A Multi-Factor Model for Energy Derivatives

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Abstract

In this paper we develop a general framework for the risk management of energy derivatives. The framework is designed to be consistent not only with the market observable forward price curve but also the volatilities and correlations of forward prices. We show how these volatilities and correlations can be estimated from the market and incorporated into the model in order to price a wide range of energy derivatives. Our framework extends and synthesises the results of Amin and Jarrow [1991a,b], Cortazar and Schwartz [1994], Amin, Ng and Pirrong [1995], Schwartz [1997], and Hilliard and Reis [1998]. We demonstrate the application of our framework to oil and gas futures data from the New York Mercantile Exchange and give numerical results for the pricing of European Caps and Swaptions.

1 Introduction

In this paper we develop a general framework for the risk management of energy derivatives. The framework is designed to be consistent not only with the market observable forward price curve but also the volatilities and correlations of forward prices. We show how these volatilities and correlations can be estimated from the market and incorporated into the model in order to price a wide range of energy derivatives.

Historically the majority of work on modelling commodity prices has been focussed on stochastic processes for the spot price and other key variables, such as the convenience yield and interest rates (see for example Gibson and Schwartz [1990], Schwartz [1997], and Hilliard and Reis [1998]). However, this approach has some fundamental disadvantages – firstly the key state variables, such as the convenience yield, are unobservable and secondly the forward price curve is an endogenous function of the model parameters and therefore will not necessarily be consistent with the market observable forward prices.

A more practical approach, based on modelling the entire forward price curve, was introduced in Cortazar and Schwartz [1994] who analysed the pricing of Copper Interest-Indexed Notes. Our framework is based on their approach but we extend the results of Cortazar and Schwartz and develop a practical and efficient framework for estimating the model and the pricing and risk management of energy derivatives. This framework also extends and synthesise the results of Amin and Jarrow [1991a,b], Amin, Ng and Pirrong [1995], Schwartz [1997], and Hilliard and Reis [1998]. The paper is organised as follows; In section 2 we introduce the multi-factor model and discuss its relationship to spot price models. Sections 3 and 4 present the results for pricing standard and general European options. We outline our proposed technique for pricing American options in section 5 and in section 6 we present numerical results for pricing Natural Gas Caps and Crude Oil Swaptions. Finally, section 7 contains our conclusions.

2 The Multi-Factor Model

We propose the following model for the evolution of the forward curve[1]:

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{n} s_i(t,T)dz_i(t) \tag{2.1}$$

where $F(t,T)$ is the forward price at time t for maturity date T and the $s_i(t,T)$ are volatility functions associated with the independent Brownian motions $z_i(t)$. We follow Cortazar and Schwartz [1994] and assume that interest rates are deterministic so that futures prices are equal to forward prices (see Cox, Ingersoll and Ross [1981]). The model described by equation (2.1) assumes there are n independent sources of uncertainty which drive the evolution of the forward curve. Each source of uncertainty has associated with it a volatility function which determines by how much and in which direction the arrival of information associated with a particular source of uncertainty moves each point of the forward curve.

Equation (2.1) can be integrated to give:

$$F(t,T) = F(0,T)\exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_0^t s_i(u,T)^2\,du + \int_0^t s_i(u,T)dz_i(u)\right\}\right] \tag{2.2}$$

The process for the spot price can be obtained by setting $T=t$:

$$S(t) = F(0,t)\exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_0^t s_i(u,t)^2\,du + \int_0^t s_i(u,t)dz_i(u)\right\}\right] \tag{2.3}$$

From this we can see that the natural logarithm of the spot price is normally distributed at time T given the forward price $F(0,T)$ at time 0 for maturity at time T, as follows:

[1] We make the standard assumptions regarding the stochastic setting and we work in the risk neutral measure throughout.

$$\ln\left(S(T)\right) \sim N\left[\ln\left(F(0,T)\right) - \frac{1}{2}\sum_{i=1}^{n}\left\{\int_o^T s_i(u,T)^2\,du\right\}, \sum_{i=1}^{n}\left\{\int_o^T s_i(u,T)^2\,du\right\}\right] \tag{2.4}$$

Equation (2.3) can be differentiated to give the stochastic differential equation for the spot price:

