Multiscale unfolding of real networks by geometric renormalization

FaMAF - UNC

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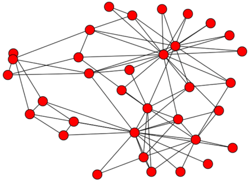
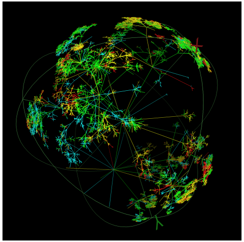
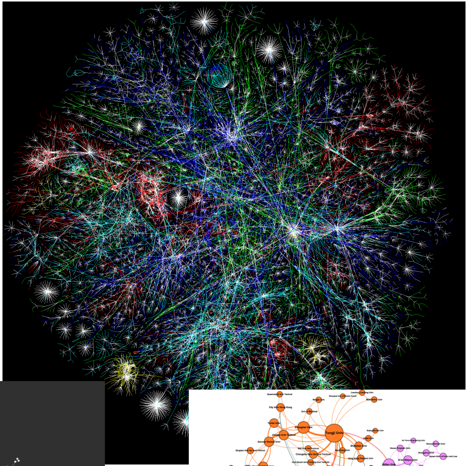
Outline

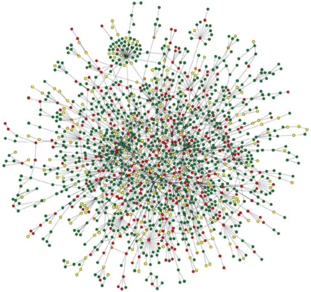
- Redes complejas
- Hyperbolic geometry: the Poincaré disk.
- $\mathbb{H}^2$ hidden metric space model.
- Geometric Renormalization
-

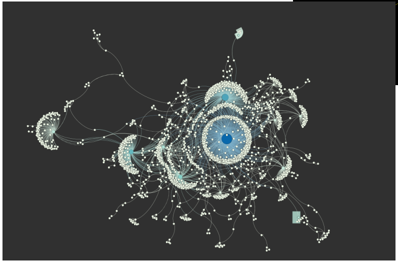
Complex Networks

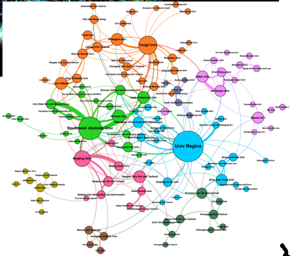




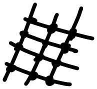




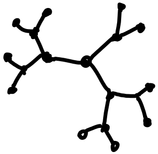




Lattice $N \sim L^d$



Complex Networks $N \sim e^L$



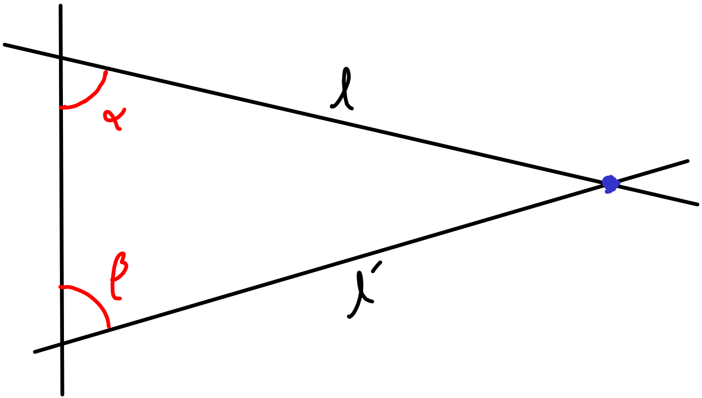
$P(k) \sim k^{-\gamma}$

loops, modules, etc.

Euclid's postulates

1. There exist a line through any two points.
2. Any line may be extended indefinitely.
3. Any center and radius determines an unique circle.
4. All right angles are congruent.
5. ...

5.



"If $\alpha+\beta < \llcorner + \llcorner$, then ℓ and ℓ' intersect."

For two millenia, many attempts to prove it from 1.–4. failed.

