MIT Sloan Finance Research Practicum

Weekly Memorandum - SRR

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SUBJECT:	Valuation Models for Spread Options

I. Introduction

Through the initial project overview with the sponsors, the first part of the assignment is to construct a spread option function with the syntax of SOPT(value_date, expiry, F1, W1, F2, W2, K, V1, V2, C12, Rf) with defined variables:
value_date = Valuation date
expiry = Expiration date
F1 = Forward price of commodity 1
W1 = Weight of commodity 1 in the payoff (see below)
F2 = Forward price of commodity 2
W2 = Weight of commodity 2 in the payoff (see below)
K = Strike of the spread-option
V1 = Black-implied volatility of commodity 1
V2 = Black-implied volatility of commodity 2
C12 = Correlation coefficient between commodity 1 and commodity 2
Rf = Risk-free rate
and the payoff of the function at maturity to be:

Payoff = Max(W1*F1 + W2*F2 – K, 0)

The function is able to output the price of spread options given two of any type of energy derivatives, each with a given price, weight and any value of correlation between -1 to 1. To successfully value the spread option model with the given inputs, various existing pricing models were explored. First, we generated the Monte Carlo simulation of the payoff function and used the results from the simulation as the benchmark for other pricing models. Our goal is to find a model that gives the best approximation to the Marlo Carlo simulation. We researched and implemented five other models, Pearson Model, Carmona-Durrleman Procedure, Bachelier Model, Kirk Model and Bjerksund and Stensland Model, and compared the results with the Monte Carlo model for different constant values and presented the error analysis for these values.

II. Pricing Models

Pearson Model

The Pearson Model computes the price of a spread option by taking the double integral of the option payoffs based on the risk-neutral joint distribution of the terminal prices of both of the underlying assets. The basic assumption for the Pearson Model is that risk-neutral distribution of the underlying assets are lognormal. Thus, to calculate the value of the double integral payoff function, the Pearson model reduces the double integral to single integral by setting the price for asset two fixed using Black-Scholes closed form equation and taking the expected value of the random prices for asset one. According to this approach, we used the following single integral formula to calculate the spread option prices where $F(S_{1T})$ is the payoff function of S_{1T} and $f(S_{1T})$ is the marginal density of S_{1T}:

$$C = e^{-r(T-t)} \int_0^\infty F(S_{1T}) f(S_{1T}) dS_{1T}$$

The Pearson model further simplifies the single integral approach of calculating the spread option prices by using the linear approximation approach. First, it recognizes that f(S_{1T}) is the risk-neutral density function. Second, if the payoff function F is a piecewise linear function of S_{1T} , we can simply replicate the payoff using portfolios of bonds and ordinary put and call options with the same payoffs as S_{1T} . Thus, a variant of the Black-Scholes formula with continuous dividend yield can be used to calculate the put and call option values. Third, as the payoff function F is not a piecewise linear function of S_{1T} , we can still use linear approximation as the payoff of bonds, puts and calls. Using the Pearson linear approximation method, we use the following equation as the payoff function:

$$F(S_{1T}) = e^A S_{2t} \left(\frac{S_{1T}}{S_{1t}}\right)^{\frac{\rho\sigma_2}{\sigma_1}} N(x_1) - (S_{1T} + K) N(x_2)$$

where A is a constant and x_1 and x_2 are functions of S_{1T} . Using the equation above, we can define any S_{1T} value to find the payoff. To form piecewise linear approximation, we find the constant value h using k positive constant k and 2*n number of areas under the curve:

$$h = \frac{k\sigma_1\sqrt{T-t}}{\sqrt{n}}$$

We then calculate the payoff value F at points $e^{m_1 - nh}$, $e^{m_1-(n-1)h}$ ... e^{m_1-h}, e^{m_1}, e^{m_1+h}, ... $e^{m_1+(n-1)h}$, e^{m_1+nh} where the approximation is centered on $e^{m_1} = S_{1t} e^{(r-\delta_1-\sigma_1^2/2)(T-t)}$. Then the approximated value of portfolio can be found using the equation:

$$C_n = e^{-r(T-t)} F(e^{m_1}) + c(e^{m_1})\Delta_0 + \sum_{j=1}^{n-1} c(e^{m_1+jh})(\Delta_j - \Delta_{j-1}) - c(e^{m_1+nh})\Delta_{n-1} - p(e^{m_1})\Delta_{-1}$$
$$- \sum_{j=1}^{n-1} p(e^{m_1-jh})\left(\Delta_{-(j+1)} - \Delta_j\right) + p\left(e^{m_1-nh}\right)\Delta_{-n}$$

where Δ is the slope value between contingent points and c and p functions follow the Black-Scholes calculation for put and call options.

