Contour integral and Laplace transform

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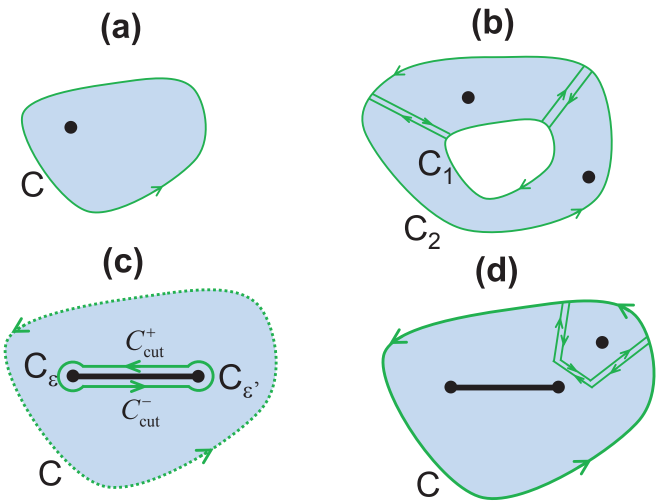

FIG. 1 Residue theorem for contour integral in a finite analytical region (shaded region).

CONTENTS

I. CONTOUR INTEGRAL

For the function $f(z)$ of a complex variable $z \equiv x + iy$ to be differentiable or equivalently analytic, the dependence of f on x and y must come entirely from its dependence on z, thus

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

This is **Cauchy-Riemann equation** – the necessary condition for $f(z)$ to be analytic. When $\partial f/\partial x$ and $\partial f/\partial y$ are continuous, Cauchy-Riemann equation ensures $f(z)$ is analytic. When a function is multi-valued, we can define a principal branch, e.g., for $\alpha \neq$ integers, $(\cdots)^\alpha$ is multi-valued. We can define the branch cut along the $-x$ axis, then the principal branch is $\arg(\cdots) \in [-\pi, \pi]$ and $(\cdots)^\alpha$ is analytic and single-valued in the complex plane excluding the cut. On the cut, $(\cdots)^\alpha$ is two-valued. For example, in the principal branch, we have $(\pm i)^\alpha = (e^{\pm i\pi/2})^\alpha = e^{\pm i\alpha\pi/2}$. For $\arg z_1$ and $\arg z_2$ in the principal branch, we have $z^\alpha z^\beta = z^{\alpha+\beta}$, but $(z_1 z_2)^\alpha$ is not necessarily equal to $z_1^\alpha z_2^\alpha$, unless $\arg z_1 + \arg z_2$ is also in the principal branch. For example, when $z_1 = z_2 = e^{i\pi/4}$, we have $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha = e^{i\pi\alpha/2}$. However, when $z_1 = z_2 = e^{i3\pi/4}$, we have $(z_1 z_2)^\alpha = e^{-i\alpha\pi/2}$, but $z_1^\alpha z_2^\alpha = e^{i3\alpha\pi/2}$.

A. Residue theorem for individual poles

The arc with radius R around z_0 is parametrized by $z - z_0 = Re^{i\varphi}$ with polar angle $\varphi \in [\varphi_1, \varphi_2]$, so $\int dz = \int ie^{i\varphi}Rd\varphi$. For a counter-clockwise loop, we have

$$\oint \frac{dz}{(z - z_0)^n} = 2\pi i \cdot \delta_{n,1}. \tag{1}$$

This leads to the residue theorem [Fig. 1(a)]: the contour integral along a *counter-clockwise* loop C enclosing an analytical region of $f(z)$ is determined by all the residues in this region (shaded region):

$$\oint_C f(z)dz = 2\pi i \sum_{\text{enclosed by } C} \mathrm{Res} f(z).$$

For the loop in Fig. 1(b), the integral is determined by all the residues enclosed between C_1 and C_2 (shaded region):

$$\left(\oint_{C_2} + \oint_{C_1}\right) f(z)dz = 2\pi i \sum_{\text{enclosed by } C_1 + C2} \mathrm{Res} f(z),$$

because $C_2 + C_1$ can be continuously deformed to individual counter-clockwise loops around each pole enclosed by C_1

and C_2. The pole z_p is of the nth order if $\lim_{z\to z_\mathrm{p}}(z-z_\mathrm{p})^n f(z)$ is nonzero and finite. The residue of a nth-order pole z_p is calculated from

$$\mathrm{Res} f(z_\mathrm{p}) = \left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_\mathrm{p})^n f(z)\right]_{z\to z_\mathrm{p}}.$$

B. Contour integral around branch cuts

As shown in Fig. 1(c), the contour integral along a counter-clockwise loop C enclosing a branch cut C_cut can be continuously deformed into the integral along the two edges of the cut and around the two branch points:

$$\oint_C f(z)dz = \left(\int_{C_\mathrm{cut}^+} + \int_{C_\mathrm{cut}^-} + \int_{C_\varepsilon} + \int_{C_{\varepsilon'}}\right) f(z)dz.$$

We take the direction of C_cut to be the same as C_cut^-, then

$$\left(\int_{C_\mathrm{cut}^-} + \int_{C_\mathrm{cut}^+}\right) f(z)dz = \int_{C_\mathrm{cut}} [f_-(z) - f_+(z)]\, dz,$$

where $f_+(z)$ $[f_-(z)]$ is the analytic branch of $f(z)$ above (below) the branch cut. Namely, the integral along the edges of the branch cut is determined by the discontinuity of $f(z)$ across this cut. When the counter-clockwise loop C also encloses discrete poles [Fig. 1(d)], it can be deformed into loops around each branch cut and around each pole.

In the presence of branch points, the poles and their residues may depend on the choice of the branch cuts. As an example, we consider $f(z) = \sqrt{z}/(z+i)$. If we define the branch cut as $x \in [0, +\infty)$, then the principal branch is $\arg z \in [0, 2\pi]$, the pole is $z_\mathrm{p} = e^{i3\pi/2}$ and $\mathrm{Res} f(z_\mathrm{p}) = \sqrt{z_\mathrm{p}} = e^{i3\pi/4}$. By contrast, if we define the branch cut as $x \in (-\infty, 0]$, then the principal branch is $\arg z \in [-\pi, \pi]$, the pole is $z_\mathrm{p} = e^{-i\pi/2}$ and $\mathrm{Res} f(z_\mathrm{p}) = \sqrt{z_\mathrm{p}} = e^{-i\pi/4}$. Therefore, before applying the residue theorem, we must first define all branch cuts to make the integrand analytic on the principal plane.

C. Residue at infinity

When $f(z)$ is analytic near $z \equiv \infty$ (i.e., outside a sufficiently large circle), we can expand $f(z)$ near $z \equiv \infty$ as

$$f(z) = \sum_n a_n^{(\infty)} z^n.$$

The integral along a *clockwise* contour C_∞ (radius $R \to \infty$) in the vicinity of $z = \infty$ follows from Eq. (1) as

$$\oint_{C_\infty} f(z)dz = -2\pi i a_{-1}^{(\infty)} \equiv 2\pi i \mathrm{Res} f(\infty), \tag{2}$$

where $\mathrm{Res} f(\infty) \equiv -a_{-1}^{(\infty)}$ is the residue at $z = \infty$. If, in addition, $f(z)$ only has discrete poles $\{z_\mathrm{p}\}$ and no branch cuts [see Fig. 2(a)], then

$$\oint_{C_\infty} f(z)dz = 2\pi i \mathrm{Res} f(\infty) = -2\pi i \sum_\mathrm{p} \mathrm{Res} f(z_\mathrm{p}).$$

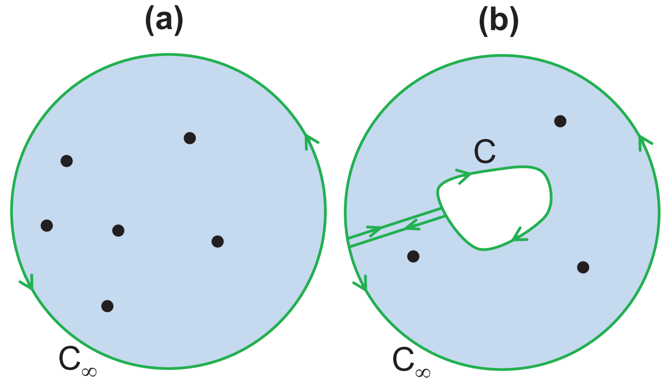

FIG. 2 Residue theorem for contour integral enclosing an infinite analytical region (shaded).

Therefore, if $f(z)$ is analytic over the entire plane (except for discrete poles), then the sum of all the residues (including the one at infinity) is equal to zero.

As shown in Fig. 2(b), if $f(z)$ is analytic outside contour C, then we can add an infinite contour C_∞ (with $R \to \infty$), so that

$$\left(\oint_C + \oint_{C_\infty}\right) f(z)dz = 2\pi i \sum_\mathrm{p} \mathrm{Res} f(z_\mathrm{p}),$$

because $C + C_\infty$ can be continuously deformed into contours around each pole. On the other hand, since $f(z)$ is analytic at $z = \infty$, we have $\oint_{C_\infty} f(z)dz = -2\pi i \mathrm{Res} f(\infty)$, thus

$$\oint_C f(z)dz = 2\pi i \sum_\mathrm{p} \mathrm{Res} f(z_\mathrm{p}) + 2\pi i \mathrm{Res} f(\infty).$$

This the residue theorem for an infinite analytical domain.

To calculate $a_{-1}^{(\infty)}$, we let $\zeta \equiv 1/z$, then

$$f(z) = \sum_n a_{-n}^{(\infty)} \zeta^n, \tag{3}$$

i.e., $a_{-1}^{(\infty)}$ is the coefficient of the linear ζ term. If $\zeta = 0$ is a mth-order pole, i.e., the sum runs over $n \geq -m$, then

$$a_{-1}^{(\infty)} = \frac{1}{(m+1)!}\left(\frac{d^{m+1}}{d\zeta^{m+1}}\zeta^m f(z)\right)_{\zeta=0}.$$

To understand the residue theorem at infinity, we notice that if $z = e^{i\varphi}R$, then $\zeta = 1/z = (1/R)e^{-i\varphi}$, i.e., the large clockwise contour C_∞ in the complex z plane becomes a small *counter-clockwise* contour C_ε in the complex ζ plane. Using $dz = -d\zeta/\zeta^2$, Eq. (2) becomes

$$\oint_C f(z)dz = -\oint_{C_\varepsilon} \frac{f(z)}{\zeta^2} d\zeta = -2\pi i a_{-1}^{(\infty)}$$

since the residue of $f(z)/\zeta^2$ at $\zeta = 0$ is $a_{-1}^{(\infty)}$.

