Geometría Diferencial Discreta

FaMAF - UNC

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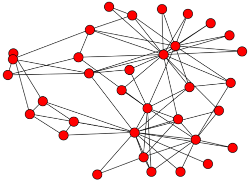
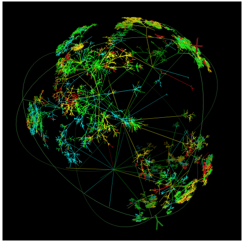
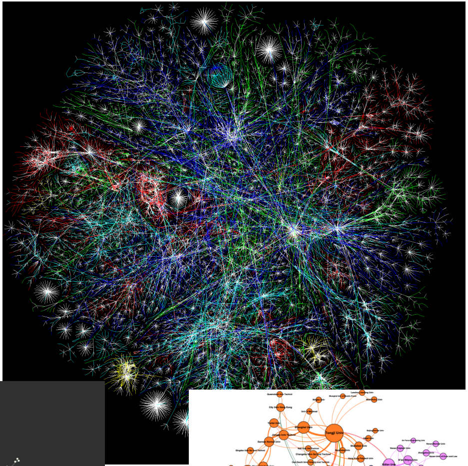
Outline

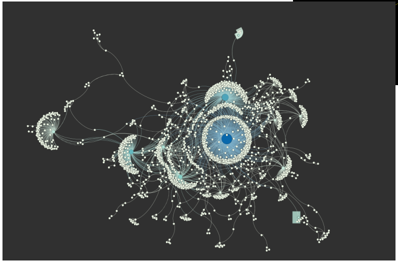
- Redes complejas
- Hipergrafos (higher order networks)
- Grupos de Renormalización
- Repaso de álgebra lineal

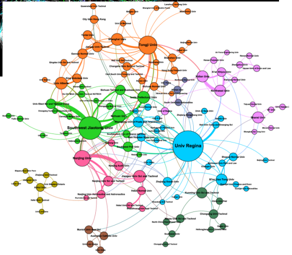
Complex Networks





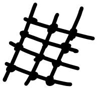






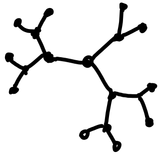
Lattice

$N \sim L^d$



Complex Networks

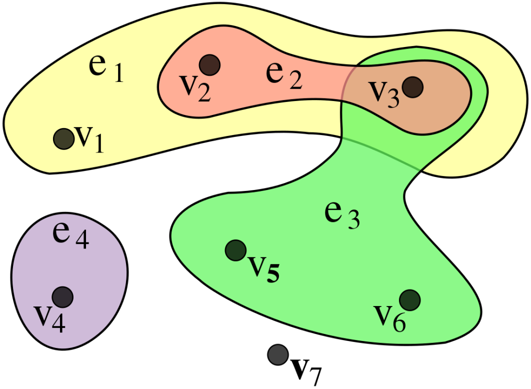
$N \sim e^L$



$P(k) \sim k^{-\gamma}$

loops, modules, etc.

Hypergrafos



No tiene la estructura de un espacio topológico.

GR

regularizadores

campo (Fourier)

$$Z(\mu) = \int [d\tilde{\phi}]_{\lambda_{min}}^{\lambda_{max}} \, e^{-\tilde{S}_{\mu}[\tilde{\phi}]}$$

acción

parámetros

$$[d\tilde{\phi}]_{\lambda_{min}}^{\lambda_{max}} = \prod_{k} d\tilde{\phi}(k)$$

$$S_{\mu}[\phi] = \int \mathcal{L}_{\mu}(\phi, \nabla\phi, \dot{\phi}, x, t)\, dx\, dt$$

densidad lagrangiana

Acción de teoría efectiva (mesoscópica) a escala $\varepsilon\Lambda_{max}$

$$e^{-\tilde{S}_{\varepsilon\mu}[\tilde{\phi}]} := \int [d\tilde{\phi}]_{\Lambda_{min}}^{\varepsilon\Lambda_{max}} e^{-\tilde{S}_{\mu}[\tilde{\phi}]} \qquad 0 \lesssim \varepsilon < 1$$

