A. Rotation of 3D real vectors

A *proper* rotation is completely described by the rotation axis $\mathbf{n}$ (a real unit vector) and the rotation angle $\theta \in \mathbb{R}$, or equivalently described by a real vector $\boldsymbol{\omega}$, corresponding to $\mathbf{n} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ and $\theta \equiv |\boldsymbol{\omega}|$. The natural basis of the 3D space are

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

An arbitrary real vector in 3D space can be written as

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z.$$

The proper rotation of a 3D real vector is described by a 3×3 matrix $\mathbf{g}(\boldsymbol{\omega})$, which brings $\mathbf{v}$ to

$$\mathbf{g}(\boldsymbol{\omega})\mathbf{v} = \mathbf{g}(\boldsymbol{\omega}) \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = v_x\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_x + v_y\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_y + v_z\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_z.$$

Let

$$\mathbf{g}(\boldsymbol{\omega}) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix},$$

we have

$$\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_x = \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix}, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_y = \begin{bmatrix} g_{12} \\ g_{22} \\ g_{32} \end{bmatrix}, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_z = \begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix}.$$

or concisely

$$[\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_x, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_y, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_z] = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]\mathbf{g}(\boldsymbol{\omega}).$$

This gives a physical interpretation for the 3×3 matrix $\mathbf{g}(\boldsymbol{\omega})$, i.e., its ith column gives $\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_i$ ($i = 1, 2, 3$):

$$\mathbf{g}(\boldsymbol{\omega}) = [\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_x, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_y, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_z] = \mathbf{g}(\boldsymbol{\omega})[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z].$$

This allows us to write down the 3×3 rotation matrix of simple rotations. For example, a θ-rotation about the z axis is described by a real vector $\boldsymbol{\omega} = \theta\mathbf{e}_z$. To obtain its 3×3 rotation matrix, we use

$$\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_x = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_y = -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_z = \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

to obtain

$$\mathbf{g}(\theta\mathbf{e}_z) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, a θ-rotation about the x axis described by a real vector $\boldsymbol{\omega} = \theta\mathbf{e}_x$. To obtain its 3×3 rotation matrix, we use

$$\mathbf{g}(\theta\mathbf{e}_x)\mathbf{e}_x = \mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_x)\mathbf{e}_y = \cos\theta\mathbf{e}_y + \sin\theta\mathbf{e}_z = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_x)\mathbf{e}_z = -\sin\theta\mathbf{e}_y + \cos\theta\mathbf{e}_z = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix},$$

to obtain

$$\mathbf{g}(\theta\mathbf{e}_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

A θ-rotation about the y axis described by a real vector $\boldsymbol{\omega} = \theta \mathbf{e}_y$. To obtain its 3×3 rotation matrix, we use

$$\mathbf{g}(\theta\mathbf{e}_y)\mathbf{e}_x = \cos\theta\mathbf{e}_x - \sin\theta\mathbf{e}_z = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_y)\mathbf{e}_y = \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{g}(\theta\mathbf{e}_y)\mathbf{e}_z = \sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_z = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix},$$

to obtain

$$\mathbf{g}(\theta\mathbf{e}_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

A special rotation is

$$\mathbf{g}(\varphi\mathbf{e}_z)\mathbf{g}(\theta\mathbf{e}_y) = \begin{bmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

which brings $\mathbf{e}_z$ to a unit vector with polar angle θ and azimuth angle φ. For a $\pi/2$-rotation about the z axis $\boldsymbol{\omega} = \mathbf{e}_z\pi/2$, we have

$$\mathbf{g}(\frac{\pi}{2}\mathbf{e}_z) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence

$$\begin{aligned} \mathbf{g}(\frac{\pi}{2}\mathbf{e}_z)\mathbf{e}_x &= \mathbf{e}_y, \\ \mathbf{g}(\frac{\pi}{2}\mathbf{e}_z)\mathbf{e}_y &= -\mathbf{e}_x, \\ \mathbf{g}(\frac{\pi}{2}\mathbf{e}_z)\mathbf{e}_z &= \mathbf{e}_z. \end{aligned}$$

