

PS1

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15:38

Q1, 6, 7, 9

1a) Fair coin tossed repeatedly

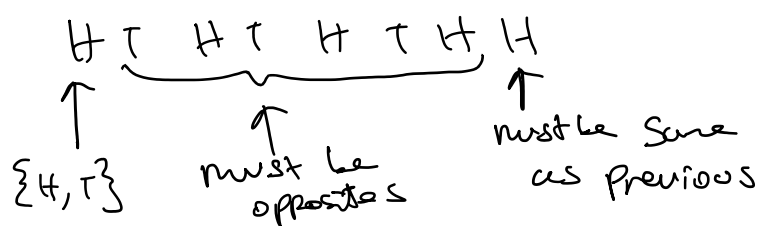
N number of tosses up to and including the first time 2 consecutive tosses are the same.

IP mass function of N ?

$$P(N=1) = 0$$

$$P(N=2) = \frac{1}{2}$$

$$P(N=3) = \frac{1}{4}$$



$$P(N=n) = 1 \times \frac{1}{2^{n-2}} \times \frac{1}{2} = \frac{1}{2^{n-1}} \text{ if } n \geq 2$$

$$P(N=n) = \frac{1}{2^{n-1}} \mathbb{I}(n \geq 2)$$

b) P N is even?

$$\{N \text{ even}\} = A$$

$$\text{Let } A_n = \{N=2\} \cup \{N=4\} \cup \dots \cup \{N=n\}.$$

So A_n is increasing and $A_n \uparrow A$.

So by continuity of P and Axiom 3 we have

$$\begin{aligned}
 P(A) &= \lim_{n \rightarrow \infty} P(A_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n P(A_i = 2^i) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2^{2^i-1}} = \frac{\sqrt{2}}{1-\sqrt{2}} = \frac{2}{3}
 \end{aligned}$$

c) P N is odd?

$$P(N \text{ odd}) = 1 - P(N \text{ even}) = 1 - \frac{2}{3} = \frac{1}{3}$$

6) X_1, \dots iid taking values 0, 1 each with $P \frac{1}{2}$.

$$S_n = \sum_{i=1}^n X_i \quad N = \min \{n: X_n = 1\} \leftarrow \text{First time we get } 1$$

Find $E(X_i)$, $E(N)$, $E(S_N)$, Does $E(S_N) = E(N)E(X_1)$ hold?

Are N , $(X_i)_{i \geq 1}$ independent?

$$E(X_i) = (0)P(X_i=0) + (1)P(X_i=1) = 0 + 1 \times \frac{1}{2} = \frac{1}{2}$$

$$\begin{aligned}
 E(N) &= (1)P(X_1=1) + \frac{E(N)}{2}P(X_1=0) \\
 &= 1 + \frac{1}{2}E(N)
 \end{aligned}$$

$$\Rightarrow E(N) = 2$$

$$E(S_N | N=n) = E(\sum_{i=1}^n X_i | N=n) = \sum_{i=1}^n E(X_i) = \frac{n}{2}$$

$$E(S_N) = E(E(S_N | N)) = E(\frac{N}{2}) = 1 \text{ by the tower rule}$$

$$E(N)E(X_1) = 2 \times \frac{1}{2} = 1 = E(S_N)$$

$$P(N=n \cap X_1=1) = \mathbb{I}(n=1)$$

$$P(N=n)P(X_1=1) = \frac{1}{2}P(N=n)$$

Taking $n=2$ we have $P(N=2 \cap X_1=1) = 0 < \frac{1}{2}P(N=2)$ since $P(N=2) > 0$. So N and $(X_i)_{i \geq 1}$ are not independent.

7) Same setup as 6). Let $M = N-1$. Find $E(M)$, $E(S_M)$. Does $E(S_M) = E(M)E(X_1)$. Are M , $(X_i)_{i \geq 1}$ independent.

$$\begin{aligned}
 E(M) &= E(N-1) = E(N) - 1 = 2 - 1 = 1 \\
 E(S_M) &= E(S_{N-1}) = E(S_N - X_N) = 1 - \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

$$E(M)E(X_1) = 1 \times \frac{1}{2} = \frac{1}{2} = E(S_M)$$

$$P(M=m \cap X_1=1) = \mathbb{I}(m=0)$$

$$P(M=m)P(X_1=1) = \frac{1}{2}P(M=m)$$

Again taking $m=2$ we see $P(M=2 \cap X_1=1) = 0 < \frac{1}{2}P(M=2)$. So M and $(X_i)_{i \geq 1}$ are not independent.

9) Parrot types random letters A-Z each with $P \frac{1}{26}$. Key presses are independent. Prove the P it eventually types PARROT is $\frac{1}{26}$.

Split the parrots text into blocks of 6 letters. Let A be the event the parrot never types a block that equals "PARROT". Let

$$A_n = \{ \text{first } n \text{ blocks don't equal "PARROT"} \}.$$

So $A_n \downarrow A$. So by the continuity of probability

$$P(A) = \lim_{n \rightarrow \infty} P(A_n) = \lim_{n \rightarrow \infty} \left(1 - \left(\frac{1}{26}\right)^6\right)^n = 0.$$

Hence the parrot types "PARROT" eventually almost surely.