Measurement of Gravitational Constant with Cavendish Balance

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The Cavendish experiment was the first method for calculating the gravitational constant G. The experimental apparatus includes a torsion pendulum with two large masses placed nearby which exert a gravitational force on the masses attached to the pendulum, causing the torsion wire to twist. From the torque induced by the twisting and the known mass of the large spheres, a value for the gravitational constant can be calculated through a few different methods. The methods used in this report were: final deflection from equilibrium of the torsion pendulum, graphical analysis of the pendulum's oscillation period, and initial acceleration of the pendulum. A value of G= $(6.74\pm0.19)\times10^{-11}\ N\cdot m^2/kg^2$, G= $(6.65\pm0.19)\times10^{-11}\ N\cdot m^2/kg^2$, and G= $(4.51\pm0.23)\times10^{-11}\ N\cdot m^2/kg^2$ were calculated with their respective methods.

I. Introduction

The gravitational constant G is the least accurately known fundamental constants in all of physics. The first relatively accurate measurement of the gravitational constant was done by Henry Cavendish by measuring the force between masses in a laboratory setting. He viewed his measurement of the gravitational constant as a way to calculate the average density of Earth, yielding a value of G=$6.74\times10^{-11}\ N\cdot m^2/kg^2$. Surprisingly enough, this estimate is within 1% of our accepted value today, which can be credited to our lack of advancement beyond Cavendish's original experiment. Many physicists have attempted to more accurately determine the gravitational constant through a variety of techniques, all resulting in inconsistent values to the accepted value of G. A more accurate technique may be discovered in the future, but for now we continue to use the original method done by Cavendish over two-hundred years ago.

In order to perform the Cavendish experiment we must use a torsion balance. Originally invented by Coulomb in his pursuit to measure the electrostatic force between objects, this apparatus allows us to observe the effect of very weak forces that would normally be negligible in an ordinary setting. It works by suspending a bar with a thin fiber and enclosing it in a metal casing to reduce noise from other forces. When a weak, unknown force is applied to the ends of the bar, the fiber twists until it reaches an equilibrium position. The unknown force has applied a torque to this system which is now balanced by the torque due to the twisting of the fiber, thus giving us a way to determine the magnitude of this applied force. For this report, we will apply a gravitational force to the torsion balance using two large spheres and utilize Newton's Law of Universal Gravitation in order to experimentally determine a value for the gravitational constant G.

II. Experiment and Methods of Analysis

Shown in Fig. 1 is a diagram of the experimental setup, and a separate diagram of just

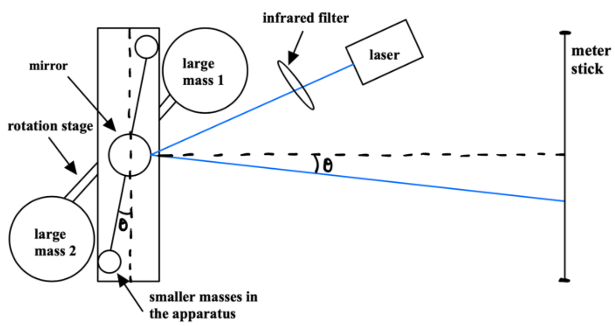

Figure 1. Digital sketch of the experimental setup from an above perspective. Shows the large masses, rotation stage, torsion balance (mirror and smaller masses), laser with infrared filter, and meter stick.

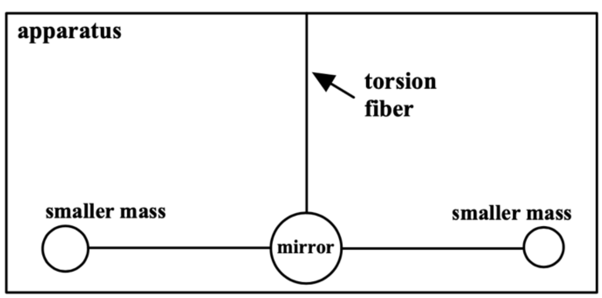

Figure 2. Digital sketch of the torsion balance from the front. Includes the torsion wire, mirror, and smaller masses attached to the suspended bar.

