ODEs HW5 (Assessed)

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1i) We start with the ODE

$$(\dot{x},\dot{y}) = (-\frac{1}{2}y(1+x) + x(1-4x^2-y^2), 2x(1+x) + y(1-4x^2-y^2)) \tag{1}$$

on $\mathbb{R}^2$. Take $(x,y) = (0,0)$. Then $(\dot{x},\dot{y}) = (0,0)$ so $(0,0)$ is an equilibrium. Now suppose $(x,y) \neq (0,0)$. We know

$$(\dot{x},\dot{y}) = (0,0) \Rightarrow y\dot{x} - x\dot{y} = 0$$

and

$$y\dot{x} - x\dot{x} = -\frac{1}{2}y^2(1+x) - 2x^2(1+x) = -(1+x)[2x^2 + \frac{1}{2}y^2].$$

Since $(x,y) \neq (0,0)$, $x = -1$ is our only possible equilibrium but this leads to the ODE

$$(\dot{x},\dot{y}) = (y^2 - 3, y(1-4-y^2))$$

which has no equilibria since

$$\dot{x} = 0 \Rightarrow y = \pm\sqrt{3} \Rightarrow \dot{y} = \mp 6\sqrt{3} \neq 0.$$

So $(0,0)$ is the unique equilibrium for ODE (1).

ii) Define $V(x,y) = (1-4x^2-y^2)^2$ for $(x,y) \in \mathbb{R}^2$. We have by the chain rule

$$\dot{V}(x,y) = \partial_x V\dot{x} + \partial_y V\dot{y} = -4(1-4x^2-y^2)[4x\dot{x} + y\dot{y}].$$

Substituting ODE (1) we get

$$4x\dot{x} + y\dot{y} = (1-4x^2-y^2)[y^2+4x^2]$$

so

$$\dot{V}(x,y) = -4(1-4x^2-y^2)^2[y^2+4x^2].$$

iii) Define $\Gamma = \{(x,y) \in \mathbb{R}^2 : 4x^2 + y^2 = 1\}$. We know that Γ will be positive invariant if it is tangent to the vector field. Let $(x,y) \in \Gamma$. So $4x^2 + y^2 = 1$. By implicit differentiation we have $8xdx + 2ydy = 0$ so

$$\frac{dy}{dx} = -4\frac{x}{y}.$$

If we are on Γ then the vector field becomes

$$(\dot{x},\dot{y}) = (-\frac{1}{2}y(1+x), 2x(1+x))$$

so

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2x(1+x)}{-\frac{1}{2}y(1+x)} = -\frac{4x}{y}$$

assuming $x \neq -1$. However $x = -1$ implies $y^2 = -3$ so is not in Γ. So for all $(x, y) \in \Gamma$, Γ is tangent to the vector field so Γ is positive invariant.

iv) Define $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : V(x, y) \leq C\}$ for some $C > 0$. We want to show that there exists an $r \in \mathbb{R}_{>0}$ such that $\mathcal{M} \subseteq B(0, r)$. Let $(x, y) \in \mathcal{M}$. Then we have

$$V(x, y) = (1 - 4x^2 - y^2)^2 \leq C \Rightarrow -\sqrt{C} \leq 1 - 4x^2 - y^2 \leq \sqrt{C}.$$

So

$$1 + \sqrt{C} - 3x^2 \geq x^2 + y^2 \geq 1 - \sqrt{C} - 3x^2 \Rightarrow x^2 + y^2 \leq 1 + \sqrt{C} - 3x^2 \leq 1 + \sqrt{C}.$$

Hence $\mathcal{M} \subseteq B(0, \sqrt{1 + \sqrt{C}})$ so $\mathcal{M}$ is bounded.