D. Contour integral: lemma

In contour integral, the general principle is to combine the target contour with *auxiliary* contours to form a closed contour in the analytic domain of the integrand and then use the

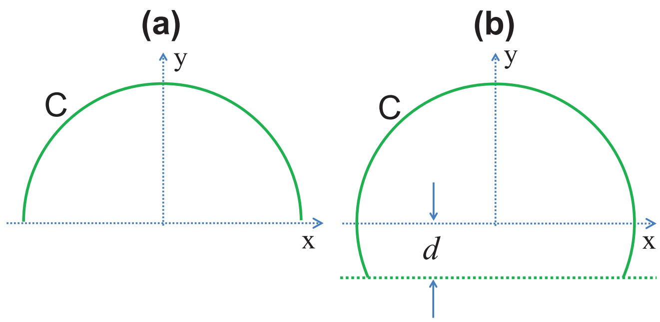

FIG. 3 (a) Jordan lemma and (b) generalized Jordan lemma.

residue theorem to evaluate the integral along this closed contour. Auxiliary contours include those that can be easily evaluated (e.g., infinite arcs) or those connected to the target integral via simple relations (e.g., integral along branch cuts). Below we introduce some rules for the integral to vanish along certain contours.

For $zf(z) \to 0$ on an arc C,

$$\int_C f(z)dz = \int_C zf(z)\frac{dz}{z} \le |zf(z)|_{\max} \int_C \frac{|dz|}{|z|} \to 0,$$

i.e., the contour integral vanishes if $f(z)$ decays to zero faster than $1/z$. Essentially, $f(z) = zf(z) \times 1/z$ can provide a decay factor $1/z$ to make the integral measure $\int_C (1/|z|)|dz|$ bounded, so the integral vanishes when $zf(z) \to 0$ on the contour.

Jordan lemma: if $f(z) \to 0$ and $\mathrm{Im}\beta z \ge 0$ ($\beta \in \mathbb{C}$ is *nonzero*) on an infinite arc C (i.e., radius $R \to \infty$), then

$$\begin{aligned}\int_C f(z)e^{i\beta z}dz &\le |f(z)|_{\max}\left|\int_{\mathrm{Im}\beta z\ge 0} e^{-\mathrm{Im}\beta z}dz\right| \\ &= |f(z)|_{\max}\left|\int_{\sin\varphi\ge 0} e^{-|\beta|R\sin\varphi}Rd\varphi\right| \to 0,\end{aligned}$$

where the dominant contribution to the φ integral comes from $\sin\varphi \approx 0$. Essentially, the exponential decay factor $e^{-\mathrm{Im}\beta z}$ makes the integral measure $\int_C e^{-\mathrm{Im}\beta z}dz$ bounded, so the contour integral vanishes when $f(z) \to 0$ on the contour. For example, if β is positive, then Jordan lemma is applicable for C being an infinite arc in the upper half plane [Fig. 3(a)].

Generalized Jordan lemma: the above result remains valid even when C includes a *finite* segment with negative but *finite* $\mathrm{Im}\beta z$, because these two conditions makes the contribution from this segment to the integral measure $\int e^{-\mathrm{Im}\beta z}dz$ bounded. For example, if β is positive, then Jordan lemma is applicable to C shown in Fig. 3(b) if $|\mathrm{Im}\, z|_{\max} = d$ is finite.

Before constructing the closed contour, we should analyze the analytic behavior of the integrand $f(z)$, including its poles, branch cuts, and in particular, the region at which $f(z)$ decays quickly to zero at infinity.

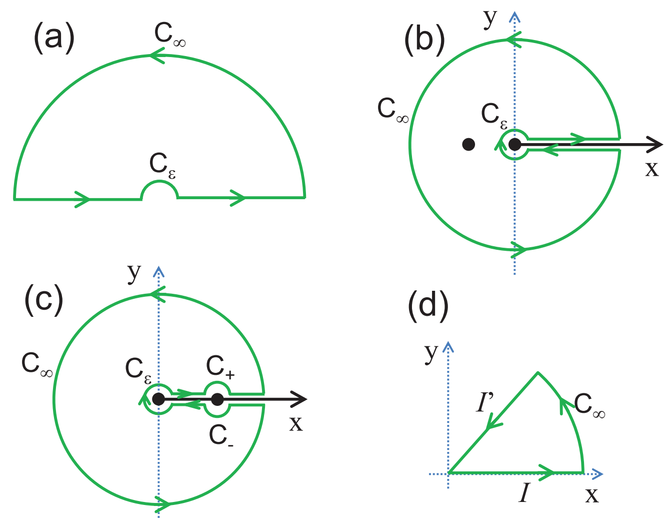

FIG. 4 Contour for several examples.

E. Contour integral: examples

$$\textbf{Example 1}: \quad I \equiv \int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx.$$

Here $f(z) = e^{iz}/z$ quickly decays to zero in the upper half plane, so an arbitrary infinite arc in the upper half plane can be used as auxiliary contours (with vanishing contribution). This motivates the contour in Fig. 4(a):

$$0 = \oint e^{iz}\frac{dz}{z} = I + \frac{1}{2i}\int_{C_\varepsilon} e^{iz}\frac{dz}{z} = I + \frac{1}{2i}\int_\pi^0 id\varphi = I - \frac{\pi}{2},$$

thus $I = \pi/2$.

$$\textbf{Example 2}: \quad I \equiv \int_0^\infty \frac{x^\alpha}{x(x+1)}dx \;\; (0 < \alpha < 1).$$

Here $f(z) = z^\alpha/[z(z+1)]$ decays faster than $1/z$ in the whole plane, so an arbitrary infinite arc can be used as an auxiliary contour with vanishing contribution. $f(z)$ also has a branch point $z = 0$ and a first-order pole on the x axis. We choose the branch cut $x \in [0, +\infty)$ [Fig. 4(b)], then the principle branch is $\arg z \in [0, 2\pi]$. In the upper edge $\arg z = 0$ and $f_+(x) = f(x)$. In the lower edge $\arg z = 2\pi$ and $f_-(x) = e^{2\pi\alpha i}f(x)$. Thus the integral along the two edges of the cut gives $I - e^{2\pi\alpha i}I$. This motivates the contour in Fig. 4(b). This contour encloses a first-order pole $z_\mathrm{p} = e^{\pi i}$ with $\mathrm{Res}f(z_\mathrm{p}) = -e^{\pi\alpha i}$, so we can use the residue theorem to obtain

$$\oint f(z)dz = 2\pi i\mathrm{Res}f(z_\mathrm{p}).$$

On the other hand, the entire contour consists of C_∞, C_ε and the edges of the branch cuts. The integral along C_∞ and C_ε vanish, so $\oint f(z)dz = I(1 - e^{2\pi\alpha i})$, which gives $I = \pi/\sin(\pi\alpha)$.

Example 3: $\quad I \equiv \int_0^\infty \frac{x^\alpha}{x(x-1)}dx \quad (0 < \alpha < 1).$

Here $f(z) = z^\alpha/[z(z-1)]$ is similar to the previous example, so we still define the branch cut as $x \in [0, +\infty)$ and choose the contour in Fig. 4(c). By the residue theorem, the entire contour integral vanishes. On the other hand, this integral consists of $C_\varepsilon, C_\infty, C_+, C_-$, and the two edges of the branch cut. The integral along C_∞ and C_ε vanish. The integral along the two edges gives $I - e^{2\pi\alpha i}I$. Along the contour C_+, we have $z = e^{i0}$, so $\int_{C_+} f(z)dz = -\pi i$. Along the contour C_-, we have $z = e^{i2\pi}$, so $\int_{C_-} f(z)dz = -\pi i e^{2\pi i\alpha}$. Therefore, we obtain $I = -\pi \cot(\pi\alpha)$.

Example 4: $\quad I \equiv \int_{-\infty}^{+\infty} \frac{e^{\alpha x}}{1+e^x}dx \quad (0 < \alpha < 1).$

Here $f(z) = e^{\alpha z}/(1 + e^z)$ decays to zero exponentially when $\mathrm{Re}\, z \to \pm\infty$ and has an infinite number of poles at $z = (2n + 1)\pi i$ $(n \in \mathbb{Z})$ along the y axis. Further, $f(z+2\pi i) = e^{2\pi i\alpha} f(z)$, so we choose a rectangular contour consisting of the right-going $y = 0$ line, the left-going $y = 2\pi$ line, and two vertical lines at $\mathrm{Re}\, z \to \pm\infty$. The entire contour integral is

$$2\pi i \mathrm{Res} f(\pi i) = I - e^{2\pi i\alpha}I,$$

where $\mathrm{Res} f(\pi i) = -e^{\alpha\pi i}$. Thus we obtain $I = \pi/\sin(\pi\alpha)$.

Example 5: $\quad I \equiv \int_0^\infty e^{ix^2}dx.$

Here $f(z) = e^{iz^2}$ decays to zero as $|f(z)| = e^{-|z|^2\sin(2\varphi)}$ at infinity for $\varphi \equiv \arg z \in [0, \pi/2]$. The integral along the ray $z = Re^{i\pi/4}$ $(R \in [0, \infty])$ can be easily obtained:

$$I' = -e^{i\pi/4}\int_0^\infty e^{-R^2}dR = -e^{i\pi/4}\sqrt{\frac{\pi}{2}}.$$

This motivates the contour in Fig. 4(d). The entire contour integral vanishes. The integral along C_∞ is dominated by contributions near the x axis:

$$\int_{C_\infty} f(z)dz \le \left|\int_0^{\pi/4} e^{-R^2\sin(2\varphi)}Rd\varphi\right| = \left|\int_0^{\pi/4} e^{-2R^2\varphi}Rd\varphi\right| \to 0.$$

Therefore, we obtain $I = e^{i\pi/4}\sqrt{\pi/2}$.

