Analysis of the inference of ratings and rankings on Higher Order Networks with complex topologies

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(Dated: October 18, 2025)

Bla bla...

I. INTRODUCTION

See [1, 2].

II. THEORY

A. Least-squares inference of team ratings and rankings

Consider a set of $N = n+1$ teams labeled $\{0, 1, \ldots, n\}$. For each distinct pair of teams i and j, assume a pairwise antisymmetric comparison score $f_{ij} = -f_{ji}$ is available. Informally, $f_{ij} > 0$ indicates that team i is rated higher than team j, whereas $f_{ij} < 0$ suggests the opposite.

A natural approach to inferring consistent ratings is to assign scores w_i $(i = 0, 1, \ldots, n)$ that minimize the total squared discrepancy

$$\sum_{i<j} \big(f_{ij} - (w_i - w_j)\big)^2. \tag{1}$$

Once the team ratings $w_0, w_1, \ldots, w_n$ have been obtained, the ranking $r_0, r_1, \ldots, r_n$ follows from the ordering condition $r_i < r_j$ whenever $w_i > w_j$. This least-squares formulation defines a well-posed optimization problem that admits an elegant reformulation in terms of cochains over simplicial complexes. The resulting framework for rating and ranking inference is known as *HodgeRank* [1, 3].

B. Discrete exterior calculus on simplicial complexes

To review the HodgeRank method, we recall the basic elements of discrete exterior calculus on simplicial complexes. Let $V = \{0, 1, \ldots, n\}$ denote a set of vertices, and define a simplicial complex as a collection

$$K = K_0 \cup K_1 \cup \cdots \cup K_n,$$

where

$$K_k \subseteq \big\{\{i_0, \ldots, i_k\} : 0 \le i_0 < \cdots < i_k \le n\big\}$$

is the set of k-simplices, i.e., unordered subsets $\{i_0, \ldots, i_k\}$ of vertices satisfying the closure condition: if $s \in K$ and $r \subset s$ with $r \neq \emptyset$, then $r \in K$.

The set K of simplices is associated with a vector space C of *chains*, with canonical basis $\{e_s : s \in K\}$. We equip C with the inner product

$$e_r \cdot e_s = \delta_{rs}, \qquad r, s \in K.$$

With Einstein summation convention, any chain $c \in C$ can be written as $c = c^s e_s$ with coefficients $c^s \in \mathbb{R}$. The chain space decomposes as

$$C = C_0 \oplus C_1 \oplus C_2,$$

where C_k is spanned by $\{e_s : s \in K_k\}$.

The dual space C^* of *cochains* is spanned by $\{e^s : s \in K\}$, where $e^r(e_s) = \delta^r{}_s$. Any $f \in C^*$ decomposes as $f = f_s e^s$ with coefficients $f_s \in \mathbb{R}$, and

$$C^* = C_0^* \oplus C_1^* \oplus C_2^*.$$

By Riesz's theorem, every cochain $f \in C^*$ corresponds uniquely to a chain $f^\flat \in C$ such that $f(c) = f^\flat \cdot c$ for all $c \in C$. This induces an inner product on C^*:

$$f \cdot g = f^\flat \cdot g^\flat.$$

The map $f \mapsto f^\flat$ is an isomorphism, with inverse denoted $c \mapsto c^\sharp$.

Orientation and boundary. An orientation is fixed by requiring

$$e_{i_0 \ldots i_k} = (-1)^p e_{j_0 \ldots j_k},$$

for each simplex $s = \{i_0, \ldots, i_k\} \in K$, where p is the parity of the permutation mapping $(i_0, \ldots, i_k)$ to $(j_0, \ldots, j_k)$.