We implemented the Pearson integration model in Matlab and tested the results against the Monte Carlo simulation and other model outputs. We also implemented the linear approximation model in VBA and Matlab and optimized the n and k constant values in Matlab that would produce the best approximation results.

Carmona-Durrleman Procedure

Carmona-Durrleman procedure is a rather new theory on spread option valuation, compared to other models. It is built on Geometric Brownian motion of the two underlying forward prices, and utilizes trigonometric functions to evaluate the option. As a result, a model similar to the Black-Scholes formula is sufficient to find the approximated option value. When strike price $K = 0$, this model reduces to the Magrabe model.

This model first defines a "lower bound approximation," Π , and proposes that the value of the spread option to be $sup\ \Pi(\theta, d)$, by optimizing over both θ and d. In order to find these two crucial parameters, we have to solve a system of two equations, namely the equation $\Pi' = 0$, and $\widehat{\Pi} = 0$,

$$\begin{cases} 0 & = & \alpha e^{-\beta\cos(\theta^*+\phi)\ d^*-\beta^2\cos^2(\theta^*+\phi)\ /2} - \gamma e^{-\delta\cos\theta^*\ d^*-\delta^2\cos^2\theta^*\ /2} - \kappa \\ 0 & = & \alpha\beta\sin(\theta^*+\phi)\ e^{-\beta\cos(\theta^*+\phi)\ d^*-\beta^2\cos^2(\theta^*+\phi)\ /2} - \gamma\delta\sin\theta^*\ e^{-\delta\cos\theta^*\ d^*-\delta^2\cos^2\theta^*\ /2}. \end{cases}$$

This system, according to the authors, can be further reduced to:

$$\begin{aligned} & \frac{1}{\delta\cos\theta}\ln\left(-\frac{\beta\kappa\sin(\theta+\phi)}{\gamma[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right)-\frac{\delta\cos\theta}{2} \\ = \;& \frac{1}{\beta\cos(\theta+\phi)}\ln\left(-\frac{\delta\kappa\sin\theta}{\alpha[\beta\sin(\theta+\phi)-\delta\sin\theta]}\right)-\frac{\beta\cos(\theta+\phi)}{2} \end{aligned}$$

As a result, the θ we derived from this equation shall be the θ that maximizes Π. To solve this equation, we first attempted to constrain θ to be within the range $[\pi, 2\pi]$. As it is possible that the logarithmic function to be 0 or even negative, we only evaluate the equation when the imaginary part is 0. Then, as we successfully find a set of θ that minimizes the above equation, we use this set of θ to compute d. Finally, we use the set of optimized d to compute Π, which now is the value of the spread option. Current challenge that we face is to correctly solve the system and derive θ and d, since this is our interpretation of the procedure. According to the paper, the authentic way of calculating each parameters remains unclear. We implemented the model in Matlab according to our interpretation of the paper. It functions properly but the result gives peculiar values for certain constraints. We has thus emailed the author regarding our concern and question. Further clarification and explanation is to be expected.

<u>Bachelier Model</u>

The premise of this pricing formula is based on the assumption that the dynamics of a spread follows an arithmetic Brownian motion, compared to the geometric Brownian motion model, which restricts the value of underlying asset to be positive. According to such assumption, the spread option price can be computed by evaluating Gaussian integrals, which can be represented by a closed form formula, similar to the Black-Scholes formula.

$$(4.4) \qquad p = \left(m(T) - Ke^{-rT}\right)\Phi\left(\frac{m(T)-Ke^{-rT}}{s(T)}\right) + s(T)\varphi\left(\frac{m(T)-Ke^{-rT}}{s(T)}\right),$$

where the functions $m(T)$ and $s(T)$ are defined by

$$m(T) = (x_2 - x_1)e^{(\mu-r)T}$$

and

$$(4.5) \quad s^2(T) = e^{2(\mu-r)T}\left[x_1^2\left(e^{\sigma_1^2 T}-1\right) - 2x_1x_2\left(e^{\rho\sigma_1\sigma_2 T}-1\right) + x_2^2\left(e^{\sigma_2^2 T}-1\right)\right].$$

Such closed form formula is only an approximation because our assumption says the distribution of $F_2(T) - F_1(T)$ follows the Gaussian distribution with the correct first two moments: expectation and variance. This suggests that the Bachelier Model may be accurate for certain set of parameters which can been illustrated by our numerical result in the appendix. We observe that the accuracy of the Bachelier approximation drifts away from our Monte Carlo simulation result when the strike price is large and the correlation between the two assets is closer to either 1 or -1.

