STSCI 7170 Homework 2

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Problem 1

Suppose that $\text{rank}(A) = a$ and $B \in \mathbb{R}^{m\times n}$. We can write the quadratic form $Y'AY = Z'DZ$ where $Z = P'\Sigma^{-\frac{1}{2}}Y \sim N_n(P'\Sigma^{-\frac{1}{2}}\mu, I_n)$ and $\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}} = PDP'$ by spectral decomposition. D is a diagonal with $d_{ii} \neq 0 \quad \forall i \in \{1, 2, \dots, a\}$.

Moreover, $BY = B\Sigma^{\frac{1}{2}}PZ = MZ$ where $M = B\Sigma^{\frac{1}{2}}P$.

$$A\Sigma B' = 0 \iff B\Sigma A = 0 \iff B\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}} = 0 \iff B\Sigma^{\frac{1}{2}}PDP' = 0 \iff MD = 0$$

$$D = \begin{bmatrix} D_a & 0 \\ 0 & 0 \end{bmatrix}; \quad M = \begin{bmatrix} M_1 & M_2 \end{bmatrix},$$

where $M_1 \in \mathbb{R}^{m\times a}$ and $M_2 \in \mathbb{R}^{m\times(n-a)}$

$$MD = 0 \iff M_1 D_a = 0 \iff M_1 = 0 \iff M = \begin{bmatrix} 0 & M_2 \end{bmatrix}$$

$$Y'AY = \sum_{j=1}^{a} \lambda_j z_j^2; \quad (BY)_i = \sum_{j=n-a+1}^{n} m_{ij} z_j$$

Because z_i's are independent, $Y'AY$ and BY are independent.

Problem 2

Suppose the positive semi-definite symmetric matrix A has rank a, then A has a non-zero eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_a > 0$ and $\lambda_i = 0 \; \forall i \in \{a+1, \dots, n\}$. We can write the spectra decomposition of A as $A = PDP'$ where the columns of P are a orthonormal vectors: $P = (p_1, p_2, \dots, p_n) \in \mathbb{R}^{n\times n}$. Therefore,

$$A = \sum_{k=1}^{n} \lambda_k p_k p_k' = \sum_{k=1}^{a} \lambda_k p_k p_k'$$

Or equivalently,

$$a_{ij} = \sum_{k=1}^{n} \lambda_k p_{ki} p_{kj} = \sum_{k=1}^{a} \lambda_k p_{ki} p_{kj}$$

where p_{ki} is the i^{th} element of the k^{th} columns of P.

2a

$$a_{ii} = \sum_{k=1}^{n} \lambda_k p_{ki}^2 \geq 0$$

because $\lambda_k \geq 0$.

2b

If

$$a_{ii} = \sum_{k=1}^{n} \lambda_k p_{ki}^2 = \sum_{k=1}^{a} \lambda_k p_{ki}^2 = 0$$

Because $\lambda_k > 0 \quad \forall k = 1, 2, \dots, a$, $p_{ki} = 0 \quad \forall k = 1, 2, \dots, a$.

$$\forall i \neq j \quad a_{ij} = \sum_{k=1}^{a} \lambda_k p_{ki} p_{kj} = 0$$

Problem 3

3a

$$Y_{ijk} = \mu_0 + B_i + \tau_j + (B\tau)_{ij} + E_{ijk}$$

In the above model, μ_0 is the average response, B_i is the random effect on subject i, τ_j is the fixed effect by drug j, $(B\tau)_{ij}$ is the random interaction effect between subject i and drug j and finally, E_{ijk} is the random error of the experiment. Note that $i = 1, 2, \ldots, 5 = s$, $j = 1, 2 = d$ and $k = 1, 2, 3, 4 = t$.

3b

1. μ_0 is a fixed factor because it is the average response of the experiment.
2. τ_j is fixed with respect to two drugs.
3. $B_i \sim N(0, \sigma_B^2)$ is random to account for the noises of different subjects.
4. $(B\tau)_{ij} \sim N(0, \sigma_{B\tau}^2)$ is random to account for different subjects' responses to the drugs.
5. $E_{ijk} \sim N(0, \sigma_E^2)$ is the noise of the experiment.

