Lecture 18: Change of Variables

1 Introduction to Change of Variables in Double Integrals

Example 1. *Calculate the area of the ellipse with semiaxes a and b.*
We can use double integrals to calculate the area. By definition of double integrals, the area of the described ellipse

$$\begin{aligned} A &= \iint_R 1 dA \\ &= \iint_{(\frac{x}{a})^2+(\frac{y}{b})^2=1} dxdy \end{aligned}$$

To simplify the equation, we can set

$$\begin{aligned} u &= \frac{x}{a} \\ v &= \frac{y}{b} \end{aligned}$$

Then

$$\begin{aligned} du &= \frac{dx}{a} \\ dv &= \frac{dy}{b} \\ dudv &= \frac{dxdy}{ab} \\ dxdy &= abdudv \end{aligned}$$

Therefore, the equation of the double integral is equivalent to

$$\begin{aligned}
A &= \iint_{(\frac{x}{a})^2+(\frac{y}{b})^2=1} dxdy \\
&= \iint_{u^2+v^2=1} abdudv \\
&= ab \iint_{u^2+v^2=1} dudv \\
&= \pi ab
\end{aligned}$$

Therefore, the area of the ellipse with semiaxes a and b is πab.

From the above example we can see, in general when we change the variables in a double integral, we need to find the scaling factor between $dA = dxdy$ and $dA' = dudv$. In order to ensure the correctness of the double integral after changing the variables, we need to multiply the scaling factor with differentials of changed variables, to ensure the differential of area is still dA rather than dA'. Also note that the scaling factor between dA and dA' might not be constant in the region R.

Question. *Why when we make sure dA is correctly expressed by $dudv$ the double integral with change of variables yields the correct result?*
Answer:
Probably think about it from the definition of double integral, and the geometric interpretation of change of variables.

2 Change of Variables with Linear Transformation

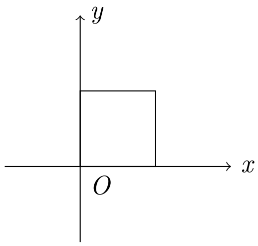
Example 2. *Find the scaling factor between $dA = dxdy$ and $dA' = dudv$ with the following linear transformation:*

$$\begin{cases} u = 3x - 2y \\ v = x + y \end{cases}$$

In the xy-coordinates, $dA = dxdy$ represents a small rectangle. Regarding the geometric interpretation of the linear transformation, we have the following two observations:

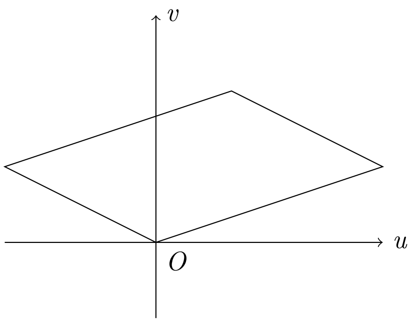
- *A rectangle in the xy-coordinates corresponds to a parallelogram in the uv-coordinates. The ratio of the area of the parallelogram in the uv-coordinates to the area of the rectangle in the xy-coordinates is the area scaling factor.*
- *The area scaling factor of the linear transformation doesn't depend on the choice of rectangles, either the position or the area.*

Based on the above two observations, we can study a special case to find the scaling factor of the linear transformation. The special case we choose is a unit rectangle with the left bottom corner on the origin.



After the linear transformation,

$$\begin{aligned}(0,0) &\to (0,0)\\ (1,0) &\to (3,1)\\ (1,1) &\to (1,2)\\ (0,1) &\to (-2,1)\end{aligned}$$



We can use the determinant to calculate the area of a parallelogram in a plane

$$\begin{aligned}A &= \det(< 3, 1 >, < -2, 1 >)\\ &= \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}\\ &= 5\end{aligned}$$

Therefore, the scaling factor of the linear transformation is 5*:*

$$\begin{aligned}dA' &= 5dA\\ dA &= \frac{1}{5}dA'\\ dxdy &= \frac{1}{5}dudv\end{aligned}$$

Question. *Why the scaling factor in the above example is* 5 *rather than* $\frac{1}{5}$*?*
Answer:

3 Change of Variables in General

For general cases of change of variables:

$$\begin{aligned} u &= u(x, y) \\ v &= v(x, y) \end{aligned}$$

By linear approximation, we have

$$\begin{aligned} \Delta u &\approx u_x \Delta x + u_y \Delta y \\ \Delta v &\approx v_x \Delta x + v_y \Delta y \\ \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} &\approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \end{aligned}$$

