PS3

Sunday, 21 February 2021 15:42

Q 6, 8, 10

6) Let $X_1, X_2, \ldots$ be iid taking $+1, -1$ with equal probability. Let $S_n = \sum_{i=1}^n X_i$. Determine whether the following are stopping times for X.

(a) $T = \min\{n > 0 : \text{Exactly 3 of } X_1, \ldots, X_n \text{ are } +1\}$

Yes. $\{T \leq n\}$ clearly fully determined by $X_1, \ldots, X_n$

(b) $T = \min\{n > 0 : X_n = X_{n+1} = X_{n+2}\}$

No. Need X_{n+1}, X_{n+2} not in $X_1, \ldots, X_n$.

(c) $T = \min\{n \geq 3 : X_{n-2} = X_{n-1} = X_n\}$

Yes, as for $n \geq 3$, X_{n-2}, X_{n-1} are in $X_1 \ldots X_n$.

(d) $T = \min\{n > 0 : S_n > n/2\}$

Yes. $S_n = \sum_{i=1}^n X_i$ which is fully determined by $X_1, \ldots, X_n$.

(e) $T = \min\{n > 0 : S_n > 0 \text{ and } n \text{ is a multiple of } 10\}$

Yes as n being a multiple of 10 can determined from $X_1, \ldots, X_n$.

(f) $T = \min\{n > 0 : S_{10n} > 0\}$

No. $S_{10n} = \sum_{i=1}^{10n} X_i$. Contains X_{n+1} not in $X_1, \ldots, X_n$.

(g) $T = 7$.

Yes.

8) Suppose each child born equally likely to be a girl or a boy. A couple decide to keep having children until they get 3 consecutive boys. Then stop.
G number of girls.
B number of boys.
Show $\mathbb{E}(G) = \mathbb{E}(B)$. Is this surprising.

Let $X_1, X_2, \ldots$ be iid taking values $+1$ in the case of a boy, -1 in the case of a girl. So $S_n = \sum_{i=1}^n X_i$ is the number of boys minus the number of girls. Note that

$$\mathbb{E}(S_n) = \mathbb{E}(B_n - G_n) = \mathbb{E}(B_n) - \mathbb{E}(G_n)$$

So

$$[\mathbb{E}(S_n) = 0 \iff \mathbb{E}(B_n) = \mathbb{E}(G_n)].$$

We have stopping condition given by (6c).

By chunking born children into independent blocks of 3 we have the time until the first BBB block distributed to geometric (1/8). So

$$\mathbb{E}(N) \leq 3(1/8) = 24 < \infty.$$

Then as

$$\mathbb{E}(X_1) = \tfrac{1}{2}(1) + \tfrac{1}{2}(-1) = 0,$$

by Wald's identity

$$\mathbb{E}(S_T) = \mathbb{E}(T) \times 0 = 0$$

$$\implies \mathbb{E}(B) = \mathbb{E}(G)$$

Bit surprising since we only stop after 3 boys which intuitively means we'd expect expected number of boys to be higher.

10) Bank has initial wealth k.

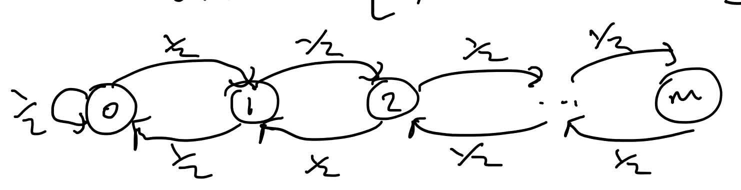
Plays independent game each day. Gains 1 with P p; Loses 1 with P $q = 1-p$

If wealth equals 0 can still play.
↳ Loses then bailed out - stays on 0
wins then increases to 1.

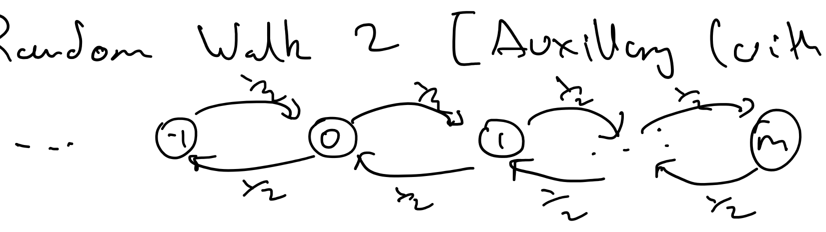
Let $m > k$, assume $p = 1/2$.

Find expected number of times bank bailed out when its wealth first reaches m.

Random Walk 1 [Actual Bank]:



Random Walk 2 [Auxillary (with debt)]



Notice that the number of times the bank is bailed out when its wealth first reaches m is the difference in the time taken for each random walk to reach m. For the first random walk this time was computed in PS2, Q11 to be

$$m(m+1) - k(k+1) \in \mathbb{Z}.$$

By a similar argument to the final example of chapter 2 in the notes the expectation of the time taken to reach m is ∞. So the difference and thus the number of times the bank is bailed out is ∞.