PS2

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1a) $\underline{X} \overset{iid}{\sim} \text{Poisson}(\lambda)$

$$\ell(\lambda;\underline{x}) = n\bar{x}\log\lambda - \lambda n - \sum_{i=1}^{n}\log x_i!$$

$$\partial_\lambda \ell(\lambda;x) = n\bar{x}\left(\frac{1}{\lambda}\right) - n \overset{\text{SET}}{=} 0$$
$$\implies \hat{\lambda} = \bar{x}$$

$$\partial_\lambda \ell(\lambda;x) = n\bar{x}\left(-\frac{1}{\lambda^2}\right)$$
At $\lambda = \bar{x}$ we have
$$\partial_\lambda^2 \ell(\lambda;x) = -\frac{n}{\bar{x}} < 0$$

Since for all $x \in \underline{x}$, $x \geq 0$.

b) $X \sim \text{Binomial}(n,p)$ with n specified

$$\ell(n,p;x) = \log\binom{n}{x} + x\log p + (n-x)\log(1-p)$$
$$\partial_p \ell(n,p;x) = x\left(\frac{1}{p}\right) + (n-x)\cdot -\frac{1}{1-p}$$
$$= x/p + \frac{x-n}{1-p}$$
$$\overset{\text{SET}}{=} 0$$
$$\implies (1-p)x + (x-n)p = 0$$
$$\implies x - px + xp - np = 0$$
$$\implies \hat{p} = x/n$$

$$\partial^2_p \ell(n,p;x) = x\left(-\frac{1}{p^2}\right) + (x-n)\cdot\left(\frac{1}{(1-p)^2}\right).$$
At $p = x/n$ we have
$$\partial^2_p \ell(n,p;x) = x\left(-\frac{n^2}{x^2}\right) + (x-n)\cdot\left(\frac{n^2}{(n-x)^2}\right)$$
$$= -n^2/x - \frac{n^2}{(n-x)^2} < 0$$

as $x \geq 0$.

c) $\underline{X} \overset{iid}{\sim} \text{Normal}(\mu,\sigma^2)$, $\theta = (\mu,\sigma^2)$

$$\ell(\theta;\underline{x}) = -n\log\sigma\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$
$$= -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\partial_\mu \ell(\theta;\underline{x}) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}-2(x_i-\mu) = \frac{n}{\sigma^2}(\bar{x}-\mu).$$

So
$$\partial_\mu \ell(\theta;\underline{x}) = 0 \implies \hat{\mu} = \bar{x}$$

$$\partial_{\sigma^2} \ell(\theta;\underline{x}) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$
$$= \frac{1}{\sigma^2}\left(\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2}\right).$$

So

$$\partial_{\sigma^2}(\theta;\underline{x}) = 0 \implies \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu).$$

To check let's compute the hessian

$$H(\underline{\theta};\underline{x}) = \begin{pmatrix} -n/\sigma^2 & -n/\sigma^4(\bar{x}-\mu) \\ -n/\sigma^4(\bar{x}-\mu) & -\frac{1}{\sigma^4}\left(\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2 - \frac{n}{2}\right) + \frac{1}{\sigma^2}\left(-\frac{1}{2\sigma^4}\sum_{i=1}^n (x_i-\mu)^2\right) \end{pmatrix}$$

$$\Rightarrow H(\hat{\underline{\theta}};\underline{x}) = \begin{pmatrix} -n/\hat{\sigma}^2 & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} - \frac{1}{\hat{\sigma}^6}\sum_{i=1}^n (x_i-\hat{\mu})^2 \end{pmatrix}$$

$$= \begin{pmatrix} -n/\hat{\sigma}^2 & 0 \\ 0 & \frac{n}{2\hat{\sigma}^4} - \frac{n}{\hat{\sigma}^4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{n}{2\hat{\sigma}^4} \end{pmatrix}.$$

Now

$$\det H(\hat{\underline{\theta}};\underline{x}) = \frac{n^2}{2\hat{\sigma}^6} > 0$$

and

$$\partial^2_\mu \ell(\hat{\theta};\underline{x}) = -n/\hat{\sigma}^2 < 0$$

So $\hat{\underline{\theta}}$ is a maximum of ℓ.

