AN EFFICIENT APPROACH FOR PRICING SPREAD OPTIONS

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Spread options are options whose payoff is based on the difference in the prices of two underlying assets. The price of a spread option is the (discounted) double integral of the option payoffs over the risk-neutral joint distribution of the terminal prices of the two underlying assets. Analytic expressions for the values of spread puts and calls in a Black-Scholes environment are not known, and various numerical algorithms must be used.

This article presents an accurate and efficient approach for pricing European-style spread options on equities, foreign currencies, and commodities. The key to the approach is to recognize that the joint density of the terminal prices of the underlying assets can be factored into the product of univariate marginal and conditional densities, and that an analytic expression for the integral of the option payoffs over the conditional density is available. The remaining integration amounts to valuing the payoff function given by the results of the first integration. This payoff function is approximated by a portfolio of ordinary puts and calls, and valued accordingly.

The approach is more accurate than existing bivariate binomial schemes, and fast enough for practical applications. It also allows for accurate and efficient computation of the partial derivatives of the option price, i.e., the Greek letter risks.

Spread options are options whose payoff is based on the difference in the prices of two underlying assets. At the expiration date T the payoff of a spread call option is

$$C_T = \max\,[S_{2T} - S_{1T} - K, 0]$$

where S_{2T} and S_{1T} are the spot prices of the two underlying assets, and K is the exercise price, which may be negative. The payoff of a spread put is

$$P_T = \max\,[K - (S_{2T} - S_{1T}), 0]$$

When the exercise price K = 0, the spread options reduce to exchange options, for which pricing formulas are given by Margrabe [1978], Johnson [1981], and Stulz [1982].

Spread options allow investors to take a position on the relative performance of two assets or indexes. Alternatively, spread options allow an investor to hedge the risk that the performance of the two underlying assets or indexes will differ. For example, a portfolio manager who has outperformed her benchmark by taking a position in a particular industry might lock in the outperformance with a spread option on the difference between a marketwide index and an index for that industry. Examples of spread options and some of their uses are discussed in Falloon [1992]. In addition, spread options are sometimes embedded in swaps and debt securities traded in the over-the-counter markets.

The price of a spread option is the (discounted) double integral of the option payoffs over the risk-neutral joint distribution of the terminal prices of the two underlying assets. The standard assumption underlying most option pricing models is that the risk-neutral distribution of the prices of the underlying asset or assets is lognormal. In this setting, analytic expressions for the values of spread puts and calls are not known, and various numerical algorithms must be used.[1]

This article presents an accurate and efficient approach for pricing general European-style spread options on equities, foreign currencies, and commodities. The approach can be adapted to handle basket options on portfolios of two assets, dual-strike options, and bivariate analogues of path-independent digital options.

The key to the approach is that the joint density of the terminal prices of the underlying assets can be factored into the product of univariate marginal and conditional densities, and that an analytic expression for the integral of the option payoffs over the conditional density is available. That is, letting $g(S_{2T}|S_{1T})$ denote the conditional density of the terminal price of the second underlying asset, the integral $\int_0^\infty \max[S_{2T} - S_{1T} - K, 0]\, g(S_{2T}|S_{1T})\, dS_{2T} = F(S_{1T})$ is known. This reduces the problem to the integration $e^{-r(T-t)} \int_0^\infty F(S_{1T})\, f(S_{1T})\, dS_{1T}$, which amounts to the discounted expected value of the payoff function $F(S_{1T})$ given by the results of the first integration. This payoff function is approximated by a portfolio of ordinary puts and calls, and valued accordingly.[2]

This approach turns out to be more accurate than existing bivariate binomial schemes, and fast enough for practical applications. It also allows for accurate and efficient computation of the partial derivatives of the option price, i.e., the Greek letter risks.

An alternative approach taken by Wilcox [1990] is to assume that the spread is a normal random variable. Drawbacks of this approach are that it is inconsistent with the lognormal framework used for pricing ordinary options, and leads to errors in computing option prices and their derivatives. Other approaches to pricing spread options that assume or are consistent with lognormality of the underlying asset prices include numerical quadrature (Heenk, Kemna, and Vorst [1990]) and bivariate binomial and trinomial approximations (Boyle [1988], Boyle, Evnine, and Gibbs [1989], Madan, Milne, and Shefrin [1989], He [1990], Rubinstein [1991a, 1991b]). It is also possible to price European spread options using simulation. Drawbacks of these approaches are their somewhat limited accuracy and the computational effort involved.

I. CHARACTERIZING THE OPTION PRICE

The prices of the underlying assets are assumed to follow the process

$$\begin{bmatrix} dS_{1t} \\ dS_{2t} \end{bmatrix} = \begin{bmatrix} (\mu_1 - \delta_1) S_{1t} \\ (\mu_2 - \delta_2) S_{2t} \end{bmatrix} dt + \begin{bmatrix} \sigma_1 S_{1t} & 0 \\ \rho\sigma_2 S_{2t} & \sqrt{1-\rho^2}\,\sigma_2 S_{2t} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}$$

where W_{1t} and W_{2t} are standard Brownian motions. The parameters μ_1 and μ_2 are the instantaneous expected rates of return on the two assets; δ_1 and δ_2 are the instantaneous dividend yields; σ_1 and σ_2 are the instantaneous standard deviations of the rates of return, and ρ is the correlation coefficient. With this process, if μ_1 and μ_2 are constant, the joint distribution of the prices of the two underlying assets is lognormal. There is also available for trading a riskless bond with a constant rate of return r. There are no taxes, transaction costs, or other market frictions.

In this setup, a standard result is that if there are no arbitrage opportunities then there exists a martingale or risk-neutral probability measure $\mathcal{Q}$ that has the properties that, under $\mathcal{Q}$: 1) the instantaneous expected rate of return on each asset is equal to the riskless rate of interest r; and 2) the price of an option or other contingent claim is equal to the discounted expected value of its payoffs, where the discounting is done using the riskless rate of interest r.[3]

Using this result, the value of the spread call at the current date t is given by

$$C = e^{-r(T-t)} E^{\mathcal{Q}} \max[S_{2T} - S_{1T} - K, 0]$$

$$= e^{-r(T-t)} \int_0^\infty \int_0^\infty \max[S_{2T} - S_{1T} - K, 0] \times g(S_{2T}|S_{1T})\, f(S_{1T})\, dS_{2T} dS_{1T}$$

$$= e^{-r(T-t)} \int_0^{\infty} \{\int_{\max(S_{1T}+K,0)}^{\infty} \times$$

$$[S_{2T} - (S_{1T} + K)]\, g\,(S_{2T} \mid S_{1T})\, dS_{2T}\} \times$$

$$f\,(S_{1T})\, dS_{1T} \tag{1}$$

where $g\,(S_{2T} \mid S_{1T})$ is the conditional density of S_{2T} given S_{1T}, $f\,(S_{1T})$ is the marginal density of S_{1T}, and the product $g\,(S_{2T} \mid S_{1T})\, f\,(S_{1T})$ is the joint density. Under $\mathfrak{Q}$, the distribution of S_{1T} and S_{2T} is bivariate lognormal, and the conditional distribution of S_{2T} given S_{1T} is lognormal. These density functions are shown in the appendix.

