PS4

Thursday, 26 November 2020 15:55

W: PS6 Q3 SE: Ch6 Q2,3

W3a) $\dot{x} = \alpha x$
$\dot{y} = \beta y + \gamma x^{n+1}$ $\alpha < 0, \beta > 0, \gamma \in \mathbb{R}, n \in \mathbb{Z}_{>0}$

Equilibrium points:

$$(\alpha x, \beta y + \gamma x^{n+1}) = 0 \implies x = 0, \ \beta y + \gamma x^{n+1} = 0$$

$$\implies (0,0) \text{ is an equilibrium point}$$

Linearised ODE

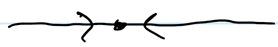
$$J = \begin{pmatrix} \alpha & 0 \\ (n+1)\gamma x^n & \beta \end{pmatrix} \implies J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

$\implies$ Eigenvalues of $J(0,0)$ are α and β

$\implies$ hyperbolic saddle at $(0,0)$.

b) We have $x=0$ clearly an invariant set. If $y > 0$ then $\dot{y} > 0$ and $y < 0 \implies \dot{y} < 0$ so the y-axis is the unstable subspace.

Consider $y = 0$. Now if we're close to the origin $x^{n+1} \to 0$, further $x > 0 \implies \dot{x} < 0$ and $x < 0 \implies \dot{x} > 0$ so $y = 0$ is locally our global stable manifold.



c) Our linearised system about the origin is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \implies \left.\begin{matrix} \dot{u} = \alpha u \\ \dot{v} = \beta v \end{matrix}\right\}.$$

So for the unstable subspace we have $v = 0 \implies \dot{v} = 0$ so we have invariance. Similarly for the stable subspace we have $u = 0 \implies \dot{u} = 0$.

d) First

$$\dot{x} = \alpha x \implies \int_{x_0}^{x} \frac{1}{\alpha x'}\, dx' = t \implies \ln x' \Big|_{x_0}^{x} = t\alpha \implies x = x_0 e^{\alpha t}.$$

Now write $\dot{y} = \beta y + A$ for $A = \gamma x^{n+1} = \gamma x_0^{n+1} e^{\alpha(n+1)t}$. So

$$\dot{y} = \beta y + A \implies \int_{y_0}^{y} \frac{1}{\beta y' + A}\, dx' = t \implies \frac{1}{\beta}\ln(\beta y' + A)\Big|_{y_0}^{y} = t \implies \frac{\beta y + A}{\beta y_0 + A} = e^{\beta t}.$$

Hence

Hence

$$\beta y = \beta y_0 e^{\beta t} + A e^{\beta t} - A$$

$$= \beta y_0 e^{\beta t} + \gamma x_0^{n+1} e^{\alpha(n+1)t} e^{\beta t} - \gamma x_0^{n+1} e^{\alpha(n+1)t}$$

$$\Rightarrow \quad y = y_0 e^{\beta t} + \frac{\gamma}{\beta} x_0^{n+1} e^{\alpha(n+1)t} e^{\beta t} - \frac{\gamma}{\beta} x_0^{n+1} e^{\alpha(n+1)t}.$$

CAN'T REARANG THIS INTO THE REQUIRED FORM ☹

e) Unstable manifold clearly $x=0$ so

$$W^u((0,0)) = \{(0,y) : y \in \mathbb{R}\}.$$

For the stable manifold our seed is the x-axis near the origin. We need as $t \to \infty$, $(x,y) \to (0,0)$ hence

$$y_0 - \frac{\gamma x_0^{n+1}}{\alpha(n+1)-\beta} = 0 \Rightarrow y_0 = \frac{\gamma x_0^{n+1}}{\alpha(n+1)-\beta}.$$

So

$$y = \frac{\gamma x^{n+1}}{\alpha(n+1)-\beta}$$

is our global stable manifold.