The boundary operator $\partial : C \to C$ is defined on basis elements as

$$\partial e_{i_0 \ldots i_k} = \sum_{j=0}^{k} (-1)^j e_{i_0 \ldots i_{j-1} i_{j+1} \ldots i_k}.$$

Thus $\partial(C_k) \subseteq C_{k-1}$, with $C_{-1} = \{0\}$, yielding the decomposition

$$\partial = \partial_0 \oplus \partial_1 \oplus \partial_2, \qquad \partial_k(C_k) \subseteq C_{k-1},\ \partial_0 = 0.$$

Coboundary. The coboundary operator $d : C^* \to C^*$ is defined by

$$df(c) = f(\partial c), \qquad f \in C^*,\ c \in C.$$

Hence $d(C^*_k) \subseteq C^*_{k+1}$, with $C^*_3 = \{0\}$, and

$$d = d_0 \oplus d_1 \oplus d_2, \qquad d_2 = 0.$$

Adjoint operators. The adjoints $\partial^* : C \to C$ and $d^* : C^* \to C^*$ are defined by

$$\partial^* c \cdot q = c \cdot \partial q, \qquad d^* f \cdot g = f \cdot dg,$$

for all $c, q \in C$ and $f, g \in C^*$. They decompose as

$$\partial^* = \partial^*_0 \oplus \partial^*_1 \oplus \partial^*_2, \qquad d^* = d^*_0 \oplus d^*_1 \oplus d^*_2.$$

Hodge Laplacians. The boundary and coboundary operators define the Hodge Laplacians

$$\begin{aligned} \mathcal{L} &= (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial : C \to C, \text{and} \\ L &= (d + d^*)^2 = dd^* + d^*d : C^* \to C^*, \end{aligned}$$

with $\partial^2 = (\partial^*)^2 = d^2 = (d^*)^2 = 0$. These decompose as

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2, \qquad L = L_0 \oplus L_1 \oplus L_2,$$

with

$$\mathcal{L}_k = \mathcal{L}^\downarrow_k \oplus \mathcal{L}^\uparrow_k, \qquad L_k = L^\downarrow_k \oplus L^\uparrow_k,$$

$$\mathcal{L}^\downarrow_k = \partial^*_k \partial_k, \quad \mathcal{L}^\uparrow_k = \partial_{k+1}\partial^*_{k+1},$$

$$L^\uparrow_k = d^*_k d_k, \quad L^\downarrow_k = d_{k-1} d^*_{k-1}.$$

Hodge decomposition. Every k-cochain f admits the Hodge decomposition

$$f = g \oplus h \oplus s,$$

where $g \in \operatorname{im} d_{k-1}$ (gradient), $h \in \ker L_k$ (harmonic), and $s \in \operatorname{im} d^*_k$ (solenoidal). Equivalently, there exist $w \in C^*_{k-1}$ and $u \in C^*_{k+1}$ such that

$$g = d_{k-1} w, \qquad s = d^*_k u,$$

with w and u determined by

$$d_k f = L^\downarrow_{k+1} u, \tag{2}$$

$$d^*_{k-1} f = L^\uparrow_{k-1} w, \tag{3}$$

while $h = f - g - s$.

The least-squares problem (1) admits a reformulation in the language of discrete calculus:

$$\|f - d_0(w)\|, \tag{4}$$

where the scalar entries f_{ij} of the 1-cochain $f = f_{ij} e^{ij}$ encode the pairwise comparison scores, and the components w_i of the 0-cochain $w = w_i e^i$ represent the team ratings. The minimizer of (4) coincides with the solution of Eq. (3) for $k = 1$.

In this setting, the set of 0-simplices K_0 corresponds to the teams V, the set of 1-simplices K_1 to the pairwise comparisons E, and the set of 2-simplices K_2 to the closed triples of comparisons:

$$K_2 = \big\{\{i,j,k\} : \{i,j\}, \{j,k\}, \{i,k\} \in E\big\}.$$

For $k > 2$, $K_k = \emptyset$.

III. METHODS

The dataset covers five major soccer leagues: England, France, Germany, Italy, and Spain. Each league consists of $N = 20$ teams, except Germany, which has $N = 18$.

