ES3(Q21)

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R ring. $x \in R$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.

(1) Prove the set of nilpotents of R is an ideal of R.

Let N be the set of nilpotents of R. First, $0 = 0^1 \in N$. Let $n_1, n_2 \in N$. So $\exists i_1, i_2 \in \mathbb{N}$ such that $n_1^{i_1} = 0$, $n_2^{i_2} = 0$. We have

$$(n_1 + n_2)^{i_1+i_2} = \sum_{k=0}^{i_1+i_2} \binom{i_1+i_2}{k} n_1^{i_1+i_2-k} n_2^k$$

by the binomial theorem. Now for $0 \leq k \leq i_2$, $i_1 + i_2 - k \geq i_1$ so $n_1^{i_1+i_2-k} = 0$. For $i_2 \leq k \leq i_1 + i_2$, $n_2^k = 0$. Hence

$$(n_1 + n_2)^{i_1+i_2} = 0 .$$

Also

$$(n_1 - n_2)^{i_1+i_2} = (n_1 + (-1)n_2)^{i_1+i_2} = \sum_{k=0}^{i_1+i_2} \binom{i_1+i_2}{k} n_1^{i_1+i_2-k} (-1)^k n_2^k = 0$$

by the same argument. So N is an additive subgroup of R. Let $r \in R$. Then

$$(r n_1)^{i_1} = r^{i_1} n_1^{i_1} = r^{i_1} 0 = 0 .$$

So N is an ideal of R.

(2) Which of the following have non-zero nilpotent elements? Explain?

$$\mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{R} \times \mathbb{R},\ \mathbb{Z}/77\mathbb{Z},\ \mathbb{Z}/8\mathbb{Z}$$

First note that any nilpotent element is a zero-divisor since for any nilpotent $x \in R$, $\exists n \in \mathbb{N}$ such that

$$0 = x^n = x \times x^{n-1}$$

and $x^{n-1} \in R$.

Now $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are integral domains which means they have no zero-divisors and hence no nilpotent elements.

$\mathbb{R} \times \mathbb{R}$ is not an integral domain. Its zero-divisors take the form $(x, 0)$ or $(0, x)$ for $x \in \mathbb{R}$. However these can't be nilpotent since the reals contain no nilpotents.

$\mathbb{Z}/77\mathbb{Z}$ has zero-divisors $\overline{7}$, $\overline{11}$ but neither of these are nilpotent.

$\mathbb{Z}/8\mathbb{Z}$ has zero-divisors $\overline{2}$, $\overline{4}$. We have $(\overline{2})^3 = \overline{0}$ so $\mathbb{Z}/8\mathbb{Z}$ contains a non-zero nilpotent.