PS4

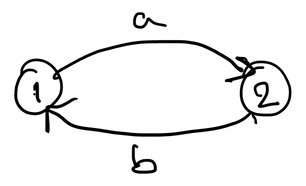
Monday, 1 March 2021 15:00

Q3, 6, 8a

3) Consider a Markov chain with state space $\{1,2\}$ such that $P_{1,2} = a$, $P_{2,1} = b$ where $a, b \in (0,1)$. Write down the full transition matrix. Using the Chapman – Kolmogorov equations show by induction that

$$P^n_{1,1} = \frac{a(1-a-b)^n + b}{a+b}.$$

Find the other n-step transition probabilities.



$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

<u>$n=1$</u>

$$P^1_{1,1} = \frac{a(1-a-b)+b}{a+b} = \frac{a - a^2 - ab + b}{a+b} = \frac{(-a+1)(a+b)}{a+b} = 1-a$$

<u>$n = k+1$</u> $(k \in \mathbb{N})$

$$\begin{aligned}
P^{(k+1)}_{1,1} &= P^k_{1,1} P_{1,1} + P^k_{1,2} P_{2,1} \qquad \text{[Chapman-Kolmogorov]} \\
&= P^k_{1,1} P_{1,1} + (1 - P^k_{1,1}) P_{2,1} \qquad [P^k_{1,1} + P^k_{1,2} = 1] \\
&= P^k_{1,1}(P_{1,1} - P_{2,1}) + P_{2,1} \\
&= P^k_{1,1}(1-a-b) + b \\
&= \frac{a(1-a-b)^k + b}{a+b}(1-a-b) + b \\
&= \frac{a(1-a-b)^{k+1} + b(1-a-b) + b(a+b)}{a+b} \\
&= \frac{a(1-a-b)^{k+1} + b}{a+b}.
\end{aligned}$$

So assuming the statement holds for k, it holds for $k+1$. As also true for $k=1$, statement true for all $k \in \mathbb{N}$ by induction.

By symmetry,

$$P^n_{2,2} = \frac{b(1-a-b)^n + a}{a+b}.$$

Then using the fact that rows must sum to 1 we get

$$P^n = \begin{pmatrix} \frac{a(1-a-b)^n + b}{a+b} & 1 - \frac{a(1-a-b)^n + b}{a+b} \\ 1 - \frac{b(1-a-b)^n + a}{a+b} & \frac{b(1-a-b)^n + a}{a+b} \end{pmatrix}.$$

6) You have €3 but want €8. Play sequence of independent games each of which has $p = 0.4$ probability of winning. You can stake any amount €k up to your current wealth. Win then gain extra €k, otherwise lose your stake.

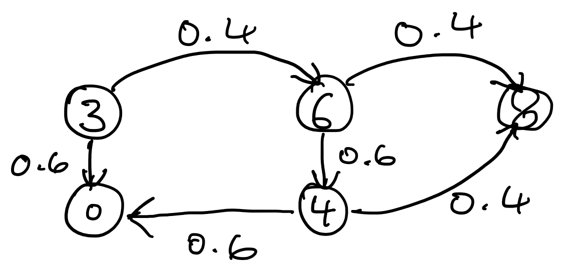
Bold strategy: Stake max amount not taking wealth strictly above €8 if win.

Find P you reach €8 before bankrupt.

Cautious strategy: Always stake €1.

Compare.

<u>Bold</u>



$$\mathbb{P}(\text{reach €8 before bankrupt}) = 0.4 \times 0.4 + 0.4 \times 0.6 \times 0.4$$
$$= 0.256$$

<u>Cautious</u>

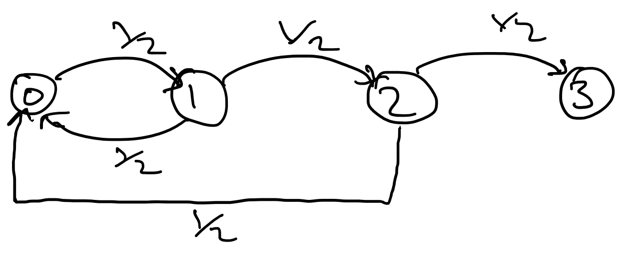
This is a SRW with absorbing barriers at 0 and 8 starting at 3. By the standard formula

$$\mathbb{P}(\text{reach €8 before bankrupt}) = \frac{1 - \left(\frac{0.6}{0.4}\right)^3}{1 - \left(\frac{0.6}{0.4}\right)^8} = 0.0964 \text{ (3sf)}.$$

So there is a much greater probability of reaching €8 before going bankrupt by using the bold strategy over the cautious strategy.

8a) A fair coin is tossed repeatedly. By formulating the appropriate Markov chain find the expected number of tosses until the first occurence of HHH.

Let X_n be the number of consecutive correct guesses we've just had ending at the n'th toss.



Assume $P_{3,3} = 1$. Let

$$t_i = \mathbb{E}_i(\text{time to hit } 3)$$

the expected time to hit 3 given we're in state i. So

$$\left.\begin{aligned}
t_0 &= 1 + \tfrac{1}{2} t_0 + \tfrac{1}{2} t_1 \\
t_1 &= 1 + \tfrac{1}{2} t_0 + \tfrac{1}{2} t_2 \\
t_2 &= 1 + \tfrac{1}{2} t_0 + \tfrac{1}{2} t_3 \\
t_3 &= 0
\end{aligned}\right\}$$

$$\Rightarrow \tfrac{1}{2} t_0 = 1 + \tfrac{1}{2}\left(1 + \tfrac{1}{2} t_0 + \tfrac{1}{2}\left(1 + \tfrac{1}{2} t_0 + \tfrac{1}{2}(0)\right)\right)$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{4} t_0 + \tfrac{1}{4} + \tfrac{1}{8} t_0$$
$$= \tfrac{3}{8} t_0 + \tfrac{7}{4}$$
$$\Rightarrow t_0 = 14.$$