PS3

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Q3 a-c, Q4

3a) $x = a\cosh\mu\cos\nu$
$y = a\sinh\mu\sin\nu$

$\mu \in [0,\infty)$, $\nu \in [0, 2\pi)$

$$\left(\frac{x}{a\cosh\mu}\right)^2 + \left(\frac{y}{a\sinh\mu}\right)^2 = \cos^2\nu + \sin^2\nu = 1$$

hence curves of constant μ form ellipses.

$$\left(\frac{x}{a\cos\nu}\right)^2 - \left(\frac{y}{a\sin\nu}\right)^2 = \cosh^2\mu - \sinh^2\mu = 1$$

hence curves of constant ν form hyperbolae.

b) $(x,y) = \underline{r}(\mu,\nu) = (a\cosh\mu\cos\nu,\ a\sinh\mu\sin\nu)$

$\underline{h}_\mu = \partial_\mu \underline{r} = (a\sinh\mu\cos\nu,\ a\cosh\mu\sin\nu)$

$$\Rightarrow |\underline{h}_\mu| = \sqrt{a^2\sinh^2\mu\cos^2\nu + a^2\cosh^2\mu\sin^2\nu}$$
$$= \sqrt{a^2\sinh^2\mu\cos^2\nu + a^2(1+\sinh^2\mu)\sin^2\nu}$$
$$= a\sqrt{\sinh^2\mu(\cos^2\nu + \sin^2\nu) + \sin^2\nu}$$
$$= a\sqrt{\sinh^2\mu + \sin^2\nu}$$

$\underline{h}_\nu = \partial_\nu \underline{r} = (-a\cosh\mu\sin\nu,\ a\sinh\mu\cos\nu)$

$$\Rightarrow |\underline{h}_\nu| = \sqrt{a^2\cosh^2\mu\sin^2\nu + a^2\sinh^2\mu\cos^2\nu}$$
$$= \sqrt{a^2(1+\sinh^2\mu)\sin^2\nu + a^2\sinh^2\mu\cos^2\nu}$$
$$= a\sqrt{\sinh^2\mu(\sin^2\nu + \cos^2\nu) + \sin^2\nu}$$
$$= a\sqrt{\sinh^2\mu + \sin^2\nu}$$

So

$$\underline{\hat{\mu}} = \frac{1}{\sqrt{\sinh^2\mu + \sin^2\nu}}(\sinh\mu\cos\nu,\ \cosh\mu\sin\nu)$$
$$\underline{\hat{\nu}} = \frac{1}{\sqrt{\sinh^2\mu\ \sin^2\nu}}(-\cosh\mu\sin\nu,\ \sinh\mu\cos\nu).$$
$$\underline{\hat{\mu}}\cdot\underline{\hat{\nu}} = \frac{1}{\sqrt{\sinh^2\mu + \sin^2\nu}}\left[-\sinh\mu\cosh\mu\sin\nu\cos\nu + \sinh\mu\cosh\mu\sin\nu\cos\nu\right] = 0$$

So they are orthogonal.

c)

$$J_{\underline{r}} = \begin{vmatrix} a\sinh\mu\cos\nu & -a\cosh\mu\sin\nu \\ a\cosh\mu\sin\nu & a\sinh\mu\cos\nu \end{vmatrix}$$
$$= a^2\sinh^2\mu\cos^2\nu + a^2\cosh^2\mu\sin^2\nu$$
$$= a^2\sinh^2\mu\cos^2\nu + a^2(1+\sinh^2\mu)\sin^2\nu$$

$= a^2 (\sin^2 \nu + 1)$

$\sin^2 \nu \geq 0$ so providing $a \neq 0$, $J_{\underline{r}}$ is non-vanishing and the mapping of co-ordinates is invertible.

It is non-invertible if $a = 0$.

4a) $\Delta(fg) = \underline{\nabla} \cdot \underline{\nabla}(fg)$

$= (\partial_x, \partial_y, \partial_z) \cdot (g\partial_x f + f\partial_x g,\ g\partial_y f + f\partial_y g,\ g\partial_y f + f\partial_y g)$

$= \partial_x g\, \partial_x f + g\partial_x^2 f + \partial_x f \partial_x g + f\partial_x^2 g$
$+ \partial_y g\, \partial_y f + g\partial_y^2 f + \partial_y f \partial_y g + f\partial_y^2 g$
$+ \partial_z g\, \partial_z f + g\partial_z^2 f + \partial_z g \partial_z f + f\partial_z^2 g$

$= f(\partial_x^2 g + \partial_y^2 g + \partial_z^2 g) + g(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f)$
$+ 2(\partial_x g \partial_x f + \partial_y g \partial_y f + \partial_z g \partial_z f)$

$= f\Delta g + g\Delta f + 2\underline{\nabla} f \cdot \underline{\nabla} g$

b) $\Delta(r^2 \log r) = \Delta((x^2+y^2)\log\sqrt{x^2+y^2})$

$= \Delta\left(\frac{1}{2}(x^2+y^2)\log(x^2+y^2)\right)$

Let $f(\underline{r}) = \frac{1}{2}(x^2+y^2)$, $\quad g(\underline{r}) = \log(x^2+y^2)$

$\underline{\nabla} f(\underline{r}) = (x, y)$ $\quad \underline{\nabla} g(\underline{r}) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$

$\Delta f(\underline{r}) = 2$

$$\Delta g(\underline{r}) = \frac{(x^2+y^2)(2) - (2x)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(2) - (2y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(y^2 - x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)}$$

$= 0$

so $\Delta(r^2\log r) = 2\log(x^2+y^2) + 2\left(\frac{2x^2+2y^2}{x^2+y^2}\right)$

$= 4 + 2\log(x^2+y^2)$

$= 4 + 4\log(r)$.

c) $\Delta^2(r^2\log r) = \Delta(4 + 2\log(x^2+y^2))$

$= \Delta(4 + 2g(\underline{r}))$

$= 0$

as the laplacian is a linear operator.