Factoring the density into the form $g\,(S_{2T} \mid S_{1T}) \times f\,(S_{1T})$ and rearranging to obtain the last expression in Equation (1) is one of the key steps, for the integral inside the braces is straightforward to evaluate. Computations carried out in the appendix establish that the integral is given by

$$F\,(S_{1T}) = e^{A} S_{2t} \left(\frac{S_{1T}}{S_{1t}}\right)^{\frac{\rho\sigma_2}{\sigma_1}} N\,(x_1) -$$

$$(S_{1T} + K)\, N\,(x_2)$$

where

$$A = [r\left(1 - \frac{\rho\sigma_2}{\sigma_1}\right) - \left(\delta_2 - \delta_1 \frac{\rho\sigma_2}{\sigma_1}\right) +$$

$$\rho\sigma_2\,(\sigma_1 - \rho\sigma_2)/2](T - t)$$

$$x_1 = \frac{M_2 + \sigma^2 - \ln \max\,(S_{1T} + K,\ 0)}{\sigma}$$

$$x_2 = \frac{M_2 - \ln \max\,(S_{1T} + K,\ 0)}{\sigma}$$

$$M_2 = m_2 + \frac{\rho\sigma_2}{\sigma_1}(\ln S_{1T} - m_1)$$

m_1 and m_2 are the means of $\ln S_{1T}$ and $\ln S_{2T}$ under the risk-neutral distribution, as defined in the appendix, and $N\,(\cdot)$ is the cumulative standard normal distribution function. Substituting this into (1), the problem now is to evaluate

$$C = e^{-r(T-t)} \int_0^{\infty} F\,(S_{1T})\, f\,(S_{1T})\, dS_{1T} \tag{2}$$

In going from (1) to (2), the dimension of the integration is reduced from two to one, simplifying the problem and reducing the computational burden of computing the option price. Moreover, the right-hand side of (2) can be interpreted as the value of a security with payoffs given by $F\,(S_{1T})$. This observation is the basis of the approximation below.

II. APPROXIMATING THE OPTION VALUE AND ITS PARTIAL DERIVATIVES

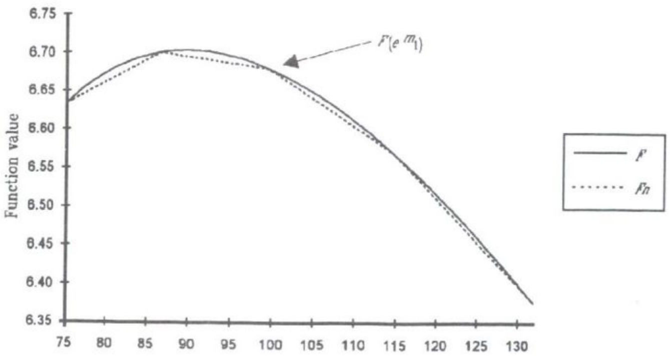
The trick to evaluating the right-hand side of Equation (2) is to recognize three things. First, recalling that $f\,(S_{1T})$ is the risk-neutral density, the right-hand side of (2) is the value of a security with payoffs given by $F\,(S_{1T})$.

Second, if the function F is piecewise linear in S_{1T}, then the problem would be solved, for piecewise linear payoffs can be replicated by portfolios of bonds and ordinary put and call options with payoffs contingent on S_{1T}, and the value of a security with piecewise linear payoffs is simply the value of the replicating portfolio of bonds, puts, and calls. In the setting of this article, the values of these ordinary put and call options are given by the variant of the Black-Scholes formula allowing for a continuous dividend yield equal to δ_1.

Third, while the function F is not piecewise linear in S_{1T}, it can be approximated arbitrarily closely by piecewise linear functions, and the approximating function can be interpreted as the payoff of a portfolio of bonds, puts, and calls.

The main idea is illustrated in Exhibit 1, which shows the function F and piecewise linear approximation F_n in a small interval. Letting $m_1 \equiv E^{\mathfrak{Q}}\,(\ln S_{1T}) = \ln S_{1t} + (r - \delta_1 - \sigma_1^2/2)\,(T - t)$, the approximation is "centered" on $e^{m_1} = S_{1t}e^{(r-\delta_1-\sigma_1^2/2)(T-t)}$, and covers the interval $[e^{m_1-2h},\ e^{m_1+2h}]$. The first compo-

EXHIBIT 1
EXAMPLE OF FUNCTION F AND PIECEWISE LINEAR APPROXIMATION F_n



The figure shows the function F and the piecewise linear approximation F_n in the interval $[e^{m_1-2h}, e^{m_1+2h}]$. In this interval, the approximation is given by

$$F_n(S_{1T}) = F(e^{m_1}) + c_0\Delta_0 + c_1(\Delta_1 - \Delta_0) - p_0\Delta_{-1} - p_{-1}(\Delta_{-2} - \Delta_{-1})$$

For example, if $e^{m_1+h} \le S_{1T} < e^{m_1+2h}$, then $p_{-1} = 0$, $p_0 = 0$, $c_0 = S_{1T} - e^{m_1}$, $c_1 = S_{1T} - e^{m_1+h}$, and

$$\begin{aligned} F_n(S_{1T}) &= F(e^{m_1}) + (S_{1T} - e^{m_1})\Delta_0 + \\ &\quad (S_{1T} - e^{m_1+h})(\Delta_1 - \Delta_0) \\ &= F(e^{m_1+h}) + (S_{1T} - e^{m_1+h})\Delta_1 \end{aligned}$$

Here $m_1 = 4.60$, and the parameters k and n are chosen so that $h = 0.141$, $e^{m_1-2h} = 74.99$, $e^{m_1-h} = 86.4$, $e^{m_1} = 99.5$, $e^{m_1+h} = 114.6$, and $e^{m_1+2h} = 132.0$.

nent of the approximating portfolio is a zero-coupon bond with a face value of $F(e^{m_1})$. This matches the value of the approximation F_n to the value of the function F in the middle of the interval at $S_{1T} = e^{m_1}$, i.e., it assures $F_n(e^{m_1}) = F(e^{m_1})$.

In addition to agreeing at $S_{1T} = e^{m_1}$, F_n and F also agree at the points e^{m_1-2h}, e^{m_1-h}, e^{m_1+h}, and e^{m_1+2h} where h is a small increment. F_n and F are made to agree at e^{m_1+h}, the first point greater than e^{m_1}, by holding Δ_0 call options with an exercise price of e^{m_1}, where Δ_0 is the slope of the chord connecting e^{m_1} and e^{m_1+h}. F_n and F are made to agree at e^{m_1+2h} by in addition holding $\Delta_1 - \Delta_0$ call options with an exercise price of e^{m_1+h}, where Δ_1 is the slope of the chord connecting e^{m_1+h} and e^{m_1+2h}. The choice of $\Delta_1 - \Delta_0$ call options with an exercise price of e^{m_1+h} makes the slope of the approximation equal Δ_1 between e^{m_1+h} and e^{m_1+2h}, because the payoff of a portfolio of Δ_0 call options with an exercise price of e^{m_1} and $\Delta_1 - \Delta_0$ call options with an exercise price of e^{m_1+h} has a slope of $\Delta_0 + (\Delta_1 - \Delta_0) = \Delta_1$ between e^{m_1+h} and e^{m_1+2h}.

Finally, F_n and F are made to agree at points less than e^{m_1-h} and e^{m_1-2h} by holding appropriate numbers of put options with exercise prices of e^{m_1} and e^{m_1-h}. Letting Δ_{-1} denote the slope of the chord connecting e^{m_1-h} and e^{m_1} and Δ_{-2} denote the slope of the chord connecting e^{m_1-2h} and e^{m_1-h}, this involves holding $-\Delta_{-1}$ put options with an exercise price of e^{m_1} and $-(\Delta_{-2} - \Delta_{-1})$ put options with an exercise price of e^{m_1-h}.

Putting this together, the function F_n is given by

$$F_n(S_{1T}) = F(e^{m_1}) + c_0\Delta_0 + c_1(\Delta_1 - \Delta_0) - p_0\Delta_{-1} - p_1(\Delta_{-2} - \Delta_{-1})$$

where c_0 denotes the payoff of a call option with exercise price e^{m_1}, c_1 denotes the payoff of a call option with exercise price e^{m_1+h}, p_0 denotes the payoff of a put option with exercise price e^{m_1}, and p_1 denotes the payoff of a put option with exercise price e^{m_1-h}.

