STSCI 7170 Homework 4

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Problem 1

Let X and Y be two random variables and a and b be two scalars, we have

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y)$$

Therefore, given two unbiased estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ and $\hat{\theta} = w\hat{\theta}_1 + (1-w)\hat{\theta}_2$, the variance of the new estimate $\hat{\theta}$ is

$$\mathrm{Var}(\hat{\theta}) = w^2\mathrm{Var}(\hat{\theta}_1) + (1-w)^2\mathrm{Var}(\hat{\theta}_2) + 2w(1-w)\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2) = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

We want to minimize $\mathrm{Var}(\hat{\theta})$.

$$\frac{d\mathrm{Var}(\hat{\theta})}{dw} = 2w\sigma_1^2 - 2(1-w)\sigma_2^2 + 2(1-2w)\sigma_{12} = 0$$

$$w = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

If $\hat{\theta}_1$ and $\hat{\theta}_2$ are uncorrelated, i.e. $\sigma_{12} = 0$,

$$w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Problem 2

$y \sim N_n(\mu, \Sigma)$ where $\Sigma \in \mathbb{R}^{n\times n}$ and $\mathrm{rank}(\Sigma) = m < n$. We can rewrite $\Sigma = Q\Lambda Q'$ where

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix} \quad \Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^{m\times m} \quad \lambda_i > 0$$

The Moore-Penrose inverse of Σ is $\Sigma^- = Q\Lambda^- Q'$ where

$$\Lambda^- = \begin{bmatrix} \Lambda_1^{-1} & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix} \quad \Lambda_1^{-1} = \mathrm{diag}(1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_m)$$

Define

$$\sqrt{\Sigma^-} = Q \begin{bmatrix} \Lambda_1^{-1/2} & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix} Q'$$

and $z = \sqrt{\Sigma^-}y \sim N_n(\sqrt{\Sigma^-}\mu, \sqrt{\Sigma^-}\Sigma\sqrt{\Sigma^-}) \sim N_n(\sqrt{\Sigma^-}\mu, K)$

$$K = \sqrt{\Sigma^-}\Sigma\sqrt{\Sigma^-} = Q \begin{bmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix} Q' = \begin{bmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{bmatrix}$$

K is idenpotent with rank m. Therefore, $z'Kz = z'z = y'\sqrt{\Sigma^-}\sqrt{\Sigma^-}y = y'\Sigma^- y \sim \chi_m^2(\delta)$ where

$$\delta = (\sqrt{\Sigma^-}\mu)'K\sqrt{\Sigma^-}\mu = \mu'\Sigma^-\mu$$

Problem 3

The model is

$$Y_{ij} = \mu_0 + B_i + \tau_j + \gamma z_{ij} + E_{ij}$$

where μ_0, τ_j, z_{ij} are fixed while $B_i \sim N(0, \sigma_b^2)$ and $E_{ij} \sim N(0, \sigma_e^2)$ are random. Re-write the model in the vector form.

$$Y = \mu_0(\mathbf{1}_b \otimes \mathbf{1}_t) + (I_b \otimes \mathbf{1}_t)\tilde{B} + (\mathbf{1}_b \otimes I_t)\tilde{\tau} + (I_b \otimes I_t)\tilde{E}$$

Let $Y_i = (Y_{i1}, Y_{i2}, \dots, Y_{it})'$, $z_i = (z_{i1}, z_{i2}, \dots, z_{it})'$ and $t \times t$ Helmert matrix P

$$P = \begin{bmatrix} \frac{1}{\sqrt{t}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{t(t-1)}} \\ \frac{1}{\sqrt{t}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{t(t-1)}} \\ \frac{1}{\sqrt{t}} & 0 & -\frac{2}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{t(t-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{t}} & 0 & 0 & \cdots & -\frac{t-1}{\sqrt{t(t-1)}} \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \dots & p_t \end{bmatrix}$$

the transformed responses are defined as $Y_{ij}^* = p_j' Y_i$. Note that $p_i' p_j = \delta_{ij}$

3a

We can write Y_i as the following.