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \left[\frac{\partial \ln F(0,t)}{\partial t} - \sum_{i=1}^{n}\left\{\int_0^t s_i(u,t)\frac{\partial s_i(u,t)}{\partial t}\,du + \int_0^t \frac{\partial s_i(u,t)}{\partial t}\,dz_i(u)\right\}\right]dt \\ &+ \left[\sum_{i=1}^{n} s_i(t,t)\,dz_i(t)\right]\end{aligned} \tag{2.5}$$

The term in square parentheses in the drift can be interpreted as being equivalent to the sum of the deterministic riskless rate of interest $r(t)$ and a convenience yield $d(t)$ which in general will be stochastic. Also, since the last component of the drift term involves the integration over the Brownian motions, the spot price process will, in general, be non-Markovian - that is the evolution of the spot price will depend upon its past evolution.

Many well-known models are special cases of this general framework. For example, Schwartz [1997] describes two simple Markovian models for the behaviour of commodity prices – a single factor simple mean-reverting model and a two factor stochastic convenience yield model. The single factor model can be represented by the following SDE

$$dS = a\left(m - l - \ln(S)\right)S\,dt + s\,S\,dz \tag{2.6}$$

where l is the market price of risk. Schwartz shows that for the case of this single factor model (n=1) the term structure of forward price volatilities has the following form:

$$s_1(t,T) = s\,e^{-a(T-t)} \tag{2.7}$$

By substituting equation (2.7) into equation (2.5) and simplifying we obtain the spot price process:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} + a\left(\ln F(0,t) - \ln S(t)\right) + \frac{s^2}{4}\left(1 - e^{-2at}\right)\right]dt + s\,dz_1(t) \quad (2.8)$$

Schwartz also shows that under the model described by equation (2.6) forward prices are given analytically by

$$F(0,T) = \exp\left[e^{-aT}\ln(S) + \left(1 - e^{-aT}\right)\left(m - \frac{s^2}{2a} - l\right) + \frac{s^2}{4a}\left(1 - e^{-2aT}\right)\right] \qquad (2.9)$$

Substituting the expression for forward prices given by (2.9) into equation (2.8) leads to equation (2.6). Therefore equation (2.8) is an important generalisation of equation (2.6) which allows the model to be consistent with the initial forward price curve. In Clewlow and Strickland [1999] we show that the model defined by equation (2.8) has a great deal of analytical tractability. Using our results it is possible to derive a future forward curve, consistent with the observed curve, as well as price standard European options on both the spot energy price and a forward contract. We also show how to construct trinomial trees for the spot price consistent with the initial forward curve, allowing the efficient pricing of American-style and path dependent options.

The two factor model described in Schwartz [1997] can be defined by the following SDE's

$$dS = \left(r - d\right)Sdt + s_1 S dz_1 \qquad (2.10)$$

$$dd = \left[a\left(\bar{d} - d\right) - l\right]dt + s_2 dz_2 \qquad (2.11)$$

where $dz_1 dz_2 = r dt$. Schwartz shows that under this joint process forward prices are again given analytically. We show in appendix A that this model also fits in our general framework if we chose the volatility functions have the following form

$$s_1(t,T) = s_1 - r s_2 \frac{(1-e^{-a(T-t)})}{a}$$
$$s_2(t,T) = -s_2\sqrt{1-r^2}\frac{(1-e^{-a(T-t)})}{a} \qquad (2.12)$$

Although special cases, such as these, of the general model are often chosen for analytically tractability we show in the following sections that it is not necessary to restrict the functional form of the volatility functions or indeed the number of volatility functions in order to obtain a tractable model.

3 Pricing Standard European Options

In this section we discuss the pricing of standard European options on both forward contracts and the spot energy price for the general framework. Similar results have previously appeared in Amin and Jarrow [1991a,b] and Amin, Ng and Pirrong [1995].

