Enhancing Least Squares Monte Carlo with Diffusion Bridges: an Application to Energy Facilities*†

Tommaso Pellegrino and Piergiacomo Sabino ‡
tommaso.pellegrino@eon.com, piergiacomo.sabino@eon.com

CRM, Risk Methods and Model Governance,
E.ON Global Commodities SE,
Holzstrasse 6, Düsseldorf, Germany

March 24, 2014

Abstract

The aim of this study is to present an efficient and easy framework for the application of the Least Squares Monte Carlo methodology to the pricing of gas or power facilities as detailed in Boogert and de Jong [7]. As mentioned in the seminal paper by Longstaff and Schwartz [18], the convergence of the Least Squares Monte Carlo depends on the convergence of the optimization combined with the convergence of the pure Monte Carlo method. In the context of the energy facilities, the convergence of the algorithm is more challenging in particular for the computation of sensitivities and optimal dispatched quantities. To our knowledge, an extensive study of the convergence and hence, of the reliability of the algorithm has not been performed yet, in our opinion because of the apparent infeasibility and complexity to use a very high number of simulations. We present then an easy way to simulate random trajectories by means of diffusion bridges similar to the one proposed by Kutt and Welke [13] that is equivalent to generate a time reversal Itô diffusion. Our approach permits to perform a backward dynamic programming strategy based on a huge number of simulations without storing the whole simulated trajectory. Generally, in the valuation of energy facilities one is also interested in the forward recursion. We then design the backward and forward recursions algorithm such that one can produce the same random trajectories by the use of multiple independent random streams without storing at intermediate time steps. Finally, we show the advantages of our methodology for the valuation of virtual hydro power plants and gas storages.

Keywords: Least Squares Monte Carlo, Markov Bridges, Stochastic Optimization, Energy Derivatives, Computational Finance.

1 Introduction and Motivation

The aim of this study is to present an efficient and easy framework for the application of the Least Squares Monte Carlo (LSMC) methodology in order to price gas or power facilities as detailed in Boogert and de Jong [7]. Although Boogert and de Jong [7] and the sequel [8] discuss the method and its applicability to gas storages, to our knowledge, an extensive study of the convergence and hence, of the reliability of the algorithm has not been performed yet, in our opinion because of the apparent infeasibility and complexity to use a very high number of simulations.

As mentioned in the seminal paper by Longstaff and Schwartz [18], the convergence of the LSMC depends on the convergence of the optimization combined with the convergence of the pure Monte

*The technique here discussed does not reflect EGC view.

†Both authors are grateful to Christoph Waidacher for his support as well as the rest of CRM team members and German Ramirez-Espinosa for the useful discussions.

‡Corresponding author.

Carlo (MC) method. In the context of the energy facilities, the optimization is more challenging and its convergence in particular for the computation of sensitivities and optimal dispatched quantities.

In abstract terms, the MC dimension d of a problem is equal to the number of random variables to simulate (see Glasserman [16]). It means that for a gas storage or virtual hydro power plant, d equals the number of days of the contract if we assume a one-factor spot model (even more if one uses a finer time grid for the optimization). A common rule of thumb in MC estimations is to consider the number of simulations N_S at least hundreds of times higher than d, that implies $d \approx 365 \times 100 \times F = 36500F$, for a one year contract facility, where F is the number of factors selected for the spot dynamics.

In this paper we describe an easy simulation strategy where one can potentially employ a huge number of simulations by means of diffusion bridges. In fact, the main limitation to the use of a high number of simulations might seem the generation and the storing of the $N_S \times N_T$ matrix of the simulated spot prices paths, where N_T is the number of time steps.

As presented in Dutt and Welke [13], the computational complexity of the LSMC can be remarkably reduced using a modified version of the common Brownian Bridge (BB) construction. Indeed, for American options what matters in the dynamic programming is the comparison between the intrinsic value and the continuation value at a given time step t. Hence, one needs to know the simulated prices at time t and time $t+1$ only, nothing more.

The BB construction provides an easy and flexible way to obtain this requirement because it consists of generating a Brownian trajectory by filling the intermediate points of the grid knowing the value at the end points (we refer the reader to Blandt and Sørensen [6] and Barczy and P. Kern [5] for an extensive study of diffusion bridges). In the common BB construction one fills iteratively the mid-points of the grid; however in the iteration for the LSMC one must always anchor the first point and iteratively fill the grid starting from the last point (see Figure 1).

The same construction can be employed for other Markov dynamics, intuitively the only mathematical requirement for a practical implementation is the knowledge of the transition density of the process in closed form. Another way to tackle the problem is, in mathematical terms, to apply the theorem of Anderson [3], Haussmann and Pardoux [17] for time-reversal Itô diffusions. Finally, everything can be also extended to Lévy-driven processes as described by Hoyle et al. [14] in the context of Lévy bridges.

The construction outlined above can reduce the computational complexity of the backward recursion of the LSMC that returns the price of the contract only. In case of energy facilities one is often interested in the optimal dispatched strategy that is obtained by the so-called forward recursion. One might be tempted to generate another set of random trajectories once having stored the coefficients of the regression or think to fall back to the choice of simulating all the price trajectories. However, these choices may lead to biased or unreliable calculations especially with a low number of simulations.

The solution of this problem is another contribution of our study. Most programming languages provide a quite high control to the generation of the random variables via the use of random streams. One can think at the generation of a random variable as the extraction of a certain index of a very long array whose values emulate the distribution of an abstract random variable. Once known and stored the index (the state) of the array, one can repeat exactly the same sequence of random draws.

We propose then, to base the construction on multiple and independent random streams, one for each point of the grid. In practice, each factor at time t is always obtained by generating a random variable extracted by the random stream at time t (ϵ_t is drawn from the t^{th} random stream). The iteration and the stream at time t will determine the random extractions at time t only, again one needs to save only one state per time and not at all the random innovations or factors. In the implementation of the forward recursion then, the solution is simply to reset all the states of the streams and revert the iteration of the backward construction.