En algunos casos (GR)

$$\tilde{S}_{\varepsilon\mu}[\tilde{\phi}] \approx \tilde{S}_{\mu_\varepsilon}[\tilde{\phi}_\varepsilon]$$

$\tilde{\phi}_\varepsilon$ campo reescalado

μ_ε param. reescalados

$$S(\psi) = \int_{\Omega} \underbrace{\left(\frac{1}{2} (\nabla \psi)^2 - f \psi \right)}_{L} d\Omega \qquad \psi, f, g \in (\Omega \to \mathbb{R})$$

Ec. Poisson

$$\begin{cases} -\Delta \phi = f & \text{en } \Omega \\ \phi = g & \text{en } \Omega \end{cases} \iff \begin{array}{l} \phi = \operatorname{argmin} S(\psi) \\ \psi = g \text{ en } \Omega \end{array}$$

$(\nabla \psi)^2 \approx$ rugosidad de $\psi \approx$ regularizador

$\Delta \psi = 0 \iff \psi$ es func. armónica.

Repaso Algebra Lineal

Espacio vectorial

$$v + cv' \in V \qquad v, v' \in V$$

$$c \in \mathbb{R}$$

$$V^* = [V \to \mathbb{R}] \qquad \text{dual}$$

$$f \in V^* \qquad f(v + cv') = f(v) + c f(v')$$

Base

$\mathcal{B} = \{ b_1, \dots, b_n \}$ base de V

$$v = \sum_{i=1}^{n} v^i b_i = v^i b_i$$

Notación de Einstein

$v^1, \dots, v^n \in \mathbb{R}$

Base dual

$\mathcal{B}^* = \{ b^1, \ldots, b^n \}$ base de V^* dual a $\mathcal{B}$

$$b^i(b_j) = \delta^i_j$$

Doble dual

$$V^{**} := (V^*)^* = [V^* \longrightarrow \mathbb{R}]$$

$$V \simeq V^{**}$$ isomorfismo natural

$$V \ni v \longleftrightarrow \hat{v} \in V^{**} \qquad \hat{v}(f) := f(v)$$

V y V^* No son naturalmente isomorfos

$b_i \not\longleftrightarrow b^i$ ya que b^i depende de b_j

Producto interno

$v \cdot v \geq 0$ $\qquad$ $v \cdot v = 0 \Rightarrow v = 0$

$v \cdot v' = v' \cdot v$

$v \cdot (v' + c v'') = v \cdot v' + c\, v \cdot v''$

Teorema de Riesz e Isom. Musicales

Un prod. interno induce isomorfismos

$$V \ni v \underset{\sharp}{\overset{\flat}{\rightleftarrows}} f \in V^* \qquad \sharp = \flat^{-1}$$

$$f(v') := v \cdot v'$$

Operador Adjunto

$$a, a^* \in [V \to V]$$

$$a^*(v) \cdot v' = v \cdot a(v')$$

Prod. interno

$$m_{ij} = e_i \cdot e_j$$

$$m^{ij} = e^i \cdot e^j := \#(e^i) \cdot \#(e^j)$$

$$v = v^i b_i \qquad \flat(v) = v_i b^i \qquad v_i = m_{ij} v^j$$

$$f = f_i b^i \qquad \#(f) = f^i b_i \qquad f^i = m^{ij} f_j$$

$$\flat(b_i) \neq b^i \qquad \#(b^i) \neq b_i$$

En resumen

$$
\begin{tikzcd}
v \in V \arrow[dr] & & V^* \ni f \arrow[dl] \\
& \cdot \arrow[d, "\text{geom.}"'] & \\
& V \simeq V^* &
\end{tikzcd}
$$

prod. interno

$\cancel{V^{**}, V^{***}, \ldots}$

no hacen falta

$\flat$ y $\sharp$ isom. music.