Note to self: we're finding the initial conditions that result in a trajectory ending up at the origin.

f) For $x=0$, $\dot{x} = \alpha(0) = 0 \Rightarrow W^u((0,0))$ is an invariant set. We need to show that the global stable manifold is tangent to the vector space. We have

$$\dot{y} = \frac{\gamma}{\alpha(n+1)-\beta}(n+1)x^n \dot{x}$$

by differentiation. Additionally we have by substitution

$$(\dot{x},\dot{y}) = \left(\alpha x, \beta\left(\frac{\gamma x^{n+1}}{\alpha(n+1)-\beta}\right) + \gamma x^{n+1}\right)$$

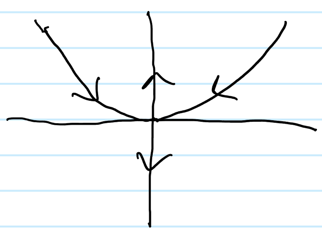
$$= \left(\alpha x, \gamma x^{n+1}\left[\frac{\beta}{\alpha(n+1)-\beta} + 1\right]\right)$$

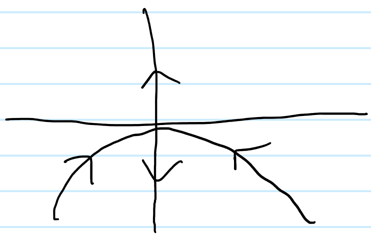
$$= \left(\alpha x, \gamma x^{n+1}\left[\frac{\alpha(n+1)}{\alpha(n+1)-\beta}\right]\right)$$

$$\Rightarrow \dot{y} = \frac{\gamma}{\alpha(n+1)-\beta}(n+1)\alpha x x^n = \frac{\gamma}{\alpha(n+1)-\beta}(n+1)x^n\dot{x}.$$

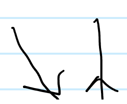
So they are tangent and thus the global stable manifold is invariant.

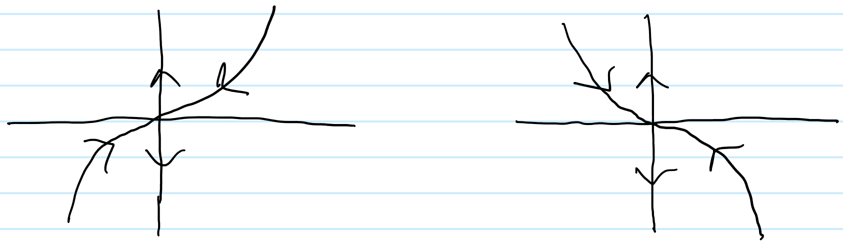
g) $\gamma > 0$, $n+1$ even $\qquad\qquad \gamma < 0$, $n+1$ even





$\gamma > 0$, $n+1$ odd $\qquad\qquad \gamma < 0$, $n+1$ odd





SB 2i) $\dot{x} = x + 3y + xy$
$\dot{y} = 3x + y + x^2$

Equilibria: $(\dot{x}, \dot{y}) = (0,0) \implies \left.\begin{array}{l} x + 3y + xy = 0 \\ 3x + y + x^2 = 0 \end{array}\right\}$

$$\begin{aligned}
3x + y + x^2 = 0 \implies y &= -x^2 - 3x \\
\implies x + 3(-x^2 - 3x) + x(-x^2 - 3x) &= 0 \\
\implies x - 3x^2 - 9x - x^3 - 3x^2 &= 0 \\
\implies -x^3 - 6x^2 - 8x &= 0 \\
\implies -x(x^2 + 6x + 8) &= 0 \\
\implies -x(x+4)(x+2) &= 0
\end{aligned}$$

So equilibria are $(0,0), (-4,-4), (-2,2)$

ii) $J = \begin{pmatrix} 1+y & 3+x \\ 3+2x & 1 \end{pmatrix} \implies J(0,0) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$P_J(\lambda) = \lambda^2 - 2\lambda - 8 \overset{\text{SET}}{=} 0 \implies \lambda = 4, \lambda = -2$

$\implies 4 > 0, -2 < 0$ So hyperbolic saddle at $(0,0)$.

iii) Linearised ODE:

$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \implies \left.\begin{array}{l} \dot{U} = U + 3V \\ \dot{V} = 3U + V \end{array}\right\}$

If $U = V$ then $(\dot{U}, \dot{V}) = (4U, 4U)$ so $\dot{U} = \dot{V}$. Hence $U = V$ is an invariant set. Then also $U = -V \implies (\dot{U}, \dot{V}) = (2V, 2U)$ so $\dot{U} = -\dot{V}$ So $U = -V$ is also an invariant set.