the apparatus is displayed in Fig. 2. The two larger masses that exert the gravitational force are placed on a rotation stage level with the torsion balance. Inside the torsion balance, two smaller masses are attached to the bar, along with a mirror placed at the midpoint which is exposed to a laser. The laser passes through an infrared filter to reduce the effects of heat before being reflected onto a meter stick located a distance away from the torsion balance. When the large masses are on the rotation stage, they each generate a torque on the smaller masses, thus making the torsion fiber twist and the mirror rotate to produce a deflection angle θ.

Using this setup, there are three methods for determining the gravitational constant: final deflection, equilibrium positions using graphical analysis, and initial acceleration. The second method is an extension of the first and utilizes the same formula for calculating G.

To complete the first method, set up the experiment identically to figure 1, allow the laser projection to settle and record its equilibrium position. Next, rotate the stage to the mirror image of figure 1 and the laser projection will begin to oscillate, record its period of oscillation and when it settles record its equilibrium position. The period is necessary to determine the torsion constant of the wire and torque produced. This is all the data needed to complete the first method, and the second method is identical to this except it uses a video recording of the oscillations to calculate the period by plotting a displacement vs time graph in logger pro. Utilizing the equation below (Eq. 1) and its list of parameters (Table 1) we can successfully determine a value of G.

$$G = \frac{\pi^2 \Delta S (d^2 + \frac{2}{5} r^2)}{T^2 L M d} \left(\frac{1}{b^2} - \frac{b}{(b^2 + 4d^2)^{3/2}} \right)^{-1}$$

Equation 1.

Parameter	Description
d	Distance from torsion wire to small sphere
r	Radius of small spheres
L	Distance from torsion balance to laser projection
M	Mass of large spheres
b	Distance between centers of small and large mass

ΔS	Equilibrium displacement from the two positions
T	Period of oscillation

Table 1. List of Equation 1 parameters.

This equation was derived by setting the gravitational torque equal to the torque in the wire, $2\tau_{grav} = \tau_{wire}$ (2 because there are two large masses twisting the wire in the same direction). τ_{grav} is calculated using Newton's Universal Law of Gravitation $F = \frac{GMm}{b^2}$, and $\tau_{wire} = k\theta$ where $k = \frac{4\pi^2 I}{T^2} = \frac{8\pi^2 m(d^2+\frac{2}{5}r^2)}{T^2}$ is the torsion constant and $\theta = \frac{\Delta S}{4L}$ is the deflection angle. Plugging these terms in to the torque balancing equation and solving for G will result in equation 1.

The final method eliminates the dependence on the torque from the wire and is set up similarly to the first two methods except only the first minute of oscillation is needed. We calculate the initial acceleration of this oscillation by video recording and analyzing in logger pro to produce a displacement vs time graph. After that we plug into the simple equation $2\tau_{grav} = I\alpha$, where α is rotational acceleration and solve for G. The resulting equation is:

$$G = \frac{\alpha}{M}(d^2 + \frac{2}{5}r^2)\left(\frac{1}{b^2} - \frac{b}{(b^2+4d^2)^{3/2}}\right)^{-1}$$

Equation 2.

III. Results

The primary results for the first method of final deflection were calculated using the data below (Table 2). The two measurements made here that are not physical parameters are ΔS and T. When plugged into equation 1 these values give us G = (6.74±0.19) ×10^{-11} $N \cdot m^2/kg^2$.

Parameter	Value	Error	Units
d	.0499	±0.0005	m
r	.0076	±0.00005	m
L	7.16	±0.05	m
M	1.495	±0.00005	kg
b	.0519	±0.0005	m
ΔS	.195	±0.005	m
T	560	±3	s

Table 2. Parameter values and uncertainties used in first method calculation.