Example 6: $\quad I = \int_0^\infty \frac{\ln x}{(1+x^2)^2}dx.$

Here $f(z) = (\ln z)/(1+z^2)^2$ has a branch point $z = 0$ and two 2nd-order poles on the y axis. Normally, we would define the branch cut as $x \in [0, +\infty)$ and choose the contour in Fig. 4(b), so the principal branch is $\arg z \in [0, 2\pi]$. In the upper edge $\arg z = 0$ and $f_+(x) = f(x)$. In the lower edge $\arg z = 2\pi$ and

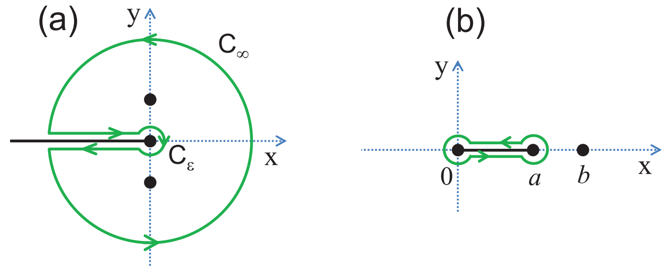

FIG. 5 Contour for further examples.

$f_-(x) = 2\pi i/(1 + x^2)^2 + f(x)$. The discontinuity $f_-(x) - f_+(x)$ is not proportional to $f(x)$, so the integral of $f(z)$ along the edges of the branch cut does not contain the target integeral I. To calculate I, we should find a function $F(z)$ such that its discontinuity across its branch cut is proportional to $f(z)$. This function turns out to be

$$F(z) \equiv \frac{(\ln z)^2}{(1+z^2)^2},$$

which has the same poles and branch points as $f(z)$.

We define the branch cut $x \in [0, +\infty)$. The principal branch is $\arg z \in [0, 2\pi]$. On the upper edge of the cut $\arg z = 0$ and

$$F_+(x) = \frac{(\ln x)^2}{(1+x^2)^2}.$$

On the lower edge of the cut $\arg z = 2\pi$ and

$$F_-(x) = \frac{(\ln x + 2\pi i)^2}{(1+x^2)^2},$$

so the discontinuity across the branch cut

$$F_-(x) - F_+(x) = \frac{(2\pi i)^2}{(1+x^2)^2} + 4\pi i f(x)$$

contains $f(x)$. This motivates the closed contour in Fig. 4(b):

$$\oint F(z)dz = 2\pi i[\mathrm{Res} f(z_+) + \mathrm{Res} f(z_-)],$$

where $z_+ = e^{i\pi/2}$ and $z_- = e^{i3\pi/2}$ are 2nd-order poles and

$$\mathrm{Res} f(z_+) = \left[\frac{d}{dz}(z-z_+)^2 f(z)\right]_{z\to z_+} = \frac{i\pi}{8}\left(2i + \frac{\pi}{2}\right),$$
$$\mathrm{Res} f(z_-) = \left[\frac{d}{dz}(z-z_-)^2 f(z)\right]_{z\to z_-} = -i\frac{3\pi}{8}\left(2i + \frac{3\pi}{2}\right).$$

On the other hand, the entire contour consists of several parts: C_∞, C_ε, and the two edges of the branch cut. The integral along C_∞ and C_ε both vanish, thus

$$\oint F(z)dz = \int_0^{+\infty} [F_+(x) - F_-(x)]\, dx = \pi^3 - 4\pi i I,$$

where we have used $\int_0^{+\infty} dx/(1 + x^2)^2 = \pi/4$ by closing the contour in the upper plane. Thus we obtain $I = -\pi/4$.

A simpler choice is to define the branch cut as $x \in (-\infty, 0]$, so the principal branch is $\arg z \in [-\pi, \pi]$. Along the upper edge of the cut $\arg z = \pi$ and

$$F_+(x) = \frac{(\ln|x| + \pi i)^2}{(1+x^2)^2}.$$

Along the lower edge of the cut $\arg z = -\pi$ and

$$F_-(x) = \frac{(\ln|x| - \pi i)^2}{(1+x^2)^2}.$$

The discontinuity across the branch cut, $F_+(x) - F_-(x) = 4\pi i f(|x|)$, is proportional to $f(|x|)$. This motivates the closed contour shown in Fig. 5(a). This contour encloses two 2nd-order poles at $z_\pm = e^{\pm i\pi/2}$:

$$\mathrm{Res} f(z_\pm) = \frac{i\pi}{8}\left(2i \pm \frac{\pi}{2}\right),$$

so we can use the residue theorem to obtain

$$\oint_C F(z)dz = -i\pi^2.$$

On the other hand, the entire closed contour consist of C_ε, C_∞, and the two edges of the branch cut. The integral along C_∞ and C_ε both vanish. The integral along the two edges gives

$$\left(\int_{C^+_{\mathrm{cut}}} + \int_{C^-_{\mathrm{cut}}}\right) F(z)dz = \int_{-\infty}^{0} [F_+(x) - F_-(x)]\, dx = 4\pi i I.$$

Therefore, we obtain $I = -\pi/4$.

Example 7: $I = \int_0^a \frac{x^{3/4}(a-x)^{1/4}}{b-x} dx \equiv \int_0^a f(x)dx \ (b > a > 0).$

We consider the function

$$F(z) \equiv \frac{z^{3/4}(z-a)^{1/4}}{b-z},$$

which has two branch points at $z = 0$ and $z = a$: the first is associated with $z^{3/4}$ and the second is associated with $(z-a)^{1/4}$. We define the branch cut of $(\cdots)^\nu$ as $x \in (-\infty, 0]$, so the principal branch of $z^{3/4}$ is $\arg z \in [-\pi, \pi]$ and the principal branch of $(z-a)^{1/4}$ is $\arg(z-a) \in [-\pi, \pi]$. On the upper edge of $x \in (-\infty, 0]$, we have $\arg z = \arg(z-a) = \pi$. On the lower edge of $x \in (-\infty, 0]$, both $\arg z$ and $\arg(z-a)$ change from π to $-\pi$, so $F(z)$ is continuous across this cut. On the upper edge of $x \in [0, a]$, we have $\arg z = 0$ and $\arg(z-a) = \pi$. On the lower edge of $x \in [0, a]$, we have $\arg z = 0$ and $\arg(z-a) = -\pi$, so $F(z)$ is discontinuous across this cut:

$$F_\pm(0 \le x \le a) = f(x)\frac{1 \pm i}{\sqrt{2}}.$$

Moreover, $F(z)$ approaches a constant $F(\infty) = -1$ at infinity, in sharp contrast to previous examples where $f(z)$ decays to zero. This motivates the closed contour in Fig. 5(b). The entire contour consists of the upper and lower edges of the cut $x \in [0, a]$ and the small circles at the two branch points. The latter vanishes because $F(z) \to 0$ at these points. Thus the entire contour integeral is

$$\oint F(z)dz = \int_0^a [F_-(x) - F_+(x)]\, dx = -i\sqrt{2}I.$$

On the other hand, $F(z)$ is analytic outside the entire closed contour except for a first-order pole $z = b$, so we can use the residue theorem for this infinite analytic region to obtain

$$\oint F(z)dz = -2\pi i \mathrm{Res} f(b) - 2\pi i \mathrm{Res} f(\infty),$$

where $\mathrm{Res} f(b) = -b^{3/4}(b-a)^{1/4}$. To calculate $\mathrm{Res} f(\infty) = -a^{(\infty)}_{-1}$, we let $\zeta = 1/z$ and expand $F(z)$ about ζ near $\zeta = 0$ [cf. Eq. (3)]:

$$F(z) = \frac{(1-a\zeta)^{1/4}}{b\zeta - 1} = -1 + a^{(\infty)}_{-1}\zeta + a^{(\infty)}_{-2}\zeta^2 + \cdots,$$

thus $a^{(\infty)}_{-1} = [\partial_\zeta F(z)]_{\zeta=0} = (a/4 - b)$. Therefore, we obtain $I = -\sqrt{2}\pi\left[b^{3/4}(b-a)^{1/4} + (a/4 - b)\right]$.

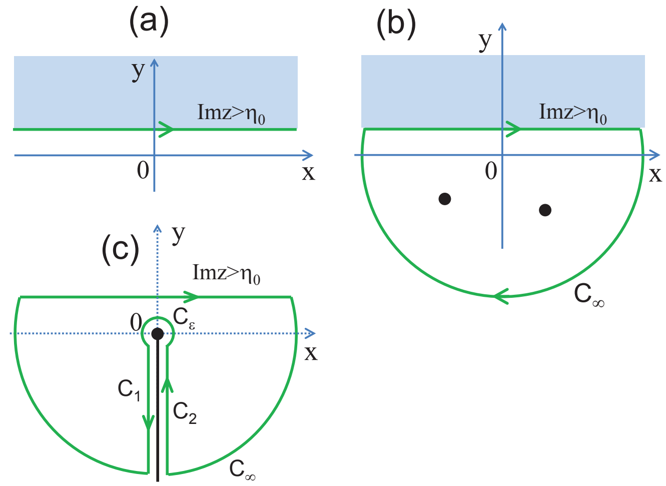

FIG. 6 Contours for Laplace transform.

II. LAPLACE TRANSFORM

For an arbitrary function $f(t)$ that vanishes at $t < 0$, there exists a minimum positive number η_0 so that $e^{-\eta_0 t} f(t)$ has a fourier transform. For any $\eta \ge \eta_0$, we can define

$$\bar{f}(\omega + i\eta) \equiv \int_0^\infty f(t)e^{i(\omega+i\eta)t}dt \Leftrightarrow f(t) = \int_{-\infty}^\infty \bar{f}(\omega+i\eta)e^{-i(\omega+i\eta)t}\frac{d\omega}{2\pi}.$$

Let $z \equiv \omega + i\eta$, the Laplace transform becomes

$$\bar{f}(z) \equiv \int_0^\infty f(t)e^{izt}dt \Leftrightarrow f(t) = \int_{-\infty+i\eta}^{\infty+i\eta} \bar{f}(z)e^{-izt}\frac{dz}{2\pi}, \qquad (4)$$

where $\eta > \eta_0$ in the second integral. The contour for the integral over z is shown in Fig. 6(a). The first integral converges at $\mathrm{Im}\, z > \eta_0$.