From the standard risk-neutral pricing results (Cox and Ross [1976], Harrison and Pliska [1981]) the price of any contingent claim, maturing at time T and dependent on the s-maturity forward price, $C(t, F(t,s);\Theta)$, is given by the expectation of the discounted payoff at time T under the risk neutral measure

$$C(t,F(t,s);\Theta)=\mathrm{E}_t\left[P(t,T)C(T,F(T,s);\Theta)\right] \qquad (3.1)$$

where $P(t,T) = \exp\left(-\int_t^T r(u)du\right)$ and Θ is a vector of constant parameters. Therefore for a standard European call option $c(t,F(t,s);K,T)$ with strike price K and maturity date T on the forward price $F(t,s)$ we have

$$c(t,F(t,s);K,T)=\mathrm{E}_t\left[P(t,T)\max(0,F(T,s)-K)\right] \qquad (3.2)$$

From equation (2.2) we see that the natural log of the forward price is normally distributed

$$\ln F(T,s) \sim N\left(\ln F(t,s) - \frac{1}{2}\sum_{i=1}^{n}\left\{\int_t^T s_{\,i}(u,s)^2 du\right\} \sum_{i=1}^{n}\left\{\int_t^T s_{\,i}(u,s)^2 du\right\}\right) \qquad (3.3)$$

Since we have assumed interest rates are deterministic we can use standard results to obtain the following analytical formula for a standard European call option

$$c(t, F(t,s); K, T) = P(t,T)\left[F(t,s)N(h) - KN(h - \sqrt{w})\right] \qquad (3.4)$$

where

$$h = \frac{\ln\left(\dfrac{F(t,s)}{K}\right) + \dfrac{1}{2}w}{\sqrt{w}} \quad , \quad w = \sum_{i=1}^{n}\left\{\int_t^T s_{\,i}(u,s)^2 du\right\},$$

The formula for standard European put options can be easily obtained by put-call parity. Options on the spot energy price are similarly valued simply by setting $s=T$.

The implication of equation (3.4) is that regardless of the number of factors standard European options on both the spot and a forward contract can be priced with a Black and Scholes [1973] like analytical formula. The calculation requires only univariate integrations involving the volatility functions of the forward prices.

4 Pricing General European Contingent Claims

We now turn our attention to pricing general European contingent claims whose payoff can depend on a finite set of spot or forward prices. We consider any contingent claim to be a series of possibly contingent cashflows $C_k(s_k, \Theta, \{c_l F(s_k, s_l)\})$ occurring on dates s_k, k = 1, ..., m and depending on forward prices with maturity dates also on the set of dates s_k with face values c_k and also on the parameter vector Θ. The price of this contingent claim is given by

$$C(t,\Theta,\{c_k\},\{s_k\}) = \mathrm{E}_t\left[\sum_{k=1}^{m} P(t,s_k)C_k\left(s_k,\Theta,\{c_l F(s_k,s_l)\}\right)\right] \tag{4.1}$$

The expectation is taken over the $m*m$-dimensional normal distribution of the correlated natural logarithms of the forward prices $\ln\left(F(s_k,s_l)\right)$.

In order to perform Monte-Carlo simulation to evaluate (4.1) we must therefore compute the $m'*m'$ covariance matrix Σ (where $m' = m^2$);

$$\Sigma_{jk} = Cov\left[\ln\left(F(t_j,s_j)\right), \ln\left(F(t_k,s_k)\right)\right] = \sum_{i=1}^{n}\left\{\int_t^{\min(t_j,t_k)} s_i(u,s_j)s_i(u,s_k)du\right\} \tag{4.2}$$

In order to efficiently sample under this covariance matrix we compute the orthogonal representation of the covariance matrix which gives us the m' eigenvectors $\underline{w}_i$ and associated m' eigenvalues λ_i such that

$$\Sigma = \Gamma\Lambda\Gamma' \tag{4.3}$$

where $$\Gamma = \begin{vmatrix} w_{11} & w_{21} & \cdots & w_{m'1} \\ w_{12} & w_{22} & \cdots & w_{m'2} \\ \cdots & \cdots & \cdots & \cdots \\ w_{1m'} & w_{2m'} & \cdots & w_{m'm'} \end{vmatrix} \text{ and } \Lambda = \begin{vmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & l_{m'} \end{vmatrix}$$

The columns of Γ are the eigenvectors. Let M be the number of samples or simulations and e_i, $i = 1,\ldots,m'$ be independent standard normal random numbers. Therefore, we have;

$$C(t,\Theta,\{c_k\},\{s_k\}) = \frac{1}{M}\sum_{j=1}^{M}\left[\sum_{k=1}^{m} P(t,s_k)C_k\left(s_k,\Theta,\{c_l F(t,s_l)Y_j(t,s_k,s_l)\}\right)\right] \tag{4.4}$$

where $Y_j(t,s_k,s_l) = \exp\left[-\frac{1}{2}\sum_{i=1}^{m'}\left\{w_{il'}{}^2 l_i\right\}+\sum_{i=1}^{m'}\left\{w_{il'}\sqrt{l_i}e_i\right\}\right], \; l' = k\,l$

Notice that because of the Gaussian nature of our framework we effectively jump straight to the end of the life of the option under each simulation, rather than the more usual practice of simulating at a large number of small discrete time steps until maturity.

