School of Applied Sciences and Engineering
EAFIT - Medellín

From Error to Learning: How Backpropagation Trains Neural Networks

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Class Topics

Class Overview

Basic Concepts

Computational Graphs and Propagation

Examples

Exercises and activities

Class Objective

To explain the fundamental concepts, mathematical formulation, and practical considerations of the Backpropagation algorithm in neural networks.

Learning Outcomes

- **Explain** the fundamental principles of Backpropagation.

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- **Derive** and analyze the mathematical foundations of Backpropagation.

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- **Explain** the fundamental principles of Backpropagation.
- **Derive** and analyze the mathematical foundations of Backpropagation.
- **Apply** the Backpropagation algorithm to practical neural network problems.

Target Audience

Undergraduate, Master's, and PhD students, as well as postgraduate students in applied sciences and engineering, who have basic knowledge of neural networks, calculus, and proficiency in Python or similar.

Class Structure

- Conceptual Explanation
- Mathematical Derivation
- Examples
- Exercises and activities

Basic Concepts - Optimization

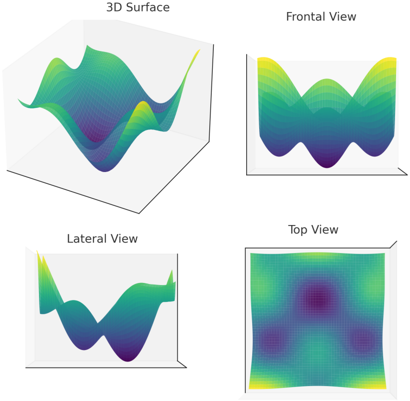


Figure: Multivariate function

Figure: General concept

Basic Concepts

Derivative: Measures the rate of change of a function.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

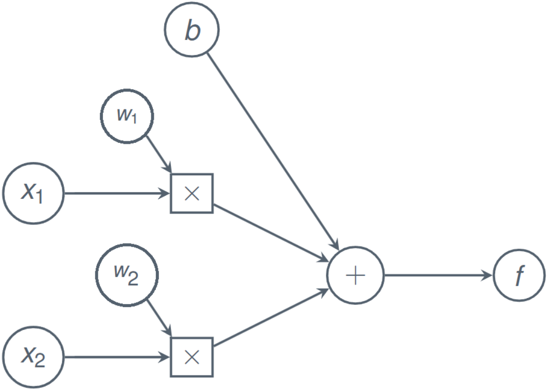
Gradient (Generalized Derivative): Used in multivariable functions.

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)^T \tag{2}$$

Computational Graph

Definition: A computational graph represents a mathematical function in a structured form. Here, we compute:

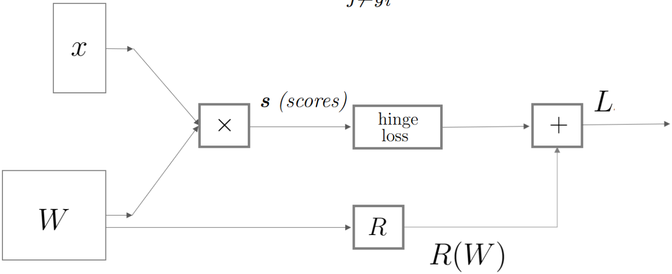
$$f = (x_1 w_1 + x_2 w_2) + b$$



Computational Graph

$$f = Wx$$

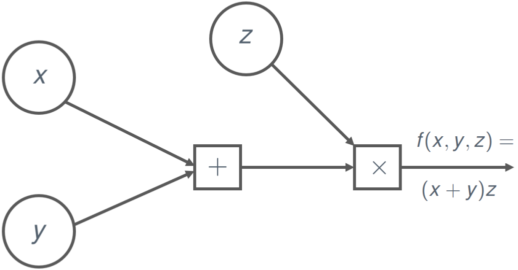
$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$



Example: We define the function:

$$f(x, y, z) = (x + y)z$$

Computational Graph:



Example 01:

The function can be rewritten in steps:

$$f(x, y, z) = (x + y)z$$
$$q = x + y, \quad f = q \cdot z$$

Partial derivatives:

$$\frac{\partial f}{\partial q} = z, \quad \frac{\partial f}{\partial z} = q, \quad \frac{\partial q}{\partial x} = 1, \quad \frac{\partial q}{\partial y} = 1$$

Using the **chain rule**, we compute:

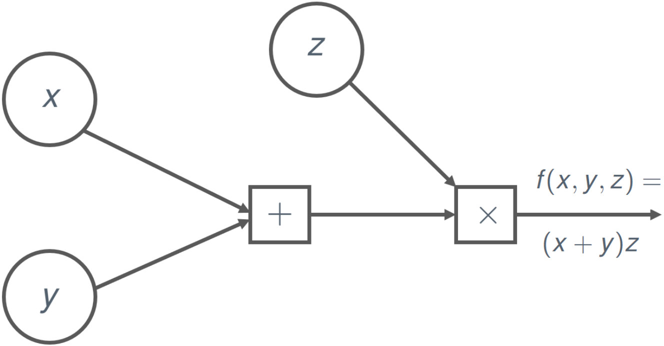
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = z$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = z$$

$$\frac{\partial f}{\partial z} = q$$

Forward Propagation :

$$f(x, y, x) = (x + y)z = qz$$



Example 01:

Forward Propagation :

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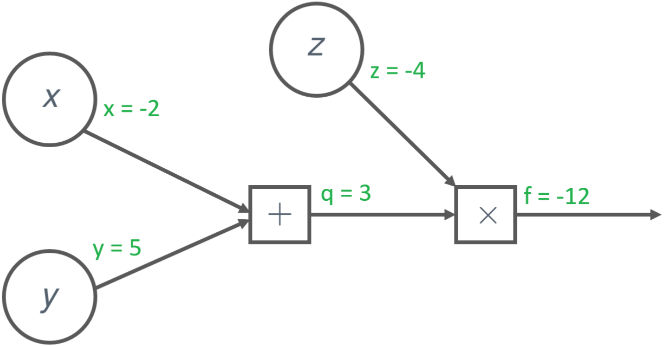

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Backward Propagation :

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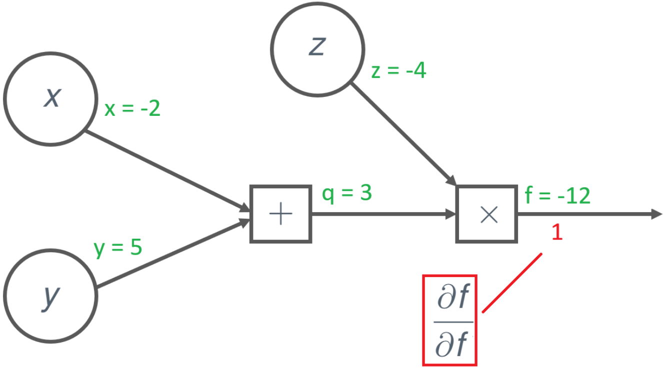

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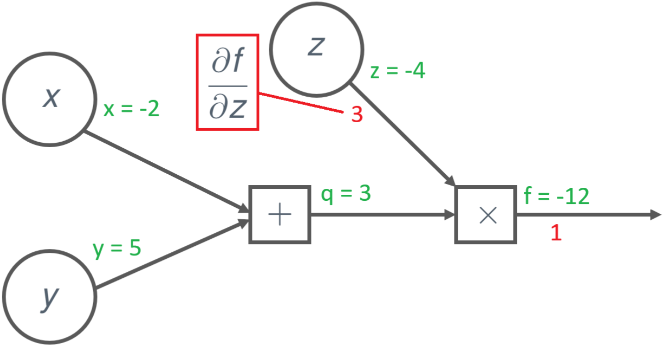

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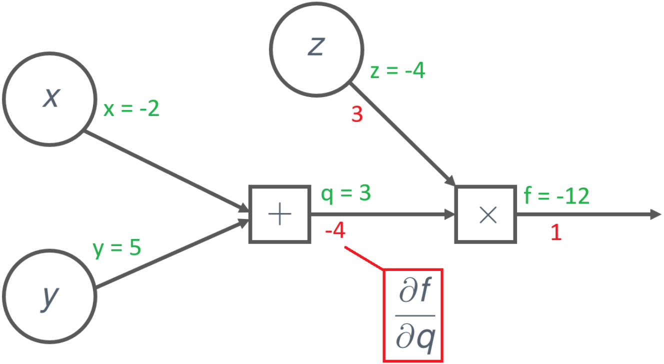

