Multi-Factor Models of the Forward Price Curve

This article is the second part of a two part series investigating multi-factor models of the forward price curve in energy markets. The first article (*Commodities Now* June 2002) explored principal component analysis (PCA) as a tool for reducing the dimensionality of the covariance matrix, and proposed the use of 'seasonal' PCA as a method for capturing the complex seasonality inherent in forward curve evolution. In this article **CARLOS BLANCO**, **DAVID SORONOW** & **PAUL STEFISZYN** of Financial Engineering Associates examine the inner workings of the PCA based multi-factor simulation, propose a mean reverting model for the spot price within the context of this multi-factor framework, and look at issues related to hedging and risk management.

IN ENERGY MARKETS, the forward price curve is of paramount importance. It dictates pricing and production decisions, and is a building block of any energy trading and risk management methodology. Risk managers face the challenge of modelling forward curves for earnings, profits, cash flow or value at risk calculations. As any practitioner will tell you, this is no easy task as the forward price curve evolves with extraordinary complexity including multiple layers of seasonality coupled with sophisticated stochastic variation. Add to this the challenge of modelling a remarkably complex spot price process exhibiting such behaviour as mean reversion and price spikes; and it is easy to see that derivatives pricing and risk management is no easy task. That said, a realistic and sound model of the evolution of the forward price curve is necessary, and can be intuitive if viewed from the correct perspective.

Why use a multi-factor model? In our first article, we discussed why a multi-factor model is essential for pricing and risk management of energy derivatives that depend on more than one forward price. These types of derivatives are common in OTC energy markets and include calendar spread options, swap options, swing options, natural gas storage, average price options and average strike options. As an example, consider a natural gas calendar spread option. Exercising a call option results in a long position in the prompt month contract and a short position in a later maturity contract. This instrument derives its value directly from the relative levels of two points along a forward price curve. When pricing this derivative, how do we assess the joint probability of the two prices moving favourably to the option holder?

> Risk managers face the challenge of modelling forward curves for earnings, profits, cash flow or value at risk calculations <

Where Single-Factor Models Fall Short

Single-factor forward curve models leave very little room for complex movements of the forward price curve due to the ever-present assumption of perfect correlation between forward prices. This is inherently unrealistic, and becomes a critical fault when the derivative in question depends on the relative values of points along the forward price curve.

Another problem with the single-factor model of the forward price curve manifests itself in pricing long-term contracts on the spot price. Most practitioners agree that the spot price exhibits 'mean reversion' and this has led to a proliferation of one-factor mean reverting models where the mean reversion 'rate' dictates the speed at which prices revert to mean price levels over time. In the one-factor geometric mean-reverting model, the mean reversion rate has two related effects:

1. It controls the speed at which the spot price reverts to the mean reversion levels over time.
2. It controls the rate at which the instantaneous variance of forward contracts increases as time to maturity decreases (the so-called 'time to maturity' effect).

As many traders and analysts can attest, there can be drastically inconsistent mean reversion rates associated with each of these two effects, especially in natural gas and power markets. The commodity delivery provisions are a contributing factor to this inconsistency. In natural gas markets for example, the futures contract delivery provision calls for a uniform, rate-of-flow delivery over the entire contract delivery month. Contrast this to the spot price, which represents delivery over the next day. Thus, the spot price is for a single delivery day, while the futures price is an 'average' price for several delivery days.

A mean reversion rate calibrated from spot prices can be much higher than a mean reversion rate calibrated from futures price data. In fact, the spot mean reversion rate might be so high as to bring the instantaneous variance of a forward price to virtually zero for the majority of the life of the contract. This is where pricing long-term contracts becomes problematic since we are left with the dubious assumption that the long-term mean price levels remain unchanged for the majority life of the contract.

The PCA Based Simulation

Once we have performed the seasonal principal component analysis and calculated the factor scores and factor loadings, we can use the results to simulate new hypothetical evolutions of the forward curve. In our previous article we presented the PCA in the context of energy forward curve modelling. To review, PCA reduces the covariance matrix into an importance-ordered subset of uncorrelated variables representing the most salient constructs underlying the covariance matrix, namely, the principal components. The analysis explicitly reveals the importance of each principal component (its 'factor score'), which is an expression of the contribution of that source of risk to the volatility of the forward price. Also associated with each principal component is a set of 'factor loadings' which define how the price of each forward contract will change in response to a shock to the component.

