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# Pricing and Hedging Spread Options*

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**Abstract.** We survey theoretical and computational problems associated with the pricing and hedging of spread options. These options are ubiquitous in the financial markets, whether they be equity, fixed income, foreign exchange, commodities, or energy markets. As a matter of introduction, we present a general overview of the common features of all spread options by discussing in detail their roles as speculation devices and risk management tools. We describe the mathematical framework used to model them, and we review the numerical algorithms actually used to price and hedge them. There is already extensive literature on the pricing of spread options in the equity and fixed income markets, and our contribution is mostly to put together material scattered across a wide spectrum of recent textbooks and journal articles. On the other hand, information about the various numerical procedures that can be used to price and hedge spread options on physical commodities is more difficult to find. For this reason, we make a systematic effort to choose examples from the energy markets in order to illustrate the numerical challenges associated with these instruments. This gives us a chance to discuss an interesting application of spread options to an asset valuation problem after it is recast in the framework of real options. This approach is currently the object of intense mathematical research. In this spirit, we review the two major avenues to modeling energy price dynamics. We explain how the pricing and hedging algorithms can be implemented in the framework of models for both the spot price dynamics and the forward curve dynamics.

**Key words.** spread options, energy markets, derivative pricing theory, closed form approximations

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## 1. Introduction.

The main goal of this survey is to address one of the most fundamental challenges facing applied mathematicians in the financial arena: pricing and hedging financial instruments for which closed form formulae cannot be derived. Following the pioneering work of Black, Scholes and Merton, pricing and hedging schemes have been developed for all sorts of financial derivatives, and their implementations are responsible for some of the most remarkable successes of the applications of mathematics to an important sector of the world economy, although some would argue that they are also responsible for some of the most spectacular failures. Despite the well-publicized success of this theory, many liquid instruments remain without efficient and reliable pricing and hedging methodologies. We use the example of spread options to review recent attempts to overcome the difficult problems created by the lack of

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closed form formulae for prices and sensitivities, and we illustrate the implementation of the mathematical results for the case of energy markets.

A *spread* option is an option written on the difference of two underling assets, whose values at time $t$ we denote by $S_1(t)$ and $S_2(t)$. We consider only options of the European type for which the buyer has the right to be paid, at the maturity date $T$, the difference $S_2(T) - S_1(T)$, known as the spread. To exercise the option, the buyer must pay at maturity a prespecified price $K$, known as the *strike*, or the *exercise price* of the option. In other words, the payoff of a spread option at maturity $T$ is $(S_2(T) - S_1(T) - K)^+$, where we use the usual notation $x^+$ for the positive part of $x$, i.e., $x^+ = \max\{x, 0\}$. The fundamental question of pricing is then: How much is such an option currently worth, i.e., what should the *purchase price* of this option be?

Such options are one of the simplest generalizations of the classical call and put options (options to buy or sell a *single* asset at a prespecified price at a future date). The latter are the major building blocks of modern finance theory, and any newcomer to this field should read the fundamental papers of Black and Scholes [6] and Merton [49]. These represent a spectacular breakthrough, well deserving of a Nobel prize. Surprisingly, the state of affairs for the multivariate extensions that we are considering in this paper is still rather unsatisfactory.

Whether the motivation comes from speculation, basis risk mitigation, or even asset valuation, the use of spread options is widespread despite the fact that the development of pricing and hedging techniques has not followed at the same pace. These options can be traded on an exchange, but the bulk of the volume comes from over-the-counter trades. They are designed to mitigate adverse movements of several indexes, hence their popularity. Because of their generic nature, spread options are used in markets as varied as fixed income markets, currency and foreign exchange markets, commodity futures markets, and energy markets.

Spread options are extremely simple in nature. However, they already contain features at the source of some of the most acute problems in contemporary mathematical finance. They offer a good test bed to understand the foundations of the theory and the behavior of market participants. One of our goals is to review existing literature on the subject. In doing so, we include a self-contained discussion of all the pricing and hedging methodologies known to us. We implemented most of the pricing algorithms whose existence we are aware of, and for the purpose of comparison, we report their numerical performances and give evidence of their relative accuracy and computing times.

Another interesting feature of spread options is that they became a risk mitigation tool of crucial importance in the energy markets. These markets bear some resemblance to the equity and fixed income markets, but important differences make their analysis quite unique. Also, dealing with energy markets will allow us to discuss other contemporary mathematical modeling challenges. For example, we shall systematically address the consistency issue, namely, we shall repeatedly ask the question: Is the model consistent with the observed forward curves? These curves are given by the graphs of the prices of the forward contracts when viewed as functions of the contract maturities. So on any given day, the forward curve gives the expectations of the market for the future prices of the commodity.

Standard stock market theory relies on probabilistic models for the dynamics of stock prices and uses arbitrage arguments to price derivatives. In most models, futures and forward contract prices are simply the current (spot) price of the stock corrected for growth at the current interest rate. This simple relationship between spot and forward prices does not hold in the commodity markets, and we will repeatedly mention seasonality and mean reversion as the main culprits. In order to reconcile the physical commodity market models with its equity relatives, researchers have used several tricks to resolve this apparent anomaly, and consistency with the no-arbitrage theory is restored most often by adding other factors such as the cost of storage and convenience yields to the stochastic factors driving the models; see, e.g., [31], [15], [58], [37], [52], and the more recent papers [51] and [5]. Still, the main limitation of these methods remains the inherent difficulties in modeling these unobserved factors, and the proposal to use stochastic filtering techniques to estimate them, though very attractive, has not yet fully succeeded in resolving these problems.

The energy markets have seen rapid changes in the last decade, mostly because of the introduction of electricity trading and the restructuring of the power markets. The diversity in the statistical characteristics of the underlying indexes on which financial instruments are written, together with the extreme complexity of the derivatives traded, makes the analysis of these markets an exciting challenge to mathematically inclined observers. This paper resulted in part from our study of these markets and our desire to better understand their idiosyncrasies. The reader is referred to [25] for a clear introduction to the intricacies of the energy markets, and to the recent texts [39], [60], and [7] for the economic and public policy issues specific to the electricity markets. Our emphasis here will be different since we are concerned only with the technical aspects of energy trading and risk management. Several textbooks are devoted to mathematical models of, and risk management issues in, the energy markets. The most frequently quoted are [56] and [14], but this may change with the recent publication of the book [26], which devotes an entire section to spread options and a chapter to correlation. We shall use many of the models and procedures presented in [14], even though this paper concentrates on spread options and [14] does not deal with cross-commodity instruments.

We start with a review of the various forms of spread options in section 2. We give examples of instruments traded in the equity, fixed income, and commodity markets. Understanding this diversity is paramount to understanding the great variety of mathematical models and of pricing recipes that have appeared in the technical literature.

Like many financial instruments, a derivative transaction involves two agents or counter-parties that, for the sake of simplicity, we shall call the buyer and the seller. Financial theory assumes that both agents act rationally, and that they both include the future payments associated with the derivative in the cash flow analysis of their respective portfolios. These agents behave optimally in order to maximize their respective expected utilities, taking into account all the possible (random) scenarios for the future evolutions of the instruments in their portfolios. In this way both the buyer and the seller determine what the derivative is worth to them, arriving at a *buyer price* and a *seller price*, which are not necessarily the same in general. The reasons for this gap include imbalances between offer and demand, illiquidity of the instrument, transaction costs, etc., but the transaction can still take place when buyer and seller decide to settle on a common price. In fact, when modeling such a transaction, if all the sources of randomness can be hedged perfectly, which is the case when the number of independent sources of randomness in the model is equal to the number of tradable liquid instruments, then arbitrage arguments force these two prices to be the same. In this case we say that the market model is complete. It is in this framework of complete market models that Black, Scholes and Merton produced their groundbreaking analysis.

The mathematical framework for *risk-neutral pricing* of spread options is introduced in section 3. Even though most of the spread options require only the statistics of the underlying indexes at one single point in time, namely, the time-of-maturity of the option, these statistics are usually derived from a model of the time evolution of the values of the indexes. We also give a short discussion of calendar spreads, where the joint distribution of the same underlying index at different times is needed. We introduce the *stochastic differential equations* used to model the dynamics of the underlying indexes. The price of a spread option is given by an expectation over the sample paths of the solution of this system of stochastic differential equations. One usually assumes that the coefficients of the stochastic differential equations are Markovian. In this case, the price is easily seen to be the solution of a parabolic partial differential equation (PDE). This connection between solutions of stochastic differential equations and solutions of PDEs is the cornerstone of Itô's stochastic calculus, and it has been exploited in many financial applications. Only in very exceptional situations do these equations have solutions given by closed form formulae. *PDE solvers*, *tree methods*, and *Monte Carlo methods* are most commonly used to produce numerical values approximating the price of a spread option. Because the applied mathematics community is more familiar with PDE solvers than with the other two, we spend more time reviewing the tree and Monte Carlo methods and the specifics of their implementations in the pricing of spread options. We also address the issue of the quantification of the dependencies of the price upon the various parameters of the model. We emphasize the crucial role of these sensitivities in a risk management context by explaining their roles as hedging tools.

As we will see in the discussion of section 3, the model of utmost importance is *Samuelson's model*, in which the distributions at the time-of-maturity of the indexes underlying the spread option are log-normal. This model is also known as the *Black–Scholes model* because these authors used it in their groundbreaking work on option pricing. We shall concentrate most of our efforts on understanding the underpinnings of the model assumptions and their consequences on the statistics of the indexes underlying the options. The core analysis concerns the pricing of spread options on two correlated geometric Brownian motions, and from there, we derive applications to more general situations. This is not the only possible approach. For example, Duan and Pliska recently proposed in [21] to price spread options on co-integrated time series. Co-integration could provide an attractive alternative, but this line of research is still in an embryonic stage, and it will take some time to overcome the technical difficulties inherent in its implementation.

Section 4 presents the first approximation procedure leading to a full battery of closed form expressions for the price and the hedging portfolios of a spread option. It is based on a simpleminded remark: as evidenced by a quick look at empirical samples, the distribution of the difference between two random variables with log-normal distributions looks very much like a normal distribution. This is the rationale for the first of the three approximation methods that we review. In this approach, one refrains from modeling the distributions of the indexes separately and instead models the distribution of their difference. As we just argued, it is then reasonable to assume that the latter is normally distributed. This model is called *Bachelier's model* because it is consistent with a model of the spread dynamics based on a single Brownian motion, in the same way Bachelier originally proposed to model the dynamics of the value of a stock by a continuous time process generalizing the notion of random walk. Little did he know that he was ahead of Einstein introducing the process of Brownian motion. We give a complete analysis of this model. We derive explicit formulae for the option prices in the original form of the model, as well as after the model is adjusted for consistency with observed forward curves. In this section, we also examine in detail the numerical performance of the pricing formula by comparing its results to the exact values when the driving dynamics are actually given by *geometric Brownian motions*, the model that we study next.

In section 5 we turn our attention to the particular case of a spread option with log-normal underlying indexes and strike $K = 0$. As in the case of the Bachelier model, it is possible to give a Black–Scholes-type formula for the price of the option. This formula was first derived by Margrabe in [47]. It cannot be extended to the general case $K \neq 0$, and this is the main reason for the investigations that we review in this paper. Besides the fact that the case $K = 0$ leads to a solution in closed form, it has also a practical appeal to the market participants. Indeed, it can be viewed as an option to exchange one product for another at no cost. Let us imagine for the sake of illustration that we are interested in owning, at a given time $T$ in the future, either one of two instruments whose prices at time $t$ we denote by $S_1(t)$ and $S_2(t)$, and that choosing which one to buy has to be done now at time $t = 0$. Naturally we would like to own the one that will be worth more. If we fear that the difference in value may be significant at time $T$, choosing the first instrument and also purchasing a spread option with strike $K = 0$ is the best way to guarantee that we will end up financially in the same situation as if we had chosen the more valuable instrument. Indeed, if the second instrument ends up being the more valuable, i.e., if $S_2(T) > S_1(T)$, then the payoff $S_2(T) - S_1(T)$ of the option will exactly compensate us for our wrong choice. The only cost to us will be the purchase of the spread option.

The main mathematical thrust of the paper is section 6, where we review the application to the case of spread options of the recent results of [10] on the pricing of basket options, and where we compare the numerical performance of the algorithms introduced in [10] to the approximations based on Bachelier's approach and a lesser-known procedure called Kirk's approximate pricing formula. The basic problem is the pricing and hedging of the simplest spread option (i.e., a European call option on the difference of two underlying indexes) when the risk-neutral dynamics of the values of the underlying indexes are given by correlated geometric Brownian motions. The results of [10] are based on a systematic analysis of expectations of functions of linear combinations of log-normal random variables. The motivation for this analysis comes from the growing interest in basket options, whose pricing involves the computation of these expectations when the number of log-normal random variables is large. These products are extremely popular, as they are perceived as a safe diversification tool. But a rigorous pricing methodology is still missing. The authors of [10] derive lower bounds in closed form, and they propose an approximation to the exact value of these expectations by optimizing over these lower bounds. The performance of their numerical scheme when applied to spread options is always as good as the results of Kirk's formula. But beyond the greater generality of basket options, the main advantage of their approach is the fact that it also provides a set of approximations for all the sensitivities of the spread option price, an added bonus that at the same time makes risk management possible. We review the properties of these approximations, from both a theoretical and a numerical point of view, by quantifying their accuracy on numerical simulations. The reader interested in detailed proofs and extensive numerical tests is referred to [10].

In preparation for our discussion of the practical examples discussed in the last sections of the paper, we devote section 7 to a detailed presentation of the kind of data available to the energy market participants. The special characteristics of these data not only justify the kind of notation and assumptions we use but also serve as a yardstick to quantify how well the pricing algorithms do. The geometric Brownian motion assumption of Samuelson's theory is not realistic for most of the spread options traded in the energy markets. Indeed, most energy commodity indexes have a strong seasonal component, and they tend to revert to a long-term mean level, this mean level having the interpretation of *cost of production.* These features are not accounted for by the plain geometric Brownian motion model of Samuelson. Section 8 deals with the extension of the results of section 6 to the case of spread options on the difference of indexes whose risk-neutral dynamics include these features. We also show how to include jumps in the dynamics of these indexes. This is motivated by the pricing of spark spread options that involve electricity as one of the two underlying indexes, or the pricing of calendar spread options on electric power.

Up until section 9, we work only with stochastic differential equation models for the indexes underlying the spread. In the case of the energy markets, the natural candidates for these underlying indexes are the commodity spot prices, and these models are usually called spot price models. See, e.g., Chapters 6 and 7 of [14]. According to the prevailing terminology, they are one-factor models for the term structure of forward prices. But it should be emphasized that our analysis extends easily, and without major changes, to the multifactor models, at least as long as the distributions of the underlying indexes can be constructed from log-normal building blocks. This is the case for most of the models used in the literature on commodity markets. See, e.g., [31], [15], [58], [37], [52], [51] or the textbooks [56] and [14].

In general, energy commodities do not behave much differently from other physical commodities. For example, they exhibit mean reversion, as we will mention quite often in this paper. Moreover, their prices tend to be subject to strong seasonal factors. Surprisingly enough, some energy commodities do not exhibit much seasonality. This is the case for crude oil, for example. Natural gas has become one of the favorites of analysts and research in academia because clean historical data are readily available in large quantities, and they offer a test bed for the analysis of seasonality and mean reversion. However, one of the energy commodities stands out because of its unique features: electricity. Indeed, its price is a function of factors as diverse as (1) instant perishableness, (2) strong demand variations due to seasonality and geographic location, (3) extreme volatility and sudden fluctuations caused by changes in temperature, precipitation, etc., and (4) physical constraints in production (start-ups, ramp-ups) and transmission (capacity constraints). It is by far the most difficult commodity index to model and predict. Derivative pricing and risk management present challenges of a new dimension. However, what appears to be a nightmare for policy makers and business executives is also a tremendous opportunity for the academic community. The need for realistic mathematical models and rigorous analytics is a very attractive proposition for the scientific community at large.

The last section of the paper is concerned with *forward curve models.* Using ideas from the Heath–Jarrow–Morton theory (HJM) developed for the fixed income markets in [35], the starting point of section 9 is a set of equations for the stochastic time evolution of the entire forward curve. This is a departure from the approach used in the previous sections, where the dynamics of the spot prices were modeled, and where consistency with the existing forward curves was only an afterthought. We give a detailed account of the fitting procedure based on principal components analysis (PCA). The importance of this data analysis technique for forward curve model fitting was pointed out by Litterman and Scheinkman [46] in the case of the fixed income markets. Its usefulness in the analysis of energy markets has been subsequently recognized, and applications to crude oil are given, for example, in Alexander's book [1]. We illustrate the use of PCA in the more challenging case of commodities like natural gas and electric power, whose forward curves contain strong seasonal components. In any case, restricting the coefficients of the stochastic differential equations to be deterministic leads again to log-normal distributions, and the results reviewed in this paper can be applied. We show how to price calendar spreads and spark spreads in this framework.

## 2. Zoology of the Spread Options.

Even though the term *spread* is sometimes understood as the difference between the bid and ask prices (for example, one often says that liquid markets are characterized by narrow bid/ask spreads), the term is most frequently used for the difference between two indexes: the spread between the yield of a corporate bond and the yield of a Treasury bond, the spread between two rates of returns, etc., are typical examples. Naturally, a spread option is an option written on the difference between the values of two indexes. But as we are about to see, its definition has been loosened to include all forms of options written as a linear combination of a finite set of indexes. In the currency and fixed income markets, spread options are based on the difference between two interest or swap rates, two yields, etc. In the commodity markets, spread options are based on the differences between the prices of the same commodity at two different locations (location spreads) or between the prices of the same commodity at two different points in time (calendar spreads), or between the prices of inputs to, and outputs from, a production process (processing spreads), as well as between the prices of different grades of the same commodity (quality spreads). The New York Mercantile Exchange (NYMEX) offers the only exchange-traded options on energy spreads: the heating oil/crude oil and gasoline/crude oil crack spread options.

The following review is far from exhaustive. It is merely intended to give a flavor of the diversity of spread instruments in order to justify the variety of mathematical models and pricing algorithms found in the technical literature on spreads. In this paper, most of the emphasis is placed on cross-commodity spreads because of the tougher mathematical challenges they present. As we shall see, single commodity spreads (typically calendar spreads) are usually easier to price.

### 2.1. Spread Options in Currency and Fixed Income Markets.

Spread options are quite common in the foreign exchange markets where spreads involve interest rates in different countries. The French–German and Dutch–German bond spreads are used because the economies of these countries are intimately related. A typical example is the standard cross-currency spread option, which pays at maturity $T$ the amount $(\alpha Y_1(T) - \beta Y_2(T) - K)^+$ in currency 1. Here $\alpha$, $\beta$, and $K$ are positive constants. The underlying indexes $Y_1$ and $Y_2$ are swap rates in possibly different currencies, say 2 and 3. The pricing of these spread options is usually done under some form of log-normality assumption via numerical integration of the Margrabe formulae derived in section 5 below. More elaborate forms of this approach are used to price quanto-swaptions. See, for example, [8].

In the U.S. fixed income market, the most liquid spread instruments are spreads between maturities, such as the NOB spread (notes - bonds) and spreads between quality levels, such as the TED spread (treasury bills - Eurodollars). The MOB spread measures the difference between municipal bonds and treasury bonds. See [2] for an econometric analysis of the market efficiency of these instruments. Spreads between treasury bills and treasury bonds have been studied in [42] and [23]. A detailed analysis of a spread option between the three-month and the six-month LIBORs (London Interbank Offered Rate) is given in [10], where some of the mathematical tools reviewed in this paper were introduced.

**2.2. Spread Options in Agricultural Futures Markets.** Several spread options are frequently traded in agricultural futures markets. For the sake of definiteness, we concentrate on the so-called *crush spread* traded on the Chicago Board of Trade (CBOT). It is also known as the *soybean complex spread.* The underlying indexes comprise futures contracts of soybean, soybean oil, and soybean meal. The unrefined product is the soybean, and the derivative products are meal and oil. This spread is known as the *crush spread* because soybeans are processed by crushing. The soybean crush spread is defined as the value of meal and oil extracted from a bushel of soybeans minus the price of a bushel of soybeans. Notice that the computation of the spread requires three prices as well as the yield of oil and meal per bushel. The crush spread gives market participants an indication of the average gross processing margin. It is used by processors to hedge cash positions, or for pure speculation by market participants.

The crush spread relates the cash market price of the soybean products (meal and oil) to the cash market price of soybeans. Since soybeans, soybean meal, and soybean oil are priced differently, conversion factors are needed to equate them when calculating the spread. On the average, crushing one bushel (i.e., 60 pounds) of soybeans produces 48 pounds of meal and 11 pounds of oil. Consequently, the value $[CS]_t$ at time $t$ of the crush spread in dollars per bushel can be defined as

$$[CS]_t = 48[SM]_t/2000 + 11[SO]_t/100 - [S]_t, \tag{2.1}$$

where $[S]_t$ is the futures price at time $t$ of a soybean contract in dollars per bushel, $[SO]_t$ is the futures price at time $t$ of a contract of soy oil in dollars per 100 pounds, and $[SM]_t$ is the price at time $t$ of a soy meal contract in dollars per ton. If we think of the crushing cost as a real constant, then crushing soybeans is profitable when the spread $[CS]_t$ is greater than that real constant. The crush spread was analyzed from the point of view of market efficiency in [41].

**2.3. Spread Options in Energy Markets.** In the energy markets, beside the *temporal spread* traders, who try to take advantage of the differences in prices of the same commodity at two different dates in the future, and the *locational spread* traders, who try to hedge transportation/transmission risk exposure from futures contracts on the same commodity with physical deliveries at two different locations, most of the spread traders deal with at least two different physical commodities. In the energy markets spreads are typically used as a way to quantify the cost of production of refined products from the complex of raw material used to produce them. The most frequently quoted spread options are the crack spread options and the spark spread options, which we review in detail in this section. Crack spreads are often called paper refineries while spark spreads are sometimes called paper plants.