This formulation also gives a natural way to trade off accuracy with speed by truncating the integration space dependent on the sizes of the eigenvalues. The size of eigenvalues indicate the relative importance of the corresponding eigenvector factors in reproducing the covariance matrix Σ. Therefore, by truncating the eigenvector representation, we reduce the dimensionality of the Monte Carlo integration and thus reduce the computation time required but at a cost of reduced accuracy.

5 Pricing General American Contingent Claims

In order to price diverse American style contingent claims in our general multi-factor setting we use an approximation approach pioneered by Clewlow and Strickland [1998][2]. The approach involves numerically solving for an early exercise strategy in a low-dimensional model similar to the higher dimensional model of interest. Specifically the technique requires the construction of a one or two factor Markovian approximation to the non-Markovian general model. Under this approximating model it is possible to build a recombining tree for the state variables – for example the spot price and convenience yield. Using this tree we can numerically solve for the early exercise strategy for the American claim. This approximate strategy can then be used in the Monte Carlo simulation of the full model to obtain an accurate approximation to the price of the American claim. Furthermore the approximation can easily be improved by maximising the price with respect to the early exercise boundary.

2 See also Clewlow, Pang, and Strickland [1996]

Equation (4.1) can be generalised to reflect the optimal choice of early exercise strategy;

$$C(t,\Theta,\{c_k\},\{s_k\}) = \max_{\Theta}\left\{ \mathrm{E}_t\left[\sum_{k=1}^{m} P(t,s_k) C_k\left(s_k,\Theta,\{c_l F(s_k,s_l)\}\right)\right]\right\} \tag{5.1}$$

where the parameter vector Θ now includes the early exercise strategy.

A one factor Markovian restriction of the full multi-factor model can be represented by (c.f. equation (2.8);

$$dx(t) = [q(t) - a\,x(t)]dt + s\,dz(t) \tag{5.2}$$

where $x(t) = \ln\left(S(t)\right)$ and $\frac{q(t)}{a}$ represents the level to which $x(t)$ mean reverts with mean-reversion rate a. The time dependent function is determined by the initial forward price curve as shown in equation (2.8). Clewlow and Strickland [1999] show how a trinomial tree representing equation (5.2) can be efficiently constructed and how general contingent claims can be priced using the tree. In particular the tree can be used to obtain the early exercise strategy for a general American contingent claim under the model represented by equation (5.2). We represent this early exercise strategy by the parameter vector $\tilde{\Theta}$. We can now obtain a lower bound approximation to the price of the American contingent claim under the full model using

$$\tilde{C}(t,\tilde{\Theta},\{c_k\},\{s_k\}) = \mathrm{E}_t\left[\sum_{k=1}^{m} P(t,s_k) C_k\left(s_k,\tilde{\Theta},\{c_l F(s_k,s_l)\}\right)\right] \tag{5.3}$$

Furthermore we can obtain an arbitrarily good approximation by maximising the value of the contingent claim with respect to the early exercise strategy using equation (5.3) as an initial condition.

6 Numerical Results

In this section we firstly estimate the model using Crude Oil and Natural Gas futures contract closing prices on NYMEX. We then go on to illustrate the numerical implementation by pricing European Caps on Natural Gas and European Swaptions on Crude Oil.

6.1 The Data

The data consists of daily closing futures prices for the nearest 24 monthly contracts for both Light, Sweet Crude Oil and Henry Hub Natural Gas from November 1995 to December 1997. The contracts for each commodity were used to construct continuous series of daily returns for the nearest maturity contract (with a maturity of between zero and 1 month) out to a contract with between 23 and 24 months to maturity. As the forward curve input for pricing derivatives we taken the forward curves implied by the futures prices on 17 December 1997.