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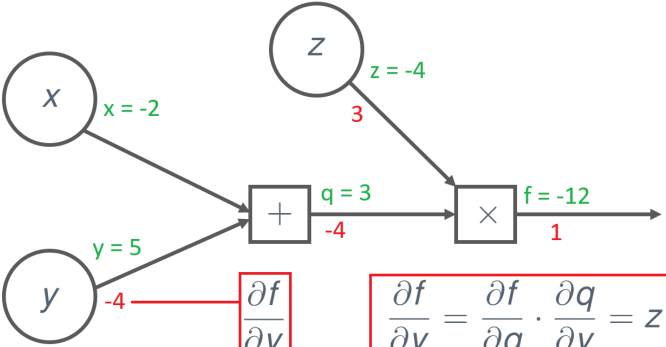


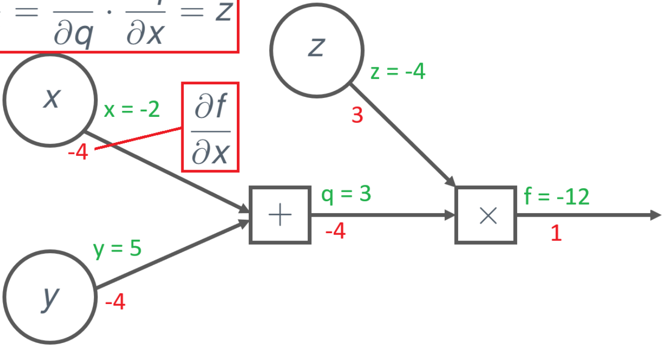
$$\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = z$$

Backward Propagation :

$$f(x, y, x) = (x + y)z = qz$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = z}$$



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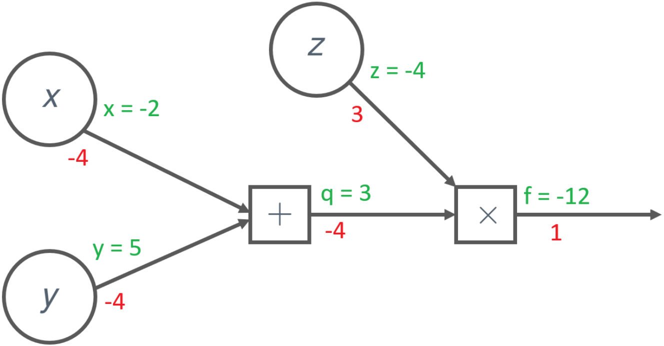
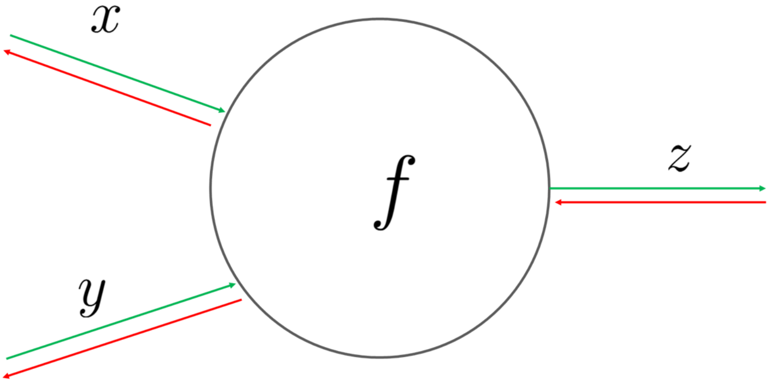


Activations:



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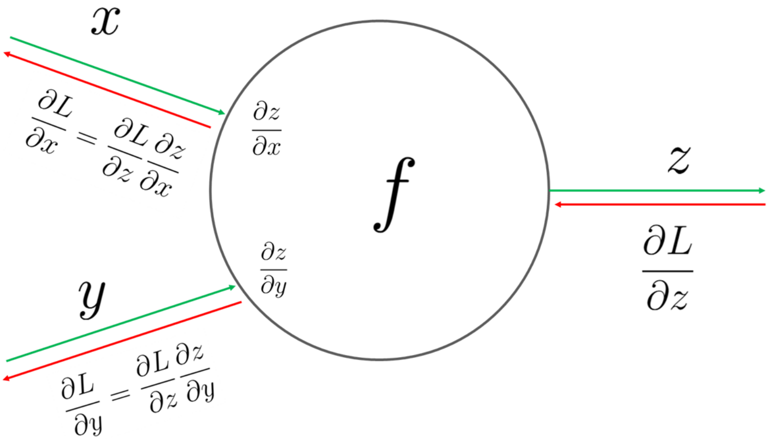