**Crack Spreads.** A *crack spread* is the simultaneous purchase or sale of crude against the sale or purchase of refined petroleum products. These spread differentials, which represent refining margins, are normally quoted in dollars per barrel by converting the product prices into dollars per barrel and subtracting the crude price. They were introduced in October 1994 by the NYMEX with the intent of offering a new risk management tool to oil refiners.

For the sake of illustration, we describe the detailed structure of the most popular crack spread contracts. These spreads are computed on the daily futures prices of crude oil, heating oil, and unleaded gasoline.

- The $3:2:1$ *crack spread* involves three contracts of crude oil, two contracts of unleaded gasoline, and one contract of heating oil. Using self-explanatory notation, the defining formula for such a spread can be written as

$$[CS]_t = \frac{2}{3}[UG]_t + \frac{1}{3}[HO]_t - [CO]_t, \tag{2.2}$$

  which means that at any given time $t$, the value (in U.S. dollars) $[CS]_t$ of the $3:2:1$ crack spread underlying index is given by the right-hand side of formula (2.2), where $[UG]_t$, $[HO]_t$, and $[CO]_t$ denote the prices at time $t$ of a futures contract of unleaded gasoline, heating oil, and crude oil, respectively. A modicum of care should be taken in numerical implementations of formula (2.2) with real data. Indeed, crude oil prices are usually quoted in "dollars per barrel," while unleaded gasoline and heating oil prices are quoted in "dollars per gallon." A simple conversion needs to be applied to the data using the fact that there are 42 gallons per barrel. This point is of no consequence from the mathematical point of view. However, it should not be overlooked in practice.
- The $1:1:0$ *gasoline crack spread* involves one contract of crude oil and one contract of unleaded gasoline. Its value is given by the formula

$$[GCS]_t = [UG]_t - [CO]_t.$$

- The $1:0:1$ *heating oil crack spread* involves one contract of crude oil and one contract of heating oil. It is defined by the formula

$$[HOCS]_t = [HO]_t - [CO]_t.$$

Notice that the first example is computed from three underlying indexes, while the remaining two examples involve only two underlying indexes. Most of our analysis will concentrate on spread options written on two underlying indexes.

Crack spread options are the subject of a large number of papers attempting to demonstrate the stationarity of the spread time series by means of a statistical quantification of the co-integration properties of the underlying index time series from which the spread is computed. Most of these papers are also concerned with the profitability of spread-based trading strategies, a subject that we would not dare to consider here. The interested reader is referred, for example, to [32], [33], and the references therein for further information on these topics.

**Spark Spreads.** A *spark spread* is a proxy for the cost of converting a specific fuel (usually natural gas) into electricity at a specific facility. It is the primary cross-commodity transaction in the electricity markets. Mathematically, it can be defined as the difference between the price of electricity sold by a generator and the price of the fuel used to generate it, provided these prices are expressed in appropriate units. The most commonly traded contracts include the following:

- The $4:3$ *spark spread* that involves four electric contracts and three contracts of natural gas. Its value is given by

$$[SS]_t^{4,3} = 4[E]_t - 3[NG]_t.$$

- The 5 : 3 *spark spread* involves five electric contracts and three contracts of natural gas. Its value is given by

$$[SS]_t^{5,3} = 5[E]_t - 3[NG]_t.$$

But whether or not they are traded in this form, the most interesting spark spread options are European calls on an underlying index of the form

$$S_t = F_e(t) - H_{\text{eff}} F_g(t),$$

where $F_e(t)$ and $F_g(t)$ denote the prices of futures contracts on electric power and natural gas, respectively, and where $H_{\text{eff}}$ is the heat rate, or the efficiency factor, of a power plant. The spark spread can be expressed in \$/MWh (U.S. dollars per megawatt hour) or any other applicable unit. It is calculated by multiplying the price of gas (for example, in \$/MMBtu) by the *heat rate* (in Btu/KWh), dividing by 1,000, and then subtracting the electricity price (in \$/MWh). One of the most intriguing use of spark spread options is in real asset valuation or capacity valuation. This encapsulates the economic value of the generation asset used to produce the electricity. The heat rate is often called the *efficiency*. Indeed, a natural gas–fired unit can be viewed as a series of spark spread options:

- when the heat rate implied by the spot prices of power and gas is above the operating heat rate of the plant, then the plant owner should buy gas, produce power, and sell it for profit;
- otherwise, the plant owner should shut down its operation, i.e., when the heat rate implied by the spot prices of power and gas is below the operating heat rate of its plant.

If an investor/producer wonders how much to bid for a power plant, he can easily estimate and predict the real estate and the hardware values of the plant with standard methods. But the operational value of the plant is better captured by the sum of the prices of spark spread options than with the present value method based on the computation of discounted future cash flows (the so-called DCF method in the jargon of the business). This *real option* approach to plant valuation is one of the strongest incentives to develop a better understanding of the risk of spark spread options.

## 3. Spread Option Pricing: Mathematical Setup.

We described actual examples of spread options, and we saw that many involve more than two underlying instruments. Moreover, some of the mathematical results that we review in this paper can be applied to large baskets of instruments. Nevertheless, for the sake of simplicity we restrict ourselves to the case of spreads between two underlying asset prices. We consider two indexes $S_1 = \{S_1(t)\}_{t\geq 0}$ and $S_2 = \{S_2(t)\}_{t\geq 0}$ evolving in time. We call them indexes instead of prices because, even though $S_1(t)$ and $S_2(t)$ will usually be the prices of stocks or commodities at time $t$, they could as well be interest rates, exchange rates, or compound indexes computed from the aggregation of other financial instruments. The spread is naturally defined as the instrument $S = \{S(t)\}_{t\geq 0}$, whose value at time $t$ is given by the difference

$$S(t) = S_2(t) - S_1(t), \qquad t \geq 0. \tag{3.1}$$

Buying such a spread is buying $S_2$ and selling $S_1$. It is instructive to view a spread option as a standard derivative on the underlying instrument whose value at time $t$ is the spread $S(t)$ so defined.

**3.1. The Black–Scholes Pricing Paradigm.** Our goal is to price European options on the spread $S(t)$. We briefly review the modern way of pricing derivatives, and for the purposes of the present discussion, we should not limit ourselves to the case of the spread defined by (3.1), and instead we should think of $S(t)$ as the price of a traded asset or a more general financial instrument. The reader is referred to standard textbooks [38], [22], [43], [53], and [45] for a more in-depth treatment of that question.

A European call option is defined by a date $T$, called the date of maturity, and a positive number $K$, called the strike or exercise price, and it gives the right to its owner to acquire at time $T$ one unit of the underlying instrument at the unit price $K$. Assuming that this underlying instrument can be resold immediately on the market for its price $S(T)$ at that time, this means that the owner of the option will secure the amount $S(T) - K$ when the value of the underlying instrument at time $T$ is greater than $K$, i.e., when $S(T) > K$, and nothing otherwise since, in that case, he will act rationally and will not exercise the option. So the owner of the option is guaranteed to receive the payout

$$(S(T) - K)^+ = (S(T) - K)\mathbf{1}_{\{S(T)>K\}} \tag{3.2}$$

at maturity $T$. We denote by $p$ the price at time 0 of this European call option with date of maturity $T$ and strike $K$. More generally, we shall denote by $p_t$ its price at time $t < T$. The Black–Scholes pricing paradigm gives, in an amazingly neat fashion, a rational way to compute this price. Even thirty years later, there is still nothing better than going back to the original papers by Black and Scholes [6] and Merton [49] to appreciate the depth of their contributions. Their idea is that uncertain future cash flows (like the payoff of our spread option) can be replicated by a self-financing strategy. Following this paradigm, today's price of an option has to be the cost of setting up the replicating strategy. If this were not the case, selling the more expensive of the two and buying the other one would result in an arbitrage (i.e., the possibility of making money starting from nothing). Arbitrage must be ruled out in a viable market. The original derivation proposed by Black and Scholes is relying heavily on the solution of a parabolic PDE which is known as the Black–Scholes PDE. It states that, in the absence of arbitrage, the price $p(t, x)$ of a call option at a time $t$ when the underlying index is $S(t) = x$ is given by the solution of the backward parabolic PDE

$$\partial_t p(t, x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} p(t, x) + rx\partial_x p(t, x) - rp(t, x) = 0 \tag{3.3}$$

with terminal condition $p(T, x) = (x - K)^+$. Here $T$ is the maturity date of the call option, $K$ is the strike, $r$ denotes the short rate of interest giving the time value of money, and $\sigma$ denotes the volatility of the underlying asset. The mathematical definition of this notion is given below.

Simultaneously, it was also shown that the price $p(t, x)$ could be represented both as the expectation of the discounted future cash flows for a probability structure called *risk-neutral*. This last representation is extremely useful in practice because it lends itself to natural generalization to more complex situations involving more general underlying instruments with more sophisticated stochastic dynamics. Also, computing an expectation is often more elementary and transparent than solving a PDE, especially in higher dimension. The price to pay—so to speak—is to understand the very important concept of *risk-neutral* probability measure. To compute the discounted future cash flows under this measure, we need to know the dynamics of our assets under that measure. Without going into detail, it turns out that this risk-neutral probability measure makes the drift of these price dynamics equal to the interest rate. For the sake of completeness, we should mention that the mathematical theory giving the rigorous framework for the switch from the historical probability (quantifying the statistics of the prices observed historically) to the risk-neutral probability (governing the statistics of the derivative prices) is known as the Cameron–Martin–Girsanov theory. This covers the transformations of probability measures that preserve the measure class, i.e., which take a probability measure into an equivalent one. The theory of these transformations was originally developed for stochastic analysis purposes and filtering applications. However, it appears that it is tailor-made for the mathematical framework of risk-neutral pricing of financial instruments. We shall see an example of such a transform in section 5.

The Black–Scholes formula gives a value for $p$ when $S(T)$ has a log-normal distribution under the risk-neutral measure. In general, following the previous discussion, it is given by

$$p = \mathbb{E}\{e^{-rT}(S(T) - K)^+\},$$

where, as before, $r$ is the short rate of interest. The exponential factor takes care of the discounting, and the probability measure under which the expectation is to be computed is risk-neutral. When $S(T)$ has the log-normal distribution, this expectation can be computed explicitly, yielding

$$p = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{3.4}$$

where

$$d_1 = \frac{\ln\left(S(0)e^{rT}/K\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{3.5}$$

Here and throughout the paper, we use the notation $\varphi(x)$ and $\Phi(x)$ for the density and the cumulative distribution function of the standard normal $N(0,1)$ distribution, i.e.,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \text{and} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

The quantity $S(0)e^{rT}/K$ appearing in the Black–Scholes formula is often called the moneyness of the option. This ratio compares the forward price of the stock to the strike $K$. The option is said to be *in the money* when this ratio is greater than one, and *out of the money* otherwise. Notice also that $\sigma$ is the only parameter appearing in formula (3.4) that was not introduced before. It is called the volatility for reasons that will become clear later. For now, note that it is related to the variance of the logarithm of $S(T)$ through

$$\mathrm{var}\{\ln(S(T))\} = \sigma^2 T.$$

The Black–Scholes pricing paradigm was later extended to more general classes of random variables, and even to situations where the dynamics of the underlying index $S(t)$ are given by a stochastic differential equation in the sense of Itô, leading to a stochastic process $S(t)$, possibly with jumps as long as it remains a semimartingale. In any case, the price of the option is given by the risk-neutral expectation of the discounted payout of the option at maturity. In general, the discounting rate $r \geq 0$ is nothing but the short interest rate. But in some cases it can contain corrections taking into account the rate of dividend payments, or the convenience yield in the case of physical commodities. For the sake of the present discussion, we shall assume that the discounting rate is the short interest rate, which is assumed to be deterministic and constant throughout the life of the option (i.e., until the maturity date $T$).

The last element of Black–Scholes pricing theory that we will refer to is the call-put parity. Using the fact that a real number is equal to the difference between its positive and negative parts (i.e., $x = x^+ - (-x)^+$), and the fact that the payout of a European put option with maturity $T$ and strike $K$ is given by $(K - S(T))^+$, the difference between the price $p$ of a European call option with maturity $T$ and strike $K$ and the price $p'$ of a European put option with same maturity and strike is given by

$$\begin{aligned} p - p' &= e^{-rT}\mathbb{E}\{(S(T) - K)^+\} - e^{-rT}\mathbb{E}\{(K - S(T))^+\} \\ &= e^{-rT}\mathbb{E}\{(S(T) - K)^+ - (K - S(T))^+\} \\ &= e^{-rT}\mathbb{E}\{S(T) - K\} \\ &= S(0) - e^{-rT}K, \end{aligned}$$

where we used the fact that $S(0) = \mathbb{E}\{e^{-rT}S(T)\}$ for risk-neutral expectation. This is a consequence of the fact that, in a risk-neutral world, all the discounted price processes of tradable instruments are martingales when applied to $\{e^{-rt}S(t)\}_t$. The formula

$$p - p' = S(0) - e^{-rT}K \tag{3.6}$$

that we just proved is called call-put parity. It shows that the analysis of European put option prices can be reduced to the analysis of the call option prices. We shall use a bivariate extension of this parity argument later in the paper.

So, coming back to our spread option pricing concern, according to the Black–Scholes pricing paradigm, the price $p$ of our spread option is given by the risk-neutral expectation:

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\}. \tag{3.7}$$

**Pricing by Computing a Double Integral.** It is important to remark that, even though we always define mathematical models by sets of prescriptions for the dynamics of the indexes $S_1$ and $S_2$, the fact that we are considering options of the European type implies that we do not really need the full dynamics when it comes to actually computing the price of a spread option. Indeed, the payout at maturity depends only upon the values of the indexes at time $T$, i.e., upon the values $S_1(T)$ and $S_2(T)$, regardless of the history of prices. So in order to compute the risk-neutral expectation (3.7) giving the price $p$, the only thing we need is the joint density of the couple $(S_1(T), S_2(T))$ of random variables under that particular risk-neutral measure. This density is usually called the state price density. So, ignoring momentarily the dynamics of the underlying indexes, we can write the price of a spread option as a double integral. More precisely,

$$e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} = e^{-rT}\int\int(s_2 - s_1 - K)^+ f_T(s_1, s_2)\, ds_1 ds_2$$

if we denote by $f_T(s_1, s_2)$ the joint density of the random variables $S_1(T)$ and $S_2(T)$. Computing the expectation by conditioning first by the knowledge of $S_1(T)$, we get

$$\begin{aligned}
&e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} \\
&\quad = e^{-rT}\mathbb{E}\{\mathbb{E}\{(S_2(T) - S_1(T) - K)^+|S_1(T)\}\} \\
&\quad = \int \mathbb{E}\{(S_2(T) - (s_1 + K))^+|S_1(T) = s_1\}f_{1,T}(s_1)ds_1 \\
&\quad = \int \left(\int (s_2 - s_1 - K)^+ f_{2,T|S_1(T)=s_1}(s_2)ds_2\right) f_{1,T}(s_1)ds_1,
\end{aligned}$$

where we used the notation $f_{1,T}(s_1)$ for the density of the first index $S_1(T)$ at the time $T$ of maturity, and the notation $f_{2,T|S_1(T)=s_1}(s_2)$ for the conditional density of the second index $S_2(T)$ at maturity, given that the first index is equal to $s_1$ at that time. The intermediate result shows that the price of the spread option is the integral over $s_1$ of the prices of European calls on the second index with strikes $s_1 + K$.

In the log-normal models, the conditional density $f_{2,T|S_1(T)=s_1}(s_2)$ is still log-normal, so the value of the innermost integral is given by the classical Black–Scholes formula with $s_1 + K$ as strike. This shows that the price of the spread option is an integral of Black–Scholes formulae with respect to the (log-normal) density of the first index. Pricing the spread option by computing these integrals numerically can *always* be done. But even a good approximation of the price $p$ is not sufficient in practice. Indeed—and this fact is too often ignored by the newcomers to financial mathematics—a pricing algorithm has to produce much more than a price if it is to be of any practical use, and this is the main reason why the search for closed form formulae is still such an active research area, even in these days of fast and inexpensive computers. It is difficult to explain why without getting into details that would sidetrack our presentation, but we nevertheless justify our claim by a few remarks, leaving the details to asides that we will sprinkle throughout the rest of the paper when appropriate.

**Why Do Practitioners Like Closed Form Formulae?** Both obvious and fundamental reasons can be given as answers. Among the obvious ones, we must stress the fact that closed form formulae are easily and quickly implemented. This may appear to be laziness but makes a great deal of sense to anyone who has actually seen the stress of the trading desk! Also a closed form formula will always give the same result, whereas Monte Carlo approximations, as we shall review later, always contain some randomness. It is also much faster. Besides these *natural* reasons, we would like to emphasize three more important points.

1. Let us for a moment put ourselves in the shoes of the seller of the option. From the time of the sale, he or she is exposed to the risk of having to pay (3.2) at the date of maturity $T$. This payout is random and cannot be predicted with certainty. As we already explained, the crux of the Black–Scholes pricing paradigm is to set up a portfolio and to devise a trading strategy that will have the same exact value as the payout at the time of maturity $T$, whatever the outcome is. The thrust of the discovery of Black and Scholes lies in proving that such a replication of the payoff was possible, and once this stunning statement was proved, the price of the option had to be the cost of initially setting up such a replicating portfolio. Replication of the payout of an option is obviously the best way to get a perfect hedge for the risk associated with the sale of this option. But what is even more remarkable is the fact that the components of the replicating portfolio are explicitly given by the derivatives of the price with

respect to the initial value of the underlying index. The partial derivatives of the price of the option with respect to the parameters of the model (initial value of the underlying instrument, interest rate, volatility, or instantaneous standard deviation, etc.) give the sensitivities of the price with respect to these parameters, and as such they quantify the sizes of the price fluctuations produced by infinitesimal changes in these economic parameters. These partial derivatives are of great importance to the trader and the risk manager, who both rely on their values. For this reason they are given special names, *delta, gamma, rho, vega,* etc., and are generically called the *Greeks.* Having a closed form formula for the price of the option usually yields closed formulae for the Greeks, which can then be evaluated rapidly and accurately. This is of great value to the practitioners, and this is one of the reasons, alluded to earlier, why people are searching so frantically for pricing formulae in closed forms.

2. Hedging is not the only reason why a pricing formula in closed form is far superior to a numerical algorithm. When a pricing formula can be inverted, one can infer values of the parameters (volatility, correlations, etc.) of the pricing model from price quotes for options with different maturities and strikes traded on the market. The values inferred in this way are called *implied.* They are of great significance, and are used by the market makers to price new instruments. They are also good financial indicators of how markets see these variables. The reader is referred to [10] for an illustration in the present context.

3. Finally, even the most sophisticated models are very crude approximations of reality, and traders prefer having simple and reliable closed form formulae whose behavior can be well understood. As far as the Black–Scholes formula is concerned, the reader is referred to [24], where widespread use of this particular closed form formula is justified because of its *robustness* with respect to model mis-specification.

**3.2. Markovian Models and PDEs.** In the previous section we saw that the price $p$ of a spread option is given by a risk-neutral expectation (3.7). In order to compute such an expectation, we need to specify the risk-neutral dynamics of the underlying indexes. Let us assume for the sake of definiteness that they satisfy a two-dimensional system of Itô's stochastic differential equations of the type

$$\begin{cases} dS_1(t) = S_1(t)\Big[\mu_1(t,\mathbf{S}(t))dt + \sigma_1(t,\mathbf{S}(t))[\rho(t,\mathbf{S}(t))d\tilde{W}_1(t) \\ \qquad\qquad\qquad\qquad + \sqrt{1-\rho(t,\mathbf{S}(t))}d\tilde{W}_2(t)]\Big], \\ dS_2(t) = S_2(t)\Big[\mu_2(t,\mathbf{S}(t))dt + \sigma_2(t,\mathbf{S}(t))d\tilde{W}_2(t)\Big], \end{cases} \tag{3.8}$$

where we use the notation $\mathbf{S}$ for the couple $(S_1, S_2)$, and where $\{\tilde{W}_1(t)\}_t$ and $\{\tilde{W}_2(t)\}_t$ are independent standard real-valued Wiener processes also called processes of Brownian motions. We shall use both terminologies alternatively. The specific form of the stochastic equations (3.8) comes from the fact that one implicitly tries to find an equation for the returns, and since the returns are of the form $[S(t+\Delta t) - S(t)]/S(t)$, such a form of (3.8) becomes natural when one thinks of letting the time interval $\Delta t$ go to zero. The intuitive interpretation of equations (3.8) is as follows: at each time $t$, the infinitesimal changes in the return on $S_i(t)$ are normally distributed with means $\mu_i(t,\mathbf{S}(t))\,dt$ and variance $\sigma_i^2(t,\mathbf{S}(t))\,dt$, $\rho(t,\mathbf{S}(t))$ giving the instantaneous correlation between these two conditionally normal random variables. We also assume that the coefficients $\mu_i$, $\sigma_i$, and $\rho$ are smooth enough for the existence and uniqueness of a strong solution. It is well known that a Lipschitz assumption with linear growth will do. See, for example, [43], [45], or [53]. But rather than giving technical conditions under which these assumptions are satisfied, we go on to explain how one can compute the expectation (3.7) giving the price of the spread option. This can be done by solving a PDE. This link is known as *Feynman–Kac representation*. Even though we shall not need this level of generality in what follows, we state it in the general case of a time-dependent stochastic short interest rate $r = r(t, \mathbf{S}(t))$ given by a deterministic function of $(t, \mathbf{S}(t))$.