A. Properties

We denote the Laplace transform by $f(t) \longleftrightarrow \bar{f}(z)$, then

$$\partial_t f(t) \longleftrightarrow -iz\bar{f}(z) - f(0), \tag{5}$$

$$\int_0^t f(t')dt' \longleftrightarrow \frac{1}{-iz}\bar{f}(z), \tag{6}$$

$$t^n f(t) \longleftrightarrow (-i\partial_z)^n \bar{f}(z), \tag{7}$$

$$e^{-iz_0 t} f(t) \longleftrightarrow \bar{f}(z - z_0), \tag{8}$$

$$f(t - t_0) \longleftrightarrow e^{izt_0}\bar{f}(z), \tag{9}$$

$$f_1(t) * f_2(t) \longleftrightarrow \bar{f}_1(z)\bar{f}_2(z), \tag{10}$$

where $f_1(t) * f_2(t) \equiv \int_0^t f_1(t-t')f_2(t')dt' = \int_0^t f_2(t-t')f_1(t')dt'$ is the convolution. When $z \to 0$, we have $e^{izt} \to 1$, so Eq. (5) becomes

$$-i \lim_{z\to 0} z\bar{f}(z) - f(0) = \int_0^\infty [\partial_t f(t)]dt = f(\infty) - f(0).$$

When $f(\infty)$ exists, we have

$$if(\infty) = \lim_{z\to 0} z\bar{f}(z). \tag{11}$$

Similarly, when $\operatorname{Im} z \to +\infty$, we have $e^{izt} \to 0$, Eq. (5) gives

$$if(0) = \lim_{\operatorname{Im} z\to +\infty} z\bar{f}(z). \tag{12}$$

The image function $\bar{f}(z)$ is analytic within its domain of definition $\operatorname{Re} z > \eta_0$, i.e., on the upper half plane above η_0 [shaded region in Fig. 6(a)], but it can be analytically continued to the whole plane. Moreover, $\bar{f}(z) \to 0$ when z tends to infinity in the upper half plane (not including the x axis) because in this case $\operatorname{Im} z \to +\infty$ in Eq. (4). For example,

$$-i\frac{(-it)^n}{n!} \longleftrightarrow \frac{1}{z^{n+1}},$$

$$-ie^{-iz_0 t} \longleftrightarrow \frac{1}{z - z_0},$$

$$-i\frac{(-it)^n}{n!}e^{-iz_0 t} \longleftrightarrow \frac{1}{(z - z_0)^{n+1}}.$$

For $f(t) \sim t^n$, the image $\bar{f}(z)$ is analytic within its definition domain $\operatorname{Im} z > 0$. For $f(t) \sim e^{-iz_0 t}$, the image $\bar{f}(z)$ is also analytic within its definition domain $\operatorname{Im}(z - z_0) > 0$. The above examples satisfy Eq. (12), but not Eq. (11) since $f(\infty)$ does not exist.

B. Inversion formula

Generally, we can calculate $f(t)$ by Eq. (4). When $t < 0$, the integrand $\bar{f}(z)e^{-izt}$ decays exponentially in the upper half plane above η_0, thus we can close the contour in the upper half plane and obtain $f(t) = 0$. When $t > 0$, the integrand $\bar{f}(z)e^{-izt}$ decays to zero rapidly in the lower half plane, so we close the contour using an infinite arc in the lower half plane below $\operatorname{Im} z = \eta_0$, as shown in Fig. 6(b). When $\bar{f}(z)$ is single-valued (i.e., no branch points), the entire contour integral is determined by all the residues in the complex plane:

$$-i \sum \operatorname{Res}[\bar{f}(z)e^{-izt}] = \int_{C_\infty} \bar{f}(z)e^{-izt}\frac{dz}{2\pi} + f(t).$$

According to the Jordan lemma, the integral along C_∞ below the x axis vanishes due to the exponential decay factor $|e^{-izt}| = e^{t \operatorname{Im} z}$. Along the two extra segments connecting $y = 0$ to $y = \eta_0$, the integral measure $\int |e^{-izt}| \times |dz|/(2\pi)$ is finite and $\bar{f}(z) \to 0$, so the integral along the whole C_∞ vanishes. Therefore,

$$f(t) = -i\theta(t) \sum \operatorname{Res}[\bar{f}(z)e^{-izt}],$$

where $\theta(t) = 1$ for $t > 0$ and vanishes otherwise. For $\bar{f}(z) = 1/(z - z_0)$, the function $\bar{f}(z)e^{-izt}$ has a residue 1 at the first-order pole z_0, so $f(t) = -ie^{-iz_0 t}$. For $\bar{f}(z) = 1/(z - z_0)^{n+1}$, the function $\bar{f}(z)e^{-izt}$ has a $(n+1)$th-order pole at z_0. Its residue is

$$\frac{1}{n!}\frac{d^n}{dz^n}\left[(z - z_0)^{n+1}\bar{f}(z)e^{-izt}\right] = \frac{1}{n!}(-it)^n,$$

so we obtain $f(t) = -i(-it)^n/n!$.

When $\bar{f}(z)$ has branch points, we should choose the contour within its analytic region, e.g., $\bar{f}(z) = 1/\sqrt{z}$ gives

$$f(t) = \int_{-\infty+i\eta}^{+\infty+i\eta} \frac{e^{-izt}}{\sqrt{z}}\frac{dz}{2\pi}.$$

For $t > 0$, the integrand decays to zero in the lower half plane below the x axis and has a branch point at $z = 0$. Along $-y$ axis $z = -iR$ with $R \in [0, +\infty]$, the integral $\sim \int_0^\infty e^{-Rt}/\sqrt{R}dR = 1/(2\sqrt{\pi t})$. This motivates the contour in Fig. 6(c). The entire contour integral vanishes:

$$0 = f(t) + \left(\int_{C_1} + \int_{C_2} + \int_{C_\varepsilon}\right)\frac{e^{-izt}}{\sqrt{z}}\frac{dz}{2\pi}.$$

The integral along C_ε vanishes. We let $z = e^{i3\pi/2}R$ along C_1 and $z = e^{-i\pi/2}R$ along C_2, so

$$\left(\int_{C_1} + \int_{C_2}\right)\frac{e^{-izt}}{\sqrt{z}}\frac{dz}{2\pi} = (e^{i3\pi/4} - e^{-i\pi/4})\int_0^\infty \frac{e^{-Rt}}{\sqrt{R}}\frac{dR}{2\pi} = -\frac{e^{-i\pi/4}}{\sqrt{\pi t}},$$

thus we obtain $f(t) = e^{-i\pi/4}/\sqrt{\pi t}$.

III. BESSEL FUNCTION IN THE COMPLEX PLANE

A. Definition by series expansion

The solutions to $z^2 f'' + zf' + (z^2 - \nu^2)f = 0$ are call Bessel functions of order $\nu \in \mathbb{R}$. The first solution is the Bessel function of the first kind:

$$J_\nu(z) \equiv \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-z^2/4)^k}{k!\Gamma(\nu + k + 1)}, \tag{13}$$

where $\Gamma(z)$ is the Gamma function. When ν is not an integer, $J_{-\nu}(z)$ is the second independent solution. When $\nu = n$ is an integer, we have $\Gamma(n) = (n-1)!$ (with $0! \equiv 1$ and $n! = \infty$ for negative n), so $J_{-n}(z) = (-1)^n J_n(z)$ is no longer independent, so we consider the Bessel function of the second kind:

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)},$$

which does not have a regular series expansion about $z = 0$ and $\lim_{z\to 0} Y_\nu(z) = \infty$. When ν is not an integer, $Y_\nu(z)$ is just a linear combination of $J_{\pm\nu}(z)$. When $\nu = n$ is an integer, however, $Y_n(z) \equiv \lim_{\nu\to n} Y_\nu(z)$ is the second independent solution and $Y_{-n}(z) = (-1)^n Y_n(z)$. We can also define the Bessel functions of the third kind (Hankel functions):

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) = \frac{i}{\sin(\nu\pi)}\left[e^{-i\nu\pi}J_\nu(z) - J_{-\nu}(z)\right], \tag{14a}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) = -\frac{i}{\sin(\nu\pi)}\left[e^{i\nu\pi}J_\nu(z) - J_{-\nu}(z)\right], \tag{14b}$$

which obey $H_{-\nu}^{(1)}(z) = e^{i\nu\pi}H_\nu^{(1)}(z)$ and $H_{-\nu}^{(2)}(z) = e^{-i\nu\pi}H_\nu^{(2)}(z)$.

The solutions to the equation $z^2 f'' + zf' - (z^2+\nu^2)f = 0$ are called modified Bessel functions. The first solution is the modified Bessel function of the first kind:

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)}. \tag{15}$$

When ν is not an integer, $I_{-\nu}(z)$ is the second independent solution. When $\nu = n$ is an integer, then $I_{-n}(z) = I_n(z)$ is not independent, so we consider the modified Bessel function of the second kind:

$$K_\nu(z) \equiv \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} = K_{-\nu}(z),$$

When ν is not an integer, $K_\nu(z)$ is just a linear combination of $I_{\pm\nu}(z)$. When $\nu = n$ is an integer, $K_n(z) \equiv \lim_{\nu\to n} K_\nu(z)$ is the second independent solution.

The spherical Bessel function of the first and second kind is

$$j_n(z) \equiv \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z),$$
$$y_n(z) \equiv \sqrt{\frac{\pi}{2z}}Y_{n+1/2}(z).$$

The spherical Bessel function of the third kind (spherical Hankel functions) are

$$h_n^{(1)}(z) \equiv j_n(z) + iy_n(z) = \sqrt{\frac{\pi}{2z}}H_{n+1/2}^{(1)}(z),$$
$$h_n^{(2)}(z) \equiv j_n(z) - iy_n(z) = \sqrt{\frac{\pi}{2z}}H_{n+1/2}^{(2)}(z).$$

The spherical Bessel functions can be expressed in terms of a finite number of trigonometric functions, e.g., $j_0(z) = (1/z)\sin z$, $y_0(z) = -(1/z)\cos z$, and $h_0^{(1)}(z) = (-i/z)e^{iz}$, $h_0^{(2)}(z) = (i/z)e^{-iz}$.

B. Analytical behavior in the complex plane

When ν is not an integer, all the Bessel functions $J_\nu, Y_\nu, H_\nu^{(1,2)}$ and modified Bessel functions I_ν, K_ν have a branch point $z = 0$ since they contain z^ν, so we need to define the principal branch of $(\cdots)^\nu$.

We define the principal branch as $\arg(\cdots) \in [-\pi,\pi]$ and use $z \in$ D for $\arg z \in [-\pi, 0]$ and $z \in$ U for $\arg z \in [0,\pi]$. The Bessel functions in the upper and lower half planes are connected via

$$J_\nu(z\in \mathrm{D}) = e^{-i\nu\pi}J_\nu(-z),$$
$$I_\nu(z\in \mathrm{D}) = e^{-i\nu\pi}I_\nu(-z),$$
$$H_\nu^{(2)}(z\in \mathrm{D}) = -e^{i\nu\pi}H_\nu^{(1)}(-z),$$

or equivalently

$$J_\nu(z\in \mathrm{U}) = e^{i\nu\pi}J_\nu(-z),$$
$$I_\nu(z\in \mathrm{U}) = e^{i\nu\pi}I_\nu(-z),$$
$$H_\nu^{(1)}(z\in \mathrm{U}) = -e^{-i\nu\pi}H_\nu^{(2)}(-z),$$

because $-z = e^{i\pi}z$ when $z \in$ D and $-z = e^{-i\pi}z$ when $z \in$ U.