3c

1. μ: $A_1 = \bar{J}_s \otimes \bar{J}_d \otimes \bar{J}_t$
2. B: $A_2 = C_s \otimes \bar{J}_d \otimes \bar{J}_t$
3. τ: $A_3 = \bar{J}_s \otimes C_d \otimes \bar{J}_t$
4. $(B\tau)_{ij}$: $A_4 = C_s \otimes C_d \otimes \bar{J}_t$
5. E_{ijk}: $A_5 = I_s \otimes I_d \otimes C_t$

$$\sum_{i=1}^{5} A_i = I_{sdt} \quad \sum_{i=1}^{5} \text{rank}(A_i) = 1 + (s-1) + (d-1) + (d-1)(s-1) + sd(t-1) = sdt$$

3d

$$Y = (1_s \otimes 1_d \otimes 1_t)\mu_0 + (I_s \otimes 1_d \otimes 1_t)\tilde{B} + (1_s \otimes I_d \otimes 1_t)\tilde{\tau} + (I_s \otimes I_d \otimes 1_t)\tilde{B\tau} + (I_s \otimes I_d \otimes I_t)\tilde{E}$$

where $\tilde{B} \sim N_s(0_s, \sigma_B^2 I_s)$, $\tilde{B\tau} \sim N_{sd}(0_{sd}, \sigma_{B\tau}^2 I_{sd})$ and $\tilde{E} \sim N_{sdt}(0_{sdt}, \sigma_E^2 I_{sdt})$

3e

$$E[Y] = (1_s \otimes 1_d \otimes 1_t)\mu_0 + (1_s \otimes \tilde{\tau} \otimes 1_t)$$

where $\tilde{\tau} = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}'$.

$$\text{Var}(Y) = \Sigma = (dt\sigma_B^2)(I_s \otimes \bar{J}_d \otimes \bar{J}_t) + (t\sigma_{B\tau}^2)(I_s \otimes I_d \otimes \bar{J}_t) + \sigma_E^2(I_s \otimes I_d \otimes I_t)$$

3f

$$\Sigma = (dt\sigma_B^2)(A_1 + A_2) + (t\sigma_{B\tau}^2)\left(\sum_{i=1}^{4} A_i\right) + \sigma_E^2\left(\sum_{i=1}^{5} A_i\right)$$

$$\Sigma = (dt\sigma_B^2 + t\sigma_{B\tau}^2 + \sigma_E^2)(A_1 + A_2) + (t\sigma_{B\tau}^2 + \sigma_E^2)(A_3 + A_4) + \sigma_E^2 A_5$$

Moser 2.5

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_A)'$ and since $Y_{ijk} = \mu_i + S_{ij} + T_{ijk}$ we can write

$$Y = (\mu \otimes 1_s \otimes 1_t) + (I_a \otimes I_s \otimes 1_t)S + (I_a \otimes I_s \otimes I_t)T$$

where $S \sim N_{as}(0_{as}, \sigma_S^2 I_{as})$ and $T \sim N_{ast}(0_{ast}, \sigma_T^2 I_{ast})$. Therefore, we have the following.

$$E[Y] = \mu \otimes 1_s \otimes 1_t$$

$$\Sigma = (I_a \otimes I_s \otimes 1_t)\sigma_S^2 I_{as}(I_a \otimes I_s \otimes 1_t') + (I_a \otimes I_s \otimes I_t)\sigma_T^2 I_{ast}(I_a \otimes I_s \otimes I_t)$$
$$\Sigma = \sigma_S^2(I_a \otimes I_s \otimes 1_t 1_t') + \sigma_T^2 I_{ast}$$

a

$\bar{Y}_i = (\bar{Y}_{1..}, \bar{Y}_{2..}, \dots, \bar{Y}_{a..})' = MY$ where $M = I_a \otimes (\frac{1}{s}1_s') \otimes (\frac{1}{t}1_t') = \frac{1}{st} I_a \otimes 1_s' \otimes 1_t'$. Therefore, $\bar{Y}_i \sim N_a(\mu, M\Sigma M')$.

$$M\Sigma M' = \frac{\sigma_S^2}{s^2t^2}(I_a \otimes 1_s' 1_s \otimes 1_t' 1_t) + \frac{\sigma_T^2}{s^2t^2}(I_a \otimes 1_s' 1_s \otimes 1_t' 1_t)$$
$$M\Sigma M' = \left(\frac{\sigma_S^2 + \sigma_T^2}{st}\right) I_a$$

b

$(\bar{Y}_{11.} - \bar{Y}_{1..}, \dots, \bar{Y}_{1s.} - \bar{Y}_{1..}, \dots, \bar{Y}_{a1.} - \bar{Y}_{a..}, \dots, \bar{Y}_{as.} - \bar{Y}_{a.}) = KY \in \mathbb{R}^{as}$ where

$$KY = \left(I_a \otimes I_s \otimes \frac{1}{t}1_t'\right) Y - \left(I_a \otimes \frac{1}{s}1_s 1_s' \otimes \frac{1}{t}1_t'\right) Y = \left(I_a \otimes C_s \otimes \frac{1}{t}1_t'\right) Y$$

