

Multiscale unfolding of real networks by
geometric renormalization

Fa MAF - UNC

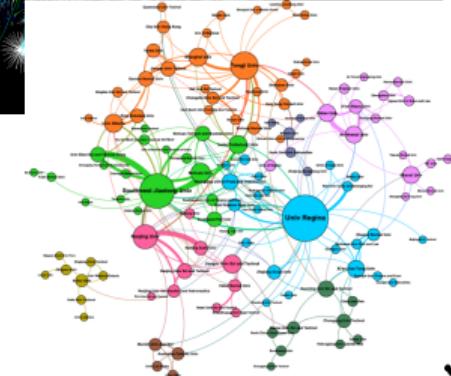
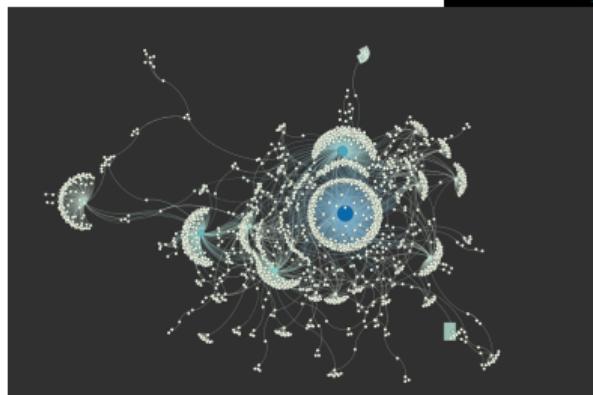
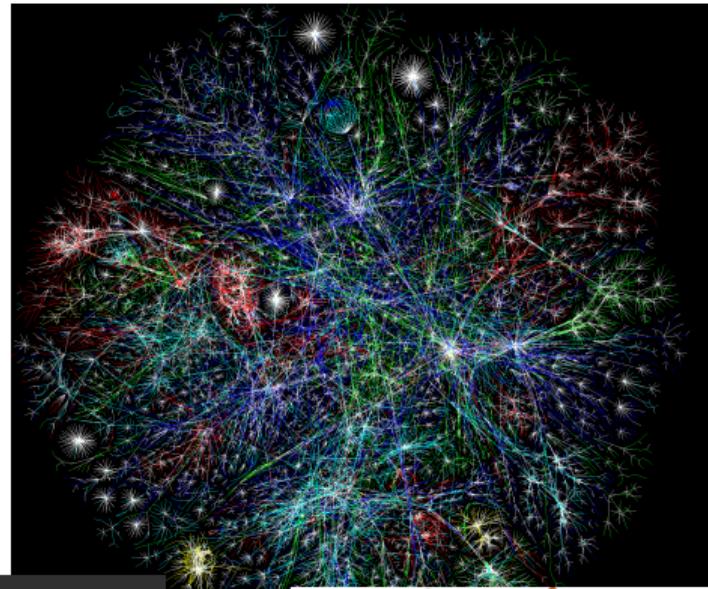
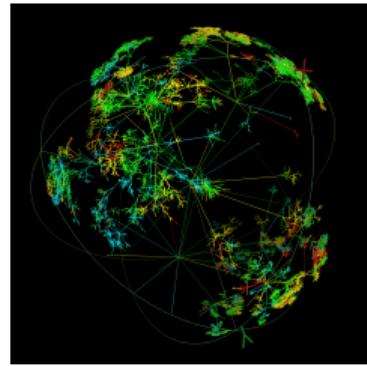
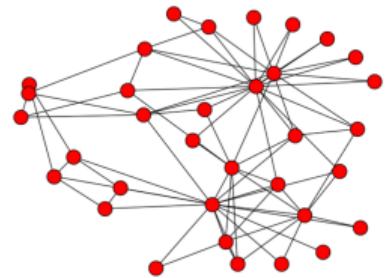
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Outline

- Riemannian complex
- Hyperbolic geometry : the Poincaré disk .
- \mathbb{H}^2 hidden metric space model .
- Geometric Renormalization
-

