PS3

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1a) $\underline{X} \overset{iid}{\sim} \text{Exp}(\lambda)$ where $\Theta = \mathbb{R}_{++}$. $\tau = \frac{1}{1+\lambda}$.

First let's calculate the likelihood and log likelihood in terms of λ.

$$L(\lambda; \underline{x}) = C f_n(\underline{x}; \lambda)$$
$$= C \prod_{i=1}^{n} f(x_i; \lambda)$$
$$= C \prod_{i=1}^{n} \lambda \exp(-\lambda x_i)$$

$$\ell(\lambda; \underline{x}) = \log C + \sum_{i=1}^{n} \log \lambda + \sum_{i=1}^{n} -\lambda x_i$$
$$= \log C + n \log \lambda - n \lambda \bar{x}$$

Now we need to write this function in terms of τ. We have

$$\tau = \frac{1}{1+\lambda} \implies \lambda = \frac{1-\tau}{\tau}.$$

So

$$\ell(\lambda; \underline{x}) = \log C + n \log \frac{1-\tau}{\tau} - n\bar{x} \frac{1-\tau}{\tau}.$$

Now we can take the partial derivative w.r.t τ to get our MLE estimate. We have

$$\partial_\tau \ell(\lambda; \underline{x}) = n\left(\frac{\tau}{1-\tau}\right)\left(-\frac{1}{\tau^2}\right) + \frac{n\bar{x}}{\tau^2}$$
$$= -\frac{n}{\tau(1-\tau)} + \frac{n\bar{x}}{\tau^2}.$$

So

$$\partial_\tau \ell(\lambda; \underline{x}) = 0 \implies \frac{\bar{x}}{\tau^2} = \frac{1}{\tau(1-\tau)}$$
$$\implies \bar{x} = \frac{\tau}{(1-\tau)}$$
$$\implies \bar{x} - \tau\bar{x} = \tau$$
$$\implies \tau = \frac{\bar{x}}{1+\bar{x}}.$$

Since $\hat{\lambda} = \frac{1}{\bar{x}}$, we have

$$\hat{\tau} = \frac{\hat{\lambda}^{-1}}{1+\hat{\lambda}^{-1}} = \frac{1}{1+\hat{\lambda}}.$$

b) $X \sim \text{Poisson}(\lambda)$ where $\Theta = \mathbb{R}_{++}$. We already know $\hat{\lambda}$ is the ML estimate of λ. By the invariance under bijective transformation theorem, as $g(\lambda) = \exp(-\lambda)$ is a bijective function $\hat{P}_x(0; \lambda) = g(\hat{\lambda}) = \exp(-\hat{\lambda})$.

2a) $X_1, X_2, \ldots \overset{iid}{\sim} \text{Uniform}(0,1)$. Each has density $f(x) = \mathbb{I}(0 \le x \le 1)$.

We have that

$$\mathbb{E}(\log(X_1)) = \int_{-\infty}^{\infty} \log(x)\, \mathbb{I}(0 \le x \le 1)\, dx$$
$$= \int_0^1 \log(x)\, dx$$
$$= x \log x - x \Big|_0^1$$

$$= x\log x - x \Big|_0^1$$
$$= -1 - \lim_{x\to 0^+} \frac{x}{\log x} .$$

Now by L'hôpital's rule

$$\lim_{x\to 0^+} \frac{x}{\log x} = \lim_{x\to 0^+} \frac{1}{1/x} = \lim_{x\to 0^+} x = 0 .$$

Hence

$$\mathbb{E}(\log(X_1)) = -1 .$$

Now

$$\mathbb{E}(\log(X_1)^2) = \int_{-\infty}^{\infty} \log(x)^2 \, \mathbb{1}(0 \leq x \leq 1)\, dx$$
$$= \int_0^1 \log(x)^2 \, dx$$
$$= x\log(x)^2 \Big|_0^1 - \int_0^1 2\log x \, dx$$
$$= x\log(x)^2 \Big|_0^1 + 2$$
$$= -\lim_{x\to 0^+} \frac{x}{(\log(x))^2} + 2 .$$

Again by L'hôpital

$$\lim_{x\to 0^+} \frac{x}{((\log x))^2} = \lim_{x\to 0^+} \frac{1}{2\log(x)\frac{1}{x}} = \frac{1}{2} \lim_{x\to 0^+} \frac{x}{\log x} = \frac{1}{2}\cdot 0 = 0 .$$

So

$$\begin{aligned} \mathrm{Var}(\log X_1) &= \mathbb{E}(\log(X_1)^2) - \mathbb{E}(\log(X_1))^2 \\ &= 2 - (-1)^2 \\ &= 1 . \end{aligned}$$

b) We have $(X_1 \cdots X_n)^{e^{\sqrt{n}}/\sqrt{n}}$. Taking logs we get $\frac{e^{\sqrt{n}}}{\sqrt{n}} \sum_{i=1}^{n} \log X_i$.
We know by the CLT