The Bessel function and the modified Bessel functions are also connected. When z and $iz \equiv e^{i\pi/2}z$ both lie in the principal branch, i.e., $\arg z \in [-\pi,\pi/2]$, we have

$$I_\nu(z) = e^{-i\pi\nu/2}J_\nu(iz), \tag{16a}$$
$$K_\nu(z) = \frac{i\pi}{2}e^{i\pi\nu/2}H_\nu^{(1)}(iz), \tag{16b}$$

otherwise $I_\nu(z) = e^{i3\pi\nu/2}J_\nu(iz)$. When z and $-iz \equiv e^{-i\pi/2}z$ both lie in the principal branch, i.e., $\arg z \in [-\pi/2,\pi]$, we have

$$I_\nu(z) = e^{i\pi\nu/2}J_\nu(-iz), \tag{17a}$$
$$K_\nu(z) = -\frac{i\pi}{2}e^{-i\nu\pi/2}H_\nu^{(2)}(-iz), \tag{17b}$$

otherwise $I_\nu(z) = e^{-i3\pi\nu/2}J_\nu(-iz)$.

Finally, if we define the principal branch as $\arg(\cdots) \in [0,2\pi]$ and use $z \in$ U for $\arg z \in [0,\pi]$ and $z \in$ D for $\arg z \in [\pi, 2\pi]$, then $-z = e^{i\pi}z$ when $z \in$ U and $-z = e^{-i\pi}z$ when $z \in$ D, so we have

$$J_\nu(z\in \mathrm{U}) = e^{-i\nu\pi}J_\nu(-z),$$
$$I_\nu(z\in \mathrm{U}) = e^{-i\nu\pi}I_\nu(-z),$$
$$H_\nu^{(2)}(z\in \mathrm{U}) = -e^{i\nu\pi}H_\nu^{(1)}(-z),$$

or equivalently

$$J_\nu(z\in \mathrm{D}) = e^{i\nu\pi}J_\nu(-z),$$
$$I_\nu(z\in \mathrm{D}) = e^{i\nu\pi}I_\nu(-z),$$
$$H_\nu^{(1)}(z\in \mathrm{D}) = -e^{-i\nu\pi}H_\nu^{(2)}(-z).$$

These results are obtained from the case $\arg(\cdots) \in [-\pi,\pi]$ by interchanging "U" and "D". When z and $iz \equiv e^{i\pi/2}z$ both lie in the principal branch, i.e., $\arg z \in [0, 3\pi/2]$, we have Eq. (16). When z and $-iz \equiv e^{-i\pi/2}z$ both lie in the principal branch, i.e., $\arg z \in [\pi/2, 2\pi]$, we have Eq. (17).

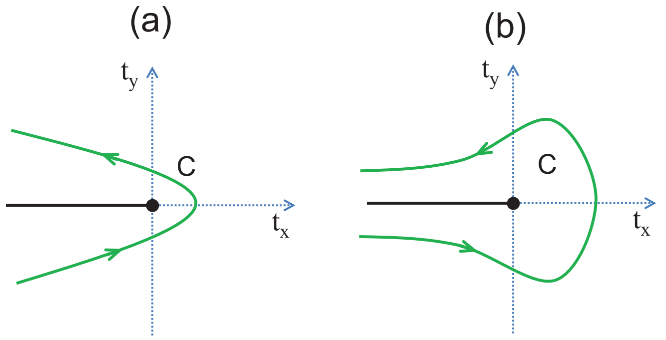

FIG. 7 Contours integral definition for Bessel functions.

C. Definition by contour integral

The integral definition for Bessel function:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{e^{[t-z^2/(4t)]}}{t^{\nu+1}} dt, \tag{18a}$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{e^{[t+z^2/(4t)]}}{t^{\nu+1}} dt. \tag{18b}$$

As shown in Fig. 7(a), the counter-clockwise contour C encircles the branch cut $(-\infty, 0]$ of $1/t^{\nu+1}$, comes from infinity in the third quadrant, and go to infinity in the second quadrant, which ensures the integrand to vanish at the end of C. Expanding $e^{\pm z^2/(4t)}$ in power of z gives

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!} \times \frac{1}{2\pi i} \int_C \frac{e^t}{t^{k+\nu+1}} dt,$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!} \times \frac{1}{2\pi i} \int_C \frac{e^t}{t^{k+\nu+1}} dt.$$

The last term is the Hankel definition of $\Gamma(n+\nu+1)$, so we recover Eqs. (13) and (15). Equation (18) also gives

$$\left[\partial_z^2 + \frac{1}{z}\partial_z + \left(1 - \frac{\nu^2}{z^2}\right)\right] J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{d}{dt}\left(\frac{e^{[t-z^2/(4t)]}}{t^{\nu+1}}\right) dt,$$

$$\left[\partial_z^2 + \frac{1}{z}\partial_z - \left(1 + \frac{\nu^2}{z^2}\right)\right] I_\nu(z) = -\left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{d}{dt}\left(\frac{e^{[t+z^2/(4t)]}}{t^{\nu+1}}\right) dt,$$

which vanishes since $e^{[t\pm z^2/(4t)]}/t^{\nu+1}$ vanishes at the end of the contour. Therefore, $J_\nu(z)$ and $I_\nu(z)$ indeed obey the Bessel and modified Bessel equations, respectively.

When $\arg z \in (-\pi/2, \pi/2)$, we can let $t \to zt/2$ and transform Eq. (18) into

$$J_\nu(z) = \frac{1}{2\pi i} \int_C \frac{e^{(z/2)(t-1/t)}}{t^{\nu+1}} dt, \tag{19a}$$

$$I_\nu(z) = \frac{1}{2\pi i} \int_C \frac{e^{(z/2)(t+1/t)}}{t^{\nu+1}} dt. \tag{19b}$$

As shown in Fig. 7(b), the contour C comes from infinity at $\arg t \to -\pi$ and goes to infinity at $\arg t \to \pi$ to make the integrand vanish at the end of C. When $\nu = n$ is an integer, the integrand has no branch point, so C can be closed. Let $e^{(z/2)(t-1/t)} = \sum_k c_k t^k$, then the residue theorem dictates $J_n(z) = c_n$ since c_n is the residue of the integrand at $t = 0$. This gives the generating function of Bessel functions of integer order:

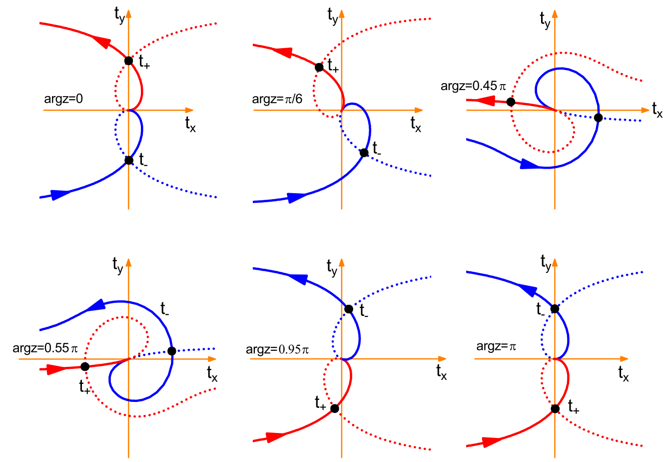

FIG. 8 Paths in the complex t plane that makes $\delta f(t)$ real for $\arg z \in [0, \pi]$: $\delta f(t) > 0$ on dashed lines and $\delta f(t) < 0$ on solid lines. Red paths for $\delta f(t)$ relative to the saddle point t_+ and blue paths for $\delta f(t)$ relative to t_-.

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z) \Rightarrow \begin{cases} e^{iz\sin\varphi} = \sum_{n=-\infty}^{\infty} e^{in\varphi} J_n(z), \\ e^{iz\cos\varphi} = \sum_{n=-\infty}^{\infty} i^n e^{-in\varphi} J_n(z). \end{cases} \tag{20}$$

If we choose C as a circle of unit radius and let $t = e^{i\theta}$, then

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} e^{iz\sin\theta} d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta) d\theta.$$

Similarly, for $I_\nu(z)$ with $\nu = n$ being an integer, we can close the contour and use the residue theorem to obtain $I_n(z)$ as the coefficient of t^n in the expansion of $e^{(z/2)(t+1/t)}$, so that

$$e^{(z/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} t^n I_n(z).$$

If we choose C as a circle of unit radius and let $t = e^{i\theta}$, then

$$I_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} e^{z\cos\theta} d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta) e^{z\cos\theta} d\theta.$$

D. Asymptotic behavior

From Eq. (18a), we can also derive the asymptotic behavior of the Bessel functions at $|z| \to \infty$. Let

$$f(t) = t - \frac{z^2}{4t}.$$

The saddle point is determined by $f'(t) = 0$ as $t_\pm \equiv \pm iz/2$ and $f(t_\pm) = \pm iz$. The saddle-point approximation amounts

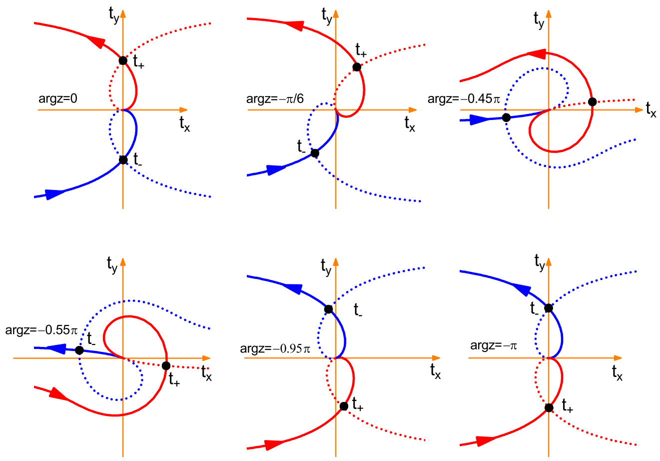

FIG. 9 The same as Fig. 8, except that $\arg z \in [-\pi, 0]$.

to choosing the contour along the steepest descent path across the saddle points, i.e., C should go cross each saddle point and make the deviation

$$\delta f(t) \equiv f(t) - f(t_\pm) = t - \frac{z}{4t} \mp iz$$

along the contour be *real* and *negative*, i.e., $\delta f < 0$. When $|t| \to 0$, the second term in $\delta f(t)$ dominates, so the steepest descent path approaches $\arg t = 2\arg z$. When $|t| \to \infty$, the second term is negligible, so we have $\operatorname{Im} t = \pm \operatorname{Re} z$ and $\operatorname{Re} t < \mp \operatorname{Im} z$, i.e., the steepest descent path approaches a straight line parallel to the real axis on the *left* half plane. The steepest descent paths for different $\arg z$ are shown in Figs. 8 and 9, e.g., for $z = x > 0$, the steepest descent path near t_+ (t_-) is shown as the solid red (blue) line in the first pannel of Fig. 8.

