PS1

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1a) $\underline{X} \overset{iid}{\sim} \text{Poisson}(\lambda)$ $\qquad f_X(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$L(\lambda; \underline{x}) \propto \prod_{i=1}^n f_X(x_i;\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$\ell(\lambda; \underline{x}) = \log \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^n \log \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \sum_{i=1}^n \left(-\lambda + \log \frac{\lambda^{x_i}}{x_i!}\right) = -\lambda n + \sum_{i=1}^n (x_i \log\lambda - \log x_i!)$$

$$= -\lambda n + \log\lambda \cdot (n\bar{x}) - \sum_{i=1}^n \log x_i!$$

$$= n\bar{x}\log\lambda - \lambda n - \sum_{i=1}^n \log x_i!$$

b) $X \sim \text{Binomial}(n, p)$ $\qquad P(x; p, n) = \binom{n}{x} p^x (1-p)^{n-x}$

$$L(n, p; x) \propto \binom{n}{x} p^x (1-p)^{n-x}$$

$$\ell(n, p; x) = \log \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \log\binom{n}{x} + x\log p + (n-x)\log(1-p)$$

c) $\underline{X} \overset{iid}{\sim} N(\mu, \sigma^2)$ $\qquad f_X(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$

$$L(\mu, \sigma^2; \underline{x}) \propto \prod_{i=1}^n f_X(x_i; \mu, \sigma^2)$$

$$= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp \sum_{i=1}^n -\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$$

$$\ell(\mu, \sigma^2; \underline{x}) = -n\log\sigma\sqrt{2\pi} - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$$

d) $\underline{X} \overset{iid}{\sim} U(a, b)$ $\qquad f_X(x; a, b) = \mathbb{I}\{x \in [a,b]\} \cdot \frac{1}{b-a}$

$$L(a, b; \underline{x}) \propto \prod_{i=1}^n \mathbb{I}\{x \in [a,b]\} \frac{1}{b-a}$$

$$= \left(\frac{1}{b-a}\right)^n \prod_{i=1}^n \mathbb{I}\{x \in [a,b]\}$$

$$\ell(a, b; \underline{x}) = -n\log(b-a) + \sum_{i=1}^n \log \mathbb{I}\{x \in [a,b]\}$$

2) $X_1, X_2, X_3 \overset{iid}{\sim} \text{Exp}(\lambda)$.

Consider $Y = \min\{X_1, X_2, X_3\}$. Now

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\min\{X_1, X_2, X_3\} \le y)$$
$$= 1 - \mathbb{P}(X_1 > y \cap X_2 > y \cap X_3 > y).$$

As X_1, X_2, X_3 are iid we have

$$F_Y(y) = 1 - \mathbb{P}(X_1 > y)\mathbb{P}(X_2 > y)\mathbb{P}(X_3 > y)$$
$$= 1 - (1 - F_{X_1}(y))^3$$
$$= 1 - (1 - (1 - e^{-\lambda y}))^3$$
$$= 1 - e^{-3\lambda y}.$$

Hence

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y}(1 - e^{-3\lambda y}) = 3\lambda e^{-3\lambda y}.$$

So

$$L(\lambda; 2.5) \propto 3\lambda e^{-7.5\lambda} \propto \lambda e^{-7.5\lambda}.$$

$\square$

3) For $X=0$ there is p probability plus $(1-p)$ times the probability that a poisson variable returns 0. Otherwise there's just $(1-p)$ times the poisson variable.

$$P_X(x) = \mathbb{I}\{x=0\}p + (1-p)\frac{\lambda^x e^{-\lambda}}{x!}.$$

Now for $\underline{x}$ from $\underline{X} \overset{iid}{\sim} ZIP(p,\lambda)$ we have

$$L(p,\lambda; \underline{x}) \propto \prod_{i=1}^{n} \mathbb{I}\{x_i=0\}p + (1-p)\frac{\lambda^x e^{-\lambda}}{x!}.$$