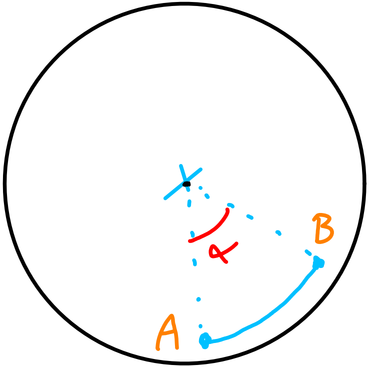
In 19th century mathematicians created geometries where the 5. postulate fails:

- Gauss
- Lobachevsky
- Riemann
- Beltrami $\longleftarrow$ $x^2 + y^2 - z^2 = -1$
- Klein
- Poincaré

Elliptic (Spheric) geometry

$$x^2 + y^2 + z^2 = K$$

The cut of the sphere by the plane defines a "great circle" or geodesic.



<u>Metric</u> $d(A,B) = \arccos\left(\frac{A.B}{|A||B|}\right)$

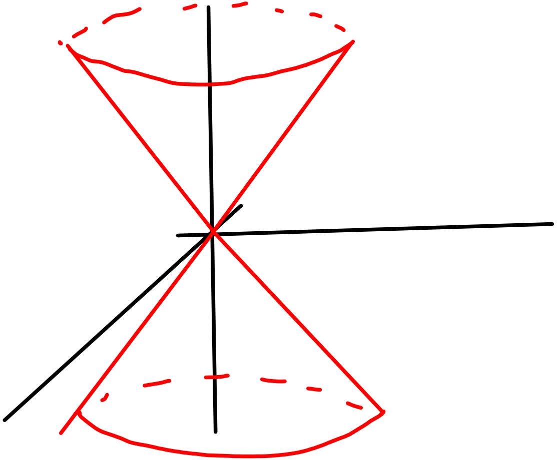
1) $d(x,x) = 0$
2) $d(x,y) = d(y,x) > 0 \quad x \neq y$
3) $d(x,y) \leq d(x,z) + d(z,y)$

