PS4

Wednesday, 16 December 2020 11:58

Q2, 4a,b, 6, 7, 8

2.1) $f(z) = \sin^2 z$ about $z=0$.

$f(z) = (\sin z)(\sin z)$

$= (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots)(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots)$

$= z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \frac{z^4}{3!} + \frac{z^6}{(3!)^2} + \frac{z^6}{5!} + \dots$

$= z^2 - \frac{2z^4}{3!} + \frac{2z^6}{5!} + \frac{z^6}{3!} + \dots$

$= z^2 - \frac{2}{3!}z^4 + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)z^6 + \dots$

$= z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 + \dots$

.2) $f(z) = 1/1+z$ about i.

$f(z) = \frac{1}{1-(-z)} = \sum_{i=0}^{\infty} (-z-i)^n = 1 - (z+i) + (z+i)^2 - \dots$

.3) $f(z) = e^z$ about 1

$f(z) = e^z = \sum_{i=0}^{\infty} \frac{1}{n!}(z-1)^n = 1 + (z-1) + \frac{1}{2}(z-1)^2 + \dots$

4a) $f(z) = \frac{\cos z}{(z-1)^3(z^2-1)^2}$

Let $h(z) = (z-1)^3(z^2-1)^2$. We have $h(z)$ has roots at $z=\pm 1$. Furthermore clearly $h'(\pm 1) = 0$, $h''(\pm 1) = 0$ and $h'''(1) = 0$. So f has a pole of order 3 at $z=1$ and a pole of order 2 at $z=-1$.

b) $f(z) = 1/\sin z$.

Now f has poles at πk for all $k \in \mathbb{Z}$. Further these are all simple poles since $\frac{d}{dz}\sin z = \cos z$ and

$$\cos \pi k = \begin{cases} 1 & \text{even } k \\ -1 & \text{odd } k \end{cases}.$$

6a) $f(z) = e^z/z^3 = \frac{1}{z^3}\left[1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots\right] = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \dots$

$A = \{ z : |z| > 0 \}$

b) $f(z) = \frac{1}{z^3(1-z^2)} = \frac{1}{z^3}\left[1 + z^2 + (z^2)^2 + \dots\right] = \frac{1}{z^3} + \frac{1}{z} + z + \dots$

$A = \{ z : 0 < |z| < 1 \}$

Not sure how to do it for $|z| > 1$.

7a) $f(z) = \frac{z^3+1}{z^2(z-1)}$

Poles: $a_1 = 0$ order 2
$a_2 = 1$ simple

-3 . 1

Poles: $a_1 = 0$ order 2
$a_2 = 1$ simples

$$\text{Res}\{f; a_1\} = \lim_{z\to 0} \frac{d}{dz}\left(z^2 \frac{z^3+1}{z^2(z-1)}\right)$$
$$= \lim_{z\to 0} \frac{d}{dz} \frac{z^3+1}{z-1}$$
$$= \lim_{z\to 0} \frac{(z-1)(3z^2+1) - (z^3+1)}{(z-1)^2}$$
$$= \frac{-1-1}{1}$$
$$= -2$$

$$\text{Res}\{f; a_2\} = \frac{2}{2\cdot 0 + 1^2} = 2$$

b) $f(z) = z/\cos z$

Poles at $z = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$
All simple

$$\text{Res}\{f; \tfrac{\pi}{2} + \pi k\} = \frac{\pi/2 + \pi k}{-\sin(\frac{\pi}{2} + \pi k)} = \begin{cases} -\frac{\pi}{2} - \pi k & k \text{ even} \\ \frac{\pi}{2} + \pi k & k \text{ odd} \end{cases}$$

c) $f(z) = \frac{\sin 3z}{z^2}$

Pole at $z = 0$ order 2

$$\text{Res}\{f; 0\} = \lim_{z\to 0} \frac{d}{dz} z^2 \frac{\sin 3z}{z^2} = \lim_{z\to 0} 3\cos 3z = 3$$

8a) $\gamma = \gamma(0; 2)$

$f(z) = \frac{5z-2}{z(z-1)}$ Simple poles at $z=0$ and $z=1$

$$\text{Res}\{f; 0\} = \frac{-2}{-1+0} = 2$$
$$\text{Res}\{f; 1\} = \frac{3}{0+1} = 3$$
$$\int_\gamma f\, dz = 2\pi i\,[2+3] = 10\pi i$$

b) $f(z) = \frac{1}{z^3 - z^2} = \frac{1}{z^2(z-1)}$

Simple pole at $z = 1$
pole order 2 at $z = 0$

$$\text{Res}\{f; 1\} = \frac{1}{3-2} = 1$$
$$\text{Res}\{f; 0\} = \lim_{z\to 0} \frac{d}{dz} z^2 \frac{1}{z^2(z-1)} = \lim_{z\to 0} \frac{d}{dz} \frac{1}{z-1}$$
$$= \lim_{z\to 0} \frac{(z-1)(0) - (1)(1)}{(z-1)^2} = \frac{-1}{1} = -1$$

$$\int_\gamma f\, dz = 2\pi i\,[1-1] = 0$$

c) $f(z) = e^{-z}/z^2$

pole order 2 at $z=0$

$$\operatorname{Res}\{f;0\} = \lim_{z\to 0} \frac{\partial}{\partial z} z^2 \frac{e^{-z}}{z^2} = \lim_{z\to 0} -e^{-z} = -1$$

$$\int_\gamma \frac{e^{-z}}{z^2}\, \partial z = 2\pi i\,[-1] = -2\pi i$$