The 3×3 rotation matrix for a general rotation $\boldsymbol{\omega}$ is an 3×3 *real orthogonal* matrix:

$$\mathbf{g}(\boldsymbol{\omega}) \equiv e^{-i\mathbf{I}\cdot\boldsymbol{\omega}}$$

satisfying $\mathbf{g}^T(\boldsymbol{\omega})\mathbf{g}(\boldsymbol{\omega}) = \mathbf{g}(\boldsymbol{\omega})\mathbf{g}^T(\boldsymbol{\omega}) = \mathbf{1}$, where

$$\mathbf{I}_x \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \mathbf{I}_y \equiv \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_z \equiv \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

or $(\mathbf{I}_\alpha)_{\beta\gamma} = -i\varepsilon_{\alpha\beta\gamma}$ in short are Hermitian generators. For example, we can readily verify

$$\mathbf{g}(\theta\mathbf{e}_\alpha) = e^{-i\theta\mathbf{I}_\alpha} \ \ (\alpha = x, y, z).$$

In the above we only considered proper rotations, for which $\det \mathbf{g}(\boldsymbol{\omega}) = +1$. In addition to proper rotations, another operation on vectors is spatial inversion $\mathcal{I}$. If a vector $\mathbf{v} = [v_x, v_y, v_z]^T$ changes sign under spatial inversion, then we say $\mathbf{v}$ is a polar vector; if $\mathbf{v}$ remains invariant under spatial inversion, then we call $\mathbf{v}$ an axial vector (or pseudo vector). For example, if $\mathbf{a} = [a_x, a_y, a_z]^T$ and $\mathbf{b} = [b_x, b_y, b_z]^T$ are polar vectors, then $\mathbf{a} \times \mathbf{b}$ is a axial vector.

B. Rotation of operators in orbital space

A proper rotation $\boldsymbol{\omega}$ in the orbital space brings an arbitrary operator A to

$$e^{-i\mathbf{L}\cdot\boldsymbol{\omega}} A e^{i\mathbf{L}\cdot\boldsymbol{\omega}} = \hat{g}(\boldsymbol{\omega}) A \hat{g}^{-1}(\boldsymbol{\omega}),$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital angular momentum operator and $\hat{g}(\boldsymbol{\omega}) = e^{-i\mathbf{L}\cdot\boldsymbol{\omega}}$ is the operator describing the rotation $\boldsymbol{\omega}$ in the orbital space. The position operator

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

has three components and obeys

$$[L_\alpha, r_\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} r_\gamma.$$

The momentum operator

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

has three components and obeys

$$[L_\alpha, p_\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} p_\gamma.$$

The position operator $\mathbf{r}$ and the momentum operator $\mathbf{p}$ are both vector operators in the orbital space: when a three-component operator

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

satisfies

$$[L_\alpha, V_\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} V_\gamma,$$

we call $\mathbf{V}$ a vector operator in the orbital space, such as $\mathbf{r}, \mathbf{p}$, and $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

An arbitrary rotation $\boldsymbol{\omega}$ brings a vector operator $\mathbf{V}$ to

$$\hat{g}(\boldsymbol{\omega}) \mathbf{V} \hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega}) \mathbf{V} \Leftrightarrow \hat{g}(\boldsymbol{\omega}) \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega}) \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}, \tag{1}$$

where $\mathbf{g}(\boldsymbol{\omega})$ is the 3×3 rotation matrix acting on 3D real vectors. Taking matrix transpose gives

$$\hat{g}(\boldsymbol{\omega}) [V_x, V_y, V_z] \hat{g}^{-1}(\boldsymbol{\omega}) = [V_x, V_y, V_z] \mathbf{g}(\boldsymbol{\omega}).$$

Compared with

$$[\mathbf{g}(\boldsymbol{\omega}) \mathbf{e}_x, \mathbf{g}(\boldsymbol{\omega}) \mathbf{e}_y, \mathbf{g}(\boldsymbol{\omega}) \mathbf{e}_z] = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] \mathbf{g}(\boldsymbol{\omega}),$$

we see that $[V_x, V_y, V_z]$ obey exactly the same transformation rule as $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$:

$$[V_x, V_y, V_z] \xrightarrow{\hat{g}(\boldsymbol{\omega})} [V_x, V_y, V_z] \mathbf{g}(\boldsymbol{\omega}),$$

$$[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] \xrightarrow{\mathbf{g}(\boldsymbol{\omega})} [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] \mathbf{g}(\boldsymbol{\omega}).$$

