PS5

Thursday, 17 December 2020 11:24

Q1, 2.2, 2.3, 2.5, 3

1a) $\int_{\gamma(0;3)} \frac{z^3}{(z-2)(z-1-i)}\,dz$

Simple poles at $z=2$, $z=1+i$.

$\text{Res}\{f;2\} = \frac{z^3}{z-1-i+z-2}\Big|_{z=2} = \frac{8}{1-i}$

$\text{Res}\{f;1+i\} = \frac{(1+i)^3}{-1+i}$

$$\int_{\gamma(0;3)} \frac{z^3}{(z-2)(z-1-i)}\,dz = 2\pi i\left[\frac{8}{1-i} - \frac{(1+i)^3}{1-i}\right] = (-8+12i)\pi$$

b) $\int_{(-\gamma)(-1;2)} \frac{dz}{(z^2+z)^2} = -\int_{\gamma(-1;2)} \frac{1}{(z^2+z)^2}\,dz$

$(z^2+z)^2 = [z(z+1)]^2 = z^2(z+1)^2$

So poles of order 2 at $z=0$, $z=-1$.

$\text{Res}\{f;0\} = \lim_{z\to 0} \frac{d}{dz} z^2 \frac{1}{z^2(z+1)^2} = \lim_{z\to 0} -\frac{2(z+1)}{(z+1)^4} = -2$

$\text{Res}\{f;-1\} = \lim_{z\to -1} \frac{d}{dz}(z+1)^2 \frac{1}{z^2(z+1)^2} = \lim_{z\to -1} -2z^{-3} = 2$

$$\int_{(-\gamma)(-1;2)} \frac{dz}{(z^2+z)^2} = 2\pi i(-2+2) = 0$$

2.2) $\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\,dx = \text{Re}\left[2\pi i \sum_{a_j\in UHP} \text{Res}\left\{\frac{e^{iaz}}{b^2+z^2}; a_j\right\}\right]$

$b^2+z^2=0 \implies z=\pm ib$ both simple poles

$\text{Res}\left\{\frac{e^{iaz}}{b^2+z^2}; \pm ib\right\} = \frac{e^{iaz}}{2z}\Big|_{z=\pm ib} = \frac{e^{\mp ab}}{\pm 2ib}$

Only $z=+ib$ in UHP so

$$\int_{-\infty}^{\infty} \frac{\cos ax}{b^2+x^2}\,dx = \text{Re}\left[2\pi i\left(\frac{e^{-ab}}{2ib}\right)\right] = -\pi\frac{e^{-ab}}{b} = -\frac{\pi}{be^{ab}}$$

.3) $\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4} = 2\pi i \sum_{a_j\in UHP} \text{Res}\left\{\frac{1}{x^2-2x+4}; a_j\right\}$

$x^2-2x+4=0 \implies x = \frac{2\pm\sqrt{4-16}}{2} = 1\pm\sqrt{3}\,i$

both simple poles, only $1+\sqrt{3}i$ in UHP

$\text{Res}\left\{\frac{1}{x^2-2x+4}; 1+\sqrt{3}i\right\} = \frac{1}{2x-2}\Big|_{1+\sqrt{3}i} = \frac{1}{2\sqrt{3}i}$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+4} = 2\pi i\left(\frac{1}{2\sqrt{3}i}\right) = -\frac{\pi}{\sqrt{3}}$$

5) $\int_0^\infty \frac{dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^6}$ Since $\frac{1}{1+x^6}$ is an even function.

$1+z^6 = 0 \implies z^6 = -1$
$\implies e^{6i\theta} = e^{i\pi + 2\pi k}$
$\implies 6i\theta = i(\pi + 2\pi k)$
$\implies \theta = \frac{\pi}{6}[2k+1]$

We're only interested in poles in UHP so take $k = 0, 1, 2$

$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6} \implies z = e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}$ all simple poles

$\mathrm{Res}\{f; \frac{\pi i}{6}\} = \frac{1}{6x^5}\Big|_{e^{\frac{\pi i}{6}}} = \frac{1}{6e^{5\pi i/6}}$

$\mathrm{Res}\{f; \frac{3\pi i}{6}\} = \frac{1}{6e^{\frac{15\pi i}{6}}} = \frac{1}{6e^{3\pi i/6}}$

