STSCI 7170 Homework 3

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Problem 1

1a

Let $Y_i = X_i^2$ and $Y = \sum_{i=1}^n Y_i$. If we can find the mgf of Y_i, $m_{Y_i}(t)$, the mgf of Y is simply $m_Y(t) = \prod_{i=1}^n m_{Y_i}(t)$.

$$m_{Y_i}(t) = E\left[e^{tY_i}\right] = E\left[e^{tX_i^2}\right] = \int e^{tx^2}\frac{1}{\sqrt{2\pi}}e^{-(x-\mu_i)^2/2}dx = \frac{1}{\sqrt{2\pi}}\int \exp\left[\frac{-1}{2}\left((1-2t)x^2 - 2\mu_i x\right) - \frac{1}{2}\mu_i^2\right]dx$$

$$= \frac{1}{\sqrt{2\pi}}\int \exp\left[\frac{-1}{2}(1-2t)\left(x - \frac{\mu_i}{1-2t}\right)^2\right]\exp\left(\frac{t\mu_i^2}{1-2t}\right)dx = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{t\mu_i^2}{1-2t}\right)\sqrt{\frac{2\pi}{1-2t}}$$

$$m_{Y_i}(t) = (1-2t)^{-1/2}\exp\left(\frac{t\mu_i^2}{1-2t}\right)$$

$$m_Y(t) = \prod_{i=1}^n m_{Y_i}(t) = (1-2t)^{-n/2}\exp\left(\frac{t\delta}{1-2t}\right) \quad \delta = \sum_{i=1}^n \mu_i^2 \quad t < \frac{1}{2}$$

1b

We first show that $f_Y(y,\delta)$ is a proper pdf and then $E[e^{tY}]$ is the same as the result in 1a.

$$\int_0^\infty f_Y(y,\delta)dy = \int_0^\infty \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{y^{(n+2k)/2}e^{-y/2}}{\Gamma[(n+2k)/2]2^{(n+2k)/2}}$$

$$= \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{1}{\Gamma[(n+2k)/2]2^{(n+2k)/2}}\int_0^\infty y^{(n+2k)/2}e^{-y/2}dy$$

$$= \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{1}{\Gamma[(n+2k)/2]2^{(n+2k)/2}}\Gamma[(n+2k)/2]2^{(n+2k)/2}$$

$$= \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!} = e^{\delta/2}e^{-\delta/2} = 1$$

Moreover,

$$E[e^{tY}] = \int_0^\infty \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{y^{(n+2k)/2}e^{-(\frac{1}{2}-t)y}}{\Gamma[(n+2k)/2]2^{(n+2k)/2}} \quad t < \frac{1}{2}$$

$$= \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{1}{\Gamma[(n+2k)/2]2^{(n+2k)/2}}\frac{\Gamma[(n+2k)/2]}{(\frac{1}{2}-t)^{(n+2k)/2}}$$

$$= \sum_{k=0}^\infty \frac{(\frac{\delta}{2})^k e^{-\delta/2}}{k!}\frac{1}{(1-2t)^{\frac{n}{2}+k}} = e^{-\delta/2}(1-2t)^{-n/2}\sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\delta}{2(1-2t)}\right)^k$$

$$= e^{-\delta/2}(1-2t)^{-n/2}e^{\frac{\delta}{2(1-2t)}} = (1-2t)^{-n/2}\exp\left(\frac{t\delta}{1-2t}\right)$$

Therefore, $f_Y(y,\delta)$ is the pdf for the variable $Y = \sum_{i=1}^n X_i^2$.

Problem 2

$$Y_{ijk} = \mu_0 + S_i + D_j + SD_{ij} + T_k + ST_{ik} + DT_{jk} + E_{ijk}$$

where $i \in \{1, 2, \ldots, n = 5\}$, $j \in \{1, d = 2\}$ and $k \in \{1, 2, 3, t = 4\}$. The subject is the random factor. We can rewrite the model as

$$Y = \mu_0(\mathbf{1}_n \otimes \mathbf{1}_d \otimes \mathbf{1}_t) + (I_n \otimes \mathbf{1}_d \otimes \mathbf{1}_t)\tilde{S} + (\mathbf{1}_n \otimes I_d \otimes \mathbf{1}_t)\tilde{D} + (I_n \otimes I_d \otimes \mathbf{1}_t)\tilde{SD}$$

