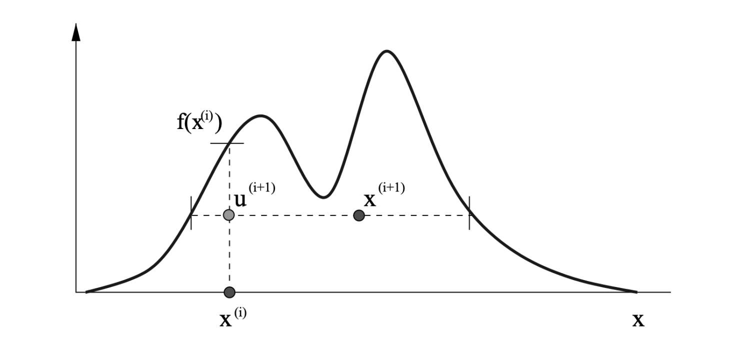
# A Literature Review

# Elliptical Slice Sampling

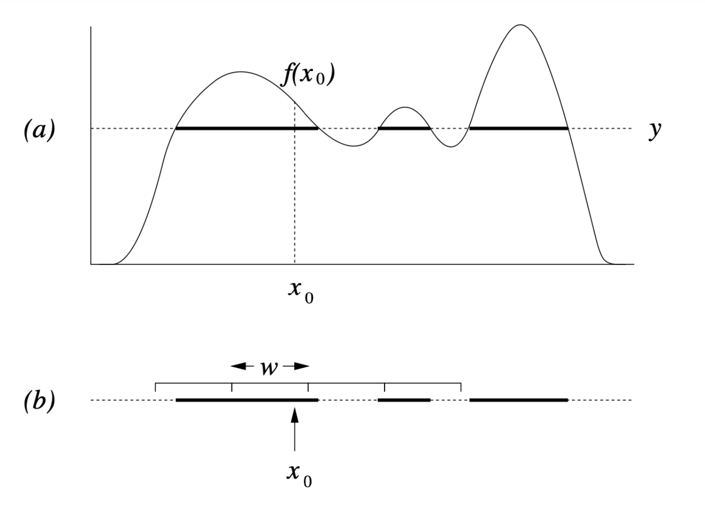
- Title: Elliptical Slice Sampling
- Authors: Iain Murray, Ryan Prescott Adams, David J.C. Mackay
- Year: 2010
- Conference: *Proceedings of the thirteenth international conference on artificial intelligence and statistics* (pp. 541-548)

# Slice Sampling




1. Choosing a start value $x_0$ for which $f(x_0) > 0$ .
2. Sample a $y$ value uniformly between 0 and $f(x_0)$.
3. Draw a horizontal line across the curve at this $y$ position.
4. Sample a point $(x, y)$ from the line segments within the curve.
5. Repeat from step 2 using the new x value.

# Slice Sampling with width parameter w



# Prior beliefs about dependencies

- Multivariate Gaussian distribution is commonly used for a prior belief

$$\mathcal{N}(\mathbf{f}; 0, \Sigma) \equiv |2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{f}^\top \Sigma^{-1} \mathbf{f}\right)$$

- Target distribution for the MCMC sampler is

$$p^\star(\mathbf{f}) = \frac{1}{Z}\mathcal{N}(\mathbf{f}; 0, \Sigma) L(\mathbf{f})$$

# Metropolis-Hastings Algorithm

- By metropolis hastings algorithm given an initial state $\mathbf{f}$, new state is proposed as $\mathbf{f}' = \sqrt{1-\epsilon^2}\mathbf{f} + \epsilon\boldsymbol{\nu}$ where $\boldsymbol{\nu} \sim \mathcal{N}(0, \Sigma)$
- The acceptance probability is $p(accep\,t) = \min\left(1, L\left(\mathbf{f}'\right)/L(\mathbf{f})\right)$
- $\epsilon \in [-1, 1]$ is a step-size parameter which needs to be updated as model parameter
- To choose appropriate step-size $\epsilon$, preliminary runs required

# Elliptical Slice Sampling (ESS)

- Define a full ellipse passing through the current state $\mathbf{f}$,

$$\mathbf{f}' = \boldsymbol{\nu} \sin\theta + \mathbf{f} \cos\theta$$

- Replace the original variable as

$$\begin{aligned} \boldsymbol{\nu}_0 &\sim \mathcal{N}(0, \Sigma) \\ \boldsymbol{\nu}_1 &\sim \mathcal{N}(0, \Sigma) \\ \theta &\sim \text{Uniform}[0, 2\pi] \\ \mathbf{f} &= \boldsymbol{\nu}_0 \sin\theta + \boldsymbol{\nu}_1 \cos\theta \end{aligned}$$

- Sample from the posterior

$$p^\star(\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \theta) \propto \mathcal{N}(\boldsymbol{\nu}_0; 0, \Sigma)\, \mathcal{N}(\boldsymbol{\nu}_1; 0, \Sigma)\, L(\mathbf{f}(\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \theta))$$

- Naively transforming $\theta$ does not result in a Markov chain transition operator with the correct stationary distribution.

