Final Exam: ECE 5412
December 2019

Return your answers by 11 am on Friday Dec 20
Upload your answers to Canvas as a single pdf file.

This final exam is worth 44%.

- Include all your Matlab/Python source codes with your answers. Make sure your code is sufficiently well commented and understandable! Unreadable/uncommented code will lose marks.

  Present your simulation results using illustrative figures. Simply plotting a random sample path contains zero information - you need to think carefully about how to present your results usefully in figures.

- If a question asks you to derive something, please do the careful proper mathematical derivation!

- This is an exam – collaborative work or plagiarizing some one else's work will be considered as cheating.

- Your thought process (clearly documented) is more important than the final answer. The onus is on you to demonstrate that you understand how to solve the problems - so make sure your writeup convinces us of this.

Question 1 [22 marks] Consider a sensor that measures a Markov chain corrupted by noise and a sinusoid:

$$y[k] = s[k] + n[k] + A \sin wk, \quad k = 1, 2, \ldots$$

where $y[k]$ denotes the measurement, $n[k] \sim N(0,1)$ is white Gaussian noise independent of $s[k]$. Here A denotes amplitude and w denotes the angular frequency of a sinusoid.

Assume that $s[k]$ is a 2 state Markov chain with states -1 and $+1$ and transition probability matrix

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

(a) Give a real life example where the above model arises.

(b) Assuming full knowledge of the model parameters (namely, transition probabilities, state levels, noise variance, sinusoid amplitude and frequency), derive an optimal filter for estimating the Markov chain $s[k]$ given the observations $y[1], \ldots, y[k]$.

(c) Implement this optimal filter in Matlab/Python by generating a 1000 point sequence of the Markov chain $s[k]$, etc. The output of your program should be the conditional mean estimate. (You also need to scale/normalize the recursions to avoid numerical ill-conditioning).

(d) Evaluate the accuracy of the filter as follows: Do 50 independent simulations of the above question and use that to compute the mean square error of the filtered estimate.

(e) Given the observation sequence $Y_{1000} = (y_1, \ldots, y_{1000})$, derive an expression for the smoothed density $P(s[k] = q_i | Y_{1000})$, for $k = 1, \ldots, 1000$. (This needs to be derived in terms of a forward and backward recursion. State all assumptions made).

(f) Again evaluate the smoother by doing 50 independent simulations and use that to compute the mean square error of the smoothed estimate. If you implemented it correctly, the smoother should have a smaller mean square error.

(g) Derive a maximum likelihood estimator using the EM algorithm for amplitude of the sinusoid A. Assume that the frequency w is known. Also assume that the transition probabilities P and levels of the Markov chain are known; and also the variance of the noise is known. *Derive* means start with the definition of the auxillary (complete) likelihood, and derive expressions for the E-Step and M-Step of the EM algorithm for the above model. Carefully state all the assumptions involved in your derivation. (Simply writing down the final equations gets you zero marks).

(h) Simulate the EM algorithm in Question 1 (g). Choose a suitable value for frequency w that nicely illustrates your answer. Let $A^{(I)}$ denote the estimates generated by the EM algorithm at iteration I, $I = 1, 2, \ldots$. Plot the log-likelihood of the model estimate, i.e., $p(Y_{1000}|A^{(I)})$ versus iterations $I = 1, 2 \ldots$. If your simulation is correct, then the EM algorithm should generate estimates with likelihoods that are increasing vs iterations I.

Question 2 [22 marks]

(a) Use the Metropolis Hastings algorithm to simulate from pdf

$$\pi(x) \propto \cos^2(x) \times \sin^2(2x) \times \psi(x)$$

where $\psi(x) = N(0, 1)$. Plot the empirical histogram and compare with the true cdf.

(b) Suppose iid noise $v[k]$ with the above pdf was the measurement noise in a linear state space model:

$$\begin{aligned} x[k+1] &= x[k] + w[k], \quad w[k] \sim N(0,1) \quad \text{i.i.d.} \\ y[k] &= x[k] + (v[k] - \mu)] \end{aligned}$$

where $\mu = \mathbf{E}\{v[k]\}$. Write down the Kalman filter equations for estimating the state $x[k]$ given the observation sequence $y[1], \ldots, y[k]$.

(c) Simulate the performance of the Kalman filter for this model. Evaluate the mean square estimation error from the simulation.

(d) What can you say about the optimality of the Kalman filter for this model?

(e) Simulate a bootstrap particle filter for estimating the state $x[k]$ for the above model. Compare the performance with the Kalman filter.

The END

Il n'est pas certain que tout soit incertain.