# STSCI 7170 Homework 1

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### 1.a

Suppose a quadratic form

$$x'Ax = \sum_i \sum_j x_i a_{ij} x_j = \sum_i x_i^2 a_{ii} + \sum_{i \neq j} x_i a_{ij} x_j$$

where $A \in \mathbb{R}^n$ is an arbitrary matrix therefore $a_{ij} \neq a_{ji}$. We can construct another matrix $A^*$ with $a^*_{ii} = a_{ii}$ and $a^*_{ij} = a^*_{ji} = \frac{1}{2}(a_{ij} + a_{ji})$. As a result,

$$x'Ax = x'A^*x$$

where $A^*$ is a symmetric matrix.

### 1.b

Suppose that non-zero $x, y \in \mathbb{R}^n$ with $x'y = 0$ are linearly dependent, there exist a non-zero scalar $c$ such that $x = cy$. Then

$$||x||^2 = x'x = x'(cy) = cx'y = 0$$

which contradict with the assumption. So $x, y$ must be linearly independent.

### 1.c

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ji} = \sum_{j=1}^{m} \sum_{i=1}^{n} b_{ji} a_{ij} = tr(BA)$$

### 1.5

Let $I \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$

$$(I + vv') \left( I - \frac{vv'}{1 + v'v} \right) = I + vv' - \frac{vv'}{1 + v'v} - \frac{v'v}{1 + v'v} vv' = I + vv' - vv' = I$$

### 1.6

$$M = (a - b)I + bJ = (a - b)(I + \frac{b}{a - b}\mathbf{1}\mathbf{1}')$$

using the result in 1.5 and the fact that $a > b$, we can set

$$v = \sqrt{\frac{b}{a - b}}\mathbf{1}$$

Therefore,

$$M^{-1} = \frac{1}{a - b} \left( I - \frac{\frac{b}{a-b}\mathbf{1}\mathbf{1}'}{1 + \frac{a}{a-b}\mathbf{1}'\mathbf{1}} \right) = \frac{1}{a - b} \left( I - \frac{b\mathbf{1}\mathbf{1}'}{a - b + nb} \right) = \frac{1}{a - b} \left( I - \frac{bJ}{a - b + nb} \right)$$

## 1.9

Let $G = (a-b)I + bJ = (a-b)I + b\mathbf{1}\mathbf{1}'$. We have $G\mathbf{1} = (a+(n-1)b)\mathbf{1}$. And $\forall x$ satisfying $\mathbf{1}'x = 0$, $x$ is the eigenvectors of $G$ with eigenvalue $a-b$.

$$Gx = (a-b)x + b\mathbf{1}\mathbf{1}'x = (a-b)x$$

Therefore, $G = PDP'$ where $D$ is a diagonal matrix with $d_{11} = a+(n-1)b$ and $d_{ii} = a-b \forall i \in \{2,3,...,n\}$. $P$ is the Helmert matrix with each column normalized, i.e. $a_1 = (1,1,...,1)'/\sqrt{n}$, $a_2 = (1,-1,0,...,0)'/\sqrt{2}$, $a_3 = (1,1,-2,0,...,0)'/\sqrt{6}$ and so on. $a_n = (1,1,...,-(n-1))'/\sqrt{n(n-1)}$.

## 1.10

$\Sigma$ is symmetric so $\Sigma_{12} = \Sigma_{21}'$, $\Sigma_{11} = \Sigma_{11}'$ and $\Sigma_{22} = \Sigma_{22}'$. And if $\Sigma_{22}^{-1}$ exists, it is also symmetric.

$$B\Sigma B' = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' & 0 \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{12}' & I \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' & 0 \\ 0 & \Sigma_{22} \end{bmatrix}$$

## 1.13

$$cov(P'Y) = P'cov(Y)P = P'(\sigma_1^2 I + \sigma_2^2 \mathbf{1}\mathbf{1}')P$$
$$= \begin{bmatrix} \mathbf{1}' \\ P_n' \end{bmatrix} (\sigma_1^2 I + \sigma_2^2 \mathbf{1}\mathbf{1}') \begin{bmatrix} \mathbf{1} & P_n \end{bmatrix} = \sigma_1^2 \begin{bmatrix} \mathbf{1}' \\ P_n' \end{bmatrix} \begin{bmatrix} \mathbf{1} & P_n \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{1}' \\ P_n' \end{bmatrix} \mathbf{1}\mathbf{1}' \begin{bmatrix} \mathbf{1} & P_n \end{bmatrix}$$
$$= \sigma_1^2 \begin{bmatrix} \mathbf{1}\mathbf{1}' & \mathbf{1}'P_n \\ P_n'\mathbf{1} & P_n'P_n \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{1}' \\ P_n' \end{bmatrix} \begin{bmatrix} \mathbf{1}\mathbf{1}'\mathbf{1} & \mathbf{1}\mathbf{1}'P_n \end{bmatrix} = \sigma_1^2 \begin{bmatrix} n & 0 \\ 0 & I_{(n-1)} \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{1}' \\ P_n' \end{bmatrix} \begin{bmatrix} n\mathbf{1} & 0 \end{bmatrix}$$
$$= \sigma_1^2 \begin{bmatrix} n & 0 \\ 0 & I_{n-1} \end{bmatrix} + \sigma_2^2 \begin{bmatrix} n^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} n(\sigma_1^2 + n\sigma_2^2) & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