To complete the second method, I recorded the oscillations and used logger pro to track the laser and plot a damped harmonic for displacement vs time, which is shown in Figure 3. From this plot we obtained a period of oscillation of 556.4 seconds and ΔS of .19 meters. Using these values along with Table 2 and equation 1, we obtain G= (6.65±0.19) ×10^{-11} $N \cdot m^2/kg^2$. Both values for G shared a propagation of uncertainty which was defined as:

$$\delta G = \sqrt{\left(\frac{\partial G}{\partial d}\delta d\right)^2 + \left(\frac{\partial G}{\partial r}\delta r\right)^2 + \left(\frac{\partial G}{\partial L}\delta L\right)^2 + \left(\frac{\partial G}{\partial M}\delta M\right)^2 + \left(\frac{\partial G}{\partial b}\delta b\right)^2 + \left(\frac{\partial G}{\partial S}\delta S\right)^2 + \left(\frac{\partial G}{\partial T}\delta T\right)^2}$$

Where the del value for each term is located in Table 2 under the error column. The largest sources of uncertainty were ΔS and T, which can be credited to issues while performing the experiment since they were the only two values obtained experimentally. The other sources of uncertainty are human error and limitations when completing a measurement.

The final method was done by recording the first minute of motion when the large masses are rotated to the opposite side. I then generated a plot of displacement vs time (Figure 4) and calculated the initial acceleration by taking the second derivative of the line of best fit. Using the

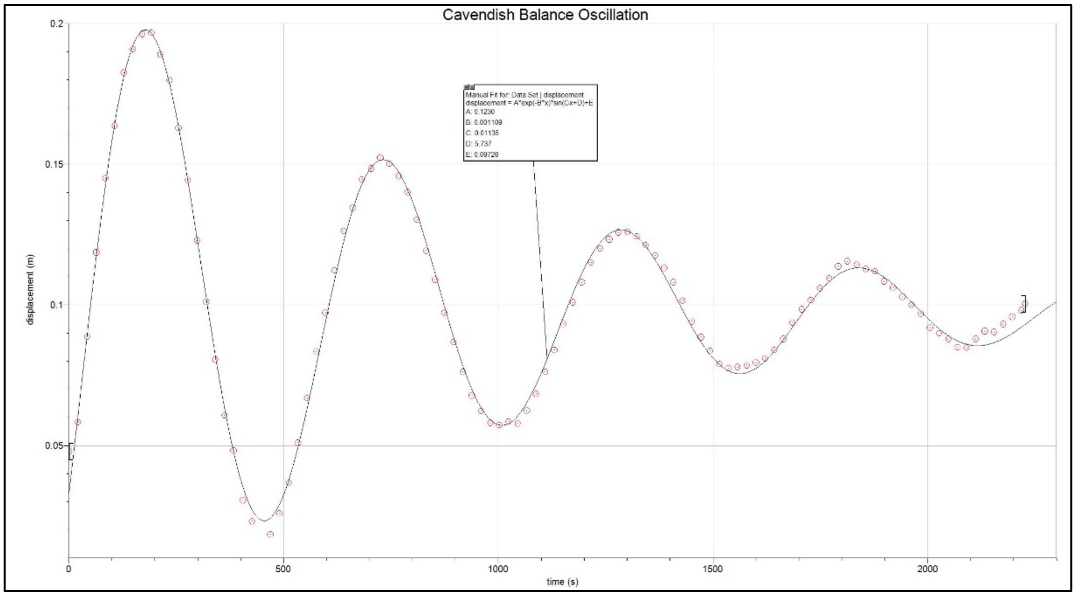

Figure 3. Oscillation of laser projection over an hour when the positions of the large masses are moved from position the position in figure 1 to its mirror image. Fit with a damped harmonic with a period of 556.4 seconds.