To describe the approximation completely, introduce a positive constant k, define

$$h \equiv \frac{k\sigma_1\sqrt{T-t}}{\sqrt{n}}$$

and consider 2n intervals $[e^{m_1-nh}, e^{m_1-(n-1)h}]$, $[e^{m_1-(n-1)h}, e^{m_1-(n-2)h}]$, ..., $[e^{m_1-h}, e^{m_1}]$, $[e^{m_1}, e^{m_1+h}]$,

..., $[e^{m_1+(n-1)h}, e^{m_1+nh}]$. This set of intervals covers the range from $e^{m_1-\sqrt{n}k\sigma_1\sqrt{T-t}}$ to $e^{m_1+\sqrt{n}k\sigma_1\sqrt{T-t}}$. Also, use c_j and p_j to denote the payoffs of ordinary European call and put options on S_{1T} with exercise prices of e^{m_1+jh}:

$$c_j = \begin{cases} 0 & \text{if } S_{1T} < e^{m_1+jh} \\ S_{1T} - e^{m_1+jh} & \text{if } S_{1T} \geq e^{m_1+jh} \end{cases}$$

$$p_j = \begin{cases} e^{m_1+jh} - S_{1T} & \text{if } S_{1T} \leq e^{m_1+jh} \\ 0 & \text{if } S_{1T} > e^{m_1+jh} \end{cases}$$

Finally, let Δ_j denote the slope of the chord connecting $F(e^{m_1+jh})$ and $F(e^{m_1+(j+1)h})$. This notation is used to describe a sequence of functions $\{F_n\}$, where each function $F_n: \Re^+ \to \Re^+$ is given by

$$\begin{aligned} F_n(S_{1T}) = {} & F(e^{m_1}) + c_0\Delta_0 + \\ & \sum_{j=1}^{n-1} c_j(\Delta_j - \Delta_{j-1}) - \\ & c_n\Delta_{n-1} - p_0\Delta_{-1} - \\ & \sum_{j=1}^{n-1} p_{-j}(\Delta_{-(j+1)} - \Delta_{-j}) + p_{-n}\Delta_{-n} \end{aligned}$$

The function F_n is a piecewise linear approximation of F that agrees with F at the points e^{m_1-nh}, $e^{m_1-(n-1)h}$, ..., e^{m_1-h}, e^{m_1}, e^{m_1+h}, ..., $e^{m_1+(n-1)h}$, e^{m_1+nh}. It is obvious that the sequence $\{F_n\}$ converges pointwise on $\Re^+$ to F.

The function F_n gives the payoffs of a portfolio of: 1) a riskless bond paying $F(e^{m_1})$; 2) Δ_0 call options, each with an exercise price of e^{m_1}; 3) for j = 1, ..., n − 1, $\Delta_j - \Delta_{j-1}$ call options with exercise prices of e^{m_1+jh}; 4) $-\Delta_{n-1}$ call options with an exercise price of e^{m_1+nh}; 5) $-\Delta_{-1}$ put options with an exercise price of e^{m_1}; 6) for j = 1, ..., n − 1, $-(\Delta_{-(j+1)} - \Delta_{-j})$ put options with exercise prices of e^{m_1-jh}; and 7) Δ_{-n} put options with an exercise price of e^{m_1-nh}.

The value of such a portfolio is

$$\begin{aligned} C_n = {} & e^{-r(T-t)} F(e^{m_1}) + c(e^{m_1})\Delta_0 + \\ & \sum_{j=1}^{n-1} c(e^{m_1+jh})(\Delta_j - \Delta_{j-1}) - c(e^{m_1+nh}) \times \\ & \Delta_{n-1} - p(e^{m_1})\Delta_{-1} - \sum_{j=1}^{n-1} p(e^{m_1-jh}) \times \\ & (\Delta_{-(j+1)} - \Delta_{-j}) + p(e^{m_1-nh})\Delta_{-n} \end{aligned}$$

where

$$\begin{aligned} c(x) = {} & e^{-\delta_1(T-t)} S_{1t}N(d_1(x)) - \\ & xe^{-r(T-t)}N(d_2(x)) \end{aligned}$$

$$\begin{aligned} p(x) = {} & xe^{-r(T-t)}N(-d_2(x)) - \\ & e^{-\delta_1(T-t)} S_{1t}N(-d_1(x)) \end{aligned}$$

$$d_1(x) = \frac{\ln(S_{1t}/x) + (r - \delta_1 + \sigma_1^2/2)(T - t)}{\sigma_1\sqrt{T - t}}$$

$$d_2(x) = \frac{\ln(S_{1t}/x) + (r - \delta_1 - \sigma_1^2/2)(T - t)}{\sigma_1\sqrt{T - t}}$$

The functions c and p give the values of ordinary European call and put options as functions of the exercise price. It is straightforward to show that C_n approximates the value of the spread call in the sense that $\lim_{n\to\infty} C_n = C$.

Given the value of a spread call, the value of a spread put can be obtained using put-call parity. The payoffs of a spread put are identical to those of a portfolio of a spread call with the same exercise price and expiration date, short one unit of the second underlying asset, long one unit of the first underlying asset, and long a riskless bond that promises to pay the exercise price on the expiration date. The value at time t of a claim to S_{1T} is $e^{-\delta_1(T-t)} S_{1t}$, and the value of a claim to S_{2T} is $e^{-\delta_2(T-t)} S_{2t}$, so the put-call parity relation for spread options is

$$P = C - e^{-\delta_2(T-t)} S_{2t} + e^{-\delta_1(T-t)} S_{1t} + e^{-r(T-t)} K$$

The Greek Letter Risks

The analysis preceding is based on Equation (2), which characterizes the option price as the value of a security with payoffs given by $F(S_{1T})$. In going from (1) to (2), the dimension of the integration is reduced from two to one, simplifying the problem and reducing the computational burden of calculating the option price. The reduction of effort in computing the partial derivatives of the option price, i.e., the Greek letter risks, is even greater. Arbitrarily picking $\partial C/\partial \sigma_1$ to illustrate this, one can pass differentiation through the integral in (2) and write

$$\frac{\partial C}{\partial \sigma_1} = e^{-r(T-t)} \int_0^\infty \frac{\partial (F(S_{1T}) f(S_{1T}))}{\partial \sigma_1} dS_{1T} \quad (3)$$

Taking the derivative $\partial (F(S_{1T}) f(S_{1T}))/\partial \sigma_1$ is straightforward (although a bit tedious), as is differentiating with respect to the other parameters. A minor rearrangement of (3) gives

$$\frac{\partial C}{\partial \sigma_1} = e^{-r(T-t)} \int_0^\infty \left[\frac{\partial (F(S_{1T}) f(S_{1T}))}{\partial \sigma_1} \frac{1}{f(S_{1T})} \right] \times f(S_{1T}) dS_{1T}$$

$$= e^{-r(T-t)} \int_0^\infty F_{\sigma_1}(S_{1T}) f(S_{1T}) dS_{1T} \quad (4)$$

where

$$F_{\sigma_1}(S_{1T}) \equiv \frac{\partial (F(S_{1T}) f(S_{1T}))}{\partial \sigma_1} \frac{1}{f(S_{1T})}$$

This has the same form as (2), with $F_{\sigma_1}(S_{1T})$ replacing $F(S_{1T})$. This permits the option vega $\partial C/\partial \sigma_1$ to be interpreted as the value of a security with payoffs given by $F_{\sigma_1}(S_{1T})$. This vega can be computed using the procedure above simply by replacing $F(S_{1T})$ with $F_{\sigma_1}(S_{1T})$.

Expressions analogous to (4) can be obtained for the other derivatives. These let us use the approach above to evaluate all the partial derivatives or "Greeks." The advantage is that each partial derivative (including each second derivative) can be computed by a single one-dimensional integration. In contrast, computing the option price using either two-dimensional numerical quadrature or a standard bivariate binomial approximation forces us to approximate the derivative $\partial C/\partial \sigma_1$ using a finite difference, e.g.,

$$\frac{\partial C}{\partial \sigma_1} \approx \frac{C(\sigma_1 + \varepsilon) - C(\sigma_1 - \varepsilon)}{2\varepsilon}$$

As a result, computing this one derivative requires two evaluations of the option price, each involving a two-dimensional numerical integration. Avoiding the approximation of derivatives by finite differences also avoids the difficulties in obtaining accurate estimates of derivatives sometimes encountered in doing this (see, e.g., Ralston and Rabinowitz [1978, p. 93]).