In these competitions (e.g., Premier League, La Liga, Serie A, Bundesliga, Ligue 1), the final standings are determined by a points system based on match results throughout the season. In the standard system, a win yields 3 points, a draw 1 point, and a loss 0 points. Each team plays every other team twice (home and away) in a round-robin format. Over the season, the i-th team accumulates π_i total points. At the end of the season, teams are ranked in descending order of π_i. If two or more teams tie on points, tiebreakers are applied, typically in the following order (with minor variations across leagues): goal difference (GD: goals scored minus goals conceded), goals scored (GF), head-to-head record, and, in rare cases, a playoff or extra match. Thus, the final table is determined primarily by total points, with tiebreakers resolving ties.

For each pair $\{i, j\}$ of teams in a given league, several pairwise antisymmetric comparison scores f_{ij} are defined:

1. **crossings:** ...
2. **counterattacks:** ...
3. **pressure loss:** ...
4. **build-up time:** ...
5. **direct play:** ...
6. **pressure points:** ...
7. **shots:** ...
8. **flow rate:** ...
9. **maintenance time:** ...
10. **middle-zone time:** ...

11. **style:** the Singular Value Decomposition (SVD) of previous metrics is used to define higher-level descriptors called *styles*, so $f_{ij} = ...$ [2].

12. **rank based ground-truth:** $f_{ij} = i - j$.

13. **point based ground-truth:** $f_{ij} = \pi_j - \pi_i$.

A. HodgeRank inference via metrics

Given a metric, the coefficients

$$f_{ij} = -\frac{1}{2}(m_{ij} + m_{ji})$$

are defined for each $i < j$. The minus sign is imposed to identify ratings as potentials [1]. To test the HodgeRank inference method, two types of ground-truth coefficients are also introduced: (i) *rank-based* coefficients $f_{ij} = i - j$, and (ii) *points-based* coefficients $f_{ij} = \pi_j - \pi_i$.

In all cases, the underlying structure is a fully connected graph with N nodes, promoted to a simplicial complex composed of 0-simplices labeled by i, 1-simplices labeled by ij with $i < j$, and 2-simplices labeled by ijk with $i < j < k$. Correspondingly, the coefficients f_{ij} form the components of a 1-cochain f, where the anti-symmetry $f_{ji} = -f_{ij}$ is imposed by definition. There are no k-simplices for $k > 2$.

IV. RESULTS

The following experiments are carried on.

1. Compute ratings and rankings from the full weighted network of scores ϕ_{ij}.

2. Compute ratings and rankings from sparsified versiond of the weighted networks of schores ϕ_{ij}, by randomly removing a varying fraction of links.

3. Compute ratings and rankings from sparsified versiond of the weighted networks, by removing a varying fraction of the weighted links of the weighted matrix of scores ϕ_{ij} sorted in decreasing order of absolute weight.

4. Compute ratings defining the 1-cochain coordiantes $f'_{ij} = \phi_j - \phi_i$, where the strength $\phi_i = \sum_j w_{ij}$ defines a local potential for the i-th team.

V. CONCLUSIONS

The author acknowledges partial support from CONICET under grant PIP2021-2023 number 11220200101100, and from SeCyT (Universidad Nacional de Córdoba, Argentina). Special thanks go to O.V. Billoni for a careful reading of the article, and to the Centro de Cómputo de Alto Desempeño (CCAD) at the Universidad Nacional de Córdoba (UNC), `http://ccad.unc.edu.ar/`, which is part of SNCAD – MinCyT, Argentina, for providing computational resources.

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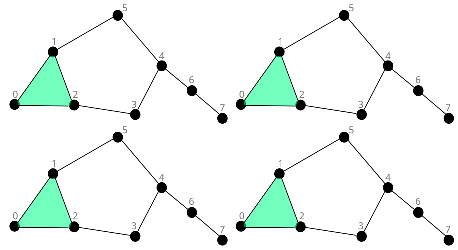

FIG. 1. (color online).