Hyperbolic geometry

$$x^2 + y^2 - z^2 = K$$

$K = 0$

Cones

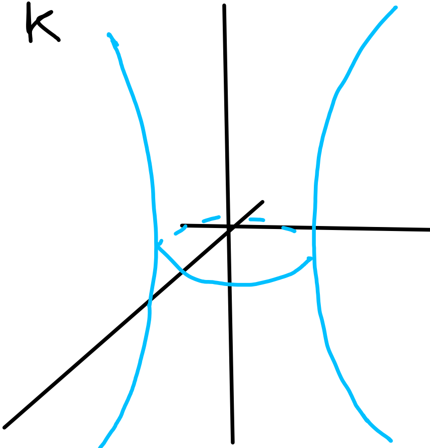


Hyperbolic geometry

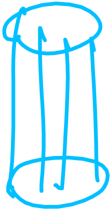
$$x^2 + y^2 - z^2 = K$$

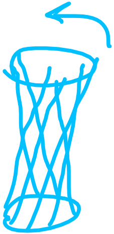
$K = 1$

Elliptic hyperboloid



hyperbola



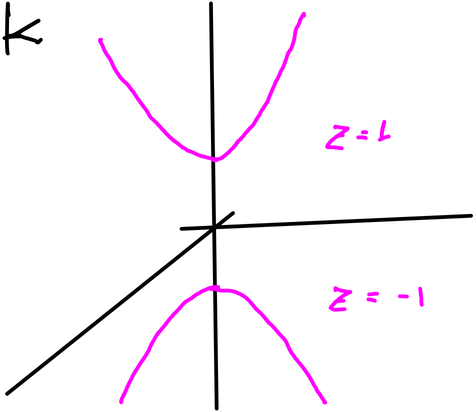


Hyperbolic geometry

$$x^2 + y^2 - z^2 = K$$

$K = -1$

hyperbolic hyperboloid



Beltrami

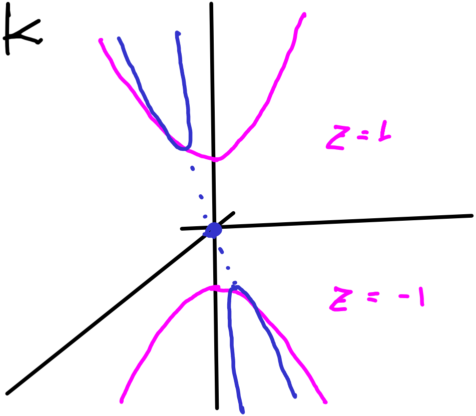
has two components

Hyperbolic geometry

$$x^2 + y^2 - z^2 = K$$

$K = -1$

hyperbolic hyperboloid



Beltrami

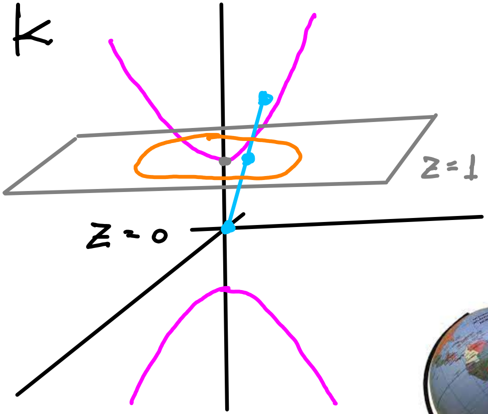
geodesic

(plane crossing the origin)