The Replicating Portfolio

Using Itô's lemma to compute the (risk-neutral) dynamics of the call price, one obtains

$$dC_t = \frac{\partial C}{\partial S_{1t}} dS_{1t} + \frac{\partial C}{\partial S_{2t}} dS_{2t} + \left[\frac{\partial C}{\partial t} + \frac{\sigma_1^2 S_{1t}^2}{2} \frac{\partial^2 C}{\partial S_{1t}^2} + \frac{\sigma_2^2 S_{2t}^2}{2} \frac{\partial^2 C}{\partial S_{2t}^2} + \rho \sigma_1 \sigma_2 S_{1t} S_{2t} \frac{\partial^2 C}{\partial S_{1t} \partial S_{2t}} \right] dt$$

From this it is clear that, analogous to the case of an option on a single underlying asset, the hedge or replicating portfolio contains $\partial C/\partial S_{1t}$ units of the

EXHIBIT 2

SPREAD CALL OPTION PRICES FOR DIFFERENT VOLATILITIES, ASSET 1 PRICES, CORRELATION COEFFICIENTS, AND TIMES TO EXPIRATION

Volatility of First Underlying	Price of First Underlying	Correlation −0.5, T − t: 0.02	0.083	1.00	5.00	Correlation 0.0, T − t: 0.02	0.083	1.00	5.00	Correlation 0.5, T − t: 0.02	0.083	1.00	5.00
0.1	92	4.25	5.36	11.82	19.48	4.14	4.98	10.37	16.88	4.05	4.56	8.62	13.67
	96	1.48	3.00	9.99	18.23	1.25	2.55	8.49	15.56	0.98	1.99	6.65	12.26
	100	0.28	1.47	8.39	17.06	0.16	1.08	6.88	14.34	0.06	0.63	5.04	10.97
	104	0.03	0.63	7.00	15.96	0.01	0.37	5.52	13.21	0.00	0.14	3.74	9.81
0.2	92	4.52	6.13	14.65	24.49	4.29	5.51	12.39	20.52	4.08	4.75	9.44	15.22
	96	1.92	3.89	12.91	23.36	1.56	3.18	10.58	19.29	1.11	2.25	7.52	13.86
	100	0.57	2.29	11.35	22.29	0.33	1.63	8.99	18.14	0.10	0.83	5.91	12.61
	104	0.12	1.25	9.96	21.28	0.04	0.74	7.60	17.07	0.00	0.24	4.58	11.46
0.3	92	4.87	7.02	17.75	29.78	4.56	6.25	15.06	25.21	4.23	5.30	11.61	19.61
	96	2.40	4.87	16.10	28.77	1.98	4.02	13.33	24.10	1.44	2.93	9.77	17.90
	100	0.95	3.23	14.59	27.81	0.62	2.41	11.78	23.05	0.26	1.41	8.18	16.73
	104	0.30	2.05	13.21	26.90	0.14	1.35	10.39	22.06	0.02	0.59	6.80	15.64

The payoff of the spread call option is max $[S_{2T} - S_{1T} - K, 0]$. The price of the second underlying asset is $S_{2t} = 100$, the exercise price is $K = 4$, the continuously compounded interest rate is $r = 0.1$, the dividend yields are $\delta_1 = \delta_2 = 0.05$, the volatility of standard deviation of the rate of return of the second underlying asset is $\sigma_2 = 0.2$, and $n = 100$.

first underlying asset and $\partial C/\partial S_{2t}$ units of the second, and these can be labeled the deltas with respect to the first and second underlying assets. The replicating portfolio also contains a dollar investment in riskless bonds B_t given by

$$B_t = C_t - \frac{\partial C}{\partial S_{1t}} S_{1t} - \frac{\partial C}{\partial S_{2t}} S_{2t}$$

For a spread call with payoff max $[S_{2T} - S_{1T} - K, 0]$, the delta or partial derivative with respect to the first underlying asset is negative, and that with respect to the second underlying asset is positive. As the option becomes increasingly in the money, the first delta decreases, eventually approaching −1 if the first underlying asset is a stock that pays no dividends. The second delta increases, eventually approaching 1 if the second underlying asset is a stock that pays no dividends, while the dollar investment in bonds approaches $-e^{-r(T-t)}K$ (that is, borrowing of $e^{-r(T-t)}K$).

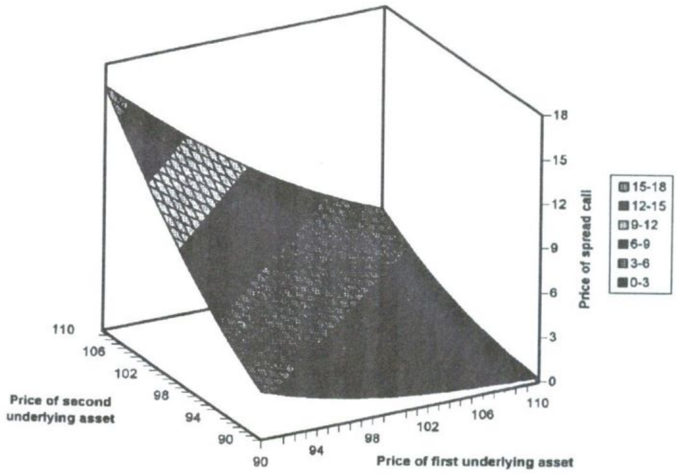
To understand this behavior of the deltas, note that as the option becomes in the money, the option holder is increasingly likely to receive S_{2T} in exchange for $S_{1T} + K$. When $S_{2T} - S_{1T} - K$ is large, the spread option is like a long position in the second underlying asset combined with a short position in the first and borrowing. When the option is well out of the money, both the stock and bond positions in the replicating portfolio approach zero.

III. EXAMPLE AND COMPARISON TO THE APPROACH OF WILCOX

Exhibit 2 shows some spread call option prices for a spread call with payoff max $[S_{2T} - S_{1T} - K, 0]$ computed using this approach with $n = 100$ and h chosen so that F is integrated over a range extending from the mean of ln (S_{1T}) less five standard deviations to the mean plus five standard deviations. The table shows the prices of spread call options for three different values of the volatility or standard deviation of the instantaneous rate of return of the first underlying asset $\sigma_1 = 0.1, 0.2, 0.3$; four different prices of the

EXHIBIT 3

VALUE FUNCTION OF SPREAD CALL OPTION AS A FUNCTION OF THE PRICES OF THE TWO UNDERLYING ASSETS



The payoff of the spread call option is max $[S_{2T} - S_{1T} - K, 0]$. The continuously compounded interest rate is $r = 0.1$, the dividend yields are $\delta_1 = \delta_2 = 0.05$, the volatilities or standard deviations of the rates of return of the two underlying assets are $\sigma_1 = \sigma_2 = 0.2$, the correlation coefficient is $\rho = 0.5$, the time to expiration is $T - t = 0.5$, and the exercise price is $K = 4$.

underlying asset $S_{1t} = 92, 96, 100, 104$; three different correlation coefficients $\rho = -0.5, 0, 0.5$; and four different times to expiration $T - t = 0.02$ (one week), 0.083 (one month), 1, and 5.

The initial price of the second underlying asset is $S_{2t} = 100$, the exercise price is $K = 4$, the continuously compounded interest rate is $r = 0.1$, the instantaneous dividend yields are $\delta_1 = \delta_2 = 0.05$, and the volatility or standard deviation of the instantaneous rate of return of the second underlying asset is $\sigma_2 = 0.2$. As one would anticipate, the prices are increasing in the volatility of the first underlying asset and the remaining time to expiration, and decreasing in the correlation coefficient and the price of the first underlying asset.