Complex Networks



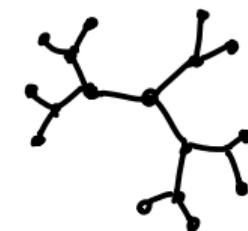
Lattice

$$N \sim L^d$$



Complex Networks

$$N \sim e^L$$



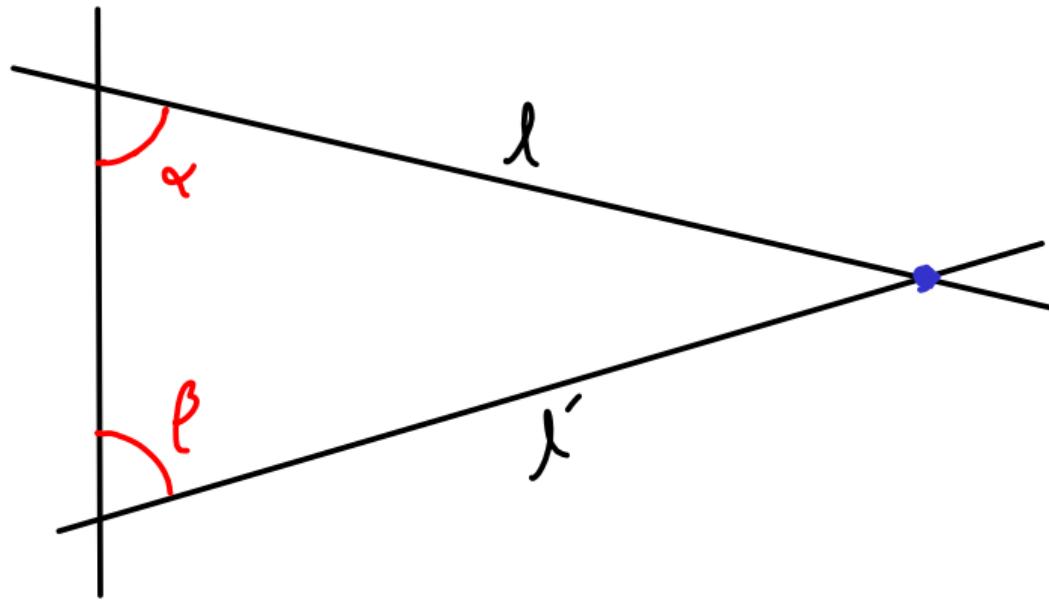
$$P(k) \sim k^{-\gamma}$$

loops, modules, etc.

Euclid's postulates

1. There exist a line through any two points.
2. Any line may be extended indefinitely.
3. Any center and radius determines an unique circle.
4. All right angles are congruent.
5. ...

S.



"If $\alpha + \beta < L + L$, then l and l' intersect."

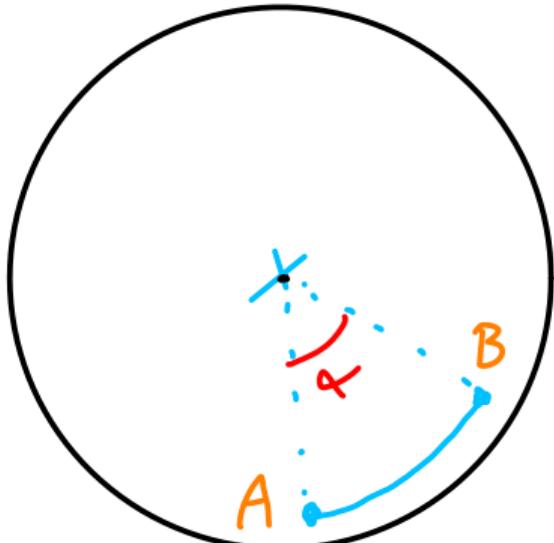
For two millennia, many attempts to prove it from 1.-4. failed.

In 19th century mathematicians created geometries where the 5. postulate fails:

- Gauss
- Lobachevsky
- Riemann
- Beltrami $\leftarrow \quad x^2 + y^2 - z^2 = -1$
- Klein
- Poincaré'

Elliptic (Spherical) geometry

$$x^2 + y^2 + z^2 = K$$



The cut of the sphere by the plane defines a "great circle" or geodesic.

Metric $d(A, B) = \arccos \left(\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} \right)$

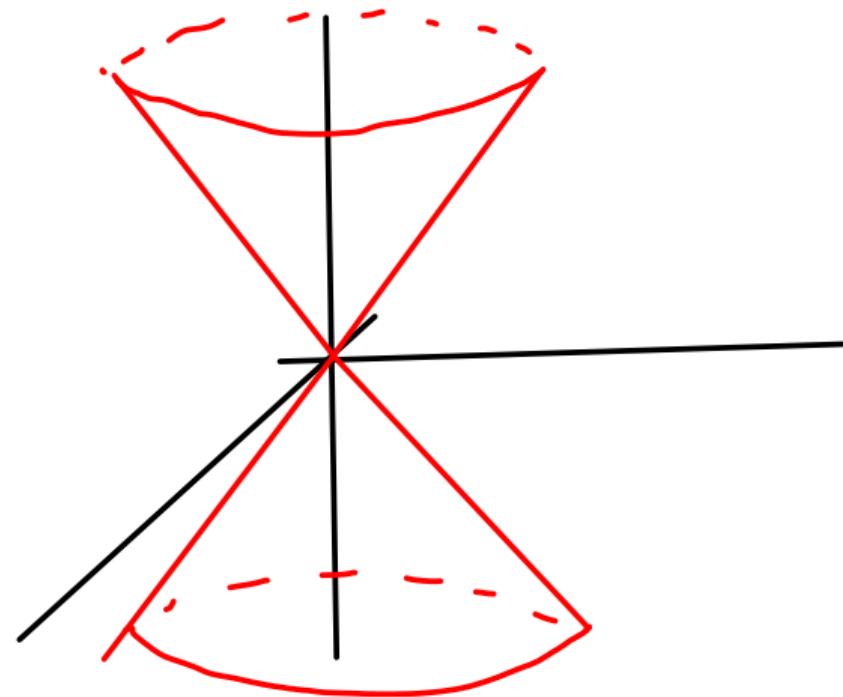
- 1) $d(x, x) = 0$
- 2) $d(x, y) = d(y, x) > 0 \quad x \neq y$
- 3) $d(x, y) \leq d(x, z) + d(z, y)$

