PS1

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TB 1a-e , SE 1,2,5

TB 1a) $\ddot{\theta} + \delta\dot{\theta} + \sin\theta = F\cos\omega t \qquad \theta \in S^1$

$\dot{\theta} = V$
$\dot{V} = F\cos\omega t - \sin\theta - \delta V$

Dependant variable – θ
Independ variable – t
parameters – δ, F, ω

Non-Linear
Non-Autonomous

b) $\ddot{\theta} + \delta\dot{\theta} + \theta = F\cos\omega t$ $\qquad \ddot{\theta} = \underbrace{(-\delta\dot{\theta} - \theta)}_{\text{So linear}} + F\cos\omega t$

$\dot{\theta} = V$
$\dot{V} = F\cos\omega t - \theta - \delta V$

Dependant Variable – θ
Independant Variable – t
parameters – δ, F, ω

Linear
Non-autonomous

c) $y''' + x^2 y y' + y = 0 \qquad x \in \mathbb{R}$

$y' = V \implies V'' = -y - x^2 y V$
$V' = W$
$W' = -y - x^2 y V$

Dependent Variable – y
Independent Variable – x
parameters – none

Non-Linear
Non-autonomous

d) $\ddot{x} + \delta\dot{x} + x - x^3 = \theta$
$\ddot{\theta} + \sin\theta = 0 \qquad (x,\theta) \in \mathbb{R} \times S^1$

$\dot{x} = V$
$\dot{V} = \theta + x^3 - x - \delta V$
$\dot{\theta} = W$
$\dot{W} = -\sin\theta$

Dependent Variables – x, θ
Independent Variable – t
parameters – δ

Linear
Autonomous

e) $\ddot{\theta} + \delta\dot{\theta} + \sin\theta = x$
$\ddot{x} - x + x^3 = 0 \qquad (\theta, x) \in S \times \mathbb{R}$

$\dot{\theta} = V$
$\dot{V} = x - \sin\theta - \delta V$
$\dot{x} = W$
$\dot{W} = -x^3 + x$

Dependent variables – θ, x
Independent variables – t
parameters – δ

Non Linear
Autonomous

Dependent variables - θ, ...   Non Linear
Independent variables - t Autonomous
parameters - δ

SE 1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $a = (b, c) \in \mathbb{R}^2$. So $\exists A \in \mathbb{R}^{1\times 2}$ s.t.

$$f(\underline{x}) = f(\underline{a}) + A(\underline{x} - \underline{a}) + \alpha(\underline{x})|\underline{x} - \underline{a}|$$

for $\underline{x} \neq \underline{a}$ where $\alpha(\underline{x}) \to 0$ as $\underline{x} \to \underline{a}$. So

$$\begin{aligned}
\partial_x f(a) &= \lim_{h\to 0} \frac{f(b+h, c) - f(b, c)}{h} \\
&= \lim_{h\to 0} \frac{f(\underline{a}) + A(h, 0) + \alpha(b+h, c)|(h, 0)| - f(\underline{a})}{h} \\
&= \lim_{h\to 0} A(1, 0) + \alpha(b+h, c) \\
&= A(1, 0)
\end{aligned}$$

Not entirely sure how to phrase this

as when $h \to 0$, $\alpha(b+h, c) \to \alpha(\underline{a}) \to 0$. Similarly

$$\begin{aligned}
\partial_y f(\underline{a}) &= \lim_{h\to 0} \frac{f(b, c+h) - f(\underline{a})}{h} \\
&= \lim_{h\to 0} \frac{f(\underline{a}) + A(0, h) + \alpha(b, c+h)|(0, h)| - f(\underline{a})}{h} \\
&= \lim_{h\to 0} A(0, 1) + \alpha(b, c+h) \\
&= A(0, 1).
\end{aligned}$$

So

$$\left.\begin{aligned}
\partial_x f(\underline{a}) &= (A_{11} \quad A_{12})\begin{pmatrix}1\\0\end{pmatrix} = A_{11} \\
\partial_y f(\underline{a}) &= (A_{11} \quad A_{12})\begin{pmatrix}0\\1\end{pmatrix} = A_{12}
\end{aligned}\right\}$$

$$\Longrightarrow \quad A = (\partial_x f(\underline{a}) \quad \partial_y f(\underline{a})).$$

2) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable on $\mathbb{R}^2$. Suppose $(x(t), y(t))$ is a differentiable curve in $\mathbb{R}^2$ defined on an open interval I. We know

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h}.$$

By question 1 with $\underline{a} = (x(t), y(t))$ and $\underline{x} = (x(t+h), y(t+h))$

$$\begin{aligned}
f(x(t+h), y(t+h)) = f(x(t), y(t)) &+ A(x(t+h) - x(t), y(t+h) - y(t)) \\
&+ \alpha(x(t+h), y(t+h))|(x(t+h) - x(t), y(t+h) - y(t))|.
\end{aligned}$$

Hence

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{h\to 0} \frac{1}{h}\left[(f_x \quad f_y)\begin{pmatrix} x(t+h) - x(t) \\ y(t+h) - y(t)\end{pmatrix} + \alpha(x(t+h), y(t+h))\left|\begin{pmatrix} x(t+h) - x(t) \\ y(t+h) - y(t)\end{pmatrix}\right|\right].$$

Now as $h \to 0$ $\alpha(x(t+h), y(t+h)) \to 0$ So

$$\begin{aligned}
\frac{d}{dt} f(x(t), y(t)) &= \lim_{h\to 0} \frac{f_x(x(t+h) - x(t)) + f_y(y(t+h) - y(t))}{h} \\
&= f_x \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} + f_y \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \\
&= f_x(x(t), y(t))\,\dot{x}(t) + f_y(x(t), y(t))\,\dot{y}(t)
\end{aligned}$$

$= f_x \lim_{h\to 0} \frac{\cdots}{h} + f_y \lim_{h\to 0} \frac{\quad}{h}$

$= f_x(x(t), y(t))\, \dot{x}(t) + f_y(x(t), y(t))\, \dot{y}(t)$

$= \nabla f(\underline{x}(t)) \cdot \dot{\underline{x}}(t)$.

5) Let $m, n \in \mathbb{N}$ with $m \geq 2$. Consider vector ODE

$$\underline{x}^{(n)} = \underline{f}(t, \underline{x}, \dots, \underline{x}^{(n-1)})$$

where $f: \mathbb{R}^{1+nm} \longrightarrow \mathbb{R}^m$ Write $x_j^{(i)}$ for the jth element of $\underline{x}^{(i)}$, note $j = 1, \dots, m$, $i = 1, \dots, n$. Now define $V_{j1} = x_j$, $V_{j2} = \dot{x}_j, \dots, V_{jn} = x_j^{(n-1)}$.
So our system of equations becomes

$$\begin{cases} \dot{V}_{11} = V_{12}, \ \dot{V}_{12} = V_{13}, \dots, \dot{V}_{1n} = f_1(t, V_{11}, \dots, V_{1n}) \\ \dot{V}_{21} = V_{22}, \ \dot{V}_{22} = V_{23}, \dots, \dot{V}_{2n} = f_2(t, V_{21}, \dots, V_{2n}) \\ \vdots \\ \dot{V}_{m1} = V_{m2}, \ \dot{V}_{m2} = V_{m3}, \dots, \dot{V}_{mn} = f_m(t, V_{m1}, \dots, V_{mn}) \end{cases}$$

which is of first-order.