ES1 (Q7)

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Write $i = \sqrt{-1} \in \mathbb{C}$. Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$.

1) Show $\mathbb{Z}[i]^\times = \{1, -1, i, -i\}$ a cyclic group order 4.

Let $z, \omega \in \mathbb{Z}[i]$ such that $z\omega = 1$. Then we have that $1 = |z\omega|^2 = |z|^2|\omega|^2$. Let $z = a+bi$, $\omega = c + di$ where $a, b, c, d \in \mathbb{Z}$. We have

$$(a^2 + b^2)(c^2 + d^2) = 1.$$

We must have $a^2 + b^2 = 1 = c^2 + d^2$ since a, b, c, d are integers. Assuming $c^2 + d^2 = 1$ we get solutions for (a, b):

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1)$$

Corresponding to $z = 1, -1, i, -i$ which are all units by taking $\omega = 1, -1, -i, i$ respectively. Hence

$$\mathbb{Z}[i]^\times = \{1, -1, i, -i\} = \langle i \rangle \cong C_4$$

the cyclic group of order 4 since

$$i \longrightarrow i^2 = -1 \longrightarrow -i \longrightarrow -i^2 = 1 \longrightarrow i.$$

2) Which elements of $\mathbb{Z} \times \mathbb{Z}$ satisfy $x^2 = 1$.

Let $x \in \mathbb{Z} \times \mathbb{Z} = \{(a, b) : a, b \in \mathbb{Z}\}$. So

$$x^2 = 1 \iff (a^2, b^2) = (1, 1)$$

for some $a, b \in \mathbb{Z}$. Only integers satisfying this property are 1 and -1. So the elements are

$$\{(1,1), (1,-1), (-1,1), (-1,-1)\} \subseteq \mathbb{Z} \times \mathbb{Z}.$$

3) Prove $\mathbb{Z} \times \mathbb{Z} \ncong \mathbb{Z}[i]$.

$\mathbb{Z} \times \mathbb{Z}$ has 4 elements that square to 1. Since $\mathbb{Z}[i]^\times = \{1, -1, i, -i\}$ we only have to check these.

$$1^2 = 1, \quad (-1)^2 = 1, \quad i^2 = -1, \quad (-i)^2 = -1.$$

So $\mathbb{Z}[i]$ only has 2 elements that square to 1.

Let $\varphi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}[i]$ be an isomorphism. We have

$$\varphi((a,b)^2) = \varphi((a,b)(a,b)) = \varphi((a,b))\varphi((a,b)) = \varphi((a,b))^2$$

for $a, b \in \mathbb{Z}$. Further as φ is bijective we should have the same number of elements that square to 1 in both rings.

As this is not the case $\mathbb{Z} \times \mathbb{Z} \ncong \mathbb{Z}[i]$.