Hyperbolic geometry

$$x^2 + y^2 - z^2 = K$$

$$K=0$$

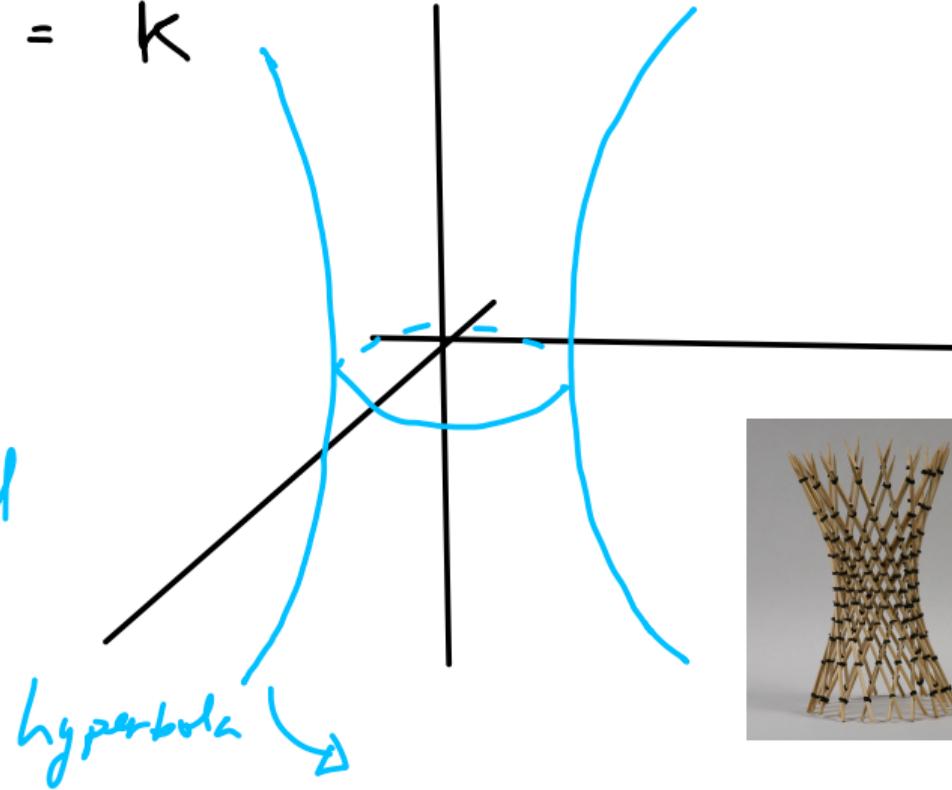
Cone?



Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$k = 1$
elliptic
hyperboloid

