**CE 3890/5999 SCS**

# Homework 2: Introduction to Systems Analysis

## Due: Tuesday 09/25/2018

*Professor D. Work* *Fall 2018*

## 1 Assignment overview

This is an individual assignment which extends the ideas we discussed in lecture on systems modeling and analysis. Your solutions to the following questions should be typed in a suitable editor (e.g. Word or LaTex) or markdown text in a Jupyter notebook (if you'd like to combine text and code), with appropriate references (if you used them), figures, and text, as needed to explain your results. Submit an electronic copy on Brightspace, and bring a printed copy to class.

Presentation of your work is important. Please keep in mind figures should have a title, labels on the $x$ and $y$ axis, and a legend if needed. Figures should be numbered and contain a brief caption (e.g. "**Figure 1** Relationship between time in seconds ($x$-axis) and money in $ ($y$-axis) from 2013 survey data"). Additionally, every figure should be explained in the text of your write-up. (e.g. "The tradeoff between time and and money is illustrated in Figure 1, and are shown to be linearly related. Our results confirm that time is indeed equal to money, at least over the range investigated in this study...").

Your Python code should be included with your report. Your report should clearly describe your code, and explain how to run the appropriate files. Finally, you will need to upload your `.py` or `.ipynb` file(s) so we can run your code.

## 2 Calculus warm-up

**Question 2.1** Recall separation of variables is an important method for solving *ordinary differential equations* (ODEs). A convenient method for solving many ODEs is *separation of variables.* Roughly speaking, this technique involves separating the functions of the variable $x$ from the functions of $t$, and integrating to obtain an explicit equation for $x$. You may wish to consult Mathworld `http://mathworld.wolfram.com/SeparationofVariables.html` if you need a refresher. Solve the following ODEs for the initial condition $x(0) = x_0$:

$$\frac{dx}{dt} = x \sin(t)$$

and

$$\frac{dx}{dt} = e^x \left(a + t^2\right)$$

# 3 Population dynamics

Let $r(t)$ denote the number of rabbits in the woods at time $t$. In the absence of any predators, the population dynamics of the rabbits grows according to the following equation:

$$\frac{dr}{dt} = (b_r - d_r)\, r \tag{1}$$

where the parameter $b_r$ denotes the birth rate of rabbits (i.e. *number of rabbits born per rabbit*), and the parameter $d_r$ denotes the natural death rate of rabbits. The next few questions will explore the simple model

**Question 3.2** Note the change in the population due to births (and/or deaths) is proportional to the population $r(t)$. Is this a realistic assumption on the growth rate? Explain why or why not.

**Question 3.3** Assuming an initial population of rabbits at time $t = 0$ is given by $r(0) = r_0$, solve the ODE (1).

**Question 3.4** Assume the birth rate $b_r$ is larger than the death rate of rabbits $d_r$. What happens to the population of rabbits? Pick a few values of $b_r$ and $d_r$ such that $b_r > d_r$, and generate one plot of the population of rabbits as a function of time. Explain what you observe.

**Question 3.5** Assume the birth rate $b_r$ is smaller than the death rate of rabbits $d_r$. What happens to the population of rabbits? Pick a few values of $b_r$ and $d_r$ such that $b_r < d_r$, and generate one plot of the population of rabbits as a function of time. Explain what you observe.

**Question 3.6** Repeat your analysis a final time, this time assuming the birth rate and death rates are equal. Generate a plot of the population of rabbits as a function of time and explain what you observe. Why does this make sense?