$$+(\mathbf{1}_n \otimes \mathbf{1}_d \otimes I_t)\tilde{T} + (I_n \otimes \mathbf{1}_d \otimes I_t)\tilde{ST} + (\mathbf{1}_n \otimes I_d \otimes I_t)\tilde{DT} + (I_n \otimes I_d \otimes I_t)\tilde{E}$$

where $\tilde{S} \sim N_n(0, \sigma_S^2 I_n)$, $\tilde{SD} \sim N_{nd}(0, \sigma_{SD}^2 I_{nd})$, $\tilde{ST} \sim N_{nt}(0, \sigma_{ST}^2 I_{nt})$ and $\tilde{E} \sim N_{ndt}(0, \sigma_E^2 I_{ndt})$.

The A-matrices are:

1. μ_0: $A_1 = \bar{J}_n \otimes \bar{J}_d \otimes \bar{J}_t = A_1^f$, $\text{rank}(A_1) = 1$
2. $\tilde{S}$: $A_2 = C_n \otimes \bar{J}_d \otimes \bar{J}_t = A_1^r$, $\text{rank}(A_2) = n - 1$
3. $\tilde{D}$: $A_3 = \bar{J}_n \otimes C_d \otimes \bar{J}_t = A_2^f$, $\text{rank}(A_3) = d - 1$
4. $\tilde{SD}$: $A_4 = C_n \otimes C_d \otimes \bar{J}_t = A_2^r$, $\text{rank}(A_4) = (n-1)(d-1)$
5. $\tilde{T}$: $A_5 = \bar{J}_n \otimes \bar{J}_d \otimes C_t = A_3^f$, $\text{rank}(A_5) = t - 1$
6. $\tilde{ST}$: $A_6 = C_n \otimes \bar{J}_d \otimes C_t = A_3^r$, $\text{rank}(A_6) = (n-1)(t-1)$
7. $\tilde{DT}$: $A_7 = \bar{J}_n \otimes C_d \otimes C_t = A_4^f$, $\text{rank}(A_7) = (d-1)(t-1)$
8. $\tilde{E}$: $A_7 = C_n \otimes C_d \otimes C_t = A_4^r$, $\text{rank}(A_8) = (n-1)(d-1)(t-1)$

Let $Z_i = A_i^f + A_i^r$.

$$I_{ndt} = \sum_{i=1}^{8} A_i = \sum_{i=1}^{4} \left(A_i^r + A_i^f\right) = \sum_{i=1}^{4} Z_i$$

$$\Sigma = dt\sigma_S^2(I_n \otimes \bar{J}_d \otimes \bar{J}_t) + t\sigma_{SD}^2(I_n \otimes I_d \otimes \bar{J}_t) + d\sigma_{ST}^2(I_n \otimes \bar{J}_d \otimes I_t) + \sigma_E^2(I_n \otimes I_d \otimes I_t)$$

$$= \left(dt\sigma_S^2 + t\sigma_{SD}^2 + d\sigma_{ST}^2 + \sigma_E^2\right)(A_1 + A_2) + \left(t\sigma_{SD}^2 + \sigma_E^2\right)(A_3 + A_4) + \left(d\sigma_{ST}^2 + \sigma_E^2\right)(A_5 + A_6) + \sigma_E^2(A_7 + A_8)$$

$$\Sigma = \left(dt\sigma_S^2 + t\sigma_{SD}^2 + d\sigma_{ST}^2 + \sigma_E^2\right) Z_1 + \left(t\sigma_{SD}^2 + \sigma_E^2\right) Z_2 + \left(d\sigma_{ST}^2 + \sigma_E^2\right) Z_3 + \sigma_E^2 Z_4$$

$$\mu = E[Y] = Y = \mu_0(\mathbf{1}_n \otimes \mathbf{1}_d \otimes \mathbf{1}_t) + (\mathbf{1}_n \otimes I_d \otimes \mathbf{1}_t)\tilde{D} + (\mathbf{1}_n \otimes \mathbf{1}_d \otimes I_t)\tilde{T} + (\mathbf{1}_n \otimes I_d \otimes I_t)\tilde{DT}$$