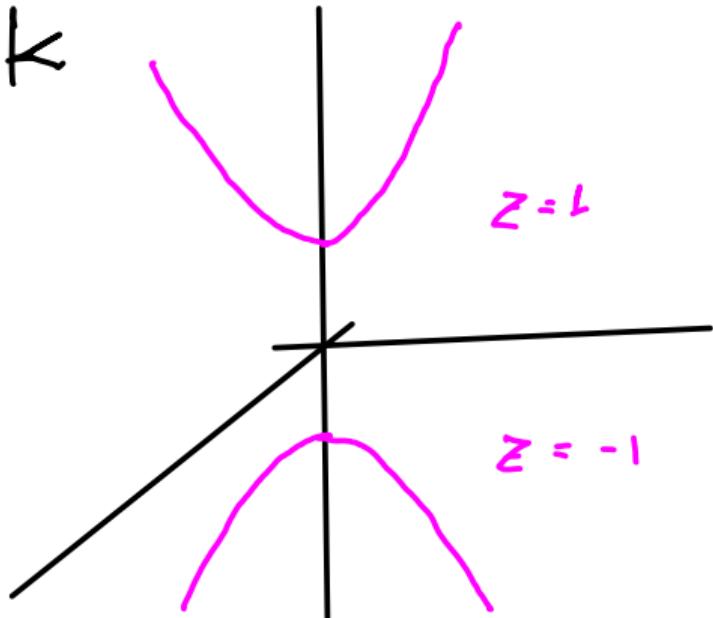


Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$$k = -1$$

hyperbolic
hyperboloid



Beltrami

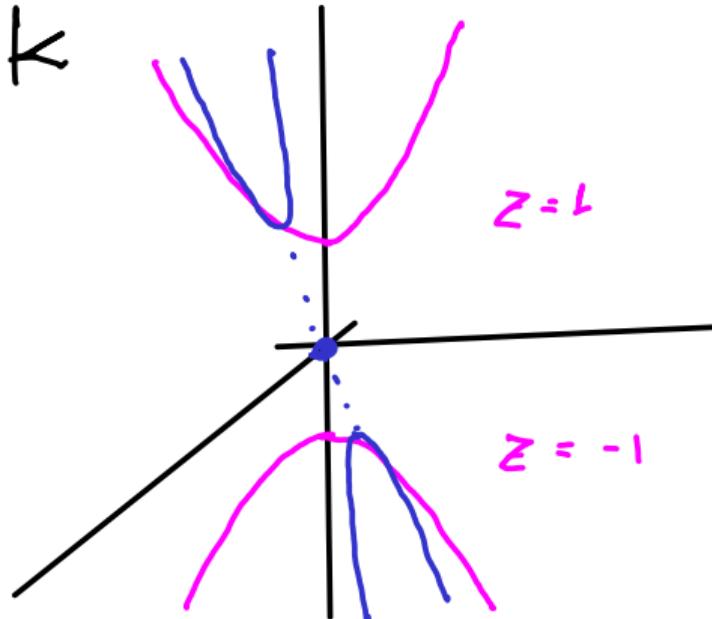
has two
components

Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$$k = -1$$

hyperbolic
hyperboloid



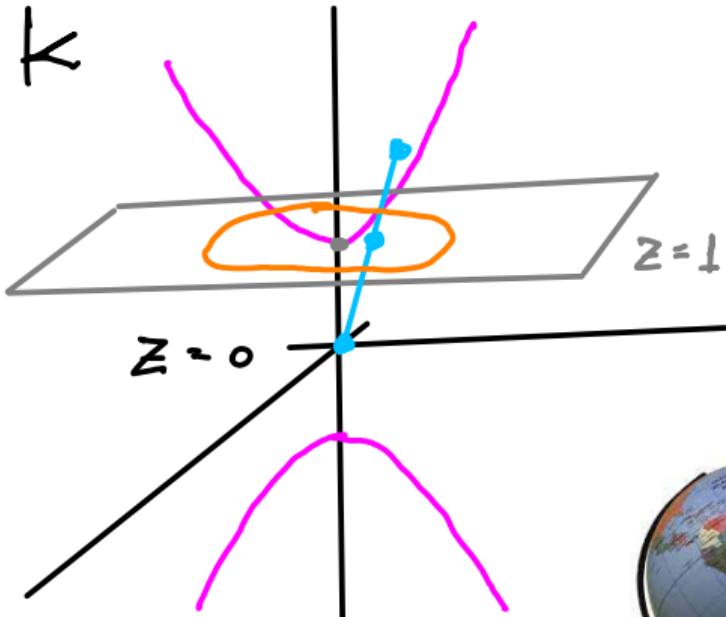
Beltrami

geodesic
(plane crossing
the origin)

Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$$k = -1$$



Beltrami
Klein
Model

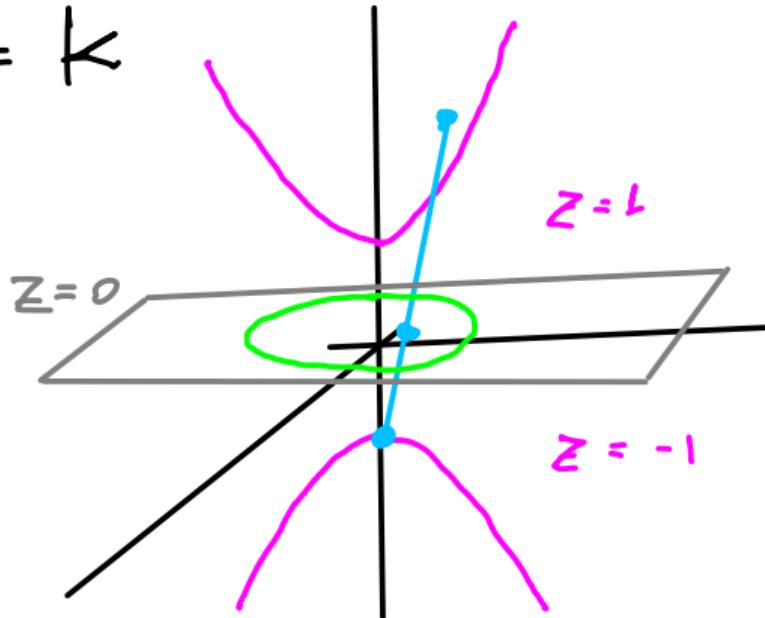


Projection

Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$$k = -1$$

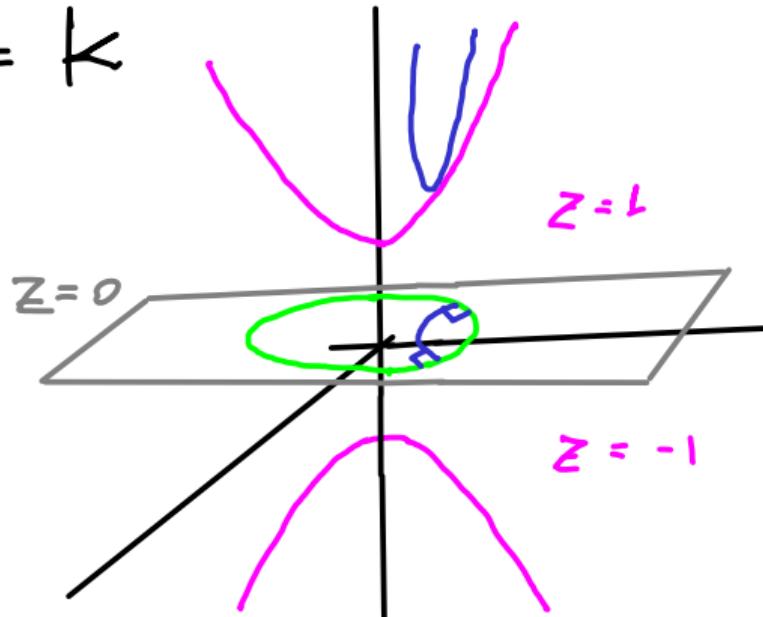


Beltrami
Poincaré
model

Hyperbolic geometry