1d)

$$\ell(a,b;\underline{x}) = \begin{cases} c - n\log(b-a) & \text{all } x_i \in [a,b] \\ -\infty & \text{otherwise} \end{cases}$$

If $\exists x_i \notin [a,b]$ then $\ell(a,b;\underline{x}) = -\infty$. Hence we need $a \leqslant \min\{x_1,\dots,x_n\}$ and $b \geqslant \max\{x_1,\dots,x_n\}$. Note

$$\partial_a \ell = \frac{n}{b-a} > 0$$

and

$$\partial_b \ell = \frac{n}{a-b} < 0$$

as $b > a$. So ℓ is increasing in a and decreasing in b hence

$$\hat{a} = \min\{x_1,\dots,x_n\}$$

and

$$\hat{b} = \max\{x_1,\dots,x_n\}.$$

2a) $f_X(x;x_0,\theta) = \frac{\theta x_0^\theta}{x^{\theta+1}} \mathbb{I}(x \geqslant x_0)$ for $x_0 > 0$, $\theta > 0$. So

$$F_X(x;x_0,\theta) = \int_{-\infty}^{x} \frac{\theta x_0^\theta}{t^{\theta+1}} \mathbb{I}(t \geqslant x_0)\, dt$$

$$= \mathbb{I}(x \geqslant x_0)\,\theta x_0^\theta \int_{x_0}^{x} t^{-(\theta+1)}\, dt$$

$$= \mathbb{I}(x \geqslant x_0)\,\theta x_0^\theta \left(\frac{t^{-\theta}}{-\theta}\right)\Big|_{x_0}^{x}$$

$$= \mathbb{I}(x \geq x_0)\, \theta x_0 \int_{x_0} t \quad dt$$

$$= \mathbb{I}(x \geq x_0)\, \theta x_0^{\theta} \left(\frac{t^{-\theta}}{-\theta} \right) \Big|_{x_0}^{x}$$

$$= \mathbb{I}(x \geq x_0)\, x_0^{\theta} \left(x_0^{-\theta} - x^{-\theta} \right)$$

$$= \left(1 - \left(\frac{x_0}{x} \right)^{\theta} \right) \mathbb{I}(x \geq x_0).$$

b) For the quantile function we want the smallest $x \in \mathbb{R}$ such that

$$F_X(x) \geq u$$

for any $u \in (0,1)$. If $x < x_0$ then $F_X(x) = 0 < u$ as $0 < u < 1$. So assume $x \geq x_0$. Then

$$\begin{aligned}
& F_X(x) \geq u \\
\iff \quad & 1 - \left(\frac{x_0}{x}\right)^{\theta} \geq u \\
\iff \quad & -\left(\frac{x_0}{x}\right)^{\theta} \geq u - 1 \\
\iff \quad & \left(\frac{x_0}{x}\right)^{\theta} \leq 1 - u \\
\iff \quad & \frac{x_0}{x} \leq (1-u)^{1/\theta} \\
\iff \quad & \frac{x}{x_0} \geq ((1-u))^{-1/\theta} \\
\iff \quad & x \geq x_0 (1-u)^{-1/\theta}
\end{aligned}$$

as $x_0 > 0$ and $\theta > 0$. Now

$$\begin{aligned}
& 0 < u < 1 \\
\implies \quad & 0 > -u > -1 \\
\implies \quad & 1 > 1 - u > 0 \\
\implies \quad & 1 > (1-u)^{1/\theta} > 0 \\
\implies \quad & (1-u)^{-1/\theta} > 1
\end{aligned}$$

for all $u \in (0,1)$. So $F_X^{-1}(u; x_0, \theta) = x_0 (1-u)^{-1/\theta}$.

c) We can generate random numbers on $U(0,1)$. Let $U \sim U(0,1)$ and

$$Y = x_0 (1 - U)^{-\frac{1}{\theta}}.$$

Now let $g : (0,1) \longrightarrow (0, \infty)$ be defined by $g(u) = x_0 (1-u)^{1/\theta}$. Let $h = g^{-1}$ so $h(y) = 1 - \left(\frac{x_0}{y}\right)^{\theta}$. Hence

$$\begin{aligned}
f_Y(y) &= f_U(h(y))\, |h'(y)| \\
&= 1 \cdot \left| -(x_0^{\theta}) \cdot -\theta \cdot \frac{1}{y^{\theta+1}} \right| \\
&= \frac{\theta x_0^{\theta}}{y^{\theta+1}}
\end{aligned}$$

$y = x_0 ((1-u))^{-1/\theta}$
$(y/x_0)^{\theta} = 1 - u$
$u = 1 - \left(\frac{x_0}{y}\right)^{\theta}$

which is the Pareto distribution.