I. POLAR COORDINATE

In 2D, the Cartesian coordinate is $\boldsymbol{\rho} = (x, y)$ with unit vectors $\mathbf{e}_x, \mathbf{e}_y$. The polar coordinate is $\boldsymbol{\rho} = (\rho, \theta)$ ($\rho \geq 0$, $\theta \in [0, 2\pi]$) with unit vector $\mathbf{e}_\rho(\boldsymbol{\rho}), \mathbf{e}_\theta(\boldsymbol{\rho})$. The Cartesian coordinate variables can be expressed via the polar coordinate variables:

$$x = \rho \cos\theta, \;\; y = \rho \sin\theta.$$

From

$$\begin{aligned}
\partial_\rho &= \frac{\partial x}{\partial \rho}\partial_x + \frac{\partial y}{\partial \rho}\partial_y = \cos\theta\partial_x + \sin\theta\partial_y, \\
\partial_\theta &= \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y = -\rho\sin\theta\partial_x + \rho\cos\theta\partial_y.
\end{aligned}$$

or

$$\begin{aligned}
\partial_\rho &= \cos\theta\partial_x + \sin\theta\partial_y, \\
\frac{1}{\rho}\partial_\theta &= -\sin\theta\partial_x + \cos\theta\partial_y.
\end{aligned}$$

We can solve the equation above for ∂_x and ∂_y and obtain

$$\begin{aligned}
\partial_x &= \cos\theta\partial_\rho - \sin\theta\frac{1}{\rho}\partial_\theta, \\
\partial_y &= \sin\theta\partial_\rho + \cos\theta\frac{1}{\rho}\partial_\theta.
\end{aligned}$$

From

$$\begin{aligned}
\mathbf{e}_\rho &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \\
\mathbf{e}_\theta &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y,
\end{aligned}$$

$$d\mathbf{e}_\rho = \mathbf{e}_\theta d\theta$$

we can solve for $\mathbf{e}_x$ and $\mathbf{e}_y$ to obtain

$$\begin{aligned}
\mathbf{e}_x &= \cos\theta\mathbf{e}_\rho - \sin\theta\mathbf{e}_\theta, \\
\mathbf{e}_y &= \sin\theta\mathbf{e}_\rho + \cos\theta\mathbf{e}_\theta.
\end{aligned}$$

The

$$\nabla = \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y = \mathbf{e}_\rho\partial_\rho + \mathbf{e}_\theta\frac{1}{\rho}\partial_\theta.$$

$$\mathbf{A} = \nabla\theta = \frac{\mathbf{e}_\theta}{\rho},$$

$$d\boldsymbol{\rho} = Rd\mathbf{e}_\rho = R\mathbf{e}_\theta d\theta.$$

$$d\boldsymbol{\rho} = d(\rho\mathbf{e}_\rho) = \mathbf{e}_\rho d\rho + \rho d\mathbf{e}_\rho$$

$$\oint_{\text{counterclockwise}} \mathbf{A} \cdot d\boldsymbol{\rho} = \int_0^{2\pi} \frac{\mathbf{e}_\theta}{R} \cdot R\mathbf{e}_\theta d\theta = \int_0^{2\pi} d\theta = 2\pi.$$