PS7

Sunday, 22 November 2020 14:39

1a) Consider random variables X_i for $i=1,\dots,12$ representing the 12 bulbs. We have $\underline{X} \overset{iid}{\sim}$ Bernoulli(p) such that $X_i = 1$ means the ith bulb is red and $X_i = 0$ means the ith bulb is yellow. Then $p \in \Theta = \{0.6, 0.25\}$. We have hypotheses

$$H_0: p = 0.25$$
$$H_1: p = 0.6$$

and critical region

$$R = \{\underline{x} : T(\underline{x}) \geq 8\}$$

where our critical value is 8 and $T(\underline{x})$ is the number of red bulbs which is the sum $\sum_{i=1}^{12} X_i$

b) We have the power function

$$\pi(p; T, c) = \mathbb{P}(T(\underline{x}) \geq c; p)$$
$$= \mathbb{P}(\textstyle\sum_{i=1}^{12} X_i \geq 8; p).$$

By the definition of the binomial distribution $Y = \sum_{i=1}^{12} X_i \sim$ Binomial$(12, p)$. So

$$\pi(p; T, c) = \mathbb{P}(Y \geq 8; p)$$
$$= 1 - \mathbb{P}(Y \leq 7; p)$$
$$= 1 - \text{pbinom}(7, 12, p).$$

The significance level is given by

$$\alpha = \sup_{\theta \in \Theta_0} \pi(p; T, C) = \pi(0.25) = 0.0027815 1.$$

The type II error is given by

$$1 - \pi(0.6) = 0.5618218$$

c) In R

d) We have

$$T_{NP}(\underline{x}) = f_n(\underline{x}; 0.6)/f_n(\underline{x}; 0.25) = \prod_{i=1}^{n} f(x_i; 0.6)/f(x_i; 0.25)$$

as $\underline{X} \overset{iid}{\sim} f_n(\underline{x}; p)$. Let $t(x_i) = f(x_i; 0.6)/f(x_i; 0.25)$. So

$$t(x_i) = 0.6^{x_i} 0.4^{1-x_i} / 0.25^{x_i} 0.75^{1-x_i}$$
$$= \left(\frac{0.6}{0.25}\right)^{x_i} \left(\frac{0.4}{0.75}\right)^{1-x_i}$$
$$= \left(\frac{12}{5}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$
$$= \left(\frac{9}{2}\right)^{x_i} \left(\frac{8}{15}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$

$$= \left(\frac{9}{2}\right)^{x_i} \left(\frac{8}{15}\right)^{x_i} \left(\frac{8}{15}\right)^{1-x_i}$$

$$= \frac{8}{15} \left(\frac{9}{2}\right)^{x_i}.$$

So

$$T_{NP}(\underline{x}) = \prod_{i=1}^{12} \frac{8}{15} \left(\frac{9}{2}\right)^{x_i} = \left(\frac{8}{15}\right)^{12} \left(\frac{9}{2}\right)^{\sum_{i=1}^{12} x_i}.$$

NOT SURE WHAT TO DO FROM HERE

e) Let $T(\underline{x}) = \sum_{i=1}^{n} X_i$ and $\pi(p, T, c) = P(T(\underline{X}) \geq c; p)$. We know $X = \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$. So we need

$$\pi(0.25) = 0.05, \; [1 - \pi(0.6) = 0.1 \Leftrightarrow \pi(0.6) = 0.9].$$

or

$$\pi(0.6) - \pi(0.25) \approx 0.85$$

$$\Rightarrow g(n, c) := \pi(0.6) - \pi(0.25) - 0.85 \approx 0.$$

Using R we find our solution to be $n = 15$, $c = 7$.

2a) $f(x) = \frac{\lambda_1^x e^{-\lambda_1}}{x!} \Big/ \frac{\lambda_0^x e^{-\lambda_0}}{x!} = \left(\frac{\lambda_1}{\lambda_0}\right)^x e^{-\frac{\lambda_1}{\lambda_0}}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} \left(\frac{\lambda_1}{\lambda_0}\right)^{x_i} e^{-\frac{\lambda_1}{\lambda_0}} = \left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{x}} e^{-n\frac{\lambda_1}{\lambda_0}}$

b) $f(x) = \lambda_1 e^{-\lambda_1 x_i} / \lambda_0 e^{-\lambda_0 x_i} = \frac{\lambda_1}{\lambda_0} e^{\frac{-\lambda_1}{\lambda_0}}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} \frac{\lambda_1}{\lambda_0} e^{-\frac{\lambda_1}{\lambda_0}} = \left(\frac{\lambda_1}{\lambda_0} e^{-\frac{\lambda_1}{\lambda_0}}\right)^n$

c) $f(\underline{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma}\right)^2} \Big/ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_0}{\sigma}\right)^2} = e^{\left(\frac{x-\mu_1}{x-\mu_0}\right)^2}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} e^{\left(\frac{x_i-\mu_1}{x_i-\mu_0}\right)^2}$

d) $f(\underline{x}) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2} \Big/ \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_0}\right)^2} = \frac{\sigma_0}{\sigma_1} e^{\left(\frac{\sigma_0}{\sigma_1}\right)^2}$