Using $f''(t) = -z^2/(2t^3)$, we make a Taylor expansion

$$\delta f(t) \approx \mp \frac{2i}{z}(t - t_\pm)^2 = -\frac{2}{|z|} R^2 e^{i(\pm\pi/2 - \arg z + 2\theta)},$$

where we have let $t - t_\pm = e^{i\theta} R$. The direction $\theta_{\rm sd}^\pm$ of the steepest descent path near the saddle point $t_\pm$ is determined by $\delta f(t) < 0$, i.e., $\theta_{\rm sd}^\pm = (1/2)\arg z \mp \pi/4 + n\pi$ ($n \in \mathbb{Z}$), and by requesting R go from $-\infty$ to $+\infty$ when we go across the saddle point along the positive direction of the contour C. For example, if $z = x > 0$, then $\theta_{\rm sd}^+ = 3\pi/4$ and $\theta_{\rm sd}^- = \pi/4$. With the help of Figs. 8 and 9, we find

$$\theta_{\rm sd}^+ = \begin{cases} \frac{1}{2}\arg z + \frac{3\pi}{4} & \text{for } \arg z \in [-\pi, \pi/2], \\ \frac{1}{2}\arg z - \frac{\pi}{4} & \text{for } \arg z \in [\pi/2, \pi], \end{cases}$$

and

$$\theta_{\rm sd}^- = \begin{cases} \frac{1}{2}\arg z + \frac{\pi}{4} & \text{for } \arg z \in [-\pi/2, \pi], \\ \frac{1}{2}\arg z + \frac{5\pi}{4} & \text{for } \arg z \in [-\pi, -\pi/2], \end{cases}$$

The contribution from the steepest descent path near t_+ is

$$\begin{aligned} J_\nu^{(+)}(z) &= \left(\frac{z}{2}\right)^\nu \frac{e^{f(t_+)}}{t_+^{\nu+1}} \frac{e^{i\theta_{\rm sd}^+}}{2\pi i} \int_{-\infty}^{\infty} e^{-(2/|z|)R^2} dR \\ &= \frac{e^{i(z-\nu\pi/2-\pi/4)}}{\sqrt{2\pi z}} \times \begin{cases} 1 & \text{for } \arg z \in [-\pi, \pi/2] \\ -e^{i2\pi\nu} & \text{for } \arg z \in [\pi/2, \pi] \end{cases}. \end{aligned}$$

The contribution from the steepest descent path near t_- is

$$\begin{aligned} J_\nu^{(-)}(z) &= \left(\frac{z}{2}\right)^\nu \frac{e^{f(t_-)}}{t_-^{\nu+1}} \frac{e^{i\theta_{\rm sd}^-}}{2\pi i} \int_{-\infty}^{\infty} e^{-(2/|z|)R^2} dR \\ &= \frac{e^{-i(z-\nu\pi/2-\pi/4)}}{\sqrt{2\pi z}} \times \begin{cases} 1 & \text{for } \arg z \in [-\pi/2, \pi] \\ -e^{-i2\pi\nu} & \text{for } \arg z \in [-\pi, -\pi/2] \end{cases}. \end{aligned}$$

Using the relation between Bessel and Hankel functions, we immediately obtain

$$H_\nu^{(1)}(z) \approx H_\nu^{(1,+)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\nu\pi/2-\pi/4)}, \tag{21a}$$

$$H_\nu^{(2)}(z) \approx H_\nu^{(2,-)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z-\nu\pi/2-\pi/4)}, \tag{21b}$$

for $|\arg z| < \pi$, where we have neglected an exponentially small contribution $\sim e^{iz}$ from t_+ to $H_\nu^{(2)}(z)$ at $\arg z \in [\pi/2, \pi]$ (i.e., in the upper half plane) and an exponentially small contribution $\sim e^{-iz}$ from t_- to $H_\nu^{(1)}(z)$ at $\arg z \in [-\pi, -\pi/2]$ (i.e., in the lower half plane). This also gives

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z - \nu\pi/2 - \pi/4), \tag{22a}$$

$$Y_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \sin(z - \nu\pi/2 - \pi/4). \tag{22b}$$

Using the connection between I_ν and J_ν gives

$$I_\nu(z) \approx \begin{cases} \frac{e^z}{\sqrt{2\pi z}} & \text{for } |\arg z| < \pi/2, \\ \frac{-i}{\sqrt{2\pi z}} e^{-z} e^{-i\nu\pi} & \text{for } \arg z \in [-\pi, -\pi/2], \\ \frac{i}{\sqrt{2\pi z}} e^{-z} e^{i\nu\pi} & \text{for } \arg z \in [\pi/2, \pi]. \end{cases}$$

Using the connection between K_ν and $H_\nu^{(1,2)}$ gives

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

When $|z| \to \infty$, $H_\nu^{(1)}(z)$ decays (increases) exponentially in the upper (lower) half plane, $H_\nu^{(2)}(z)$ decays (increases) exponentially in the lower (upper) half plane, $K_\nu(z)$ decays (increases) exponentially in the right (left) half plane, while $J_\nu(z), Y_\nu(z), I_\nu(z)$ increase exponentially in the whole plane.

In the above, we have neglected the contribution of the term $1/t^{\nu+1}$ in the integrand to (i) the location of the saddle point and (ii) the second-order Taylor expansion near the saddle point. Strickly speaking, we should write the integrand as $e^{F(t)}$ with $F(t) = f(t) - (\nu+1)\ln t$, determine the saddle points $T_\pm$ from $F'(t) = 0$, and expand $F(t)$ near $T_\pm$ up to the second order. Fortunately, when $|z| \to \infty$, the difference $|t_\pm - T_\pm| = O(1)$ is much smaller than $|t_\pm| = O(z)$. The difference between $F''(t)$ and $f''(t)$ are also much smaller than $|f''(t_\pm)| = O(z^2)$. Therefore, the results above are valid up to the leading order of $1/z$.

IV. GREEN'S FUNCTION FOR FREE ELECTRONS

In the momentum space, the GF in grand canonical ensemble is defined as $g(\zeta, \mathbf{k}) \equiv [\zeta + \mu - k^2/(2m_0)]^{-1}$, where μ is the chemical potential. The GF in n-dimensional real space is

$$g(\zeta, \mathbf{R}) = \int g(\zeta, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{R}} \frac{d^n\mathbf{k}}{(2\pi)^n} = -2m_0 \int \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2 - k_\zeta^2} \frac{d\mathbf{k}}{(2\pi)^n},$$

where $k_\zeta \equiv \sqrt{2m_0(\zeta+\mu)}$. If we define the principal branch $\arg(\zeta+\mu) \in [-\pi, \pi]$ by the branch cut $(-\infty, -\mu]$, then k_ζ lies in the same (i.e., upper or lower) half plane as ζ. If we define the principal branch $\arg(\zeta+\mu) \in [0, 2\pi]$ by the branch cut $[-\mu, +\infty)$, then k_ζ always lies in the upper half plane. The retarded (advanced) GF corresponds to $\zeta = \omega + i0^+$ ($\zeta = \omega - i0^+$), and the imaginary-time GF corresponds to $\zeta = i\omega_m$, where $\omega_m = (2m+1)\pi/\beta$ and $\beta = 1/(k_B T)$. For convenience, we define $k_\omega \equiv k_{\zeta\to\omega+i0^+}$, which always lies in the upper half plane for either the cut $(-\infty, -\mu]$ or the cut $[-\mu, +\infty)$.

A. 3D free electrons

$$g(\zeta, \mathbf{R}) = \frac{im_0}{2\pi^2 R} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - (k_\zeta R)^2} x dx. \tag{23}$$

We define the principal branch by the cut $(-\infty, -\mu]$ and close the contour in the upper half plane to obtain

$$g(\zeta, \mathbf{R}) = -\frac{m_0}{2\pi R} e^{\pm i k_\zeta R},$$

with upper (lower) sign for $\zeta \in \mathrm{U}$ ($\zeta \in \mathrm{D}$). Taking $\zeta \to \omega \pm i0^+$ gives $k_\zeta \to k_\omega$ and recovers the retarded and advanced GFs:

$$G^{R,A}(\omega, \mathbf{R}) = -\frac{m_0}{2\pi R} e^{\pm i k_\omega R}.$$

Notice that if we define the principal branch $\arg(\zeta+\mu) \in [0, 2\pi]$ by the branch cut $[-\mu, +\infty)$, then k_ζ always lies in the upper half plane, so $g(\zeta, \mathbf{R})$ always takes the upper sign. When $\zeta = \omega \pm i0^+$, we have $k_\zeta = \pm k_\omega$ and recover the retarded and advanced GFs.

B. 2D free electrons

$$g(\zeta, \mathbf{R}) = -\frac{m_0}{2\pi} \int_0^{\infty} \frac{z[H_0^{(1)}(z) + H_0^{(2)}(z)]}{z^2 - (k_\zeta R)^2} dz.$$

where we have used Eq. (20) and $J_\nu = (H_\nu^{(1)} + H_\nu^{(2)})/2$. We define the principal branch by the cut $(-\infty, -\mu]$, then the Hankel functions are analytical on the right half plane, so we can deform the contour slightly above (below) the real axis for the $H_0^{(1)}(z)$ term [$H_0^{(2)}(z)$ term] and then use $H_\nu^{(2)}(z \in \mathrm{D}) = -e^{i\nu\pi} H_\nu^{(1)}(-z)$ to obtain

$$\begin{aligned} g(\zeta, \mathbf{R}) &= -\frac{m_0}{2\pi} \left(\int_{i0^+}^{\infty + i0^+} \frac{z H_0^{(1)}(z)}{(\cdots)} dz + \int_{-i0^+}^{\infty - i0^+} \frac{z H_0^{(2)}(z)}{(\cdots)} dz \right) \\ &= -\frac{m_0}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} \frac{z H_0^{(1)}(z)}{z^2 - (k_\zeta R)^2} dz. \end{aligned}$$

Since $H_0^{(1)}(z) \to 0$ at infinity in the upper half plane, we close the contour in the upper half plane to obtain

$$g(\zeta, \mathbf{R}) = \frac{m_0}{2i} H_0^{(1)}(\pm k_\zeta R),$$

with upper (lower) sign for $\zeta \in \mathrm{U}$ ($\zeta \in \mathrm{D}$).