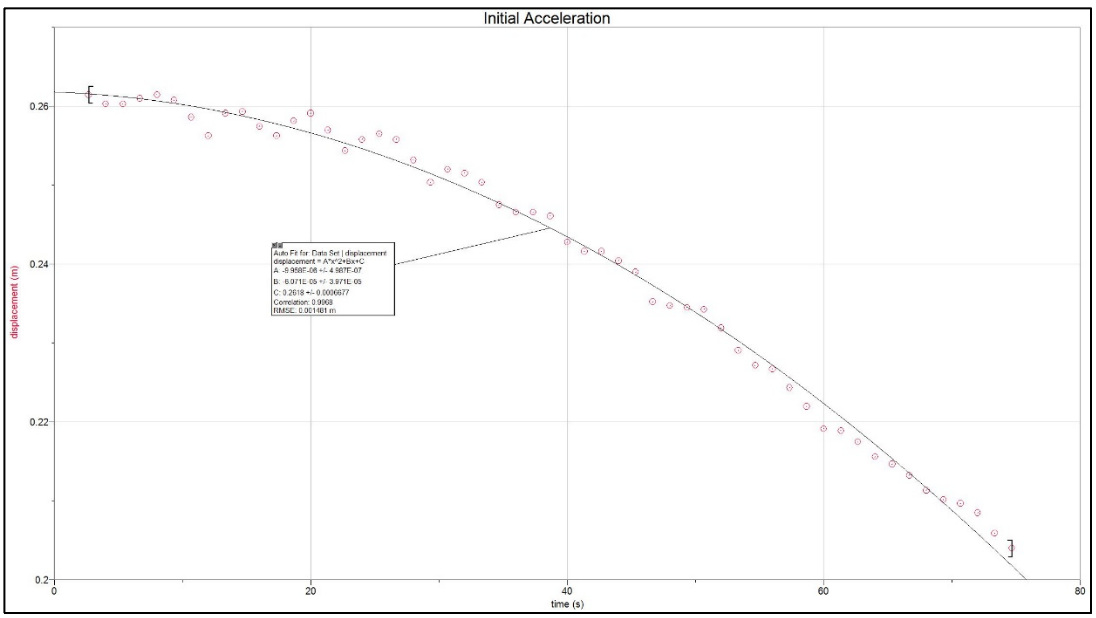

Figure 4. Initial acceleration of laser projection when the positions of the large masses are moved from position the position in figure 1 to its mirror image. Fit with a quadratic with an initial acceleration of $\alpha = 9.96 \times 10^{-6}\ m/s$.

initial acceleration $\alpha = 9.96 \times 10^{-6}\ m/s$ from Figure 4 and parameters from Table 2, we can plug into Equation 2 and get G= (4.51±0.23) ×10^{-11} $N \cdot m^2/kg^2$. Similar to before, the propagation of uncertainty was calculated using:

$$\delta G = \sqrt{\left(\frac{\partial G}{\partial d}\delta d\right)^2 + \left(\frac{\partial G}{\partial r}\delta r\right)^2 + \left(\frac{\partial G}{\partial M}\delta M\right)^2 + \left(\frac{\partial G}{\partial b}\delta b\right)^2 + \left(\frac{\partial G}{\partial \alpha}\delta \alpha\right)^2}$$

The del values are the same as Table 2 and $\delta\alpha = \pm 0.000005$. The large deviation from the theoretically accepted value of G contained in the literature can be credited to systematic error since tracking the laser projection in the early portion of movement was very difficult due to unwanted vertical oscillations.

IV. Conclusion

By applying a small gravitational force to a torsion balance, we were able to measure the gravitational constant using three different methods: final deflection from equilibrium, graphical analysis of the pendulum's oscillation period, and initial acceleration. Their respective values were:

$$G= (6.74 \pm 0.19) \times 10^{-11}\ N \cdot m^2/kg^2$$

$$G= (6.65 \pm 0.19) \times 10^{-11}\ N \cdot m^2/kg^2$$

$$G= (4.51 \pm 0.23) \times 10^{-11}\ N \cdot m^2/kg^2.$$

The first two methods produced similar values, which is logical since their set ups were identical. The torsion balance is extremely sensitive, I found out the hard way, which made obtaining accurate data for the third method extremely difficult when trying to track the initial acceleration, although I am pleased that the order of magnitude was the same as the literature value for G. One refinement to gathering more accurate data would be a more isolated setting. The sensitivity of the Cavendish balance makes any force nearby a potential source of error.

References

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