$T(\underline{x}) = \prod_{i=1}^{n} f(\underline{x}) = \prod_{i=1}^{n} \frac{\sigma_0}{\sigma_1} e^{\left(\frac{\sigma_0}{\sigma_1}\right)^2} = \left(\frac{\sigma_0}{\sigma_1} e^{\left(\frac{\sigma_0}{\sigma_1}\right)^2}\right)^n$

Problem Sheet 7

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Question 1b

```r
power <- function (p) 1-pbinom(7,12,p)

alpha <- power(0.25)
prob.type.II.error <- 1 - power(0.6)

alpha
```

```
## [1] 0.00278151
```

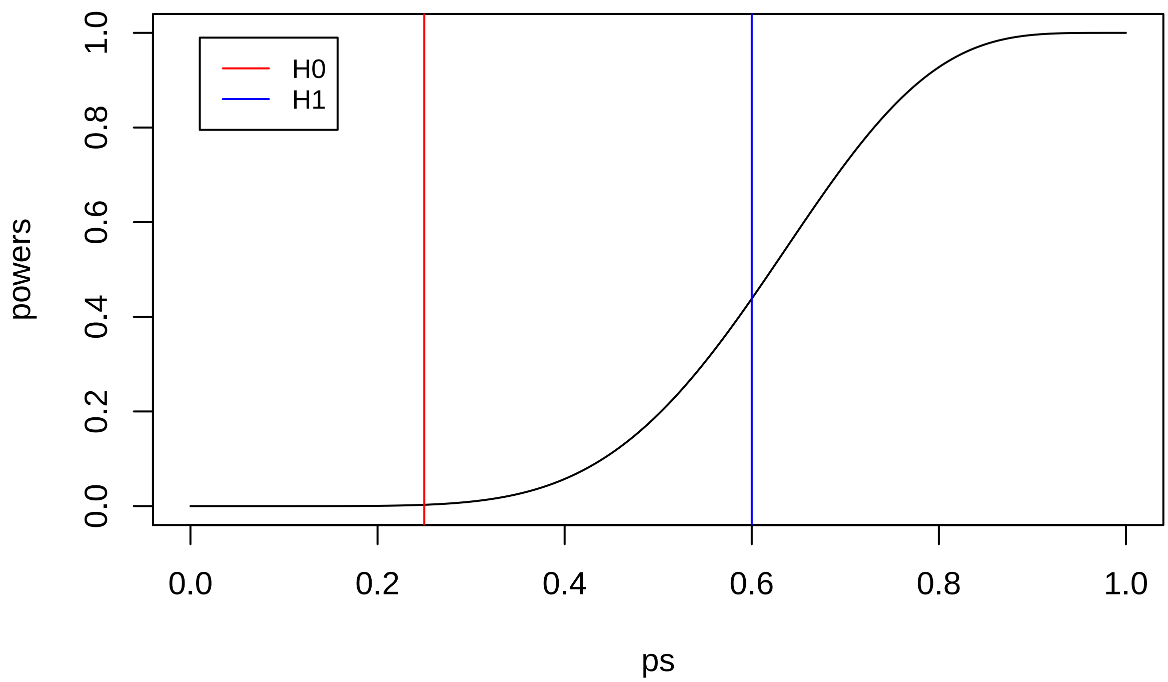
```r
prob.type.II.error
```

```
## [1] 0.5618218
```

Question 1c

```r
ps <- seq(0,1,0.001)
powers <- power(ps)

plot(ps, powers, type="l")
abline(v=c(0.25, 0.6), col=c("red", "blue"))
legend(x=0.01, y=0.99, legend=c("H0", "H1"),
       col=c("red", "blue"), lty=1, cex=0.8)
```



Question 1e

```r
power <- function (p, n, c) 1-pbinom(c-1,n,p)
g <- function (n, c) power(0.6, n, c) - power(0.25, n, c) - 0.85

n.final <- 0
c.final <- 0
e.final <- Inf

for (n in seq(1, 50, 1)) {
  for (c in seq(1, 50, 1)) {
    e <- abs(g(n, c))
    if (e < e.final) {
      n.final <- n
      c.final <- c
      e.final <- e
    }
  }
}

c(n.final, c.final, e.final)
```

```
## [1] 15.000000000  7.000000000  0.001667718
```