Therefore, for any transformation or change of variables, a small rectangle in the original coordinate system corresponds to a parallelogram in the new coordinate system.
The correspondence of the vertices between the small rectange in the original coordinate system and the parallelogram in the new coordinate system is

$$\begin{aligned} (0, 0) &\to (0, 0) \\ (\Delta x, 0) &\to (u_x \Delta x, v_x \Delta x) \\ (0, \Delta y) &\to (u_y \Delta y, v_y \Delta y) \\ (\Delta x, \Delta y) &\to (u_x \Delta x + u_y \Delta y, v_x \Delta x + v_y \Delta y) \end{aligned}$$

Therefore, the area of the parallelogram is

$$\begin{aligned} A &= \det(< u_x \Delta x, v_x \Delta x >, < u_y \Delta y, v_y \Delta y >) \\ &= \begin{vmatrix} u_x \Delta x & v_x \Delta x \\ u_y \Delta y & v_y \Delta y \end{vmatrix} \\ &= (u_x v_y - u_y v_x) \Delta x \Delta y \\ &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \Delta x \Delta y \end{aligned}$$

Therefore, the scaling factor between two coordinates is $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Since the scaling factor consists of partial derivatives, it varies with different positions of the differential of area.
Here we define the Jacobian determinant as

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Then

$$dudv = |J|dxdy = |\frac{\partial(u,v)}{\partial(x,y)}|dxdy$$

Example 3. *Find the scaling factor between the Cartesian coordinates and the polar coordinates.*
According to the definition of the polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Then we can find the Jacobian determinant as

$$\begin{aligned}
J &= \frac{\partial(x,y)}{\partial(r,\theta)} \\
&= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r
\end{aligned}$$

Therefore,

$$dxdy = |\frac{\partial(x,y)}{\partial(r,\theta)}|drd\theta = rdrd\theta$$

Note that the Jacobian determinant of a change of variables works in both ways, which means

$$J_1 = \frac{\partial(x,y)}{\partial(u,v)} = \frac{dxdy}{dudv}$$
$$J_2 = \frac{\partial(u,v)}{\partial(x,y)} = \frac{dudv}{dxdy}$$
$$\frac{dxdy}{dudv} \cdot \frac{dudv}{dxdy} = 1$$
$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

Therefore, the two Jacobian determinants are inverse to each other, and we can choose the easier one to calculate.

Example 4. *Calculate* $\int_0^1 \int_0^1 x^2 y dxdy$ *by changing the variables as*

$$u = x$$
$$v = xy$$

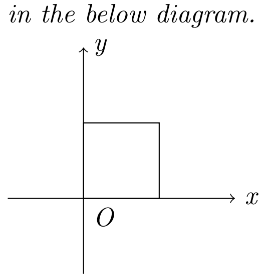
First we need to calculate the differential of area.

$$
\begin{aligned}
J &= \frac{\partial(u,v)}{\partial(x,y)} \\
&= \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} \\
&= x \\
dudv &= |J|dxdy \\
&= |x|dxdy \\
&= xdxdy \ (x \geq 0)
\end{aligned}
$$

Then we need to calculate the integrand

$$
\begin{aligned}
x^2ydxdy &= x^2y\frac{1}{x}dudv \\
&= xydudv \\
&= vdudv
\end{aligned}
$$

Then we need to determine the region of the double integral in the new coordinates.
In the xy-coordinates, the region of the double integral is a rectangle as shown in the below diagram.



After changing the variables, if we calculate the double integral in the order of $dudv$, it means for each possible value of v, we will keep it constant, and vary the value of u.
Therefore, the way we determine the boundary of the region is to find the range of v, and for each value of v, find the possible values of u based on the region in xy-coordinates.
According to the diagram, we can find that the range of v is $(0, 1)$, and for each value of v, the range of u is $(v, 1)$.

Therefore, the double integral after changing the variables is

$$\begin{aligned}
\int_0^1 \int_0^1 x^2 y dx dy &= \int_0^1 \int_v^1 v du dv \\
&= \int_0^1 (uv)|_{u=v}^{u=1} dv \\
&= \int_0^1 v - v^2 dv \\
&= (\frac{v^2}{2} - \frac{v^3}{3})|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$