Free Vector Space

X conjunto $\qquad e_x \in (X \to \mathbb{R})$

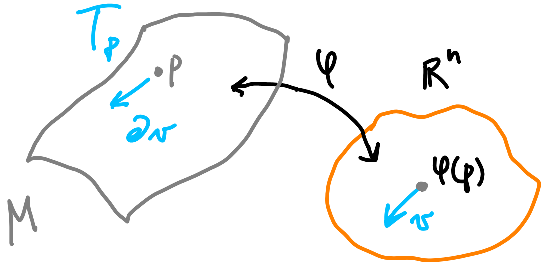
$$e_x(x') = \begin{cases} 1 & x = x' \\ 0 & \text{c.c.} \end{cases}$$

$B_X = \{ e_x : x \in X \}$ es base de espacio vectorial

$$F_X = \operatorname{span}(B_X)$$

Geometría Diferencial Continua

deriv. direccional



$\mathbb{R}^n \simeq T_p$

bases espacio

$e_i \longleftrightarrow \partial_i \in T_p$ tang.

$e^i \longleftrightarrow \varepsilon^i \in T_p^*$ cotang.

$\varepsilon^i(\partial_j) = \delta^i_j$

$m_{ij} = \partial_i \cdot \partial_j$

métrica

M es Riemm.

Teorema de Stokes Generalizado

cálculo $\qquad \displaystyle\int_M d\omega = \int_{\partial M} \omega$

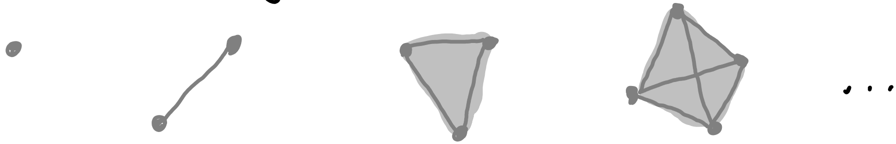
topología

álgebra

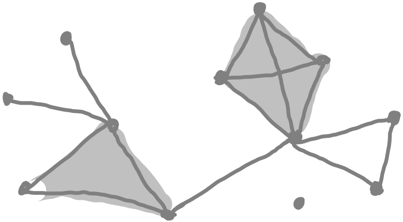
geometría ?

Geometría Diferencial Discreta

Complejos Simpliciales

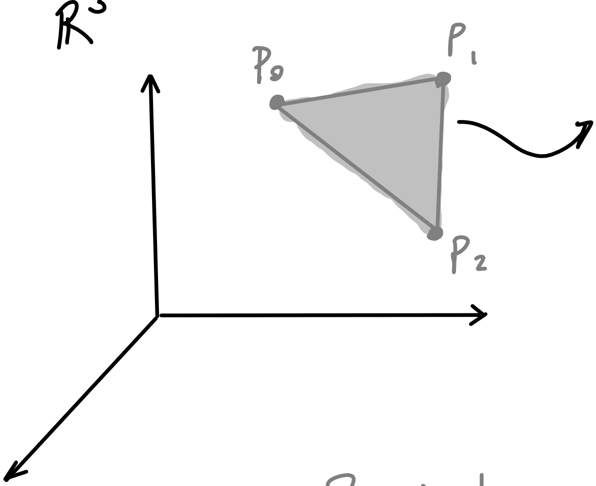


0-simplex K_0 | 1-simplex K_1 | 2-simplex K_2 | 3-simplex K_3 | …



triang. de región en $\mathbb{R}^n$

$K := K_0 \cup \dots \cup K_n$



$\lambda^i p_i$

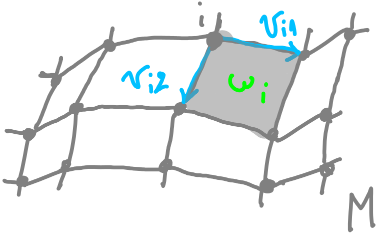
$\Sigma_i \lambda^i = 1$

$\lambda^i \geq 0$

2-simplex

Integrales y Formas

$$\int_M \omega \approx \sum_i \omega_i(v_{i1}, \dots, v_{ik})$$



$$\left.\begin{array}{l} * \ \omega \quad \text{multilineal} \\ \\ * \ \omega(\dots, v, \dots, v, \dots) = 0 \end{array}\right\} \Rightarrow \omega \quad \text{alternante}$$

