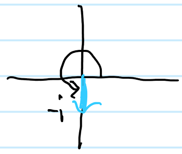
PS1

Wednesday, 25 November 2020 20:28

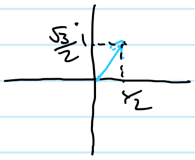
Q 1b,d , 2a,c , 5 , 6, 7

1b)



$$-i = e^{i\frac{3\pi}{2}}$$

d)



$$r = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2} = \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} = \sqrt{1} = 1$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}(\sqrt{3}) = \pi/3$$
$$\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\frac{\pi}{3}}$$

2a) $z^3 = 4$

Let $z = re^{i\theta}$. Then

$$z^3 = 4 \implies (re^{i\theta})^3 = 4$$
$$\implies r^3 e^{i3\theta} = 4(1) = 4e^{i2\pi k}$$

for $k \in \mathbb{Z}$. So $r = \sqrt[3]{4}$ and $\theta = \frac{2\pi k}{3}$. Taking $k = 0,1,2$ gives us solutions

$$z = \sqrt[3]{4},\ \sqrt[3]{4}e^{i\frac{2\pi}{3}},\ \sqrt[3]{4}e^{i\frac{4\pi}{3}}.$$

c) $(az+b)^3 = c \qquad a,b,c \in \mathbb{R}_{>0}$.

Define $w = az+b \in \mathbb{C}$. Now we solve $w^3 = c$ using the same method as in part a to get

$$w = \sqrt[3]{c},\ \sqrt[3]{c}e^{i\frac{2\pi}{3}},\ \sqrt[3]{c}e^{i\frac{4\pi}{3}}$$
$$\implies z = \frac{\sqrt[3]{c} - b}{a},\ \frac{\sqrt[3]{c}e^{i\frac{2\pi}{3}} - b}{a},\ \frac{\sqrt[3]{c}e^{i\frac{4\pi}{3}} - b}{a}.$$

5i) $f(z) = z^3$.

Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then

$$f(z) = f(x+iy) = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3.$$

So

$$\left.\begin{aligned} \mathrm{Re}(z) &= x^3 - 3xy^2 \\ \mathrm{Im}(z) &= 3x^2y - y^3 \end{aligned}\right\}.$$

Now the Cauchy-Riemann equations require

$$\left.\begin{aligned} \partial_x(x^3 - 3xy^2) &= \partial_y(3x^2y - y^3) \\ \partial_x(3x^2y - y^3) &= -\partial_y(x^3 - 3xy^2) \end{aligned}\right\}$$

$$\implies \left.\begin{aligned} 3x^2 - 3y^2 &= 3x^2 - 3y^2 \\ 6xy &= 6xy \end{aligned}\right\}.$$

$$\Longrightarrow \quad 3x^2 - 3y^2 = 3x^2 - 3y^2, \quad 6xy = 6xy \quad \Big\}.$$

So the Cauchy-Riemann equations and the partial derivatives are continuous $\forall (x,y) \in \mathbb{R}^2$ so f is differentiable on the domain $\mathbb{C}$.

(ii) $f(z) = (z + z^{-1}), \quad z \neq 0$

Let $z = x + iy$ for $x, y \in \mathbb{R}$. Then

$$\begin{aligned} f(z) &= f(x+iy) \\ &= x + iy + \frac{1}{x+iy} \\ &= x + iy + \frac{x - iy}{(x+iy)(x-iy)} \\ &= x + iy + \frac{x - iy}{x^2+y^2} \\ &= \left(x + \frac{x}{x^2+y^2}\right) + \left(y - \frac{y}{x^2+y^2}\right)i. \end{aligned}$$

So

$$\left.\begin{aligned} Re(z) &= x + \frac{x}{x^2+y^2} \\ Im(z) &= y - \frac{y}{x^2+y^2} \end{aligned}\right\}.$$

Now the Cauchy-Riemann equations require

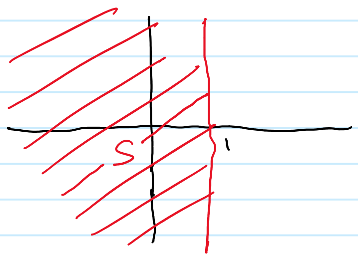
$$\left.\begin{aligned} 1 + \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} &= 1 - \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} &= \frac{2xy}{(x^2+y^2)^2} \end{aligned}\right\}.$$

Clearly the second equation is satisfied so we just need to work with the first. We have

$$\begin{aligned} 1 + \frac{y^2 - x^2}{x^2+y^2} &= 1 - \frac{x^2 - y^2}{x^2+y^2} \\ \Longrightarrow \quad 1 + \frac{y^2 - x^2}{x^2+y^2} &= 1 + \frac{y^2 - x^2}{x^2+y^2} \end{aligned}$$

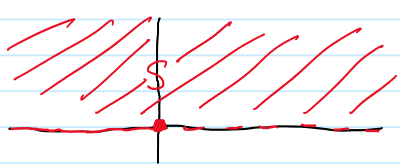
and so the second equation is also true $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Hence f is differentiable on $\mathbb{C} \setminus \{0\}$.

6a) $S = \{z \in \mathbb{C} : Im(z) \leq 1\}$



Not open as no disc exists for $z = 1$.
Closed as complement $Im(z) > 1$ is open.
Connected - just take the euclidean distance as the curve.
Not a region as not open.

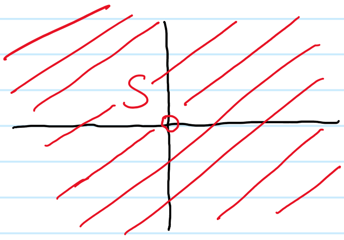
b) $S = \{z \in \mathbb{C} : z = |z| e^{i\theta}, 0 < \theta \leq \pi\}$



Not open as no disc exists for $z = -1$,
Not closed as no disc exists for $z = 1 \in \mathbb{C} \setminus S$.
Connected - again by euclidean distance.

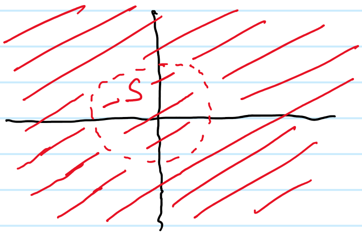
Not closed as no disc exists
for $z = 1 \in \mathbb{C} \setminus S$.
Connected - again by euclidean distance.
Not a region as not open.

c) $S = \{ z \in \mathbb{C} : |z| \neq 0 \}$



Open - take $0 < r < |z|$
Not closed as complement $\{0\}$
is not open.
Connected - take arc on a disc
to match angle followed
the euclidean distance.
Region as open, connected and $1 \in S$
so not empty.

d) $S = \{ z \in \mathbb{C} : |z| \neq 2 \}$



Open - not sure how to justify
Not closed as complement not open
Not connected - no curve between
$z = 0 \in S$ and $z = 3 \in S$ lying
in S.
Not a region as not connected

7i) $f(z) = \text{Im}(z)$

Let $z = x + iy$ for $x, y \in \mathbb{R}$. We have $f(z) = y$. We would need $0 = 1$ for the second equation which is never true.

ii) $g(z) = \overline{z}$

Let $z = x + iy$ for $x, y \in \mathbb{R}$. We have $f(z) = x - iy$. We would need $1 = -1$ for the first equation which is never true