Exhibit 3 shows the value of one particular spread call option as a function of the prices of the two underlying assets. The parameters used in the figure are $r = 0.1$, $\delta_1 = \delta_2 = 0.05$, $\sigma_1 = \sigma_2 = 0.2$, $\rho = 0.5$, $T - t = 0.5$, and $K = 4$. As one would anticipate, the option price is highest when the price of the second underlying asset is high and the price of the first underlying asset is low.

Exhibit 4 shows the value of a portfolio of the same spread call option and one unit of the first underlying asset as a function of the prices of the two underlying assets. This is a portfolio that will do well if either of the underlying assets performs well.

An alternative approach to pricing spread options comes from Wilcox [1990], who assumes that under the risk-neutral probability the spread $S_2 - S_1$ follows an arithmetic Brownian motion

$$d\,(S_{2t} - S_{1t}) = \alpha dt + \nu dW_t$$

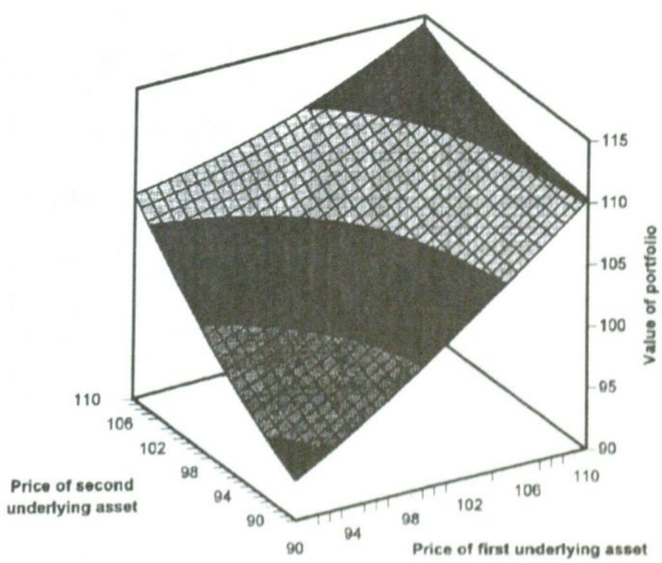
where α and ν are constant. While this assumption about the spread is inconsistent with the standard assumption that S_1 and S_2 follow geometric Brownian motion processes under the risk-neutral probability, it has the advantage of leading to a simple formula:

$$C = \nu\sqrt{T - t}\,\exp[-r(T - t)] \times$$

$$\left(dN(d) + \frac{1}{\sqrt{2\pi}}\exp[-d^2/2]\right)$$

EXHIBIT 4

VALUE FUNCTION OF PORTFOLIO OF SPREAD CALL OPTION AND ONE UNIT OF THE FIRST UNDERLYING ASSET AS A FUNCTION OF THE PRICES OF THE TWO UNDERLYING ASSETS



Parameters are identical to those used in Exhibit 3.

EXHIBIT 5
SPREAD CALL OPTION PRICES COMPUTED USING THE WILCOX FORMULA

Volatility of First Underlying	Price of First Underlying	Correlation −0.5 T − t 0.02	0.083	1.00	5.00	Correlation 0.0 T − t 0.02	0.083	1.00	5.00	Correlation 0.5 T − t 0.02	0.083	1.00	5.00
0.1	92	4.25	5.37	11.46	15.89	4.15	5.00	10.10	13.82	4.05	4.59	8.45	11.31
	96	1.47	2.99	9.53	14.46	1.25	2.54	8.10	12.31	0.98	1.98	6.34	9.68
	100	0.27	1.44	7.85	13.13	0.15	1.04	6.39	10.92	0.05	0.59	4.61	8.21
	104	0.02	0.59	6.40	11.91	0.01	0.34	4.96	9.66	0.00	0.11	3.23	6.90
0.2	92	4.51	6.12	14.10	19.88	4.29	5.50	11.93	16.60	4.08	4.75	9.12	12.34
	96	1.91	3.88	12.35	18.67	1.56	3.17	10.10	15.31	1.11	2.24	7.17	10.92
	100	0.57	2.28	10.78	17.55	0.33	1.62	8.50	14.12	0.10	0.82	5.55	9.65
	104	0.12	1.24	9.39	16.51	0.04	0.73	7.12	13.03	0.00	0.23	4.25	8.53
0.3	92	4.86	6.99	17.02	24.27	4.55	6.21	14.40	20.32	4.22	5.26	11.05	15.26
	96	2.39	4.85	15.43	23.29	1.98	4.00	12.75	19.28	1.44	2.92	9.32	14.14
	100	0.96	3.23	13.99	22.39	0.63	2.43	11.29	18.32	0.27	1.44	7.85	13.13
	104	0.31	2.08	12.69	21.55	0.14	1.39	9.99	17.44	0.03	0.63	6.61	12.24

Parameters are identical to those used in Exhibit 2.

where $d \equiv (S_{2t} - S_{1t} + \alpha (T - t) - K)/(\nu\sqrt{T - t})$. A reasonable approach in implementing the Wilcox formula is to set α and ν equal to the (local) drift and standard deviation of the actual spread process

$$d(S_{2t} - S_{1t}) = ((r - \delta_2) S_{2t} - (r - \delta_1) S_{1t})\,dt + (\rho\sigma_2 S_{2t} - \sigma_1 S_{1t})\,dW_{1t} + \sqrt{1 - \rho^2}\,\sigma_2 S_{2t}\,dW_{2t}$$

That is, set $\alpha = (r - \delta_2) S_{2t} - (r - \delta_1) S_{1t}$ and $\nu = \sqrt{(\rho\sigma_2 S_{2t} - \sigma_1 S_{1t})^2 + (1 - \rho^2)\sigma_2^2 S_{2t}^2}$. Exhibit 5 shows the prices of spread call options computed using the Wilcox formula with these choices of α and ν for the parameters used in Exhibit 2. The Wilcox prices are quite close to those in Exhibit 2 for the options with 0.02 and 0.083 years to expiration, but below those in Exhibit 2 for the longer-term options with one and five years to expiration. For the options with one year to expiration, most of the Wilcox prices differ from those in Exhibit 2 by more than 5%, with one price (that for which $\sigma_1 = 0.1$, $S_{1t} = 104$, $\rho = 0.5$, and $T - t = 1.00$) differing by more than 10%. For the options with five years to expiration, the Wilcox prices differ from those in Exhibit 2 by up to 30%, with most differing by about 20%.

In some situations, these errors may not be important because option pricing formulas are typically used with implied volatilities chosen so that model prices approximate actual prices. Spread options are not actively traded, however, making it difficult to infer the volatility ν used in the Wilcox formula directly from the prices of spread options. More important, the Wilcox formula leads to systematic errors in computing the partial derivatives of option prices.

For example, Exhibit 6 shows the percentage differences in the partial derivative $\partial C/\partial S_{2t}$ computed from the Wilcox formula and using the approach in Section II for the parameters used in Exhibits 2 and 5. The percentage differences are the differences between the derivatives of the Wilcox formula and the "true" derivatives, divided by the "true" derivatives. Such percentage differences are a reasonable metric for evaluating an option pricing model, for they measure the extent to which hedge ratios will be miscalculated.

While the Wilcox model performs relatively well for the options with 0.02 and 0.083 years to expiration, it performs poorly for those with one year to expiration and volatilities of 0.1 and 0.3, and also for those with five years to expiration for all three volatilities. The Wilcox model seems to perform best when the volatility of the first underlying asset is 0.2, equal to the volatility of the second underlying asset.

This appears to be because the arithmetic Brownian motion process underlying the Wilcox model implies that the distribution of the spread is symmetric. The distribution of the spread between two lognormal prices is symmetric if the volatilities are identical, but not when the volatilities differ.