6.2 Eigenvalue Analysis

Using the returns series described in section 6.1 we construct the sample covariance matrix of the returns and then perform an eigenvalue decomposition on the resulting covariance matrix. Figures 1 and 2 show the results in terms of the most important eigenvectors (measured in terms of the size of the eigenvalues). The eigenvectors have been multiplied by their corresponding eigenvalue in order to obtain discrete versions of the volatility functions in the sense of equation (2.1). Figures 1 and 2 also show the overall futures volatility obtained as the square root of the sum of the squares of the volatility functions. Analysis of the eigenvalues resulting from the analysis shows that in the case of the Crude Oil contracts only the first three volatility functions are significant whereas for the Natural Gas contracts the eigenvectors are significant out to the sixth.

Figure 1 : Volatility Functions for NYMEX Light Sweet Crude Oil

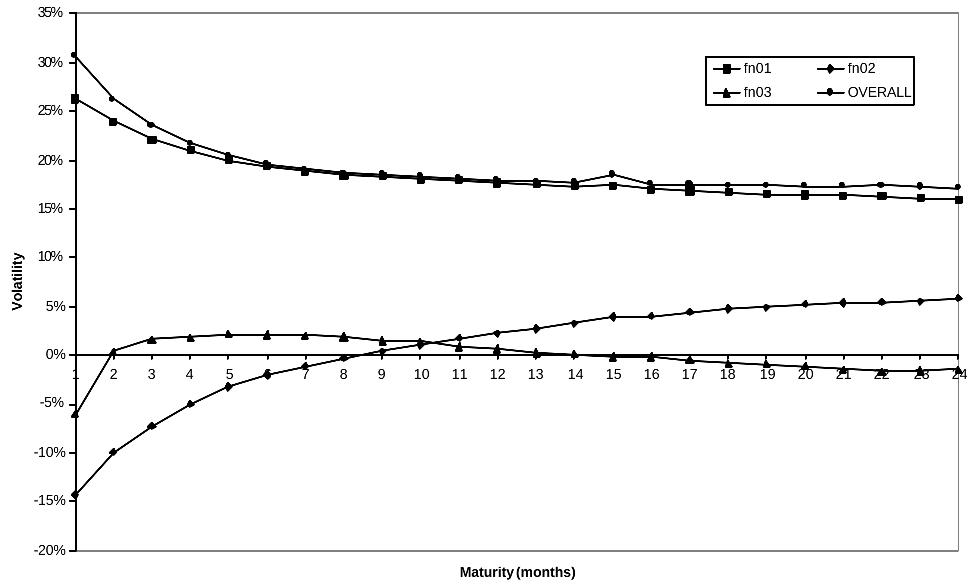
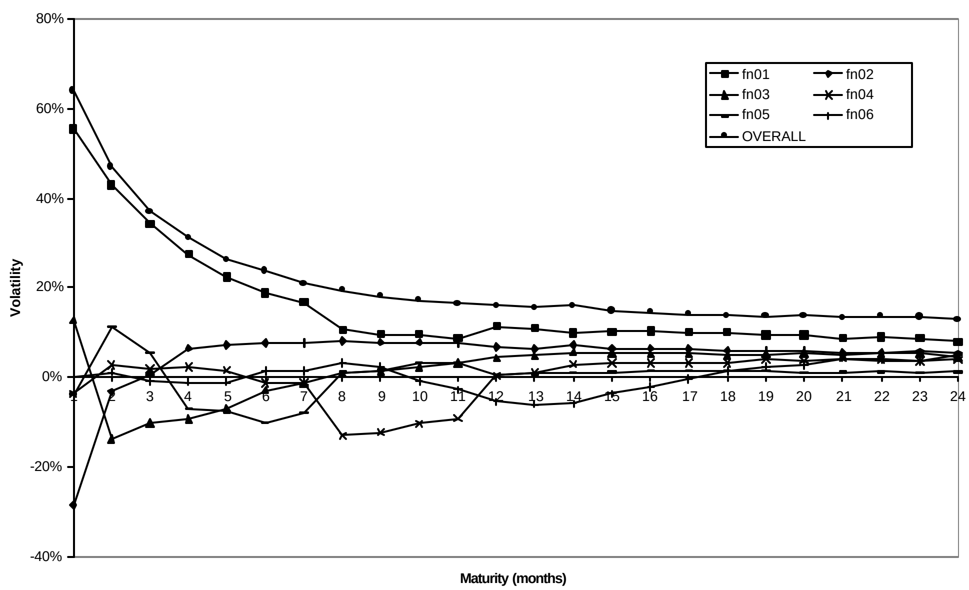