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \log X_i + 1\right) \xrightarrow{\mathcal{D}} N(0,1)$$

$$\Rightarrow \quad \sum_{i=1}^n \log X_i \xrightarrow{\mathcal{D}} n\left(\frac{1}{\sqrt{n}} N(0,1) - 1\right)$$

$$\Rightarrow \quad \frac{e^{\sqrt{n}}}{\sqrt{n}} \sum_{i=1}^n \log X_i \xrightarrow{\mathcal{D}} \frac{n e^{\sqrt{n}}}{\sqrt{n}}\left(\frac{1}{\sqrt{n}} N(0,1) - 1\right)$$

$$\Rightarrow \quad \frac{e^{\sqrt{n}}}{\sqrt{n}} \sum_{i=1}^n \log X_i \xrightarrow{\mathcal{D}} e^{\sqrt{n}}\left(N(0,1) - \sqrt{n}\right)$$

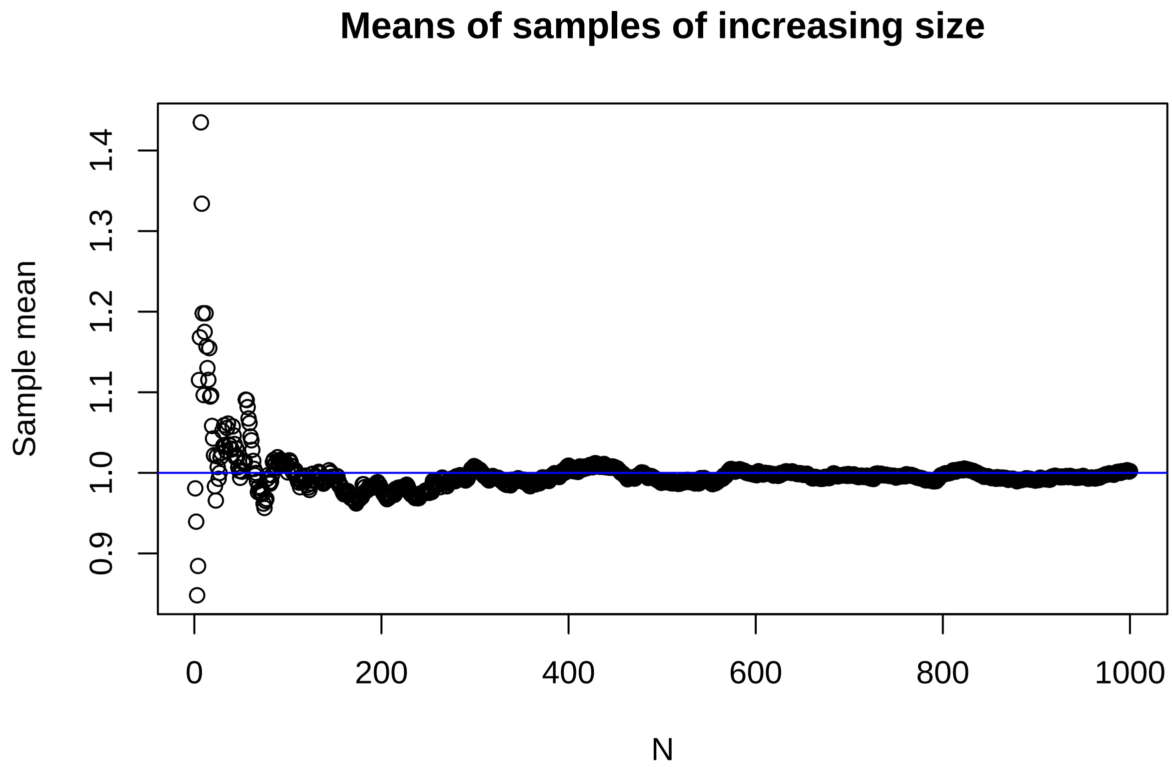
$$\Rightarrow \quad (X_1 \cdots X_n)^{\frac{e^{\sqrt{n}}}{\sqrt{n}}} \xrightarrow{\mathcal{D}} \exp\left(e^{\sqrt{n}}(N(0,1) - \sqrt{n})\right) .$$

Problem Sheet 3

Problem Sheet 3

Question 2

```r
N <- 1000
alpha <- 2
lambda <- 2
X <- rgamma(N, rate = lambda, shape=alpha)
Xbar <- cumsum(X) / 1:N
plot(1:N, Xbar, main = "Means of samples of increasing size",
     xlab = "N", ylab = "Sample mean")
abline(h=alpha/lambda, col = "blue")
```



Question 3

```r
N <- 1000
alpha <- 2
lambda <- 2
X <- rgamma(N, shape = alpha, rate = lambda)
Fx <- mean(X <= 1.5)
print(Fx)
```

```
## [1] 0.807
```

```r
pgamma(1.5, shape = alpha, rate = lambda)
```

```
## [1] 0.8008517
```

```r
print("---------------")
```

```
## [1] "---------------"
```

```r
Fx <- mean(0.5 <= X & X <= 1.5)
Fx
```

```
## [1] 0.53
```

```r
pgamma(1.5, shape = alpha, rate = lambda) - pgamma(0.5, shape = alpha, rate = lambda)
```

```
## [1] 0.5366106
```