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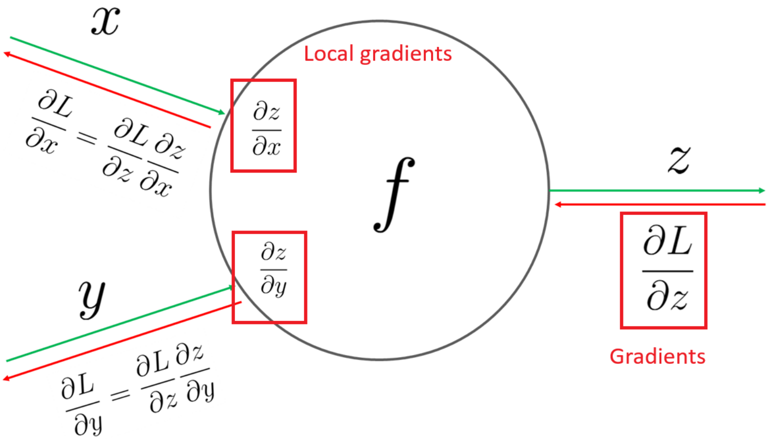
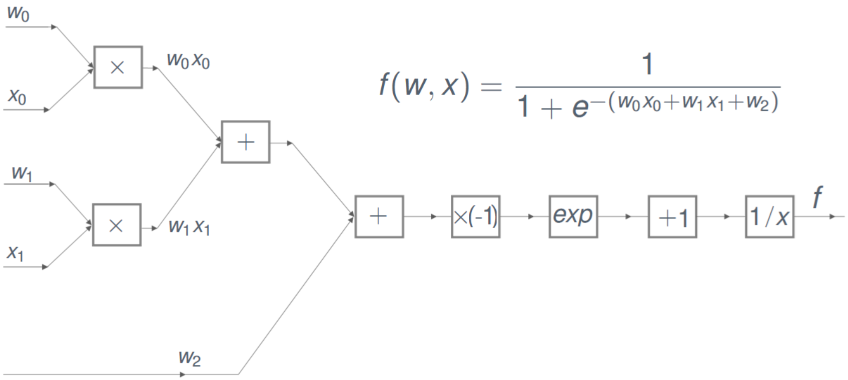


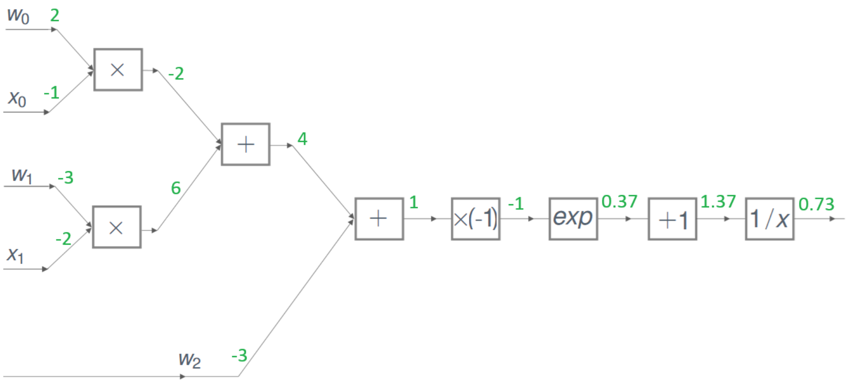
Forward Propagation :



$$f(w,x) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$

Example 02:

Forward Propagation :



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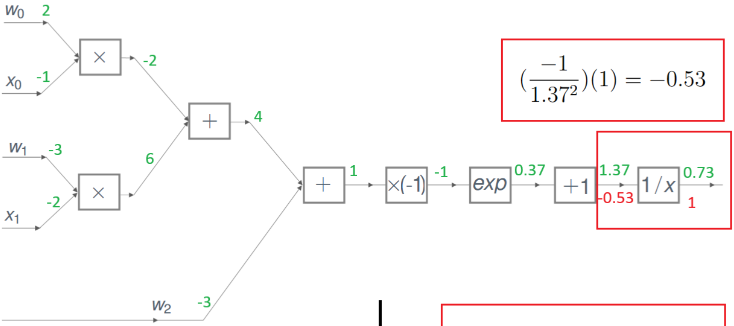