These differences between the Wilcox and true derivatives are not eliminated by the use of implied volatilities, with some errors increasing (in absolute value) and others decreasing. I illustrate this using the options with $\sigma_1 = 0.1$ and $\rho = 0.5$ (the upper right-hand corner of the tables), which are reasonable values for some pairs of exchange rates. If the volatility is adjusted so that the Wilcox formula correctly prices the at-the-money option ($S_{1t} = 96$) with one year to expiration, the percentage error in the partial derivative for this option goes from −5.9 to −6.3. The percentage errors for the other options with one year to expiration and S_{1t} equal to 92, 100, and 104 go to −5.7, −5.6, and −3.8, respectively, with all the changes representing increases in the absolute value of the error.

IV. ACCURACY OF THE APPROXIMATION

The approach described in Section II yields greater accuracy than bivariate binomial schemes, the most efficient of which is described in Rubinstein [1991a, 1991b]. To show this, I construct a bound on the absolute value of the approximation error $|C_n - C|$. The bound is based on the second derivative of the function F. Recalling that C and C_n are integrals of F and the piecewise linear approximation F_n, respectively, a bound based on the second derivative of F makes intuitive sense because the maximum difference between F and the piecewise linear approximation F_n depends on the curvature of F and the lengths of the intervals used in constructing F_n. (For example, were F linear, F_n would agree with F.)

Specifically, for each of the 2n intervals

EXHIBIT 6

PERCENTAGE DIFFERENCES IN $\frac{\partial C}{\partial S_{2t}}$ COMPUTED USING THE WILCOX FORMULA AND THE APPROACH IN SECTION II

Volatility of First Underlying	Price of First Underlying	Correlation −0.5 T − t 0.02	0.083	1.00	5.00	Correlation 0.0 T − t 0.02	0.083	1.00	5.00	Correlation 0.5 T − t 0.02	0.083	1.00	5.00
0.1	92	0.0	−0.7	−3.6	−1.6	0.1	−0.6	−4.3	−3.6	0.2	−0.5	−5.4	−6.7
	96	−0.6	−1.3	−3.5	−0.6	−0.8	−1.5	−4.3	−2.4	−1.0	−2.0	−5.9	−5.6
	100	0.4	−1.0	−3.0	0.7	1.2	−1.0	−3.8	−0.9	4.4	−0.6	−5.2	−3.8
	104	7.0	1.0	−2.1	2.2	13.4	2.6	−2.6	0.9	33.0	8.5	−3.1	−1.3
0.2	92	0.0	−0.2	−0.2	5.0	0.0	−0.2	−0.5	4.0	0.0	−0.2	−1.1	2.1
	96	0.0	0.0	0.4	6.3	0.0	−0.1	0.3	5.6	−0.1	−0.2	−0.1	4.4
	100	0.1	0.2	1.1	7.5	0.2	0.3	1.2	7.3	0.4	0.4	1.4	7.0
	104	0.4	0.6	1.9	8.8	0.6	0.8	2.2	9.0	1.0	1.3	3.1	9.8
0.3	92	0.1	0.7	3.4	11.2	0.1	0.7	3.8	11.8	−0.1	0.6	4.7	13.1
	96	0.6	1.2	4.2	12.4	0.7	1.4	4.9	13.4	1.0	2.0	6.5	15.7
	100	0.7	1.6	5.0	13.6	0.6	1.9	6.0	15.1	−0.1	2.5	8.2	18.2
	104	−0.6	1.6	5.8	14.8	−2.3	1.6	7.0	16.7	−11.1	0.6	9.7	20.6

Parameters are identical to those used in Exhibits 2 and 5.

$[e^{m_1-nh}, e^{m_1-(n-1)h}]$, $[e^{m_1-(n-1)h}, e^{m_1-(n-2)h}]$, ..., $[e^{m_1-h}, e^{m_1}]$, $[e^{m_1}, e^{m_1+h}]$, ..., $[e^{m_1+(n-1)h}, e^{m_1+nh}]$ I construct a bound on the maximum difference $\max_x |F_n(x) - F(x)|$, where x is restricted to the interval under consideration. Using the extended mean value theorem, and neglecting terms involving the third and fourth derivatives, the bound on the difference between F_n and F in the j-th interval $[e^{m_1+jh}, e^{m_1+(j+1)h}]$ is $B_j = F''(e^{m_1+jh})\,\eta_j^2/8$, where $\eta_j = e^{m_1+(j+1)h} - e^{m_1+jh}$ is the length of the interval.

Consider the function B defined by $B(x) = B_j$ for $x \in [e^{m_1+jh}, e^{m_1+(j+1)h}]$). The bound on the approximation error in the option price is the integral of this function over the density of S_{1T}, i.e., $\int_{e^{m_1-nh}}^{e^{m_1+nh}} B(S_{1T})\, f(S_{1T})\, dS_{1T}$. This bound overestimates the actual approximation error, for the difference $|F_n(x) - F(x)|$ is often considerably less than the bound.

For example, at the endpoints of each of the intervals the difference is zero. While this bound ignores the behavior of F_n and F outside the interval $[e^{m_1-nh}, e^{m_1+nh}]$, given n the parameter h can be chosen to guarantee that the probability that S_{1T} is outside this interval is trivial.

Exhibit 7 shows the bounds on the approximation errors for the spread call prices shown in Exhibit 2. For these prices and bounds, n = 100, and the piecewise linear approximation F_n has 2n = 200 segments in the interval $[e^{m_1-nh}, e^{m_1+nh}]$. Also, h is chosen so that F is integrated over a range extending from the mean of ln (S_{1T}) less five standard deviations to the mean plus five standard deviations.

For the options with time to expiration of one year or less, the largest value shown in Exhibit 7 is 0.00542, or about one-half of one cent if the values in Exhibits 2 and 7 are taken to be dollars. As indicated above, these

EXHIBIT 7

Error Bounds on Spread Call Option Prices in Exhibit 2 with n = 100

Volatility of First Underlying	Price of First Underlying	Correlation −0.5, T − t: 0.02	0.083	1.00	5.00	Correlation 0.0, T − t: 0.02	0.083	1.00	5.00	Correlation 0.5, T − t: 0.02	0.083	1.00	5.00
0.1	92	0.00015	0.00048	0.00226	0.00760	0.00003	0.00012	0.00052	0.00143	0.00000	0.00000	0.00000	0.00000
	96	0.00027	0.00056	0.00228	0.00762	0.00007	0.00015	0.00054	0.00147	0.00000	0.00000	0.00000	0.00000
	100	0.00015	0.00049	0.00227	0.00761	0.00003	0.00013	0.00055	0.00150	0.00000	0.00000	0.00000	0.00000
	104	0.00031	0.00034	0.00221	0.00758	0.00000	0.00008	0.00054	0.00152	0.00000	0.00000	0.00000	0.00000
0.2	92	0.00032	0.00087	0.00380	0.01249	0.00013	0.00041	0.00169	0.00455	0.00002	0.00010	0.00034	0.00064
	96	0.00045	0.00095	0.00384	0.01257	0.00023	0.00048	0.00174	0.00466	0.00007	0.00014	0.00039	0.00066
	100	0.00033	0.00088	0.00385	0.01262	0.00015	0.00044	0.00176	0.00476	0.00003	0.00012	0.00042	0.00068
	104	0.00013	0.00071	0.00381	0.01265	0.00004	0.00032	0.00175	0.00483	0.00000	0.00006	0.00043	0.00071
0.3	92	0.00051	0.00125	0.00532	0.01725	0.00029	0.00077	0.00301	0.00730	0.00012	0.00039	0.00144	0.00346
	96	0.00064	0.00134	0.00539	0.01739	0.00042	0.00086	0.00310	0.00812	0.00024	0.00047	0.00152	0.00353
	100	0.00052	0.00128	0.00542	0.01750	0.00031	0.00081	0.00315	0.00828	0.00014	0.00043	0.00156	0.00358
	104	0.00029	0.00113	0.00541	0.01759	0.00013	0.00067	0.00316	0.00842	0.00003	0.00030	0.00156	0.00363

The price of the second underlying asset is $S_{2t} = 100$, the exercise price is K = 4, the continuously compounded interest rate is r = 0.1, the dividend yields are $\delta_1 = \delta_2 = 0.05$, and the volatility or standard deviation of the rate of return of the second underlying asset is $\sigma_2 = 0.2$.

bounds are likely to be considerably larger than the actual magnitudes of the approximation errors. Given these error bounds, one can be confident that the spread call prices in Exhibit 2 are correct to the precision reported for the options with $T - t$ less than or equal to one year.