Finally, we tested our methodology on concrete examples. First, we test the correctness of our implementation computing the value of American call option driven by a geometric BM (GBM) knowing that their prices must coincide to their European counterparts. We show here that the use of LSMC may need many simulations to get an acceptable convergence. In a second step, we adopt our methodology to price virtual hydro power plants with a given inflow and gas storages with one year maturity. The benefit of our methodology is evident in all the cases. The number of simulations one can use is by far higher than the one used when saving all the trajectories. The benefit would become even more remarkable for longer maturities.

The paper is organized as follows. Section 2 briefly presents the original LSMC algorithm introduced in Longstaff and Schwartz [18] and its application to energy facilities as detailed in Boogert and de Jong [7]. In Section 3 we show how to modify the BB to perform the backward simulation and how to obtain the Ornstein-Uhlenbeck bridge (OUB) construction. In addition, we compare the bridge strategy with the time-reversal Itô diffusion strategy and give the references for the bridge construction for other stochastic processes. Section 4 shows the numerical results and describes how to perform the forward recursion without storing the whole price path. Finally, Section 5 concludes the paper and suggests future studies.

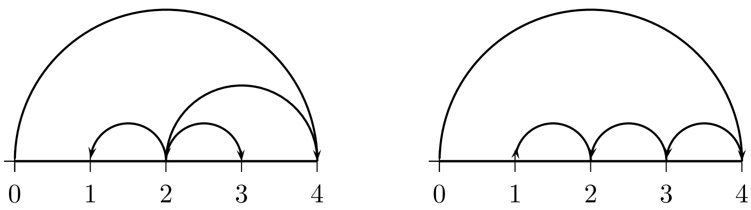


Figure 1: The figure on the left shows the common bridge construction, the figure on the right shows the backward construction. In the latter one the point 0 is the anchored point.

2 A brief description of the LSMC Algorithm

In this section we briefly describe the general LSMC algorithm originally introduced in Longstaff and Schwartz [18] and its application to energy facilities as done in Boogert and de Jong [7], we refer the reader to the original papers for more on the topic.

In the following we define the usual probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}(t))_{t\leq T}$ which represents the continuous filtration generated by a (multi)-dimensional Brownian motion $W(t)$ on a finite time horizon $[0, T]$. In addition, we consider that the price process(es) are driven by Itô diffusions and, consistent with the no-arbitrage argument, we assume the existence of an equivalent martingale measure $\mathbb{Q}$.

2.1 The original Longstaff-Schwartz Algorithm

Let $S(t)$ be the price of the underlying security, the value $A(t)$ at time t of an American option on S with payoff g at maturity T is (we consider for simplicity $r = 0$):

$$A(t) = \text{esssup}_{\tau\in\mathcal{T}(t,T)}\mathbb{E}\left[g(S(\tau))|\mathcal{F}(t)\right], \quad t \leq T,$$

where $\mathcal{T}(t, T)$ is the set of all stopping times in $[t, T]$ and $\mathbb{E}$ is taken w.r.t. the risk-neutral measure $\mathbb{Q}$ (for simplicity we drop $\mathbb{Q}$ in our notation).

The classical approach is to turn the American into a Bermudan problem over a time grid of N points, $t_1, \ldots, t_N$, by comparing at each time step the intrinsic and the continuation value:

$$A(t_i) = \max\left(S(t_i) - K, \mathbb{E}\left[A(t_{i+1})|\mathcal{F}(t_i)\right]\right) \quad i = 1, \ldots, N. \tag{1}$$

The estimation of the American option price and the implementation of a MC-based algorithm for the recursion above is challenging. In fact, given a certain underlying price at time t_i, $S(t_i, \omega)$, we have that the intrinsic value depends only on one ω, while the calculation of the continuation value depends on the entire set of random events from t_{i+1} to t_N. Just sitting on one price trajectory will produce perfect foresight and hence an overestimation of the American option price (that can be considered as a MC upper bound).

The LSMC approach is based on a dynamic programming backward recursion where the expected value of the continuation value is approximated by a least-squares regression:

$$\mathbb{E}\left[A(t_{i+1})|\mathcal{F}(t_i)\right](\omega) \simeq a_0 + a_1 S(t_i, \omega) + \ldots, +a_B S^B(t_i, \omega), \tag{2}$$

as usual, the regression may be performed on a different set of basis functions and the coefficients $a_0, a_1, \dots, a_B$ depends on the number of simulations and the choice of the regressors. Several works have been devoted to study the efficiency of the method including convergence and over-fitting (see for instance Bouchard and Warin [9]). We will not discuss this topic and focus on the computational aspects of the approach. In fact, one naive implementation of the algorithm may rely on the simulation of N_S trajectories per each point of the grid. This means that this strategy needs to store $N_S \times N$ numbers that can limit the performance of the MC estimation because of memory and speed problems. Based on the work of Dutt and Welke [13] we show how the use of diffusion bridges can enhance the performance of the LSMC approach for several financial problems that rely on the simulation of a wide range of price dynamics.

2.2 The LSMC Algorithm for Energy Facilities

Denote by $C(t)$ the volume of a (virtual) gas storage or a hydro storage (VPS) or virtual hydro plant (VHP) at time t and suppose that $C_{min} \leq C(t) \leq C_{max}$. The holder of such an energy asset is faced with a timing problem that consists in deciding when to inject, to withdraw gas or water or do-nothing. This problem is then similar to the choice of the trading strategy of an American option, although in this case the decision is more challenging because of the several volumetric constrains and the possibility of multiple actions. For example VHPs generally have an exogenous inflow that is either described in the contract or dependent on the nature of the assets. This particular feature prevents the do-nothing action because one is forced to inject. VPSs may have an external inflow but it is not always the case.