III. Results & Conclusion

After developing the Pearson Model, Carmona-Durrleman Procedure, and Bachelier Model, we also implemented two other models namely the Kirk Model and Bjerksund and Stensland Model by using the built-in spread option functions in

MATLAB. To compare these five models numerically, we use the most standard spread option with W1 = -1 and W2 = 1. Simplifying our original payoff function, the option payoff at maturity becomes:

Payoff = Max(F2 - F1 – K, 0)

In our calculation, we have used forward prices F1 = 103.05, F2 = 112.22, annualized volatilities V1 = 15%, V2 = 10%, annualized risk free rate Rf = 5%, and time to maturity of the spread option T = 1. We then generate a table according to different combinations of strike K ranging from -20 to 25 and correlation C12 ranging from -1 to 1.

The results of these five models are presented in the Appendix. Table 1 shows the results of the spread option value approximation. The top number is obtained from the Monte Carlo simulation with 1,000,000 trials. The second, third, fourth, and fifth number represent the spread option value from the Pearson Integration, Bachelier Model, Kirk Approximation, and Bjerksund and Stensland Model respectively. Table 2 shows the approximation error associated with each model compared to the Monte Carlo Simulation. Table 3 shows the errors in percentage terms normalized by the size of the spread option value. Each line in Table 2 and Table 3 is produced by the following order: Pearson Integration, Bachelier Model, Kirk Approximation, and Bjerksund and Stensland Model. Observe that the Pearson Integration and Bachelier Model generally underprice the option except when the strike is large and correlation is close to 1. On the other hand, the Kirk Approximation seems to overprice the option when strikes is away from zero, and underprice when the strike is closer to zero. The Bjerksund and Stensland Model appears to always underprice the spread option except when incidence when strike is -20 and correlation is 1. Overall, observe that the Pearson approach is the most accurate model to our spread option price when the strike is high and the two assets are highly positively correlated. The Bjerksund and Stensland performs well when strike is negative and assets are negatively correlated. Note that we have excluded the Carmona-Durrleman Procedure result from our table because this procedure did poorly in numerical terms due the lack of an explicit solution.

IV. Further Tasks and Questions

One factor yet to be taken into consideration is the conversion factor. Based on our understanding of the spread option function, W1, W2 are considered to be conversion coefficients, such as the heat rate. However, because the desired spread option function does not serve to calculate one particular type of spread option in the energy market, we are still in the research process to figure out the correct W1 and W2.

We have compiled a list of questions for our next meeting.

1) It would be greatly helpful to understand the practical purpose of the spread option function, as we have replicated a few generic models with arbitrary inputs to calculate spread option price but does not take into consideration specific types of energy derivatives and their pricing dynamics. We wonder if we are required to fit any parameters in the model.
2) As the spread option function is nearly finished, we decided to proceed to the multi-factor model for forward curves. We wonder if we are required to create a general model for all underlying commodities, or rather specific ones on gas, oil and electricity. It would be great if you can provide more insight in constructing the multi-factor model.
3) We would also like to know what will be our deliverables for both the option spread model analysis and for the multi-factor model at the end of the project.

V. Appendix

Table 1. Testing results

	p = -1	-0.5	0	0.3	0.8	1
k = -20	30.35912347	29.63331737	28.94579916	28.59001869	28.22747008	28.20216754
	30.35969064	29.62155998	28.93022464	28.57314167	28.21463799	28.19310012
	30.33279448	29.60696468	28.92385046	28.57060277	28.21602054	28.19467866
	29.51477134	28.85709569	28.25818505	27.97038481	27.74982972	27.74736231
	30.40609092	29.67240427	28.97828044	28.61567954	28.23689963	28.20687625
	30.34279617	29.61714735	28.934182	28.58099863	28.22658387	28.20600744
-10	22.62982537	21.60253231	20.51109705	19.83976995	18.8558406	18.69277945
	22.63141318	21.60224256	20.50319128	19.82973331	18.84445506	18.68394843
	22.61139654	21.58300336	20.49294194	19.82395846	18.84454974	18.68538218
	21.75280647	20.78330004	19.75864625	19.13803839	18.30495208	18.23507194
	22.63970899	21.6101498	20.51841575	19.8479288	18.86193915	18.69338488
	22.61581303	21.5876036	20.49776725	19.82896388	18.84994478	18.69087847
0	15.92076083	14.64443764	13.17491763	12.16131613	10.1287946	9.273613785
	15.9119516	14.64987303	13.17470721	12.15870632	10.12364501	9.266029214
	15.90499184	14.62728179	13.16128402	12.14985262	10.12093754	9.267135133
	15.08312383	13.86539153	12.4592582	11.48297072	9.511049103	8.747744253
	15.90499196	14.62728179	13.16128402	12.14985262	10.12093754	9.267130951
	15.90499196	14.62728179	13.16128402	12.14985262	10.12093754	9.26713095
5	13.02393067	11.67107281	10.0829253	8.952113996	6.44375175	4.887530586
	13.01711588	11.67976877	10.08636489	8.952808043	6.441948547	4.887107018
	13.00799132	11.65652811	10.07225954	8.943212022	6.438032787	4.883817951
	12.23649316	10.95158183	9.436435678	8.3499928	5.91E+00	4.33E+00
	13.00447378	11.65269445	10.06805597	8.938683085	6.431581036	4.868988672
	13.0062733	11.65479888	10.07047745	8.941373598	6.435947361	4.88132272
15	8.234747071	6.870332244	5.27940448	4.14575396	1.623180678	0.104699802
	8.232130462	6.885359146	5.288377581	4.150910477	1.624984592	0.105297916
	8.216944584	6.863466486	5.275447554	4.142368946	1.621890634	0.104258001
	7.597636842	6.329785572	4.842545147	3.784638104	1.469769776	0.1682662
	8.227663703	6.87229756	5.282660907	4.148923601	1.630524847	0.133009736
	8.213031109	6.859658478	5.271765732	4.138795827	1.618810039	0.103056113
25	4.78039603	3.609946594	2.324771549	1.492791323	0.159995897	0
	4.780821589	3.621890813	2.331309655	1.496311396	0.160536194	0
	4.766122249	3.604451031	2.322220767	1.491182164	0.159946997	0
	4.331655838	3.266604652	2.099234578	1.351273231	0.173093954	0.000123581
	4.806531058	3.638367408	2.349081643	1.513244573	0.1697232	4.61E-09
	4.761691472	3.600363512	2.318589378	1.48796219	0.158253733	-7.89E-08