For the options with five years to expiration, the largest error bound is 0.01759 or almost 2 cents, which occurs with $\rho = -0.5$. While this is larger than a penny, the option price corresponding to this error bound is 26.90, so the error bound is about 6 one-hundredths of 1% of the option price.

Comparisons of results obtained using the approach in this article and the spread option prices reported in Rubinstein [1991b] indicate that Rubinstein's bivariate binomial scheme can result in approximation errors of up to 0.02 for options with times to expiration of 0.5 and 0.95 and other parameters identical or similar to those in Exhibit 2. For the same parameters, the error bounds for the approach in this article are less than 0.00525. Given that Rubinstein uses a tree with 100 time steps, and the approximation error in binomial schemes declines approximately with the number of time steps, the accuracy of the approximation in this article could be attained with the bivariate binomial approximation only by using a much larger number of time steps (and therefore a large number of terminal nodes).

Exhibit 8 shows spread call prices computed with $n = 40$ using the same parameter values used in Exhibit 2. The bounds on the approximation errors, while not shown, are somewhat more than six times as large as the bounds in Exhibit 7, and for some of the prices the error bounds are considerably larger than 0.01.

Nonetheless, comparison of the prices in Exhibits 2 and 8 reveals that the prices in Exhibit 8 differ from those in Exhibit 2 by at most 0.02 for options with times to expiration of one year or less, which is equal to the largest error in the prices reported in Rubinstein [1991b]. The prices in Exhibit 8 differ from those in Exhibit 2 by at most 0.04 for options with five years to expiration.

Moreover, the largest differences in the prices in Exhibit 8 occur for the options with correlation equal to −0.5 and volatility equal to 0.2 or 0.3. These pricing errors are relatively unimportant, since there simply aren't many stock pairs with correlation coefficients of −0.5, and the leading currencies have volatilities of less than 0.2. For the other parameter combinations, performance of the approach in this article with $n = 40$ generally is equal to or better

EXHIBIT 8
Spread Call Option Prices with n = 40

Volatility of First Underlying	Price of First Underlying	Correlation −0.5 T − t 0.02	0.083	1.00	5.00	Correlation 0.0 T − t 0.02	0.083	1.00	5.00	Correlation 0.5 T − t 0.02	0.083	1.00	5.00
0.1	92	4.25	5.36	11.82	19.50	4.14	4.98	10.38	16.88	4.05	4.56	8.62	13.67
	96	1.48	3.00	9.99	18.24	1.25	2.55	8.49	15.56	0.98	1.99	6.65	12.26
	100	0.28	1.47	8.40	17.07	0.16	1.08	6.88	14.34	0.06	0.63	5.04	10.97
	104	0.03	0.63	7.01	15.98	0.01	0.37	5.52	13.21	0.00	0.14	3.74	9.81
0.2	92	4.52	6.14	14.66	24.51	4.29	5.51	12.39	20.52	4.08	4.75	9.45	15.22
	96	1.92	3.90	12.93	23.39	1.57	3.18	10.59	19.30	1.11	2.25	7.52	13.86
	100	0.57	2.30	11.37	22.32	0.33	1.63	8.99	18.15	0.10	0.83	5.91	12.61
	104	0.12	1.25	9.97	21.31	0.04	0.74	7.61	17.08	0.00	0.24	4.58	11.46
0.3	92	4.87	7.03	17.76	29.81	4.56	6.25	15.07	25.23	4.23	5.30	11.61	19.16
	96	2.40	4.87	16.11	28.81	1.98	4.02	13.34	24.12	1.44	2.94	9.78	17.91
	100	0.95	3.23	14.60	27.85	0.62	2.42	11.79	23.07	0.26	1.41	8.18	16.74
	104	0.30	2.05	13.23	26.94	0.14	1.35	10.40	22.07	0.02	0.59	6.80	15.65

Parameters are identical to those used in Exhibit 2.

than performance of the bivariate binomial approach with 100 time steps. At least for the parameter values used in Exhibits 2 and 8, reasonably accurate estimates of the prices of spread calls can be obtained with n = 40.

In addition to being very accurate, the approximation described in Section II can also be computed reasonably quickly. An apparent drawback of the approximation is that it requires a large number of evaluations of the functions c and p, and therefore a large number of evaluations of the standard normal distribution function. This is not an insurmountable difficulty, for c and p can be evaluated using the approximation for the standard normal distribution function given by formula 26.2.17 of Zelen and Severo [1965], which has a maximum error of less than 7.5×10^{-8}. As can be seen in Hastings [1955, p. 169], the error in evaluating the normal distribution function using this approximation is usually considerably less than this bound.

Using the approximation for the standard normal distribution function, the approximation described in Section II was programmed in C^{++} as an "add-in function" for Microsoft Excel 4.0. The computation of an array of 625 spread call prices with n = 100 (the prices used in Exhibit 3) requires thirty-four seconds on a personal computer equipped with a 66 MHz 486 DX2 processor, or about 0.054 seconds per option price. This computational speed is fast enough for practical applications.[4] The computational times with n = 40 are of course only 40% as long.

Moreover, it is possible to reduce computation time further using the technique of Richardson extrapolation (see, e.g., Ralston and Rabinowitz [1978, Section 4.2]). In Section II I construct an approximating sequence $\{C_n\}$ in which as $1/n \to 0$ the error has the asymptotic form

$$E_n = \sum_{j=1}^{\infty} a_j (1/n)^2 \tag{5}$$

which is a special case of when Richardson extrapolation may be applied.

The idea behind Richardson extrapolation is that the first term in the sum on the right-hand side of Equation (5), $a_1(1/n)^2$, can be eliminated by an appropriate linear combination of two elements of the approximating sequence $\{C_n\}$. Intuitively, using knowledge of the relatively magnitudes of the errors, one extrapolates from two elements of the approximating sequence to obtain a better estimate of the limit. Letting n_1 and n_2 denote the two different values of n, with $n_2 = bn_1$ and $b > 1$, in the special case given by Equation (5) the extrapolated estimate C_{12} is $C_{12} = (C_2 - b^2C_1)/(1 - b_2)$.

Computations using this formula with $n_1 = 6$ and $n_2 = 12$ produce prices that agree with those in Exhibit 2 to the precision reported. This requires only about 20% of the work involved in computing the prices in Exhibit 2.

V. CONCLUSION

This article presents a new approach for valuing spread options. The main idea of the approach is that in a Black-Scholes environment an analytic expression for the conditional expectation $F(S_{1T}) = E^{Q}(\max[S_{2T} - S_{1T} - K, 0] \mid S_{1T})$ of the option payoffs given the terminal price of the first asset, S_{1T}, is available. Then it is necessary only to value a security with payoffs equal to the conditional expectation $F(S_{1T})$. The approach taken here is to approximate $F(S_{1T})$ with a piecewise linear function of S_{1T} and then value the approximating function as a portfolio of bonds and ordinary puts and calls. Alternatively, an instrument with payoffs $F^{*}(S_{1T})$ can be valued using numerical quadrature or a binomial approximation.