To value these energy assets we follow a spot-price based strategy as described in Boogert and de Jong [7]. Denote by $J(t, x, c)$ the value of such an energy asset at time t given $C(t) = x$, one can write:

$$J(t, x, c) = \sup_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^T \phi_u\left(S(s)\right) ds + q((S(T), C(T))\,|S(t) = x, C(t) = c\right], \tag{3}$$

$\mathcal{U}$ denotes the set of admissible strategies and $u(t) \in \{-1, 0, 1\}$ is the regime at each time t such that ϕ is:

$$\begin{cases} \phi_{-1}(S(t)) & = & -S(t)a_{in} - K_{in}, & \text{injection} \\ \phi_0(S(t)) & = & -K_N, & \text{do nothing} \\ \phi_{-1}(S(t)) & = & S(t)a_w - K_{out} & \text{withdrawal} \end{cases} \tag{4}$$

$S(t)$ is the spot (day-ahead) price at time t, a_{in} and a_w are the injection (or pump for VPS) and withdrawal rates, respectively and K_{in}, K_{out} and K_N represent the costs of injection or pump, do-nothing and withdrawal. q takes into account the possibility of final penalties.

As already mentioned, VHPs do not offer the flexibility of pure do-nothing actions because of the exogenous inflow. In addition, the pump action plays the role of the injection in VPSs, while for VHPs there is no pump flexibility.

In the spirit of the Bellman principle, one can perform the following backward recursion for $i = 1, \dots, N$:

$$J(t_i, x, c) = \sup_{k \in \{-1,0,1\}} \left\{\phi_k S(t_i) + \mathbb{E}\left[J\left(t_{i+1}, S(t_{i+1}), \tilde{c}_k\right)|S(t_i) = x, C(t_i) = c\right]\right\}, i = 1, \dots, N, \tag{5}$$

where

$$\begin{cases} \tilde{c}_{-1} & = & \min(c + a_{in}, C_{max}) \\ \tilde{c}_0 & = & c \\ \tilde{c}_1 & = & \min(c - a_w, C_{min}). \end{cases} \tag{6}$$

VHPs have an exogenous inflow that is generally modeled by a deterministic function. This leads to the inclusion of an additional deterministic term $v(t)$ in the Bellman recursion. v_t represents the known volume that will be injected in the time interval $[t_i, t_{i+1}]$. In the general case, the rates can also be time or volume dependent quantities; we do not analyze this configuration because it a is complication that does not affect our approach.

As proposed in Boogert and de Jong [7], the backward recursion is then obtained by defining a finite grid of G steps for the admissible capacities c of the plant and then apply the LSMC methodology per volume step to the continuation value.

From the computational perspective, the implementation of the LSMC algorithm for energy assets is even more challenging than for American options. Following the naive approach described in the introduction, one would need to store $N_S \times N$ numbers for the spot price trajectories and also store $N_S \times N \times G$ numbers for the continuation values. This solution may computationally unfeasible.

The approach we describe and propose in this study reduces remarkably the computational costs because it relies on an implementation of the price trajectories based on diffusion bridges. Briefly, starting form the knowledge of the simulated prices at time T, at each time step t_i the backward recursion requires the knowledge of the simulated at time t_i and t_{i+1}, nothing more. Hence, it will be computationally convenient to simulate prices backward in time. Our approach relies on the construction of diffusion bridges that in some cases can be seen as the construction of path trajectories backward in time as we show in the following section.

3 An overview of Markov Bridges

In this section we describe some Markov bridges and the construction of random trajectories backward in time, in particular we focus on the BB and OUB. We also discuss the relationship between the use of bridges with time-reversal Itô diffusions.

3.1 The Brownian Bridge

Let $(W(t))_{t\geq 0}$ be a standard one-dimensional Wiener process on a filtered probability space introduced in section 2. Following Barczy and Kern [5], one can consider several equivalent versions of the BB from a to b over the time-interval $[0,T]$, where $a, b \in \mathbb{R}$:

$$B(t)^{\mathrm{ir}} = \begin{cases} a + (b-a)\dfrac{t}{T} + \displaystyle\int_0^t \dfrac{T-t}{T-s} dW(s), \text{ if } 0 \leq t < T, \\ \\ b, \text{ if } t = T. \end{cases} \tag{7}$$

or

$$B(t)^{\mathrm{st}} = \begin{cases} a + (b-a)\dfrac{t}{T} + \dfrac{T-t}{T} W\left(\dfrac{tT}{T-t}\right), \text{ if } 0 \leq t < T, \\ \\ b, \text{ if } t = T. \end{cases} \tag{8}$$

The above versions $B(t)^{\mathrm{ir}}$ and $B(t)^{\mathrm{st}}$ are called integral representation or space-time representations. The BB is not an adapted process, however one has the operative tool to generate the trajectory of a Wiener process that is indeed, given $W(t_b) = x_b$ and $W(t_a) = x_a$ with $t_a \leq t_b$ and $t \in [t_a, t_b]$:

$$W(t) \stackrel{d}{=} \frac{(t_b - t)x_a + (t - t_a)x_b}{t_b - t_a} + \sqrt{\frac{(t_b - t)(t - t_a)}{t_b - t_a}}\epsilon \tag{9}$$

with $\epsilon \sim \mathcal{N}(0,1)$.

The most common way is to first generate $W(t_N)$ and then sample $W(t_{\lfloor N/2 \rfloor})$ conditional to $W(t_0)$ and $W(t_N)$ and proceed iteratively as in Figure 1 (left). The construction can be performed in any order, in particular it can be designed such that the Wiener process is generated backward in time. We need to look at the bridge between two points where one of the two is always anchored at time $t = 0$ with $W(0) = 0$. For simplicity of notation we assume an equally spaced grid ($\Delta t = 1$) and just show the time indexes $t_i = \Delta t \times i$:

$$W(N-i) \stackrel{d}{=} \frac{N-i}{N-i+1} W(N-i+1) + \sqrt{\frac{N-i}{N-i+1}}\epsilon_i, \quad i = 1, \ldots, N-1 \tag{10}$$

with ϵ_i i.i.d. $\sim \mathcal{N}(0,1)$.

