Generating Function and Others

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Catalogue

Outline

Introduction

What is a Generating Function?

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> *A generating function is a clothesline on which we hang up a sequence of numbers for display.*

Definition (Generating Function)

For a series of information a_n, our generating function $A(x)$ satisfies,

$$A(x) = \sum_{i=1}^{n} b_i * x^i$$

where b_i is only related to a_i and i itself.

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Question (Zeitz)

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If we represent our standard die as follows,

$$\mathcal{D}(x) = x + x^2 + x^3 + x^4 + x^5 + x^6$$

Then we have only one pair of nonstandard dice that meet our restriction,

$$\mathcal{A}(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$$

$$\mathcal{B}(x) = x + 2x^2 + 2x^3 + x^4$$

Namely,

$$\mathcal{A}(x) * \mathcal{B}(x) = \mathcal{D}(x)^2$$

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Validation:From Algebraic to Analytic

Let's focus on the generating function of Fibonacci numbers

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$$\mathcal{F}(x) = \frac{x}{1 - x - x^2}$$

Naturally, we would try to subtitute $x = \frac{1}{3}$ or $x = 1$ into this identity and then we get

$$\mathcal{F}(\frac{1}{3}) = \frac{3}{5}, \;\; \mathcal{F}(1) = -1$$

The former might be right, but the latter is totally absurd.

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So what went wrong?

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Notice the x in our generating function is not a variable, but rather merely a placeholder, though we could intuitive define the same laws of addition, multiplication and differentiation.

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which means $\mathcal{F}(x)$ is not a function of x, Therefore functional operations such as evaluating $\mathcal{F}(1)$ make absolutely no sense in the realm of formal generating functions.

However, it is often desirable to use generating functions for more than their place-holding capabilities, and actually, $\mathcal{F}(\frac{1}{3}) = \frac{3}{5}$ is absolutely right!

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For example, Fibonacci generating function $\mathcal{F}(x)$ converge to $\frac{x}{1-x-x^2}$ on domain $[-\frac{1}{\phi}, \frac{1}{\phi}]$, where ϕ stands for golden ratio.

And because $x = \frac{1}{3}$ lies in that domain while $x = 1$ do not, thus only

$$\mathcal{F}(\frac{1}{3}) = \sum_{i=1}^{\infty} f_i * (\frac{1}{3})^i = \frac{3}{5}$$

got the right answer.

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This form of the generating function only store the sequence information.

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Since a number is divisible by 3 if and only if the sum of its digits is divisible by 3, wei'll find how many of those numbers have a digital sum divisible by 3.
So let's try to make a generating function $\mathcal{F}(x) = \sum_{i=0}^{9n} f_i x^i$ where f_k is the number of n-digit numbers with digits in {2, 3, 7, 9} and a digital sum of k. then our Answer

$$A = \sum_{i=0}^{3n} f_{3i}$$

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Intuitively, we have

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But how do we extract every third coeffcient of the polynomial? The ingenious answer comes from complex numbers. DeOfine $\varepsilon = e^{2\pi i/3}$ as one of the cube roots of unity, i.e. one of the solutions to $x^3 = 1$. This number has the simple property that $1 + \varepsilon + \varepsilon^2 = \frac{1-\varepsilon^3}{1-\varepsilon} = 0$. This property allows us to single out every third coeffcient of $\mathcal{F}$ quite easily. We have

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$$\mathcal{F}(1) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots$$

$$\mathcal{F}(\varepsilon) = f_0 + f_1\varepsilon + f_2\varepsilon^2 + f_3 + f_4\varepsilon + \cdots$$

$$\mathcal{F}(\varepsilon^2) = f_0 + f_1\varepsilon^2 + f_2\varepsilon + f_3 + f_4\varepsilon^2 + \cdots$$

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$$\mathcal{F}(1)+\mathcal{F}(\varepsilon)+\mathcal{F}(\varepsilon^2)=3f_0+(1+\varepsilon+\varepsilon^2)f_1+(1+\varepsilon+\varepsilon^2)f_2$$
$$+3f_3+(1+\varepsilon+\varepsilon^2)f_4+\cdots$$
$$=3A$$

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$$+3f_3 + (1 + \varepsilon + \varepsilon^2)f_4 + \cdots$$
$$= 3A$$

With this we can easily calculate the answer of this problem:

$$\begin{aligned} A &= \frac{1}{3}(\mathcal{F}(1) + \mathcal{F}(\varepsilon) + \mathcal{F}(\varepsilon^2)) \\ &= \frac{1}{3}((1 + 1 + 1 + 1)^n + (\varepsilon^2 + 1 + \varepsilon + 1)^n + (\varepsilon + 1 + \varepsilon^2 + 1)^n) \\ &= \frac{1}{3}(4^n + 2) \end{aligned}$$

Root of Unity Filter

Theorem

Define $\varepsilon = e^{2\pi i/n}$ *for a positive integer* n. *For any polynomial* $\mathcal{F}(x) = \sum_{i=0}^{\infty} f_i x^i$, *the* $\sum_{i=0}^{\infty} f_{ni}$ *is given by*

$$\sum_{i=0}^{\infty} f_{ni} = \frac{1}{n}\left(\sum_{i=0}^{n-1} \mathcal{F}(\varepsilon^i)\right)$$

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Proof.