PROPOSITION 3.1. *Let $u$ be a $C^{1,2,2}$-function of $(t, x_1, x_2)$ with bounded partial derivatives in $t$, $x_1$, and $x_2$ satisfying the terminal condition*

$$\forall x_1, x_2 \in \mathbb{R}, \quad u(T, x_1, x_2) = f(x_1, x_2)$$

*for some nonnegative function $f$, and the PDE*

$$\left(\partial_t + \frac{1}{2}\sigma_1^2 x_1^2 \partial^2_{x_1x_1} + \rho\sigma_1\sigma_2 x_1 x_2 \partial^2_{x_1x_2} + \frac{1}{2}\sigma_2^2 x_2^2 \partial^2_{x_2x_2} + \mu_1 x_1 \partial_{x_1} + \mu_2 x_2 \partial_{x_2} - r\right) u = 0 \tag{3.9}$$

*on $[0,T] \times \mathbb{R} \times \mathbb{R}$. Then for all $(t, x_1, x_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ one has the representation*

$$u(t, x_1, x_2) = \mathbb{E}\left\{ e^{-\int_t^T r(s,\mathbf{S}(s))ds} f(\mathbf{S}(T)) \middle| \, \mathbf{S}(0) = (x_1, x_2) \right\}.$$

Obviously, this result is a generalization of the Black–Scholes PDE as stated in (3.3). It is a classical example of the representation of solutions of parabolic PDEs as expectations over diffusion processes. Even though pure semigroup proofs can be provided, the most general ones rely on Itô's calculus and the so-called Feynman–Kac formula. We refer the reader interested in a detailed proof in the context of financial applications to [43] and [45].

In the case of interest to us (recall formula (3.7) giving the price of the spread option), we shall assume that the interest rate is a constant $r(t, (x_1, x_2)) \equiv r$, and we shall use the function $f(x_1, x_2) = (x_2 - x_1 - K)^+$ for the terminal condition.

**3.3. Geometric Brownian Motion Model and Black–Scholes Framework.** The system (3.8) gives a reasonably general setup for the pricing of the spread options. Indeed, most of the abstract theory (see, for example, [43]) can be applied. Unfortunately, this setup is too general for explicit computations, and especially the derivation of pricing formulae in closed form. Thus, we often restrict ourselves to more tractable specific cases. The most natural one is presumably the model obtained by assuming that the coefficients $\mu_i$, $\sigma_i$, and $\rho$ are constants independent of time and of the underlying indexes $S_1$ and $S_2$. Setting $W_1(t) = \rho\tilde{W}_1(t) + \sqrt{1-\rho^2}\tilde{W}_2(t)$ and $W_2(t) = \tilde{W}_2(t)$, we have that

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t), \qquad i = 1, 2, \tag{3.10}$$

where $\{W_1(t)\}_t$ and $\{W_2(t)\}_t$ are two Brownian motions with correlation $\rho$. The two equations can be solved separately. Indeed, they are coupled only through the statistical correlation of the two driving Brownian motions. The solutions are given explicitly by

$$S_i(t) = S_i(0) \exp\left[(\mu_i - \sigma_i^2/2)t + \sigma_i W_i(t)\right], \qquad i = 1, 2. \tag{3.11}$$

In this way each index process $\{S_i(t)\}_t$ appears as a geometric Brownian motion. The unexpected $-(\sigma_i^2/2)t$ appearing in the deterministic part of the exponent is due to the idiosyncrasies of Itô's stochastic calculus and is called Itô's correction. If the initial conditions $S_i(0) = x_i$ are assumed to be deterministic, then the distribution of the values $S_i(t)$ of the indexes are log-normal, and we can explicitly compute their densities. This log-normality of the distribution was first advocated by Samuelson, but it is often associated with the names of Black and Scholes because it is in this framework that these two authors derived their famous pricing formula for European call and put options on a single stock. Dynamics given by stochastic differential equations of the form (3.10) are at the core of the analyses reviewed in this paper. Let us emphasize once more that our interest in pricing forces us to consider risk-neutral dynamics. For this reason, the drifts $\mu_i$ need to be replaced by $r$ (the rate of interest) or $r - q_i$ (where $q_i$ is the dividend payment rate in case we are considering stocks bearing dividends). But whatever the drift choice, the major modeling choice here is that of continuous processes with specified volatilities $\sigma_i$.

**3.4. Numerics.** This subsection is devoted to the discussion of the most commonly used numerical methods to price and hedge financial instruments in the absence of explicit formulae in closed forms. We restrict ourselves to the Markovian models described above, and we review methods that are prevalent in the industry by illustrating their implementations in the case of the spread option valuation problem. Proposition 3.1 states that, at least in the Markovian case, the price of the spread option is the solution of a PDE. Consequently, valuing a spread option can be done by solving a PDE, and the first two of the four subsections below describe possible implementations of this general idea.

**Using PDE Solvers.** As explained earlier, the Feynman–Kac representation given in Proposition 3.1 suggests the use of a PDE solver to get a numerical value for the price of a spread option. Because of the special form of the stochastic dynamics of the underlying indexes, the coefficients in the second order terms of the PDE (3.9) can vanish and it appears as if the PDE is degenerate. For this reason, the change of variables $(x_1, x_2) = (\log u_1, \log u_2)$ is often used to reduce (3.9) to a nondegenerate parabolic equation. This PDE is three-dimensional: one time dimension and two space dimensions. There is extensive literature on the stability properties of the various numerical algorithms capable of solving these PDEs, and we shall refrain from going into these technicalities. We shall concentrate only on one very special tree-based numerical scheme. This explicit finite difference method was made popular by Hull in his book [38] in the case of a single underlying index. We present the details of the generalization necessary for the implementation in the case of cross-commodity spreads.

**Trinomial Trees.** Trees offer a very convenient way to visualize simplified models of stochastic dynamics for the prices of underlying instruments. Such stochastic time evolution of prices, together with the corresponding pricing procedures, has been very popular in finance because of their appealing intuitive nature. This is especially true for the so-called binomial tree, for they provide the best setup to highlight the various steps of the pricing of American options. Details on pricing procedures on binomial trees can be found in most introductory textbooks on derivative pricing.

Here, we choose to consider only trinomial tree models. Originally, these models were introduced to accommodate incomplete models for which the risk-neutral probability structure is not uniquely determined. Indeed, binomial tree models are complete in the sense that there is uniqueness of the risk-neutral probability measure. Moreover, there are other advantages to using trinomial trees: not only do they offer a chance to model more stylized properties of real markets such as incompleteness, but they also provide an appealing bridge with the pricing procedures based on PDE solvers. As pointed out by Hull and White, pricing on a binomial tree is equivalent to pricing solving a pricing PDE with an explicit finite difference scheme. We refer the reader interested in the use of a classical trinomial tree for the pricing of an option on a single underlying stock to Hull's book [38]. We proceed with a discussion of a generalization of this classical one-dimensional approach to the two-dimensional setting of spread options. This part of the paper is directly inspired by [18]. Since we have two underlying processes, we need a tree spanning two directions. More precisely, even though we keep the terminology of trinomial tree, each node leads to nine new nodes at the next time step. The computations are local, and we can assume, without any loss of generality, that all the coefficients of the diffusion equations are constant. Recall equation (3.11),

$$S_i(t) = S_i(0)\exp\left[\mu_i t - \frac{1}{2}\sigma_i^2 t + \sigma_i W_i(t)\right],$$

where the two Brownian motions $W_1$ and $W_2$ satisfy $\mathbb{E}\{W_1(t)W_2(t)\} = \rho t$. The basic idea behind the tree's construction is to discretize the mean-zero Gaussian vector $(\sigma_1 W_1(t), \sigma_2 W_2(t))$ with covariance matrix $\Sigma t$, where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

We first compute the eigenvalues and the corresponding eigenvectors of this covariance matrix and write it as

$$\Sigma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^*,$$

where $*$ denotes the transpose and

$$\lambda_1 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2+\sigma_2^2)^2 - 4(1-\rho^2)\sigma_1^2\sigma_2^2}\right),$$

$$\lambda_2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2+\sigma_2^2)^2 - 4(1-\rho^2)\sigma_1^2\sigma_2^2}\right),$$

$$\theta = \arctan\left(\frac{\lambda_1 - \sigma_1^2}{\rho\sigma_1\sigma_2}\right).$$

We can now write our Gaussian vector as follows:

$$\begin{pmatrix} \sigma_1 W_1(t) \\ \sigma_2 W_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta\sqrt{\lambda_1}B_1(t) - \sin\theta\sqrt{\lambda_2}B_2(t) \\ \sin\theta\sqrt{\lambda_1}B_1(t) + \cos\theta\sqrt{\lambda_2}B_2(t) \end{pmatrix},$$

where $(B_1, B_2)$ is a two-dimensional standard Brownian motion (i.e., with independent components). The discretization of our diffusion is then equivalent to the discretization of two independent standard Brownian motions. Let $X_1$ and $X_2$ be two independent and identically distributed random variables taking three values $(-h, 0, h)$ with probabilities $(p, 1-2p, p)$, respectively. The idea is to use $(X_1, X_2)$ to approximate the increment $(B_1(t+dt) - B_1(t), B_2(t+dt) - B_2(t))$. It is well known in financial engineering that if we take $p = 1/6$ and $h = \sqrt{3dt}$, then $X_i$ and $B_i(t+dt) - B_i(t)$ have the same first four moments. Thanks to the independence of $X_1$ and $X_2$ the probabilities for getting to each one of the nine new nodes follow. We organize them in a matrix:

$$\frac{1}{36}\begin{pmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{pmatrix}.$$

From this point on, the approximation of the price is done in exactly the same way as in the one-dimensional tree. We simulate the diffusion up to the terminal date and compute the probabilities attached to all the terminal nodes. We then compute the payout of the option at these nodes, i.e., at the terminal time and in the different states of the world, and we compute the desired expectation by computing the weighted average of the discounted payout with the probability weights.

**Monte Carlo Computations.** Most often, a good way to compute an expectation is to use a traditional Monte Carlo method. The idea is to generate a large number of sample paths of the process $\mathbf{S} = (S_1, S_2)$ over the interval $[0, T]$ for each of these sample paths to compute the value of the function of the path whose expectation we wish to evaluate, and then to average these values over the sample paths. The principle of the method is simple, and the computation of the sample average is usually quite straightforward; the only difficulty is in quantifying and controlling the error. Various methods of random sampling (stratification being one of them) and variance reduction (for example, importance sampling and introduction of antithetic variables) are used to improve the reliability of the results. Notice that, even when the function to integrate is only a function of the terminal value $\mathbf{S}(T)$ of the process $\mathbf{S}$, one generally has to generate samples from the entire path $\mathbf{S}(t)$ for $0 \le t \le T$, just because one does not know the distribution of $\mathbf{S}(T)$. Random simulation based on the dynamical equation of the process is often the only way to get at it. This requires the choice of a discretization time step $\Delta t$, and the generation of discrete time samples $\mathbf{S}(t_0 + j\Delta t)$ for $j = 0, 1, \ldots, N$ with $t_0 = 0$ and $t_0 + N\Delta t = T$, and these steps should be taken with great care to make sure that the (stochastic) numerical scheme used to generate these discrete samples produce reasonable approximations.

The situation is much simpler when we assume that all the coefficients are deterministic. Indeed, in such a case the joint distribution of the underlying indexes at the terminal time can be computed explicitly, and there is no need to simulate entire sample paths: one can simulate samples from the terminal distribution directly. This distribution is well known. As explained earlier, it is a joint log-normal distribution. The simulation of samples from the underlying index values at maturity is very easy. Indeed, if the coefficients are constant, the couple $(S_1(t), S_2(t))$ of indexes at maturity can be written in the form

$$S_1(T) = S_1(0)\exp\left[\left(\mu_1 - \sigma_1^2/2\right)T + \sigma_1\rho\sqrt{T}U + \sigma_2\sqrt{1-\rho^2}\sqrt{T}V\right],$$

$$S_2(T) = S_2(0)\exp\left[\left(\mu_2 - \sigma_2^2/2\right)T + \sigma_2\sqrt{T}U\right],$$

where $U$ and $V$ are two independent standard Gaussian random variables. The simulation of samples of $(U, V)$ is quite easy.

Note that the same conclusion holds true when the coefficients are deterministic, and possibly varying with time. Indeed, one can show that the terminal joint distribution of two underlying indexes is still log-normal. The difference is that the parameters are now functions (typically time averages) of the time-dependent coefficients. We shall present the details of these computations several times in what follows, so we refrain from dwelling on them at this stage.

**Quasi–Monte Carlo Methods.** Traditionally Monte Carlo methods use random number generators, deterministic algorithms that attempt to produce values that appear truly random by various measures. Instead, better results can often be obtained by using quasi–random numbers that are generated by algorithms developed with different criteria. They efficiently cover the interval of interest with points from the proper distribution, without necessarily appearing random. This can improve the rate of convergence of the Monte Carlo methods considerably. These methods have gained great popularity in the last few years, and the standard reference is Niederreiter [54]. Examples that support the idea that these methods are superior to the classical Monte Carlo methods in applications to finance can be found in, e.g., [55].

In order to shed more light on the differences with the standard Monte Carlo methods, it is possible to be a little more specific about the rationale of the quasi–Monte Carlo methods, without becoming too technical. For the purposes of the present discussion, we can assume, without any loss of generality, that the probability space over which our integral is to be computed is the hypercube $[0,1]^d$. With Monte Carlo methods we generate uniform random points in $[0,1]^d$. Quasi–Monte Carlo methods, however, use nonrandom points in order to have a more nicely uniform distribution. How nice it is can be quantified by the (star) discrepancy $D^*$. For a sequence $\mathbf{u} = \{u_i\}_{1\le i\le n}$ it is defined as

$$D^* = \sup_{x\in[0,1]^d} \left| \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \mathbf{1}_{\{u_i^j \le x^j\}} - \prod_{j=1}^{d} x^j \right|.$$

This number must be interpreted as a distance between the theoretical continuous uniform distribution and the discrete empirical distribution of the sequence $\mathbf{u}$. Moreover, it is exactly related to the error made by using quasi–Monte Carlo methods. There exist sequences (called low discrepancy sequences) that have a discrepancy of order $(\log n)^d/n$. It is the optimal bound on the discrepancy. This order of magnitude has to be compared to the order $1/\sqrt{n}$ of Monte Carlo methods. It is now clear that, at least for low-dimension problems, quasi–Monte Carlo methods should be favored.

**Fourier Transform Approximations.** The Fourier transform has become a popular tool in option pricing when the coefficients of the dynamical equations are deterministic and constant. This popularity is partially due to the contributions of Stein and Stein [59], Heston [36], Carr and Madan [13], and many others who followed in their footsteps. In [19], Dempster and Hong use the fast Fourier transform to approximate numerically the two-dimensional integral introduced earlier in our discussion of the spread valuation by multiple integrals. We refer the reader interested in Fourier methods to this paper and to the references therein.

**Concluding Remarks.** We conclude this section on numerical methods by pointing out some of their (well-known) flaws, and, in so doing, we advocate the cause of analytical approximations in closed form. Although simple implementations of Monte Carlo methods can give very good approximations for prices, unfortunately, they require a modicum of care to be used in the computation of the sensitivities of the prices with respect to the different parameters—the Greeks as we defined them earlier. Among these sensitivities, those partial derivatives with respect to the current values of the underlying index values are of crucial importance. First, they give the weights needed to build self-financing replicating portfolios to hedge the risk associated with the options. Second, they enter in the computations of value at risk (VaR) of portfolios containing the options. To compute numerically these partial derivatives, one should in principle recompute the price of the option with a slightly different value of the underlying index, and this should be done many times, which makes the use of Monte Carlo methods unreasonable. To be more specific, the first derivative is typically approximated by computing the limit as $\epsilon \searrow 0$ of an expression of the form

$$\Delta_1(\epsilon) \approx \frac{1}{\epsilon}\left[p(x_1+\epsilon, x_2) - p(x_1, x_2)\right].$$

In practice, we need to compute $\Delta_1(\epsilon)$ for a few values of $\epsilon$, hoping for stable results indicating that these $\epsilon$'s are small enough. We need to generate Monte Carlo samples starting from $x_1$ in order to compute $p(x_1, x_2)$, and in order to compute each single $\Delta_1(\epsilon)$, we need to redo the Monte Carlo simulation with samples starting from $x_1+\epsilon$. From the point of view of computing time, such a procedure can be costly, especially when the dynamics of the underlying indexes are given by general diffusion processes. Fortunately, the situation is more favorable in the case of underlying indexes with geometric Brownian motion dynamics. Indeed, in this case, the dependence with respect to the initial values is extremely simple, as one can see from formula (3.11). One side effect of this simple dependence is the fact that we can use the same Monte Carlo samples to compute the various $\Delta_1(\epsilon)$. Obviously, this is a source of significant savings.

When computing sensitivities upon the initial values of the indexes in the case of geometric Brownian motions, the simplicity of formula (3.11) is an invitation to interchange the (partial) derivatives and the expectation sign. This approach is advocated in Glasserman's recent book [34], and it can be viewed in a certain sense as a precursor of the application of the Malliavin calculus to the computation of the Greeks as introduced in the remarkable papers [27] and [28]. See also [12] for a *pedestrian version* adapted to the geometric Brownian motion case. This approach is especially appealing in the case of options written on a single geometric Brownian motion. Indeed in this case, assuming that the volatility $\sigma$ is nonzero to avoid artificial results, the Black–Scholes PDE (3.3) can be used to express the Gamma (second derivative of the option price with respect to the initial value of the underlying index) in terms of first order Greeks, namely, the first order derivatives with respect to the initial value of the index and time. Unfortunately, this trick cannot be used in the case of spread options. Indeed the corresponding PDE (3.9) involves the three second order partial derivatives, and there is no obvious way to use it to compute one of the second order Greeks unless we have already computed the other two.

So it seems that there is no way to avoid the computation of second order derivatives in the case of spread options. This remark is a serious concern because, even when the convergence of the numerical algorithm is good for the price itself, it is poorer for the approximations of the partial derivatives, and it becomes even worse as the order of the derivation increases. This is also the source of undesirable instabilities which most of the computational algorithms based on closed form formulae do not have, since it is in general not too difficult to derive closed form formulae for the Greeks as well.

In order to give a fairly exhaustive review of the Monte Carlo methods used in practice to compute the sensitivities of option prices, we need to mention a set of methods which have been introduced as applications of the Malliavin calculus. They were presented in a set of two very elegant papers based on the formula of integration by parts with respect to an infinite-dimensional Gaussian measure. See [27] and [28]. Even though the prescriptions derived from these theoretical considerations seem very attractive when the underlying indexes are general diffusion processes, and despite the fact that the gain over the brute force Monte Carlo approach described above seems to be significant, we believe that they are still extremely involved. In fact, we believe that in the present situation of geometric Brownian motion underlying indexes, it may even be less efficient than the methods reviewed later in this survey. For a streamlined version of the Malliavin calculus approach as specialized to the particular case of Samuelson's geometric Brownian motion model, the interested reader is referred to [12].

It is also fair to emphasize that, contrary to the Monte Carlo method, the trinomial tree method allows one to compute the partial derivatives along with the price. But its main shortcoming remains its slow rate of convergence: so precision in the approximation is traded for reasonable computing times. The other major problem with the trinomial tree method is the fact that it *blows up* exponentially with the dimension. It is still feasible with two assets, as we are considering here, but it is very unlikely to succeed in any higher dimension.

Finally, it is important to stress the fact that none of these methods allow one to efficiently compute implied volatilities or implied correlations from a set of market prices. An implied parameter (whether it is a volatility, a correlation, etc.) is the value of the parameter in question that reproduces best the prices actually quoted on the market. So in order to compute these implied parameters, one needs to be able to invert the pricing formula/algorithm, and recover an input, say the value of the volatility parameter for example, from a value of the output, i.e., a market quote. None of the numerical methods reviewed above can do that in a reasonable fashion. On the other hand, most of the numerical methods based on the evaluation of closed form formulae can provide values for the implied parameters for most of these formulae that can easily be inverted numerically. Implied parameters have a great appeal to the traders and other market makers, and being able to estimate them in an efficient manner is a very desirable property of a computational method.

**4. The Bachelier Model for Spread Options.** In most applications to equity markets, the underlying instruments are stock prices that are modeled by means of log-normal distributions as prescribed by the geometric Brownian motion model. One of the most noticeable features of this model is that it produces underlying prices that are inherently positive. But the positivity restriction does not apply to the spreads themselves, since the latter can be negative as differences of positive quantities. Indeed, computing histograms of historical spread values shows that the marginal distribution of a spread at a given time extends on both sides of zero and, surprisingly enough, that the normal distribution can give a reasonable fit. This simple remark is the starting point of a series of papers proposing the use of arithmetic Brownian motion (as opposed to the geometric Brownian motion leading to the log-normal distribution) for the dynamics of spreads. In so doing, prices of options can be derived by computing Gaussian integrals leading to simple closed form formulae. This approach was originally advocated by Shimko in the early nineties. See [57] for a detailed exposition of this method. Extensions can be found, for instance, in [40] and [48]. For the sake of completeness, we devote this section to a review of this approach. We quantify numerically the departures of its results from those provided by the log-normal model studied later in the paper, which is an extension in the same vein as Samuelson's model extended Bachelier's original model for the dynamics of the price of a single stock price.

