ES4(Q25)

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A ring R is a local ring if it has a unique maximum ideal.

(1) Prove that $\mathbb{Q}$ and $\mathbb{Z}/8\mathbb{Z}$ are local rings, and $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}$ are not.

Let I be an ideal of $\mathbb{Q}$. By the definition of an ideal $ri \in I$ for all $r \in \mathbb{Q}$ and $i \in I$. Let $r \neq 0$. Then

$$\mathbb{Q} \supseteq I \supseteq \{ri : r \in \mathbb{Q}\} = \mathbb{Q} \quad \Rightarrow \quad I = \mathbb{Q}.$$

So the only ideals of $\mathbb{Q}$ are $\mathbb{Q}$ and $\{0\}$. So $\{0\}$ is the unique maximal ideal of $\mathbb{Q}$, so $\mathbb{Q}$ is a local ring.

The ideals of $\mathbb{Z}/8\mathbb{Z}$ are clearly

$$\{\bar{0}\}, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}, \{\bar{0}, \bar{4}\}, \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}.$$

Therefore $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is the unique maximal ideal so $\mathbb{Z}/8\mathbb{Z}$ is a local ring.

The ideals of $\mathbb{Z}/6\mathbb{Z}$ are

$$\{\bar{0}\}, \{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{0}, \bar{3}\}, \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}.$$

Note that $\{\bar{0}, \bar{2}, \bar{4}\}$, $\{\bar{0}, \bar{3}\}$ are both maximal so $\mathbb{Z}/6\mathbb{Z}$ is not a local ring.

For $\mathbb{Z}$ consider ideals $2\mathbb{Z}$ and $3\mathbb{Z}$. Note that the only ideals of $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ are $\{0\}$ and itself so $\mathbb{Z}/2\mathbb{Z}$ is a field so $2\mathbb{Z}$ is maximal. Similar argument holds for $3\mathbb{Z}$. So $\mathbb{Z}$ has no unique maximal ideal so it is not a local ring.

(2) Prove that R is a local ring iff all elements of R that are not units form an ideal.

($\Rightarrow$)
Suppose R is a local ring. So $\exists M \subset R$ a unique maximum ideal. We claim $R \setminus R^\times$ is an ideal. To justify note that $0 \in R \setminus R^\times$, if $x, y \in R \setminus R^\times$ then ideals $(x) \subset M$, $(y) \subset M$ since M is maximal. So $x + y \in M$. Now as $M \neq R$ by definition, it contains no units so $x + y \in R \setminus R^\times$. Similarly we have $x - y \in R \setminus R^\times$ so $R \setminus R^\times$ forms and additive subgroup. Finally if $r \in R$ then $rx \in (x) \Rightarrow rx \in M$ so by the same argument as before $rx \in R \setminus R^\times$. So $R \setminus R^\times$ is an ideal.

($\Leftarrow$)
Suppose $R \setminus R^\times$ is an ideal. Since $1 \in R^\times$, $R \setminus R^\times \neq R$. Suppose M is a maximum ideal of R. Since $M \neq R$ all elements of M are non-units, so $M \subseteq R \setminus R^\times$. But M is maximal so $M = R \setminus R^\times$. So $R \setminus R^\times$ is the unique maximal ideal so R is a local ring.