Simulating the Forward Price Curve Using Principal Components

$$F_i(t+\Delta t) = F_i(t)\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(c_{ij}\lambda_j\right)^2\Delta t + \sum_{j=1}^{N} c_{ij}\lambda_j\sqrt{\Delta t}\,\varepsilon_j\right]$$

Where:

$F_i(t)$	is thew forward price at time
ε's	are the drawings from a standard normal distribution N(0,1)
λ's	denote the factor scores for each principal component
c's	are the factor-loadings

The PCA based multi-factor simulation itself is relatively straightforward:

1. Determine the number of relevant factors. In most cases the first three components account for well over 95% of the variance.
2. Draw an uncorrelated random variate from a standard normal distribution for each component retained in step 1.
3. Scale each random variate by its associated factor score and by the square root of the time step.
4. 'Load' the scaled values from step 3 onto each forward price using the 'factor loadings'. The return for each forward price is the sum of the product of each scaled value from step 3 multiplied by the 'factor loading'.

Example

Suppose we want to generate a simulated price path for the current prompt month (1st nearby) forward price using our multi-factor PCA model and the principal component analysis reveals that over 90% of the variance is attributable to the first two principal components. As a result, we decide to concern ourselves only with the first two components. The factor scores and factor loadings are listed below:

Factor-Loadings

	1st component	2nd component	3rd component
Factor scores	1.3248	0.3928	0.2278
% of the total variance	88.31%	7.76%	2.61%
1st nearby	0.4739	-0.4394	-0.2613

The current forward price $F(t)$ equals $3.20, and we want to simulate a new forward price for tomorrow (Δt = 1/365).

We draw two random variates from a standard normal distribution:

$$\varepsilon_1 = -0.4160$$
$$\varepsilon_2 = 0.2691$$

$$F(t+\Delta t) = F(t)\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\left(c_j\lambda_j\right)^2\Delta t + \sum_{j=1}^{N} c_j\lambda_j\sqrt{\Delta t}\,\varepsilon_j\right]$$

Plugging the numbers into our propagation equation we obtain:

F($t+\Delta t$) = 3.2*EXP((-0.5*((1.3248*0.4739)^2+(0.3928*-0.4394)^2)*(1/365))+((1.3248*0.4739*-0.416)+(0.3928*-0.4394*0.2691))*(1/365)^0.5)

F($t+\Delta t$) =3.1470

The simulated forward price is $3.15

A Joint Model for the Evolution of Spot & Forward Prices

Until now, we have only addressed issues concerning the modelling of the forward curve dynamically through PCA, but we have not introduced spot prices in this model. Many practitioners will agree with the fact that spot and forward prices follow very distinct patterns, but they are not uncorrelated altogether.

In this section, we present a unified framework to model spot and forward prices.

FEA's Mean Reverting to Prompt Month Spot Price Process

$$d\log\left\{S(t)/F_1\right\} = -a\log\left\{S(t)/F_1\right\}dt + \sigma_s(t)\left\{\rho\,dw_1(t) + \sqrt{1-\rho^2}\,dw_s(t)\right\}$$

Where:

a	the spot mean-reversion rate
ρ	the spot-to-prompt-month correlation
$\sigma_s(t)$	the volatility of the spot price at time "t"
$dw_1(t), dw_s(t)$	the (uncorrelated) random-walk variables for the prompt month forward and spot respectively.
t	the time of observation,
F_1	prompt-month forward price

The multi-factor framework frees us from the limited set of assumptions used in the one-factor model. One approach is to assume that the spot price follows its own stochastic process but with mean reversion towards the prompt-month futures price. In other words, the spot price exhibits stochastic diffusion around a mean level, where the mean level itself evolves according to a diffusive process. If the spot price is much higher than the prompt month price, it will be pulled back towards the prompt month price, at a speed defined by the mean reversion rate. When the initial prompt month contract stops trading, the spot price then chases the new prompt month contract, and the sequence continues.

The joint evolution of spot and prompt month forward prices is driven by two parameters that govern short term and medium term dynamics. On one side, the correlation coefficient can be thought of as the parameter that determines the co-movement of the two price series on the short run. On the other side, the mean reversion parameter acts as an error correction term, whenever spot prices drift away from the prompt month forward price. For those readers

FEA
FPFC

with an Econometrics background, this term is similar to the Error Correction Term used in Cointegrated systems.

In Figure 1. we can see a potential evolution of spot and different forward prices under the model described above. It is interesting to notice that for this particular case we have assumed that spot and the prompt month forward are highly correlated, and as the graph reveals, these prices are indeed tracking each other.

By disaggregating the spot price from the forward curve evolution, we can focus on calibrating a mean reversion rate controlling only the speed at which the spot price reverts to the mean reversion levels over time.

In Figure 2. we can see simulated spot and prompt forward price changes using a correlation coefficient of 90%.