The premise of the pricing formula proposed in this section is to assume that the risk-neutral dynamics of the spread $S(t)$ is given by a stochastic differential equation of the form

$$dS(t) = \mu S(t)dt + \sigma dW(t) \tag{4.1}$$

for some standard Brownian motion $\{W(t)\}_{t\geq 0}$ and some positive constant $\sigma$. Here and in the following, $\mu$ stands for the short interest rate $r$, or $r-\delta$, where $\delta$ denotes the continuous rate of dividend payments, or some form of cost of carry or convenience yield, as we explain later in the paper. In any case, $\mu$ is assumed to be a deterministic constant. Equation (4.1) is appropriate when the spread is defined as $S(t) = \alpha_2 S_2(t) - \alpha_1 S_1(t)$ for some coefficients $\alpha_1$ and $\alpha_2$, and when the dynamics of the individual component indexes $S_1(t)$ and $S_2(t)$ are given by stochastic differential equations of the form

$$\begin{aligned} dS_1(t) &= \mu S_1(t)dt + \sigma_1 dW_1(t), \\ dS_2(t) &= \mu S_2(t)dt + \sigma_2 dW_2(t), \end{aligned}$$

with positive constants $\sigma_1$ and $\sigma_2$ and two Brownian motions $W_1$ and $W_2$ with correlation $\rho$. As usual, the initial values of the indexes will be denoted by $S_1(0) = x_1$ and $S_2(0) = x_2$. Indeed, choosing

$$\sigma = \sqrt{\alpha_1^2\sigma_1^2 - 2\rho\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2} \tag{4.2}$$

and

$$W(t) = \frac{\alpha_2\sigma_2}{\sigma}W_2(t) - \frac{\alpha_1\sigma_1}{\sigma}W_1(t)$$

gives the dynamics (4.1) for $S$. However, such a rationale cannot be taken very seriously because it implies dynamics for $S_1$ and $S_2$ that are totally unrealistic since their marginal distributions are Gaussian, and therefore $S_1(t)$ and $S_2(t)$ can be negative with positive probability.

**4.1. Pricing Formulae.** The main interest of the arithmetic Brownian motion model is that it leads to a closed form formula, akin to the Black–Scholes formula, for the price of spread options. In order to arrive at such a formula, one assumes that the dynamics of $S_1$ and $S_2$ are given by geometric Brownian motions, namely, by equations of the form

$$\begin{cases} dS_1(t) = S_1(t)\left[\mu dt + \sigma_1 dW_1(t)\right], \\ dS_2(t) = S_2(t)\left[\mu dt + \sigma_2 dW_2(t)\right], \end{cases} \tag{4.3}$$

to be specific, and we assume that (4.1) provides a reasonable approximation for the dynamics of the spread. Notice that equations (4.3) are nothing more than our previous definition (3.10) of the geometric Brownian motion dynamics with $\mu_1 = \mu_2 = \mu$. With all these provisos one can prove the following.

Proposition 4.1. *If the value of the spread at maturity is assumed to have a Gaussian distribution with the correct first two moments, then the price $p$ of a spread option with maturity $T$ and strike $K$ is given by*

$$p = \left(m(T) - Ke^{-rT}\right)\Phi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right) + s(T)\varphi\left(\frac{m(T) - Ke^{-rT}}{s(T)}\right), \tag{4.4}$$

*where the functions $m(T)$ and $s(T)$ are defined by*

$$m(T) = (x_2 - x_1)e^{(\mu - r)T}$$

*and*

$$s^2(T) = e^{2(\mu-r)T}\left[x_1^2\left(e^{\sigma_1^2 T}-1\right) - 2x_1x_2\left(e^{\rho\sigma_1\sigma_2 T}-1\right) + x_2^2\left(e^{\sigma_2^2 T}-1\right)\right]. \tag{4.5}$$

*Proof.* We have already given the solution of the dynamical equations (4.3) as formulae (3.11), which we recall here:

$$S_i(T) = S_i(0)\exp\left[\mu T - \frac{1}{2}\sigma_i^2 T + \sigma_i W_i(T)\right].$$

Since our assumption is to approximate the true distribution of $S(T) = S_2(T) - S_1(T)$ by the Gaussian distribution with the correct first two moments, we are in fact assuming

$$S(T) \sim N(\mathbb{E}\{S_2(T) - S_1(T)\}, \mathrm{var}\{S_2(T) - S_1(T)\}).$$

Classical formulae for the moment generating function of the Gaussian distribution give

$$\mathbb{E}\{S_2(T) - S_1(T)\} = (x_2 - x_1)e^{\mu T} \quad \text{and} \quad \mathrm{var}\{S_2(T) - S_1(T)\} = s^2(T)e^{2rT}.$$

Consequently, the price $p$ at time $T$ of the option is given by

$$\begin{aligned} p &= e^{-rT}\mathbb{E}\{(S(T) - K)^+\} \\ &= \mathbb{E}\{(m(T) - Ke^{-rT} + s(T)\xi)^+\} \end{aligned}$$

for some $N(0,1)$ random variable $\xi$. Consequently,

$$p = \frac{1}{\sqrt{2\pi}}\int_{\frac{m(T)-Ke^{-rT}}{s(T)}}^{\infty}(m(T) - Ke^{-rT} - s(T)u)e^{-u^2/2}du,$$

from which we easily get the stated result. $\square$

We elaborate on the rationale of the assumptions in force in this section. Equation (4.1) can be solved explicitly. The solution is given by

$$S(t) = e^{\mu t}S(0) + \sigma\int_0^t e^{\mu(t-u)}dW_u,$$

from which we see that $S(t)$ is indeed a Gaussian random variable, which is consistent with the assumption of Proposition 4.1. The mean of $S(t)$ is given by $e^{\mu t}S(0)$ and its variance by

$$\sigma^2\int_0^t e^{2\mu u}du = \sigma^2\frac{e^{2\mu t}-1}{2\mu}.$$

We see that the practitioners' approximation made in Proposition 4.1 is compatible with dynamics like (4.1), as long as we allow for a time-dependent volatility $\sigma_t$. In other words, we need to replace the naive dynamics (4.1) by something of the type

$$dS(t) = \mu S(t)dt + \sigma_t dW(t), \tag{4.6}$$

in which case the spread becomes

$$S(t) = e^{\mu t}S(0) + \int_0^t \sigma_u e^{\mu(t-u)}dW_u, \tag{4.7}$$

and its variance is now given by

$$\int_0^t \sigma_u^2 e^{2\mu(t-u)} du.$$

In order for this last quantity to be equal to the quantity $s^2(t)$ given in Proposition 4.1, we must take

$$\sigma_t = \sqrt{e^{2\mu t} \frac{d}{dt} \left(e^{-2\mu t} s^2(t)\right)}.$$

**4.2. Numerical Performance of Arithmetic Brownian Motion Model for Spreads.** Despite the suspicious naivete of the model, and the extreme simplicity of the above derivations, the pricing formulae obtained in this section can be surprisingly accurate for specific ranges of the parameters, even when the dynamics of the underlying indexes are not given by *arithmetic Brownian motions*. We illustrate this anticlimatic fact by comparing the results obtained by this approach to the *true* values when the underlying indexes evolve like geometric Brownian motions.

**The Parameters of the Experiment.** For the purposes of this numerical experiment, we use the parameters of a spark spread option with efficiency parameter $H_{\text{eff}} = 7.5$, and $x_1 = 2.7$ and $x_2 = 28$ for the current values of the gas and electricity contracts. We assume that their (annualized) volatilities are $\sigma_1 = 30\%$ and $\sigma_2 = 50\%$, respectively, and $\mu = 0$ since we are dealing with futures whose dividend rate is the short rate of interest. We make several runs to compare the effects of the remaining parameters. For a fixed maturity $T$, we compare the exact price when the dynamics of the underlying indexes are given by geometric Brownian motions, and the approximation given by the Bachelier model. We compute these two prices on a $21 \times 41$ grid of values of the couple $(K, \rho)$. The strike $K$ varies from $K = -5$ to $K = +5$ by increments of .5, while the correlation coefficient $\rho$ varies from $\rho = -1$ to $\rho = +1$ by increments of .05.

The first comparison uses an option maturing in 60 days, i.e., $\tau = T - t = 60/252$ yr. The results are given graphically in the left pane of Figure 4.1. The next comparison is done for options maturing in $\tau = T - t = 1.5$ yr. The results are given graphically in the right pane of this same figure.

The final comparison still uses options maturing in $\tau = T - t = 1.5$ yr, but we increase the volatility of the electricity price to $\sigma_2 = 80\%$. The results are shown in Figure 4.2.

**Experimental Results.** One can see from the left pane of Figure 4.1 that the agreement is remarkably good when the time-to-maturity is small, independently of the correlation of the underlying indexes, as long as the strike is very negative. From all the experiments we made, it seems that the normal approximation underestimates the value of the option, and that the error increases with the strike and decreases with the correlation. Comparing the two surface plots of Figure 4.1, one sees that increasing the time-to-maturity increased dramatically the error when the option is out of the money for large strikes. Notice the difference in scales on the vertical axes of the two plots. Both plots of Figure 4.2 seem to confirm this fact, with a significant deterioration of the performance of the normal approximation for more volatile lognormal indexes. We shall come back later to this model and compare it with other models in terms of hedging rather than in terms of pricing.

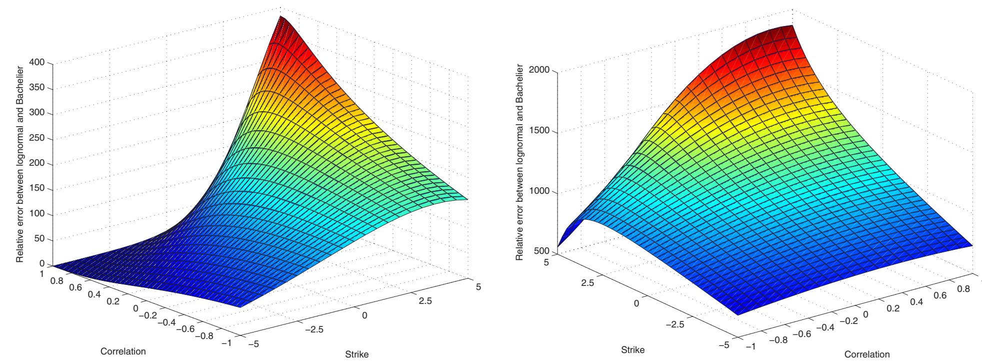


**Fig. 4.1** *Surface plots of the ratio of the exact price divided by Bachelier's approximation. In both cases, the parameters of the log-normal model were $x_1 = 2.7$, $x_2 = 28$, $H_{eff} = 7.5$, $\sigma_1 = 30\%$, $\sigma_2 = 50\%$, and the time-to-maturity $\tau = T - t$ was chosen to be $\tau = 60/252$ years for the computations leading to the surface on the left and $\tau = 1.5$ for the surface on the right. Ratios are given in basis points ($1bp = 10^{-4}$).*

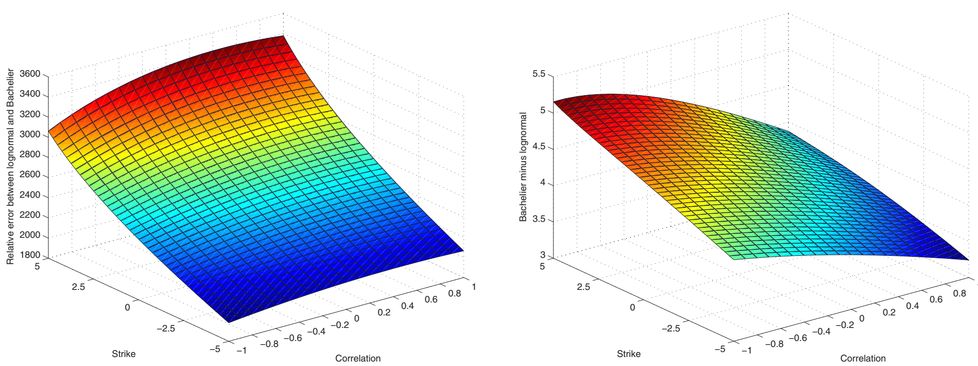


**Fig. 4.2** *Surface plots of the ratio of the exact price divided by Bachelier's approximation (left) and of the difference between the exact price and Bachelier's approximation (right). The parameters of the spark spread option are $x_1 = 2.7$, $x_2 = 28$, $H_{eff} = 7.5$, $\sigma_1 = 30\%$, $\sigma_2 = 80\%$, $r = 8\%$, and $T = 1.5$.*

**4.3. Consistency with the Forward Curve.** This section is concerned with a practice intended to correct spot models in order to make them consistent with the observed forward prices. We present these ideas in the case of the model (4.6) and its solution (4.7). They have been made popular in the fixed income markets where the most tractable short interest models fail to be compatible with the observed yield curves. We adapt them to the present framework.

In the rest of the paper, we often discuss the consistency of a spot price model with the observed forward curves. This is done by computing the theoretical values of the forward curve from the model: indeed since we assume deterministic interest rates, and since we shall not model the convenience yield as a stochastic factor, the values of the forward contracts on any given day should be given by the conditional expectation of the future values of the spot prices under the *pricing* measure. See, for example, [16] and [52] for details. Note that this is in contradiction with the old economic belief that the values of the forward curve are nothing but the best predictors/guesses for the future values of the spot. Indeed, this would imply that forward prices are the conditional expectation of the future values of the spot prices under the *objective* (also called historical) measure. Why is this so? Again, it is good to recall that modern asset pricing theory gives prices of future cash flows as today's price of any self-financing portfolio that would give the same cash flows in all circumstances. In the case of a forward contract, the cash flows at maturity $T$ are: receive the commodity with price $S(T)$ and pay the forward price $F(t,T)$. Replicating this cash flow from today is extremely simple: buy the commodity now for $S(t)$ and borrow $e^{-r(T-t)}F(t,T)$ from the bank to be sure to have $F(t,T)$ at time $T$. This implies that these two quantities are equal at time $t$, and hence

$$F(t,T) = e^{r(T-t)}S_t = \mathbb{E}_t\{S_T\},$$

where we used the notation $\mathbb{E}_t\{\,\cdot\,\}$ for the risk-neutral conditional expectation given the past up to time $t$. The last equality holds because $(e^{-rt}S_t)_{t\geq 0}$ is a martingale under the pricing measure. Notice also that the above argument requires the ability to store the commodity and ignores the cost of carrying the commodity to maturity.

We now suppose that, on the day $t$ when we value the spread option, we have information on other instruments derived from the underlying assets $S_1$ and $S_2$. Let us assume, for example, that we have a finite set of future dates $T_1, T_2, \ldots, T_n$, and that we have the prices at time $t$, say $f_1, f_2, \ldots, f_n$, of instruments maturing at these dates. In such a situation, it is very likely that if we use the model at hand to price these instruments, then we would find prices different from $f_1, f_2, \ldots, f_n$. This fact alone should be enough for us to lose confidence in the model, and we may not want to price spread options using this model. In order to reconciliate our model with these observed price data, one usually adds parameters to the model and uses these extra degrees of freedom to calibrate to the data. In other words, one chooses the extra parameters in order to replicate the prices quoted on the market.

As we already pointed out, this practice is used in the fixed income markets where the prices $f_i$ are the prices of bonds, swaps, and other liquid options whose prices are available through various financial services and brokerage houses. These prices are used to infer a curve $T \hookrightarrow f(T)$ giving the term structure of interest rates, whether it is given by all the future values of the instantaneous yield, or of the instantaneous forward rates, or even by the mere future discount curve. This initial term structure curve is then used to calibrate the model.

Because of the crucial importance of spread options in the commodity markets, we illustrate this calibration philosophy in the case of these markets. On day $t$, we usually have access to the prices $f_1(t,T_1), f_1(t,T_2), \ldots, f_1(t,T_n)$ of forward contracts on the first commodity, and the prices $f_2(t,T_1), f_2(t,T_2), \ldots, f_2(t,T_n)$ of forward contracts on the second commodity. Notice that it is quite possible that the maturity dates $T_1, T_2, \ldots, T_n$ may not be the same for both commodities. This is typically the case for the spark spread on power and gas. Indeed, as we shall see later, the structure of the maturity dates of these contracts differs wildly. In this case, massaging the data appropriately may easily get us where we want to be: one can indeed interpolate or smooth the values of the available forward contract prices to obtain a continuous curve that is then easy to sample at the desired times $T_1, T_2, \ldots, T_n$. Details are given in section 7. From this point on, it is easy to see that one can get a set of prices for the differences, or the appropriate linear combinations in the case of more

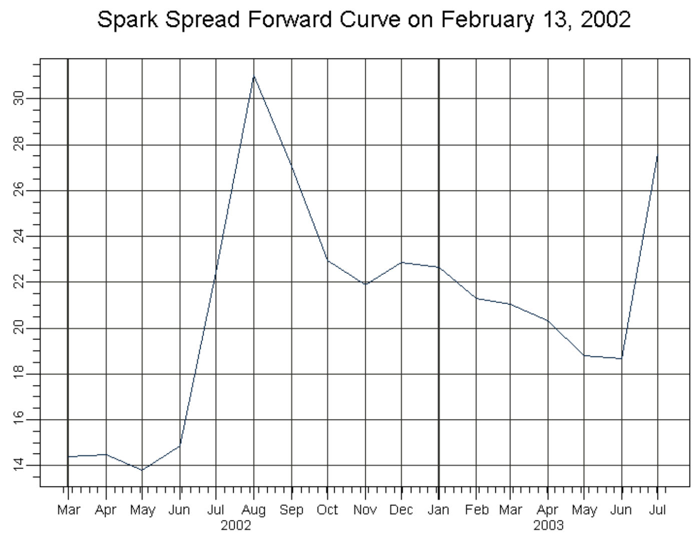


**Fig. 4.3** *Forward curve for the spark spread on February* 13, 2002*. We used the California–Oregon border forward price curve for the electricity and the Henry Hub natural gas forward curve on the same day; we ignored the basis risk due to the difference in location, and we used an efficiency factor of* 2.5 *for the sake of definiteness.*

general commodity spreads. See Figure 4.3 for an example of a spark spread forward curve. In this case, the first thing we need to do is to check that the pricing model used above is consistent with the forward curve at hand. In order to understand the consistency issue, we need first to identify the kind of forward curves implied by the assumptions of the model.

We now come back to our mathematical model, and we try to find out what kind of forward curves are supported by the model. Notice that, because of the independence of the increments of the Brownian motion, the stochastic integral $\int_t^T \sigma_s e^{\mu(T-s)} dW_s$ is independent of the values of the Brownian motion before time $t$, and hence of the values of $S$ before time $t$. Consequently, recalling formula (4.7) and the fact that the interest rate is constant, the price $F(t,T)$ of the forward contract with date of maturity $T$ is given by

$$\begin{aligned}F(t,T) = \mathbb{E}_t\{S(T)\} &= \mathbb{E}_t\left\{e^{\mu(T-t)}S(t) + \int_t^T \sigma_s e^{\mu(T-s)} dW_s\right\}\\ &= e^{\mu(T-t)}S(t).\end{aligned}$$

This shows that the forward curve ought to be exponential starting from the current value of the spread. This is highly unrealistic, as one can see from Figure 4.3. The fact that we did not consider stochastic interest rates does not change anything in this conclusion.

Since the class of forward curves implied by the model (4.6) is too small, we attempt to extend the model in order to allow for forward curves that can be observed in real life. So, for the rest of this subsection, we assume that we are given a forward curve $T \hookrightarrow F_0(T)$. One can think of $F_0$ as the result of the smoothing of a set of observed sample forward values of the spread as they appear in Figure 4.3, for example. We look for a stochastic differential equation similar to (4.1), which would imply a forward curve $T \hookrightarrow F(0,T) = \mathbb{E}_0\{S(T)\}$ equal to $F_0$. We assume that the equation is of the same form as (4.1), and we merely assume that the rate of growth $\mu$ is now time dependent, i.e., $\mu = \mu_t$, while still remaining deterministic. In this case, the solution formula (4.7) becomes

$$S(t) = e^{\int_0^t \mu_s ds} S(0) + \int_0^t \sigma_s e^{\int_s^t \mu_u du} dW_s. \tag{4.8}$$

The same argument as above gives (recall that interest rates are deterministic)

$$F(0,T) = \mathbb{E}_0\{S(T)\} = S(0) e^{\int_0^T \mu_u du},$$

which shows that, if we want the model to be consistent with the forward curve observed today at time $t = 0$, we should choose

$$\mu_T = \frac{d}{dT} \log F_0(T).$$

In other words, by choosing the time-dependent drift coefficient as the logarithmic derivative of the observed forward curve, the model becomes consistent in the sense that the current forward curve computed out of this model is exactly the current market curve.

This implies new expressions for the mean and variance of $S_2(T) - S_1(T)$:

$$m(T) = (x_2 - x_1) e^{\int_0^T (\mu_u - r) du},$$
$$s^2(T) = e^{2\int_0^T (\mu_u - r) du} \left( x_1^2 \left( e^{\sigma_1^2 T} - 1 \right) - 2 x_1 x_2 \left( e^{\rho \sigma_1 \sigma_2 T} - 1 \right) + x_2^2 \left( e^{\sigma_2^2 T} - 1 \right) \right).$$

Also, to be consistent with our pricing formula (4.4) we need to take

$$\sigma_t = \sqrt{ e^{2\int_0^t \mu_u du} \frac{d}{dt} \left( e^{-2\int_0^t \mu_u du} s^2(t) \right) }.$$

The price $p$ of the spread option will still be given by formula (4.4) as long as we use the above expressions for the constants $m(T)$ and $s(T)$.

## 5. The Case $K = 0$, or the Option to Exchange.

The textbook treatment of spread options is usually restricted to the special case $K = 0$. When the distributions of the underlying indexes $S_1$ and $S_2$ are log-normal, this is the only case for which one has a pricing formula in closed form à la Black–Scholes. This formula was first derived by Margrabe in [47] as early as 1978, and it bears his name. We present this derivation in full for the sake of completeness, and because of its importance, we elaborate on some of its consequences.

### 5.1. Exchange of One Asset for Another.

The case $K = 0$ corresponds to an *exchange* since the payoff $(S_2(T) - S_1(T))^+$ provides the holder of the option with the difference $S_2(T) - S_1(T)$ at time $T$ whenever $S_2(T) > S_1(T)$.