$$x^2 + y^2 - z^2 = k$$

$$k = -1$$

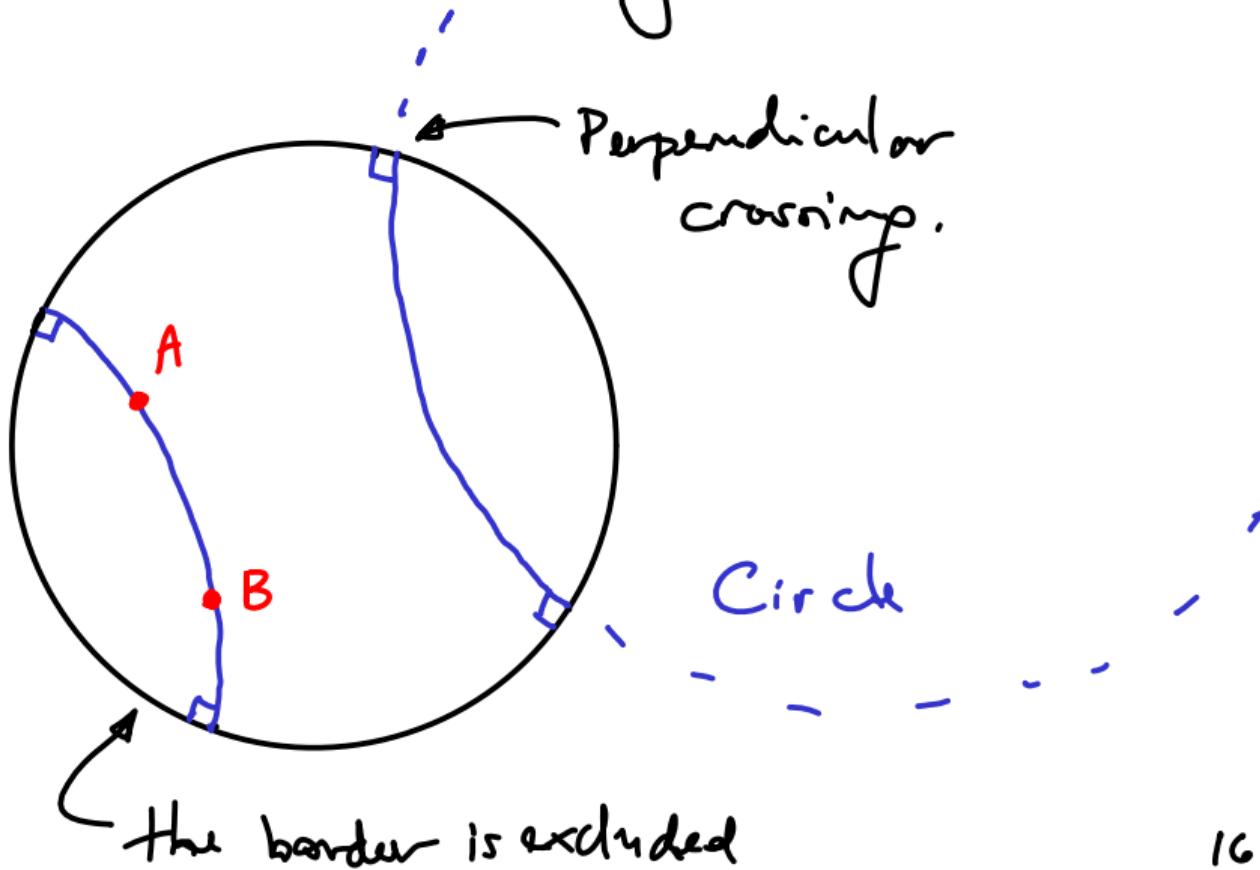


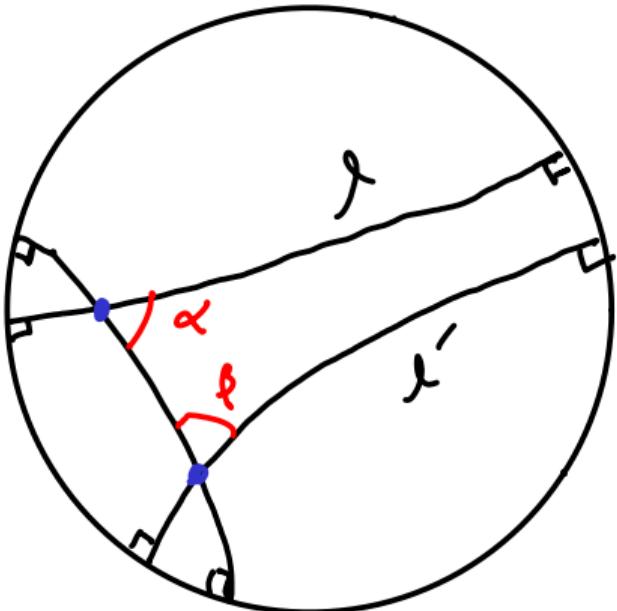
Beltrami
Poincaré
model

Geodesic

Poincaré disk model of the hyperbolic plane

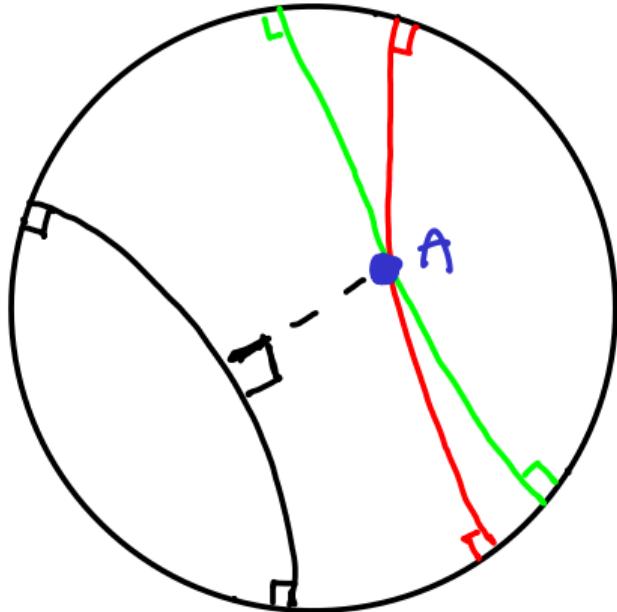
1. "straight" line passing through A and B.
(geodesic)





