PS1

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Q1, Q3, Q4

1i) $\underline{F}(\underline{x}) = (x_3, x_1, x_2)$

$$\underline{F}(\lambda \underline{x} + \mu \underline{y}) = \underline{F}(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \lambda x_3 + \mu y_3)$$
$$= (\lambda x_3 + \mu y_3, \lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2)$$
$$= \lambda (x_3, x_1, x_2) + \mu (y_3, y_1, y_2)$$
$$= \lambda \underline{F}(\underline{x}) + \mu \underline{F}(\underline{y})$$

So $\underline{F}$ is a linear map. Now

$$A = \underline{F}'(\underline{x}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Check:
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$$

ii) $\underline{F}(\underline{x}) = (x_3 x_1, x_1 x_2, x_2 x_3)$

let $\underline{x} = (1,1,1)$, $\underline{y} = (2,2,2)$

$\underline{F}(\underline{x} + \underline{y}) = \underline{F}(3,3,3) = (9,9,9) \neq (1,1,1) + (4,4,4)$.

So $\underline{F}$ is not a linear map.

iii) $\underline{F}(\underline{x}) = (x_3 + x_1, x_2 + x_3)$

$$\underline{F}(\lambda \underline{x} + \mu \underline{y}) = \underline{F}(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \lambda x_3 + \mu y_3)$$
$$= (\lambda x_3 + \mu y_3 + \lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2 + \lambda x_3 + \mu y_3)$$
$$= (\lambda(x_3 + x_1) + \mu(y_3 + y_1), \lambda(x_2 + x_3) + \mu(y_2 + y_3))$$
$$= \lambda F(\underline{x}) + \mu F(\underline{y})$$

So $\underline{F}$ is a linear map. Now

$$A = \underline{F}'(\underline{x}) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Check:
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \end{pmatrix}$$

3a) $\underline{F}(\underline{x}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\underline{x} = (x_1 + 2x_2, 2x_1 + x_2, x_1)$

$\underline{G}(\underline{x}) = (x_1 x_2, x_2 x_3, \sin(x_1 x_2 x_3))$

$\underline{H}(\underline{x}) = \underline{G} \circ \underline{F}$

$\underline{H}'(\underline{x}) = \underline{G}'(\underline{F}(\underline{x}))\underline{F}'(\underline{x})$ So $\underline{H}'(1,1) = \underline{G}'(\underline{F}(1,1))\underline{F}'(1,1)$

$G'(x) = \Big(\quad x_2 \quad x_1 \quad 0$

$\underline{H}'(\underline{x}) = \underline{G}'(\underline{F}(\underline{x}))\underline{F}'(\underline{x})$ So $\underline{H}'(1,1) = \underline{G}'(\underline{F}(1,1))\underline{F}'(1,1)$

$$\underline{G}'(\underline{x}) = \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ \cos(x_1x_2x_3)x_2x_3 & \cos(\cdots)x_1x_3 & \cos(\cdots)x_1x_2 \end{pmatrix}$$

$$\underline{F}'(\underline{x}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{F}(1,1) = (3,3,1)$$

$$\underline{G}'(\underline{F}(1,1)) = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos 9 & 3\cos 9 & 9\cos 9 \end{pmatrix}$$

$$\underline{H}'(1,1) = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos 9 & 3\cos 9 & 9\cos 9 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 5 & 1 \\ 18\cos 9 & 9\cos 9 \end{pmatrix}$$

b) $\underline{H} = \underline{G} \circ \underline{F}$ So

$$H(\underline{x}) = \begin{pmatrix} (x_1+2x_2)(2x_1+x_2) \\ (2x_1+x_2)x_1 \\ \sin((x_1+2x_2)(2x_1+x_2)x_1) \end{pmatrix} = \begin{pmatrix} 2x_1^2 + 5x_1x_2 + 2x_2^2 \\ 2x_1^2 + x_2x_1 \\ \sin(2x_1^3 + 5x_1^2x_2 + 2x_2^2x_1) \end{pmatrix}.$$

Hence

$$H'(\underline{x}) = \begin{pmatrix} 4x_1 + 5x_2 & 5x_1 + 4x_2 \\ 4x_1 + x_2 & x_1 \\ \cos(2x_1^3 + 5x_1^2x_2 + 2x_2^2x_1)[6x_1^2 + 10x_1x_2 + 2x_2^2] & \cos(\cdots)[5x_1^2 + 4x_2x_1] \end{pmatrix}$$

So

$$H'(1,1) = \begin{pmatrix} 9 & 9 \\ 5 & 1 \\ 18\cos 9 & 9\cos 9 \end{pmatrix}.$$

4) $x^3 + e^y = s, \quad \cos x + xy = t$

Define $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(s,t) = \underline{F}(\underline{x}) = (x^3 + e^y, \cos x + xy).$$

Now

$$\underline{F}'(\underline{x}) = \begin{pmatrix} 3x^2 & e^y \\ -\sin x + y & x \end{pmatrix}$$

$$\Rightarrow J_{\underline{F}}(\underline{x}) = 3x^3 + e^y \sin x - ye^y.$$

At $(x,y) = (0,0)$ we have

$$J_{\underline{F}}(0,0) = 0$$

So far we can't garantee unique solutions near $(s,t) = (1,1)$.