ES2(Q15)

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Let R be a ring and $I, J \subset R$ ideals.

1) Prove $I+J = \{i+j : i \in I, j \in J\}$ is an ideal of R.

First since I and J are subsets of R and R is closed under addition, $I+J$ is a subset of R. Next note that $0 = 0+0 \in I+J$. Now let $x, y \in I+J$. So $\exists i_1, i_2 \in I$, $\exists j_1, j_2 \in J$ such that $x = i_1+j_1$, $y = i_2+j_2$. Then

$$x+y = i_1+j_1+i_2+j_2 = i_1+i_2+j_1+j_2 \in I+J$$

since I and J are closed under addition. Also

$$x-y = i_1+j_1-(i_2+j_2) = i_1-i_2+j_1-j_2 \in I+J$$

since I and J are closed under subtraction. So $I+J$ is an additive subgroup of R. Let $r \in R$. We have

$$rx = r(i_1+j_1) = ri_1+rj_1 \in I+J$$

since I and J are ideals of R. So $I+J$ is an ideal of R.

2) Prove $IJ = \{\text{finite sums } \sum_{k=1}^{n} i_k j_k : n \in \mathbb{N}, i_k \in I, j_k \in I\}$ is an ideal of R.

Since I, J are subsets of R and R is closed under addition and multiplication, IJ is a subset of R. Also $0 = 0 \times 0 \in IJ$. Clearly IJ is closed under addition since the sum of two finite sums is just another finite sum. IJ will also be closed under additive inverses since

$$-\sum_{k=1}^{n} i_k j_k = \sum_{k=1}^{n} (-i_k) j_k$$

and I is closed under additive inverses. So IJ is an additive subgroup of R. Let $r \in R$, $x \in IJ$. We have

$$r\sum_{k=1}^{n} i_k j_k = \sum_{k=1}^{n} (r i_k) j_k \in IJ$$

since I is an ideal of R. So IJ is an ideal of R.

3) Suppose $R = \mathbb{Z}$, $I = m\mathbb{Z}$, $J = n\mathbb{Z}$, $h = hcf(m,n)$. Show that $I+J = h\mathbb{Z}$, $IJ = mn\mathbb{Z}$.

We have

$$\begin{aligned} I+J &= \{i+j : i \in I, j \in J\} \\ &= \{i+j : i \in m\mathbb{Z}, j \in n\mathbb{Z}\} \\ &= \{mi+nj : i,j \in \mathbb{Z}\} \\ &= \{phi+qhj : i,j \in \mathbb{Z}\} \end{aligned}$$

where $p, q \in \mathbb{Z}$ since $h|m$, $h|n$. So

$$I+J = \{h(pi+qj) : i,j \in \mathbb{Z}\} \subseteq h\mathbb{Z}.$$

Also if $x \in h\mathbb{Z}$, $\exists p \in \mathbb{Z}$ such that $x = ph$. By the euclidean algorithm $\exists s,t \in \mathbb{Z}$ such that

$$ms+nt = h \Rightarrow mps+npt = ph = x.$$

So $h\mathbb{Z} \subseteq I+J$. Hence $I+J = h\mathbb{Z}$.

We have

$$\begin{aligned} IJ &= \{ij : i \in I, j \in J\} \\ &= \{ij : i \in m\mathbb{Z}, j \in n\mathbb{Z}\} \\ &= \{minj : i,j \in \mathbb{Z}\} \\ &= mn\{ij : i,j \in \mathbb{Z}\} \\ &= mn\mathbb{Z} \end{aligned}$$

Since if $p \in \mathbb{Z}$, then $p = 1 \times p \in \{ij : i,j \in \mathbb{Z}\}$ and if $p \in \{ij : i,j \in \mathbb{Z}\}$ then $p \in \mathbb{Z}$ since the product of two integers is an integer.