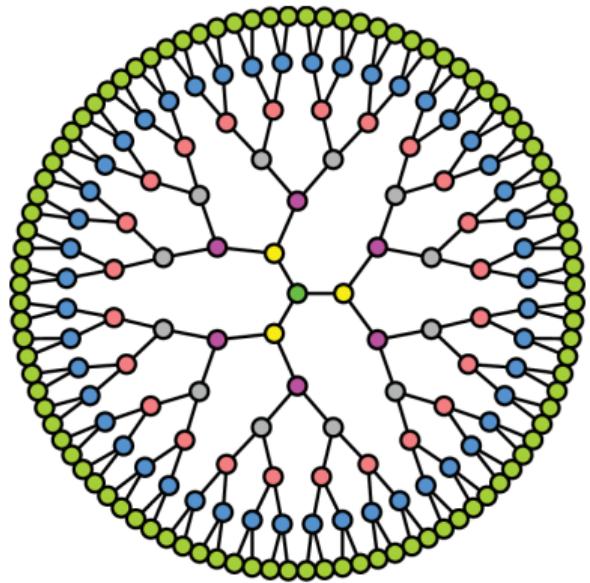
$$\alpha + \beta < L + L$$

but l and l'
do not intersect



More than
two "parallel"
lines passing
through A

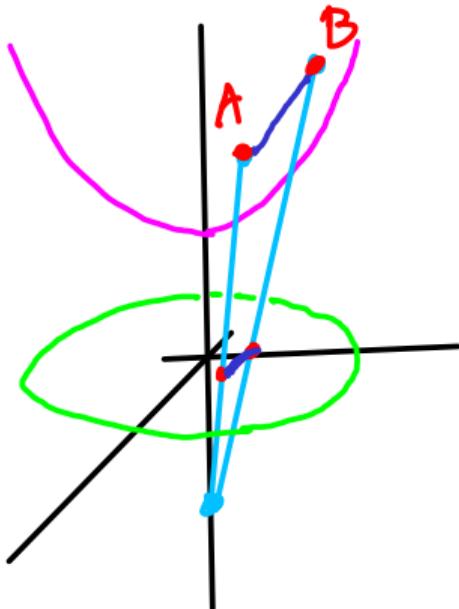
distances



Cayley tree



M.C. Escher



Minkowski
inner prod.

$$A \cdot B = -A_0 B_0 + A_1 B_1 + \dots + A_n B_n$$

$$|A|^2 = A \cdot A = \text{squared norm}$$

Hyperbolic n -space:

$$\mathbb{H}^n = \left\{ A \in \mathbb{R}^{n+1} : |A|^2 = -1 \right\}$$

Metric over \mathbb{H}^n induces metric over
the Poincaré disc.

$$d(A, B) = \operatorname{arccosh} \left(\frac{A \cdot B}{|A||B|} \right)$$

Curvature (follows from the metric)

Positive (elliptic geometry)



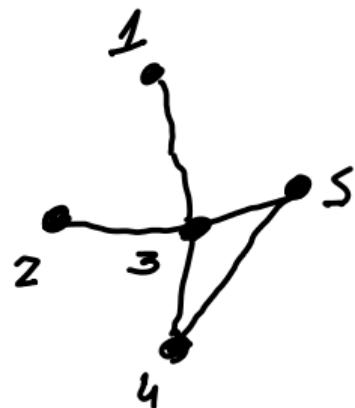
Negative (hyperbolic geom.)



Null (flat or Euclid. geom.)



Adjacency matrices



$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$A_{i,j} = n^i$ links between i and j

Generative latent space models of networks

$$P(\mathcal{G} | \alpha) = P(A | \alpha) = \prod_{i < j} P(A_{ij} | \alpha)$$

Graph Parameters

$$P(A_{ij} | \alpha) = \prod_{i < j} \alpha_{ij}^{A_{ij}} (1 - \alpha_{ij})^{1 - A_{ij}} \quad 0 \leq \alpha_{ij} \leq 1$$
$$A_{ij} \in \{0, 1\}$$

The \mathbb{H}^2 network model

$$d_{ij} \approx \frac{1}{2} (r_i + r_j + 2 \ln \frac{\Theta_{ij}}{2})$$

$$\alpha_{ij} = \left(1 + e^{\frac{\beta}{2} (d_{ij} - R)} \right)^{-1} \quad \text{Stats. Fermi-Dirac.}$$

$$R = 2 \ln n/c$$

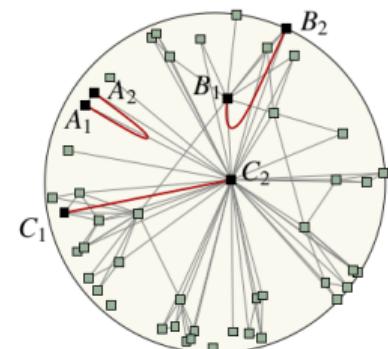
$$r_i = R - 2 \ln k_i$$

$n = n^2$ nodes

c controls the average degree

k_i = expected degree of node i .

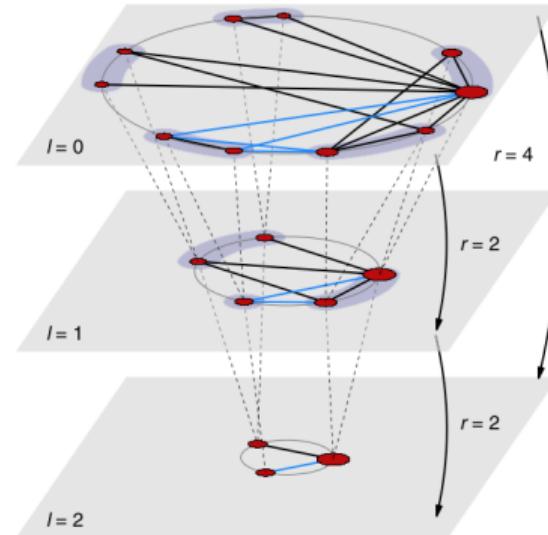
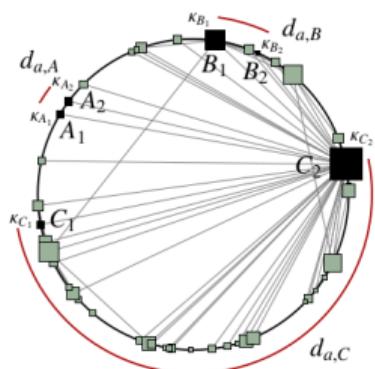
b Model \mathbb{H}^2