# 4 Predator–prey dynamics

Unfortunately for rabbits, the woods are filled with wolves, who feast off of rabbits. The population of the wolves at time $t$ is denoted $w(t)$. In this section we will study the population evolution of a group of two species, which define a *predator–prey system*. The evolution dynamics of the system are given as follows. The population of rabbits evolve according to

$$\dot{r} = (b_r - d_r - kw)\, r \tag{2}$$

Note the similarity of (2) with (1) we studied earlier, except a new term given by $kwr$, where $k$ is a parameter to be explained below. This term reflects a loss in population due to interaction (consumption) with the predatory wolves. The loss of the rabbit population is proportional to the product of the number of rabbits and the number of wolves, which model the fact that more interactions occur when the rabbit and wolf populations are large, and the interactions decrease if either the wolf population or the rabbit population decrease. The number of rabbits killed due to the interaction is regulated by the parameter $k$.

The dynamics of the wolf population is given by

$$\dot{w} = (b_w - d_w + er)\, w \tag{3}$$

Like the rabbits, the wolf population has a component associated with the nominal birth rate of wolves $b_w$ and the death rate of wolves $d_w$. The term $erw$ represents an additional growth in the population due to consumption of rabbits, where $e$ is the parameter which describes the efficiency by which consumed rabbits

| parameter | value |
|---|---|
| $b_r$ | 0.05 |
| $d_r$ | 0.01 |
| $b_w$ | 0.1 |
| $d_w$ | 0.2 |
| $k$ | 0.001 |
| $e$ | 0.002 |

Table 1: Summary of parameters

are turned into new wolves. In other words, it represents how well–fed wolves increases the reproduction rate of wolves.

You will notice that the equations (2) and (3) are two coupled ODEs, and it may not be obvious how to solve the equations by hand. Regardless, we can still get some understanding of the system by examining how it behaves in some special cases. The values of the parameters in are summarized in Table 1.

**Question 4.7** Given some non-zero rabbit population, what do you expect to happen in the short term to the rabbit population if the wolf population becomes very small? Qualitatively explain what will happen.

**Question 4.8** What do you expect to happen to the wolf population (assuming it is nonzero initially) if the rabbit population drops to zero? Explain.

**Question 4.9** Suppose the wolf population is small, but the rabbit population is very large. What will happen in the short term to the wolf population? Why?

We will now take advantage of existing tools for approximating solutions to ODEs numerically. We will use the Runge-Kutta method of order 5(4). The method is accessed using the `scipy.integrate` package function `solve_ivp`. Documentation: `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html`.

**Question 4.10** Write a function called `pred_prey`. Your function should have the following syntax:

```
def pred_prey(t, y):
...
return y_dot
```

Where $y = [r, w]$. In other words, $y$ is a vector where the first entry is the rabbit population $r(t)$ at time $t$, the second entry is the wolf population $w(t)$ at time $t$. Note, this syntax is necessary in order to use any scipy ODE solver, which specifies that $y$ have shape (n,) denoting a vector. The function should return the variable named y_dot (i.e. the derivative of $y$), which is given by $\dot{y} = [\dot{r}, \dot{w}]$. Again, we are just stacking the derivatives of each state in a vector. Your job is to implement the function `pred_prey(t, y)`, which returns $\dot{y}(t)$ given $y(t)$.

**Question 4.11** Write a script to numerically approximate the solution of the predator prey dynamics using `solve_ivp`. Simulate the model starting from $t = 0$ through $t = 1000$. The initial populations are given by $w(0) = 10$ and $r(0) = 10$.

You will need to figure out the syntax of `solve_ivp`, for example using the command: `help(solve_ivp)`. **Hints:** The "callable function" is only the function's name, `pred_prey`, without parenthses or arguments. You will not need any of the advanced options, but you may wish to manually specify time values for which the approximation should be evaluated. You should generate (on the same plot) the population of the rabbits and the wolves as a function of time, and explain what you observe.

**Question 4.12** Experiment with various changes in the initial conditions of the rabbit and wolf populations, and describe any interesting features you discover. Similarly, you may want to play with some of the parameters in the model to understand how they influence the population evolution. There is no single or simple answer for this question, you need to study the model and present a convincing case (with figures and analysis as needed), that you understand how the system behaves.

**Question 4.13** **(Optional/Bonus)** Derive the solution to coupled equations (2) and (3).