$$Y_i = \mu_0 \mathbf{1}_t + \gamma z_i + B_i \mathbf{1}_t + E_i$$

where $B_i \sim N(0, \sigma_b^2)$ and $E_i \sim N_t(0, \sigma_e^2 I_t)$. Therefore, $Y_i \sim N_t\left[\mu_0 \mathbf{1}_t + \gamma z_i, \sigma_b^2 \mathbf{1}_t \mathbf{1}_t' + \sigma_e^2 I_t\right] \sim N_t\left[\mu_0 \mathbf{1}_t + \gamma z_i, t\sigma_b^2 \bar{J}_t + \sigma_e^2 I_t\right]$. Let $\Sigma_i = t\sigma_b^2 \bar{J}_t + \sigma_e^2 I_t$ and Σ_i's are identical for $i = 1, 2, \dots, b$.

$$j = 1: \quad \mathrm{Var}(Y_{ij}^*) = p_1' \Sigma_i p_1 = \sigma_b^2 \mathbf{1}_t' \bar{J}_t \mathbf{1}_t + \sigma_e^2 = t\sigma_b^2 + \sigma_e^2$$
$$j = 2, 3, \dots, t: \quad \mathrm{Var}(Y_{ij}^*) = p_j' \Sigma_i p_j = \sigma_e^2$$

3b

The estimates for γ are

$$\hat{\gamma}_j = \frac{\sum_{i=1}^b (z_{ij}^* - \bar{z_{.j}^*}) Y_{ij}^*}{\sum_{i=1}^b (z_{ij}^* - \bar{z_{.j}^*})^2} = \frac{\sum_{i=1}^b (p_j' z_i - m) p_j' Y_i}{\sum_{i=1}^b (p_j' z_i - m)^2} \quad m = \frac{1}{b} \sum_{i=1}^b p_j' z_i$$

where agian, $Y_i \sim N_t\left[\mu_0 \mathbf{1}_t + \gamma z_i, t\sigma_b^2 \bar{J}_t + \sigma_e^2 I_t\right]$. The only randomness lies in Y_i because p_j's are constants and z_{ij}'s are pre-determined. Note that the denominator is a scalar which can be simplified as follows.

$$\sum_{i=1}^b (p_j' z_i - m)^2 = \sum_{i=1}^b (p_j' z_i)^2 - bm^2$$

First, we want to show that $\hat{\gamma}_j$ is unbiased.

$$\mathrm{E}[\hat{\gamma}_j] = \frac{\sum_{i=1}^b (p_j' z_i - m) p_j' \mathrm{E}(Y_i)}{\sum_{i=1}^b (p_j' z_i - m)^2} = \frac{\sum_{i=1}^b (p_j' z_i - m)(\mu_0 p_j' \mathbf{1}_t + \gamma p_j' z_i)}{\sum_{i=1}^b (p_j' z_i)^2 - bm^2}$$

We consider two cases where $j = 1$ and $j = 2, 3, \dots, t$.

For $j = 1$

$$\mathrm{E}[\hat{\gamma_1}] = \frac{\sum_{i=1}^{b} \left[\sqrt{t}\mu_0 p_1' z_i - \sqrt{t}\mu_0 m + \gamma(p_1' z_i)^2 - \gamma m p_1' z_i\right]}{\sum_{i=1}^{b}(p_1' z_i)^2 - bm^2} = \gamma \frac{\sum_{i=1}^{b}(p_1' z_i)^2 - bm^2}{\sum_{i=1}^{b}(p_1' z_i)^2 - bm^2} = \gamma$$

For $j = 2, 3, \dots, t$

$$\mathrm{E}[\hat{\gamma_j}] = \frac{\sum_{i=1}^{b} \left[\gamma(p_j' z_i)^2 - \gamma m p_j' z_i\right]}{\sum_{i=1}^{b}(p_j' z_i)^2 - bm^2} = \gamma \frac{\sum_{i=1}^{b}(p_j' z_i)^2 - bm^2}{\sum_{i=1}^{b}(p_j' z_i)^2 - bm^2} = \gamma$$

Therefore, $\hat{\gamma_j}$ is unbiased. We are now showing that $\hat{\gamma_j}$'s are independent. Notice that the denominator is just a scalar and the nominator is the sum of all the transformed Y_i's, which happen to have identical covariance $\Sigma_i = t\sigma_b^2 \bar{J}_t + \sigma_e^2 I_t = \Sigma$ as in 3.a. Let the transformation matrices to be T_j and $\hat{\gamma_j}$ and $\hat{\gamma_k}$ are independent if and only if $T_j \Sigma T_k' = 0$

$$T_j = \frac{\sum_{i=1}^{b}(p_j' z_i - m)p_j'}{\sum_{i=1}^{b}(p_j' z_i - m)^2}$$