## 2.2

The random variables form three independent sets: $\{Y_1, Y_2, Y_3\}$, $\{Y_4, Y_5\}$ and $\{Y_6\}$ so $\bar{Y}_i$'s are independent. Therefore, $\bar{\Sigma} = Cov(\bar{Y})$ is diagonal. Let $B_1 = \mathbf{1}_3'/3$, $B_2 = \mathbf{1}_2'/2$ and $B_3 = 1$.

$$\bar{\Sigma}_{11} = B_1(0.5I + 0.5J)B_1' = \frac{1}{18}\mathbf{1}_3'\mathbf{1}_3 + \frac{1}{18}\mathbf{1}_3'\mathbf{1}_3\mathbf{1}_3'\mathbf{1}_3 = \frac{2}{3}$$
$$\bar{\Sigma}_{22} = B_2(0.3I + 0.7J)B_2' = \frac{17}{20}$$

Finally, $\bar{\Sigma}_{33} = 1$. Therefore, $\bar{Y} \sim N(\bar{\mu}, \bar{\Sigma})$ where

$$\bar{\mu} = (\mu_1, \mu_2, \mu_3)'$$
$$\bar{\Sigma} = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{17}{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2.3

### (a)

$$Y^* = \frac{1}{\sqrt{\sum_i w_i}} W'Y \sim N(\mu^*, \sigma^{*2})$$

where $\mu^* = 0$ and

$$\sigma^{*2} = \frac{1}{\sum_i w_i} W'\Sigma W = 1$$

Therefore, $Y^* \sim N(0,1)$.

**(b)**

We can observe that each $Y_i$ is independent and normal distributed, i.e. $Y_i \sim N(0, 1/w_i)$. Moreover, $\sqrt{w_i}Y_i \sim N(0,1)$ is a standard normal distribution. The square of the standard normal distribution follows the $\chi_1^2$ distribution with 1 degree of freedom. Therefore,

$$\sum_{i=1}^{n} w_i Y_i^2 \sim \chi_n^2$$

which is the $\chi^2$ distribution with $n$ degrees of freedom.

## 2.12

**(a)**

We would like to transform $Y = (Y_1, Y_2, Y_3, Y_4)'$ into $Z = BY = (Y_1 + Y_2, Y_3 + Y_4)'$.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$X \sim N(\mu_Z, \Sigma_Z)$ where $\mu_Z = B\mu = (5, -1)'$ and $\Sigma_Z = B\Sigma B'$

$$\Sigma_Z = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$$

We see that $Y_1 + Y_2$ and $Y_3 + Y_4$ are independent. Therefore, the conditional distribution of $Y_1 + Y_2 | Y_3 + Y_4 = 1$ is the same distribution as $Y_1 + Y_2$.

$Y_1 + Y_2 | Y_3 + Y_4 = 1 \sim Y_1 + Y_2 \sim N(5, 32)$

**(b)**

Rearrange the order of the random variables of interest: $(Y_3, Y_1, Y_2) \sim N((0, 2, 3)', \bar{\Sigma})$ where

$$\bar{\Sigma} = \begin{bmatrix} 11 & 1 & -1 \\ 1 & 11 & 5 \\ -1 & 5 & 11 \end{bmatrix}$$

The conditional mean of $Y_3 | Y_1 = y_1, Y_2 = y_2$ is

$$\mu_3 + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 11 & -5 \\ -5 & 11 \end{bmatrix}^{-1} \begin{bmatrix} y_1 - \mu_1 \\ y_2 - \mu_2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 - 2 \\ y_2 - 3 \end{bmatrix} = \frac{1}{6} + \frac{1}{6} y_1 - \frac{1}{6} y_2$$

## 2.16

$(F, G)' \sim N(\mu, \Sigma)$ where $\mu = (0, 0)'$ and $\Sigma$ is diagonal with $\Sigma_{11} = \sigma^2$ and $\Sigma_{22} = 1 - \sigma^2$. $Z = (F, F + G)' = B(F, G)'$ where

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Therefore $\Sigma_Z = B\Sigma B'$

$$\Sigma_Z = \begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & 1 \end{bmatrix}$$

$$\mu_{\{F|F+G=c\}} = \sigma^2 c$$

$$\sigma^2_{\{F|F+G=c\}} = \sigma^2 - \sigma^4$$

Therefore, $F | F + G = c \sim N(c\sigma^2, \sigma^2 - \sigma^4)$

## 2.17

Without loss of generality, we would like to compare the variances of the following distributions: $A \sim \{Y_1|Y_2 = y_2\}$ and $B \sim \{Y_1|Y_2 = y_2, Y_3 = y_3\}$.

$$\sigma_A^2 = \Sigma_{11} - \begin{bmatrix} \Sigma_{12} & \Sigma_{13} \end{bmatrix} \begin{bmatrix} \Sigma_{22}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Sigma_{12} \\ \Sigma_{13} \end{bmatrix} = \Sigma_{11} - \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \Sigma_{11} - \frac{x^2}{a}$$

Similarly,

$$\sigma_B^2 = \Sigma_{11} - \begin{bmatrix} \Sigma_{12} & \Sigma_{13} \end{bmatrix} \begin{bmatrix} \Sigma_{22} & \Sigma_{23} \\ \Sigma_{32} & \Sigma_{33} \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{12} \\ \Sigma_{13} \end{bmatrix} = \Sigma_{11} - \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \Sigma_{11} - \frac{1}{ac - b^2}(cx^2 - 2bxy + ay^2)$$

$$\sigma_B^2 - \sigma_A^2 = -\frac{1}{ac - b^2}(cx^2 - 2bxy + ay^2) + \frac{x^2}{a} = -\frac{1}{a(ac - b^2)}(bx - ay)^2 \leq 0$$