Figure 2 : Volatility Functions for NYMEX Henry Hub Natural Gas



The first three volatility functions (fn01, fn02, fn03) for both commodities have similar and typical shapes obtained from this type of analysis. In the case of Crude Oil the primary function (fn01) declines slightly over the maturity period from 1 month out to around 8 months and then declines more slowly out to our maximum maturity of 24 months. The second factor (fn02) is negative for the short maturity end of the curve out to 8 months and then positive thereafter and the third factor (fn03) is positive for very short maturities and long maturities beyond around 14 months whilst being positive for intermediate maturities. The first factor and therefore the primary dynamics of the forward curve is a roughly parallel shift of the whole curve either up or down depending on the direction of the random shock. The fact that the function is not flat indicates that the volatility of the short end of the forward curve is greater than the long end. The second and third factors represent tilts and bending of the forward curve respectively.

The first three factors for Natural Gas have similar interpretations although here the tilting and bending behaviour is concentrated towards the short end of the curve. Volatility functions 4 to 6 appear to represent seasonal volatility factors as they have a cycle of approximately six months. Further work is needed to determine the significance of these factors.

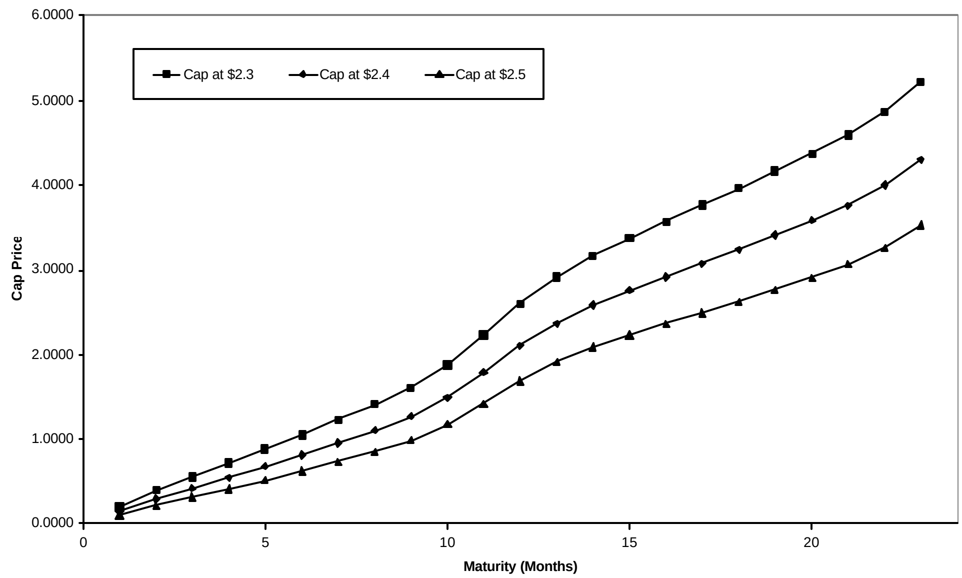
6.3 Pricing a European Energy Cap

A European energy cap which caps the spot price the holder will pay for energy at the cap level (or strike price) is a portfolio of call options on the underlying energy price. Let the dates at which the spot price is capped be denoted by $t_k, k = 1,...,m$, the spot energy price at these dates be S_k and the strike price be denoted by K. Using the results from section 3 the value of the cap, $Cap(t)$, at time t is given by

$$Cap(t) = \sum_{k=1}^{m} c(t, F(t,t_k); K, t_k) \qquad (6.1)$$

Figure 3 shows the results of pricing an option to cap the spot price of Natural Gas with monthly maturities from 1 month to 24 months, and for three cap prices - \$2.3, \$2.4, and \$2.5. Each price is the sum of the monthly caplets up to the maturity date.

Figure 3 : Natural Gas Cap Prices with Monthly Maturities



6.4 Pricing a European Energy Swaption

A European energy swaption is an option to swap a stream of floating price payments indexed to the market spot price of the energy commodity for a stream of fixed price payments. Let the dates at which the cashflows occur be denoted by $t_k, k = 1,\ldots,m$, the spot energy price at these dates be S_k and the fixed price payment (strike price) be denoted by K. The payoff of this swaption, $Swpn(T)$, at its maturity date $T < t_1$ is given by

$$Swpn(T) = \max\left(0, \left[\frac{1}{m}\sum_{k=1}^{m}(F(T,t_k) - K)\right]\right) \tag{6.2}$$

where the term in square brackets is the value of the swap at the maturity date of the option.