$$\boxed{\left(\frac{-1}{1.37^2}\right)(1) = -0.53}$$

$$f(x) = e^x \rightarrow \frac{df}{dx} = e^x$$

$$f_a(x) = ax \rightarrow \frac{df}{dx} = a$$

$$\boxed{f(x) = \frac{1}{x} \rightarrow \frac{df}{dx} = -\frac{1}{x^2}}$$

$$f_c(x) = c + x \rightarrow \frac{df}{dx} = 1$$

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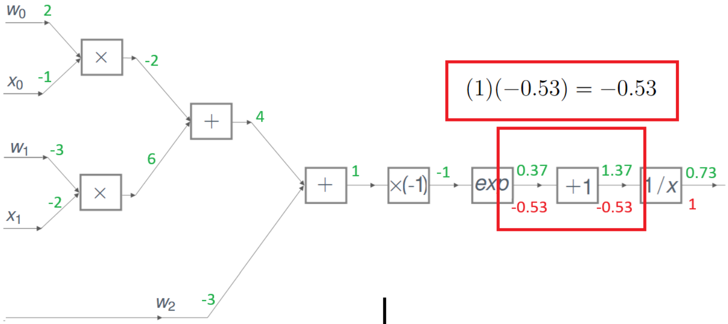

$$(1)(-0.53) = -0.53$$

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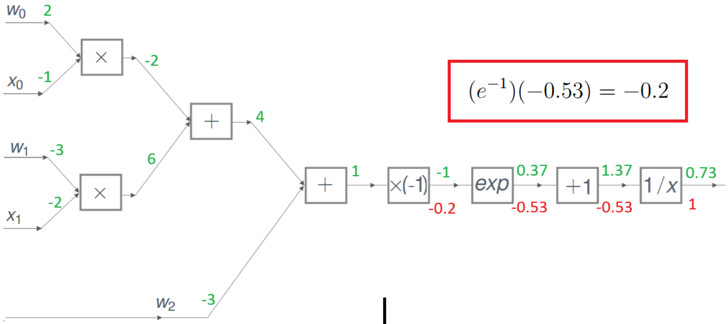

$$(e^{-1})(-0.53) = -0.2$$

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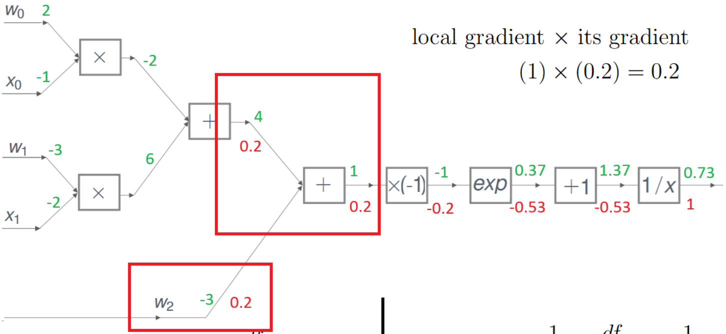

$$\text{local gradient} \times \text{its gradient}$$

$$(1) \times (0.2) = 0.2$$

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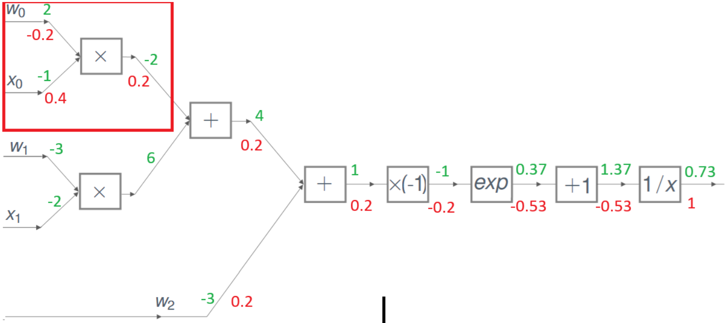

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Example 02:

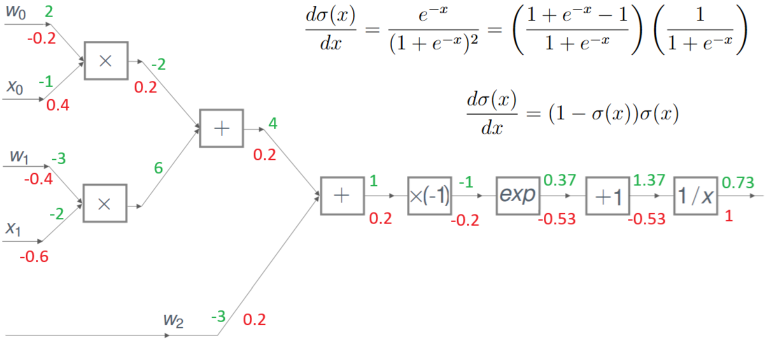
Another perspective:

