Probability

Prem Nair

Review: Probability Space, Event

Example 1.

Random Experiment: Roll two dices. What is the probability that they add up to 10?

The probability space for this experiment has two components to it:

1. The sample space: The set of all possible outcomes.

   (1, 1), (1, 2), …, (1, 6), (2, 1), …, (2, 6), (3, 1), …, (6,6).

2. The probabilities of each of these outcomes.

   Each of the outcome is equally likely and they sum up to 1.

   So the probability for each of these outcomes is 1/36.

Event: A subset of the possible outcomes.

Event A consists of all pairs of the form (x , y) such that $x + y = 10$.

$A = \{(4, 6), (5, 5), (6, 4)\}$. $P(A) = 3/36 = 1/12$.

Random variable

Roll two dice. Let X be their sum.

$$\text{outcome} = (1,1) \quad \Rightarrow \quad X = 2$$
$$\text{outcome} = (1,2) \text{ or } (2,1) \quad \Rightarrow \quad X = 3$$

Probability space:

- Sample space: $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.
- Each outcome equally likely.

Random variable X lies in $\{2,3,4,5,6,7,8,9,10,11,12\}$.

A **random variable (r.v.)** is a defined on a probability space.
It is a mapping from Ω (outcomes) to $\mathbb{R}$ (numbers).
We'll use capital letters for r.v.'s.

Expected value

Expected value, or mean

Expected value of a random variable X:

$$\mathbb{E}(X) = \sum_x x \Pr(X = x).$$

Roll a die. Let X be the number observed.
What is $\mathbb{E}(X)$?

Expected value

x	P(x)	x.P(x)
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6

$E(X) = 1/6 + 2/6 + 3/6 + 4/6 + 5/6 +6/6 = 21/6 = 3.5$

E(X) Example 2

A biased coin which turns head with probability p.

Random variable X. x = 1 if head turns up and 0 otherwise. What is E(X)?

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$E(X) = 1*p + 0*(1-p) = p.$

E(X) : Example 3

Throw two dices. X is the sum of the values.

$\mathrm{Range}(X) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$$E(X) = 2*(1/36) + 3*(2/36) + 4*(3/36)$$
$$+ 5*(4/36) + 6*(5/36) + 7*(6/36)$$
$$+ 8*(5/36) + 9*(4/36) + 10*(3/36)$$
$$+ 11*(2/36) + 12*(1/36)$$

Property of $E(X)$

Let $Y = aX + b$.

Then $E(Y) = aE(X) + b$

Independent Random Variables

Two random variables X and Y are independent if

$P(x, y) = P(x)P(y)$ for all x in X and y in Y.

Example

Pick a card from the standard deck of playing cards.

X : the suit (That is, spade, heart, and so on)

Y : value (That is, A, K, Q, J, 10, 9, …, 1)

$P(x) = 1/4$, $P(y) = 1/13$ and $P(x, y) = 1/52 = P(x)P(y)$

Independent Random Variables

Throw a fair coin 3 times.

X : number of heads

Y : 1 if last throw is head and 0 otherwise.

There are 8 cases when a coin is thrown 3 times.

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT,

$P(X = 0) = 1/8$, $P(X = 1) = 3/8$, $P(X = 2) = 3/8$ and $P(X= 3) = 1/8$.

$P(Y = 1) = 1/2$ and $P(Y = 0) = 1/2$.

$P(X = 0, Y = 1) = 0 \neq 1/16 = P(X = 0)P(Y = 1)$

X and Y are NOT independent random variables.

Bernoulli Trials

A Bernoulli trial (or **binomial trial**) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Note 1. Let Probability of Success is p. The probability of Failure is $q = (1 - p)$.

Note 2, Trials are independent.

Example : Throwing a dice. Let us define success if it turns 1 or 6. The $p = 1/3$ and $q = 2/3$.

Result 1

The Expected number (average number) of trials for a success $= \frac{1}{p}$

Proof. The expected number of trials is

$1.p + 2.q.p + 3.q^2.p + 4.q^3.p + \ldots$

Result 1 proof continued.

Let $S = 1.p + 2.q.p + 3.q^2.p + 4.q^3.p + \ldots$

$S.q \quad = \qquad\qquad 1.q.p + 2.q^2.p + 3.q^3.p + \ldots$

$(S - S.q) = p + q.p + q^2.p + q^3.p + \ldots = \frac{p}{(1-q)}$

$S.(1 - q) = 1$ (Since $1 - q = p$)

$S.p = 1.$

$S = \frac{1}{p}$

Result 2.

Expected number (average number) of trials for k successes $= \frac{k}{p}$

Result 3.

Expected number (average number) of failures before k successes $= \frac{kq}{p}$

Proof. Let x be the expected number of failures before k successes. Let y be the expected number of trials for k successes. Then

y = x + k.

By result 2, $y = \frac{k}{p}$.

$x = \frac{k}{p} - k = \frac{k-kp}{p} = \frac{k(1-p)}{p} = \frac{kq}{p}$