From the computational perspective, the construction above is neither less or more demanding than the common one. However, it permits to perform the LSMC without saving all the price trajectories $\forall t_i$ given N_S the number of simulations, because one needs the values at t_i and t_{i+1}, only. Based on the observations above, one can select a very high number of simulations and what is more important, independently of the maturity or the number of grid points. Suppose that the maximum possible memory allocation if one saves all trajectories is $M = N \times N_{S_1}$, with the BB approach one can attain a number of simulation $N_{S_2} = \frac{N \times N_{S_1}}{2}$. For instance, assume $M = 365 \times 1000$, $N_{S_1} = 1000$ while $N_{S_2} = 182500$ with the same computational burden. It is a remarkable factor especially when one needs to implement fast and efficient computations.

3.2 The Ornstein-Uhlenbeck Bridge

It is widely accepted that energy market prices or log-returns display mean reversion and several models based on the Ornstein-Uhlenbeck (OU) process have been proposed. For instance, Boogert and de Jong [8] adopt a mean-reverting OU or a multi-factor spot model; these ones are based on the combination of a OU process and a BM. This means that one must only adapt the construction of the BB to an OUB but the computational complexity and the strategy remain unchanged.

Let $(X(t))_{t \geq 0}$ be a one-dimensional canonical OU process starting in 0, i.e., it is the unique strong solution of the stochastic differential equation (SDE):

$$dX(t) = -\kappa X(t)dt + \sigma dW(t),$$

with initial condition $X(0) = 0$, $\sigma > 0$, it is well-known that the OU process has the integral representation:

$$X(t) = \sigma\, e^{-\kappa t} \int_0^t e^{\kappa s} dW(s),$$

which is a Gaussian process with mean function $\mathbb{E}[X(t)] = 0$ and variance function $Var[X(t)] = \dfrac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}) = \dfrac{\sigma^2}{\kappa} e^{-\kappa\, t} \sinh(\kappa t)$.

As for the BB, the OUB from a to b over the time-interval $[0, T]$, where $a, b \in \mathbb{R}$ can be constructed in several ways (see Barczy and P. Kern [5]):

$$X(t)^{\text{ir}} = \begin{cases} a\dfrac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} + b\dfrac{\sinh(\kappa t)}{\sinh(\kappa T)} + \sigma \displaystyle\int_0^t \dfrac{\sinh(\kappa(T-t))}{\sinh(\kappa(T-s))} dW(s), \text{ if } 0 \leq t < T, \\ \\ b, \text{ if } t = T. \end{cases} \tag{11}$$

Choosing $a = 0$ (it is only an offset) and $b = X(T)$ a more convenient notation is based on the space-time representation (for more general expressions see Barczy and P. Kern [5]):

$$X(t)^{\text{st}} = X(T)^{\text{st}} \frac{\sinh(\kappa t)}{\sinh(\kappa T)} + \sigma\, e^{-\kappa t}\, \frac{\tau(T) - \tau(t)}{\tau(T)}\, W\left(\frac{\tau(t)\tau(T)}{\tau(T) - \tau(t)}\right). \tag{12}$$

where the function $\tau(t)$ is a time-dependent function defined as:

$$\tau(t) = \int_0^t e^{2\kappa s} ds = \frac{e^{-\kappa t} \sinh(\kappa t)}{\kappa};$$

it follows that $\tau(t) = \frac{Var[X(t)]}{\sigma^2}$.

This means that the backward bridge construction of a OU trajectory at the time points defined in the grid in subsection 3.1 can be retrieved from the one of the BB by changing the time scale:

$$\begin{aligned} X(N-i) \quad &= \quad \frac{\tau(N-i)}{\tau(N-i+1)} X(N-i+1) + \\ &\sigma \sqrt{\frac{\tau(N-i)}{k\tau(N-i+1)} \left(e^{-k(N-i)}\tau(N-i+1) - e^{-k(N-i+1)})\tau(N-i)\right)}\, \epsilon_i, \quad i = 1, \ldots, N. \end{aligned} \tag{13}$$

The application to the non-canonical OU process $Y(t) = Y(0)e^{-kt} + \alpha\left(1 - e^{-kt}\right) + \sigma\, e^{-\kappa t} \int_0^t e^{\kappa s} dW(s)$ is straightforward and consists in adding a deterministic component only.

3.3 Time-reversal Itô diffusions and Bridges

The strategy that we propose in order to reduce the computational costs of the LSMC algorithm for energy facilities is based on the possibility to simulate stochastic processes backward in time where we use diffusion bridges as mathematical tool. Intuitively, the construction of a bridge requires the knowledge of the transition density of a process in a (semi)-analytical form and then play with the conditional expectation.

In the previous two subsections we rewrote the BB and OUB constructions because our focus is on energy markets that make extensive use of these dynamics. However, one may adapt the constructions available for other stochastic process.

For instance Makarov and D. Glew [19] provide the bridge construction of Bessel processes and given this result, Baldeaux and Roberts [4] extend the application to Heston volatility models, including compound Poisson jumps, making use of the of the exact simulation proposed by Broadie and Kaya [10]. However, the main goal in Baldeaux and Roberts [4] is to apply Quasi-Monte Carlo methods to the Heston model and adopt the standard bridge strategy. Finally, the strategy can be also extended to the Lévy bridges described by Hoyle et al. [14].