Hyperbolic geometry

$$x^2+y^2-z^2=K$$

$K=-1$



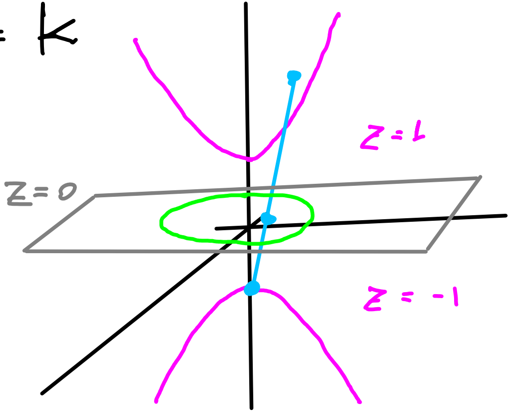
Beltrami Klein model

Projection

Hyperbolic geometry

$$x^2 + y^2 - z^2 = K$$

$K = -1$

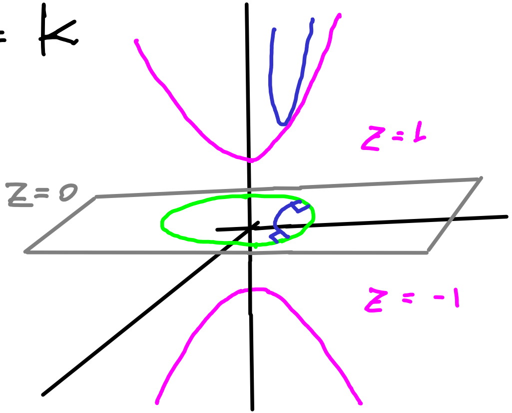


Beltrami
Poincaré
model

Hyperbolic geometry

$x^2 + y^2 - z^2 = K$

$K = -1$



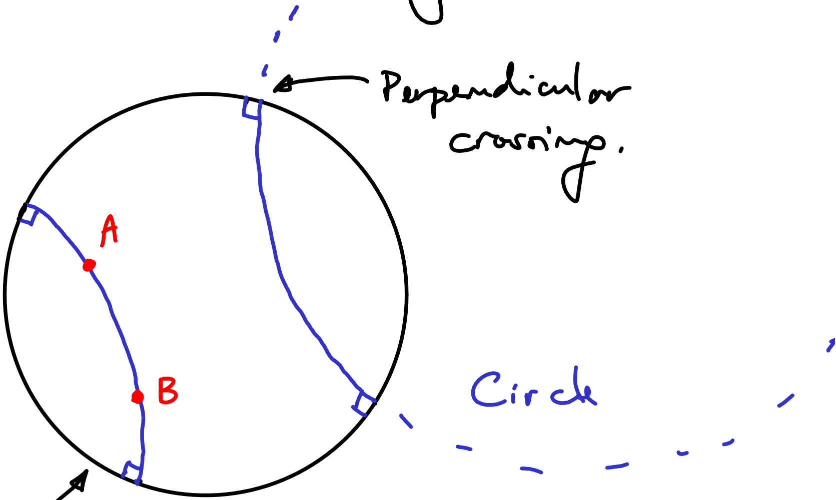
Beltrami
Poincaré
model

Geodesic

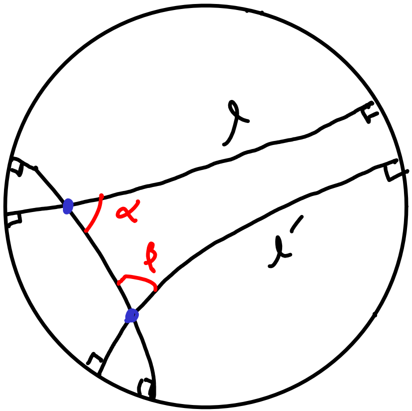
Poincaré disk model of the hyperbolic plane

1. "straight" line passing through A and B.

(geodesic)



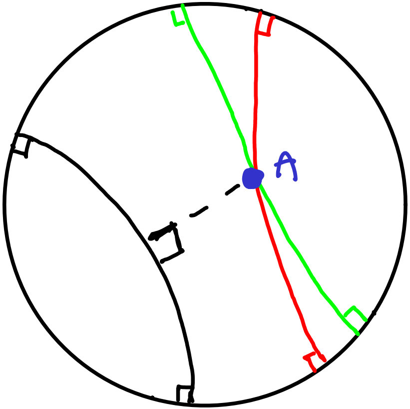
the border is excluded



$\alpha + \beta < \llcorner + \llcorner$

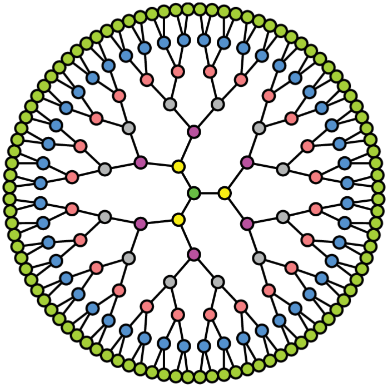
but ℓ and ℓ'

do not intersect



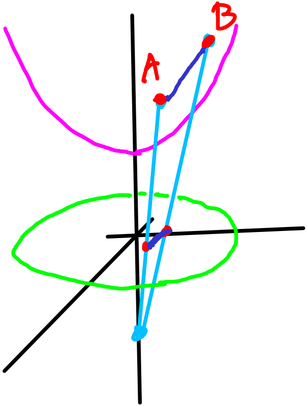
More than two "parallel" lines passing through A

Distances



Cayley tree

M.C. Escher



Minkowski inner prod.

$$A.B = -A_0B_0 + A_1B_1 + \cdots + A_nB_n$$

$$|A|^2 = A.A = \text{squared norm}$$

Hyperbolic n-space:

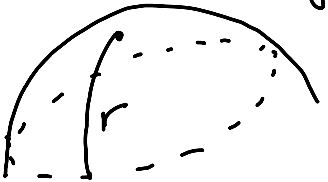
$$\mathbb{H}^n = \{ A \in \mathbb{R}^{n+1} : |A|^2 = -1 \}$$

Metric over $\mathbb{H}^n$ induces metric over the Poincaré disc.

$$d(A,B) = \operatorname{arcosh}\left(\frac{A.B}{|A||B|} \right)$$

Curvature (follows from the metric)

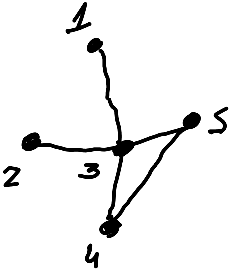
Positive (elliptic geometry)



Negative (hyperbolic geom.)

Null (flat or Euclid. geom.)