In order to illustrate this fact, let us suppose that we want to buy one of two stocks $S_1$ or $S_2$, that we are indifferent to which one we own, but that at the end of a six-month period we would like to be in the position of someone who owns the better performing stock of the two. Obviously we cannot tell which one will perform better over the next six months. We finally decided that $S_1$ is a better bet, but we are not quite sure of our pick, so we buy an option to exchange $S_1$ for $S_2$ in case $S_2$ outperforms $S_1$ after six months. This does the trick since we have synthesized exactly the payoff of the exchange option! Short of the premium (i.e., the price we have to pay to own the option), our investment is the better of the two stocks over the next six months.

**5.2. Margrabe Formula.** In this subsection we assume that the risk-neutral dynamics of the two underlying indexes are given by geometric Brownian motions, i.e., by stochastic differential equations of the form (3.11) with $\mu_i = r$ the discount rate, and the volatilities $\sigma_i$ are constant. We assume that the two indexes are correlated through the driving Brownian motions. To be more specific we assume that $\mathbb{E}\{dW_1(t)dW_2(t)\} = \rho dt$. In other words, $\rho$ is the parameter controlling the correlation between the two indexes.

PROPOSITION 5.1. *The price $p$ of a spread option with strike $K = 0$ and maturity $T$ is given by*

$$p = x_2\Phi(d_1) - x_1\Phi(d_0), \tag{5.1}$$

*where*

$$d_1 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \qquad d_0 = \frac{\ln(x_2/x_1)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T},$$

*and*

$$x_1 = S_1(0), \quad x_2 = S_2(0), \quad \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

*Proof.* In order to prove formula (5.1), we define two new independent Brownian motions $\{\hat{W}_1(t)\}_t$ and $\{\hat{W}_2(t)\}_t$ by

$$\begin{aligned} dW_2(t) &= \rho d\hat{W}_1(t) + \rho' d\hat{W}_2(t), \\ dW_1(t) &= d\hat{W}_1(t), \end{aligned}$$

where $\rho'^2 + \rho^2 = 1$. $dW_1(t)$ and $dW_2(t)$ are well-defined as long as $|\rho| < 1$. The risk-neutral valuation rule gives us the price of a zero-exercise price spread option as follows:

$$\begin{aligned} p &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\{\max(S_2(T) - S_1(T), 0)\} \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\max\left(\frac{S_2(T)}{S_1(T)} - 1, 0\right) S_1(T)\right\}. \end{aligned} \tag{5.2}$$

We use the subscript $\mathbb{Q}$ to emphasize the fact that the expectations are computed under the risk-neutral probability measure $\mathbb{Q}$. The price of $S_2$ expressed in the numéraire $S_1$ (i.e., in units of $S_1$) remains a geometric Brownian motion since Itô's formula gives

$$\begin{aligned} \frac{d(S_2(t)/S_1(t))}{S_2(t)/S_1(t)} &= \frac{dS_2(t)}{S_2(t)} - \frac{dS_1(t)}{S_1(t)} - \operatorname{cov}\left(\frac{dS_2(t)}{S_2(t)} - \frac{dS_1(t)}{S_1(t)}; \frac{dS_1(t)}{S_1(t)}\right) \\ &= (\rho\sigma_2 - \sigma_1)d\hat{W}_1(t) + \rho'\sigma_2 d\hat{W}_2(t) + \sigma_1(\sigma_1 - \rho\sigma_2)dt. \end{aligned}$$

Using Girsanov's theorem (see, for example, [45]), we define a new probability measure $\mathbb{P}$ with Radon–Nikodým derivative with respect to $\mathbb{Q}$ given on the $\sigma$-algebra $\mathcal{F}_T$ by

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \exp\left(-\frac{1}{2}\sigma_1^2 T + \sigma_1\hat{W}_1(T)\right).$$

Under $\mathbb{P}$, $\hat{W}_1(t) - \sigma_1 t$ and $\hat{W}_2(t)$ are Brownian motions. So

$$\begin{aligned} p &= e^{-rT} S_1(0)\mathbb{E}_{\mathbb{P}} \left\{ \max\left( \frac{S_2(T)}{S_1(T)} - 1, 0 \right) \right. \\ &\qquad \left. \times \exp\left( \frac{1}{2}\sigma_1^2 T - \sigma_1 \hat{W}_1(T) + \left( r - \frac{1}{2}\sigma_1^2 \right) T + \sigma_1 \hat{W}_1(T) \right) \right\} \\ &= S_1(0)\mathbb{E}_{\mathbb{P}} \left\{ \max\left( \frac{S_2(T)}{S_1(T)} - 1, 0 \right) \right\}, \end{aligned}$$

where $S_2/S_1$ is a geometric Brownian motion under $\mathbb{P}$ with volatility

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$

Using this fact, the last expression can be viewed as the price of a European call option with no interest rate, strike 1, and volatility $\sigma$. The value is thus given by the classical Black–Scholes formula (3.4) with the appropriate parameters. □

*Remarks.*

1. The above proof is a good example of the use of Girsanov's transforms (in the language of probability theory) or changes of numéraire techniques (in the language of financial engineering) to price options. These ideas were introduced in [30]. The reader is also referred to [4] for other examples of the use of these powerful techniques.

2. The above result easily extends to allow for dividend payments at a constant rate. Suppose that instead of letting $\mu_i = r$ in (3.11), we let $\mu_i = r - q_i$, where $q_i$ is the dividend payment rate for the $i$th asset:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i(t) \quad \text{for} \quad i = 1, 2.$$

The method remains the same as long as we use $S_1(t)e^{q_1 t}$ and $S_2(t)e^{q_2 t}$ instead of $S_1(t)$ and $S_2(t)$. In that case, the price of a zero strike spread option is given by

$$\begin{aligned} p &= x_2 e^{-q_2 T}\Phi(d_1) - x_1 e^{-q_1 T}\Phi(d_0), \\ d_1 &= \frac{\ln(x_2/x_1) - (q_2 - q_1)T}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \\ d_0 &= \frac{\ln(x_2/x_1) - (q_2 - q_1)T}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}. \end{aligned}$$

3. As pointed out in [38], it is interesting to note that these formulae are independent of the risk-free rate $r$. This is because after risk adjustment (some would say *in a risk-neutral world*), both underlying indexes increase at the same rate, the drifts offsetting each other in the computation of the difference appearing in the definition of the spread.

4. The above closed form formula is very nice, but unfortunately the case where $K \neq 0$ cannot be treated with the same success. Reviewing the approach outlined in section 3 where we discussed the use of multiple integrals, we can first condition by $W_1(T)$ in (5.2). This gives

$$p = e^{-rT}\mathbb{E}_{\mathbb{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\{(S_2(T) - S_1(T) - K)^+ | W_1(T)\} \right\}.$$

The innermost expectation can again be evaluated using the Black–Scholes formula with strike $S_1(T) + K$ and a slightly modified spot. But the outer expectation (the integration of the Black–Scholes formula) cannot be done explicitly.

## 6. Pricing Options on the Spread of Geometric Brownian Motions.

We resume our analysis of the valuation of spread options with nonzero strike prices.

Throughout this section we assume as above that, besides a risk-free bank account with constant interest rate $r$, our arbitrage-free market consists of two assets whose risk-neutral price dynamics are given by the following stochastic differential equations:

$$dS_1(t) = S_1(t)[(r-q_1)dt + \sigma_1 dW_1(t)],$$
$$dS_2(t) = S_2(t)[(r-q_2)dt + \sigma_2 dW_2(t)],$$

where $q_1$ and $q_2$ are the instantaneous dividend yields, the volatilities $\sigma_1$ and $\sigma_2$ are positive constants, and $W_1$ and $W_2$ are two Brownian motions with correlation $\rho$. The initial conditions will be denoted by $S_1(0) = x_1$ and $S_2(0) = x_2$. The discussion of this section focuses on the pricing of spread options on two *stocks.* The case of spread options on two futures contracts follows immediately by taking $q_1 = q_2 = r$.

The price $p$ of formula (3.7) can be rewritten in the form

$$p = e^{-rT}\mathbb{E}\left\{\left(x_2 e^{(r-q_2-\sigma_2^2/2)T+\sigma_2 W_2(T)} - x_1 e^{(r-q_1-\sigma_1^2/2)T+\sigma_1 W_1(T)} - K\right)^+\right\},$$

which shows that the price $p$ is given by the integral of a function of two variables with respect to a bivariate Gaussian distribution, namely, the joint distribution of $W_1(T)$ and $W_2(T)$. This expectation is of the form

$$\Pi(\alpha,\beta,\gamma,\delta,\kappa,\rho) = \mathbb{E}\left\{\left(\alpha e^{\beta X_1 - \beta^2/2} - \gamma e^{\delta X_2 - \delta^2/2} - \kappa\right)^+\right\}, \tag{6.1}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, and $\kappa$ are real constants and $X_1$ and $X_2$ are jointly Gaussian $N(0,1)$ random variables with correlation $\rho$. These expectations were studied in [10]. We can therefore apply the results of [10] to approximate the price of a spread option, provided we set

$$\alpha = x_2 e^{-q_2 T}, \quad \beta = \sigma_2\sqrt{T}, \quad \gamma = x_1 e^{-q_1 T}, \quad \delta = \sigma_1\sqrt{T}, \quad \text{and} \quad \kappa = Ke^{-rT}.$$

### 6.1. A Pricing Formula.

The analysis of [10] is based on simple properties of the multivariate normal distribution and elementary convexity inequalities. Combining the two, the authors derived a family of upper and lower bounds for the price $p$. Among other things, they show that the supremum $\hat{p}$ of their lower bounds provides a very precise approximation to the exact price $p$. Before we can state the result of [10] relevant to the present review, we introduce the notation $\theta^*$ for the solution of the equation

$$\begin{aligned}\frac{1}{\delta\cos\theta}\ln\left(-\frac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\delta\cos\theta}{2}\\ = \frac{1}{\beta\cos(\theta+\phi)}\ln\left(-\frac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right) - \frac{\beta\cos(\theta+\phi)}{2},\end{aligned} \tag{6.2}$$

where the angle $\phi$ is defined by setting $\rho = \cos\phi$. The following proposition gives the closed form formula derived in [10] for the approximate price $\hat{p}$.

PROPOSITION 6.1. *Let us set*

$$d^* = \frac{1}{\sigma\cos(\theta^*-\psi)\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}\sigma_2\sin(\theta^*+\phi)}{x_1 e^{-q_1 T}\sigma_1\sin\theta^*}\right) - \frac{1}{2}(\sigma_2\cos(\theta^*+\phi)+\sigma_1\cos\theta^*)\sqrt{T},$$

*where the angles $\phi$ and $\psi$ are chosen in* $[0, \pi]$ *such that*

$$\cos\phi = \rho \qquad \textit{and} \qquad \cos\psi = \frac{\sigma_1 - \rho\sigma_2}{\sigma},$$

*and $\theta^*$ is the solution of* (6.2). *Then*

(6.3)

$$\hat{p} = x_2 e^{-q_2 T}\Phi\Big(d^* + \sigma_2\cos(\theta^* + \phi)\sqrt{T}\Big) - x_1 e^{-q_1 T}\Phi\Big(d^* + \sigma_1 \sin\theta^* \sqrt{T}\Big) - K e^{-rT}\Phi(d^*).$$

Note that this formula is as close to the Black–Scholes formula (3.4) as we could hope for. Moreover, as documented in [10], it provides an approximation of the exact price of the spread option with extreme precision. One of the possible reasons for its accuracy is the fact that it reproduces exactly all known cases of exact formulae. Indeed, it has the following properties.

Proposition 6.2. *The approximation $\hat{p}$ is equal to the true price $p$ when $K = 0$, or $x_1 = 0$, or $x_2 = 0$, or $\rho = -1$, or $\rho = +1$. In particular, $\hat{p}$ reduces to Margrabe formula when $K = 0$, and to the classical Black–Scholes formula when $x_1 = 0$, or when $x_2 = 0$.*

*Proof.* We refer to [10] for a complete proof. As an example, we show how we recover the Margrabe formula in the case $K = 0$. First we notice that $\theta^*$ is given by

$$\theta^* = \pi + \psi = \pi + \arccos\left(\frac{\sigma_1 - \rho\sigma_2}{\sigma}\right),$$

with

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2},$$

which implies that

$$\sigma_2 \sin(\theta^* + \phi) = \sigma_1 \sin\theta^*$$

and

$$d^* = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{1}{2}(\sigma_2\cos(\theta^* + \phi) + \sigma_1\cos\theta^*)\sqrt{T}.$$

Consequently,

$$\sigma_2\cos(\theta^* + \phi) - \sigma_1\cos\theta^* = \sigma$$

and

$$d^* + \sigma_2\cos(\theta^* + \phi)\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2},$$

$$d^* + \sigma_1\cos\theta^*\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}$$

hold true. Finally, we have

$$\hat{p} = x_2 e^{-q_2 T}\Phi\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) + \frac{\sigma\sqrt{T}}{2}\right)$$
$$- x_1 e^{-q_1 T}\Phi\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{x_2 e^{-q_2 T}}{x_1 e^{-q_1 T}}\right) - \frac{\sigma\sqrt{T}}{2}\right),$$

which is exactly the Margrabe formula. $\square$

**6.2. Hedging and the Computation of the Greeks.** In section 3.4 we discussed some of the shortcomings of the existing numerical approximations to the price of a spread option. There, we emphasized the importance of hedging, and we explained that most of the pricing algorithms did not address this issue, failing to provide efficient methods to evaluate the so-called Greeks. Following [10], we show that hedging strategies can be computed and implemented in a very efficient way using formula (6.3) giving $\hat{p}$.

We first consider the replication issue. The derivation of formula (6.3) cannot provide an exact replication of the spread option payoff. Because it is based on lower bounds, it gives only a subhedge for the option.

PROPOSITION 6.3 (see [10]). *The portfolio formed at time $t \le T$ by*

$$\Delta_1 = -e^{-q_1 T}\Phi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)$$

*and*

$$\Delta_2 = e^{-q_2 T}\Phi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)$$

*units of the underlying assets $S_1$ and $S_2$, respectively, is a subhedge for the option. In other words, its value at the time of maturity $T$ is almost surely a lower bound for the payoff.*

But as we mentioned earlier, beyond the first partial derivatives with respect to the initial values of the underlying assets, which give the portfolio of the previous proposition, all the sensitivities of the price with respect to the various parameters are of crucial importance. The closed form formula derived for $\hat{p}$ can be used to compute explicitly the other partial derivatives of the price. We give some of them in the following proposition. Notice that here and in the following, sensitivity is synonymous with partial derivative.

PROPOSITION 6.4 (see [10]). *Let $\vartheta_1$ and $\vartheta_2$ denote the sensitivities of the price functional* (6.3) *with respect to the volatilities of each asset, let $\chi$ be the sensitivity with respect to their correlation parameter $\rho$, let $\kappa$ be the sensitivity with respect to the strike price $K$, and let $\Theta$ be the sensitivity with respect to the maturity time $T$. They are given by the formulae*

$$\begin{aligned}
\vartheta_1 &= x_1 e^{-q_1 T}\varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)\cos\theta^* \sqrt{T},\\
\vartheta_2 &= -x_2 e^{-q_2 T}\varphi\left(d^* + \sigma_2 \cos(\theta^* + \phi)\sqrt{T}\right)\cos(\theta^* + \phi)\sqrt{T},\\
\chi &= -x_1 e^{-q_1 T}\varphi\left(d^* + \sigma_1 \cos\theta^* \sqrt{T}\right)\sigma_1 \frac{\sin\theta^*}{\sin\phi}\sqrt{T},\\
\kappa &= -\Phi\left(d^*\right)e^{-rT},\\
\Theta &= \frac{\sigma_1\vartheta_1 + \sigma_2\vartheta_2}{2T} - q_1 x_1 \Delta_1 - q_2 x_2 \Delta_2 - rK\kappa.
\end{aligned}$$

These formulae are of great practical value. The price of an option is a determining factor for a buyer and a seller to get into a contract. But by indicating in which direction the price will change when some of the parameters change, the partial derivatives are also of crucial importance, and for this reason they are carefully monitored by investors, traders, and risk managers throughout the life of the option, i.e., up until maturity. Also, since parameters like volatilities or correlations are not directly observable, the corresponding sensitivities show how errors on these parameters affect price and hedging strategies. The second order partial derivatives can also be approximated in the same way. We refrain from reproducing explicit formulae here, mostly because of their technical nature. Instead, we refer the interested reader to [10].

**6.3. Kirk's Formula.** Recently, Kirk [44] proposed a closed form approximation for the price of a spread option. It reads as follows:

$$\hat{p}^K = x_2\Phi\left(\frac{\ln\left(\frac{x_2}{x_1+Ke^{-rT}}\right)}{\sigma^K}+\frac{\sigma^K}{2}\right)-(x_1+Ke^{-rT})\Phi\left(\frac{\ln\left(\frac{x_2}{x_1+Ke^{-rT}}\right)}{\sigma^K}-\frac{\sigma^K}{2}\right),$$

where

$$\sigma^K=\sqrt{\sigma_2^2-2\rho\sigma_1\sigma_2\frac{x_1}{x_1+Ke^{-rT}}+\sigma_1^2\left(\frac{x_1}{x_1+Ke^{-rT}}\right)^2}.$$

It is interesting to see how this approximation performs in comparison with our own result (6.3), as shown in the next section. A refinement of this approximation can be found in [26] and the references therein.

**6.4. Comparison of the Three Approximations.** We conclude this section with a short comparative analysis of the three approximation methods that we recommend for the actual pricing and hedging of spread options.

• Let us first compare the results of Bachelier's model with the results obtained with the approximation $\hat{p}$ given by formula (6.3). Numerical experiments show that Bachelier's price is *always* smaller than $\hat{p}$, and this strongly supports the use of $\hat{p}$. Given the simplicity of the formulae provided by Bachelier's approximation, the overhead caused by the use of formula (6.3) needs to be justified. This is quite easy since the latter requires only the numerical computation of the zero of a given function. This is done very efficiently by a Newton–Raphson method. It is very fast.

• Let us now compare the approximation given by $\hat{p}$ with Kirk's approximation. As we have already pointed out, the case in favor of $\hat{p}$ is based mostly on the easy computations of the Greeks, which in turn give sensible hedging portfolios. Since Kirk's formula also leads to two *delta hedges* (given by the partial derivatives of the pricing functional $\hat{p}^K$ with respect to $x_1$ and $x_2$), one can wonder how the performances of the two hedging strategies compare along a given scenario.

In order to illustrate this point, we reproduce the results of a simulation study done in [10]. For a given set of scenarios representing possible sample realizations of the time evolutions of the underlying assets $S_1(t)$ and $S_2(t)$, we compare the payoff of the spread option at maturity with the terminal value of the portfolio obtained by $n$ hedging operations throughout the life of the option. So for each scenario, we compute three tracking errors computed at maturity after trying to replicate the payoff of the option by the values of portfolios obtained by rebalancing $n$ times using the *delta hedge* prescriptions given by Bachelier's method, Kirk's formula, and Proposition 6.3. To avoid an artificial dependence upon a particular scenario, we repeat this operation for a large number of scenarios, and we compare the standard deviations of the tracking errors. The plots of Figure 6.1 were obtained by varying $n$ from 10 to 10,000, using two different sets of parameters. However, the results appear to be very stable. We found that in most cases, the hedges provided by Proposition 6.3

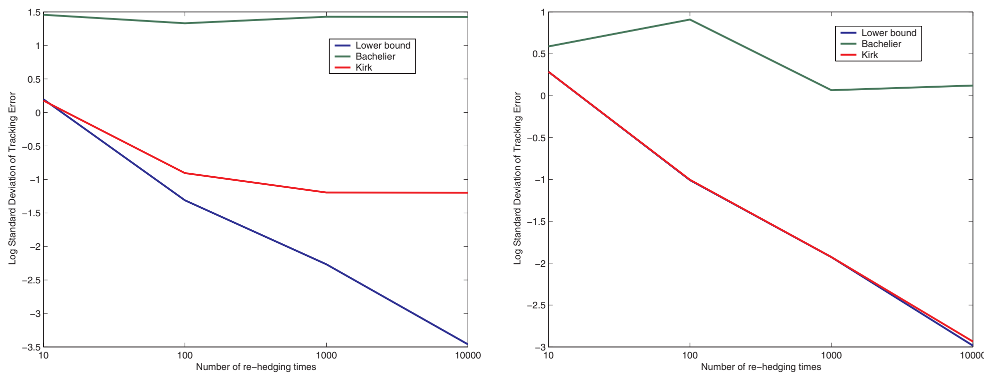


**Fig. 6.1** *Behavior of the tracking error as the number of rehedging times increases. The model data are* $x_1 = 100$*,* $x_2 = 110$*,* $\sigma_1 = 10\%$*,* $\sigma_2 = 15\%$ *and* $T = 1$*.* $\rho = 0.9$*,* $K = 30$ *(left) and* $\rho = 0.6$*,* $K = 20$ *(right).*

and Kirk's model are equally very good, with in some cases a significant advantage for the former, which seems to perform much better. In any case, these two models clearly out-perform Bachelier's model. The reasons for the poor hedging performance of Bachelier's model are twofold. First, Bachelier's model is based on a distribution assumption that stands the test of the pricing formula, but which cannot survive differentiation. Second, it is intrinsically a one-factor model trying to directly model the distribution of the difference $S_2(T) - S_1(T)$, and as such, it cannot capture the subtle structure of the true nature of the two-factor (log-normal) model that we are dealing with.

**6.5. Extension to Jump-Diffusion Models.** The inclusion of jumps in the stochastic differential equations giving the dynamics of underlying assets was proposed by Merton [50] and Cox and Ross [17] almost thirty years ago. Nevertheless these jump-diffusion models remained at the level of a scientific curiosity, mostly because of the difficulties associated with their statistical calibration. A renewal of interest in these models was prompted by the severe aftermath of several spectacular market crashes, and by the extremely volatile behavior of prices of new instruments such as electric power, for example.