The solutions above can be adapted to handle spread options on foreign currencies simply by replacing δ_1 and δ_2 by the foreign interest rates; if one of the currencies is the domestic currency, then δ_1 or δ_2 is replaced by r. Commodity options can be handled by interpreting δ_1 and δ_2 as the net (of storage costs) convenience yields.

The approach can also be extended to handle other options with payoffs that depend on the prices of two underlying assets when an analytic expression for the conditional expectation of the option payoffs given the terminal price of one of the underlying assets is known. For example, the approach can be adapted to handle basket options on portfolios of two assets, dual-strike options, and bivariate analogues of path-independent digital options.

Finally, the approach leads to a simple stochastic volatility option pricing model. Specifically, suppose one interprets a stock as a portfolio of two projects with different volatilities. In such a situation, the volatility of the stock will change as the proportion of the stock's value due to each of the projects changes. Since the approach can be adapted to value basket options on baskets or portfolios of two assets, it can be used to value an option on the stock, and therefore leads to a model to value an option on a stock with stochastic volatility.

APPENDIX

THE DENSITY FUNCTIONS FOR S_{1T} AND S_{2T}

The joint density of S_{1T} and S_{2T} is

$$h(S_{1T}, S_{2T}) = \frac{1}{2\pi |\Sigma|^{1/2} S_{1T} S_{2T}} \times$$

$$\exp\left\{-\frac{1}{2}\begin{bmatrix}\ln S_{1T} - m_1 \\ \ln S_{2T} - m_2\end{bmatrix}' \Sigma^{-1} \times \begin{bmatrix}\ln S_{1T} - m_1 \\ \ln S_{2T} - m_2\end{bmatrix}\right\}$$

where

$$m_1 \equiv \ln S_{1t} + (r - \delta_1 - \sigma_1^2/2)(T - t)$$

$$m_2 \equiv \ln S_{2t} + (r - \delta_2 - \sigma_2^2/2)(T - t)$$

$$\Sigma \equiv \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}(T - t)$$

The marginal density of S_{1T} and the conditional density of S_{2T} given S_{1T} are

$$f(S_{1T}) = \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{T - t}\, S_{1T}} \times$$

$$\exp\left[-\frac{(\ln S_{1T} - m_1)^2}{2\sigma_1^2 (T - t)}\right]$$

$$g(S_{2T} \mid S_{1T}) = \frac{1}{\sqrt{2\pi}\,\sigma S_{2T}} \times$$

$$\exp\left[-\frac{\left(\ln S_{2T} - m_2 - \dfrac{\rho\sigma_2}{\sigma_1}(\ln S_{1T} - m_1)\right)^2}{2\sigma^2}\right]$$

where $\sigma^2 \equiv \sigma_2^2 (1 - \rho^2)(T - t)$.

THE INTEGRAL IN (1)

The integral inside the braces in (1) is

$$F(S_{1T}) = \int_{\max(S_{1T}+K,0)}^{\infty} [S_{2T} - (S_{1T} + K)] \times g(S_{2T} \mid S_{1T})\, dS_{2T}$$

$$= \int_{\max(S_{1T}+K,0)}^{\infty} S_{2T}\, g(S_{2T} \mid S_{1T})\, dS_{2T} - \int_{\max(S_{1T}+K,0)}^{\infty} (S_{1T} + K)\, g(S_{2T} \mid S_{1T})\, dS_{2T}$$

I consider the two integrals on the right-hand side separately.

The first integral is

$$\int_{\max(S_{1T}+K,0)}^{\infty} / S_{2T}\, g(S_{2T} \mid S_{1T})\, dS_{2T} =$$

$$\int_{\max(S_{1T}+K,0)}^{\infty} \frac{S_{2T}}{\sqrt{2\pi}\,\sigma\, S_{2T}} \times \exp\left[-\frac{(\ln S_{2T} - M_2)^2}{2\sigma^2}\right] dS_{2T}$$

$$= \int_{\ln\max(S_{1T}+K,0)}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^y \times \exp\left[-\frac{(y - M_2)^2}{2\sigma^2}\right] dy$$

$$= \int_{\ln\max(S_{1T}+K,0)}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \times \exp\left[-\frac{(y - (M_2 + \sigma^2))^2}{2\sigma^2}\right]$$

$$\times \exp[M_2 + \sigma^2/2]\, dy$$

$$= \int_{\frac{\ln\max(S_{1T}+K,0)-(M_2+\sigma^2)}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} \times$$

$$\exp\left[-z_1^2/2\right]\exp\left[M_2 + \sigma^2/2\right]dz_1$$

$$= \exp\left[M_2 + \sigma^2/2\right]\left[1 - N(-x_1)\right]$$

$$= \exp\left[M_2 + \sigma^2/2\right] N(x_1)$$

$$= e^A S_{2t}\left(\frac{S_{1T}}{S_{1t}}\right)^{\frac{\rho\sigma_2}{\sigma_1}} N(x_1)$$

where

$$A = \left[r\left(1 - \frac{\rho\sigma_2}{\sigma_1}\right) - \left(\delta_2 - \delta_1\frac{\rho\sigma_2}{\sigma_1}\right) + \rho\sigma_2(\sigma_1 - \rho\sigma_2)/2\right](T - t)$$

$$x_1 = \frac{M_2 + \sigma^2 - \ln\max(S_{1T} + K, 0)}{\sigma}$$

$N(\cdot)$ is the cumulative standard normal distribution function, and the second and fourth lines are obtained using the changes of variable $y = \ln S_{2T}$ and

$$z_1 = \frac{y - (M_2 + \sigma^2)}{\sigma}$$

The second integral is

$$\int_{\max(S_{1T}+K,0)}^{\infty}(S_{1T} + K)\, g(S_{2T} \mid S_{1T})\, dS_{2T} =$$

$$\int_{\max(S_{1T}+K,0)}^{\infty}\frac{S_{1T} + K}{\sqrt{2\pi}\,\sigma S_{2T}} \times \exp\left[-\frac{(\ln S_{2T} - M_2)^2}{2\sigma^2}\right]dS_{2T}$$

$$= \int_{\ln\max(S_{1T}+K,0)}^{\infty}\frac{S_{1T} + K}{\sqrt{2\pi}\,\sigma} \times \exp\left[-\frac{(y - M_2)^2}{2\sigma^2}\right]dy$$

$$= \int_{\frac{\ln\max(S_{1T}+K,0)-M_2}{\sigma}}^{\infty}\frac{S_{1T} + K}{\sqrt{2\pi}} \times \exp\left[-z_2^2/2\right]dz_2$$

$$= (S_{1T} + K)\left[1 - N(-x_2)\right]$$

$$= (S_{1T} + K)\, N(x_2)$$

where

$$x_2 = \frac{M_2 - \ln\max(S_{1T} + K, 0)}{\sigma}$$

and the second and third lines are obtained using the changes of variable $y = \ln S_{2T}$ and $z2 = y - M_2/\sigma$.

Combining the results of the two integrations:

$$F(S_{1T}) = e^A S_{2t}\left(\frac{S_{1T}}{S_{1t}}\right)^{\frac{\rho\sigma_2}{\sigma_1}} N(x_1) - (S_{1T} + K)\, N(x_2)$$

ENDNOTES

The author thanks Vincent Case and Bruno Amadei for programming the computations, and Steve Thomas for finding an error in an earlier version.

[1]A difficulty is that no analytic expression for the probability density function of linear combinations of lognormal random variables is known.

[2]This idea is described in Cox and Rubinstein [1985, Section 7-2], in a setting in which there is only one underlying asset.

[3]See, for example, Cox and Huang [1989] and Duffie [1992, Chapters 5-6].

[4]For comparison, it took 33 seconds to recompute an array of 625 ordinary American call prices using the binomial function with 100 time steps in the *@nalyst* function library available from Tech Hackers, a financial software vendor. It took 18 seconds to recompute an array of 625 ordinary European call prices using the binomial function with 100 time steps, and 0.6 seconds to recompute an array of 625 ordinary European call prices using the Black-Scholes formula.

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