sqrt{30}}{15}, \frac{\sqrt{30}}{6} \rangle\end{aligned}$$

4. Verify that the cross product \times does not in general satisfy the associative law, by showing that for the particular vectors $\vec{i}, \vec{j}, \vec{k}$, we have $(\vec{i} \times \vec{j}) \times \vec{k} \neq \vec{i} \times (\vec{j} \times \vec{k})$.

Solution:

Suppose that $\vec{i} = \langle 1, 0, 1 \rangle$, $\vec{j} = \langle 1, 1, 0 \rangle$, and $\vec{k} = \langle 0, 1, 1 \rangle$. Then

$$\begin{aligned}(\vec{i} \times \vec{j}) \times \vec{k} &= \langle 0, 1, -1 \rangle \\ \vec{i} \times (\vec{j} \times \vec{k}) &= \langle 1, 0, -1 \rangle\end{aligned}$$

Hence, the cross product does not in general satisfy the associative law.

5. What can you conclude about \vec{A} and \vec{B}

a) if $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|$;

b) if $|\vec{A} \times \vec{B}| = \vec{A} \cdot \vec{B}$.

Solution:

a) According to the geometric interpretation of cross product, $|\vec{A} \times \vec{B}|$ is the area of the parallelogram formed by the vectors \vec{A} and \vec{B} . Therefore,

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta$$

where θ is the angle between the vectors \vec{A} and \vec{B} .

Then if $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|$,

$$\begin{aligned} |\vec{A} \times \vec{B}| &= |\vec{A}||\vec{B}| \\ |\vec{A}||\vec{B}| \sin \theta &= |\vec{A}||\vec{B}| \\ \sin \theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

We can conclude that the angle between the vectors \vec{A} and \vec{B} is $\frac{\pi}{2}$.

b) Similar to Exercise a), if $|\vec{A} \times \vec{B}| = \vec{A} \cdot \vec{B}$,

$$\begin{aligned} |\vec{A} \times \vec{B}| &= \vec{A} \cdot \vec{B} \\ |\vec{A}||\vec{B}| \sin \theta &= |\vec{A}||\vec{B}| \cos \theta \\ \sin \theta &= \cos \theta \\ \theta &= \frac{\pi}{4} \end{aligned}$$

We can conclude that the angle between the vectors \vec{A} and \vec{B} is $\frac{\pi}{4}$.

6. Find the volume of the tetrahedron having vertices at the four points

$$P : (1, 0, 1), Q : (-1, 1, 2), R : (0, 0, 2), S : (3, 1, -1).$$

Solution:

Part II

- Find the dihedral angle between two faces of a regular tetrahedron.
- Show that the 'polarization identity' $\frac{1}{4} (|\vec{u} + \vec{v}|^2 - |\vec{u} - \vec{v}|^2) = \vec{u} \cdot \vec{v}$ holds for any two n-vectors \vec{u} and \vec{v} . (Use vector algebra, not components.)
 - Given two non-zero vectors \vec{u} and \vec{v} , give the formula for the unit vector which bisects the (smaller) angle between \vec{u} and \vec{v} . (Use the notation $\hat{\vec{u}}$ for the unit vector in the \vec{u} - direction.)
- In this problem we examine tacking, which is the process sailboats use to travel against the wind. Sails are a familiar tool to harness the energy of the wind for transportation over the sea. Early ships had large fixed sails which would capture the wind blowing from behind to propel the ship forward. Even if the wind is blowing behind at an (acute) angle the component of the wind vector perpendicular to the sail will push on the sail and hence on the boat. However, these early fixed sail ships had no way to go against the wind and had to rely on oarsmen if the wind was blowing in the wrong direction. A great advance that allowed boats to sail against the wind was the invention of movable sails in combination with a rudder and a keel. By carefully positioning the sail the boat can be made to sail into the wind - this process is called *tacking*.

As noted before, the component of the wind perpendicular to the sail pushes on the sail and, through it, the boat. The keel only allows the boat to move along its axis. (The rudder is used to turn the boat.) That is, for any force on the boat, only the component along the boat's axis actually pushes the boat.

Described mathematically, the wind vector is first projected on the perpendicular to the sail to get the direction of the force on the sail. This resultant force is projected on the axis of the boat to find the direction the boat is being pushed. By orienting the sail correctly this double projection can result in a vector with a component pointing into the wind.

In the picture $\vec{w} = a\vec{i}$ is the wind direction. The line l_s is perpendicular to the sail (with $0 \leq \alpha \leq \frac{\pi}{2}$). And the line l_B is along the boat's axis (with $0 \leq \beta \leq \frac{\pi}{2}$).

a) Let \vec{w}_1 be the projection of \vec{w} onto the line l_s . Show that \vec{w}_1 does not have a non-zero component in the direction opposite \vec{w} . (It is sufficient to show the projections on the sketch.)

b) Find the projection of \vec{w}_1 onto l_B . (Give an explicit formula in terms of α and β .) What is the condition on α and β that this projection has a component in the $-\vec{i}$ direction? (For warmup you might try the specific case $\alpha = \frac{\pi}{3} = \beta$.)