For example, a θ-rotation about the z axis gives

$$\mathbf{g}(\theta\mathbf{e}_z)[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or explicitly

$$\begin{aligned}
\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_x &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \\
\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_y &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y, \\
\mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_z &= \mathbf{e}_z.
\end{aligned}$$

Correspondingly, we have

$$\hat{g}(\boldsymbol{\omega})[V_x, V_y, V_z]\hat{g}^{-1}(\boldsymbol{\omega}) = [V_x, V_y, V_z]\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or explicitly

$$\begin{aligned}
\mathbf{g}(\theta\mathbf{e}_z)V_x &= \cos\theta V_x + \sin\theta V_y, \\
\mathbf{g}(\theta\mathbf{e}_z)V_y &= -\sin\theta V_x + \cos\theta V_y, \\
\mathbf{g}(\theta\mathbf{e}_z)V_z &= V_z.
\end{aligned}$$

This can be written as

$$\hat{g}(\boldsymbol{\omega})(\mathbf{V}\cdot\mathbf{n})\hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{V}\cdot[\mathbf{g}(\boldsymbol{\omega})\mathbf{n}],$$

which follows from $\hat{g}(\boldsymbol{\omega})\mathbf{V}\hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega})\mathbf{V}$ by multiplying $\cdot\mathbf{n}$ on both sides.

In the above we only considered proper rotations, for which $\det\mathbf{g}(\boldsymbol{\omega}) = +1$. In addition to proper rotations, another operation on operators is spatial inversion $\mathcal{I}$ (which is a unitary operator):

$$\begin{aligned}
\mathcal{I}\mathbf{r}\mathcal{I}^{-1} &= -\mathbf{r}, \\
\mathcal{I}\mathbf{p}\mathcal{I}^{-1} &= -\mathbf{p}, \\
\mathcal{I}\mathbf{L}\mathcal{I}^{-1} &= \mathbf{L}.
\end{aligned}$$

If a vector operator $\mathbf{V} = [V_x, V_y, V_z]^T$ changes sign under spatial inversion, then we say $\mathbf{V}$ is a polar vector operator; if $\mathbf{V}$ remains invariant under spatial inversion, then we call $\mathbf{V}$ an axial vector (or pseudo vector) operator. For example, if $\mathbf{A} = [A_x, A_y, A_z]^T$ and $\mathbf{B} = [B_x, B_y, B_z]^T$ are polar vectors, then $\mathbf{A} \times \mathbf{B}$ is a axial vector operator. An example is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, which is an axial vector operator.

Finally, we derive Eq. (1). The rotation of $\mathbf{V} = [V_x, V_y, V_z]^T$ by $\delta\boldsymbol{\omega}$ gives

$$e^{-i\mathbf{L}\cdot\delta\boldsymbol{\omega}}\mathbf{V}\cdot\mathbf{m}e^{i\mathbf{L}\cdot\delta\boldsymbol{\omega}} \approx \mathbf{V}\cdot\mathbf{m} - i[\mathbf{L}\cdot\delta\boldsymbol{\omega}, \mathbf{V}\cdot\mathbf{m}] = \mathbf{V}\cdot\mathbf{m} + \mathbf{V}\cdot(\delta\boldsymbol{\omega}\times\mathbf{m}) + O(\delta\omega^2),$$

where we have used

$$[\mathbf{L}\cdot\mathbf{n}, \mathbf{V}\cdot\mathbf{m}] = i(\mathbf{n}\times\mathbf{m})\cdot\mathbf{V}.$$

In particular, for $\delta\boldsymbol{\omega} = \mathbf{e}_x\delta\theta$, we have

$$\begin{aligned} e^{-i\delta\theta L_x}V_x e^{i\delta\theta L_x} &= V_x, \\ e^{-i\delta\theta L_x}V_y e^{i\delta\theta L_x} &\approx V_y + \delta\theta V_z, \\ e^{-i\delta\theta L_x}V_z e^{i\delta\theta L_x} &\approx V_z - \delta\theta V_y. \end{aligned}$$