$U = V$: $\quad U > 0$: $\dot{U} > 0, \dot{V} > 0$; $\quad U < 0$: $\dot{U} < 0, \dot{V} < 0$ $\implies U = V$ is unstable manifold

$U = -V$: $\quad \left.\begin{array}{l} U > 0: \dot{U} < 0, \dot{V} > 0 \\ U < 0: \dot{U} > 0, \dot{V} < 0 \end{array}\right\}$ $U = -V$ is stable manifold

$W^s((0,0))$ tangent line $U = -V$
$W^u((0,0))$ tangent line $U = V$

iv) Tangent lines are perpendicular so angle is $90°$.

3i For $W^s(\underline{0})$, $x = -y$ So

$$\left.\begin{array}{l} \dot{x} = x - 3x - x^2 = -2x - x^2 \\ \dot{y} = 3x - x + x^2 = 2x + x^2 \end{array}\right\}.$$

Hence $\dot{x} = -\dot{y}$ so $W^s(\underline{0})$ is invariant.

ii) For $W^u(\underline{0})$, $x = y$ so

$$\left.\begin{aligned}\dot{x} &= x + 3x + x^2 = x^2 + 4x \\ \dot{y} &= 3x + x + x^2 = x^2 + 4x\end{aligned}\right\}.$$

Again $\dot{x} = \dot{y}$ so $W^u(\underline{0})$ is invariant.

iii) For $W^s(\underline{0})$, we have $\dot{x} = -\dot{y} = -2x - x^2$. Now we have

$$\dot{x} = -2x - x^2 \implies -t = \int_{x_0}^{x} \frac{1}{x'^2 + 2x'}\, dx'$$

$$= \frac{1}{2}\left(\log x - \log(x+2)\right)\Big|_{x_0}^{x}$$

$$= \frac{1}{2}\left(\log \frac{x}{x+2} - \log \frac{x_0}{x_0+2}\right).$$

$$= \frac{1}{2} \log \frac{x(x_0+2)}{(x+2)(x_0)}$$

$$\implies \frac{x(x_0+2)}{(x+2)(x_0)} = e^{-2t}.$$

Let $A = \frac{x_0+2}{x_0}$. So we have

$$Ax = (x+2)e^{-2t} = e^{-2t}x + 2e^{-2t}$$

$$\implies x = \frac{2e^{-2t}}{A - e^{-2t}}.$$

So as $t \to \infty$, $x \to 0$ so $\phi_t(\underline{x}) \to \underline{0}$.

iv) For $W^u(\underline{0})$, we have $\dot{x} = \dot{y} = x^2 + 4x$. So we have

$$\dot{x} = x^2 + 4x \implies t = \int_{x_0}^{x} \frac{1}{x^2+4x}\, dx$$

$$= \frac{1}{4}\log\left(\frac{x}{x+4}\right)\Big|_{x_0}^{x}$$

$$= \frac{1}{4}\left(\log \frac{x}{x+4} - \log\left(\frac{x_0}{x_0+4}\right)\right)$$

$$= \frac{1}{4}\left(\log \frac{x(x_0+4)}{x_0(x+4)}\right)$$

$$\implies e^t = \frac{x(x_0+4)}{x_0(x+4)}.$$

Let $B = \frac{x_0+4}{x_0}$. So we have

$$(x+4)e^t = Bx \implies xe^t + 4e^t = Bx$$

$$\implies x = \frac{4e^t}{B - e^t}.$$

As $t \to -\infty$, $x \to 0$ so $\phi_t(\underline{x}) \to \underline{0}$.