PS5

Monday, 9 November 2020 11:10

Q4, 5, 6

4a) $\underline{F} = (x, y, z^4)$

S conical surface formed by $z = \sqrt{x^2+y^2}$ $\quad 0 < z < 1$

$S = \{(x, y, \sqrt{x^2+y^2}) : 0 < \sqrt{x^2+y^2} < 1\}$

Use cylindrical co-ordinates

$S(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$D = \{(r\cos\theta, r\sin\theta, r) : 0 < r < 1, 0 \leq \theta < 2\pi\}$

$$\underline{N} = \frac{\partial \underline{S}}{\partial r} \times \frac{\partial \underline{S}}{\partial \theta} = (\cos\theta, \sin\theta, 1) \times (-r\sin\theta, r\cos\theta, 0)$$

$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (-r\cos\theta, -r\sin\theta, r\cos^2\theta + r\sin^2\theta)$$

$$= (-r\cos\theta, -r\sin\theta, r)$$ So need minus

$$\int_S \underline{F} \cdot \partial \underline{S} = -\int_D (r\cos\theta, r\sin\theta, r^4) \cdot (-r\cos\theta, -r\sin\theta, r)\, \partial r\, \partial\theta$$

$$= -\int_0^{2\pi}\int_0^1 -r^2\cos^2\theta - r^2\sin^2\theta + r^5 \, \partial r\, \partial\theta$$

$$= -\int_0^{2\pi}\int_0^1 -r^2(\sin^2\theta + \cos^2\theta) + r^5 \, \partial r\, \partial\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2 - r^5 \, \partial r\, \partial\theta$$

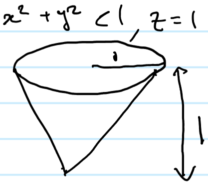
$$= \int_0^{2\pi} \frac{1}{3}r^3 - \frac{1}{6}r^6 \Big|_0^1 \, \partial\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\, \partial\theta$$

$$= \frac{\pi}{3}$$

b) $$\int_V \underline{\nabla} \cdot \underline{F} \, \partial V = \int_{\partial V} \underline{F} \cdot \partial \underline{S} = \frac{\pi}{3}$$

c) $\int_{x^2+y^2<1,\, z=1} \underline{F} \cdot \partial \underline{S}$ = Area of circle radius 1 = π



Check: lid of cone $S(r,\theta) = (r\cos\theta, r\sin\theta, 1)$

$D = \{(r,\theta) : 0 < r < 1, 0 \leq \theta < 2\pi\}$

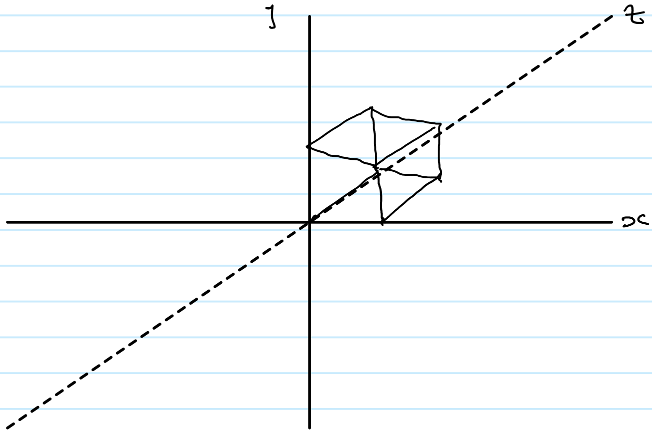
$$\underline{N} = (\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0)$$
$$= (0, 0, r)$$

$$\int \underline{F} \cdot \partial \underline{S} = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, 1) \cdot (0, 0, r)\, \partial r\, \partial\theta$$

$$\int_{x^2+y^2<1,\, z=1} \underline{F} \cdot \partial \underline{S} = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, 1)\cdot(0,0,r)\, \partial r\, \partial\theta$$
$$= \int_0^{2\pi}\int_0^1 r\, \partial r\, \partial\theta$$
$$= \int_0^{2\pi} \tfrac{1}{2} r^2 \Big|_0^1 \partial\theta$$
$$= \int_0^{2\pi} \tfrac{1}{2}\, \partial\theta$$
$$= \tfrac{1}{2}\theta \Big|_0^{2\pi}$$
$$= \pi$$

5) $\int_{\partial V} \underline{F} \cdot \partial \underline{S}$

$\partial V = \{(x,y,z) : 0<x<1,\ 0<y<1,\ 0<z<1\}$



6 planes each with different $\hat{\underline{n}}$. Assume $\hat{\underline{n}}$ points outwards.

