Contents

1. Literature Review

- Title: Estimation of COVID-19 spread curves integrating global data and borrowing information
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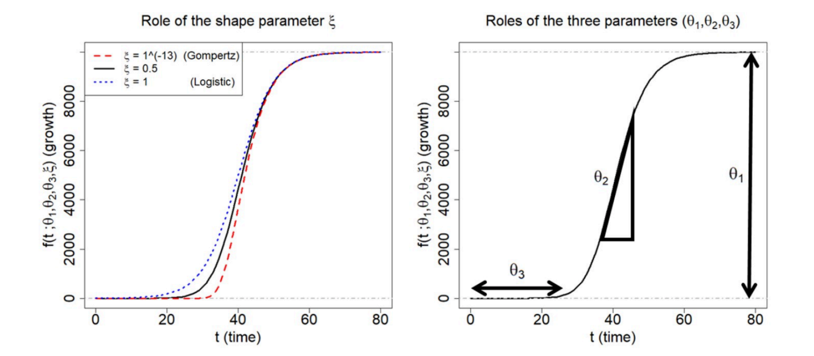

Figure 1: Description of the Richards growth curve model.

2. Scope and Purposes

- Construct the Bayesian hierarchical Richards model in order to estimate the COVID-19 curves

$$
\begin{aligned}
f\left(t;\theta_1,\theta_2,\theta_3,\xi\right) &= \theta_1\cdot\left[1+\xi\cdot\exp\left\{-\theta_2\cdot\left(t-\theta_3\right)\right\}\right]^{-1/\xi} \\
y_{it} &= f\left(t;\theta_{1i},\theta_{2i},\theta_{3i},\xi_i\right)+\epsilon_{it}, \quad \epsilon_{it}\sim\mathcal{N} \\
\theta_{li} &= \alpha_l+\mathbf{x}_i^\top\boldsymbol{\beta}_l+\varepsilon_{li}, \quad \varepsilon_{li}\sim\mathcal{N} \\
\xi_i &\sim \log\mathcal{N}(0,1), \quad (i=1,\cdots,N) \\
\beta_{lj}\mid\lambda_{lj},\tau_{lj},\sigma_l^2 &\sim \mathcal{N}\left(0,\sigma_l^2\tau_l^2\lambda_{lj}^2\right), \\
\lambda_{lj},\tau_{lj} &\sim \mathcal{C}^+(0,1), \quad (l=1,2,3, j=1,\cdots,p) \\
\alpha_l &\sim \pi(\alpha)\propto 1 \\
\sigma_l^2 &\sim \pi\left(\sigma^2\right)\propto 1/\sigma^2, \quad (l=1,2,3)
\end{aligned}
$$

3. Progress

1. Derive the full conditional distributions for interested parameters
2. Using gibbs sampling, update the parameters
3. When the full conditional distributions are not known in closed forms, use metropolis algorithm with gibbs sampling
4. When part (4) is not easy to apply, use slice sampling and elliptical slice sampling

3. Progress

Find the full conditional distributions

$$\boldsymbol{\theta}_1 \mid - \sim \mathcal{N}_N \left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}_1} \left\{ \left(1/\sigma^2\right) \mathbf{r} + \left(1/\sigma_l^2\right) \left(\mathbf{1}\alpha_1 + \mathbf{X}\boldsymbol{\beta}_1\right) \right\}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}_1} \right)$$

$$\theta_{2i} \mid - \propto \exp\left(-\frac{1}{2\sigma^2} \left\| \mathbf{y}_i - \mathbf{f}\left(\theta_{1i}, \theta_{2i}, \theta_{3i}, \xi_i\right) \right\|_2^2 - \frac{1}{2\sigma_2^2} \left(\theta_{2i} - \alpha_2 - \mathbf{x}_i^\top \boldsymbol{\beta}_2 \right)^2 \right)$$

$$\alpha_l \mid - \sim \mathcal{N}_1 \left(\mathbf{1}^\top \left(\boldsymbol{\theta}_l - \mathbf{X}\boldsymbol{\beta}_l \right) / N, \sigma_l^2 / N \right)$$

$$\boldsymbol{\beta}_l \mid - \sim \mathcal{N}_p \left(\boldsymbol{\Sigma}_{\boldsymbol{\beta}_l} \mathbf{X}^\top \left(\boldsymbol{\theta}_l - \mathbf{1}\alpha_l \right), \sigma_l^2 \boldsymbol{\Sigma}_{\boldsymbol{\beta}_l} \right)$$

$$\lambda_{lj} \mid - \sim \mathcal{N} \left(\beta_{lj} \mid 0, \sigma_l^2 \tau_l^2 \lambda_{lj}^2 \right) \cdot \left\{ 1 / \left(1 + \lambda_{lj}^2 \right) \right\}$$

$$\tau_l \mid - \sim \mathcal{N}_p \left(\boldsymbol{\beta}_l \mid \mathbf{0}, \sigma_l^2 \tau_l^2 \boldsymbol{\Lambda}_l \right) \cdot \left\{ 1 / \left(1 + \tau_l^2 \right) \right\}$$

$$\sigma_l^2 \mid - \sim \mathcal{IG} \left(\frac{N+p}{2}, \frac{\left\| \boldsymbol{\theta}_l - 1\alpha_l - \mathbf{X}\boldsymbol{\beta}_l \right\|_2^2 + \boldsymbol{\beta}_l^\top \boldsymbol{\Lambda}_{*l}^{-1} \boldsymbol{\beta}_l}{2} \right)$$

