

PS3

Monday, 26 October 2020 13:57

Q3 a-c, Q4

$$3a) \quad \begin{aligned} x &= a \cosh u \cos v \\ y &= a \sinh u \sin v \end{aligned}$$

$$u \in [0, \infty), \quad v \in [0, 2\pi)$$

$$\left(\frac{x}{a \cosh u} \right)^2 + \left(\frac{y}{a \sinh u} \right)^2 = \cos^2 v + \sin^2 v = 1$$

hence curves of constant u form ellipses.

$$\left(\frac{x}{a \cos v} \right)^2 - \left(\frac{y}{a \sin v} \right)^2 = \cosh^2 u - \sinh^2 u = 1$$

hence curves of constant v form hyperbolae.

$$b) \quad (x, y) = \Sigma(u, v) = (a \cosh u \cos v, a \sinh u \sin v)$$

$$\underline{h}_u = \partial_u \Sigma = (a \sinh u \cos v, a \cosh u \sin v)$$

$$\begin{aligned} \Rightarrow |\underline{h}_u| &= \sqrt{a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v} \\ &= \sqrt{a^2 \sinh^2 u \cos^2 v + a^2 (1 + \sinh^2 u) \sin^2 v} \\ &= a \sqrt{\sinh^2 u (\cos^2 v + \sin^2 v) + \sin^2 v} \\ &= a \sqrt{\sinh^2 u + \sin^2 v} \end{aligned}$$

$$\underline{h}_v = \partial_v \Sigma = (-a \cosh u \sin v, a \sinh u \cos v)$$

$$\begin{aligned} \Rightarrow |\underline{h}_v| &= \sqrt{a^2 \cosh^2 u \sin^2 v + a^2 \sinh^2 u \cos^2 v} \\ &= \sqrt{a^2 (1 + \sinh^2 u) \sin^2 v + a^2 \sinh^2 u \cos^2 v} \\ &= a \sqrt{\sinh^2 u (\sin^2 v + \cos^2 v) + \sin^2 v} \\ &= a \sqrt{\sinh^2 u + \sin^2 v} \end{aligned}$$

So

$$\underline{\hat{u}} = \frac{1}{\sqrt{\sinh^2 u + \sin^2 v}} (\sinh u \cos v, \cosh u \sin v)$$

$$\underline{\hat{v}} = \frac{1}{\sqrt{\sinh^2 u + \sin^2 v}} (-\cosh u \sin v, \sinh u \cos v)$$

$$\underline{\hat{u}} \cdot \underline{\hat{v}} = \frac{1}{\sqrt{\sinh^2 u + \sin^2 v}} \left[-\sinh u \cosh u \sin v \cos v + \sinh u \cosh u \sin v \cos v \right] = 0$$

So they are orthogonal.

c)

$$\underline{J}_\Sigma = \begin{vmatrix} a \sinh u \cos v & -a \cosh u \sin v \\ a \cosh u \sin v & a \sinh u \cos v \end{vmatrix}$$

$$= a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v$$

$$= a^2 \sinh^2 u \cos^2 v + a^2 (1 + \sinh^2 u) \sin^2 v$$

$$= a^2 (\sin^2 \mu + 1)$$

$\sin^2 \mu \geq 0$ so providing $a \neq 0$, Σ_c is non-vanishing and the mapping of co-ordinates is invertible.

It is non-invertible if $a=0$.

$$\begin{aligned} 4a) \quad \Delta(fg) &= \nabla \cdot \nabla(fg) \\ &= (\partial_x, \partial_y, \partial_z) \cdot (g \partial_x f + f \partial_x g, g \partial_y f + f \partial_y g, g \partial_z f + f \partial_z g) \\ &= \partial_x g \partial_x f + g \partial_x^2 f + \partial_x f \partial_x g + f \partial_x^2 g \\ &\quad + \partial_y g \partial_y f + g \partial_y^2 f + \partial_y f \partial_y g + f \partial_y^2 g \\ &\quad + \partial_z g \partial_z f + g \partial_z^2 f + \partial_z f \partial_z g + f \partial_z^2 g \\ &= f(\partial_x^2 g + \partial_y^2 g + \partial_z^2 g) + g(\partial_x^2 f + \partial_y^2 f + \partial_z^2 f) \\ &\quad + 2(\partial_x g \partial_x f + \partial_y g \partial_y f + \partial_z g \partial_z f) \\ &= f \Delta g + g \Delta f + 2 \nabla f \cdot \nabla g \end{aligned}$$

$$\begin{aligned} b) \quad \Delta(r^2 \log r) &= \Delta((x^2 + y^2) \log \sqrt{x^2 + y^2}) \\ &= \Delta\left(\frac{1}{2}(x^2 + y^2) \log(x^2 + y^2)\right) \end{aligned}$$

$$\text{Let } f(\underline{r}) = \frac{1}{2}(x^2 + y^2), \quad g(\underline{r}) = \log(x^2 + y^2)$$

$$\nabla f(\underline{r}) = (x, y) \quad \nabla g(\underline{r}) = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}\right)$$

$$\Delta f(\underline{r}) = 2$$

$$\begin{aligned} \Delta g(\underline{r}) &= \frac{(x^2 + y^2)(2) - (2x)(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2)(2) - (2y)(2y)}{(x^2 + y^2)^2} \\ &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{so } \Delta(r^2 \log r) &= 2 \log(x^2 + y^2) + 2 \left(\frac{2x^2 + 2y^2}{x^2 + y^2} \right) \\ &= 4 + 2 \log(x^2 + y^2) \\ &= 4 + 4 \log(r). \end{aligned}$$

$$\begin{aligned} c) \quad \Delta(r^2 \log r) &= \Delta(4 + 2 \log(x^2 + y^2)) \\ &= \Delta(4 + 2g(\underline{r})) \\ &= 0 \end{aligned}$$

as the Laplacian is a linear operator.