Therefore, $KY \sim N_{as}(KE[Y], K\Sigma K')$

$$KE[Y] = K(\mu \otimes 1_s \otimes 1_t) = 0_{as}$$

$$K\Sigma K' = \sigma_S^2\left(I_a \otimes C_s I_s C_s \otimes \frac{1}{t^2}1_t' 1_t 1_t' 1_t\right) + \sigma_T^2\left(I_a \otimes C_s I_s C_s \otimes \frac{1}{t^2}1_t' 1_t\right)$$
$$K\Sigma K' = \left(\sigma_S^2 + \frac{\sigma_T^2}{t}\right)(I_a \otimes C_s)$$

Moser 2.10

We have $Y = X\beta + E$ where β is an unknown constant vector and $E \sim N_n(0_n, \sigma^2 I_n)$. The only randomness comes from the vector E.

a

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'(X\beta + E) = \beta + (X'X)^{-1}X'E = \beta + ME$$

where $M = (X'X)^{-1}X'$. Therefore, since E is normal, we have $\hat{\beta} \sim N_p(\beta, \sigma^2 MM')$. Because $(X'X)^{-1}$ exists, $X'X$ must be non-singular, i.e. $\text{rank}(X'X) = p$ and $p < n$. Moreover, $(X'X)^{-1}$ is symmetric.

$$MM' = (X'X)^{-1}X'X\left((X'X)^{-1}\right)' = (X'X)^{-1}$$
$$\hat{\beta} \sim N_p\left(\beta, \sigma^2(X'X)^{-1}\right)$$

b

We can do singular value decomposition (SVD) of X, i.e. $X = U\Sigma V'$ where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{p\times p}$ with $U'U = I_n$ and $V'V = I_p$. $\Sigma \in \mathbb{R}^{n\times p}$ and since we know that $p < n$,

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$$

where Σ_1 is $p \times p$ diagonal matrix. $X'X = V\Sigma'\Sigma V' = V\Sigma_1^2 V'$ is nonsingular if and only if all the diagonal elements of Σ_1 are nonzero. Additionally, $(X'X)^{-1} = V\Sigma_1^{-2}V'$

Therefore,

$$\mathrm{Cov}(X\hat{\beta}) = \sigma^2 X(X'X)^{-1}X' = \sigma^2 U \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V'V\Sigma_1^{-2}V'V \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} U'$$

$$\mathrm{Cov}(X\hat{\beta}) = \sigma^2 U \begin{bmatrix} I_p & 0 \\ 0 & 0_{n-p} \end{bmatrix} U'$$

As a result, $\mathrm{rank}(\mathrm{Cov}(\hat{Y})) = \mathrm{rank}(\mathrm{Cov}(X\hat{\beta})) = p$

Moser 3.3

Construct the A-matrices as the following.

$$A_1 = \bar{J}_a \otimes \bar{J}_s \otimes \bar{J}_t \quad \mathrm{rank}(A_1) = n_1 = 1$$

$$A_2 = C_a \otimes \bar{J}_s \otimes \bar{J}_t \quad \mathrm{rank}(A_2) = n_2 = a - 1$$
$$A_3 = \bar{J}_a \otimes C_s \otimes \bar{J}_t \quad \mathrm{rank}(A_3) = n_3 = s - 1$$

$$A_4 = C_a \otimes C_s \otimes \bar{J}_t \quad \mathrm{rank}(A_4) = n_4 = (a-1)(s-1) = as - a - s + 1$$

$$A_5 = I_a \otimes I_s \otimes C_t \quad \mathrm{rank}(A_5) = n_5 = as(t-1) = ast - as$$

And the assumptions of Bhat's lemma are satisfied: $\sum_{i=1}^5 A_i = I_{ast}$ and $\sum_{i=1}^5 n_i = ast$.