2a

Under this setup, the expected mean squares (EMS) is defined as

$$\text{EMS}_i = \frac{Y' A_i Y}{\text{rank}(A_i)} = c_i + \frac{\mu' A_i \mu}{\text{rank}(A_i)}$$

$$\text{EMS}_2 = \frac{Y' A_2 Y}{n-1} = dt\sigma_S^2 + t\sigma_{SD}^2 + d\sigma_{ST}^2 + \sigma_E^2 \quad \because \mu' A_2 \mu = 0$$

$$\text{EMS}_4 = \frac{Y' A_4 Y}{(n-1)(d-1)} = t\sigma_{SD}^2 + \sigma_E^2 \quad \because \mu' A_4 \mu = 0$$

$$\text{EMS}_6 = \frac{Y' A_6 Y}{(n-1)(t-1)} = d\sigma_{ST}^2 + \sigma_E^2 \quad \because \mu' A_6 \mu = 0$$

$$\text{EMS}_8 = \frac{Y'A_8Y}{(n-1)(d-1)(t-1)} = \sigma_E^2 \quad \because \mu' A_8 \mu = 0$$

$$\text{EMS}(\sigma_E^2) = \frac{Y'A_8Y}{(n-1)(d-1)(t-1)}$$

$$\text{EMS}(\sigma_{ST}^2) = \frac{1}{d}\left[\frac{Y'A_6Y}{(n-1)(t-1)} - \frac{Y'A_8Y}{(n-1)(d-1)(t-1)}\right]$$

$$\text{EMS}(\sigma_{SD}^2) = \frac{1}{t}\left[\frac{Y'A_4Y}{(n-1)(d-1)} - \frac{Y'A_8Y}{(n-1)(d-1)(t-1)}\right]$$

$$\text{EMS}(\sigma_S^2) = \frac{1}{dt}\left[\frac{Y'A_2Y}{n-1} - \frac{Y'A_4Y}{(n-1)(d-1)} - \frac{Y'A_6Y}{(n-1)(t-1)} + \frac{Y'A_8Y}{(n-1)(d-1)(t-1)}\right]$$

The maximum likelihood (ML) estimate is defined as

$$\text{ML}_i = \frac{Y'A_i^rY}{\text{rank}(Z_i)} = \frac{c_i \text{rank}(A_i^r)}{\text{rank}(Z_i)} + \frac{\mu' A_i^r \mu}{\text{rank}(Z_i)} = \frac{\text{EMS}_i \text{rank}(A_i^r)}{\text{rank}(Z_i)}$$

$$\text{ML}_1 = \frac{Y'A_1^rY}{n} = \frac{n-1}{n}\left(dt\sigma_S^2 + t\sigma_{SD}^2 + d\sigma_{ST}^2 + \sigma_E^2\right)$$

$$\text{ML}_2 = \frac{Y'A_2^rY}{n(d-1)} = \frac{n-1}{n}\left(t\sigma_{SD}^2 + \sigma_E^2\right)$$

$$\text{ML}_3 = \frac{Y'A_3^rY}{n(t-1)} = \frac{n-1}{n}\left(d\sigma_{ST}^2 + \sigma_E^2\right)$$

$$\text{ML}_4 = \frac{Y'A_4^rY}{n(d-1)(t-1)} = \frac{n-1}{n}\sigma_E^2$$

$$\text{ML}(\sigma_E^2) = \frac{Y'A_4^rY}{(n-1)(d-1)(t-1)} = \text{EMS}(\sigma_E^2)$$

$$\text{ML}(\sigma_{ST}^2) = \frac{1}{d}\left[\frac{Y'A_3^rY}{(n-1)(t-1)} - \frac{Y'A_4^rY}{(n-1)(d-1)(t-1)}\right] = \text{EMS}(\sigma_{ST}^2)$$

$$\text{ML}(\sigma_{SD}^2) = \frac{1}{t}\left[\frac{Y'A_2^rY}{(n-1)(d-1)} - \frac{Y'A_4^rY}{(n-1)(d-1)(t-1)}\right] = \text{EMS}(\sigma_{SD}^2)$$

$$\text{ML}(\sigma_S^2) = \frac{1}{dt}\left[\frac{Y'A_1^rY}{n-1} - \frac{Y'A_2^rY}{(n-1)(d-1)} - \frac{Y'A_3^rY}{(n-1)(t-1)} + \frac{Y'A_4^rY}{(n-1)(d-1)(t-1)}\right] = \text{EMS}(\sigma_S^2)$$