Another approach, discussed in Kutt and Welke [13], consists in finding the time-reversal diffusion of the stochastic process. The mathematical tool relies on the following theorem (see Anderson [3], Haussmann and Pardoux [17], Russo et al. [15]):

Theorem 1. *If $X(t), 0 \leq t \leq T$ is a diffusion process such that $X(t)$ is a strong solution of*

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = x_0 \tag{14}$$

then the time-reversed process $Y(t) = X(T-t)$ is also a Markov process and it is weak solution of

$$dY(t) = \tilde{\mu}(t, Y(t))dt + \tilde{\sigma}(t, Y(t))d\tilde{W}(t), \quad Y(0) = X(T) \tag{15}$$

$\tilde{W}(t)$ is still a BM and

$$\tilde{\mu}(t, x) = \mu(T-t, x) + \frac{1}{p_X(T-t)} \frac{\partial \left(\sigma^2(T-t, x)p_X(T-t)\right)}{\partial x}, \tag{16}$$

where $p_X(T-t)$ is the law of $X(t)$.

It is possible to prove (see Kutt and Welke [13]) that the BB and consequently the OUB we presented before, can be obtained by the the theorem above because they are both Gaussian processes. However, the application to other dynamics does not imply a bridge construction. Indeed, Kutt and Welke [13] apply the theorem to a CIR dynamics and are faced to the non-trivial problem of discretizing a SDE with a complicated drift. Although some advanced discretization schemes have been proposed to simulate Bessel processes (see for instance Andersen [2] or Alfonsi [1]), to our knowledge it is not guaranteed that they can be applied to their time-reversed version. In contrast, the bridge construction (when available) is an exact simulation technique and overcomes the problem of the discretization error for time reversal diffusions. Finally, because of the fact that the filling strategy can be implemented in several ways and in particular in a backward manner, it is in our opinion the natural setting for LSMC simulations.

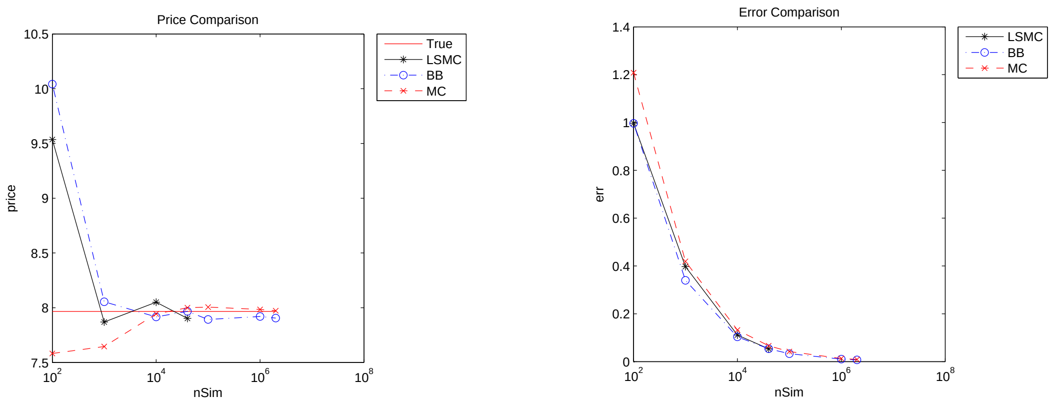
4 Numerical Experiments

In this section we describe the benefits of the use of diffusion bridges with practical applications. We consider five examples: a simple American call option in a Black-Scholes market, a VHP and a VPS with a single factor geometric-OU spot dynamics and a finally a seasonal and fast churn gas storages with a two factors geometric OU plus BB spot dynamics. In addition, everywhere in the experiments, we adopt polynomials of order two as the basis functions for the LSMC regression. Several works have been devoted to the study of the convergence and the quality of the LSMC and the regression; we here concentrate our analysis on the computational efficiency of the LSMC methodology combined with the backward bridge construction.

Given the fact that the price of an American call is equivalent to the one of its European counterpart that is known in closed form, we use this setting as a benchmark. All the experiments were performed in MATLAB on 32-bit computer with a 2.5 GHz Processor and 4 GB Memory.

S_0	K_S	N_{steps}	r	σ	T
100	100	365	0	0.20	1

Table 1: Input Parameters in the BS dynamics



(a) Call Price Comparison (b) Call Price Error Comparison

Figure 2: Comparison Across Simulation Approaches. Standard LSMC cannot be performed with more than 4×10^4 simulations.

4.1 A simple American (European) Option Case

As mentioned above, the computation of the value of an American call option by means of the LSMC method is over-engineered, however it gives us some insights for more complex applications. The configuration parameters are shown in Table 1, we explicitly analyze an at-the-money call option because the computational error refers completely to its extrinsic value component.

The benefit of the BB based LSMC method is evident in Figure 2; the maximum number of simulations that can be employed saving all the simulated trajectories is around 4×10^4 in contrast the BB-based can attain even millions of scenarios. The possibility to launch potentially a very high number of scenarios has its practical application in trading and risk management in the computation of the sensitivities. Indeed, the estimation of vega can be especially computationally demanding.

4.2 Application to Hydro Assets

In this section we turn our attention to the LSMC valuation of the energy facilities. As a first example we consider the pricing of a VHP and a VPS with a one-factor dynamic of the spot prices similar to the one proposed in Schwartz [23]:

$$\begin{aligned} dX_1(t) &= -kX_1(t)dt + \sigma_1 dW_1(t), \\ S(t) &= s_0 \exp\left(X_1(t) + h(t)\right), \end{aligned} \tag{17}$$

where $X(t)$ is a canonical OU process and $h(t)$ is a deterministic function. Indeed, it is a common practice in energy markets to assume that spot prices display mean-reversion to a deterministic level. Table 2 shows the value of the parameters of the one-factor model that we adopt in our numerical experiments; their calibration is not the focus of this paper, s_0 can be seen in euro per MWh. In addition, we assume that our calculations take place in the risk-neutral measure. Table 3 shows the contract specifications of the hydro assets under discussion, only the VHP has an external inflow (see Figure 3). We assume for simplicity that there no operational costs and that $C_{min} = 0$. We simulate prices on an equally spaces time-grid with $N = 365$ for the one-year maturity where the