Base on the property of roots of unity, we have

$$\sum_{i=0}^{n-1} \varepsilon^{ki} = [n|k] * n$$

The following proof is trivial. □

Exponential Generating Function

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$$\mathcal{A}(x) = \sum_{i=0}^{n} \frac{a_i}{i!} x^i$$

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To contrast with Original generating function, Exponential generating function stores more than the sequence a_n itself, but also the information of order.

Thus it usually points to a series of labeled-promblem.

Exponential Generating Function

Question (codeforces 891E)

You have n integers a_n and the folloing process repeats m times.
Choose an index from 1 to n uniformly at random. Name it j,
$ans = ans + \prod_{i=1,i\neq j}^{n} a_i$, *then subtract a_j by 1*
Compute $\mathbb{E}[ans], n \leq 5000, m \leq 10^9$

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let's define b_i represents the number of subtrcts ofa_i, then

$$\begin{aligned}\mathbb{E}[ans] &= \mathbb{E}[\prod_{i=1}^{n} a_i - \prod_{i=1}^{n}(a_i - b_i)] \\ &= \prod_{i=1}^{n} a_i - \frac{1}{n^m} \sum_{bi,\sum bi=m} \frac{m!}{\prod b_i!} \prod_{i=1}^{n} a_i - b_i \\ &= \prod_{i=1}^{n} a_i - \frac{m!}{n^m} \sum_{bi,\sum bi=m} \prod_{i=1}^{n} \frac{a_i - b_i}{b_i!}\end{aligned}$$

Exponential Generating Function

construct $\mathcal{F}(x)$ be the EGF of $\sum_{bi,\sum bi=m}\prod_{i=1}^{n}\frac{a_i-b_i}{b_i!}$

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$$\begin{aligned}\mathcal{F}(x) &= \prod_{i=1}^{n}\sum_{j=0}^{\infty}\frac{a_i-j}{j!}x^j \\ &= \prod_{i=1}^{n}\sum_{j=0}^{\infty}(\frac{a_ix^j}{j!}-\frac{x^j}{(j-1)!}) \\ &= \prod_{i=1}^{n}(a_i-x)e^x = e^{nx}\prod_{i=1}^{n}(a_i-x)\end{aligned}$$

Then

$$
\begin{aligned}
\mathbb{E}[ans] &= \prod_{i=1}^{n} a_i - \frac{m!}{n^m}[x^m]\mathcal{F}[x] \\
&= \prod_{i=1}^{n} a_i - \sum_{i=0}^{n}([x^i]\prod_{i=1}^{n}(a_i - x))\frac{m!}{n^i(m-i)!}
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The following things are trivial.

Notice! In our solution, we always treat $\mathcal{F}(x)$ as a algebraic object so the validation is guaranteed.

The Meaning of Exponential

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let $f : \{1, 2, \cdots, n\} \mapsto \{1, 2, \cdots, n\}$, *and* $\overbrace{f \circ \cdots \circ f(x)}^{k\ times} = \overbrace{f \circ \cdots \circ f(x)}^{k-1\ times}$,
compute the number of f.

$k \leq 10$

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let $\mathcal{F}_k(x)$ be the EGF of our answer, and $\mathcal{G}_k(x) = ln(\mathcal{F}_k(x))$, then

$$\mathcal{G}_k(x) = x\mathcal{F}_{k-1}(x)$$

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The Meaning of Exponential!

Probable Generating Function

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Definition

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By definition, we have

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} [z^i] \frac{d}{dz} \mathcal{F}(z) = \mathcal{F}'(1)$$

$$Var(X) = \sum_{i=0}^{\infty} [z^i] ((\frac{d^2}{dz^2} + \frac{d}{dz}) \mathcal{F}(z) - (\frac{d}{dz} \mathcal{F}(z))^2)$$

$$Var(X) = \mathcal{F}''(1) + \mathcal{F}'(1) - \mathcal{F}'(1)^2$$

Probable Generating Function

Question (SDOI2017)

There are n person, each one have a sequence,those n different sequences A_i consisting of only $0/1$ with length of m, and a two-faced standard die. You would like to roll the die, write down each time's result in a line, which means a result sequence s, until one A_i exist in s, namely this person win. compute each person's winning-possibility

$n, m \leq 300$

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let $f_{i,j}$ represents the possibility of i-th person wins after j-th rolling, and g_i of the possibility that no one wins after j-th rolling. And let $\mathcal{F}_i(z), \mathcal{G}(z)$ represent PGF of f_i and g

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Then we want to calculate $\mathcal{F}_i(1)$

Probable Generating Function

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$$\mathcal{G}(z)(\frac{1}{2}z)^m = \sum_{j=1}^{n}\sum_{l=1}^{m} a_{i,j}(l)\mathcal{F}_j(z)(\frac{1}{2}z)^{m-l}$$

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With

$$\sum_{i=1}^{n} \mathcal{F}_i(1) = 1$$

We have $n+1$ equations and $n+1$ variables, using Guass elimination to solve all $n+1$ variables.