3. Progress

```
#Gibbs Sampling
for (b in 1:(B-1)){
  # Step1: Sample theta1
  {
    sigma.theta1.vec <- solve((1/global.sigma[b])*H(theta2.vec[,b], theta3.vec[,b], xi.vec[,b]) +
                              ((1/local.sigma1[b])*I.N))
    mu.theta1.vec <- sigma.theta1.vec%*%((1/global.sigma[b])*r.vec(theta2.vec[,b], theta3.vec[,b], xi.vec[,b])+
                                         (1/local.sigma1[b])*(onevec.N*alpha1[b] + X%*%beta1.vec[,b]))
    theta1.vec[,b+1] <- mvtnorm::rmvnorm(n=1, mean = mu.theta1.vec,
                                         sigma = sigma.theta1.vec)
  }

  # Step2: Sample theta2 & theta3 using MH with Gaussian proposal densities
  {
    ##M-H algorithm for theta2
    for(i in 1:N){
      #Step1: generate proposals
      new.theta2.i <- rnorm(n = 1, mean = theta2.vec[i,b], sd = sqrt(pro.var.theta.2))

      #Step2: Calculate acceptance probability
      theta2.prob <- theta2.ratio(i = i, theta2.i = new.theta2.i,
                                  old.theta2. <- i = theta2.vec[i,b],
                                  theta1.i = theta1.vec[i,(b+1)],
                                  theta3.i = theta3.vec[i,b],
                                  xi.i = xi.vec[i,b],
                                  global.sigma = global.sigma[b],
                                  alpha2 = alpha2[b],
                                  local.sigma2 = local.sigma2[b],
                                  beta2.vec = beta2.vec[,b])
```

Figure 2: Update $\theta_l, \sigma^2, \alpha_l, \beta_l, \sigma_l^2$ using gibbs sampling

3. Progress

```
##ESS: delicate consideration and Gaussian prior assumed
for(i in 1:N) {
  # Step 1:  change variable
  old.eta.i <- log(xi.vec[i,b])

  #(ESS) Step 2-a: cho-ose an ellipse
  nu <- rnorm(1, mean = mu, sd = sqrt(rho.sq))
  #(ESS) Step 2-b: define a criterion function

  #(ESS) Step 2-c: choose a threshold
  u <- runif(1)
  #(ESS) Step 2-d: draw an initial proposal
  phi <- runif(1, min= -pi, max=pi)
  eta.star.i <- (old.eta.i-mu)*cos(phi)+(nu-mu)*sin(phi)+mu #assume N(0,sigma)

  eta.accept.ratio <- eta.ratio(i = i, eta.i = eta.star.i, old.eta.i = old.eta.i, theta1.i = theta1.vec[i,(b+1)],
                               theta2.i = theta2.vec[i,(b+1)],theta3.i = theta3.vec[i,(b+1)], global.sigma = global.sigma[b])
  #(ESS) Step 2-e: accept procedure
  if(u < eta.accept.ratio) {
    new.eta.i <- eta.star.i
  }
  else{
    phi.min = -pi ; phi.max = pi
    while(u >= eta.accept.ratio){
      if(phi > 0){
        phi.max = phi
      }
      else{
        phi.min = phi
      }
      phi <- runif(1, min=phi.min, max=phi.max)
      eta.star.i <- (old.eta.i-mu)*cos(phi)+(nu-mu)*sin(phi)+mu
    }
    new.eta.i <- eta.star.i
  }
  #(ESS)  step3: variable change
  xi.vec[i, (b+1)] = exp(new.eta.i)
}
```

Figure 3: Update ξ_i using Elliptical Slice Sampling

```
#Step7: Sample lambda1,2,3 with Slice sampling
{
  #lambda1
  eta1.vec = lambda1.vec[,b]^2
  updated.eta1.vec = c()
  for (j in 1:p){
    updated.eta1.vec[j] = ss.invgam(old.x = eta1.vec[j],f = 1,
                                    s = beta1.vec[j,(b+1)]^2/(2*local.sigma1[b+1]*tau1[b]^2))
  }
  lambda1.vec[,(b+1)] = sqrt(updated.eta1.vec)


  #lambda2
  eta2.vec = lambda2.vec[,b]^2
  updated.eta2.vec = c()
  for (j in 1:p){
    updated.eta2.vec[j] = ss.invgam(old.x = eta2.vec[j],f= 1,
                                    s = beta2.vec[j,(b+1)]^2/(2*local.sigma2[b+1]*tau2[b]^2))
  }
  lambda2.vec[,(b+1)] = sqrt(updated.eta2.vec)

  #lambda3
  eta3.vec = lambda3.vec[,b]^2
  updated.eta3.vec = c()
  for (j in 1:p){
    updated.eta3.vec[j] = ss.invgam(old.x = eta3.vec[j],f = 1,
                                    s = beta3.vec[j,(b+1)]^2/(2*local.sigma3[b+1]*tau3[b]^2))
  }
  lambda3.vec[,(b+1)] = sqrt(updated.eta3.vec)
}
```

Figure 4: Update horseshoe prior term λ_{ij}, τ_l using Slice Sampling

4. Remaining Works

- Apply the model to the COVID-19 infections in 2021
    1. Collect daily trajectories data for cumulative numbers of COVID-19 infections
    2. Consider new covariates to reflect the changes between 2020 and 2021 such as introduction of vaccination and vaccination rates
- Compare the COVID-19 situations between 2020 and 2021
    1. Discuss the difference of curves in between 2020 and 2021
    2. Explore the effect of new covariates to estimate the COVID-19 infections

Reference

Lee, S. Y., Lei, B., & Mallick, B. (2020). Estimation of COVID-19 spread curves integrating global data and borrowing information. PloS one, 15(7), e0236860.