$\hat{\underline{n}} = (1,0,0)$, $x=1 \Rightarrow \int_0^1\int_0^1 (1,y,-z)\cdot(1,0,0)\, \partial y\, \partial z = 1$

$\hat{\underline{n}} = (-1,0,0)$, $x=0 \Rightarrow \int_0^1\int_0^1 (0,y,-z)\cdot(-1,0,0)\, \partial y\, \partial z = 0$

$\hat{\underline{n}} = (0,1,0)$, $y=1 \Rightarrow \int_0^1\int_0^1 (x,1,-z)\cdot(0,1,0)\, \partial x\, \partial z = 1$

$\hat{\underline{n}} = (0,-1,0)$, $y=0 \Rightarrow \int_0^1\int_0^1 (x,0,-z)\cdot(0,-1,0)\, \partial x\, \partial z = 0$

$\hat{\underline{n}} = (0,0,1)$, $z=1 \Rightarrow \int_0^1\int_0^1 (x,y,-1)\cdot(0,0,1)\, \partial x\, \partial y = -1$

$\hat{\underline{n}} = (0,0,-1)$, $z=0 \Rightarrow \int_0^1\int_0^1 (x,y,0)\cdot(0,0,-1)\, \partial x\, \partial y = 0$

So $\int_{\partial V} \underline{F} \cdot \partial \underline{S} = 1+1-1 = 1$.

Check with divergence theorem:

$\underline{\nabla} \cdot \underline{F} = 1+1-1 = 1$

$$\int_{\partial V} \underline{F} \cdot \partial \underline{S} = \int_V \underline{\nabla} \cdot \underline{F}\, \partial V = \int_0^1\int_0^1\int_0^1 1\, \partial x\, \partial y\, \partial z = 1$$

6)
$$\int_V \Delta\left(\tfrac{1}{r}\right) \partial V = \int_V \nabla \cdot \nabla \cdot \left(\tfrac{1}{r}\right) \partial V$$
$$= \int_{\partial V} \nabla \cdot \left(\tfrac{1}{r}\right) \cdot \partial \underline{S}$$
$$= \int_{\partial V} -\frac{\hat{\underline{r}}}{r^2} \cdot \partial \underline{S}$$

Sperical co-ordinates. Let $R \in \mathbb{R}_{>0}$.

$$S(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$$

$$D = \{(\phi, \theta) : 0 \le \phi < \pi, \; 0 \le \theta < 2\pi\}$$

$$\underset{\sim}{N}(\phi, \theta) = \frac{\partial S}{\partial \phi} \times \frac{\partial S}{\partial \theta}$$
$$= (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi) \times (-R\sin\theta\sin\phi, R\sin\phi\cos\theta, 0)$$
$$= \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\theta\sin\phi & R\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= (R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2(\cos\phi\cos\theta\sin\phi\cos\theta + \cos\phi\sin^2\theta\sin\phi))$$
$$= (R^2\sin^2\phi\cos\theta, R^2\sin^2\phi\sin\theta, R^2\cos\phi\sin\phi)$$
$$= R^2\sin\phi\,\hat{\underline{r}}$$

$$\int_V \Delta\left(\tfrac{1}{r}\right) \partial V = \int_D -\frac{\hat{\underline{r}}}{R^2} \cdot R^2\sin\phi\,\hat{\underline{r}}\;\partial\theta\,\partial\phi = -\int_0^{\pi}\int_0^{2\pi} \sin\phi\,d\theta\,d\phi$$
$$= -\int_0^{\pi} 2\pi\sin\phi\,d\phi \qquad = 2\pi\cos\phi\Big|_0^{\pi}$$
$$= -2\pi - 2\pi \qquad = -4\pi$$

No idea - I'll trust the algebra ☺