The approximation formula for the price of a spread option that we have presented above can easily be adjusted to apply to underlying asset price dynamics with jumps. At the risk of being swept away by the tidal wave of jump-diffusion process research of the last ten years, we venture outside the class of continuous diffusion processes to prove our claim. The class of jump processes we are considering is often referred to as Merton's jump model in the financial literature. We introduce jumps in the risk-neutral dynamics of the underlying assets $S_1$ and $S_2$ by assuming that they are given by stochastic equations of the form

$$\frac{dS_i(t)}{S_i(t-)} = (r - q_i - \lambda_i\mu_i)dt + \sigma_i dW_i(t) + (e^{J_i(t)} - 1)dN_i(t), \tag{6.4}$$

where $N_1$ and $N_2$ are two independent Poisson processes with intensities $\lambda_1$ and $\lambda_2$. They are also assumed to be independent of $W_2$ and $W_1$. Finally, $(J_1(k))_{k\geq 1}$ and $(J_2(k))_{k\geq 1}$ are two independent sequences of Gaussian random variables with distributions $N(m_i, s_i^2)$. The interpretation of the jump term of (6.4) is the following:

at the time the Poisson process $N_i(t)$ jumps for the $k$th time, the index $S_i(t)$ jumps by an amount $S_i(t-)(e^{J_i(k)} - 1)$. In the same way the stochastic differential equation (3.10) giving the Samuelson's dynamics could be solved to give the explicit expressions (3.11) for the underlying assets, equation (6.4) can also be solved by a simple use of the extension of Itô's formula to processes with jumps. We get the resulting formula

$$S_i(T) = x_i \exp\left( (r - q_i - \sigma_i^2/2 - \lambda_i \mu_i)T + \sigma_i W_i(T) + \sum_{k=1}^{N_i(T)} J_i(k) \right).$$

The expectation giving the price $p$ of a spread option can be computed by first conditioning with respect to the Poisson processes and, in so doing, reduce the problem to pricing spread options for underlying assets with log-normal distributions. Indeed, given $N_1(T)$ and $N_2(T)$, the random variables $S_1(T)$ and $S_2(T)$ still have a log-normal distribution and the lower bound $\hat{p}$ can be used. This leads to the following result.

PROPOSITION 6.5 (see [10]). *If we set $\mu_i = e^{m_i + s_i^2/2} - 1$ for $i = 1, 2$, and if we denote by $\hat{p}(x_1, x_2, \sigma_1, \sigma_2, \rho)$ the price approximation given by* (6.3)*, then the price of the spread option in Merton's model can be approximated by the quantity*

$$\hat{p}^{jumps} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} e^{-(\lambda_1 + \lambda_2)T} \frac{(\lambda_1 T)^i (\lambda_2 T)^j}{i! j!} \hat{p}\left(\tilde{x}_1, \tilde{x}_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}\right), \tag{6.5}$$

*with*

$$\begin{aligned}
\tilde{x}_1 &= x_1 e^{-\lambda_1 \mu_1 T + i\left(m_1 + s_1^2/2\right)}, \\
\tilde{x}_2 &= x_2 e^{-\lambda_2 \mu_2 T + j\left(m_2 + s_2^2/2\right)}, \\
\tilde{\sigma}_1 &= \sqrt{\sigma_1^2 + i s_1^2 / T}, \\
\tilde{\sigma}_2 &= \sqrt{\sigma_2^2 + j s_2^2 / T}, \\
\tilde{\rho} &= \frac{\rho \sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + i s_1^2/T}\sqrt{\sigma_2^2 + j s_2^2/T}}.
\end{aligned}$$

The above formula involves the summation of an infinite series. The rate of convergence of this series can be estimated from a simple upper bound on $\hat{p}$, and, given any tolerance level, one easily determines how many terms we need in (6.5) to satisfy any prescribed error threshold.

## 7. Energy Market Data.

The choice of the mathematical models used in the first part of the paper was made from an historical perspective for reasons of simplicity. The arithmetic and geometric Brownian motion models, and even the jump-diffusion models discussed above, capture reasonably well the empirical nature of most of the features of equity market data. This section should open the door to a new set of models with which one could test the various spread option pricing methodologies reviewed earlier. We concentrate on markets where price discovery is done through a combination of spot and forward prices. A typical example of such a market is the bond market, also called the fixed income market. Other examples include the commodity and the energy markets. Spread options are widespread in many markets, but in the remainder of this paper we concentrate on applications to energy markets, where they are of particularly crucial importance.

One of the goals of this section is to demonstrate that some of the financial instruments used in the energy markets need to be treated with a modicum of care,

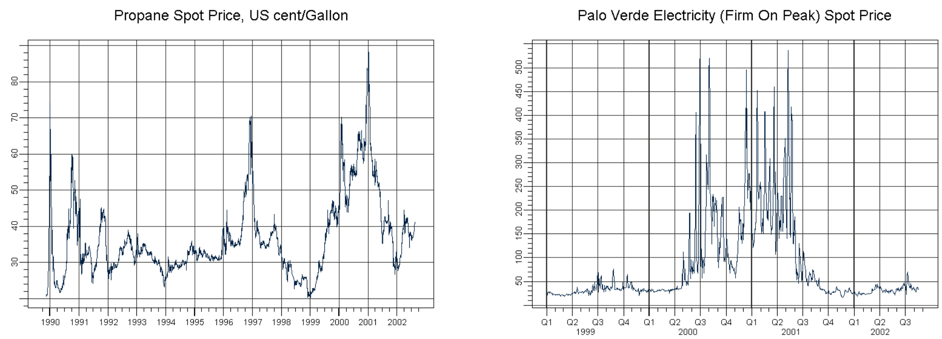


**Fig. 7.1** *Time series plots of two energy commodity daily spot prices: propane (left) and Palo Verde firm on-peak electric power (right).*

and that blindly applying tools developed for the equity and fixed income markets may not be appropriate. Most of the mathematical models used in the equity markets are based on generalizations of the geometric Brownian motion. We shall see that these models are not of great use in the applications we consider. Energy market models bear more resemblance to models for fixed income markets where there is a division between the models for the dynamics of the short interest rate and the models for the dynamics of the entire yield curve. This dichotomy will appear in our treatment of the energy markets. We divide the models into two classes, the first based on the dynamics of spot market prices, and the second based on models for the dynamics of the entire forward curve. But in order to justify the specific assumptions of our models, it is important to get a good understanding of the kind of data that analysts, risk managers, traders, and other interested parties are dealing with.

For most physical commodities, price discovery takes two different forms. The first is backward looking. It is based on the analysis of a time series of historical prices giving the values observed in the past of the so-called *spot price* of the commodity. The spot market is a market where goods are traded for immediate delivery. Figure 7.1 shows a couple of examples of energy spot prices. The left pane displays the daily values of the propane gas spot price, while the right pane contains a plot of the daily values of the Pale Verde electricity firm on-peak spot price. These series do not look anything like stock prices or equity index values. The sudden increases in value, and the high levels of volatility, set them apart. Except for that, they have more in common with plots of instantaneous interest rates. Indeed, these time series look more stationary than equity price time series. This is usually explained by appealing to the *mean reversion* property of the energy prices which tend, despite the randomness of their evolution, to return to a local or asymptotic mean level. This mean reversion property is shared with interest rates. But the latter do not have the seasonality structure that appears in Figure 7.1. Gas prices are higher during the winters because of heating needs in the northern hemisphere, and slightly higher in the summer as well. Moreover, energy prices are much more volatile than equity prices. However, as we already noticed, the specific feature that sets these data apart is the extreme nature of the fluctuations. This is obvious from the plot of the electricity spot prices given in Figure 7.1.

For the sake of simplicity, we consider only *daily* time series in this paper, but needless to say, these high levels of volatility are also found in hourly, and even higher frequency, data. Apart from the special case of the power markets, working with daily data is not a restrictive assumption. Indeed, most energy price quotes are recorded as "daily close," and using daily time series is quite appropriate. However, electricity prices are very different. They exhibit significant fluctuations at scales much smaller than one day, and as a result, prices are quoted more frequently (hourly, or even every half hour). Also, distinctions are made between *on-peak* and *off-peak* periods, weekdays and weekends, etc. Nevertheless, one of the main distinctive feature of the power markets remains its inelasticity: the fact that for all practical purposes electricity cannot be stored in a flexible manner hinders rapid responses to sudden changes in demand, and wild price fluctuations can follow. Weather is one culprit. Indeed, changes in the temperature affect the demand for power (*load*), creating sudden bursts in price volatility. The analysis of high-frequency electricity price data is a challenging problem that we will not consider in this paper. We are interested in spreads between several instruments, which most often involve several commodities, and for this reason we will restrict ourselves to daily sampling of the prices.

As in the fixed income markets, another type of data is to be reckoned with. These data encapsulate the current market expectations of future price evolutions. On any given day $t$, prices for a wide range of forward and/or futures energy contracts are available. We use forward and futures contract interchangeably for the purposes of this survey. Detailed explanations of the subtle differences between forward and futures contracts can be found in Hull's book [38], for example. We choose to ignore them because they do not play any role in our analysis. These contracts guarantee the delivery of the commodity at a given location, and at a given date, or over a given period in the future. For the sake of simplicity, we shall assume that the delivery takes place at a given date, which we denote by $T$, and which we call the date of maturity of the contract. As in the case of the yield curve, or the discount rate curve, or the instantaneous forward rate curve, used in the fixed income markets, the natural way to model the data is to assume the existence for each day $t$ of a function $T \hookrightarrow F(t,T)$ giving the price at time $t$ of a forward/futures contract with maturity date $T$. Unfortunately, the domain of definition of the mathematical function $T \hookrightarrow F(t,T)$ changes with $t$. This is very inconvenient when it comes to statistical analysis of the characteristics of the forward curves. Even the more mundane issue of plotting becomes an issue because of that fact. A natural fix to that annoying problem is to parameterize the forward curves by the "time-to-maturity" $\tau = T - t$ instead of the "time-of-maturity" $T$. This simple suggestion proved to have far reaching consequences in the case of the fixed income models. We discuss below the advantages and the shortcomings of this new parametrization in the case of the energy markets.

The existence of a continuum of maturity dates $T$ is a convenient mathematical idealization, but it is important to keep in mind that in practice, on any given day $t$, the maturity dates of the outstanding contracts form a finite set $\{T_1, T_2, \dots, T_n\}$, typically the first days of the $n$ months following $t$, for which forward/futures contracts are traded. The $T_j$'s are often regularly spaced, one contract per month, and $n$ is in the range 12 to 18 for most commodities, even though it recently became as large as $7\times 12 = 84$ in the case of natural gas. Unfortunately, available data varies dramatically from one commodity to the next, from one location to another, and even between data providers. And as one can easily imagine, historical data is often sparse and sprinkled with erroneous entries and missing values.

Despite the data integrity problems specific to the energy markets, the main challenge remains the fact that the dates at which the forward curves are sampled vary from one day to the next. Let us illustrate this simple statement with an example. On day $t = 11/10/1989$ the $T_1 = Dec.89$, $T_2 = Jan.90$, $T_3 = Feb.90$, etc., contracts are open for trading, and quotes for their prices are available. For the sake of simplicity we shall not worry about the *bid-ask* spread, and we assume that each quote gives a sharp price at which one can sell and/or buy the contract. In other words, on day $t = 11/10/1989$, we have observations of the values of the forward curve for the times-to-maturity $\tau_1 = 21$ days, $\tau_2 = 52$ days, $\tau_3 = 83$ days, etc. The following trading day is $t = 11/13/1989$, and on that day, we have observations of the prices of the same contracts with dates-of-maturity $T_1 = Dec.89$, $T_2 = Jan.90$, $T_3 = Feb.90$, etc., and we have now sample values of the forward curve for the times-to-maturity $\tau_1 = 18$ days, $\tau_2 = 49$ days, $\tau_3 = 80$ days, etc. The next trading day is $t = 11/14/1989$, and on that day, we have observations of the prices of the same contracts, but the values of the corresponding times-to-maturity are now $\tau_1 = 17$ days, $\tau_2 = 48$ days, $\tau_3 = 79$ days, etc. So the values $\tau_1$, $\tau_2$, ..., $\tau_n$ at which the forward function $\tau \hookrightarrow F(t, \tau)$ is sampled change from day to day. Even though the times-of-maturity $T_1$, $T_2$, $T_3$, ... do not vary with $t$ in the above discussion, this is not so in general. Indeed, when the date $t$ approaches the end of the month of November, the December contract suddenly stops being traded and the nearest traded contract becomes January, and an extra month is added to the list. This switch typically takes place three or four days before the end of each month, and it creates a very large jump in the values of $\tau$ at which the forward curve is sampled.

As we already mentioned, this state of affairs is especially inconvenient for plotting purposes, as well as for the statistical analysis of the forward curves. So whenever we manipulate forward curve data, it should be understood that we preprocessed the data to get samples of these forward curves computed on a fixed set $\{\tau_1, \tau_2, \dots, \tau_n\}$ that does not change with $t$. We do this by first switching to the time-to-maturity parametrization, then by *smoothing* the original data provided by the financial services, and finally by resampling the smoothed curve at the chosen sampling points. We sometimes fear that these manipulations are not completely innocent, but since we cannot clearly quantify their influence on the final results, we shall take their results for granted.

Figure 7.2 gives plots of the Henry Hub natural gas forward contract prices before and after such a processing. The left pane contains the raw data. Despite the rather poor quality of the plot, one sees clearly the structure of the data. Indeed, the domain of definition of the forward function $T \hookrightarrow F(t, T)$ is an interval of the form $[T_b(t), T_e(t)]$, where $T_b(t)$ is the date of maturity of the nearest contract, and $T_e(t)$ is the maturity of the last contract quoted on day $t$. Hence, this domain of definition changes from day to day. In principle, the left-hand point of the curve should give the spot price of the commodity, but even though we use it quite often as a proxy for the spot price, it can be significantly different because $T_b(t)$ is not exactly equal to $t$: after all, as we explained above, its values range approximately from 3 or 4, to 32 or 35. On any given day, the length of the forward curve depends upon the number of contracts traded on that day. Notice that in the case of natural gas displayed in Figure 7.2, the length recently went up to seven years. Also, the seasonality of the forward prices appears clearly on this plot. High ridges parallel to the time $t$-axis correspond to the contracts maturing in winter months when the price of gas is expected to be higher. The right pane of the figure also displays the natural gas forward surface, but the parametrization changed to the time-to-maturity $\tau = T - t$.

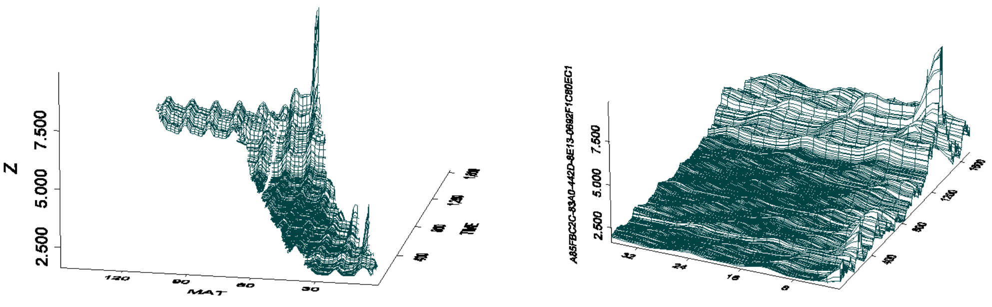


**Fig. 7.2** *Surface plots of the historical time evolution of the forward curves of the Henry Hub natural gas contracts in the time-of-maturity $T$ parametrization (left) and in the time-to-maturity $\tau$ parametrization (right).*

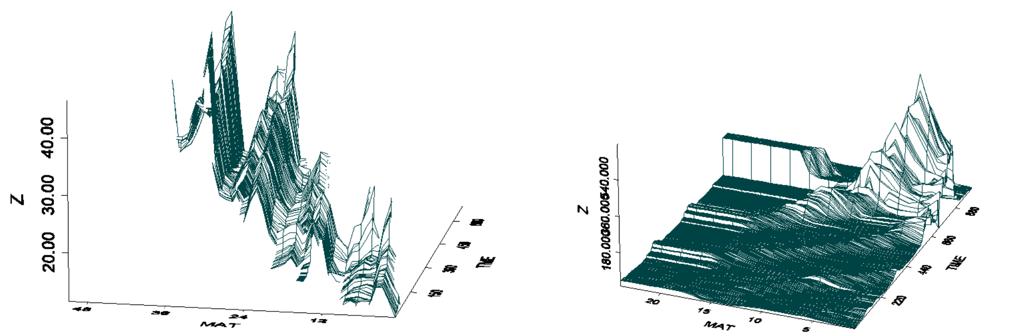


**Fig. 7.3** *Surface plots of the historical time evolution of the forward price curves of the Palo Verde forward electricity contracts when plotted as functions of the time-of-maturity (left) and when plotted as functions of the time-to-maturity variable (right). We had to take a subset of the original period because of holes in the data due to missing values. In particular, the forward ridge for the long maturities in the recent days is an artifact of our resampling method given these missing values.*

There are nevertheless several obvious points to make. First, the forward curves are defined on the same time interval and they have the same lengths, which we chose to be three years in this particular instance. But the most noticeable change comes from the different pattern of the ridges corresponding to the periods with higher prices. Because of the parametrization by the time-to-maturity $\tau$, the parallel ridges of high prices move toward the $t$-axis when $t$ increases, instead of remaining parallel to this time axis.

Figure 7.3 shows the results of a similar processing in the case of the Palo Verde forward electric contracts. The simple linear interpolation procedure that we chose does not smooth much of the erratic behavior of the data, hence the rough look of these surfaces.

We would not want the reader to believe that we are proponents of a blind implementation of the parametrization of the forward curves by the "time-to-maturity" $\tau = T - t$ instead of the "time-of-maturity" $T$. Indeed, because of their physical nature, most energy commodities exhibit strong seasonality features, and the latter are

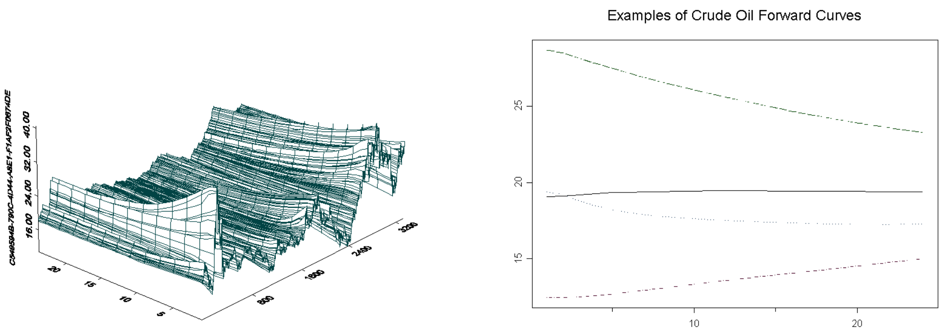


**Fig. 7.4** *Surface plot of the crude oil forward prices from* 11/10/1989 *to* 8/16/2002 *(left) and four typical individual forward curves giving examples of forward curves in contango and in backwardation (right).*

more obvious in the time-of-maturity parametrization. This temporal nature of the physical commodities makes the time-to-maturity parametrization less helpful than in the fixed income markets.

The following illustration is intended to show that despite our plea for considering seriously the effects of seasonality in the energy forward prices, it is important to keep in mind that not all the energy commodity prices have such a strong seasonal component. The left pane of Figure 7.4 gives a surface plot of the crude oil forward prices from 11/10/1989 to 8/16/2002, as parameterized by the time-to-maturity of the contracts. Clearly, the bumps and the ridges that are indicative of seasonal effects are not present in this plot. The right pane of Figure 7.4 gives line plots of four crude oil forward curves. They have been chosen at random. However, they are typical of what we should expect for crude oil forward curves: they are monotone functions of the time-to-maturity. When a forward curve is monotone decreasing, the future prices of the commodity are expected to be lower than the current (spot) price: we say the forward curve is in *backwardation.* When a forward curve is monotone increasing, the prices to come are expected to be higher than the spot price: we say the forward curve is in *contango.*

Forward prices represent market expectations of the future time evolution of spot prices. This claim, though commonly accepted, has to be taken with a grain of salt. We use the example of crude oil to illustrate this fact. We picked (essentially at random) five regularly spaced trading days separated by 200 trading days, and we superimposed the forward curves observed for these days on the plot of the spot curve. The result is given in Figure 7.5. This graph demonstrates in a dramatic fashion how poor a predictor of the spot price the forward curve can be. The situation is not always so bad, as our next illustration shows. Indeed, in stable periods of (relatively) low volatility, the forward curves can be a reasonable predictor of the future values of the spot prices. We illustrate this fact by plotting the forward curves of the Henry Hub natural gas contracts for the same five days we picked for the crude oil forward curves. As we can see from Figure 7.6, despite their greater lengths, there is a certain consistency between the forward curves in the tranquil periods. But still, they missed completely the sharp price increase of the 2000 crisis.