In terms of the 3×3 matrice $(\mathbf{I}_\alpha)_{\beta\gamma} = -i\varepsilon_{\alpha\beta\gamma}$ obeying $[\mathbf{I}_\alpha, \mathbf{I}_\beta] = i\varepsilon_{\alpha\beta\gamma}\mathbf{I}_\gamma$, we have

$$\begin{aligned} e^{-i\delta\theta L_x}[V_x, V_y, V_z]e^{i\delta\theta L_x} &\approx [V_x, V_y, V_z]e^{-i\delta\theta\mathbf{I}_x}, \\ e^{-i\delta\theta L_y}[V_x, V_y, V_z]e^{i\delta\theta L_y} &\approx [V_x, V_y, V_z]e^{-i\delta\theta\mathbf{I}_y}, \\ e^{-i\delta\theta L_z}[V_x, V_y, V_z]e^{i\delta\theta L_z} &\approx [V_x, V_y, V_z]e^{-i\delta\theta\mathbf{I}_z}, \end{aligned}$$

and hence

$$\begin{aligned} e^{-i\theta L_x}[V_x, V_y, V_z]e^{i\theta L_x} &= [V_x, V_y, V_z]e^{-i\theta\mathbf{I}_x}, \\ e^{-i\theta L_y}[V_x, V_y, V_z]e^{i\theta L_y} &= [V_x, V_y, V_z]e^{-i\theta\mathbf{I}_y}, \\ e^{-i\theta L_z}[V_x, V_y, V_z]e^{i\theta L_z} &= [V_x, V_y, V_z]e^{-i\theta\mathbf{I}_z}. \end{aligned}$$

and more generally

$$e^{-i\mathbf{L}\cdot\boldsymbol{\omega}}[V_x, V_y, V_z]e^{i\mathbf{L}\cdot\boldsymbol{\omega}} = [V_x, V_y, V_z]e^{-i\mathbf{I}\cdot\boldsymbol{\omega}}.$$

C. Rotation of operators in spin space

A rotation $\boldsymbol{\omega}$ in the orbital space brings an arbitrary operator A to

$$e^{-i\mathbf{s}\cdot\boldsymbol{\omega}}Ae^{i\mathbf{s}\cdot\boldsymbol{\omega}} = \hat{g}(\boldsymbol{\omega})A\hat{g}^{-1}(\boldsymbol{\omega}),$$

where

$$\mathbf{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

is the three-component spin operator (not necessarily spin-1/2) and $\hat{g}(\boldsymbol{\omega}) \equiv e^{-i\mathbf{s}\cdot\boldsymbol{\omega}}$ is the operator describing the rotation $\boldsymbol{\omega}$ in the spin space. The three Cartesian components of the spin operator obeys

$$[s_\alpha, s_\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} s_\gamma.$$

The spin operator is a vector operator in the spin: when a three-component operator

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

satisfies

$$[s_\alpha, V_\beta] = i \sum_\gamma \varepsilon_{\alpha\beta\gamma} V_\gamma,$$

we call $\mathbf{V}$ a vector operator in the spin space, such as $\mathbf{s}$.

An arbitrary rotation $\boldsymbol{\omega}$ brings a vector operator $\mathbf{V}$ to

$$\hat{g}(\boldsymbol{\omega})\mathbf{V}\hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega})\mathbf{V} \Leftrightarrow \hat{g}(\boldsymbol{\omega}) \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega}) \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix},$$

where $\mathbf{g}(\boldsymbol{\omega})$ is the 3×3 rotation matrix acting on 3D real vectors. Taking matrix transpose gives

$$\hat{g}(\boldsymbol{\omega})[V_x, V_y, V_z]\hat{g}^{-1}(\boldsymbol{\omega}) = [V_x, V_y, V_z]\mathbf{g}(\boldsymbol{\omega}).$$