$T_j \Sigma T_k' = 0 \iff \left[\sum_{i=1}^{b}(p_j' z_i - m)p_j'\right] \Sigma \left[\sum_{i=1}^{b}(p_k' z_i - m)p_k'\right]' = 0$. Because the p_j's are the columns of Helmert matrix, we have

$$p_j' p_k = 0$$
$$p_j' \bar{J} p_k = 0$$

if $j \neq k$. Therefore,

$$\left[\sum_{i=1}^{b}(p_j' z_i - m)p_j'\right] \Sigma \left[\sum_{l=1}^{b}(p_k' z_l - m)p_k'\right]' = \left[\sum_{i=1}^{b}(p_j' z_i - m)p_j'\right] (t\sigma_b^2 \bar{J}_t + \sigma_e^2 I_t) \left[\sum_{l=1}^{b} p_k(p_k' z_l - m)\right] = 0$$

for $j \neq k$ because of the properties of the Helmert matrix. Therefore, $\hat{\gamma_j}$ and $\hat{\gamma_k}$ are independent.

3c

Again, we discuss two cases with $j = 1$ and $j = 2, 3, \dots, t$.

For $j = 1$

$$\mathrm{Var}(\hat{\gamma_1}) = \frac{\sum_{i=1}^{b}(p_1' z_i - m)p_1' \Sigma p_1 (p_1' z_i - m)}{\sum_{i=1}^{b}(p_1' z_i - m)^2} = \frac{\sum_{i=1}^{b}(p_1' z_i - m)^2 (t\sigma_b^2 + \sigma_e^2)}{\sum_{i=1}^{b}(p_1' z_i - m)^2} = t\sigma_b^2 + \sigma_e^2$$

For $j = 2, 3, \dots, t$,

$$\mathrm{Var}(\hat{\gamma_j}) = \frac{\sum_{i=1}^{b}(p_j' z_i - m)p_j' \Sigma p_j (p_j' z_i - m)}{\sum_{i=1}^{b}(p_j' z_i - m)^2} = \sigma_e^2$$

The BLUE of γ can be written as a convex linear combination of all the $\hat{\gamma_j}$'s.

$$\hat{\gamma}_{BLUE} = \sum_{j=1}^{t} a_j \hat{\gamma}_j \quad \sum_{j=1}^{t} a_j = 1 \quad \because \mathrm{E}(\hat{\gamma}_{BLUE}) = \gamma$$

And we should pick a_j's such that $\hat{\gamma}_{BLUE}$ have the smallest variance.

$$\mathrm{Var}(\hat{\gamma}_{BLUE}) = \min_{a_j} \left[\sum_{j=1}^{t} a_j^2 \mathrm{Var}(\hat{\gamma}_j) \right] = \min_{a_j} \left[a_1^2 (t\sigma_b^2 + \sigma_e^2) + \sigma_e^2 \sum_{j=2}^{t} a_j^2 \right]$$

We can re-formulate this using $a = a_1$ and $b = a_j \ \forall j = 2, 3, \ldots, t$. Therefore our goal is now

$$\mathrm{Var}(\hat{\gamma}_{BLUE}) = \min_{a,b} \left[a^2 (t\sigma_b^2 + \sigma_e^2) + (t-1) b^2 \sigma_e^2 \right]$$

under the constraint of $a + (t-1)b = 1$. Substitute $b = \frac{1-a}{t-1}$ in the above and taking the derivative and set to 0, we have

$$f(a) = a^2 (t\sigma_b^2 + \sigma_e^2) + \frac{1}{t-1}(1-a)^2 \sigma_e^2$$

$$\frac{df}{da} = 0 = a(t\sigma_b^2 + \sigma_e^2) - \frac{1-a}{t-1}\sigma_e^2 \iff a - (1-a)\frac{1-\rho}{t-1} = 0$$

where $\rho = t\sigma_b^2/(t\sigma_b^2 + \sigma_e^2)$. Solving the above equation, we end up with

$$a = \frac{1-\rho}{t-\rho} \quad b = \frac{1}{t-\rho}$$

$$\hat{\gamma}_{BLUE} = \left(\frac{1-\rho}{t-\rho} \right) \hat{\gamma}_1 + \sum_{j=2}^{t} \left(\frac{1}{t-\rho} \right) \hat{\gamma}_j$$