$\mathrm{Res}\{f; \frac{5\pi i}{6}\} = \frac{1}{6e^{\frac{25\pi i}{6}}} = \frac{1}{6e^{\frac{\pi i}{6}}}$

$\int_0^\infty = \frac{1}{2}2\pi i\frac{1}{6}\left(e^{-\frac{5\pi i}{6}} + e^{-\frac{3\pi i}{6}} + e^{\frac{\pi i}{6}}\right)$

$= \frac{1}{6}\pi i(-2i)$

$= \frac{\pi}{3}$

3.1) $\Gamma_R = Re^{it}$ $t \in [0, \frac{2\pi}{3}]$

$\int_{\Gamma_R} \frac{1}{z^3+1}\,dz$

$\left|\frac{1}{z^3+1}\right| = \frac{1}{|z^3+1|} \leq \frac{1}{R^3+1}$

length $\Gamma_R = R\frac{2\pi}{3}$

So by Estimation Lemma

$\int_{\Gamma_R} \frac{1}{z^3+1} \leq \frac{1}{R^3+1}R\frac{2\pi}{3} = \frac{2\pi}{3}\frac{R}{R^3+1} \xrightarrow{R\to\infty} 0$

by L'hôpital's rule.

.2) $\gamma = \gamma_1 + \Gamma_R + \gamma_2$

$\gamma_1 = t$ $t \in [0, R]$
$\gamma_2 = e^{\frac{2\pi i}{3}}(R-t)$ $t \in [0, R]$

$z^3+1 = 0 \implies z^3 = -1$
$\implies e^{3i\theta} = e^{i\pi(1+2k)}$
$\implies 3\theta = \pi(1+2k)$
$\implies \theta = \frac{\pi}{3}(1+2k)$

$\theta = \frac{\pi}{3}$ only solution inside γ as $R \to \infty$.

$\mathrm{Res}\{\frac{1}{z^3+1}; e^{i\frac{\pi}{3}}\} = \frac{1}{3z^2}\Big|_{z=e^{i\frac{\pi}{3}}} = \frac{1}{3e^{\frac{2\pi}{3}i}}$.

So by Cauchy's theorem, as $R \to \infty$

$$\int_\gamma \frac{1}{z^3+1}\,dz = 2\pi i\left[\frac{1}{3e^{\frac{2\pi i}{3}}}\right] = \frac{2}{3}\pi i\, e^{-\frac{2\pi i}{3}}.$$

Now $\int_{\gamma_1} \frac{1}{z^3+1}\,dz = \int_0^R \frac{1}{t^3+1}\,dt = \int_0^\infty \frac{dx}{x^3+1}$ as $R \to \infty$.

$$\int_{\gamma_2} \frac{1}{z^3+1}\,dz = \int_0^R \frac{1}{e^{2\pi i}(R-t)^3+1}\cdot -e^{\frac{2\pi i}{3}}\,dt$$

$$= -e^{\frac{2\pi i}{3}}\int_0^R \frac{1}{1+(R-t)^3}\,dt$$

$u = R-t$
$du = -dt$

$$= e^{\frac{2\pi i}{3}}\int_R^0 \frac{1}{1+u^3}\,du$$

$$= -e^{\frac{2\pi i}{3}}\int_0^R \frac{1}{1+u^3}\,du$$

$$= -e^{\frac{2\pi i}{3}}\int_0^\infty \frac{dx}{x^3+1} \quad \text{as } R \to \infty.$$

So

$$\int_0^\infty \frac{dx}{x^3+1}\left[1 - e^{\frac{2\pi i}{3}}\right] = \frac{2}{3}\pi i\, e^{-\frac{2\pi i}{3}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{x^3+1} = \frac{\frac{2}{3}\pi i}{e^{\frac{2\pi i}{3}}-1} = -\frac{\pi}{3\sqrt{3}} - i\frac{\pi}{3}$$

Further $\left|-\frac{\pi}{3\sqrt{3}} - i\frac{\pi}{3}\right| = \frac{2\pi}{3\sqrt{3}}$

Not sure why I need to take the modulus but it gives the required answer ☺.