C. 1D free electrons

$$g(\zeta, \mathbf{R}) = -\frac{m_0 R}{\pi} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 - (k_\zeta R)^2} dx.$$

We define the principal branch by the branch cut $(-\infty, -\mu]$ and close the contour in the upper half plane to obtain

$$g(\zeta, \mathbf{R}) = \frac{m_0}{\pm i k_\zeta} e^{\pm i k_\zeta R},$$

with upper (lower) sign for $\zeta \in \mathrm{U}$ ($\zeta \in \mathrm{D}$).

D. Any dimension

$$g(\zeta, \mathbf{R}) = \int \frac{d^n\mathbf{k}}{(2\pi)^n} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{\zeta + \mu - k^2/(2m_0)}.$$

We can use $1/z = -i\int_0^\infty e^{iz\tau} d\tau$ for $\mathrm{Im}\, z > 0$ and $1/z = i\int_0^\infty e^{-iz\tau} d\tau$ for $\mathrm{Im}\, z < 0$ to obtain

$$g(\zeta, \mathbf{R}) = -2\left(\frac{m_0}{2\pi}\right)^{n/2} \left(\frac{-ik_\zeta}{m_0 R}\right)^{n/2-1} K_{n/2-1}(-ik_\zeta R),$$

where the principal branch of $(\cdots)^\alpha$ is $[-\pi, \pi]$, and the branch cut in the ζ plane is $[-\mu, +\infty)$. This makes k_ζ always in the upper half plane and $-ik_\zeta R$ always in the right half plane to ensure the exponential decay of $K_{n/2-1}(-ik_\zeta R)$ at $|\zeta| \to \infty$. For $z \in$ right half plane, $K_\nu(z) = (i\pi/2)e^{i\pi\nu/2} H_\nu^{(1)}(iz)$, so

$$g(\zeta, \mathbf{R}) = -i\pi \left(\frac{k_\zeta}{2\pi R}\right)^{n/2} \frac{m_0 R}{k_\zeta} H_{n/2-1}^{(1)}(k_\zeta R)$$

For $n = 2$, $g(\zeta, \mathbf{R}) = (m_0/2i) H_0^{(1)}(k_\zeta R)$ with k_ζ always in the upper half plane recovers the previous results. For $n = 3$, using $H_{1/2}^{(1)}(z) = -i\sqrt{2/(\pi z)} e^{iz}$ gives $g(\zeta, \mathbf{R}) = -m_0 e^{ik_\zeta R}/(2\pi R)$ and recovers the previous results.

$$g(\zeta, \mathbf{R}) = \frac{m_0}{2i} H_0^{(1)}(\pm k_\zeta R),$$

E. Discussions

In the above, all the GFs have a branch cut $[-\mu, +\infty)$ (or equivalently $(-\infty, -\mu]$). The physical origin can be seem from the general expression:

$$g(\zeta, \mathbf{R}) = \frac{1}{L^n} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{\zeta + \mu - \varepsilon(\mathbf{k})}.$$

In the discrete case, $g(\zeta, \mathbf{R})$ is analytical everywhere, except for many first-order poles at $\zeta = -\mu + \varepsilon(\mathbf{k})$ associated with the electron energy spectrum $\varepsilon(\mathbf{k})$. In the continuum limit, these poles becomes a branch cut. In other words, $g(\zeta, \mathbf{R})$ is analytical in the entire complex plane except for some poles (associated with the discrete spectrum of the electron) and branch cuts (associated with the continuous spectrum of the electron) on the real axis. In addition, $g(\zeta, \mathbf{R})$ also decays to zero exponentially at $|\zeta| \to \infty$.

V. RKKY INTERACTION

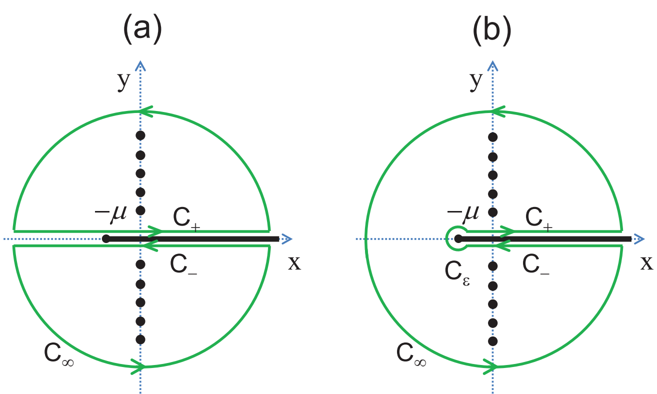

FIG. 10 Contour used to calculate the frequency summation for RKKY interaction.

For free electrons without spin-orbit coupling, the imaginary-time formalism for the RKKY range function is

$$J(\mathbf{R}) = \frac{J_{\mathrm{sd}}^2}{2} \frac{1}{\beta} \sum_m g(i\omega_m, \mathbf{R}) g(i\omega_m, -\mathbf{R}).$$

To perform the frequency summation $K \equiv (1/\beta) \sum_m (\cdots)$, we introduce an auxiliary function

$$N(z) \equiv \frac{1}{e^{\beta z} + 1}, \tag{24}$$

which is analytic and bounded over the entire complex plane except for an infinit enumber of first-order poles at $z = i\omega_m$ and $\mathrm{Res} N(i\omega_m) = -1/\beta$. The function $F(z) \equiv g(z, \mathbf{R}) g(z, -\mathbf{R}) N(z)$ is analytical over the entire complex plane, except for the poles $z = i\omega_m$ on the imaginary axis due to $N(z)$ and the branch cuts (and first-order poles at the discrete electron energy levels) on the real axis due to $g(z, \pm\mathbf{R})$.

At infinity, $F(z)$ tends to zero faster than $1/z$, so its integral along an infinite circle C_∞ vanishes. This motivates the closed contours in Fig. 10(a), thus $K = K_+ + K_-$ is the sum of the integrals along the upper and lower edges:

$$K_+ = \frac{1}{-2\pi i} \int_{C_+} F(z) dz = \frac{1}{-2\pi i} \int_{-\infty}^{\infty} G^R(\omega, \mathbf{R}) G^R(\omega, -\mathbf{R}) N(\omega) d\omega,$$

$$K_- = \frac{1}{-2\pi i} \int_{C_-} F(z) dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} G^A(\omega, \mathbf{R}) G^A(\omega, -\mathbf{R}) N(\omega) d\omega.$$

Since $G^A(\omega) = [G^R(\omega)]^\dagger$, we have $G^A(\omega, \mathbf{R}) = [G^R(\omega, -\mathbf{R})]^*$, so $K_- = -K_+^*$. This leads to the real-time formalism:

$$J(\mathbf{R}) = -\frac{J_{\mathrm{sd}}^2}{2\pi} \mathrm{Im} \int_{-\infty}^{\infty} G^R(\omega, \mathbf{R}) G^R(\omega, -\mathbf{R}) N(\omega) d\omega$$
$$\xrightarrow{T=0} -\frac{J_{\mathrm{sd}}^2}{2\pi} \mathrm{Im} \int_{-\infty}^{0} G^R(\omega, \mathbf{R}) G^R(\omega, -\mathbf{R}) d\omega.$$

Outside the (discrete or continuous) electron energy spectrum, $g(z, \mathbf{R})$ and hence $F(z)$ is continuous across the real axis, so the integral along C_+ and C_- within this range cancel each other, i.e., only the integral within the electron energy spectrum contribute to K. For free electrons, the electron energy spectrum is $[0, \infty)$ and $g(z, \mathbf{R})$ is discontinuous only within $[-\mu, +\infty)$, so $\int_{-\infty}^{\infty} d\omega$ can be replaced by $\int_{-\mu}^{\infty} d\omega$ [Fig. 10(b)]. Further, using $k_\omega^2/(2m_0) - \mu = \omega$, we have $d\omega = k_\omega dk_\omega / m_0$, so

$$J(\mathbf{R}) = -\frac{J_{\mathrm{sd}}^2}{2\pi m_0} \mathrm{Im} \int_0^{k_F} G^R(\omega, \mathbf{R}) G^R(\omega, -\mathbf{R}) k_\omega dk_\omega,$$

where $k_F \equiv \sqrt{2m_0\mu}$ is the Fermi momentum.

In the following, we always define the principal branch $\arg z \in [0, 2\pi]$ by the branch cut $[-\mu, +\infty)$.

A. 3D free electrons

Using the real-time formalism, we have

$$J(\mathbf{R}) = -\frac{J_{\mathrm{sd}}^2}{2\pi m_0} \left(\frac{m_0}{2\pi R}\right)^2 \frac{1}{4R^2} \mathrm{Im} \int_0^{2k_F R} e^{ix} x dx$$
$$= -J_{\mathrm{sd}}^2 \frac{m_0 k_F^4}{2\pi^3} \frac{\sin(2k_F R) - 2k_F R \cos(2k_F R)}{(2k_F R)^4}.$$

We may also use the imaginary-time formalism:

$$J(\mathbf{R}) = J_{\mathrm{sd}}^2 \frac{m_0^2}{8\pi^2 R^2} \frac{1}{\beta} \sum_m e^{i2k_{i\omega_m} R}.$$

In the complex z plane, the summand $e^{i2k_z R}$ has a branch cut $[-\mu, +\infty)$. In the principal branch $\arg z \in [0, 2\pi]$, k_z always lies in the upper plane, so $e^{i2k_z R}$ decays exponentially at infinity. With the auxiliary function $N(z)$ and the contour in Fig. 10(b), we obtain the frequency summation $K \equiv (1/\beta) \sum_m e^{i2k_{i\omega_m} R}$ as

$$K = \frac{1}{-2\pi i} \left(\int_{C_+} + \int_{C_-} + \int_{C_\varepsilon} \right) e^{i2k_z R} N(z) dz.$$

The integral along C_ε vanishes. Along the upper edge $e^{i2k_zR} = e^{2ik_\omega R}$ and $N(z) = N(\omega)$, so

$$K_+ = \frac{1}{-2\pi i}\int_{-\mu}^{\infty} e^{i2k_\omega R}N(\omega)d\omega.$$

Along the lower edge $e^{i2k_zR} = e^{-2ik_\omega R}$ and $N(z) = N(\omega)$, so

$$K_- = \frac{1}{2\pi i}\int_{-\mu}^{\infty} e^{-i2k_\omega R}N(\omega)d\omega$$

satisfies $K_- = -K_+^*$. The sum

$$\begin{aligned} K &= -\frac{1}{\pi}\int_{-\mu}^{\infty}\sin(2k_\omega R)N(\omega)d\omega \\ &\xrightarrow{T=0} -\frac{1}{\pi m_0}\int_0^{k_F}\sin(2k_\omega R)k_\omega dk_\omega \\ &= -\frac{\sin(2k_FR) - 2k_FR\cos(2k_FR)}{4\pi R^2 m_0} \end{aligned}$$

recovers the realt-time result.