Note: In each cell, the first line of values is the price calculated from the optimized Pearson linear approximation. The second line value is from the Pearson Integration function. Third is from the Bachelier model. Fourth line is from the Matlab built-in Kirk Approximation function and fifth line is from the Matlab built-in Bjerksund and Stensland model.

Table 2. Percentage Errors

	p = -1	-0.5	0	0.3	0.8	1
k = -20	0.002%	-0.040%	-0.054%	-0.059%	-0.045%	-0.032%
	-0.087%	-0.089%	-0.076%	-0.068%	-0.041%	-0.027%
	-2.781%	-2.619%	-2.376%	-2.167%	-1.692%	-1.613%
	0.155%	0.132%	0.112%	0.090%	0.033%	0.017%
	-0.054%	-0.055%	-0.040%	-0.032%	-0.003%	0.014%
-10	0.007%	-0.001%	-0.039%	-0.051%	-0.060%	-0.047%
	-0.081%	-0.090%	-0.089%	-0.080%	-0.060%	-0.040%
	-3.876%	-3.792%	-3.669%	-3.537%	-2.922%	-2.449%
	0.044%	0.035%	0.036%	0.041%	0.032%	0.003%
	-0.062%	-0.069%	-0.065%	-0.054%	-0.031%	-0.010%
0	-0.055%	0.037%	-0.002%	-0.021%	-0.051%	-0.082%
	-0.099%	-0.117%	-0.103%	-0.094%	-0.078%	-0.070%
	-5.261%	-5.320%	-5.432%	-5.578%	-6.099%	-5.671%
	-0.099%	-0.117%	-0.103%	-0.094%	-0.078%	-0.070%
	-0.099%	-0.117%	-0.103%	-0.094%	-0.078%	-0.070%
5	-0.052%	0.075%	0.034%	0.008%	-0.028%	-0.009%
	-0.122%	-0.125%	-0.106%	-0.099%	-0.089%	-0.076%
	-6.046%	-6.165%	-6.412%	-6.726%	-8.324%	-11.360%
	-0.149%	-0.157%	-0.147%	-0.150%	-0.189%	-0.379%
	-0.136%	-0.139%	-0.123%	-0.120%	-0.121%	-0.127%
15	-0.032%	0.219%	0.170%	0.124%	0.111%	0.571%
	-0.216%	-0.100%	-0.075%	-0.082%	-0.079%	-0.422%
	-7.737%	-7.868%	-8.275%	-8.710%	-9.451%	60.713%
	-0.086%	0.029%	0.062%	0.076%	0.452%	27.039%
	-0.264%	-0.155%	-0.145%	-0.168%	-0.269%	-1.570%
25	0.009%	0.331%	0.281%	0.236%	0.338%	
	-0.299%	-0.152%	-0.110%	-0.108%	-0.031%	
	-9.387%	-9.511%	-9.701%	-9.480%	8.186%	
	0.547%	0.787%	1.046%	1.370%	6.080%	
	-0.391%	-0.265%	-0.266%	-0.323%	-1.089%	

Note: the percentage error of spread option price for strike 25 and correlation 1 is not represented in this table because our Monte Carlo result has a value of 0, thus making the calculation result undefined when the denominator is zero. In other words, the spread (F2-F1) has never exceeded the strike price of 25.