PS2

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Q 1, 2, 3, 5, 7, 9

1i) $f(z) = z^8 + 7z^5 - \pi z^2 + 1$

Entire as polynomial.

ii) $g(z) = (z-5)^{-1}$

$z-5$ is entire so g is holomorphic on $\mathbb{C} \setminus \{5\}$

iii) $h(z) = (z(z-1)(z-2))^{-1}$

$z(z-1)(z-2)$ is entire so h is holomorphic on $\mathbb{C} \setminus \{0,1,2\}$

2i) $z = x+iy$, $x, y \in \mathbb{R}$
$f(z) = 1/|z| = (\sqrt{x^2+y^2})^{-1} = (x^2+y^2)^{-\frac{1}{2}}$

Cauchy Riemann I: $\partial_x U = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} 2x$
$\partial_y V = 0$
$\implies x(x^2+y^2)^{-3/2} = 0$
$\implies x = 0$

Cauchy Riemann II: $\partial_x V = 0$
$\partial_y U = -\frac{1}{2}(x^2+y^2)^{-3/2} 2y$
$\implies y(x^2+y^2)^{-3/2} = 0$
$\implies y = 0$

So CR satisfied at $(0,0)$ but partial derivatives are not continuous at $(0,0)$ so f not holomorphic anywhere.

ii) $z = x+iy$, $x, y \in \mathbb{R}$
$g(z) = z|z| = (x+iy)\sqrt{x^2+y^2} = \underbrace{\sqrt{x^2+y^2}\,x}_{U(x,y)} + \underbrace{\sqrt{x^2+y^2}\,y}_{V(x,y)} i$

CR I: $\partial_x U = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2x \cdot x + \sqrt{x^2+y^2}$
$\partial_y V = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} 2y \cdot y + \sqrt{x^2+y^2}$
$\implies x^2 = y^2$

CR II: $\partial_x V = y \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$
$\partial_y U = x \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$
$\implies x^2 = -y^2$

So equations are inconsistent unless $x = y = 0$ but then the partial derivatives aren't continuous so g is not holomorphic anywhere.

3) We have f holomorphic in region G. Write $f(z) = U(x,y) + iV(x,y)$ for $z = x+iy$ for $x, y \in \mathbb{R}$ and $U, V: \mathbb{R}^2 \to \mathbb{R}$. As f is holomorphic we have that all partial derivatives of U and V are continuous in G and

$$\partial_x U = \partial_y V$$
$$\partial_x V = -\partial_y U.$$

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However since $\mathrm{Re}\, f = U(x,y)$ is a constant we know that $\partial_x U = \partial_y U = 0$ which means $\partial_x V = \partial_y V = 0$. This means $V(x,y) = \mathrm{Im}(z)$ must also be a constant. Thus $f = \mathrm{Re}\, f + i\, \mathrm{Im}\, f$ must be constant.

5a) $f(z) = \frac{1}{2z+5}$

We have the geometric series, that is
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$
for $z \in D(0,1)$. Now transform $z \mapsto -2z-4$ to get
$$\sum_{n=0}^{\infty} (-2z-4)^n = \frac{1}{1-(-2z-4)} = \frac{1}{2z+5} = f(z).$$
However we now have that $-2z-4 \in D(0,1)$ for convergence so $G = D(2, \frac{1}{2})$.

b) $g(z) = \frac{1}{1+z^4}$.

Similarly to a transform the geometric series by $z \mapsto -z^4$ so
$$\sum_{n=0}^{\infty} (-z^4)^n = \frac{1}{1-(-z^4)} = \frac{1}{1+z^4} = g(z).$$
Now we need $-z^4 \in D(0,1)$ so $G = D(0,1)$.

7i) $z = x+iy, \quad x,y \in \mathbb{R}$

$$\begin{aligned} \cos z &= \cos(x+iy) \\ &= \cos x \cos iy - \sin x \sin iy \\ &= \cos x \cosh y - i \sin x \sinh x \end{aligned}$$

$$\Rightarrow \begin{cases} \mathrm{Re} \cos z = \cos x \cosh y \\ \mathrm{Im} \cos z = -\sin x \sinh x \end{cases}$$

ii) $z = x+iy, \quad x,y \in \mathbb{R}$

$$\begin{aligned} e^{z^2} &= e^{(x+iy)^2} \\ &= e^{x^2+2xyi-y^2} \\ &= e^{x^2-y^2} e^{2xyi} \\ &= e^{x^2-y^2}\left[\cos(2xy) + i\sin(2xy)\right] \end{aligned}$$

$$\Rightarrow \begin{cases} \mathrm{Re}\, e^{z^2} = e^{x^2-y^2} \cos 2xy \\ \mathrm{Im}\, e^{z^2} = e^{x^2-y^2} \sin 2xy \end{cases}$$

9) $f(z) = f(x,y) = u(x,y) + i v(x,y)$.

By the Cauchy-Riemann equations we have

By the Cauchy-Riemann equations we have

$$\begin{cases} \partial_x U(x,y) = \partial_y V(x,y) \\ \partial_x V(x,y) = -\partial_y U(x,y) \end{cases}$$

on G. Then

$$\begin{aligned} \nabla^2 U &= \partial^2_{x} U(x,y) + \partial^2_y U(x,y) \\ &= \partial_x \partial_x U(x,y) + \partial_y \partial_y U(x,y) \\ &= \partial_x \partial_y V(x,y) - \partial_y \partial_x V(x,y) \\ &= 0 \end{aligned}$$

as $\partial^2_{xy} = \partial^2_{yx}$. Similarly

$$\begin{aligned} \nabla^2 V &= \partial^2_{x} V(x,y) + \partial^2_y V(x,y) \\ &= -\partial_x \partial_y U(x,y) + \partial_y \partial_x U(x,y) \\ &= 0 \end{aligned}$$