Assessed HW 1 (ODEs)

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1(i) We have $\dot{x} = \frac{1}{x^2}$ on phase space $U = \mathbb{R} \setminus \{0\}$ and flow ϕ. Let $x_0 \in U$ and $t_0 \in \mathbb{R}$ to give the initial value problem

$$\dot{x} = \frac{1}{x^2}, \quad x(t_0) = x_0, \quad x \in U.$$

By separation of variables we get

$$\int_{x_0}^{x} x'^2 \, dx' = \int_{t_0}^{t} dt'$$

$$\implies \quad \frac{1}{3} x'^3 \Big|_{x_0}^{x} = t' \Big|_{t_0}^{t}$$

$$\implies \quad \frac{1}{3} x^3 - \frac{1}{3} x_0^3 = t - t_0$$

$$\implies \quad x = \sqrt[3]{3(t-t_0) + x_0^3}.$$

Now as $U = \mathbb{R} \setminus \{0\}$ we require $x \neq 0$ for all $t \in \mathbb{R}$. So

$$\sqrt[3]{3(t-t_0)+x_0^3} \neq 0 \implies t \neq t_0 - \frac{x_0^3}{3}.$$

Thus we obtain

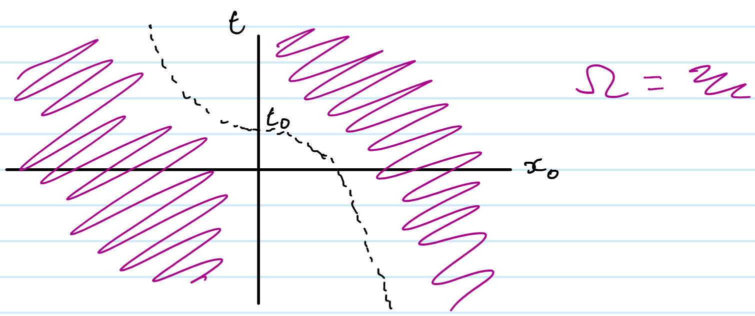
$$I(x_0) = \begin{cases} \left(-\infty, t_0 - \frac{x_0^3}{3}\right) & \text{if } x_0 < 0 \\ \left(t_0 - \frac{x_0^3}{3}, \infty\right) & \text{if } x_0 > 0 \end{cases}$$

for $x_0 \in U$.

ii) We have that

$$\Omega = \{(x_0, t) \in \mathbb{R}^2 : x_0 \in U, \, t \in I(x_0)\}.$$

Hence we obtain the sketch:



(iii) First we have that

$$\phi_t(x_0) = x(t, x_0) = x(t, 0, x_0) = \sqrt[3]{3t + x_0^3}$$

Suppose $x_0 < 0$. Then $a = -\infty$ and $b = -\frac{x_0^3}{3}$ for $t_0 = 0$. This gives us

$$\lim_{t \to a^+} \phi_t(x_0) = \lim_{t \to a^+} \sqrt[3]{3t + x_0^3} = -\infty$$

and

$$\lim_{t\to 5^-} \phi_t(x_0) = \lim_{t\to 5^-} \sqrt{3t + x_0^3} = 0 .$$

If $x_0 > 0$ then $a = -\frac{x_0^3}{3}$ and $b = \infty$ which was $\lim_{t\to a^+} \phi_t = 0$ and $\lim_{t\to b^-} \phi_t = \infty$ which agrees with the continuation theorem.

iv) Let $x_0 \in U$. Let $t \in I(x_0)$ and $s \in I(\phi_t(x_0))$. Then

$$\begin{aligned}
\phi_s(\phi_t(x_0)) &= \phi_s\left(\sqrt[3]{3t + x_0^3}\right) \\
&= \sqrt[3]{3s + \left(\sqrt[3]{3t + x_0^3}\right)^3} \\
&= \sqrt[3]{3s + 3t + x_0^3} \\
&= \sqrt[3]{3(s+t) + x_0^3} \\
&= \phi_{s+t}(x_0) .
\end{aligned}$$

with $s+t \in I(x_0)$.

2:) Let $0 < \tau < r$ and consider $(\dot{x}, \dot{y}) = (-\tau xy, \tau xy - ry)$ on $\mathbb{R}^2$. Equilibria occur when

$$\underline{0} = (\dot{x}, \dot{y}) = (-\tau xy, \tau xy - ry)$$

$$\implies \quad (xy, y(\tau x - r)) = \underline{0} .$$

So our equilibria are $(x, 0)$ for $x \in \mathbb{R}$.

i) Consider a point on the y-axis denoted $(0, y)$ for some $y \in \mathbb{R}$. We have $(\dot{x}, \dot{y}) = (0, -ry)$ which implies $x(t, \underline{x}_0)$ is a constant and thus the x co-ordinate of our point doesn't change which means the y-axis is an invariant set.

ii) Clearly the x-axis is invariant as it comprises only of equilibria. So the boundries of the first open quadrant are invariant sets. Now we have that $(\underline{x}_0, t_0) \in \mathbb{R}^2_{>0} \times I(x_0)$ so by the continuation theorem we have that $\underline{x}(t, \underline{x}_0)$ tends to the boundary of $\mathbb{R}^2_{>0}$ as t tends to the edges of the interval $I(x_0)$. Thus for all $t \in I(x_0)$ we must have $x(t, \underline{x}_0) > 0$ and $y(t, \underline{x}_0) > 0$.

iv) We have for $g(x) = y_0 - (x - x_0) + \frac{r}{\tau} \log\left(\frac{x}{x_0}\right)$ that

$$\begin{aligned}
\frac{d}{dt}\left(y(t, \underline{x}_0) - g(x(t, \underline{x}_0))\right) &= \tau xy - ry - \left(-\dot{x} + \frac{r}{\tau} \frac{x_0}{x} \frac{1}{x_0} \dot{x}\right) \\
&= \tau xy - ry - \tau xy - \frac{r}{\tau}\frac{1}{x}(-\tau xy) \\
&= -ry + ry \\
&= 0 .
\end{aligned}$$

Hence $y(t, \underline{x}_0) - g(x(t, \underline{x}_0)) = c \in \mathbb{R}$. At $t = 0$ we have

$$y(0, \underline{x}_0) - g(x(t, \underline{x}_0)) = y_0 - g(x_0) = y_0 - y_0 = 0$$

Hence $c = 0$ so $y(t, \underline{x}_0) = g(x(t, \underline{x}_0))$.

V) By the existence theorem we have a solution in the neighborhood of $t = 0$. Then as $\underline{x}(t, \underline{x}_0)$ is strictly decreasing and strictly positive for all time $t > 0$ it does not reach the boundary in finite time. Thus by the continuation theorem we have $[0, \infty) \subset I(x_0)$.

Vi) We know $\lim_{t \to \infty} \underline{x}(t, \underline{x}_0)$ must lie on the boundary which is the x-axis. So $\lim_{t \to \infty} y(t, \underline{x}_0) = 0$ and $\lim_{t \to \infty} x(t, x_0)$ is a root of $g(x)$. Then as $x > 0$ and $y > 0$ we have $\dot{x} < 0$ which implies that

$$\lim_{t \to \infty} \underline{x}(t, x_0) = (\beta, 0)$$

as solutions move strictly negative of x_0 and $\beta < x_0 < \alpha$.