PS7

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1a) Consider random variables X_i for $i=1,\dots,12$ representing the 12 bulbs. We have $\underline{X} \overset{iid}{\sim}$ Bernoulli(p) such that $X_i = 1$ means the ith bulb is red and $X_i = 0$ means the ith bulb is yellow. Then $p \in \Theta = \{0.6, 0.25\}$. We have hypotheses

$$H_0: p = 0.25$$
$$H_1: p = 0.6$$

and critical region

$$R = \{\underline{x} : T(\underline{x}) \geq 8\}$$

where our critical value is 8 and $T(\underline{x})$ is the number of red bulbs which is the sum $\sum_{i=1}^{12} X_i$

b) We have the power function

$$\pi(p; T, c) = \mathbb{P}(T(\underline{x}) \geq c; p)$$
$$= \mathbb{P}(\textstyle\sum_{i=1}^{12} X_i \geq 8; p).$$

By the definition of the binomial distribution $Y = \sum_{i=1}^{12} X_i \sim$ Binomial$(12, p)$. So

$$\pi(p; t, c) = \mathbb{P}(Y \geq 8; p)$$
$$= 1 - \mathbb{P}(Y \leq 7; p)$$
$$= 1 - \text{pbinom}(7, 12, p).$$

The significance level is given by

$$\alpha = \sup_{\theta \in \Theta_0} \pi(p; t, c) = \pi(0.25) = 0.00278151.$$

The type II error is given by

$$1 - \pi(0.6) = 0.5618218$$

c) In R

d) We have

$$T_{NP}(\underline{x}) = f_n(\underline{x}; 0.6) / f_n(\underline{x}; 0.25) = \prod_{i=1}^{n} f(x_i; 0.6)/f(x_i; 0.25)$$

as $\underline{X} \overset{iid}{\sim} f_n(\underline{x}; p)$. Let $t(x_i) = f(x_i; 0.6)/f(x_i; 0.25)$. So

$$t(x_i) = 0.6^{x_i} 0.4^{1-x_i} / 0.25^{x_i} 0.75^{1-x_i}$$
$$= \left(\frac{0.6}{0.25}\right)^{x_i} \left(\frac{0.4}{0.75}\right)^{1-x_i}$$
$$= \left(\frac{12}{5}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$
$$= \left(\frac{9}{2}\right)^{x_i} \left(\frac{8}{15}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$

$$= \left(\frac{9}{2}\right)^{x_i} \left(\frac{8}{15}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$

$$= \frac{8}{15} \left(\frac{9}{2}\right)^{x_i}.$$

So

$$T_{NP}(\underline{x}) = \prod_{i=1}^{12} \frac{8}{15} \left(\frac{9}{2}\right)^{x_i} = \left(\frac{8}{15}\right)^{12} \left(\frac{9}{2}\right)^{\sum_{i=1}^{12} x_i}$$

which is an equivilent test statistic to $\sum_{i=1}^{12} x_i$ as $\left(\frac{8}{15}\right)^{12} \in \mathbb{R}$ and $\left(\frac{9}{2}\right)^{\sum_{i=1}^{12} x_i}$ is bijective and increasing in $\sum_{i=1}^{12} x_i$.

e) Let $T(\underline{x}) = \sum_{i=1}^{n} X_i$ and $\pi(p, T, c) = \mathbb{P}(T(\underline{X}) \geq c; p)$. We know $X = \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$. So we need

$$\pi(0.25) = 0.05, \quad [1 - \pi(0.6) = 0.1 \iff \pi(0.6) = 0.9].$$

or

$$\pi(0.6) - \pi(0.25) \approx 0.85$$

$$\Rightarrow g(n, c) := \pi(0.6) - \pi(0.25) - 0.85 \approx 0.$$

Using R we find our solution to be $n = 15$, $c = 7$.

2a) $f(x) = \frac{\lambda_1^x e^{-\lambda_1}}{x!} \Big/ \frac{\lambda_0^x e^{-\lambda_0}}{x!} = \left(\frac{\lambda_1}{\lambda_0}\right)^x e^{\lambda_0 - \lambda_1}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} \left(\frac{\lambda_1}{\lambda_0}\right)^{x_i} e^{\lambda_0 - \lambda_1} = \left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{x}} e^{n(\lambda_0 - \lambda_1)}$

Now $0 < \lambda_1 < \lambda_0$, $T(\underline{x})$ is increasing in $-\bar{x}$, So take $-\bar{x}$ as an equivient test statistic.

b) $f(x) = \lambda_1 e^{-\lambda_1 x_i} / \lambda_0 e^{-\lambda_0 x_i} = \frac{\lambda_1}{\lambda_0} e^{(\lambda_0 - \lambda_1)x_i}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} \frac{\lambda_1}{\lambda_0} e^{(\lambda_0 - \lambda_1)x_i} = \left(\frac{\lambda_1}{\lambda_0}\right)^n e^{(\lambda_0 - \lambda_1) n\bar{x}}.$

Now $0 < \lambda_0 < \lambda_1$ so $T(\underline{x})$ is again increasing in $-\bar{x}$ so $-\bar{x}$ is an equivilent test statistic.

c) $f(\underline{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu_1}{\sigma}\right)^2} \Big/ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu_0}{\sigma}\right)^2} = e^{-\frac{1}{2} \frac{(x-\mu_1)^2 - (x-\mu_0)^2}{\sigma^2}}$

$$= \exp\left(-\frac{1}{2\sigma^2}\left[x^2 - 2x\mu_1 + \mu_1^2 - x^2 + 2x\mu_0 - \mu_0^2\right]\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}(-2x\mu_1 + 2x\mu_0)\right) \exp\left(-\frac{1}{2\sigma^2}(\mu_1^2 - \mu_0^2)\right)$$

$$= \exp\left(\frac{x}{\sigma^2}(\mu_1 - \mu_0)\right) \exp\left(-\frac{1}{2\sigma^2}(\mu_1^2 - \mu_0^2)\right)$$