3d

For $\rho \to 1$,

$$\hat{\gamma}_{BLUE} \to \frac{1}{t-1} \sum_{j=2}^{t} \hat{\gamma}_j = \frac{1}{t-1} \sum_{j=2}^{t} \frac{\sum_{i=1}^{b} (z_{ij}^* - \bar{z}_{.j}^*) Y_{ij}^*}{\sum_{i=1}^{b} (z_{ij}^* - \bar{z}_{.j}^*)^2} = \frac{\sum_{i=1}^{b} \sum_{j=2}^{t} (z_{ij}^* - \bar{z}_{.j}^*) Y_{ij}^*}{\sum_{i=1}^{b} \sum_{j=2}^{t} (z_{ij}^* - \bar{z}_{.j}^*)^2}$$

Rewrite the denominator and nominator as a vector form by first identifying that

$$z^* = \begin{bmatrix} I_b \otimes p_1' \\ I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} z \quad \text{and} \quad Y^* = \begin{bmatrix} I_b \otimes p_1' \\ I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} Y$$

We define the following for later convenience:

$$\tilde{z}^* = \begin{bmatrix} I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} z \quad \tilde{Y}^* = \begin{bmatrix} I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} Y$$

The demominator is

$$\sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij}^* - \bar{z}_{.j}^*)^2 = \tilde{z}^{*\prime}(C_b \otimes I_t)\tilde{z}^* = z' \begin{bmatrix} I_b \otimes p_2 & I_b \otimes p_3 & \dots & I_b \otimes p_t \end{bmatrix} (C_b \otimes I_t) \begin{bmatrix} I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} z$$

$$= z' \begin{bmatrix} C_b \otimes p_2 & C_b \otimes p_3 & \dots & C_b \otimes p_t \end{bmatrix} \begin{bmatrix} I_b \otimes p_2' \\ I_b \otimes p_3' \\ \vdots \\ I_b \otimes p_t' \end{bmatrix} z = z' \left(C_b \otimes \sum_{j=2}^{t} p_j p_j' \right) z = z'(C_b \otimes C_t) z = \sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij} - \bar{z}_{i.} - \bar{z}_{.j} + \bar{z}_{..})^2$$

Similarly, the nominator

$$\sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij}^* - \bar{z}_{.j}^*)Y_{ij}^* = z^{*\prime}(C_b \otimes I_t)Y^* = z'(C_b \otimes C_t)Y = \sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij} - \bar{z}_{i.} - \bar{z}_{.j} + \bar{z}_{..})Y_{ij}$$

Therefore,

$$\hat{\gamma}_{BLUE} \to \frac{\sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij} - \bar{z}_{i.} - \bar{z}_{.j} + \bar{z}_{..})Y_{ij}}{\sum_{i=1}^{b}\sum_{j=2}^{t}(z_{ij} - \bar{z}_{i.} - \bar{z}_{.j} + \bar{z}_{..})^2}$$

Moser 6.4

The $100(1-\gamma)\%$ confidence interval of $h'\beta$ is

$$h'\hat{\beta} \pm t_{n-p}^{\gamma/2} \sqrt{\frac{1}{n-p} \left[h'(X'X)^{-1}h\right] Y' \left[I - X(X'X)^{-1}X'\right] Y}$$

a

Let L be the length of the confidence interval, we have

$$L^2 = 4\left(t_{n-p}^{\gamma/2}\right)^2 \left[\frac{h'(X'X)^{-1}h}{n-p}\right] Y' \left[I - X(X'X)^{-1}X'\right] Y$$

The only random vector is $Y \sim N_n(X\beta, \sigma^2 I_n)$. Let $A_1 = X(X'X)^{-1}X'$ and $A_2 = I - X(X'X)^{-1}X'$, both A_1 and A_2 are idenpotent with rank p and $n-p$ respectively. Moreover, $\sigma^2 I_n = \sigma^2(A_1 + A_2)$. By Bhat's lemma, we have

$$Y'A_2Y \sim \sigma^2 \chi_{n-p}^2(\delta_2) \quad \delta_2 = (X\beta)'A_2X\beta = 0$$

