PS3

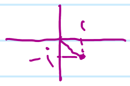
Wednesday, 9 December 2020 13:57

Q1, 3, 5.2, 5.3, 6, 7

1a) $[\![\log(-1)]\!] = \{ \log|-1| + i\theta : \theta \in [\![\arg(-1)]\!] \}$

$= \{ i\frac{3\pi}{2}k : k \in \mathbb{Z} \}$ $\log(-1) = -i\frac{\pi}{2}$

b) $[\![\log(1-i)]\!] = \{ \log|1-i| + i\theta : \theta \in [\![\arg(1-i)]\!] \}$



$= \{ \log\sqrt{2} - i\frac{\pi}{4}k : k \in \mathbb{Z} \}$ $\log(1-i) = \frac{1}{2}\log 2 - i\frac{\pi}{4}$

c) $[\![\log(e^{-\frac{2\pi i}{3}})]\!] = \{ \log(|e^{-\frac{2\pi i}{3}}|) + i\theta : \theta \in [\![\arg(e^{-\frac{2\pi i}{3}})]\!] \}$

$= \{ -i\frac{2\pi}{3}k : k \in \mathbb{Z} \}$ $\log e^{-\frac{2\pi}{3}i} = -i\frac{2\pi}{3}$

3.1) $f(z) = \mathrm{Re}(z)$
$\gamma(t) = t - it^2 \quad (t \in [0,1])$

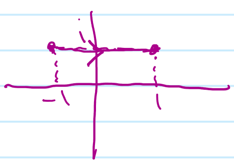
$$\int_\gamma f(z)\,dz = \int_0^1 \mathrm{Re}(t - it^2)(1 - 2it)\,dt$$
$$= \int_0^1 t - 2it^2\,dt$$
$$= \frac{1}{2}t^2 - \frac{2}{3}it^3 \Big|_0^1$$
$$= \frac{1}{2} - \frac{2}{3}i$$

.2) $f(z) = 1/z$
γ = unit circle clockwise starting at 1 traversing unit circle 4 times

$\gamma = e^{it} \quad (t \in [0, -8\pi])$

$$\int_\gamma f(z)\,dz = \int_0^{-8\pi} \frac{1}{e^{it}} ie^{it}\,dt = \int_0^{-8\pi} i\,dt = it\Big|_0^{-8\pi} = -8\pi i$$

.3) $f(z) = |z|^4$
γ = straight line from $-1+i$ to $1+i$



$\gamma(t) = t + i \quad (t \in [-1,1])$

$|t+i|^4 = (\sqrt{t^2+1})^4 = (t^2+1)^2 = t^4 + 2t^2 + 1$

$$\int_\gamma f(z)\,dz = \int_{-1}^1 |t+i|^4\,dt = \int_{-1}^1 t^4 + 2t^2 + 1\,dt = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t\Big|_{-1}^1 = \frac{56}{15}$$

5.2) Γ_R = half circle in upper half of plane radius R centred at the origin traversed CCW.

$= Re^{it} \quad (t \in [0, \pi])$

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$$|f(\Gamma_R(t))| = |Re^{it}|^{-4} = R^{-4}|e^{it}|^{-4} = R^{-4}.$$

So for $t \in [0,\pi]$, $|f(z)| \le R$. Further we have that length $Re^{it} = \pi R$ so

$$\left|\int_\gamma f(z)\,dz\right| \le R^{-4} \cdot \pi R = \pi R^{-3}$$

by the estimation lemma.

3) $\gamma(z_0;r)$ = circle radius r centred at z_0 traversed ccw.

$\gamma(0;R) = Re^{it}$ $\quad (t \in [0, 2\pi])$

Let $f(z) = \frac{z-1}{z+1}$. So $|f(z)| = \left|\frac{z-1}{z+1}\right|$. On γ we have

$$|f(\gamma(0;R))| = \left|\frac{Re^{it}-1}{Re^{it}+1}\right| = \frac{|Re^{it}-1|}{|Re^{it}+1|}.$$

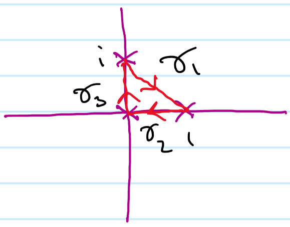
Clearly the maximum will occur when $e^{it} = -1$ so we have

$$|f(z)| \le \frac{|-1-R|}{|-R+1|} = \frac{1+R}{|1-R|}$$

providing $R \ne 1$. Further length $\gamma(0;R) = 2\pi R$ so by the estimation lemma

$$\left|\int_\gamma f(z)\,dz\right| \le \frac{1+R}{|1-R|} 2\pi R = \frac{2\pi R(R+1)}{|R-1|}.$$

6a)



$$\gamma_1(t) = t + (1-t)i \qquad t \in [0,1]$$

$$\gamma_2(t) = (1-t) \qquad t \in [0,1]$$

$$\gamma_3(t) = it \qquad t \in [0,1]$$

$\int_\gamma z\,dz = 0$ by Cauchy's theorem since γ is closed and z is holomorphic everywhere.

b) $$\int_\gamma \bar{z}\,dz = \int_0^1 ((1-t)+ti)(1-i) + (1-t)i(-1) + t(i)\,dt$$

$$= \int_0^1 (1-t)(1-i) + ti(1-i) - i + ti + it\,dt$$

$$= \int_0^1 t[\cancel{-1} + i + i \cancel{+1} + 2i) + 1 - i - i\,dt$$

$$= \int_0^1 4it + 1 - 2i\,dt$$

$$= 2it^2 + (1-2i)t \Big|^1$$

$$= \int_0 4it + 1 - 2i \, dt$$

$$= 2it^2 + (1-2i)t \Big|_0^1$$

$$= 1$$

7) Cauchy's theorem states that if f is holomorphic on a simply connected region then the path integral of f is 0 for any closed path in that region. Now γ is the circle of radius $3/2$ centered at the origin. We have:

(i) $\frac{1}{z-1}$ not holomorphic at $z = 1 < 3/2$ so path integral not equal to 0.

(ii) $z^5 e^{4z}$ holomorphic everywhere so path integral equal to 0.

(iii) $\sec^2(z) = \frac{1}{\cos^2(z)}$ not holomorphic if $\cos z = 0$. However $\cos z \neq 0$ for any $z \in D(0, 3/2)$ which is simply connected and contains γ so the path integral is equal to 0.