Compared with

$$[\mathbf{g}(\boldsymbol{\omega})\mathbf{e}_x, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_y, \mathbf{g}(\boldsymbol{\omega})\mathbf{e}_z] = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]\mathbf{g}(\boldsymbol{\omega}),$$

we see that $[V_x, V_y, V_z]$ obey exactly the same transformation rule as $[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]$:

$$[V_x, V_y, V_z] \xrightarrow{\hat{g}(\boldsymbol{\omega})} [V_x, V_y, V_z]\mathbf{g}(\boldsymbol{\omega}),$$

$$[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] \xrightarrow{\mathbf{g}(\boldsymbol{\omega})} [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z]\mathbf{g}(\boldsymbol{\omega}).$$

For example, a θ-rotation about the z axis gives

$$\mathbf{g}(\theta\mathbf{e}_z)[\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] = [\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or explicitly

$$\begin{aligned} \mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_x &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \\ \mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_y &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y, \\ \mathbf{g}(\theta\mathbf{e}_z)\mathbf{e}_z &= \mathbf{e}_z. \end{aligned}$$

Correspondingly, we have

$$\hat{g}(\boldsymbol{\omega})[V_x, V_y, V_z]\hat{g}^{-1}(\boldsymbol{\omega}) = [V_x, V_y, V_z] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

or explicitly

$$\begin{aligned} \mathbf{g}(\theta\mathbf{e}_z)V_x &= \cos\theta V_x + \sin\theta V_y, \\ \mathbf{g}(\theta\mathbf{e}_z)V_y &= -\sin\theta V_x + \cos\theta V_y, \\ \mathbf{g}(\theta\mathbf{e}_z)V_z &= V_z. \end{aligned}$$

This can be written as

$$\hat{g}(\boldsymbol{\omega})(\mathbf{V}\cdot\mathbf{n})\hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{V}\cdot[\mathbf{g}(\boldsymbol{\omega})\mathbf{n}],$$

which follows from $\hat{g}(\boldsymbol{\omega})\mathbf{V}\hat{g}^{-1}(\boldsymbol{\omega}) = \mathbf{g}^{-1}(\boldsymbol{\omega})\mathbf{V}$ by multiplying $\cdot\mathbf{n}$ on both sides.

In the above we only considered proper rotations, for which $\det\mathbf{g}(\boldsymbol{\omega}) = +1$. In addition to proper rotations, another operation on operators is spatial inversion $\mathcal{I}$ (which is a unitary operator):

$$\mathcal{I}\mathbf{s}\mathcal{I}^{-1} = \mathbf{s}.$$

If a vector operator $\mathbf{V} = [V_x, V_y, V_z]^T$ changes sign under spatial inversion, then we say $\mathbf{V}$ is a polar vector operator; if $\mathbf{V}$ remains invariant under spatial inversion, then we call $\mathbf{V}$ an axial vector (or pseudo vector) operator. For example, if $\mathbf{A} = [A_x, A_y, A_z]^T$ and $\mathbf{B} = [B_x, B_y, B_z]^T$ are polar vectors, then $\mathbf{A}\times\mathbf{B}$ is a axial vector operator. An example is $\mathbf{s}$, which is an axial vector operator.

A spin+orbital joint rotation around the axis $\boldsymbol{\omega}$ by an angle $|\boldsymbol{\omega}|$ brings an operator A to

$$e^{-i\mathbf{J}\cdot\boldsymbol{\omega}}Ae^{i\mathbf{J}\cdot\boldsymbol{\omega}} = \hat{g}_{\text{orb}}(\boldsymbol{\omega})\hat{g}_{\text{s}}(\boldsymbol{\omega})A\hat{g}_{\text{s}}^{-1}(\boldsymbol{\omega})\hat{g}_{\text{orb}}^{-1}(\boldsymbol{\omega}),$$

where $\mathbf{J} = \mathbf{L} + \mathbf{s}$.