Adjacency matrices



$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

A_{ij} = n° links between i and j

Generative latent space models of networks

$$P(G \mid \alpha) = P(A \mid \alpha) = \prod_{i<j} P(A_{ij} \mid \alpha)$$

Graph   Parameters

$$P(A_{ij} \mid \alpha) = \prod_{i<j} \alpha_{ij}^{A_{ij}} (1-\alpha_{ij})^{1-A_{ij}}$$

$0 \le \alpha_{ij} \le 1$

$A_{ij} \in \{0,1\}$

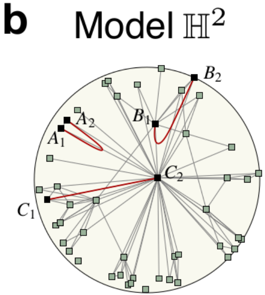
The $\mathbb{H}^2$ network model

$$d_{ij} \approx \frac{1}{2}\left(r_i + r_j + 2\ln\frac{\theta_{ij}}{2}\right)$$

$$\alpha_{ij} = \left(1 + e^{\frac{\beta}{2}(d_{ij}-R)}\right)^{-1}$$ Stats. Fermi-Dirac.

$$R = 2\ln n/c$$

$$r_i = R - 2\ln \kappa_i$$



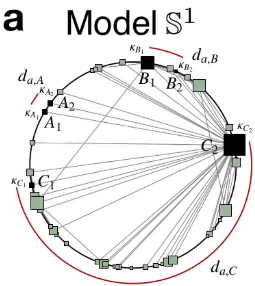
n = n° nodes

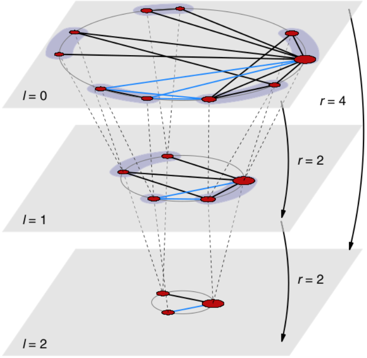
c = controls the average degree

κ_i = expected degree of node i.

RG of the $\mathbb{S}^1$ network model

The $\mathbb{H}^2$ network model is equivalent to the $\mathbb{S}^1$ network model. RG on $\mathbb{H}^2$ can be described by RG on $\mathbb{S}^1$.





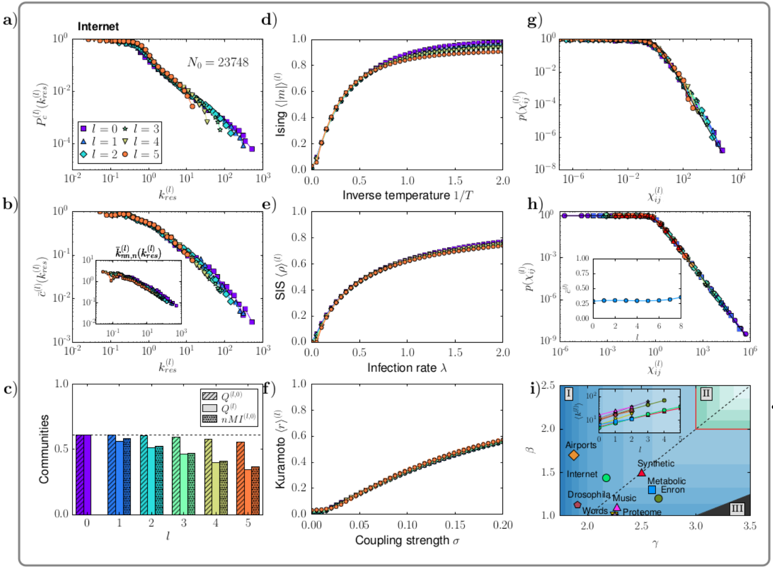
RG transform.

$$\kappa_i^{(l+1)} = \left(\sum_{j=1}^{r} \left(\kappa_j^{(l)}\right)^{\beta} \right)^{1/\beta} \qquad \theta_i^{(l+1)} = \left(\frac{\sum_{j=1}^{r} \left(\theta_j^{(l)}\kappa_j^{(l)}\right)^{\beta}}{\sum_{j=1}^{r} \left(\kappa_j^{(l)}\right)^{\beta}} \right)^{1/\beta}$$

$$\beta^{l+1} = \beta^{l}$$

$$R^{l+1} = R^{l}/r$$

Invariance of network properties under RG transform.



Phase diagram of network topology

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Gracias!