PS2

Tuesday, 20 October 2020 18:26

Questions 3, 6, 7

3a) $f(\underline{r}) = \cos(xy) + \cos(yz)$

$\underline{\nabla} f = (-y\sin(xy), -x\sin(xy) - z\sin(yz), -y\sin(yz))$

$$\underline{\nabla} \times \underline{\nabla} f = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_x & \partial_y & \partial_z \\ -y\sin(xy) & -x\sin(xy) - z\sin(yz) & -y\sin(yz) \end{vmatrix}$$

$$= \begin{pmatrix} -\sin(yz) - yz\cos(yz) + zy\cos(yz) + \sin(yz) \\ 0 \\ -\sin(xy) - xy\cos(xy) + yx\cos(xy) + \sin(xy) \end{pmatrix}$$

$$= \underline{0}$$

b) $\underline{\nabla} \cdot \underline{U}(\underline{r}) = \underline{\nabla} \cdot (x\sin z, yz, \cos z)$
$= \sin z + z - \sin z$
$= z$

c) $\underline{\nabla} \times \underline{V}(\underline{r}) = \underline{\nabla} \times (ayz, bzx, cxy)$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_x & \partial_y & \partial_z \\ ayz & bzx & cxy \end{vmatrix}$$

$$= \begin{pmatrix} cx - bx \\ ay - cy \\ bz - az \end{pmatrix}$$

$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{V}) = \underline{\nabla} \cdot (cx - bx, ay - cy, bz - az)$
$= c - b + a - c + b - a$
$= 0$

6) $\underline{\nabla}(\underline{V} \times \underline{U}) = \underline{V} \cdot (\underline{\nabla} \times \underline{U}) + \underline{U} \cdot (\underline{\nabla} \times \underline{U})$
$= \underline{\nabla} \cdot (-\underline{U} \times \underline{V})$
$= -\underline{U} \cdot (\underline{\nabla} \times \underline{V}) + \underline{V} \cdot (\underline{\nabla} \times -\underline{U})$
$= -\underline{U} \cdot (\underline{\nabla} \times \underline{V}) - \underline{V} \cdot (\underline{\nabla} \times \underline{U})$

$\Rightarrow \underline{\nabla} \cdot (\underline{V} \times \underline{U}) = 0$ which isn't true.

$\nabla \cdot (\underline{U} \times \underline{V}) = \partial_i \varepsilon_{ijk} U_j V_k$
$= \varepsilon_{ijk} (\partial_i U_j) V_k + \varepsilon_{ijk} U_j (\partial_i V_k)$
$= V_k \varepsilon_{kij} (\partial_i U_j) - U_j \varepsilon_{jik} (\partial_i V_k)$
$= V_k \cdot (\underline{\nabla} \times \underline{U}) - U_j \cdot (\underline{\nabla} \times \underline{V})$.

7) $[\underline{\nabla} \times (\underline{U} \times \underline{V})]_i = \varepsilon_{ijk} \partial_j [\underline{U} \times \underline{V}]_k$
$= \varepsilon_{ijk} \partial_j \varepsilon_{klm} U_l V_m$
$= \varepsilon_{ijk} \varepsilon_{klm} \partial_j (U_l V_m)$
$= \varepsilon_{kij} \varepsilon_{klm} \partial_j (U_l V_m)$
$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(V_m \partial_j U_l + U_l \partial_j V_m)$
$= \delta_{il}\delta_{jm} V_m \partial_j U_l + \delta_{il}\delta_{jm} U_l \partial_j V_m$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j (U_l V_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(V_m \partial_j U_l + U_l \partial_j V_m)$$

$$= \delta_{il}\delta_{jm} V_m \partial_j U_l + \delta_{il}\delta_{jm} U_l \partial_j V_m - \delta_{im}\delta_{jl} V_m \partial_j U_l - \delta_{im}\delta_{jl} U_l \partial_j V_m$$

$$= V_j \partial_j U_i + U_i \partial_j V_j - V_i \partial_j U_j - U_j \partial_j V_i$$

$$= [(\underline{V}\cdot\underline{\nabla})\underline{U}]_i + [\underline{U}(\underline{\nabla}\cdot\underline{V})]_i - [\underline{V}(\underline{\nabla}\cdot\underline{U})]_i - [(\underline{U}\cdot\underline{\nabla})\underline{V}]_i$$

$$= [(\underline{\nabla}\cdot\underline{V})\underline{U} - (\underline{\nabla}\cdot\underline{U})\underline{V} + (\underline{V}\cdot\underline{\nabla})\underline{U} - (\underline{U}\cdot\underline{\nabla}\underline{V})]_i$$

Expression true for all elements so expression holds.