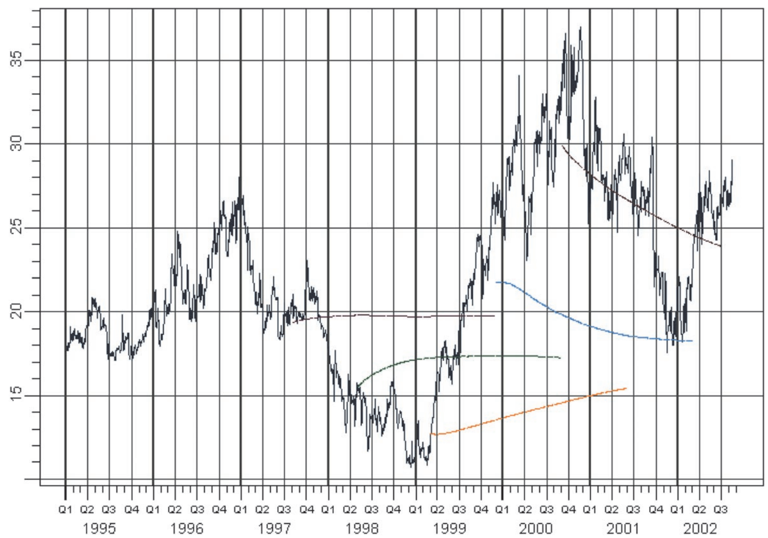


**Fig. 7.5** *Crude oil spot price with a small set of forward curves superimposed to illustrate how poor a predictor of the spot prices the forward curves can be. Notice that some forward curves do not start from the spot price curve. This happens often because the nearest contract could mature as much as thirty days ahead, which gives enough time for prices to change.*

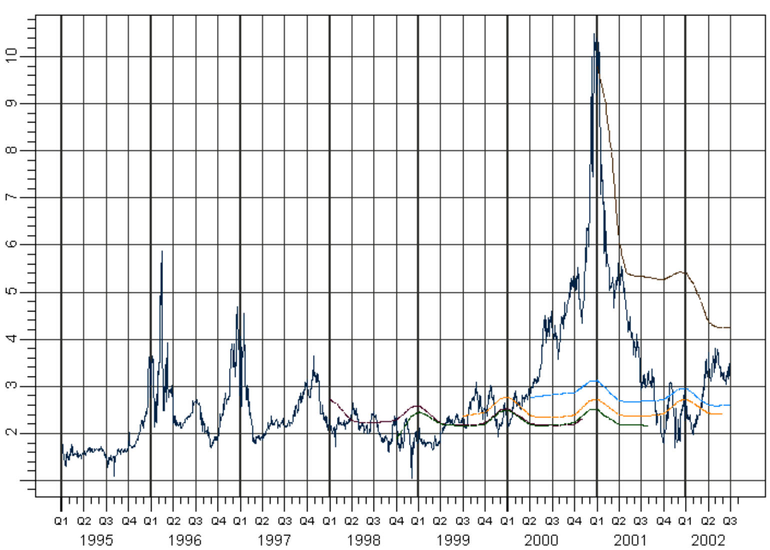


**Fig. 7.6** *Henry Hub natural gas spot price with the forward curves computed on the same days as the crude oil example illustrated in Figure* 7.5. *Again some forward curves do not start from the graph of the spot price.*

The next two sections are devoted to the implementation of the pricing formulae reviewed in this paper in the two sets of models most favored by energy market analysts. We give the details of the various model fitting steps, but in order to limit the technicalities to a minimum, we give the detailed formulae only for the prices, leaving the derivation of the hedges to the interested readers.

**8. Energy Spot Prices Models.** As in the case of fixed income markets, the mathematical models proposed for commodity markets come in two varieties: either in the form of (finite) factor models, or models for the dynamics of the entire forward curve, as those we shall review in section 9. The present section is devoted to a short review of the simplest factor models used in the industry, together with a discussion of the consequences of the assumptions of these models on the price of spread options.

The simplest factor models are the one-factor models where the factor is chosen to be the commodity spot price. So for the purpose of the present section, we assume that $S_1(t)$ and $S_2(t)$ are the spot prices at time $t$ of two commodities. It is difficult to say whether or not the models used in the previous sections are good for these indexes $S_1(t)$ and $S_2(t)$. Indeed, the stochastic differential equations used in the first part of the paper for the dynamics of the underlying indexes were given under a risk-neutral measure. These risk-adjusted dynamics cannot be observed directly. The best one can hope for is to calibrate the coefficients of the models to the observed prices, which we did consistently when making sure that our models could reproduce the observed forward curves.

In order to justify, or at least to motivate, the assumptions of the subsequent models, we take a short excursion into the real world of historical prices and stochastic models for the historical dynamics of spot prices. It is important to keep in mind that in most cases, Girsanov's theory implies that the stochastic differential equations giving the historical dynamics differ from the risk-adjusted models only through the drift part.

The geometric Brownian motion models for the time evolution of equity prices are appropriate for the dynamics of many assets, but they fail to capture one of the main characteristic feature of interest rates and physical commodity prices: *mean reversion*. This feature is included in the historical models by assuming that the dynamics of the underlying indexes $S_i(t)$ are given by geometric Ornstein–Uhlenbeck processes instead of geometric Brownian motions. These processes can be defined as the solutions of stochastic differential equations of the form

$$dS_i(t) = S_i(t)[-\lambda_i(\log S_i(t) - \alpha_i)dt + \sigma_i dW_i(t)], \tag{8.1}$$

where the constants $\lambda_i$ are positive. These constants are called the *mean reversion* coefficients. They can easily be estimated from historical data. A simple linear regression can be used to do just that. See, for example, [14] for details and examples. Notice that the other parameters, $\sigma_i$ and $\alpha_i$, can also be estimated very easily from historical data. Obviously $\sigma_i$ is the volatility, and it can be estimated empirically from the variance of the increments of the logarithms, while $\alpha_i$ is a simple function of the asymptotic mean reversion level. Historical models are useful for risk management, and nowadays for an increasing number of pricing algorithms based on replication arguments and expected utility maximization.

But the major issue is not the statistical estimation of the parameters of the historical model, as this is relatively easy. What we need in order to use the results of the theory presented in the first part of this paper is a risk-adjusted model. If we do not have any a priori information on the risk premium, the drift of the risk-adjusted dynamics can be absolutely anything. Indeed, it is easy to cook up a risk premium process so that Girsanov's theorem will turn the historical drift $S_i(t)[-\lambda_i(\log S_i(t)-\alpha_i)]dt$ into any drift prescribed a priori. For this reason, it is common practice to specify the risk-neutral dynamics directly without trying to derive them from a Girsanov transformation on a model of the historical dynamics appropriately fitted to empirical data. Then it can be argued that since we want to be working under a risk-neutral measure, the drift has to be equal to the risk-free rate, and changing it to something else is pure nonsense. However, since the risk-neutral drift determines the discounting factor used to compute present values of future cash flows, it can be different from the short interest rate. For example, we included the dividend payment rate in the risk-neutral drift of dividend paying stocks earlier in the paper. Moreover, in the case of physical commodities, especially those commodities that cannot be stored, such as electricity, the risk-adjusted drift should not only encompass the dividend payment rate but also some form of stochastic convenience yield (i.e., a stochastic dividend yield). Several recent studies have given strong empirical evidence of the presence of a term structure of convenience yield for physical commodities. In particular, a form of *spot convenience yield* is often used as a factor, joining the commodity spot price in the list of factors driving the dynamics of the forward prices. See, for example, [52].

In the rest of this section we assume that the risk-neutral dynamics are specified by a mean reverting equation of the form (8.1). This practice can only be justified by assuming that the risk adjustment preserves the mean reverting property of the historical drift. Except for mathematical convenience, we do not see clearly on what grounds this assumption can be justified. Despite this obvious lack of rationale, this assumption is nevertheless widely accepted as reasonable in the mathematical analysis of the dynamics of the commodity spot prices. See, for example, [14].

**8.1. Adding Mean Reversion to the Models.** Based on the above discussion, we assume that the risk-neutral dynamics of the underlying indexes are given by stochastic differential equations of the form

$$dS_i(t) = S_i(t)[-\lambda_i(\log S_i(t) - \mu_i)dt + \sigma_i dW_i(t)], \qquad i = 1, 2, \tag{8.2}$$

analogous to (8.1). Here, as before, the volatilities $\sigma_1$ and $\sigma_2$ are positive constants, $W_1$ and $W_2$ are two Brownian motions with correlation $\rho$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\mu_1$ and $\mu_2$ are real constants. The positive constants $\lambda_i$ give the rates of mean reversion. Indeed, as we are about to see, indexes satisfying these dynamical equations tend to revert toward the levels $e^{\mu_i^\infty}$ if we set $\mu_i^\infty = \mu_i - \sigma_i^2/2\lambda_i$. Various forms of equations (8.2) have been used as models for asset dynamics. For example, Schwartz [58] introduced them to derive closed form formulae for commodity contract prices.

Equations (8.2) can be best understood after a simple transformation leading to the dynamics of the logarithms of the underlying indexes. Setting $X_i(t) = \log S_i(t)$, a simple application of Itô's formula gives

$$dX_i(t) = -\lambda[X_i(t) - \mu_i^\infty]dt + \sigma_i dW_i(t), \qquad i = 1, 2,$$

which shows that the logarithms of the indexes are nothing but classical Ornstein–Uhlenbeck processes, whose mean reverting properties are well known.

Even though equations (8.2) are more involved than the equations giving the dynamics of the geometric Brownian motions used earlier in the paper, we can still derive explicit formulae for the indexes $S_i(T)$ in terms of exponentials of correlated Gaussian variables. Indeed, $S_i(T) = e^{X_i(T)}$ with

$$X_i(T) = \mu_i^\infty + e^{-\lambda_i(T-t)}[X_i(t) - \mu_i^\infty] + \sigma_i \int_t^T e^{-\lambda_i(T-s)} dW_i(s).$$

Hence,

$$S_i(T) = e^{\mu_i^\infty + e^{-\lambda_i T}(x_i - \mu_i^\infty) + \sigma_{i,T}\xi_i},$$

where

$$\sigma_{i,T} = \sigma_i \sqrt{\frac{1 - e^{-2\lambda_i T}}{2\lambda_i}}, \qquad i = 1, 2,$$

and $\xi_1$ and $\xi_2$ are $N(0,1)$ random variables with correlation coefficient $\tilde\rho$ given by

$$\begin{aligned}
\tilde\rho &= \frac{1}{\sigma_{1,T}\sigma_{2,T}} \mathbb{E}\{\xi_1\xi_2\} \\
&= \frac{1}{\sigma_{1,T}\sigma_{2,T}} \mathbb{E}\left\{\sigma_1 \int_0^T e^{-\lambda_1(T-s)} dW_1(s) \sigma_2 \int_0^T e^{-\lambda_2(T-s)} dW_2(s)\right\} \\
&= \frac{\rho\sigma_1\sigma_2}{\sigma_{1,T}\sigma_{2,T}} \frac{1 - e^{-(\lambda_1+\lambda_2)T}}{\lambda_1 + \lambda_2} \\
&= \rho \frac{\sqrt{\lambda_1\lambda_2}}{(\lambda_1+\lambda_2)/2} \frac{1 - e^{-(\lambda_1+\lambda_2)T}}{\sqrt{1 - e^{-2\lambda_1 T}}\sqrt{1 - e^{-2\lambda_2 T}}}.
\end{aligned}$$

Consequently, the price $p$ of a spread option, with strike $K$ and maturity $T$, on the difference between the underlying indexes $S_1$ and $S_2$ whose dynamics are given by (8.2) is given by the formula

$$p = e^{-rT}\mathbb{E}\{(S_2(T) - S_1(T) - K)^+\} = \Pi(\alpha, \beta, \gamma, \delta, \kappa, \tilde\rho)$$

with

$$\alpha = e^{-rT + \mu_2^\infty + e^{-\lambda_2 T}(x_2 - \mu_2^\infty) - \sigma_{2,T}^2/2} \qquad \text{and} \qquad \beta = \sigma_{2,T} = \sigma_2 \sqrt{\frac{1 - e^{-2\lambda_2 T}}{2\lambda_2}},$$

$$\gamma = e^{-rT + \mu_1^\infty + e^{-\lambda_1 T}(x_1 - \mu_1^\infty) - \sigma_{1,T}^2/2} \qquad \text{and} \qquad \delta = \sigma_{1,T} = \sigma_1 \sqrt{\frac{1 - e^{-2\lambda_1 T}}{2\lambda_1}},$$

with $\tilde\rho$ defined as above, and $\kappa = Ke^{-rT}$.

In particular, we can use our formula (6.3) to find an excellent approximation to its price and efficient hedging portfolio computation prescriptions.

**Calibration and Consistency with the Forward Curves.** As in the case of the Bachelier model, we can try to generalize the above mean reverting models in order to make them consistent with observed forward curves. At least in principle, this should be done in exactly the same manner. But as we are about to see, technical difficulties arise, and the computations are more involved. Indeed, if we replace the dynamics of the spot prices by the same mean reverting equations with time-dependent coefficients, we get the stochastic differential equations (we drop the subscript $i = 1, 2$ since it is irrelevant in the present discussion)

$$dS(t) = S(t)[-\lambda(\log S(t) - \mu_t)dt + \sigma_t dW_t],$$

and the logarithm $X(t) = \log S(t)$ of the underlying index satisfies the equation

$$dX(t) = -\lambda \left[ X(t) - \left( \mu_t - \frac{\sigma_t^2}{2\lambda} \right) \right] dt + \sigma_t dW_t,$$

which can be explicitly solved, giving

$$X_t = \mu_t - \frac{\sigma_t^2}{2\lambda} + e^{-\lambda t} \left[ X_0 - \mu_0 + \frac{\sigma_0^2}{2\lambda} \right] - \int_0^t e^{-\lambda(t-s)} \left[ \mu_s - \frac{\sigma_s^2}{2\lambda} \right] ds + \int_0^t e^{-\lambda(t-s)} \sigma_s dW_s.$$

Consequently, since as before

$$F(0,T) = \mathbb{E}_0\{S(T)\} = \mathbb{E}_0\{e^{X(T)}\},$$

we get the formula

$$F(0,T) = \exp \left[ \mu_T - \frac{\sigma_T^2}{2\lambda} + e^{-\lambda T} \left( X_0 - \mu_0 + \frac{\sigma_0^2}{2\lambda} \right) - \int_0^T e^{-\lambda(T-s)} \left( \mu_s - \frac{\sigma_s^2}{2\lambda} \right) ds \right.$$
$$\left. + \frac{1}{2} \int_0^T e^{-2\lambda(T-s)} \sigma_s^2 ds \right].$$

The problem is now to find, for each given forward curve $T \hookrightarrow F(0,T)$, a function $t \hookrightarrow \mu_t$ and/or a function $t \hookrightarrow \sigma_t$ satisfying the above equality. This problem does not seem to have a convenient solution in general.

**8.2. Introducing Jumps.** Even though the models of the previous subsection can easily be argued to be appropriate for most commodity spot prices, they still fall short in the case of electricity spot prices. Indeed the latter exhibit extreme volatility, and introducing jumps in the model may appear to be the best way to include sudden and extreme departures from the mean reverting level. Nevertheless, see [3] for an alternative way to achieve the same result with a continuous diffusion without jumps. At the risk of promoting the confusion caused by the coexistence of models for the historical and risk-adjusted dynamics, we refer the reader to the right pane of Figure 7.1, where one clearly sees that sudden and extreme departures from the mean do appear in the historical data. This figure illustrates that fact in the case of the Palo Verde electric power spot prices. In order to accommodate the case of spread options involving electric power (spark spreads are the typical examples), we allow for jumps in the risk-neutral dynamics of $S_2$. We assume that

$$dS_2(t) = S_2(t-)[(r - \lambda\mu)dt + \sigma_2 dW_2(t) + (e^{J_t} - 1)dN(t),$$

where $N$ is a Poisson process of intensity $\lambda$ independent of $W_2$ and $W_1$. $(J_t)_{t\geq 0}$ is a sequence of independent Gaussian random variables $(m, s^2)$, and $\mu$ is function of $m$ and $s^2$. Obviously, this model is simpler than the one studied before, since only one of the two underlying processes has jumps. We already discussed pricing in the more general setup, so we do not attempt to discuss it for this restricted model.

**Consistency with the Observed Forward Curves.** As in the case of the geometric Brownian motion model, we first derive analytic formulae for the forward curves, and we find ways to adjust the coefficients of the model to match the theoretical curves allowed by the model to the empirical curves actually observed. The strategy is the same as before. However, the presence of jumps makes it more difficult to implement because of the existence of extra integrations.

## 9. Modeling the Dynamics of the Forward Curves.

The spot price models considered in the previous section are extremely popular because of their intuitive appeal, and because of their mathematical tractability. Indeed, modeling the dynamics of spot prices seems like a reasonable thing to do, and the log-normal distribution is amenable to closed form formulae for many of the simplest single commodity derivatives. Unfortunately, these models are not always satisfactory. Indeed, given the spot price process $\{S_t\}_{t\geq 0}$ of a commodity, a forward contract with maturity $T$ should have a no-arbitrage price $F(t,T)$ at time $t$ given by

$$F(t,T) = \mathbb{E}_t\{S_T\}, \tag{9.1}$$

where, as before, we use the notation $\mathbb{E}_t\{\ \cdot\ \}$ for the conditional expectation with respect to all the information available up to time $t$. The log-normal models and their generalizations discussed in the previous section are simple enough to allow for explicit formulae for the conditional expectations appearing in the right-hand side of (9.1). Unfortunately, the forward curves $T \hookrightarrow F(t,T)$ produced in this way are very rarely consistent with the actual forward curves observed in practice, and the various fixes that we proposed in the previous sections are not satisfactory since they force the users of these models to recalibrate the model every single day! This major shortcoming is at the core of the search for more sophisticated models that can account for the observable features of the forward curves. Some of these models are considered in this section. The main departure from the previous approach is the systematic attempt to model the dynamics of the entire forward curve instead of modeling only the dynamics of its leftmost point.

Some of these forward curve models can be derived from models of the term structure of convenience yield, appropriately coupled with a model for the spot price of the commodity. This approach is advocated in [5] and [51], to which we refer the interested reader. Convenience yields are not observable, and for this reason, model calibration depends upon the solution of a difficult filtering problem. These two papers are the source of new modeling efforts, and it is very likely that interesting developments will come out of this new line of research.

### 9.1. The Model.

We postulate a stochastic differential equation (in fact, a coupled system of infinitely many equations) for the dynamics of the entire forward curve. In analogy with the HJM models of the fixed income markets, we assume that for each maturity date $T$, the dynamics of the forward prices with maturity $T$ are given by

$$\frac{dF(t,T)}{F(t,T)} = \mu(t,T)dt + \sum_{k=1}^{n} \sigma_k(t,T)dW_k(t), \qquad t \leq T, \tag{9.2}$$

where $\boldsymbol{W} = (W_1, \ldots, W_n)$ is an $n$-dimensional standard Brownian motion, and where the drift term $\mu$ and the $n$ volatilities $\sigma_k$ are deterministic functions of the current date $t$ and the time-of-maturity $T$. Such a model is called an $n$-factor forward curve model. This has been described as a promising model in the technical literature (see, e.g., Chapter 8 of [14]) although implementations in the commercial software packages available for energy risk management are still at a rather primitive stage.

Whenever we work on pricing, hedging, or asset valuation (power plant, gas storage, etc.), we assume that the dynamics have been adjusted for risk; i.e., we use risk-neutral probabilities. In that case, we set $\mu(t, T) \equiv 0$, since it guarantees that $t \hookrightarrow F(t, T)$ is a martingale for each fixed $T$. But when we work on risk management, the drift $\mu(t, T)$ needs to be kept and calibrated to historical data.

**An Explicit Solution.** Since the coefficients $\mu$ and $\sigma_k$ are assumed to be deterministic, the dynamical equation (9.2) can be solved explicitly. We obtain

(9.3)
$$F(t,T) = F(0,T) \exp\left[\int_0^t \left[\mu(s,T) - \frac{1}{2}\sum_{k=1}^n \sigma_k(s,T)^2\right] ds + \sum_{k=1}^n \int_0^t \sigma_k(s,T) dW_k(s)\right],$$

which shows that the forward prices have a log-normal distribution. More precisely, we have

$$F(t,T) = \alpha e^{\beta X - \beta^2/2}$$

for $X \sim N(0,1)$,

$$\alpha = F(0,T) \exp\left[\int_0^t \mu(s,T) ds\right], \qquad \text{and} \qquad \beta = \sqrt{\sum_{k=1}^n \int_0^t \sigma_k(s,T)^2 ds}. \tag{9.4}$$

This seemingly innocuous remark allows us to use the results derived earlier in the present situation, despite its significantly higher level of sophistication.

**Dynamics of the Spot Price.** Since the spot price should be given by the left-hand point of the forward curve (i.e., $S(t) = F(t,t)$) we can derive an explicit expression for the spot price from (9.3) above:

$$S(t) = F(0,t) \exp\left[\int_0^t [\mu(s,t) - \frac{1}{2}\sum_{k=1}^n \sigma_k(s,t)^2] ds + \sum_{k=1}^n \int_0^t \sigma_k(s,t) dW_k(s)\right];$$

differentiating both sides we get

$$\begin{aligned} dS(t) = S(t) \Bigg[ \Bigg( &\frac{1}{F(0,t)} \frac{\partial F(0,t)}{\partial t} + \mu(t,t) + \int_0^t \frac{\partial \mu(s,t)}{\partial t} ds - \frac{1}{2}\sigma_S(t)^2 \\ &- \sum_{k=1}^n \int_0^t \sigma_k(s,t) \frac{\partial \sigma_k(s,t)}{\partial t} ds + \sum_{k=1}^n \int_0^t \frac{\partial \sigma_k(s,t)}{\partial t} dW_k(s) \Bigg) dt + \sum_{k=1}^n \sigma_k(t,t) dW_k(t) \Bigg], \end{aligned}$$

where we set

$$\sigma_S(t)^2 = \sum_{k=1}^n \sigma_k(t,t)^2. \tag{9.5}$$

Consequently we have

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \log F(0,t)}{\partial t} + D(t)\right] dt + \sum_{k=1}^{n} \sigma_k(t,t) dW_k(t), \tag{9.6}$$

where the drift term $D(t)$ is given by

$$D(t) = \mu(t,t) - \frac{1}{2}\sigma_S(t)^2 + \int_0^t \frac{\partial \mu(s,t)}{\partial t} ds - \sum_{k=1}^{n} \int_0^t \sigma_k(s,t) \frac{\partial \sigma_k(s,t)}{\partial t} ds + \sum_{k=1}^{n} \int_0^t \frac{\partial \sigma_k(s,t)}{\partial t} dW_k(s).$$

From (9.6) we see that the instantaneous volatility of the spot price is $\sigma_S(t)$.