Orientación

$$\omega(\ldots, -v, \ldots) = -\omega(\ldots, v, \ldots) \qquad \text{orient.}$$

$$\omega(\ldots, v_i, \ldots, v_j, \ldots) = -\omega(\ldots, v_j, \ldots, v_i, \ldots) \qquad \text{permut.}$$

ω multilineal es ventajosa, pero introduce:

* volúmenes negativos
* orientación

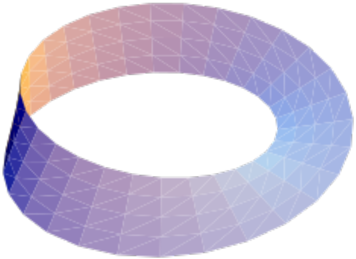
Igual, conviene multilinealidad y lidiar con los signos

$$\omega \quad \text{o} \quad -\omega$$

a usar

$$|\omega|$$

Orientabilidad



Cinta de Möbius

Una manifold M es orientable si existe una forma ω sobre M que nunca se anula.

Chains (tangent multivectors)

$$C_k := \mathbb{F}K_k$$ espai vect. de k-chains

$$e_{i_0 \dots i_k} \in C_k$$ $i_0 < \dots < i_k$

$$c = c^s e_s$$ $c^s \in \mathbb{R}$, $s \in K$

Boundary Operator

$\partial_k \in [C \to C]$ $\quad C = C_0 \oplus \cdots \oplus C_n$

$$\partial_k(e_{i_0 \cdots i_q}) = \delta_{qk} \sum_r (-1)^r e_{i_0 \cdots i_{r-1} i_{r+1} \cdots i_q}$$

$$\partial := \partial_1 \oplus \cdots \oplus \partial_n \qquad \partial_0 := 0$$

$$\partial_k(C) = \partial(C_k) \subseteq C_{k-1} \qquad \partial_k \in [C_k \to C_{k-1}]$$

Cochains (formas)

$f \in C^* = [\,C \to \mathbb{R}\,]$ análogo discreto de las formas

$f = f_s e^s \qquad s \in K,\ f_s \in \mathbb{R},\ e^s \in C^* : e^s(e_r) = \delta^s_r$

$$C^* := C^*_0 \oplus \cdots \oplus C^*_n$$

Coboundary Operator (exterior deriv.)

$$d_k \in [\, C^* \to C^* \,]$$

$$(d_k(f))(c) := f(\partial_{k+1}(c)) \qquad \forall \begin{array}{l} c \in C \\ f \in C^* \end{array}$$

$$d := d_0 \oplus \dots \oplus d_n \qquad d_n := 0$$

$$d_k(C^*) = d(C^*_k) \subseteq C^*_{k+1} \qquad d_k \in [\, C^*_k \to C^*_{k+1} \,]$$

Teorema de Stokes Generalizado

discreto $\qquad (d(\omega))(c) = \omega(\partial(c))$

Sólo topología

continuo $\qquad \int_M d\omega = \int_{\partial M} \omega$

Sin métrica!

Dual coboundary operator (codifferential)

adjunto

requiere métrica!!

$m_{rs} := e_s \cdot e_r$
$\|$
0 si s y r
son de
dist. dim.

$$d^* \in [\, C^* \to C^* \,]$$

$$d^*(f) \cdot g := f \cdot d(g) \qquad \forall\, f, g \in C^*$$

$$d^* = d_0^* \oplus \dots \oplus d_n^* \qquad d_k^*(f) \cdot g = f \cdot d_k(g)$$

$$d_k^*(C^*) = d^*(C_{k+1}^*) \subseteq C_k^* \qquad d_k^* \in [\, C_{k+1}^* \to C_k^* \,]$$

Laplace de Rham Operators

$$\Delta := d\,d^* + d^* d$$

$$\Delta = \Delta_0 \oplus \cdots \oplus \Delta_n$$

$$\Delta_k = d_{k-1} d_{k-1}^* + d_k^* d_k$$

$$=: \Delta_k^{down} + \Delta_k^{up}$$

$$\Delta(C_k^*) = \Delta_k(C^*) \subseteq C_k^*$$

$$\Delta_k \in [C_k^* \to C_k^*]$$

$$\partial_k \circ \partial_{k+1} = 0 \implies \partial^2 = 0$$

$$d_{k+1} \circ d_k = 0 \implies d^2 = 0$$

$$d_k^* \circ d_{k+1}^* = 0 \implies (d^*)^2 = 0$$

$$\Downarrow$$

$$\Delta = (\underbrace{d^* + d}_{\text{Dirac operat.}})^2$$

Physics is about 2nd order diffs. eqs. !!!