Table 1 shows the results of pricing options to swap the floating price of Crude Oil for a fixed price of $20.4 on a monthly basis. The maturity of the options ranges from 1 month to 1 year. Each of these options exercises into swaps with maturities of 1, 3, or 6 months (Swap Tenor).

Table 1: Swaption Prices For Crude Oil

		Swap Tenor		
		1m	3m	6m
Option Maturity	1m	0.0748	0.0624	0.0567
	3m	0.3628	0.3257	0.2900
	6m	0.7410	0.6755	0.6248
	1yr	1.0049	0.9355	0.8726

7 Summary and Conclusions

In this paper we have developed a general framework for the pricing and risk management of energy derivatives. Our framework extends and synthesises the results of Amin and Jarrow [1991a,b], Cortazar and Schwartz [1994], Amin, Ng and Pirrong [1995], Schwartz [1997], and Hilliard and Reis [1998]. The framework is designed to be consistent not only with the market observable forward price curve but also the volatilities and correlations of forward prices. We have shown how these volatilities and correlations can be estimated from the market and incorporated into the model in order to price a wide range of energy derivatives. To illustrate the application of our framework we estimated the model using oil and gas futures data from the New York Mercantile Exchange. We then used the model to price European Caps and Swaptions.

Appendix A : Proof of the Form of the Volatility Functions for the General Model Consistent with Schwartz's [1997] Two-Factor Model

The defining SDE's for the two factor model are repeated here;

$$dS = (r - d)Sdt + s_1 Sdz_1$$
$$dd = \left[a(\bar{d} - d) - l\right]dt + s_2 dz_2$$

Schwartz [1997] derives a closed form solution to the price of a futures contract, $F(S,d,t,s)$, under this joint process;

$$F(S,d,t,s) = S\exp\left(-d\,\frac{1-e^{-a(s-t)}}{a} + A(t,s)\right) \qquad \text{(A1)}$$

where

$$A(t,s) = \left(r - \hat{a} + \frac{1}{2}\frac{s_2^2}{a^2} - \frac{s_1 s_2 r}{a}\right)(s-t) + \frac{1}{4}s_2^2\,\frac{1-e^{-2a(s-t)}}{a^3} + \left(\hat{a}a + s_1 s_2 r - \frac{s_c^2}{a}\right)\frac{1-e^{-a(s-t)}}{a^2}$$

Assuming that the futures price function is twice continuously differentiable in S and δ. Ito's lemma gives the risk-neutralised process for futures price changes

$$dF = \left[F_t + \frac{1}{2}F_{SS}s_1^2 S^2 + \frac{1}{2}F_{dd}s_2^2 + F_{Sd}\,rSs_1 s_2 + F_S(r-d)S + F_d\left(a(\bar{d}-d) - l\right)\right]dt + F_S s_1 Sdz_1 + F_d s_2 dz_2 \qquad \text{(A2)}$$

where the subscripts on F indicate a partial derivative with respect to the subscripted variable.

Setting the drift of equation (A2) equal to zero and substituting F_S and F_d gives;

$$dF(t,s) = F(t,s)s_1 dz_1 - F(t,s)\frac{1-e^{-a(s-t)}}{a}s_2 dz_2 \qquad \text{(A3)}$$

the Brownian motions z_1 and z_2 have a correlation coefficient of r. Therefore we can write equation (A3) as follows;

$$\frac{dF(t,s)}{F(t,s)} = s_1 dz_1 - \frac{1-e^{-a(s-t)}}{a}s_2\left(r dz_1 + \sqrt{1-r^2}dz_3\right) \qquad \text{(A4)}$$

where z_1 and z_3 are independent Brownian motions. Rearranging in terms of z_1 and z_3 we obtain;

$$\begin{aligned}\frac{dF(t,s)}{F(t,s)} &= (s_1 - \frac{1-e^{-a(s-t)}}{a} r s_2)dz_1 - s_2\frac{1-e^{-a(s-t)}}{a}\sqrt{1-r^2}dz_3 \\ &= s_1(t,s)dz_1 + s_2(t,s)dz_3\end{aligned} \qquad \text{(A5)}$$

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