v) We have that ODE (1) is an autonomous vector field and $(\dot{x}, \dot{y})$ is continuously differentiable. Denote it's flow by $\phi_t(\cdot)$. Define $U = \{(x, y) \in \mathbb{R}^2 : V(x, y) < 1\}$. Now consider $\mathcal{M}$ with $C = 1 > 0$. We have already shown that $\mathcal{M}$ is bounded and we also have that it is closed since it's complement $\{(x, y) \in \mathbb{R}^2 : V(x, y) > 1\}$ is open. So $\mathcal{M}$ is compact. Furthermore as $\dot{V} \leq 0$ for all $(x, y) \in \mathbb{R}^2$, $\mathcal{M}$ is positive invariant. Define $E = \{(x, y) \in \mathcal{M} : \dot{V}(x, y) = 0\}$ and notice that $E = \Gamma \cup \{(0, 0)\}$ which is an invariant set. So by the LaSalle Invariance Principle, $\phi_t(x) \to \Gamma \cup \{(0, 0)\}$ as $t \to \infty$ for all $(x, y) \in \mathcal{M}$. Now consider $U \subset \mathcal{M}$. Since $(0, 0) \notin U$ as $V(0, 0) = 1$ and $\dot{V}(x, y) \leq 0$ for all $(x, y) \in \mathbb{R}^2$ which means U is positive invariant, we must have $\phi_t(x) \to \Gamma$ as $t \to \infty$ for all $(x, y) \in U$.

2i) Let $A \in \mathbb{R}^{2\times 2}$ with ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

on $\mathbb{R}^2$. Suppose $\operatorname{span}\{\mathbf{v}\}$ is an invariant set for ODE (2). Then $A\mathbf{v} \in \operatorname{span}\{\mathbf{v}\}$ so there exists a $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda \mathbf{v}$. Hence $\mathbf{v}$ is an eigenvector by the definition.

ii) We want to sketch the function $(x, y) = e^{\lambda t}(a + bt, b)$ for $a, b \in \mathbb{R}$, $b \neq 0$, and $\lambda > 0$. First at $t = 0$ we have $(x, y) = (a, b)$. Now as $t \to -\infty$, $(x, y) \to (0, 0)$ and as $t \to \infty$,

$$(x, y) \to \begin{cases} (\infty, \infty) & \text{if } b > 0 \\ (-\infty, -\infty) & \text{if } b < 0 \end{cases}$$

since $\lambda > 0$. Finally we have that the curve passes through the y-axis at $t = -\frac{a}{b}$. Hence we get the curves in Figure 1.

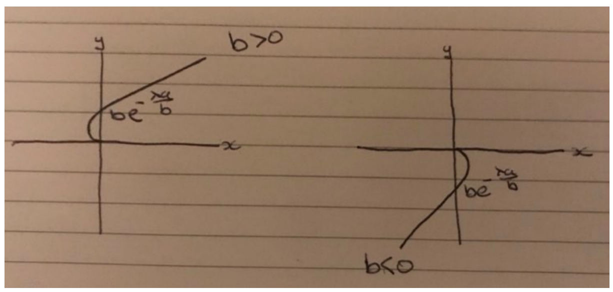

Figure 1: Curve

iii) Set

$$A = \begin{pmatrix} 8 & 4 \\ -1 & 4 \end{pmatrix}.$$

We have the characteristic polynomial $p(\lambda) = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2$ so we have repeated eigenvalues $\lambda = 6$. This gives us the Jordan matrix

$$J = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix}.$$

To form the change of basis matrices we will need independent eigenvectors. Solving $(A - \lambda I)\mathbf{v} = 0$ yields

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2v_1 + 4v_2 \\ -v_1 - 2v_2 \end{pmatrix} = 0$$

which gives us a single eigenvector $\mathbf{v} = (-2, 1)$. For the other vector we will need a generalised eigenvector $\mathbf{w}$ which we can find by solving

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \Rightarrow \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2w_1 + 4w_2 \\ -w_1 - 2w_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

This gives us $\mathbf{w} = (-1, 0)$ which allows us to write

$$A = \begin{pmatrix} 8 & 4 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$$

in JCF.