Chain & cochain complexes

$$0 = C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} = 0$$

$$\xleftarrow[\partial_{n+1}^T]{} \quad \xleftarrow[\partial_n^T]{} \quad \xleftarrow[\partial_{n-1}^T]{} \cdots \xleftarrow[\partial_2^T]{} \quad \xleftarrow[\partial_1^T]{} \quad \xleftarrow[\partial_0^T]{}$$

incidence matrices

$$0 = C^*_{n+1} \xleftarrow{d_{n+1}} C^*_n \xleftarrow{d_n} C^*_{n-1} \xleftarrow{d_{n-1}} \cdots \xleftarrow{d_2} C^*_1 \xleftarrow{d_1} C^*_0 \xleftarrow{d_0} C^*_{-1} = 0$$

$$\xrightarrow[d_n^T]{} \quad \xrightarrow[d_{n-1}^T]{} \quad \xrightarrow[d_{n-2}^T]{} \cdots \xrightarrow[d_1^T]{} \quad \xrightarrow[d_0^T]{} \quad \xrightarrow[d_{-1}^T]{}$$

$$d_n^* \quad d_{n-1}^* \quad d_{n-2}^* \quad d_1^* \quad d_0^* \quad d_{-1}^*$$

$$d_k^* \neq d_k^T$$

$c \in \ker \partial_k \Rightarrow c$ is called k-cycle

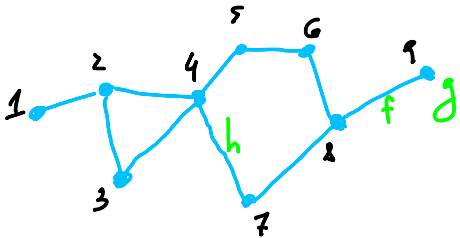
$c \in \operatorname{rng} \partial_{k+1} \Rightarrow c$ is called k-boundary

$f \in \ker d_k \Rightarrow f$ is called k-cocycle (closed form)

$f \in \operatorname{rng} d_{k-1} \Rightarrow f$ is called k-coboundary (exact form)

$g \in C_0^*$

$f, h \in C_1^*$



$f = dg$ exact.

$f_{89} = -g_8 + g_9$

Simplicial homology & cohomology (groups)

homology $\quad H_k := \ker \partial_k / \operatorname{rng} \partial_{k+1} \qquad \beta_k := \operatorname{rnk} H_k$

k-th Betti n°.

cohomology $\quad H^k := \ker d_{k+1} / \operatorname{rng} d_k$

quotient spaces

Helmholtz – Hodge decomposition

$$H^k = \ker d_{k+1} / \operatorname{rng} d_k \cong \overbrace{\operatorname{rng} d^*_{k+1} \oplus \ker\left(d^*_{k+1} d_{k+1} + d_k d^*_k\right)}^{\ker d^*_k} \oplus \underbrace{\ker\left(d^*_{k+1} d_{k+1} + d_k d^*_k\right) \oplus \operatorname{rng} d_k}_{\ker d_{k+1}}$$

$$\cong \operatorname{rng} d^*_{k+1} \oplus \ker \Delta_{k+1} \oplus \operatorname{rng} d_k$$

$f \in C^*_k$

$g \in C^*_{k-1}$

$s \in C^*_{k+1}$

$h \in \operatorname{rng} \Delta_k$

$$f = \underset{\text{gradient}}{dg} + \underset{\text{solenoidal}}{d^* s} + \underset{\text{harmonic}}{h}$$

Gracias !