Fig 1. Evolution of Spot & Forward Prices

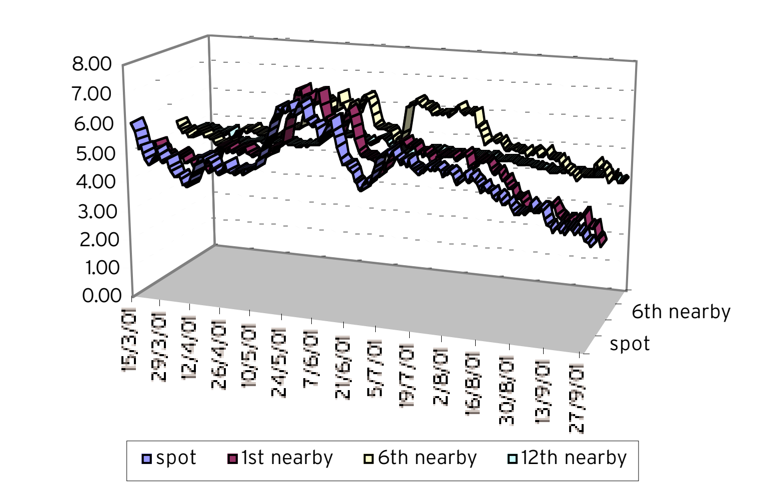
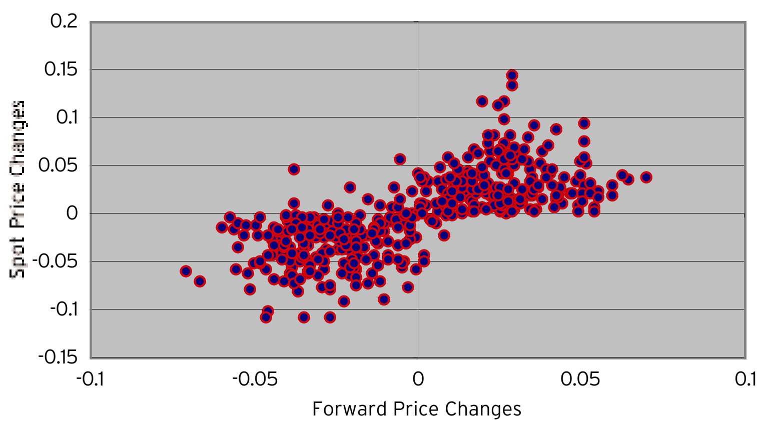


Fig 2. Simulated Spot & Prompt Forward Price Changes



Hedging & Risk Management

Although the argument is often made that one-factor models are 'good enough' for derivative pricing, few would contest that there are serious issues with respect to hedging and risk. Risk managers in markets such as commodity, energy, and interest rates have known the limitations of one-factor models for several years and have responded by developing risk and hedging models that capture forward price curve movements that are deemed highly probable in reality, but not captured explicitly in a one-factor model. This usually means resorting to a multi-factor framework that includes a risk factor for every individual forward price (36 risk factors per commodity in most cases). The forward curve PCA technique, coupled with the mean reverting to prompt month spot price process, allows us to reduce the complexity of this analysis to three or possibly four factors per commodity. Not only is this a huge improvement from a computational perspective, but it also identifies and focuses our attention on the fundamental 'constructs' within the forward curve evolution. These 'constructs' fall naturally out of the principle component analysis itself.

To illustrate the point, let's consider two risk managers facing the same situation but with different risk models:

Using the approach outlined in this and the previous article, a risk manager may discover that the company has an unacceptably large exposure to a widening of spreads between two forward prices (usually the second principal component). The PCA analysis allows the risk manager to quickly identify the source of the exposure and map it directly onto one or more principal components. In this case, the risk manager can quickly ascertain that the optimal hedge is a calendar spread option position.

Contrast this situation of a risk manager using a multi-factor risk model that doesn't make use of PCA. In this case, the risk manager may be aware that there is an unacceptably large exposure to changes in the forward price curve (perhaps due to an inordinately large Value at Risk number), but the risk framework itself lends no insight into the underlying nature of the exposure. Moreover, determining the optimal hedge becomes an illusory task and the risk manager is relegated to trial and error testing with potential hedge instruments.

> The main advantage of using this type of model is that we can link the evolution of spot prices and forward prices ... <

Conclusion

A seasonal PCA based model of the forward price curve coupled with a mean reverting to prompt month model of the spot price provides a unified framework for valuation and risk measurement of many energy contracts. The main advantage of using this type of model is that we can link the evolution of spot prices and forward prices in a realistic fashion accounting for the short term and long-term dynamics of price co-movements ■

CARLOS BLANCO directs the Global Support and Professional Services Group at FEA. **DAVID SORONOW** & **PAUL STEFISZYN** are Derivatives Specialists in the same group. Blanco, Soronow and Stefiszyn thank Cheryl Morgan, Dr. King Wang, and Dr. Angelo Barbieri for advice and assistance.

www.fea.com