Table 2: Parameters for Spot (day-ahead) dynamics

s_0	35.75	σ_1	0.75
k	0.05	T	1

Table 3: VHP and VPS specifications and Inflows

(a) VHP and VPS Contract Specifications

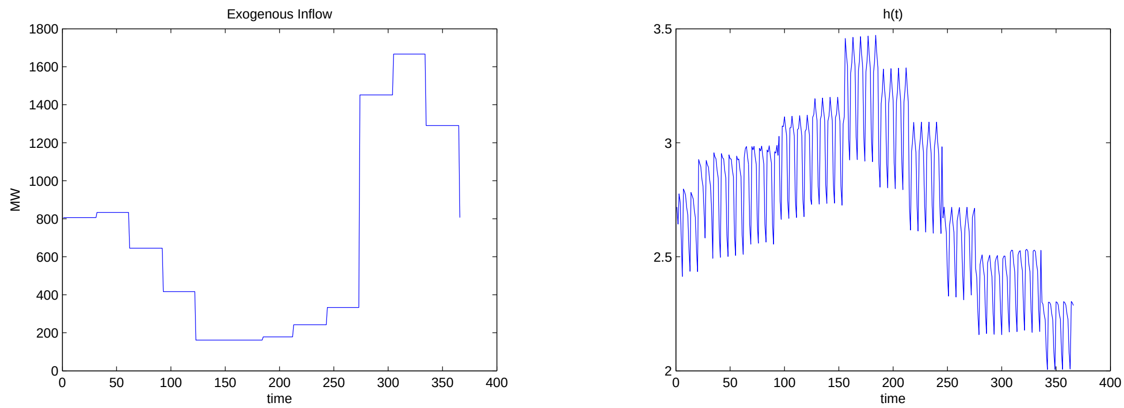
	VHP	VPS
$C(0)$(MWh)	56250	0
$C(T)$(MWh)	56250	0
a_{in}(MW)		21
a_w (MW)	100	25
C_{max}(MWh)	125000	1500

(b) VHP Inflows in MW

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
6.72	7.44	10.08	13.89	60.48	69.44	53.75	33.6	34.72	26.88	17.36	6.72

valuation day is on August 1st. Finally, we discretize the feasible capacities into $G = 100$ steps and we run the LSMC optimization with $h_S = 24$ (daily) granularity.

Figure 3: Daily Exogenous Inflow for the VHP and deterministic function $h(t)$.



(a) Exogenous Inflow.

(b) Time-varying function $h(t)$.

Figure 4: VHP Feasible Volumes.

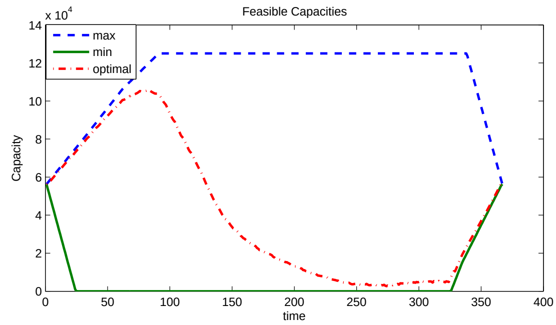


Figure 4 shows the maximum and minimum attainable volumes, the dot-dashed line represents the mean of the optimal volumes calculated with the LSMC (4×10^4 simulations). Finally Table 4 displays the numerical results we obtained combining the LSMC algorithm with the OU construction. The total value of the VHP, extrinsic value (EV) plus intrinsic value (IV), is mainly driven by the intrinsic value, $IV = 8,931,896$ that does not need to be estimated by MC simulations, we can consider that this VHP is somehow always in-the-money but we need at least a number of simulations N_S higher than 10^4 to reach the convergence. With our computer, the naive implementation that

Table 4: VHP and VPS Backward Results, values are in euro

	VHP				VPS			
N_S	Price	EV	Err	%err	Price	EV	Err	%err
10^2	10,986,044	2,054,148	790,000	39%	120,920	103,179	17,000	17%
5×10^2	9,636,049	704,153	230,000	32%	80,909	63,168	6,000	10%
10^3	9,847,144	915,248	170,000	19%	82,070	64,329	3900	6%
10^4	9,501,261	569,365	49,000	9%	72,944	55,203	920	2%
4×10^4	9,507,044	580,000	20,000	3%	72,525	54,784	460	1%
10^5	9,528,381	596,485	12,000	2%	72,463	54,722	290	1%
2×10^5	9,529,048	597,152	8,000	1%	72,483	54,742	200	0%

Table 5: Parameters for Spot (day-ahead) dynamics

s_0	67	σ_1	0.40
k	17	σ_2	0.1

relies on the generation and the storing of all the spot trajectories causes already memory issues with 7×10^3 simulations. This limitation would have prevented a correct estimation of the VHP price. In fact, the error of the estimation, (the root mean squared error divided by the square root of N_S, Err in the tables), is still high at this level of random scenarios (%err refers to Err divided by EV).

Besides that, would the granularity of the spot simulation have been hourly ($h_S = 1$) or the maturity higher that one year, our simulation strategy would have been unaffected while the naive one would have allowed much less number of simulations. Indeed, the LSMC backward recursion based on the OUB is independent of the number grid points because one needs to store only the prices at two consecutive dates.

Finally, the same observations are applicable to the VPS. Although, the extrinsic value has more weight in the price of this asset because $IV = 17,741$, once more one needs more than 10^4 scenario to reach full convergence. The possibility to run a very high number of scenario is of fundamental importance once one is interested in computing sensitivities and dispatched volumes corresponding to certain strategies.