B. 2D free electrons

Using the real-time formalism, we have

$$\begin{aligned} J(\mathbf{R}) &= J_{\rm sd}^2\frac{m_0}{8\pi R^2}\,\mathrm{Im}\int_0^{k_FR}[H_0^{(1)}(x)]^2xdx \\ &= J_{\rm sd}^2\frac{m_0}{4\pi R^2}\int_0^{k_FR}J_0(x)Y_0(x)xdx \\ &= J_{\rm sd}^2\frac{m_0k_F^2}{8\pi}\left[J_0(k_FR)Y_0(k_FR) + J_1(k_FR)Y_1(k_FR)\right]. \end{aligned}$$

We can also use the imaginary-time formalism $J(\mathbf{R}) = -(J_{\rm sd}^2m_0^2/8)K$, where $K \equiv (1/\beta)\sum_m[H_0^{(1)}(k_{i\omega_m}R)]^2$. In the complex z plane, the summand $[H_0^{(1)}(k_zR)]^2$ has a branch cut $[-\mu, +\infty)$. In the principal branch, k_z always lies in the upper half plane, so $z[H_0^{(1)}(k_zR)]^2$ and $z[H_0^{(1)}(k_zR)]^2N(z)$ tend to zero at infinity, so we choose the contour in Fig. 10 to obtain

$$K = \frac{1}{-2\pi i}\left(\int_{C_+} + \int_{C_-} + \int_{C_\varepsilon}\right)[H_0^{(1)}(k_zR)]^2N(z)dz.$$

Since $z[H_0^{(1)}(z)]^2$ tends to zero at $z \to 0$, the integral along C_ε vanishes. On the upper edge C_+, $\arg(z+\mu) = 0$ and $H_0^{(1)}(k_zR) = H_0^{(1)}(k_\omega R)$, so

$$K_+ = \frac{1}{-2\pi i}\int_{-\mu}^{\infty}[H_0^{(1)}(k_\omega R)]^2N(\omega)d\omega.$$

On the lower edge C_-, $\arg(z+\mu) = 2\pi$ and

$$H_0^{(1)}(k_zR) = H_0^{(1)}(e^{i\pi}k_\omega R) = -H_0^{(2)}(k_\omega R)$$

according to Eq. (14), so

$$K_- = \frac{1}{2\pi i}\int_{-\mu}^{\infty}[H_0^{(2)}(k_\omega R)]^2N(\omega)d\omega$$

satisfies $K_- = -K_+^*$. The sum

$$\begin{aligned} K &= -\frac{2}{\pi}\int_{-\mu}^{\infty}J_0(k_\omega R)Y_0(k_\omega R)N(\omega)d\omega \\ &\xrightarrow{T=0} -\frac{2}{\pi m_0R^2}\int_0^{k_FR}J_0(x)Y_0(x)xdx \\ &= -\frac{k_F^2}{\pi m_0}\left[J_0(k_FR)Y_0(k_FR) + J_1(k_FR)Y_1(k_FR)\right], \end{aligned}$$

recovers the real-time result.

C. 1D free electrons

Using the real-time formalism, we have

$$J(\mathbf{R}) = J_{\rm sd}^2\frac{m_0}{2\pi}\,\mathrm{Im}\int_0^{2k_FR}\frac{e^{ix}}{x+i0^+}dx = J_{\rm sd}^2\frac{m_0}{4\pi i}\int_{-2k_FR}^{2k_FR}\frac{e^{ix}}{x+i0^+}dx.$$

We add two contours $(-\infty, -2k_FR]$ and $[2k_FR, +\infty)$ and close the contour in the upper half plane to obtain

$$\begin{aligned} J(\mathbf{R}) &= -J_{\rm sd}^2\frac{m_0}{2\pi}\frac{1}{2i}\left(\int_{-\infty}^{-2k_FR}\frac{e^{ix}}{x}dx + \int_{2k_FR}^{\infty}\frac{e^{ix}}{x}dx\right) \\ &= -J_{\rm sd}^2\frac{m_0}{2\pi}\int_{2k_FR}^{\infty}\frac{\sin x}{x}dx \\ &= J_{\rm sd}^2\frac{m_0}{2\pi}\left[\mathrm{Si}(2k_FR) - \frac{\pi}{2}\right], \end{aligned} \tag{25}$$

where $\mathrm{Si}(x) \equiv \int_0^x(1/t)\sin t dt$. We can also use the imaginary-time formalism: $J(\mathbf{R}) = -(J_{\rm sd}^2m_0/4)K$, where $K \equiv (1/\beta)\sum_m F(i\omega_m)$ and the summand $F(z) \equiv e^{2ik_zR}/(z+\mu)$ has a branch cut along $[-\mu, +\infty)$ and a first-order pole at $z = -\mu$ with residue 1. Since $zF(z)N(z)$ tends to zero at infinity, we choose the contour in Fig. 10 to obtain

$$\begin{aligned} K &= \frac{1}{-2\pi i}\oint F(z)N(z)dz \\ &= \frac{1}{-2\pi i}\left(\int_{C_+} + \int_{C_-} + \int_{C_\varepsilon}\right)F(z)N(z)dz. \end{aligned}$$

The contour C_ε is parametrized by $z = -\mu + e^{i\varphi}R$ with $R \to 0$ and φ going from 2π to 0, so

$$K_\varepsilon = \frac{1}{-2\pi i}\int_{C_\varepsilon}F(z)N(z)dz = \frac{1}{e^{-\beta\mu}+1}.$$

On the upper edge C_+, $k_z = k_\omega$, so

$$K_+ = \frac{1}{-2\pi i}\int_{-\mu}^{\infty}\frac{e^{2ik_\omega R}}{\omega+\mu}N(\omega)d\omega.$$

On the lower edge C_-, $k_z = -k_\omega$, so

$$K_- = \frac{1}{2\pi i}\int_{-\mu}^{\infty}\frac{e^{-2ik_\omega R}}{\omega+\mu}N(\omega)dz$$

satisfies $K_- = -K_+^*$. The sum

$$K = \frac{1}{e^{-\beta\mu}+1} - \frac{1}{\pi}\int_{-\mu}^{\infty} \frac{\sin(2k_\omega R)}{\omega+\mu} N(\omega)d\omega \xrightarrow{T=0} 1 - \frac{2}{\pi}\,\mathrm{Si}(2k_F R)$$

recover thes real-time result.

D. Any dimension

We use the imaginary-time formalism:

$$J(\mathbf{R}) = -\pi^2 \frac{J_{\mathrm{sd}}^2}{2} \frac{m_0^2}{(2\pi)^n R^{2n-4}} K,$$

where $K \equiv (1/\beta)\sum_m F(i\omega_m)$ and $F(z) \equiv (k_z R)^{n-2}[H^{(1)}_{n/2-1}(k_z R)]^2$. Since k_z always lies in the upper half plane, $zF(z)$ and $zF(z)N(z)$ tend to zero at infinity, so we choose the contour in Fig. 10 to obtain

$$K = \frac{1}{-2\pi i}\left(\int_{C_+} + \int_{C_-} + \int_{C_\varepsilon}\right) F(z)N(z)dz.$$

At $z \to 0$, we have $F(z) \to \ln z$ for $n = 2$ and $F(z) \to$ constant for $n \geq 3$. For both cases, the integral along C_ε vanish. On the upper edge C_+, we have $k_z = k_\omega$, so

$$K_+ = \frac{1}{-2\pi i}\int_{-\mu}^{\infty} (k_\omega R)^{n-2}[H^{(1)}_{n/2-1}(k_\omega R)]^2 N(\omega)d\omega.$$

On the lower edge C_-, we have $k_z = e^{i\pi}k_\omega$ and $H^{(1)}_{n/2-1}(e^{i\pi}k_\omega R) = e^{-in\pi/2}H^{(2)}_{n/2-1}(k_\omega R)$, so

$$K_- = \frac{1}{2\pi i}\int_{-\mu}^{\infty} (k_\omega R)^{n-2}[H^{(2)}_{n/2-1}(k_\omega R)]^2 N(\omega)d\omega$$

satisfies $K_- = -K_+^*$. The sum

$$K = -\frac{2}{\pi}\int_{-\mu}^{\infty} (k_\omega R)^{n-2} J_{n/2-1}(k_\omega R) Y_{n/2-1}(k_\omega R) N(\omega)d\omega \xrightarrow{T=0} -\frac{2}{\pi m_0 R^2}\int_0^{k_F R} x^{n-1} J_{n/2-1}(x) Y_{n/2-1}(x)dx.$$

For two cylinder functions A_μ and B_ν of orders μ and ν, using

$$\int^z t^{\mu+\nu+1} A_\mu(t)B_\nu(t)dt = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)}\left[A_\mu(z)B_\nu(z) + A_{\mu+1}(z)B_{\nu+1}(z)\right],$$

gives

$$K \xrightarrow{T=0} -\frac{1}{\pi m_0 R^2}\frac{(k_F R)^n}{n-1}\Phi_n(k_F R),$$

where $\Phi_n(x) \equiv J_{n/2-1}(x)Y_{n/2-1}(x) + J_{n/2}(x)Y_{n/2}(x)$. Therefore,

$$J(\mathbf{R}) = \frac{J_{\mathrm{sd}}^2}{2}\left(\frac{k_F}{2\pi R}\right)^n \frac{m_0\pi}{n-1} R^2 \Phi_n(k_F R).$$