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$$\mathcal{F}(x) * \mathcal{G}(x) = \sum_{n=1}^{\infty} \frac{\sum_{d|n} f_d g_{\frac{n}{d}}}{n^x}$$

and if $f(n)$ is a multiplicative function, then

$$\mathcal{F}(x) = \prod_{p \in \mathbb{P}} (1 + \frac{f(p)}{p^x} + \frac{f(p^2)}{p^{2x}} + \cdots)$$

Dirichlet Generating Function

Question (Project Euler 639)

A multiplicative function $f(n)$ *satisfies* $f(1) = 1, f(p^k) = p^2, p \in \mathbb{P}$, *calculate* $\sum_{i=1}^{n} f(i) mod 10^9 + 7$
$n \leq 10^{12}$

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$$\begin{aligned}\mathcal{F}(x) &= \prod_{p\in\mathbb{P}}(1 + \frac{f(p)}{p^x} + \frac{f(p^2)}{p^{2x}} + \cdots) \\ &= \prod_{p\in\mathbb{P}}(1 + p^2(\frac{1}{p^x} + \frac{1}{p^{2x}} + \cdots))\end{aligned}$$

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&= (\sum_{n=1}^{\infty} n^{2-x})\prod_{p\in\mathbb{P}}(1+(p^2-p^4)(p^{-2x}+p^{-3x}+\cdots))
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$$= (\sum_{n=1}^{\infty} n^{2-x})\prod_{p\in\mathbb{P}}(1+(p^2-p^4)(p^{-2x}+p^{-3x}+\cdots))$$

The following solution only need wome basic trick.

Outline

My reverie

Finally we've gone so far......

My reverie

Finally we've gone so far......

Sharelist:

- Horizon.
- Health.
- Mentality.

Outline

Multi-Variables Generating Function

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Multi-variables generating function has nothing special but more powerful.

let's start with an simple IMO problem, but we'll treat it rather slow, so the solution might be lengthy. but remember the generating function is for storage! And different kinds of generating function only have some subtle difference.

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Question (IMO '95/6)

let p be an odd prime number. Find the number of subsets A of the set $1, 2, \cdots, 2p$ such that

1. *A has exactly p elements*
2. *the sum of all elements in A is dibisible by p*

Multi-Variables Generating Function

Of course, we'll use a generating function to store information about the subsets. But what information might we want to keep track of? The problem talks about the size and the sum of the subsets, so we probably want to have information on both of these. To this end, wei'll design a generating function

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$$\mathcal{G}(x, y) = \sum_{n,k \geq 0} g_{n,k} x^n y^k$$

such that $g_{n,k}$ is the number of k-element subsets of $1, 2, \cdots, 2p$ with a sum of n. the answer to the problem is therefore $A = \sum_{i=1}^{\infty} g_{ip,p}$.
The construction of the generating function G is very straightforward. We ask ourselves, how could the number t affect a subset? If it is not in the subset, then it affects neither the sum nor the size. But if it is in the subset, then it increases the sum by t and the size by 1. So for each t we should have the term $(1 + x^t y)$ in $\mathcal{G}$.

Multi-Variables Generating Function

Therefore,

$$\mathcal{G}(x, y) = (1 + xy)(1 + x^2 y) \cdots (1 + x^{2p} y)$$

is the generating function weíre looking for. To get to our answer A, we need to extract coeffcients from two types of terms of $\mathcal{G} : y^p$, and powers of x^p.

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$$\sum_{n,k \geq 0, p|n} g_{n,k} y^k = \frac{1}{p} \sum_{i=0}^{p-1} \mathcal{G}(\varepsilon^i, y) \tag{1}$$

Multi-Variables Generating Function

So we need to calculate $\mathcal{G}(\varepsilon^k, y)$ for $0 \leq k \leq p-1$. When $k = 0$, $\mathcal{G}(1, y) = (1+y)^{2p}$. For $1 \leq k \leq p-1$, since $gcd(p, k) = 1$, the numbers $\{k, 2k, \cdots, pk\}$ form a complete list of residues modulo p, so

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Putting these values back into eaustion (1) produces

$$\sum_{n,k\ge 0, p|n} g_{n,k} y^k = \frac{1}{p}((1+y)^{2p} + (p-1)(1+y^p)^2)$$

Thus we got the answer

$$\frac{1}{p}(2(p-1) + \frac{(2p)!}{p!^2})$$

Multi-Variables Generating Function

Question

By a partition π of an integer $n \geq 1$, we mean here a representation of n as a sum of one or more positive integers where the summands must be put in nondecreasing order.
For any partition π, define A(π) to be the number of 1's which appear in π, and B(π) to be the number of distinct integers which appear in π.

Prove that, for any fixed n, the sum of A(π) over all partitions π of n is equal to the sum of B(π) over all partitions π of n.

Think! How namy information need to be stored....?

Q&A

Ask anything whitout hesitation!

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Thanks for listening!
Vielen Danke!