Therefore,

$$L^2 \sim C\chi_{n-p}^2(0) \quad \text{where} \quad C = 4\sigma^2 \left(t_{n-p}^{\gamma/2}\right)^2 \left[\frac{h'(X'X)^{-1}h}{n-p}\right]$$

b

First, we find $\hat{\beta} = (X'X)^{-1}X'Y = X'Y = (5, 10, 15)' = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. We know that $100(1-\gamma)\%$ condifence interval of $h'\beta$ is

$$h'\hat{\beta} \pm t_{n-p}^{\gamma/2}\sqrt{\frac{1}{n-p}\left[h'(X'X)^{-1}h\right]Y'\left[I - X(X'X)^{-1}X'\right]Y}$$

$$= h'\hat{\beta} \pm t_{17}^{\gamma/2}\sqrt{\frac{1}{17}h'h\left[Y'Y - (X'Y)'(X'Y)\right]} = h'\hat{\beta} \pm t_{17}^{\gamma/2}\sqrt{\frac{68}{17}h'h} = h'\hat{\beta} \pm 2\left(t_{17}^{\gamma/2}\right)\sqrt{h'h}$$

Now using 95% condifence interval, $t_{17}^{\gamma/2}$ is the 97.5 quantile of t-distribution with 17 degrees of freedom. $t_{17}^{2.5} = 2.1098$. The 95% condifence interval of $h'\beta$ is now

$$h'\hat{\beta} \pm 4.2196\sqrt{h'h}$$

The only thing we need to determine is h.

$$h' = (1,0,0) \quad h'\beta = \beta_1 = \hat{\beta}_1 \pm 4.2196 \Longrightarrow \mathrm{CI}_{95}(\beta_1) = [0.7804, 9.2196]$$
$$h' = (1,1,0) \quad h'\beta = \beta_1 + \beta_2 = \hat{\beta}_1 + \hat{\beta}_2 \pm 4.2196\sqrt{2} \Longrightarrow \mathrm{CI}_{95}(\beta_1 + \beta_2) = [9.0326, 20.9674]$$
$$h' = (1,0,-1) \quad h'\beta = \beta_1 - \beta_3 = \hat{\beta}_1 - \hat{\beta}_3 \pm 4.2196\sqrt{2} \Longrightarrow \mathrm{CI}_{95}(\beta_1 - \beta_3) = [-15.9674, -4.0326]$$

Moser 6.7

$$Y_{ij} = \mu_j + B_i + (BT)_{ij} \quad i = 1, 2, \ldots, n \quad j = 1, 2$$
$$Y = (\mathbf{1}_n \otimes I_2)\tilde{\mu} + (I_n \otimes \mathbf{1}_2)\tilde{B} + (I_n \otimes I_2)\tilde{BT}$$

where $\tilde{B} \sim N_n(0, \sigma_B^2 I_n)$ and $\tilde{BT} \sim N_{2n}(0, \sigma_{BT}^2 I_{2n})$. We can write $Y = X\beta + E$ where $X = \mathbf{1}_n \otimes I_2$, $\beta = \tilde{\mu} = (\mu_1, \mu_2)'$ and $E \sim N_{2n}(0, \Sigma)$.

$$\Sigma = 2\sigma_B^2(I_n \otimes \bar{J}_2) + \sigma_{BT}^2(I_n \otimes I_2)$$

Define the A-matrices for fixed and random factors:

$$A_1^f = \bar{J}_n \otimes \bar{J}_2 \quad \text{rank} = 1$$
$$A_1^r = C_n \otimes \bar{J}_2 \quad \text{rank} = n - 1$$
$$A_2^f = \bar{J}_n \otimes C_2 \quad \text{rank} = 1$$
$$A_2^r = C_n \otimes C_2 \quad \text{rank} = n - 1$$

$$\Sigma = 2\sigma_B^2(I_n \otimes \bar{J}_2) + \sigma_{BT}^2(I_n \otimes I_2) = (2\sigma_B^2 + \sigma_{BT}^2)(A_1^f + A_1^r) + \sigma_{BT}^2(A_2^f + A_2^r)$$

a

The maximum likelihood estimates are the following.