2b

Construct the A matrices.

```r
n = 5;
d = 2;
t = 4;

Jn = rep(1, n) %*% t(rep(1, n)) / n
Cn = diag(n) - Jn
Jd = rep(1, d) %*% t(rep(1, d)) / d
Cd = diag(d) - Jd
Jt = rep(1, t) %*% t(rep(1, t)) / t
Ct = diag(t) - Jt

A1 = Jn %x% Jd %x% Jt
A2 = Cn %x% Jd %x% Jt
A3 = Jn %x% Cd %x% Jt
A4 = Cn %x% Cd %x% Jt
A5 = Jn %x% Jd %x% Ct
A6 = Cn %x% Jd %x% Ct
A7 = Jn %x% Cd %x% Ct
A8 = Cn %x% Cd %x% Ct
```

$$\mathrm{ML}(\sigma_E^2) = \frac{Y' A_4^r Y}{(n-1)(d-1)(t-1)} = 0.09944708$$

$$\mathrm{ML}(\sigma_{ST}^2) = \frac{1}{d}\left[\frac{Y' A_3^r Y}{(n-1)(t-1)} - \frac{Y' A_4^r Y}{(n-1)(d-1)(t-1)}\right] = 0.001075417$$

$$\mathrm{ML}(\sigma_{SD}^2) = \frac{1}{t}\left[\frac{Y' A_2^r Y}{(n-1)(d-1)} - \frac{Y' A_4^r Y}{(n-1)(d-1)(t-1)}\right] = 0.1274217$$

$$\mathrm{ML}(\sigma_S^2) = \frac{1}{dt}\left[\frac{Y' A_1^r Y}{n-1} - \frac{Y' A_2^r Y}{(n-1)(d-1)} - \frac{Y' A_3^r Y}{(n-1)(t-1)} + \frac{Y' A_4^r Y}{(n-1)(d-1)(t-1)}\right] = 0.06218708$$

2c

```r
serum =  data.frame(matrix(ncol = 4, nrow = n*d*t))
colnames(serum) = c("response", "subject", "drug", "time")
serum$response = Y
serum$subject = 1:n %x% rep(1, d) %x% rep(1,t)
serum$drug = rep(1, n) %x% 1:d %x% rep(1,t)
serum$time = rep(1, n) %x% rep(1, d) %x% 1:t

ml = lmer(response ~ 1 + (1|subject) + (drug) + (time)
                       + (1|subject:drug)
                       + (1|subject:time)
                       + (drug:time)
                       , data=serum, REML=FALSE)
summary(ml)
```

```
Warning message:
In checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv,  :
  Model failed to converge with max|grad| = 0.0178682 (tol = 0.002, component 1)

Linear mixed model fit by maximum likelihood  ['lmerMod']
```

```
Formula: response ~ 1 + (1 | subject) + (drug) + (time) + (1 | subject:drug) +      (1 | subject:time) +
   Data: serum

     AIC      BIC   logLik deviance df.resid
    69.5     83.0    -26.7     53.5       32

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.59168 -0.41244 -0.03616  0.28838  1.70536

Random effects:
 Groups       Name        Variance Std.Dev.
 subject:time (Intercept) 0.1068   0.3268
 subject:drug (Intercept) 0.1023   0.3198
 subject      (Intercept) 0.0229   0.1513
 Residual                 0.0817   0.2858
Number of obs: 40, groups:  subject:time, 20; subject:drug, 10; subject, 5

Fixed effects:
            Estimate Std. Error t value
(Intercept)  1.42500    0.51134   2.787
drug        -0.10700    0.29990  -0.357
time        -0.14180    0.14357  -0.988
drug:time    0.07100    0.08085   0.878

Correlation of Fixed Effects:
          (Intr) drug   time
drug      -0.880
time      -0.702  0.569
drug:time  0.593 -0.674 -0.845
convergence code: 0
Model failed to converge with max|grad| = 0.0178682 (tol = 0.002, component 1)
```

Moser 5.2

a

$\hat{\beta} = (X'X)^{-1}X'Y$ is an unbiased estimator for β because

$$E[\hat{\beta}] = (X'X)^{-1}X'E[Y] = (X'X)^{-1}X'X\beta = \beta$$

b

Since $VX = XF$ where F is non-singular, we have $X'V = F'X'$ and $X' = F'X'V^{-1}$. Therefore,

$$\hat{\beta} = (X'X)^{-1}X'Y = (F'X'V^{-1}X)^{-1}F'X'V^{-1}Y = (X'V^{-1}X)^{-1}(F')^{-1}F'X'V^{-1}Y = (X'V^{-1}X)^{-1}X'V^{-1}Y = \hat{\beta}_W$$