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$\mathbf{K}_+$ valley:

$$\hat{H} = v_x p_x \sigma_x + v_y p_y \sigma_y + v_t p_z \sigma_0.$$

Isotropic case:

$$\hat{H} = v(p_x\sigma_x + p_y\sigma_y).$$

$\hat{H}$ is invariiant under any joint rotation of spin+orbital around the z axis:

$$\hat{H} = v\mathbf{p}\cdot\sigma$$

$$\mathbf{r}|\mathbf{R}\rangle = \mathbf{R}|\mathbf{R}\rangle,$$

$$\hat{g}|\mathbf{R}\rangle = |g\mathbf{R}\rangle,$$

$$f(\mathbf{r})|\mathbf{R}\rangle =$$

$$\mathbf{r}(\hat{g}_{\text{orb}}|\mathbf{R}\rangle) = \hat{g}_{\text{orb}}(\hat{g}_{\text{orb}}^{-1}\mathbf{r}\hat{g}_{\text{orb}})|\mathbf{R}\rangle = \hat{g}_{\text{orb}}(g\mathbf{r})|\mathbf{R}\rangle = \hat{g}_{\text{orb}}(g\mathbf{r})|\mathbf{R}\rangle = (g\mathbf{R})(\hat{g}_{\text{orb}}|\mathbf{R}\rangle) \Rightarrow \hat{g}_{\text{orb}}|\mathbf{R}\rangle = |g\mathbf{R}\rangle.$$

$$\hat{g}_{orb}\hat{g}_s\hat{H}\hat{g}_s^{-1}\hat{g}_{orb}^{-1} = v\hat{g}_{orb}\mathbf{p}\hat{g}_{orb}^{-1}\cdot\hat{g}_s\sigma\hat{g}_s^{-1} = v(g^{-1}\mathbf{p})\cdot(g^{-1}\sigma) = v\mathbf{p}\cdot\sigma$$

Define $\hat{g}(\boldsymbol{\omega}) = \hat{g}_{orb}(\boldsymbol{\omega})\hat{g}_s(\boldsymbol{\omega})$, then

$$\hat{g}(\boldsymbol{\omega})\hat{H}\hat{g}^{-1}(\boldsymbol{\omega}) = \hat{H}$$

for any $\boldsymbol{\omega} = \theta\mathbf{e}_z$ **for any** θ.

$$g_s = e^{-i\theta s_z} = e^{-i\theta(\sigma_z/2)}.$$

$$e^{-i\theta(\sigma_z/2)}|\uparrow\rangle = e^{-i\theta/2}|\uparrow\rangle$$

$$\begin{aligned}\langle\mathbf{R}_1,\alpha|G|\mathbf{R}_2,\beta\rangle &= \left\langle\hat{g}(\mathbf{R}_1,\alpha)|\hat{g}G\hat{g}^{-1}\hat{g}|\mathbf{R}_2,\beta\right\rangle \\ &= \left\langle(g\mathbf{R}_1),(\hat{g}_s\alpha)|\hat{g}G\hat{g}^{-1}|g\mathbf{R}_2,(\hat{g}_s\beta)\right\rangle\end{aligned}$$

$$\langle\mathbf{R}_1,A|G|\mathbf{R}_2,A\rangle = \langle(g\mathbf{R}_1),A|G|g\mathbf{R}_2,A)\rangle$$

$$\langle\mathbf{R}_1,\alpha|G|\mathbf{R}_2,\alpha\rangle = \langle(g\mathbf{R}_1),\alpha|G|g\mathbf{R}_2,\alpha)\rangle$$

$$G_{\alpha\alpha}(\mathbf{R}) = G_{\alpha\alpha}(g\mathbf{R})$$

$$\langle \mathbf{R}_1, A|G|\mathbf{R}_2, B\rangle = e^{i\theta} \langle (g\mathbf{R}_1), A|G|g\mathbf{R}_2, B\rangle \,,$$

$$e^{-i\theta} G_{AB}(R\mathbf{e}_x) = G_{AB}(Rg\mathbf{e}_x).$$

$$G_{AB}(\mathbf{R}) = G_{AB}(R\mathbf{e}_x) e^{-i\theta_{\mathbf{R}}},$$

$$\mathbf{R} \equiv Rg\mathbf{e}_x$$

$$G_{AB}(\mathbf{R}) = H_1(k_F R) e^{-i\theta_{\mathbf{R}}}$$

Suppose $\mathbf{R}$ is obtained from $\mathbf{R}_0$ by a θ-rotation about z axis, then

$$G_{AB}(\mathbf{R}) = G_{AB}(\mathbf{R}_0) e^{-i\theta}.$$

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