$$\hat{\beta} = (X'X)^{-1}X'Y = \frac{1}{n}(\mathbf{1}_n' \otimes I_2)Y = (\hat{\mu_1}, \hat{\mu_2})'$$

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^{n} Y_{i1} \quad \hat{\mu_2} = \frac{1}{n}\sum_{i=1}^{n} Y_{i2}$$

$$\hat{a_2} = \hat{\sigma}_{BT}^2 = \frac{Y'A_2^rY}{n} = \frac{1}{n}\sum_i\sum_j \left(Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..}\right)^2$$

$$\hat{a_1} = 2\hat{\sigma}_B^2 + \hat{\sigma}_{BT}^2 = \frac{Y'A_1^rY}{n} = \frac{1}{n}\sum_i \left(\bar{Y}_{i.} - \bar{Y}_{..}\right)^2$$

$$\hat{\sigma}_B^2 = \frac{\hat{a_1} - \hat{a_2}}{2} = \frac{1}{2n}\left[\sum_i \left(\bar{Y}_{i.} - \bar{Y}_{..}\right)^2 - \sum_i\sum_j \left(Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..}\right)^2\right]$$

b

Under $H_0: \mu_1 = \mu_2$, the constraint is $H\beta = h = 0$ where $H = (1, -1)$. Let $M = \frac{1}{n}(\mathbf{1}_n' \otimes I_2)$. $Y \sim N_{2n}(X\beta, \Sigma)$. Therefore

$$\hat{\beta} \sim N_2(\beta, M\Sigma M') \Longrightarrow H\hat{\beta} = \hat{\mu_1} - \hat{\mu_2} \sim N(H\beta, HM\Sigma M'H') \sim N(\mu_1 - \mu_2, HM\Sigma M'H')$$

$$HM\Sigma M'H' = \frac{1}{n^2}\left[(\mathbf{1}_n' \otimes (1,-1))2\sigma_B^2(I_n \otimes \bar{J}_2)(\mathbf{1}_n \otimes (1,-1)') + (\mathbf{1}_n' \otimes (1,-1))\sigma_{BT}^2(I_n \otimes I_2)(\mathbf{1}_n \otimes (1,-1)')\right]$$

$$HM\Sigma M'H' = \frac{1}{n^2}\left(2\sigma_B^2 \times n \times 0 + 2n\sigma_{BT}^2\right) = \frac{2\sigma_{BT}^2}{n}$$

Therefore, $H\hat{\beta} \sim N(\mu_1 - \mu_2, 2\sigma_{BT}^2/n)$. We can construct the following statistics:

$$V = \frac{(H\hat{\beta})'[HM\Sigma M'H']^{-1}(H\hat{\beta})/1}{Y'(I - X(X'X)^{-1}X)Y/(2n-2)} = \frac{Y'(HM)'[HM\Sigma M'H']^{-1}HMY}{Y'(I - X(X'X)^{-1}X)Y/(2n-2)} \sim F_{1,2n-2}(\delta)$$

We have

$$(HM)'HM = \frac{1}{n^2}\left(\mathbf{1}_n\mathbf{1}_n' \otimes 2C_2\right) = \frac{2}{n}(\bar{J}_n \otimes C_2)$$

$$I - X(X'X)^{-1}X' = I - XM = I - (\bar{J}_n \otimes I_2) = C_n \otimes I_2$$

Combining the avove 4 quantities, we have

$$V = \frac{Y'(HM)'[HM\Sigma M'H']^{-1}HMY}{Y'(I - X(X'X)^{-1}X)Y/(2n-2)} = \frac{Y'(\bar{J}_n \otimes C_2)Y/\sigma_{BT}^2}{Y'(C_n \otimes I_2)Y/(2n-2)} \sim F_{1,2n-2}(\delta)$$

Therefore,

$$\frac{Y'(\bar{J}_n \otimes C_2)Y}{Y'(C_n \otimes I_2)Y/(2n-2)} = \sigma_{BT}^2 V \sim \sigma_{BT}^2 F_{1,2n-2}(\delta)$$

Under $H_0: \mu_1 = \mu_2$, $\delta = 0$.