# Elliptical Slice Sampling (ESS)

1. Sample from $p\left(\boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \theta \mid (\boldsymbol{\nu}_0 \sin\theta + \boldsymbol{\nu}_1 \cos\theta = \mathbf{f})\right)$ :

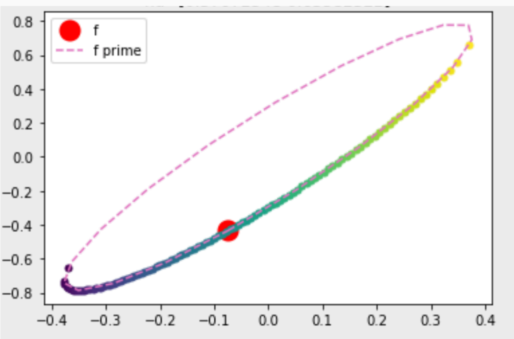
$$\begin{aligned} \theta &\sim \text{ Uniform } [0, 2\pi] \\ \boldsymbol{\nu} &\sim \mathcal{N}(0, \Sigma) \\ \boldsymbol{\nu}_0 &\leftarrow \mathbf{f} \sin\theta + \boldsymbol{\nu} \cos\theta \\ \boldsymbol{\nu}_1 &\leftarrow \mathbf{f} \cos\theta - \boldsymbol{\nu} \sin\theta \end{aligned}$$

2. Update $\theta \in [0, 2\pi]$ using slice sampling on:

$$p^\star\left(\theta \mid \boldsymbol{\nu}_0, \boldsymbol{\nu}_1\right) \propto L\left(\boldsymbol{\nu}_0 \sin\theta + \boldsymbol{\nu}_1 \cos\theta\right)$$

3. return $\mathbf{f}' = \boldsymbol{\nu}_0 \sin\theta + \boldsymbol{\nu}_1 \cos\theta$

# Elliptical Slice Sampling (ESS)




- The range of slice sampling is $\theta \in [0, 2\pi]$
- The algorithm can be neater

# ESS algorithm

1. Choose ellipse: $\boldsymbol{\nu} \sim \mathcal{N}(0, \Sigma)$
2. Log-likelihood threshold:

$$u \sim \text{ Uniform } [0, 1]$$
$$\log y \leftarrow \log L(\mathbf{f}) + \log u$$

3. Draw an initial proposal, also defining a bracket:

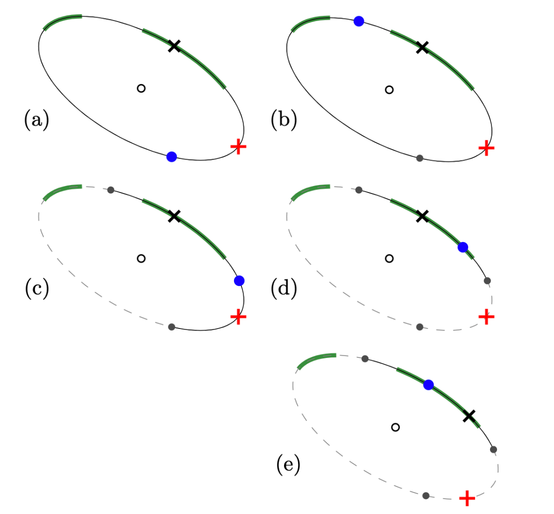
$$\theta \sim \text{ Uniform } [0, 2\pi]$$
$$[\theta_{\text{min}}, \theta_{\text{max}}] \leftarrow [\theta - 2\pi, \theta]$$

4. $\mathbf{f}' \leftarrow \mathbf{f} \cos\theta + \boldsymbol{\nu} \sin\theta$
5. if $\log L(\mathbf{f}') > \log y$ then:
   Accept: return $\mathbf{f}'$

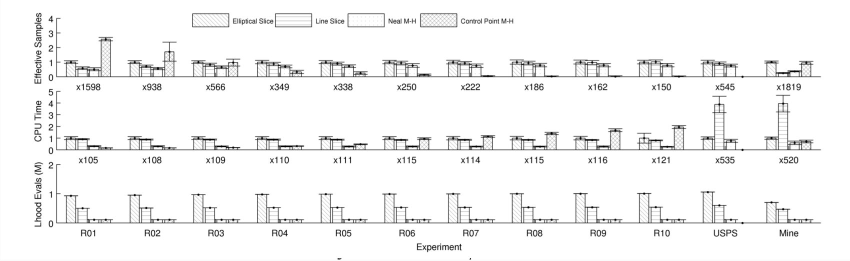
# ESS algorithm (Continued)

7. else: Shrink the bracket and try a new point:
8. if $\theta < 0$ then: $\theta_{\text{min}} \leftarrow \theta$ else: $\theta_{\text{max}} \leftarrow \theta$
9. $\theta \sim \text{Uniform}\left[\theta_{\text{min}}, \theta_{\text{max}}\right]$
10. GoTo 4.

# ESS Procedure



# Efficiency




- More effective samples than Neal's M-H method
- ESS takes less time than line slice sampling
- Performances differences are not huge

## Summary

- The time for likelihood computations are not negligible
- ESS is a simple generic algorithm with no tweak parameters
- ESS has advantage of having no free parameters
- It performs similarly to the best possible performance of related M-H scheme.
- Applied to a wide variety of applications in both low and high dimensions.

# Mini Project

# Plan of Actions

**Study**

- Bayesian Hierarchical Richard Model
- Horseshoe Prior

**Implementation**

- Implement R code
- Adapt main paper algorithm to 2021 COVID-19 data
- Convert the algorithm into JAGS's language

# Reference

Murray, I., Adams, R., & MacKay, D. (2010, March). Elliptical slice sampling. In Proceedings of the thirteenth international conference on artificial intelligence and statistics (pp. 541-548). JMLR Workshop and Conference Proceedings.

Neal, R. M. (2003). Slice sampling. The annals of statistics, 31(3), 705-767.

Andrieu, C., De Freitas, N., Doucet, A., & Jordan, M. I. (2003). An introduction to MCMC for machine learning. Machine learning, 50(1), 5-43.