Moser 5.9

a

$$Y_{ij} = a + b_i x_j + E_{ij}$$

Let $Y = (Y_{11} - a, \ldots, Y_{1n} - a, Y_{21} - a, \ldots, Y_{2n} - a)' = (\tilde{Y_1}', \tilde{Y_2}')'$, we can rewrite the linear model in the matrix form.

$$Y = (I_2 \otimes \tilde{x})\tilde{\beta} + (I_2 \otimes I_n)\tilde{E}$$

where $\tilde{x} = (x_1, x_2, \ldots, x_n)'$, $\tilde{\beta} = (b_1, b_2)'$ and $\tilde{E} \sim N_{2n}(0, \sigma^2 I_{2n})$. We can identify the regression matrix $X = I_2 \otimes \tilde{x}$. The least square estimator of $\tilde{\beta}$ is $\hat{\beta} = (X'X)^{-1}X'Y$.

$$X'X = (I_2 \otimes \tilde{x}')(I_2 \otimes \tilde{x}) = (\tilde{x}'\tilde{x})I_2 \Longrightarrow (X'X)^{-1} = (\tilde{x}'\tilde{x})^{-1}I_2$$

$$X'Y = (I_2 \otimes \tilde{x}')Y = \begin{bmatrix} \tilde{x}' & 0 \\ 0 & \tilde{x}' \end{bmatrix} \begin{bmatrix} \tilde{Y_1} \\ \tilde{Y_2} \end{bmatrix} = \begin{bmatrix} \tilde{x}'\tilde{Y_1} \\ \tilde{x}'\tilde{Y_2} \end{bmatrix}$$

$$\hat{\beta} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (\tilde{x}'\tilde{x})^{-1} \begin{bmatrix} \tilde{x}'\tilde{Y_1} \\ \tilde{x}'\tilde{Y_2} \end{bmatrix}$$

Let $t = (1, -1)'$, the BLUE of $b_1 - b_2 = t'\tilde{\beta}$ is $t'\hat{\beta}$

$$t'\hat{\beta} = (\tilde{x}'\tilde{x})^{-1}\left(\tilde{x}'\tilde{Y_1} - \tilde{x}'\tilde{Y_2}\right) = (\tilde{x}'\tilde{x})^{-1}\sum_{j=1}^{n} x_j(Y_{1j} - Y_{2j}) = \frac{\sum_{j=1}^{n} x_j(Y_{1j} - Y_{2j})}{\sum_{j=1}^{n} x_j^2}$$

The absence of constant a is because $a\sum_{j=1}^{n} x_j = 0$.

b

We know that $\text{Cov}(\hat{\beta}) = \sigma^2(X'X)^{-1} = \sigma^2(\tilde{x}'\tilde{x})^{-1}I_2$.

$$\text{Cov}(t'\hat{\beta}) = t'\text{Cov}(\hat{\beta})t = \frac{2\sigma^2}{\tilde{x}'\tilde{x}} = \frac{2\sigma^2}{\sum_{j=1}^{n} x_j^2}$$

Moser 5.12

a

Since $\mathbf{1}_a' X^* = 0$, each column of X^* is centered and $\text{rank}(X^*) = p - 1$. We know the inclusion of replicates is equivalent to averaging with respect to replicates to start with. Therefore, A_1, A_2 and $A_3 (A_{lof})$ take the form $(?? \otimes \bar{J}_n)$.

$$\beta_0 : \quad A_1 = \bar{J}_a \otimes \bar{J}_n \quad \text{rank}(A_1) = 1$$
$$\beta : \quad A_2 = X^*(X^{*\prime}X^*)^{-1}X^{*\prime} \otimes \bar{J}_n \quad \text{rank}(A_2) = p - 1$$
$$\text{Lack of Fit} : \quad A_{lof} = A_3 = \left(I_a - \bar{J}_a - X^*(X^{*\prime}X^*)^{-1}X^{*\prime}\right) \otimes \bar{J}_n \quad \text{rank}(A_3) = a - p > 1$$
$$\text{Pure Error} : \quad A_{pe} = A_4 = I_a \otimes C_n \quad \text{rank}(A_4) = a(n-1)$$