*Remarks.*

1. In a risk-neutral setting, the drift appearing in formula (9.6) has a nice interpretation. The first term, i.e., the logarithmic derivative of the forward, can be interpreted as a discount rate (i.e., the running interest rate), while the term $D(t)$ can be interpreted as a convenience yield, which many researchers in the field tried to model directly in order to find no-arbitrage models consistent with the data.

2. As emphasized in [14], this drift is generally not Markovian because of the presence of the stochastic integral, which typically depends upon the entire past of the forward curve evolution. An exception is provided by the risk-neutral one-factor model (i.e., $\mu(t,T) \equiv 0$ and $n = 1$) with volatility $\sigma_1(t,T) = \sigma e^{-\lambda(T-t)}$. Indeed, in this special case, the dynamics of the spot price are given by (9.6) with

$$D(t) = \lambda[\log F(0,t) - \log S(t)] + \frac{\sigma^2}{4}(1 - e^{-2\lambda t}),$$

which shows that the spot price dynamics are those of a geometric Ornstein–Uhlenbeck process as defined in (8.1), with time-dependent mean reverting level defined by

$$\frac{dS(t)}{S(t)} = [\mu(t) - \lambda \log S(t)]dt + \sigma dW(t), \tag{9.7}$$

which is exactly the generalization considered in section 8.1 to accommodate mean reversion. The point of this remark is to stress the fact that mean reversion of the spot price is closely related to the exponential decay of the forward volatility for large times-to-maturity, a fact observed empirically very often.

**Changing Variables.** During our discussion of the format of the data available to analysts, we made a case for the switch to the time-to-maturity variable $\tau$ as an alternative to the time-of-maturity $T$. This change of variable could deceivingly appear as a mere change of notation, without much effect on the analytic expression giving the dynamics of the forward curves. This is not the case, for computing derivatives with respect to $t$, of the function $t \hookrightarrow F(t, t+\tau)$ for $\tau$ fixed, involves partial derivatives of $F$ with respect to both of its variables, since $t$ appears in both places. This is in contrast with computing the derivative with respect to $t$ of $t \hookrightarrow F(t,T)$ for $T$ fixed, since obviously the latter involves only the first of the two first order partial derivatives of $F$. To be more specific, if we set

$$\tilde{F}(t,\tau) = F(t, t+\tau), \qquad \tilde{\mu}(t,\tau) = \mu(t, t+\tau), \quad \text{and} \quad \tilde{\sigma}_k(t,\tau) = \sigma_k(t, t+\tau),$$

then the dynamics (9.2) of the forward curve allow us to write

(9.8)
$$d\tilde{F}(t,\tau) = \tilde{F}(t,\tau)\left[\left(\tilde{\mu}(t,\tau) + \frac{\partial}{\partial\tau}\log\tilde{F}(t,\tau)\right)dt + \sum_{k=1}^{n}\tilde{\sigma}_k(t,\tau)dW_k(t)\right], \qquad \tau \geq 0.$$

So once reformulated in terms of the time-to-maturity variable $\tau$, the dynamics originally given by the system (9.2) of stochastic *ordinary* differential equations are now given by a stochastic *partial* differential equation (SPDE). SPDEs originated in filtering theory, but they are now ubiquitous in applied science, and significant progress has been made in the development of a unified theory. Equation (9.8) could be considered simpler than the usual models of SPDEs because it is driven by finitely many Wiener processes. But at the same time, it could be considered as more difficult, since it is of the hyperbolic type, which has been neglected by the probabilists who developed this theory. See [11] for the state of the art a few years ago. See also [9] for a review of the applications of SPDEs (including equations driven by infinitely many Wiener processes) to the mathematical models of the fixed income markets.

**9.2. Calibration by Principal Component Analysis.** We now present an application of a standard data analysis technique to the calibration of forward energy curve models. Our goal is twofold: to justify the assumptions behind the model (9.2) chosen for the dynamics of the forward curves, and at the same time, to show how to identify and estimate the volatility functions $\sigma_k$ from actual historical market data.

**Fundamental Assumption.** The main assumption of this subsection concerns the actual form of the volatility functions $\sigma_k$. We assume that they are of the form

$$\sigma_k(t,T) = \sigma(t)\sigma_k(T-t) = \sigma(t)\sigma_k(\tau) \tag{9.9}$$

for a function $\sigma(t)$ of the single variable $t$, to be determined shortly. Revisiting formula (9.5), giving the instantaneous volatility of the spot price in light of this new assumption, we get

$$\sigma_S(t) = \tilde{\sigma}(0)\sigma(t), \tag{9.10}$$

provided we set

$$\tilde{\sigma}(\tau) = \sqrt{\sum_{k=1}^{n}\sigma_k(\tau)^2}.$$

This shows that our assumption (9.9) implies that the function $t \hookrightarrow \sigma(t)$ is, up to a constant multiplicative factor, necessarily equal to the instantaneous volatility of the spot price. We shall use this remark to estimate $\sigma(t)$ from the data, for example, by computing the standard deviation of the spot price in a sliding window of 30 days. The plot of the instantaneous standard deviation of the Henry Hub natural gas spot price computed in this way is given in Figure 9.1.

**Rationale for the Use of Principal Components Analysis.** For the purposes of the present discussion, we fix times-to-maturity $\tau_1$, $\tau_2$, …, $\tau_N$, and we assume that on any given day $t$, quotes for the forward prices with times-of-maturity $T_1 = t + \tau_1$, $T_2 = t + \tau_2$, …, $T_N = t + \tau_N$ are available. This assumption may require a little massaging of the data, which we explain in detail below. In any case, it is a reasonable

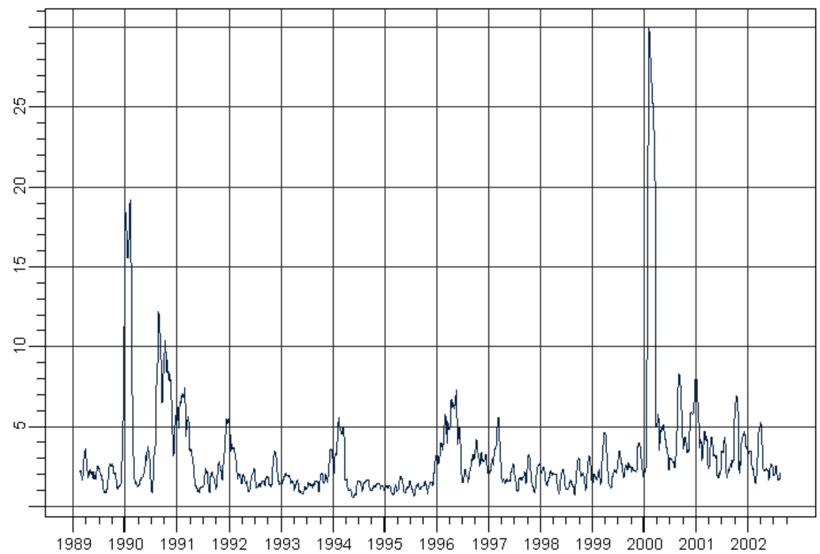


**Fig. 9.1** *Instantaneous standard deviation of the Henry Hub natural gas spot price computed in a sliding window of length* 30 *days.*

starting point for the present discussion. According to the stochastic dynamics derived in (9.8), we have

$$\frac{d\tilde F(t,\tau_i)}{\tilde F(t,\tau_i)} = \left(\tilde\mu(t,\tau_i) + \frac{\partial}{\partial\tau}\log\tilde F(t,\tau_i)\right)dt + \sigma(t)\sum_{k=1}^n \sigma_k(\tau_i)dW_k(t) \qquad i = 1,\dots,N.$$

We now define the $N\times n$ deterministic matrix $F$ by $F = [\sigma_k(\tau_i)]_{i=1,\dots,N,\ k=1,\dots,n}$. We can assume without any loss of generality that the $n$ column vectors $\{\sigma_k(\,\cdot\,)\}_{k=1,\dots,n}$ are orthonormal in $\mathbb{R}^N$. The goal of this subsection is to explain how we estimate these vectors from historical data. First, we derive the dynamics of the logarithm of the forward prices applying Itô's formula:

(9.11)
$$d\log\tilde F(t,\tau_i) = \left(\tilde\mu(t,\tau_i) + \frac{\partial}{\partial\tau_i}\log\tilde F(t,\tau_i) - \frac{1}{2}\sigma(t)^2\tilde\sigma(\tau_i)^2\right)dt + \sigma(t)\sum_{k=1}^n\sigma_k(\tau_i)dW_k(t)$$

for $i = 1,\dots,N$. Next, we compute the instantaneous variance/covariance matrix $\{M(t); t\geq 0\}$ defined by

$$d\,[\log\tilde F(\,\cdot\,,\tau_i), \log\tilde F(\,\cdot\,,\tau_j)]_t = M_{i,j}(t)dt.$$

From (9.11) we see that

$$M(t) = \sigma(t)^2\left(\sum_{k=1}^n \sigma_k(\tau_i)\sigma_k(\tau_j)\right),$$

or equivalently that

$$M(t) = \sigma(t)^2 FF^*$$

if we use the notation $F^*$ for the transpose of the matrix $F$. Our interest in the above computation is the fact that it gives a clear strategy for extracting the components of the matrix $F$ from historical data. Indeed, after estimating the instantaneous spot volatility $\sigma(t)$ in a rolling window as we explained earlier, we can estimate the matrix $FF^*$ from historical data as the empirical autocovariance of $\ln(F(t,\cdot)) - \ln(F(t-1,\cdot))$ after normalization by $\sigma(t)$. Diagonalizing this empirical variance/covariance matrix, we should be able to identify the principal components among any orthonormal basis of eigenvectors giving the columns of $F$. This singular value decomposition of the covariance matrix, together with its interpretation, are known as principal components analysis. Its relevance to the historical modeling of the dynamics of the forward curves has been recognized in the fixed income markets first, before being adopted in the analysis of the energy markets. See, for example, [46] and [14].

The gist of this derivation is that, under the assumption (9.9), if one is willing to *normalize* the log returns of the forward contract prices by the instantaneous volatility of the spot price, then the instantaneous autocovariance structure of the entire forward curve becomes time independent and hence can be estimated from the data. Moreover, diagonalizing this autocovariance matrix provides functions $\sigma_k(\tau)$, and from their relative sizes we can decide how many do contribute significantly to the dynamics, effectively choosing the order $n$ of the model.

*Remark.* PCA is based on the estimation of a covariance matrix, and as far as statistical estimation goes, some form of stationarity is needed to be able to base the estimates on time averages. In the approach described above, the reduction to stationarity was done by the division by $\sigma(t)$. Introducing the instantaneous spot volatility as a normalizing factor to capture the seasonality of the data is not the only way to use the PCA as a calibration tool. Another approach has been advocated by the scientists of Financial Engineering Associates (an energy software and consulting provider). They propose to bin the forward curves into 12 groups according to the month of the date $t$, and then to perform the PCA in each of these bins. This *seasonal principal component analysis*, as they call it, is more likely to capture the seasonality of the forward curves when present. But at the same time, it reduces drastically the size of the data sets from which the covariance matrices are estimated. Not only does it restrict the length of the forward curves modeled in this way (typically to 12 months), but it also creates serious difficulties in the assessment of the confidence in the results. Indeed, the PCA involves the inversion and the diagonalization of the covariance matrix estimate, and these operations are notoriously erratic and nonrobust, especially when the matrix is ill-conditioned, which is expected in most practical applications.

**9.3. Pricing Calendar Spreads in This Framework.** As we already explained, a calendar spread is the simplest form of spread option because it involves only one underlying commodity, the two underlying indexes being the prices of two forward contracts with different maturities, say $T_1$ and $T_2$. So using the same notation as before, we have

$$S_1(t) = F(t, T_1) \qquad \text{and} \qquad S_2(t) = F(t, T_2),$$

and since we consider only deterministic coefficients, these forward prices are lognormally distributed and we can use our pricing formula (6.1). Hence, because of

(9.4), the price at time $t$ of a calendar spread with maturity $T$ and strike $K$ on these two forward contracts is given by substitution in (6.1) of the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, and $\kappa$ given by

$$\alpha = F(t,T_2)e^{-r(T-t)}, \quad \beta = \sqrt{\sum_{k=1}^{n}\int_t^T \sigma_k(s,T_2)^2 ds},$$

$$\gamma = F(t,T_1)e^{-r(T-t)}, \quad \delta = \sqrt{\sum_{k=1}^{n}\int_t^T \sigma_k(s,T_1)^2 ds},$$

and $\kappa = Ke^{-r(T-t)}$. Recall that $\mu \equiv 0$ since we are using the risk-neutral dynamics of the forward curves for pricing purposes. Finally, the correlation coefficient $\rho$ is given by

$$\rho = \frac{1}{\beta\delta}\sum_{k=1}^{n}\int_t^T \sigma_k(s,T_1)\sigma_k(s,T_2)\, ds.$$

**9.4. Pricing Spark Spreads in This Framework.** The methodology of this subsection can be used for all the cross-commodity spreads discussed earlier in the text. For the purpose of illustration, we discuss the specific case of spark spread options, for they provide a challenging example of a cross-commodity instrument. As explained earlier, their pricing is of great importance both for risk management and asset valuation purposes. We proceed to price them in the framework introduced in this section. Because we are now dealing with two commodities, we need to adjust the notation: we choose to use the subscript $e$ for the forward prices, times-to-maturity, volatility functions, and other quantities pertaining to electric power, and the subscript $g$ for the quantities pertaining to natural gas.

**Description of a Spark Spread Option.** Let $F_e(t,T_e)$ be the price at time $t$ of an electricity forward contract with time-of-maturity $T_e$, and let $F_g(t,T_g)$ be the price at time $t$ of a natural gas forward contract with time-of-maturity $T_g$. For the purpose of this subsection, a crack spread option with maturity date $T$ is a contingent claim maturing at time $T$ that pays the amount

$$\left(F_e(T,T_e) - H_{\text{eff}}F_g(T,T_g) - K\right)^+,$$

where the efficiency $H_{\text{eff}}$ is a fixed conversion factor, and $K$ is the strike of the option. Obviously we assume that $T < \min\{T_e,T_g\}$. The buyer of such an option may be the owner of a power plant that transforms gas into electricity, seeking protection against low electricity prices and high gas prices.

**Joint Dynamics of the Commodities.** In complete analogy with (9.2), we assume that the joint dynamics of the forward prices of the two commodities are given by equations of the form

$$\begin{cases} dF_e(t,T_e) \;=\; F_e(t,T_e)[\mu_e(t,T_e)dt + \sum_{k=1}^{n}\sigma_{e,k}(t,T_e)dW_k(t)], \\ dF_g(t,T_g) \;=\; F_g(t,T_g)[\mu_g(t,T_g)dt + \sum_{k=1}^{n}\sigma_{g,k}(t,T_g)dW_k(t)]. \end{cases} \tag{9.12}$$

Notice that each commodity has its own volatility factors, and that the correlation between the two dynamics is built into the fact that they share the same driving Brownian motion processes $W_k$.

**Fitting Cross-Commodity Models.** Without going into all the gory details of the implementation, we describe the main steps of the fitting procedure.

We assume that on any given day $t$, we have electricity forward contract prices for $N^{(e)}$ times-to-maturity $\tau_1^{(e)} < \tau_2^{(e)} \cdots < \tau_{N^{(e)}}^{(e)}$, and, respectively, that we have natural gas forward contract prices for $N^{(g)}$ times-to-maturity $\tau_1^{(g)} < \tau_2^{(g)} \cdots < \tau_{N^{(g)}}^{(g)}$. As explained in our discussion of the empirical data issues, typically we have $N^{(e)} = 12$, and $N^{(g)} = 36$, and even $N^{(g)} = 84$ more recently. Assuming (9.9) for both the electricity and the natural gas forward volatilities, and estimating the instantaneous volatilities $\sigma^{(e)}(t)$ and $\sigma^{(g)}(t)$ of the electricity and gas spot prices in a rolling window of 30 days, we can consider, for each day $t$, the $N = N^{(e)} + N^{(g)}$-dimensional random vector $\mathbf{X}(t)$ defined by

$$\mathbf{X}(t) = \begin{bmatrix} \left( \frac{\log \tilde{F}_e(t+1,\tau_j^{(e)}) - \log \tilde{F}_e(t,\tau_j^{(e)})}{\sigma^{(e)}(t)} \right)_{j=1,\dots,N^{(e)}} \\ \left( \frac{\log \tilde{F}_g(t+1,\tau_j^{(g)}) - \log \tilde{F}_g(t,\tau_j^{(g)})}{\sigma^{(g)}(t)} \right)_{j=1,\dots,N^{(g)}} \end{bmatrix}.$$

Running the PCA algorithm on the historical samples of this random vector $\mathbf{X}(t)$ will provide a small number $n$ of significant factors, and for $k = 1, \dots, n$, the first $N^{(e)}$ coordinates of these factors will give the electricity volatilities $\tau \hookrightarrow \sigma_k^{(e)}(\tau)$ for $k = 1, \dots, n$, while the remaining $N^{(g)}$ coordinates will give the natural gas volatilities $\tau \hookrightarrow \sigma_k^{(g)}(\tau)$.

**Pricing a Spark Spread Option.** Risk-neutral pricing arguments imply that the price of the spark spread option at time $t$ with maturity $T$ and strike $K$ is given by

$$p_t = e^{-r(T-t)} \mathbb{E}_t \left\{ (F_e(T,T_e) - H_{\text{eff}} F_g(T,T_g) - K)^+ \right\}. \tag{9.13}$$

This pricing formula can also be evaluated with the approximation techniques advocated in this section because the distributions of $F_e(T,T_e)$ and $F_g(T,T_g)$ are log-normal under the pricing measure. Indeed, under the risk-neutral measure calibrated by PCA, the prices at time $T$ of the forward contracts are given by

$$F_e(T,T_e) = F_e(t,T_e) \exp\left[ -\frac{1}{2} \sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s,T_e)^2 ds + \sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s,T_e) dW_k(s) \right]$$

and

$$F_g(T,T_g) = F_g(t,T_g) \exp\left[ -\frac{1}{2} \sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s,T_g)^2 ds + \sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s,T_g) dW_k(s) \right].$$

As explained above, the correlation between the two markets is built into the random driving factors because we use the same Brownian motions to drive the stochastic differential equations (9.12) both for electricity and natural gas. Consequently, setting $S_1(t) = H_{\text{eff}} F_g(t,T_g)$ and $S_2(t) = F_e(t,T_e)$ we find ourselves in the log-normal world considered throughout the present review paper, and we can use formula (6.1) with the parameters

$$\alpha = F_e(t,T_e) e^{-r(T-t)} \qquad \text{and} \qquad \beta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{e,k}(s,T_e)^2 \, ds}$$

for the first log-normal distribution,

$$\gamma = H_{\text{eff}} F_g(t, T_g) e^{-r(T-t)} \qquad \text{and} \qquad \delta = \sqrt{\sum_{k=1}^{n} \int_t^T \sigma_{g,k}(s, T_g)^2 \, ds}$$

for the second one, and $\kappa = K e^{-r(T-t)}$ and

$$\rho = \frac{1}{\beta\delta} \int_t^T \sum_{k=1}^{n} \sigma_{e,k}(s, T_e) \sigma_{g,k}(s, T_g) ds$$

for the correlation coefficient.

**10. Conclusion.** Our review has concentrated on the mathematics of the pricing and hedging of spread options. This choice was motivated by the extreme importance of these financial instruments, and the rich variety of mathematical tools that have been used in their analysis. We reviewed the results published on this subject in the economics, financial, business, and mathematical literature, and we focused the spotlight on the major issue of hedging and computing accuracy in approximation schemes. Our goal was to make the applied mathematical community aware of the slue of challenging problems remaining to be solved in this area.

In so doing, we devoted a good part of our efforts to the discussion of a new pricing approach introduced recently by the authors. While giving an exhaustive review of the existing literature, we compared the performance of our algorithm to all the existing methods known to us, from both numerical and analytical points of view.

We primarily illustrated the concepts and numerical methods reviewed in this paper with examples from the energy markets. We attempted to provide a thorough review of the main technical challenges of these markets, and in particular (a) the important features of the data available to the traders and analysts, (b) the statistical models used to describe them and to build risk management systems, and (c) the pricing models used to value the complex instruments of the energy markets. The latter form a fertile ground for mathematical investigations, and we believe that applied mathematicians will be well advised to pay more attention to their analysis.

It is clear that some of the explicit formulae reviewed in the text can be inverted. This allows for efficient computations of implied quantities such as volatilities and correlations. From these implied volatilities and correlations, one should be able to build more complex models (stochastic volatility and/or stochastic correlation, jumps, etc.) that should better fit the market reality. This would be a welcome development, for it is still unclear what these implied quantities should be in the framework of spread options.

Throughout the survey, we have tried to prepare the reader for some of the challenges facing that part of the applied mathematics community interested in the practical applications of financial mathematics. Several natural extensions of some of the results reviewed in this paper are currently under active investigation. For example, some of the numerical approximations presented here have been generalized in [10] to the case of options on linear combinations of assets (basket options, rainbow options, etc.), or linear combination of prices of a single asset at different times (discrete-time average Asian option, for example). These extensions provide efficient algorithms to compute approximate prices and hedges for these options. However, more progress in this arena is sorely needed!

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