Previously, in Moser 2.5, we have

$$\mathrm{Cov}(Y) = \Sigma = t\sigma_S^2(I_a \otimes I_s \otimes \bar{J}_t) + \sigma_T^2 I_{ast}$$

$$\Sigma = (t\sigma_S^2 + \sigma_T^2)\sum_{i=1}^4 A_i + \sigma_T^2 A_5$$

a

$$V_1 = Y'(I_a \otimes C_s \otimes \bar{J}_t)Y = Y'(A_3 + A_4)Y$$

As a result, $Y'(A_3 + A_4)Y \sim (t\sigma_S^2 + \sigma_T^2)\chi^2_{a(s-1)}(\delta_1)$ where

$$\delta_1 = (t\sigma_S^2 + \sigma_T^2)^{-1}(\mu \otimes 1_s \otimes 1_t)'(I_a \otimes C_s \otimes \bar{J}_t)(\mu \otimes 1_s \otimes 1_t) = 0$$

$$V_1 \sim (t\sigma_S^2 + \sigma_T^2)\chi^2_{a(s-1)}(0)$$

b

$$V_2 = Y'(I_a \otimes I_s \otimes C_t)Y = Y'A_5Y$$

So, $Y'A_5Y \sim \sigma_T^2\chi^2_{as(t-1)}(\delta_2)$ where

$$\delta_2 = \sigma_T^{-2}(\mu \otimes 1_s \otimes 1_t)'(I_a \otimes I_s \otimes C_t)(\mu \otimes 1_s \otimes 1_t) = 0$$

$$V_2 \sim \sigma_T^2\chi^2_{as(t-1)}(0)$$

c

Combining V_1 and V_2, we have the following directly.

$$\frac{V_1/[a(s-1)]}{V_2/[as(t-1)]} \sim \frac{t\sigma_S^2 + \sigma_T^2}{\sigma_T^2}F_{a(s-1),as(t-1)}(0)$$

Moser 3.15

a

With one i fixed, and $y_i = (Y_{i1}, Y_{i2})'$ we have the following.

$$\mathrm{E}(y_i) = \begin{bmatrix}\mu_1 & \mu_2\end{bmatrix}'$$

$$\mu = 1_n \otimes \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}$$

$$\mathrm{Cov}(y_i) = \sigma^2 \begin{bmatrix}1 & \rho \\ \rho & 1\end{bmatrix} = \sigma^2\left[(1+2\rho)I_2 - 2\rho C_2\right]$$

Therefore, $\Sigma = I_n \otimes \mathrm{Cov}(y) = \sigma^2\left[(1+2\rho)(I_n \otimes I_n) - 2\rho(I_n \otimes C_2)\right]$

b

Let $y = (Y_{i1}, Y_{i2})'$ and we can write $D_i = Y_{i1} - Y_{i2} = My_i$ where $M = \begin{bmatrix} 1 & -1 \end{bmatrix}'$. The distribution of $D - i$ can be found: $D_i \sim N(\mu_2 - \mu_1, M\text{Cov}(y_i)M')$, where

$$M\text{Cov}(y_i)M' = \sigma^2 M \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} M' = 2\sigma^2(1-\rho)$$

Therefore, $D = (D_1, D_2, \ldots, D_n)'$ is a random vector $D \sim N_n((\mu_2 - \mu_1)1_n, 2\sigma^2(1-\rho)I_n)$.

$$\bar{D} = \frac{1}{n}1_n' D \sim N\left(\mu_2 - \mu_1, \frac{2}{n}\sigma^2(1-\rho)\right)$$

$$\sum_{i=1}^{n}(D_i - \bar{D})^2 = (n-1)S_D^2 = D'C_nD \sim 2\sigma^2(1-\rho)\chi^2_{n-1}(0)$$

By using the definition of t-distribution,

$$\frac{\bar{D}/\sqrt{\frac{1}{n}2\sigma^2(1-\rho)}}{\sqrt{\frac{1}{2\sigma^2(1-\rho)}\frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2}} = \frac{\bar{D}}{\left(\frac{S_D}{\sqrt{n}}\right)} \sim T_{n-1}(\mu_2 - \mu_1)$$

Moser 3.17

a

Let $\sigma_1 = \sigma_2 = \sigma$. From the corvariance matrix, Σ, Y_{ij}'s are independent. $\bar{Y}_{1.} \sim N(\mu_1, \sigma^2/n_1)$ and $\bar{Y}_{2.} \sim N(\mu_2, \sigma^2/n_2)$. Therefore,

$$\bar{Y}_{1.} - \bar{Y}_{2.} \sim N\left(\mu_2 - \mu_1, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