Moreover,

$$\sum_{i=1}^{4} A_i = I_a \otimes I_n = I_{an} \qquad \sum_{i=1}^{4} \text{rank}(A_i) = an$$

b

Let $\mu = E[Y] = \beta_0(\mathbf{1}_a \otimes \mathbf{1}_n) + (X^* \otimes \mathbf{1}_n)\beta$ and $M = X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$. Therefore, $M\bar{J} = 0$ and $M^2 = M$. The A-matrices satisfy the assumptions of Bhat's lemma with $A_iA_j = \delta_{ij}A_i$. Now, in order to apply Bhat's lemma

$$\mathrm{Cov}(Y) = \Sigma = I_a \otimes \left(\sigma_1^2 I_n + n\sigma_2 \bar{J}_n\right) = \sigma_1^2 \sum_{i=1}^{4} A_i + n\sigma_2^2 \sum_{i=1}^{3} A_i = \left(\sigma_1^2 + n\sigma_2^2\right) \sum_{i=1}^{3} A_i + \sigma_1^2 A_4$$

Therefore, by Bhat's lemma,

$$Y'A_1Y \sim \left(\sigma_1^2 + n\sigma_2^2\right) \chi_1^2(\delta_1)$$

$$\delta_1 = \left(\sigma_1^2 + n\sigma_2^2\right)^{-1} \left[\beta_0^2(\mathbf{1}_a' \otimes \mathbf{1}_n')(\bar{J}_a \otimes \bar{J}_n)(\mathbf{1}_a \otimes \mathbf{1}_n)\right] = \frac{an\beta_0^2}{\sigma_1^2 + n\sigma_2^2}$$

$$Y'A_2Y \sim \left(\sigma_1^2 + n\sigma_2^2\right) \chi_{p-1}^2(\delta_2)$$

$$\delta_2 = \left(\sigma_1^2 + n\sigma_2^2\right)^{-1} \left[\beta'(X^{*\prime} \otimes \mathbf{1}_n')(M \otimes \bar{J}_n)(X^* \otimes \mathbf{1}_n)\beta\right] = \frac{n\beta' X^{*\prime} X^* \beta}{\sigma_1^2 + n\sigma_2^2}$$

$$Y'A_3Y \sim \left(\sigma_1^2 + n\sigma_2^2\right) \chi_{a-p}^2(\delta_3) \quad \delta_3 = 0$$

$$\because \quad \mathbf{1}_a'(I_a - M - \bar{J}_a)\mathbf{1}_a = 0 \quad \text{and} \quad X^{*\prime}(I_a - M - \bar{J}_a)X^* = 0$$

$$Y'A_3Y \sim n\sigma_2^2 \chi_{a(n-1)}^2(\delta_4) \quad \delta_4 = 0$$

c

$$\Sigma = \sigma_1^2(I_a \otimes I_n) + n\sigma_2^2(I_a \otimes \bar{J}_n)$$

$Y'A_iY$ and $Y'A_jY$ are independent if and only if $A_i\Sigma A_j = 0$.

$$A_i\Sigma A_j = \sigma_1^2 A_iA_j + n\sigma_2^2 A_i(I_a \otimes \bar{J}_n)A_j$$

For $i \neq j$, the first term is always zero. The second term is zero when either $i = 4$ or $j = 4$. Moreover, for $i = 1, 2, 3$ and $i \neq j$ the second term is zero because

$$M\bar{J}_a = 0, \quad (I_a - M - \bar{J}_a)\bar{J}_a = 0, \quad (I_a - M - \bar{J}_a)M = 0$$

As a result, $Y'A_iY$ and $Y'A_jY$ are mutually independent since $\forall i, j \in \{1, 2, 3, 4\}$ and $i \neq j$, $A_i\Sigma A_j = 0$.

d

Observe that δ_2 involves the vector β, which will show up in the noncentrality of the F-distribution. Define a statistic

$$F = \frac{Y'A_2Y/(p-1)}{Y'A_3Y/(a-p)} \sim F_{p-1,a-p}(\delta_2)$$

where

$$\delta_2 = \frac{n\beta' X^{*\prime} X^* \beta}{\sigma_1^2 + n\sigma_2^2}$$

Under null hypothesis $H_0 : \beta = 0$, $F \sim F_{p-1,a-p}(0)$.