4.3 Application to Gas Storages

In this section we apply our methodology to a different setting. We consider a two-factors spot model, similar to the one proposed in Schwartz-Smith [21], and use it to price a seasonal and a fast churn gas storages whose contract specifications are shown in Table 6:

$$\begin{aligned} dX_1(t) &= -kX_1(t)dt + \sigma_1 dW_1(t), \\ dX_2(t) &= \sigma_2 dW_2(t), \\ S(t) &= s_0 \exp\left(X_1(t) + X_2(t) + h(t)\right). \end{aligned} \tag{18}$$

s_0 can be seen in pence per therms. The spot price then is driven by $X_2(t)$, that can be seen as a long-term stochastic movement and $X_1(t)$ that is a mean reverting process. As for the one-factor model, $h(t)$ is a deterministic function; the remaining model parameters, whose calibration is not the focus of this paper, are shown in Table 5. Besides that, we use the same grid discretizations as done in the previous section. The seasonal asset has already gas in the storage at the valuation date (almost 2/3 of C_{max}) and its time to maturity is $T = 1/2$ years. Given its specifications, the asset needs 100 days to be completely filled, if it initially no gas in it, that leads to the fact that its total price is mainly driven by the IV, that is $5,282,923$ (pounds). The estimated prices in Table 7 show that the EV is only a small percentage of the total value. Although one might not be interested in achieving a very high precision of the EV for this seasonal gas storage, the bridge construction allows once more to simulate a large number of scenarios that can be helpful for sensitivity analysis.

For the fast churn asset, only 6 days are necessary to completely fill the gas storage (see Figure 5; in this case we consider a one year contract ($T = 1$). The extrinsic component of the price is now relevant because the IV is $94,325$. Surprisingly, the LSMC seems to be very convergent in this

Table 6: Seasonal and Fast Storage Specification

	Seasonal	Fast
$C(0)$(Therm)	1.95×10^7	0
$C(T)$(Therm)	0	0
a_{in}(Therm/day)	3×10^5	5×10^5
a_w (Therm/day)	3×10^5	5×10^5
C_{max}(Therm)	10^7	10^7

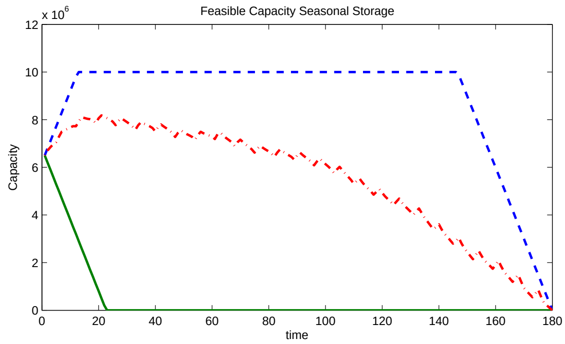
Table 7: Gas Storage Backward Results, values are in pounds

	Seasonal				Fast			
N_S	Price	EV	Err	% Err	Price	EV	Err	% Err
10^3	5,454,453	171,529	1,010	0.6%	194,922	100,596	740	0.7%
2×10^3	5,458,263	175,339	710	0.4%	196,362	102,036	470	0.5%
3×10^3	5,454,078	171,154	510	0.3%	195,866	101,541	420	0.4%
5×10^3	5,458,512	175,589	420	0.2%	194,003	99,678	310	0.3%
10^4	5,452,113	169,189	290	0.2%	194,523	100,197	220	0.2%
2×10^4	5,451,579	168,655	210	0.1%	194,586	100,260	160	0.2%
5×10^3	5,449,458	166,534	130	0.1%	194,936	100,611	98	0.1%
10^5	5,448,815	165,891	93	0.1%	194,288	99,963	69	0.1%

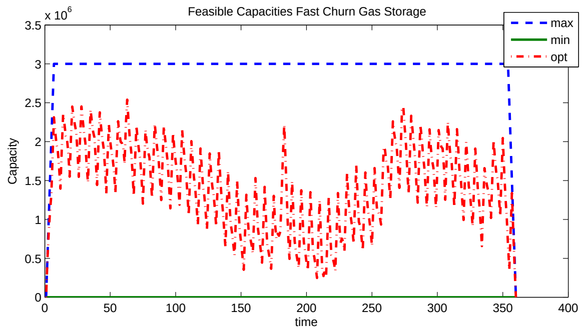
case, even $N_S = 2000$ seems already efficient. However, this conclusion is very superficial as soon as one starts looking at monthly cash-flows into more detail. In fact, at monthly granularity the convergence is very slow and one definitely needs a huge number of simulations to reach a good convergence as shown in Table 8. For simplicity Table 8 reports the values of the aggregation for the first and second month of the dispatch period of the gas storage. The total price is convergent because the variability of the higher granularity values of the fast churn storage somehow cancels out giving the impression to have reached convergence.

As shown in Table 8, at monthly level the variance of the estimator is huge compared to the expected value. The problem of pricing a gas storage generally includes the calculation of the optimal dispatching strategy. If the estimation has not reached the convergence regime and one, for instance, aggregates the cash-flows at monthly level, it may turn out that he is summing cash-flows that belong to incorrect months, and hence seeking for a hedging strategy would lead to wrong results. Running then a high number of simulations is still necessary even when it seems that a low number of scenario is sufficient. The numerical experiments that we present here clearly show the superiority of our bridge construction compared to the standard that would have been computationally unfeasible

Figure 5: Feasible Volumes.



(a) Slow Churn Gas Storage.



(b) Fast Churn Gas Storage.

Table 8: Fast Churn Gas Storage Values of Month M0 and M1.