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\sigma(x) = \frac{1}{1+e^{-(x)}}$$

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right)$$

$$\frac{d\sigma(x)}{dx} = (1-\sigma(x))\sigma(x)$$



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$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

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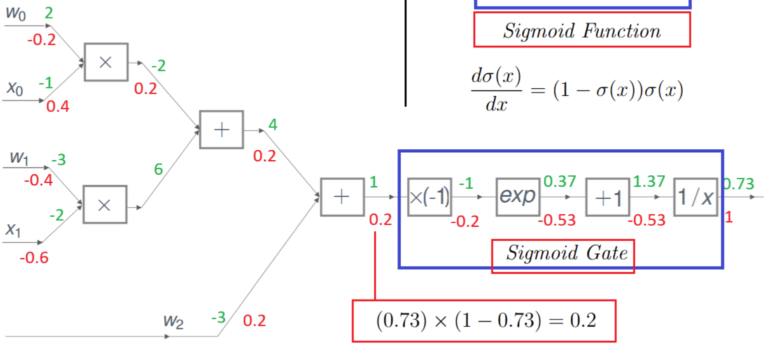
Sigmoid Function

$$\frac{d\sigma(x)}{dx} = (1-\sigma(x))\sigma(x)$$

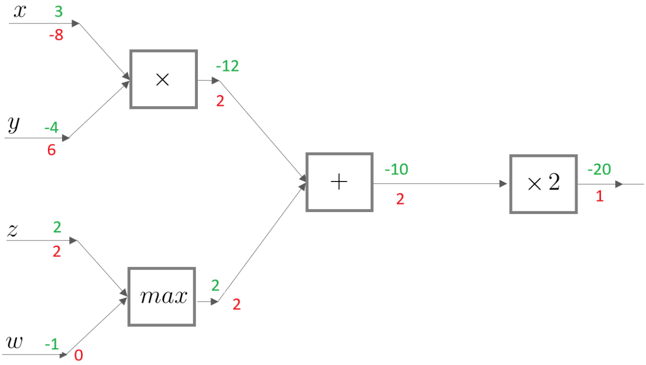


Patterns in backward flow:

Add gate: Gradient distributor

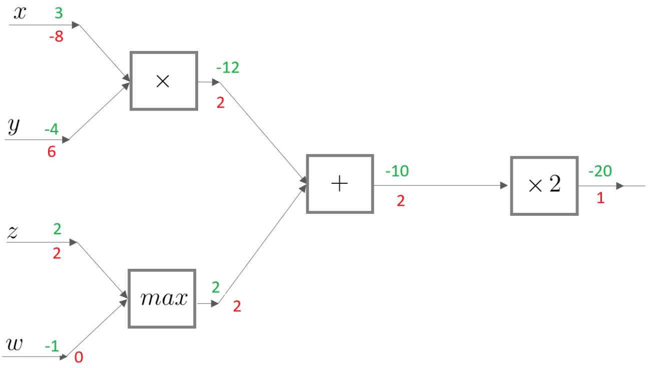
Max gate: Gradient Router

Mul gate: Gradient Switcher



Patterns in backward flow:

Gradients add at branches



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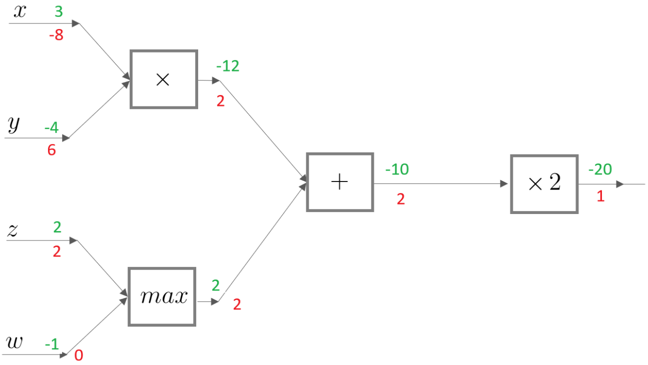

Exercises:

Task: Please complete Exercises by accessing the following Colab notebook link:

Exercises Notebook

Expected results:

- Clearly documented Python code implementation demonstrating your solution.
- Explanation of the mathematical concepts involved.
- Visual representations (plots or graphs) of your results.
- A concise interpretation and analysis of the outputs obtained.

Ensure your notebook is fully executed and your responses are thorough, clear, and well-supported by appropriate comments and explanations.

Questions?