RG of the S^1 network model

The H^2 network model is equivalent to the S^1 network model. RG on H^2 can be described by RG on S^1 .

a Model S^1



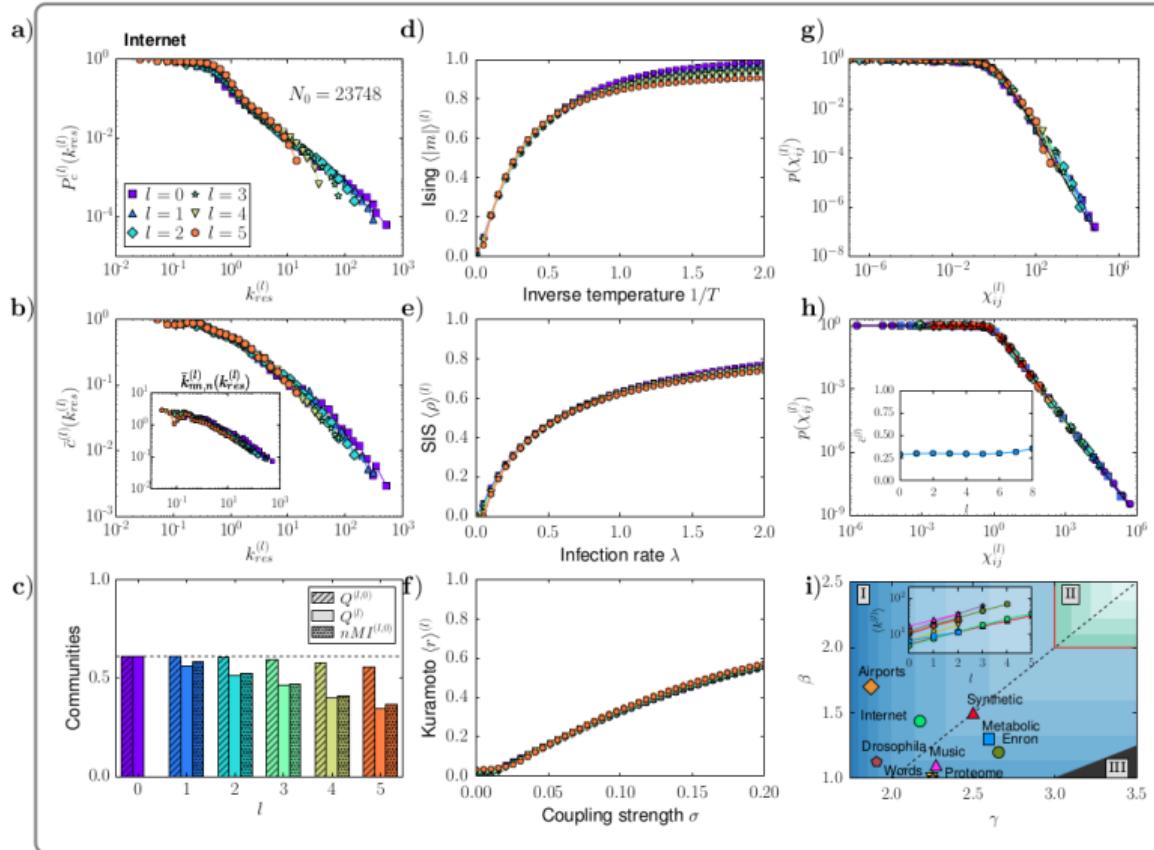
RG transform.

$$\kappa_i^{(l+1)} = \left(\sum_{j=1}^r (\kappa_j^{(l)})^\beta \right)^{1/\beta}$$
$$\theta_i^{(l+1)} = \left(\frac{\sum_{j=1}^r (\theta_j^{(l)} \kappa_j^{(l)})^\beta}{\sum_{j=1}^r (\kappa_j^{(l)})^\beta} \right)^{1/\beta}$$

$$\rho^{l+1} = \rho^l$$

$$R^{l+1} = R^l / r$$

Invariance of network properties under RG transform.



Phase diagrams of network topology 27

Gracias!