	M0			M1		
	Price	Err	%err	Price	Err	%err
10^3	8,253	37,000	448%	20,033	36,000	181%
2×10^3	14,640	26,000	180%	9,436	26,000	276%
3×10^3	22,147	21,000	95%	17,894	21,000	116%
5×10^3	13,799	16,000	118%	14,306	16,000	113%
10^4	16,635	11,000	69%	15,765	11,000	73%
2×10^4	16,046	8,100	51%	17,761	8,100	46%
5×10^4	16,015	5,100	32%	16,626	5,100	31%
10^5	16,829	3,600	22%	16,043	3,600	23%
10^6	16,244	1,100	7%	16,138	1,100	7%

t_1	t_2	$\dots$	t_N

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

π_1	π_2	$\dots$	π_N

4.4 LSMC Forward Recursion

Generally, one is also interested to perform the forward recursion that can be used to further check the convergence of the LSMC algorithm as done in Boogert and de Jong [7] or to compute the dispatched volumes. Even for the forward recursion one does not need to save the price trajectories. Indeed, it is possible to control the random simulations in a clever way.

The generation of a random variable can be seen as the extraction of a certain index of a very long array whose values emulate the distribution of an abstract random variable. Knowing the index (the state) of the array, one can repeat exactly the same sequence of random draws.

We propose then to base the construction with diffusion bridges on multiple and independent random streams, one for each point of the grid. In practice, each random factor at time t is always obtained by generating a random variable extracted by the random stream at time t. The iteration and the stream at time t will determine the random extractions at time t only, again one needs to save only one state per time and not at all the random innovations or factors (see Table 9). In the implementation of the forward recursion, the solution is simply to reset all the states of the streams and revert the iteration of the backward construction.

Adopting MATLAB we rely on the properties of the *RandStream* class and use multiple independent random streams. The principle does not change with other programming languages.

The application of our methodology is then very simple and consists in just inverting the backward generation of the BB or OU bridge and one can then regenerate the same exact prices of the backward recursion. Below we show the BB construction only, a similar recursion applies to the OU:

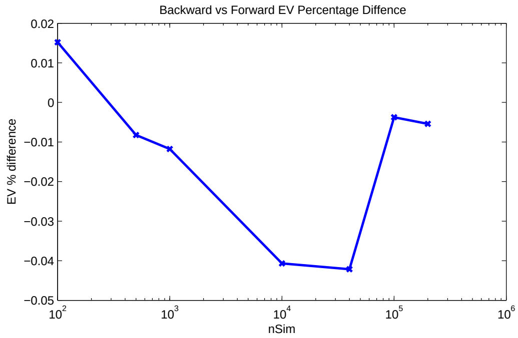
$$W(N-i+1) \stackrel{\omega}{=} \left(W(N-i) - \sqrt{\frac{N-i}{N-i+1}} \epsilon_i \right) \times \frac{N-i+1}{N-i}, \quad i = 1, \dots, N-1 \tag{19}$$

In contrast to the classical Euler scheme, here the random simulation of the BM at time $j+1$ is updated with the random innovation of time j because we have a path-wise equality (ω).

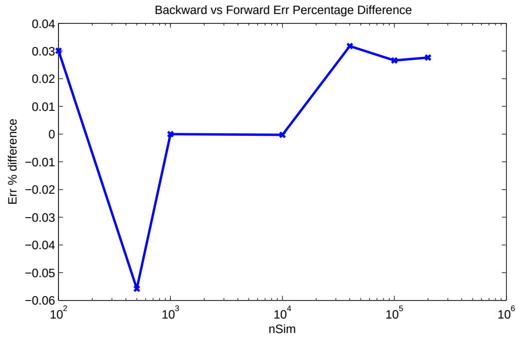
In order to measure the correctness of our implementation strategy for the forward recursion, we computed the difference between the price of the VHP with the backward and forward recursion, respectively. Figure 6 clearly shows that the forward computation is not biased and reliable. For sake of simplicity we do not report the results for the other assets.

Table 9: Random Streams Construction

Figure 6: Backward vs Forward Comparison.



(a) EV comparison.



(b) Err comparison.

5 Concluding Remarks and Future Studies

In this paper we have presented an easy simulation strategy that reduces the computational complexity of the Least Squares Monte Carlo methodology applied in order to price virtual hydro power plants and gas storages as introduced by Boogert and de Jong [7]. The approach we have discussed is based on a backward simulation of the Brownian bridge and the Ornstein-Uhlenbeck bridge because these two types of dynamics are used extensively in modeling the spot prices in energy markets. Briefly, our approach permits to run the backward recursion of the Bellman principle without saving the entire path trajectories except those ones at two consecutive time points. This leads to the possibility to employ a huge number of simulations and reduces the computational complexity. As shown by our numerical experiments a very high number of scenarios are required to reach convergence and have reliable prices. The standard approach presents memory issue already with a modest number of simulations while the bridge construction does not suffer this limitation.

Generally, in pricing energy facilities, one is also interested in performing the forward recursion. Even in this case our strategy avoids to save the price trajectories and optimizes the use of multiple independent random streams. We conclude that the diffusion bridge setting is the native strategy for the implementation of the Least Squares Monte Carlo method in order to get reliable price values.

In our opinion, the use of diffusion bridges deserves further investigations. A straightforward application is the computation of sensitivities and Greeks, as well as the introduction of higher factors in the spot dynamics.

Other than that, our approach can be adapted and be extended to other diffusions and also to Lévy-driven processes as described by Hoyle et al. [14] in the context of Lévy bridges. In case of Gaussian processes, the bridge strategy coincides with time-reversal Itô diffusions strategy, however, as shown in Kutt and Welke [13], the application of Anderson theorem to other processes (e.g. Bessel processes) may lead to complex discretization problems. This particular drawback can be avoided when the pure bridge construction is known (see Makarov and D. Glew [19] for CIR processes).

Future research might also be devoted to the combination of the bridge constructions to American options in Heston stochastic volatility models with Least Squares and Quasi-Monte Carlo methods starting from the works of Baldeaux and Roberts [4], Broadie and Kaya [10] and Cufaro-Petroni and Sabino [12]).

Finally, beyond finance, the application of time reversal Itô-Levy diffusions (see Protter [22]) to the Nelson's stochastic interpretation of quantum mechanics (see Nelson [20] and Cufaro and Pusterla [11]) may worth a closer look.

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