iv) We have for $\mathbf{x}_0 = a\mathbf{v} + b\mathbf{w} = (-2a - b, a)$

$$
\begin{aligned}
\mathbf{x}(t) &= e^{At}\mathbf{x}_0 \\
&= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} e^{Jt} \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \mathbf{x}_0 \\
&= e^{6t} \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{x}_0 \\
&= e^{6t} \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} \begin{pmatrix} -2a-b \\ a \end{pmatrix} \\
&= e^{6t} \begin{pmatrix} -2a-b-2bt \\ a+bt \end{pmatrix}.
\end{aligned}
$$

v) As $(-2,1)$ is an eigenvector, we know the line $y = -\frac{1}{2}x$ will be invariant. Furthermore as the associated eigenvalue $\lambda = 6 > 0$ we know $(0,0)$ is a hyperbolic source. Let's write ODE (2) as

$$(\dot{x}, \dot{y}) = (8x + 4y, -x + 4y).$$

If $y > -\frac{1}{2}x$ then $\dot{x} > 6x$ and $\dot{y} > -3x$ so we know for positive x we will have x increasing and y decreasing. If $y < -\frac{1}{2}x$ then $\dot{x} < 6x$ and $\dot{y} < -3x$ so we know for negative x we will have x decreasing and y increasing. This tells us on which side we will approach $y = -\frac{1}{2}x$. Hence we get the phase portrait in Figure 2.

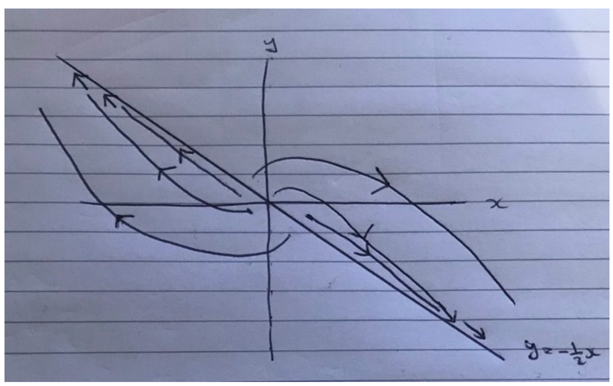

Figure 2: Phase Portrait for ODE (2)

I also wrote some Python code to display the vector field with some example trajectories computed using the equation in part **iv**. This is in Figure 3.

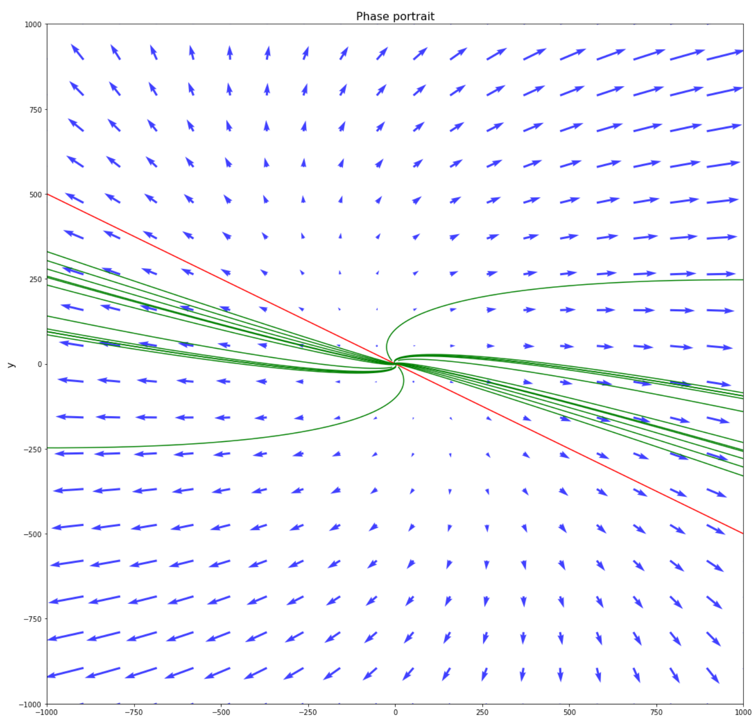

Figure 3: ODE (2) with trajectories (green) and invariant set (red)