PS1

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Q1, 6, 7, 9

1a) Fair coin tossed repeatedly

N number of tosses up to and including the first time 2 consecutive tosses are the same.

$\mathbb{P}$ mass function of N?

$\mathbb{P}(N=1) = 0$
$\mathbb{P}(N=2) = \frac{1}{2}$
$\mathbb{P}(N=3) = \frac{1}{4}$

$$\underset{\{H,T\}}{H}\ \underbrace{T\ H\ T\ H\ T\ H}_{\text{must be opposites}}\ \underset{\text{must be same as previous}}{H}$$

$\mathbb{P}(N=n) = 1 \times \frac{1}{2^{n-2}} \times \frac{1}{2} = \frac{1}{2^{n-1}}$ if $n \geq 2$

$\mathbb{P}(N=n) = \frac{1}{2^{n-1}} \mathbb{I}(n \geq 2)$

b) $\mathbb{P}$ N is even?

$\{N \text{ even}\} = A$

Let $A_n = \{N=2\} \cup \{N=4\} \cup \dots \cup \{N=n\}$.

So A_n is increasing and $A_n \uparrow A$.

So by continuity of $\mathbb{P}$ and Axiom 3 we have

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i = 2i)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^{2i-1}} = \frac{1/2}{1-1/4} = 2/3$$

c) $\mathbb{P}$ N is odd?

$\mathbb{P}(N \text{ odd}) = 1 - \mathbb{P}(N \text{ even}) = 1 - 2/3 = 1/3$

6) $X_1, \dots$ iid taking values 0, 1 each with $\mathbb{P}$ $\frac{1}{2}$.

$S_n = \sum_{i=1}^{n} X_i$ $\qquad N = \min\{n : X_n = 1\}$ ← First time we get a 1

Find $\mathbb{E}(X_1)$, $\mathbb{E}(N)$, $\mathbb{E}(S_N)$, Does $\mathbb{E}(S_N) = \mathbb{E}(N)\mathbb{E}(X_1)$ hold?

Are N, $(X_i)_{i\geq 1}$ independent?

$\mathbb{E}(X_1) = (0)\mathbb{P}(X_1 = 0) + (1)\mathbb{P}(X_1 = 1) = 0 + 1 \times \frac{1}{2} = \frac{1}{2}$

$\mathbb{E}(N) = (1)\mathbb{P}(X_1 = 1) + \mathbb{E}(N)\mathbb{P}(X_1 = 0)$
$\quad = 1 + \frac{1}{2}\mathbb{E}(N)$

$\Rightarrow \mathbb{E}(N) = 2$

$\mathbb{E}(S_N \mid N = n) = \mathbb{E}(\sum_{i=1}^{n} X_i \mid N = n) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{n}{2}$

$\mathbb{E}(S_N) = \mathbb{E}(\mathbb{E}(S_N \mid N)) = \mathbb{E}(\frac{N}{2}) = 1$ by the tower rule

$\mathbb{E}(N)\mathbb{E}(X_1) = 2 \times \frac{1}{2} = 1 = \mathbb{E}(S_N)$

$\mathbb{P}(N = n \cap X_1 = 1) = \mathbb{I}(n=1)$

$\mathbb{P}(N=n)\mathbb{P}(X_1 = 1) = \frac{1}{2}\mathbb{P}(N=n)$

Taking $n=2$ we have $\mathbb{P}(N=2 \cap X_1 = 1) = 0 < \frac{1}{2}\mathbb{P}(N=2)$ since $\mathbb{P}(N=2) > 0$. So N and $(X_i)_{i\geq 1}$ are not independent.

7) Same setup as 6). Let $M = N-1$. Find $\mathbb{E}(M)$, $\mathbb{E}(S_M)$. Does $\mathbb{E}(S_M) = \mathbb{E}(M)\mathbb{E}(X_1)$. Are M, $(X_i)_{i\geq 1}$ independent.

$\mathbb{E}(M) = \mathbb{E}(N-1) = \mathbb{E}(N) - 1 = 2 - 1 = 1$
$\mathbb{E}(S_M) = \mathbb{E}(S_{N-1}) = \mathbb{E}(S_N - X_N) = 1 - \frac{1}{2} = 1/2$

$\mathbb{E}(M)\mathbb{E}(X_1) = 1 \times \frac{1}{2} = \frac{1}{2} = \mathbb{E}(S_M)$

$\mathbb{P}(M = m \cap X_1 = 1) = \mathbb{I}(m = 0)$

$\mathbb{P}(M = m)\mathbb{P}(X_1 = 1) = \frac{1}{2}\mathbb{P}(M = m)$

Again taking $m=2$ we see $\mathbb{P}(M=2 \cap X_1 = 1) = 0 < \frac{1}{2}\mathbb{P}(M=2)$.
So M and $(X_i)_{i\geq 1}$ are not independent.

9) Parrot types random letters A-Z each with $\mathbb{P}$ $1/26$. Key presses are independent. Prove the $\mathbb{P}$ it eventually types PARROT is $1/26$.

Split the parrots text into blocks of 6 letters. Let A be the event the parrot never types a block that equals "PARROT". Let

$$A_n = \{\text{first } n \text{ blocks don't equal "PARROT"}\}.$$

So $A_n \downarrow A$. So by the continuity of probability

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \left(1 - \left(\tfrac{1}{26}\right)^6\right)^n = 0.$$

Hence the parrot types "PARROT" eventually almost surely.