$T(\underline{x}) = \exp\left(-\frac{1}{2\sigma^2}(\mu_1^2 - \mu_0^2)\right)^n \exp\left(\frac{n\bar{x}}{\sigma^2}(\mu_1 - \mu_0)\right)$

Now $\mu_1 > \mu_0$ So T increasing in $\bar{x}$ so $\bar{x}$ is an equivilent test statistic.

d) $f(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2} \Big/ \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_0}\right)^2}$

$$= \frac{\sigma_0}{\sigma_1} \exp\left(-\frac{(x-\mu)^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right)$$

$$= \frac{\sigma_0}{\sigma_1} \exp\left(-\frac{(x-\mu)^2}{2} \frac{\sigma_0^2 - \sigma_1^2}{(\sigma_0\sigma_1)^2}\right)$$

$$= \frac{\sigma_0}{\sigma_1} \exp\left(\frac{(x-\mu)^2}{2} \frac{\sigma_1^2 - \sigma_0^2}{(\sigma_0\sigma_1)^2}\right)$$

$$T(\underline{x}) = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left(\sum_{i=1}^n (x_i - \mu)^2 \cdot \frac{\sigma_1^2 - \sigma_0^2}{\sigma_0 \sigma_1}\right)$$

Now $0 < \sigma_0^2 < \sigma_1^2$ so $T(\underline{x})$ is increasing in $\sum_{i=1}^n (x_i - \mu)^2$ so $\sum_{i=1}^n (x_i - \mu)^2$ is an equivalent test statistic.

Problem Sheet 7

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Question 1b

```
power <- function (p) 1-pbinom(7,12,p)

alpha <- power(0.25)
prob.type.II.error <- 1 - power(0.6)

alpha
```

```
## [1] 0.00278151
```

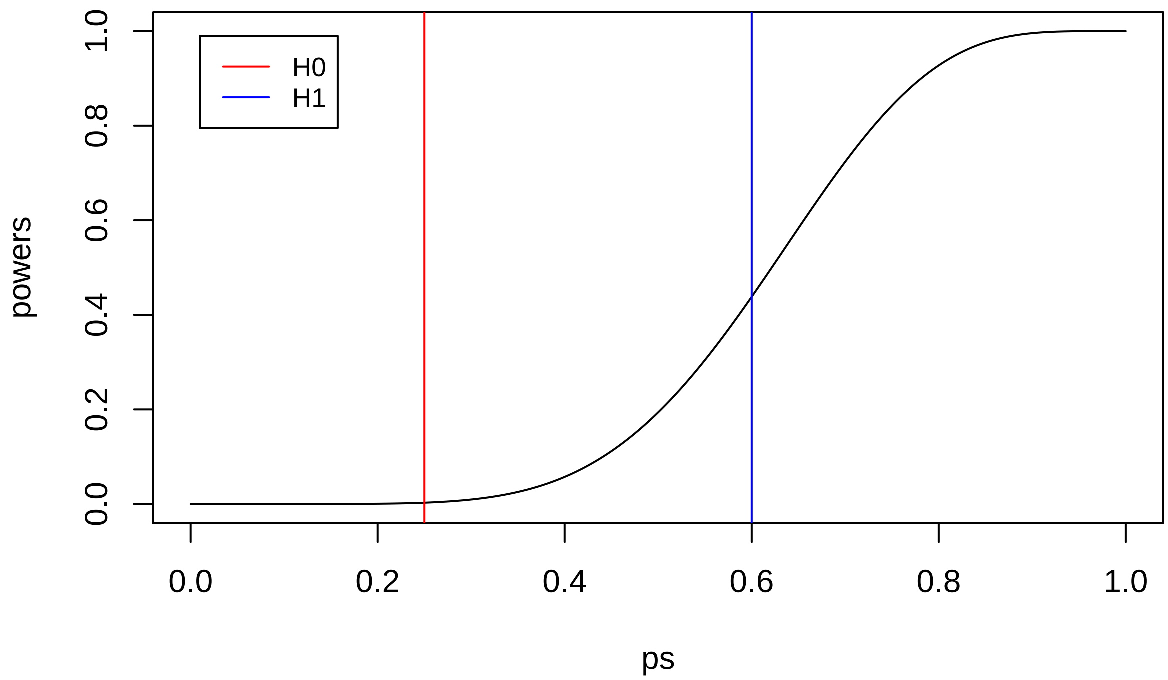
```
prob.type.II.error
```

```
## [1] 0.5618218
```

Question 1c

```
ps <- seq(0,1,0.001)
powers <- power(ps)

plot(ps, powers, type="l")
abline(v=c(0.25, 0.6), col=c("red", "blue"))
legend(x=0.01, y=0.99, legend=c("H0", "H1"),
       col=c("red", "blue"), lty=1, cex=0.8)
```



Question 1e

```
power <- function (p, n, c) 1-pbinom(c-1,n,p)
g <- function (n, c) power(0.6, n, c) - power(0.25, n, c) - 0.85

n.final <- 0
c.final <- 0
e.final <- Inf

for (n in seq(1, 50, 1)) {
  for (c in seq(1, 50, 1)) {
    e <- abs(g(n, c))
    if (e < e.final) {
      n.final <- n
      c.final <- c
      e.final <- e
    }
  }
}

c(n.final, c.final, e.final)
```

```
## [1] 15.000000000  7.000000000  0.001667718
```