Moreover, the term in the denominator can be separated into two:

$$i = 1 \quad S_1^2 = \sum_{j=1}^{n_1}(Y_{1j} - \bar{Y}_{1.})^2 \sim \sigma^2\chi^2_{n_1-1}(0)$$

$$i = 2 \quad S_2^2 = \sum_{j=1}^{n_2}(Y_{2j} - \bar{Y}_{2.})^2 \sim \sigma^2\chi^2_{n_2-1}(0)$$

Since S_1^2 and S_2^2 are independent, $S_1^2 + S_2^2 \sim \chi^2_{n_1+n_2-2}(0)$

According to the definition of F-distribution, we can write

$$\frac{(\bar{Y}_{1.} - \bar{Y}_{2.})^2/\sigma^2(\frac{1}{n_1} + \frac{1}{n_2})}{\frac{1}{\sigma^2}(S_1^2 + S_2^2)/(n_1 + n_2 - 2)} = \frac{(\bar{Y}_{1.} - \bar{Y}_{2.})^2/(\frac{1}{n_1} + \frac{1}{n_2})}{(S_1^2 + S_2^2)/(n_1 + n_2 - 2)} = V \sim F_{1,n_1+n_2-2}(\mu_1 - \mu_2)$$

Table 1: The ANOVA table for Moser 4.5.(b) and (c). δ_1 and δ_2 can be found in the main text.

	Formula	A_i	rank	$Y'A_iY$ Dist.
μ	$\sum_i \sum_j (\bar{Y}_{..})^2$	$A_1 = \bar{J}_n \otimes \bar{J}_2$	1	$\left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right) \chi_1^2(\delta_1)$
T	$\sum_i \sum_j (\bar{Y}_{i.} - \bar{Y}_{..})^2$	$A_2 = C_n \otimes \bar{J}_2$	$n-1$	$\left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right) \chi_{n-1}^2(0)$
$\tilde{B}$	$\sum_i \sum_j (\bar{Y}_{.j} - \bar{Y}_{..})^2$	$A_3 = \bar{J}_n \otimes C_2$	1	$\sigma_{BT}^2 \chi_1^2(\delta_3)$
$\tilde{BT}$	$\sum_i \sum_j (Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2$	$A_4 = C_n \otimes C_2$	$n-1$	$\sigma_{BT}^2 \chi_{n-1}^2(0)$
	$\sum_i \sum_j (Y_{ij})^2$	$A_5 = I_n \otimes I_2$	$2n$	N/A

b

With $\sigma_1^2 \neq \sigma_2^2$ and let $\sigma_1 = \sigma$, $\sigma_2 = \alpha\sigma$ where $\alpha \neq 1$, the following statistics has F-distribution.

$$V' = \frac{(\bar{Y}_{1.} - \bar{Y}_{2.})^2 / (\frac{1}{n_1} + \frac{\alpha^2}{n_2})}{(S_1^2 + S_2^2/\alpha^2)/(n_1 + n_2 - 2)} \sim F_{1, n_1+n_2-2}(\mu_1 - \mu_2)$$

We would like to investigate the distribution of original statistic V.

$$V = \frac{(\bar{Y}_{1.} - \bar{Y}_{2.})^2 / (\frac{1}{n_1} + \frac{1}{n_2})}{(S_1^2 + S_2^2)/(n_1 + n_2 - 2)} \approx\sim cF_{1,d}(\mu_1 - \mu_2)$$

where c is the approximate scaling factor of the F-distribution and d is the equivalent second degree of freedom to account for the $V' \to V$ transformation. Consider the following two cases.

$\alpha > 1$. $c > 1$ and $d < n_1 + n_2 - 2$. When $\sigma_2^2 > \sigma_1^2$, the statistic V is approximately a up-scaled F-distribution with equivalent second degrees of freedom fewer than $n_1 + n_2 - 2$.

$\alpha < 1$. $c < 1$ and $d > n_1 + n_2 - 2$. When $\sigma_2^2 < \sigma_1^2$, the statistic V is approximately a down-scaled F-distribution with equivalent second degrees of freedom more than $n_1 + n_2 - 2$.

