PS4

Wednesday, 4 November 2020 19:48

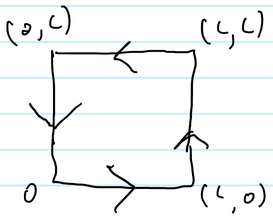
Q4, 5

4) $\underline{E} = (-x^2y,\ xy^2,\ 0)$

C square in (x, y) plane with vertices at $(0,0), (L,0), (L,L), (0,L)$

Want to find $\int_C \underline{E}\cdot d\underline{r}$. Split C into line segments

C_1 $(t, 0)$ for $t \in (0, L)$
C_2 (L, t) for $t \in (0, L)$
C_3 $(L-t, L)$ for $t \in (0, L)$
C_4 $(0, L-t)$ for $t \in (0, L)$



$$\int_C \underline{E}\cdot d\underline{r} = \int_{C_1} \underline{E}\cdot d\underline{r} + \int_{C_2} \underline{E}\cdot d\underline{r} + \int_{C_3} \underline{E}\cdot d\underline{r} + \int_{C_4} \underline{E}\cdot d\underline{r}$$

$$= 0 + \int_0^L (-L^2 t,\ L t^2)\cdot(0,1)\,dt + \int_0^L (-(L-t)^2 L,\ (L-t)L^2)\cdot(-1,0)\,dt + 0$$

$$= \int_0^L L t^2\,dt + \int_0^L (L-t)^2 L\,dt$$

$$= L\int_0^L t^2 + L^2 - 2Lt + t^2\,dt$$

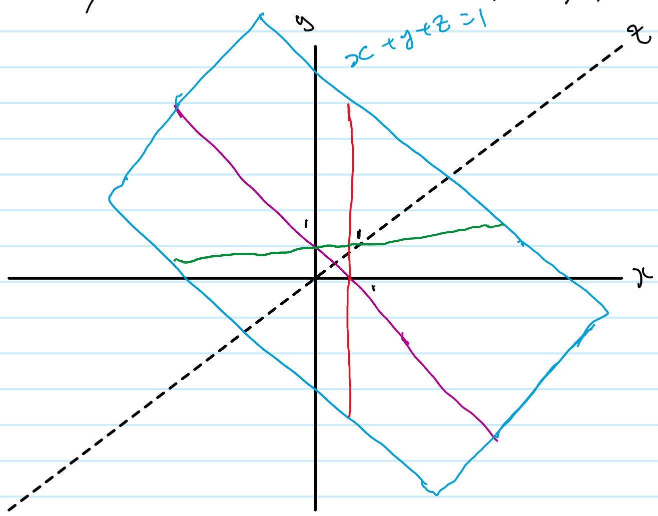
$$= L\int_0^L 2t^2 - 2Lt + L^2\,dt$$

$$= L\left(\tfrac{2}{3}t^3 - Lt^2 + L^2 t\right)\Big|_0^L$$

$$= L\left(\tfrac{2}{3}L^3 - L^3 + L^3 - 0 + 0 - 0\right)$$

$$= \tfrac{2}{3}L^4$$

5a) C_1: $x + y + z = 1$, $y = 0$ $(t, 0, 1-t)$ $t \in [0,1]$
C_2: $x + y + z = 1$, $z = 0$ $(1-t, t, 0)$ $t \in [0,1]$
C_3: $x + y + z = 1$, $x = 0$ $(0, 1-t, t)$ $t \in [0,1]$



b) $E(\underline{r}) = (x^2 z,\ x y^2,\ z^2)$

$$\int_C \underline{F}\cdot \partial \underline{r} = \int_{C_1} \underline{F}\cdot\partial\underline{r} + \int_{C_2}\underline{F}\cdot\partial\underline{r} + \int_{C_3}\underline{F}\cdot\partial\underline{r}$$

$$= \int_0^1 (t^2(1-t), 0, (1-t)^2)\cdot(1,0,-1) + (0,(1-t)t^2,0)\cdot(-1,1,0)$$

$$+ (0,0,t^2)\cdot(0,-1,1)\ \partial t$$

$$= \int_0^1 t^2(1-t) - (1-t)^2 + (1-t)t^2 + t^2 \quad \partial t$$

$$= \int_0^1 \cancel{t^2} \cancel{-t^3} - 1 + 2t \cancel{-t^2} + \cancel{t^2} \cancel{-t^3} + \cancel{t^2}\ \partial t$$

$$= \int_0^1 -2t^3 + 2t^2 + 2t - 1\ \partial t$$

$$= \left(-\tfrac{1}{2}t^4 + \tfrac{2}{3}t^3 + t^2 - t\right)\Big|_0^1$$

$$= -\tfrac{1}{2} + \tfrac{2}{3} + 1 - 1 \quad - 0$$

$$= \tfrac{1}{6}$$

c) $\int \underline{\nabla} \times \underline{F}\ \partial \underline{S} = \int_C \underline{F}\ \partial r = \tfrac{1}{6}$ by Stokes' theorem.

d) If $\underline{F} = (yz, xz, xy)$ then the parameterisation of $\underline{F}$ becomes $t(1-t)\underline{1}$ in all cases and thus the integral becomes

$$3\int_0^1 t^2(1-t) + t(1-t)^2\ \partial t$$

which will be the solution to both parts due to Stokes' theorem.