KKT 条件与约束最优解问题

首先定义

$$f(x,y,z,\cdots)$$

$$\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}+\cdots=\begin{pmatrix}\frac{\partial f}{\partial x}\\\frac{\partial f}{\partial y}\\\frac{\partial f}{\partial z}\\\vdots\end{pmatrix}$$

那么，对于f，我们称

$$\begin{cases}\nabla f+\sum\limits_{i=1}^{m}\mu_i\nabla g_i+\sum\limits_{i=1}^{n}\lambda_i\nabla h_i=\mathbf{0}\\h_i=0\ (\text{For All }i)\\g_i\leq 0\ (\text{For All }i)\\\mu_i\geq 0\ (\text{For All }i)\\\mu_i g_i=0\ (\text{For All }i)\end{cases}$$

为f的 **KKT 条件**，也就是f在约束条件下取得极值的必要条件.

现在推导拉格朗日乘数法与 KKT 条件的部分等价性.

我们取$m=n=1$，$f(x,y)$，$h(x)$作为等式约束，这个情况下的 KKT 条件就是

$$\begin{aligned}&\nabla f(x,y)+\lambda\nabla h(x,y)=\mathbf{0}\\&h(x,y)=0\end{aligned}$$

也就是

$$
\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} + \begin{pmatrix} \lambda \frac{\partial h}{\partial x} \\ \lambda \frac{\partial h}{\partial y} \end{pmatrix} = \mathbf{0}
$$

$$
h(x,y) = 0
$$

容易得到

$$
\frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x} = 0
$$

$$
\frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y} = 0
$$

$$
h(x,y) = 0
$$

现在看拉格朗日乘数法，构造拉格朗日函数

$$
F(x,y,\lambda) = f(x,y) + \lambda h(x,y)
$$

那么，取得极值的必要条件就是

$$
\frac{\partial F}{\partial x} = 0
$$

$$
\frac{\partial F}{\partial y} = 0
$$

$$
\frac{\partial F}{\partial \lambda} = 0
$$

立即得到

$$
\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x}
$$

$$
\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y}
$$

$$
\frac{\partial F}{\partial x} = \lambda \frac{\partial h}{\partial \lambda} = h(x,y)
$$

这三个偏导数必为 0，所以

$$
\frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x} = 0
$$

$$
\frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y} = 0
$$

$$
h(x,y) = 0
$$

这与 KKT 条件推导的结果完全一致.