Moser 4.5

The model for the problem is

$$Y_{ij} = \mu_j + B_i + (BT)_{ij} \quad \text{i.e.} \quad Y = (1_n \otimes \mu) + (I_n \otimes 1_2)\tilde{B} + (I_n \otimes I_2)\tilde{BT}$$

where $\tilde{B} \sim N_n(0, \sigma_B^2 I_n) \in \mathbb{R}^n$ and $\tilde{BT} \sim N_{2n}(0, \sigma_{BT}^2 I_{2n}) \in \mathbb{R}^{2n}$ are the random vectors corresponding to B and BT. The μ here is $(\mu_1, \mu_2)'$, representing the j^{th} level of fixed factors.

a

$$E[Y] = 1_n \otimes \mu$$

$$\Sigma = \text{Cov}(Y) = (I_n \otimes 1_2)(\sigma_B^2 I_n)(I_n \otimes 1_2') + (I_n \otimes I_2)(\sigma_{BT}^2 I_{2n})(I_n \otimes I_2) = \frac{1}{2}\sigma_B^2 (I_n \otimes \bar{J}_2) + \sigma_{BT}^2 (I_n \otimes I_2)$$

b

The ANOVA table is shown in Table 1.

c

In order to determine the distribution of $Y'A_iY$, we need to apply Bhat's lemma.

$$\Sigma = \frac{1}{2}\sigma_B^2(I_n \otimes \bar{J}_2) + \sigma_{BT}^2(I_n \otimes I_2) = \left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right)(A_1 + A_2) + \sigma_{BT}^2(A_3 + A_4)$$

All the A_i-matrices satisfy the assumption of Bhat's lemma $\forall i = 1, 2, 3, 4$.

$$Y'A_iY \sim c_i\chi^2_{\text{rank}(A_i)}(\delta_i) \quad \text{where} \quad \delta_i = \frac{1}{c_i}\mu' A_i \mu$$

$$Y'A_1Y \sim \left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right)\chi_1^2(\delta_1) \quad \delta_1 = \frac{n(\mu_1 + \mu_2)^2}{\sigma_B^2 + 2\sigma_{BT}^2}$$

$$Y'A_2Y \sim \left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right)\chi_{n-1}^2(\delta_2) \quad \delta_2 = 0$$

$$Y'A_3Y \sim \sigma_{BT}^2\chi_1^2(\delta_3) \quad \delta_3 = \frac{n(\mu_1 - \mu_2)^2}{2\sigma_{BT}^2}$$

$$Y'A_4Y \sim \sigma_{BT}^2\chi_{n-1}^2(\delta_4) \quad \delta_4 = 0$$

d

If a random variable X has χ^2 distribution with d degrees of freedom and noncentrality δ, $\mathrm{E}(cX) = cd + \delta$ for some constant c.

$$\mathrm{E}(Y'A_1Y) = \left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right) + \frac{n(\mu_1 + \mu_2)^2}{\sigma_B^2 + 2\sigma_{BT}^2}$$

$$\mathrm{E}(Y'A_2Y) = \left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right)(n-1)$$

$$\mathrm{E}(Y'A_3Y) = \sigma_{BT}^2 + \frac{n(\mu_1 - \mu_2)^2}{2\sigma_{BT}^2}$$

$$\mathrm{E}(Y'A_4Y) = \sigma_{BT}^2(n-1)$$

$$\mathrm{E}\left(\sum_i\sum_j Y_{ij}^2\right) = \sum_{k=1}^4 \mathrm{E}(Y'A_iY) = n\left(\frac{\sigma_B^2}{2} + \sigma_{BT}^2\right) + n\left[\frac{(\mu_1 + \mu_2)^2}{\sigma_B^2 + 2\sigma_{BT}^2} + \frac{(\mu_1 - \mu_2)^2}{2\sigma_{BT}^2}\right]$$

e

Since we have no information about σ_B and σ_{BT} for the hypothesis $H_0 : \mu_1 = \mu_2$, the distribution of the statistic must not depend on σ_B and σ_{BT} and should reflect H_0 and H_1 Therefore, define

$$U = \frac{Y'A_3Y}{Y'A_4Y/(n-1)} \sim F_{1,n-1}\left(\frac{n(\mu_1 - \mu_2)^2}{2\sigma_{BT}^2}\right)$$

f

$$U = \frac{Y'A_3Y}{Y'A_4Y/(n-1)} = \frac{n\bar{D}^2}{\frac{1}{n-1}\sum_i \sum_j (Y_{ij} - Y_{i.} - Y_{.j} + Y_{..})^2} = \frac{n\bar{D}^2}{\frac{1}{n-1}\sum_i (D_i - \bar{D}_.)^2} = \frac{\bar{D}^2}{S_D^2/n} = T^2$$