International Journal of Theoretical and Applied Finance
Vol. 12, No. 4 (2009) 491–506

PRICING OF EXOTIC ENERGY DERIVATIVES BASED ON ARITHMETIC SPOT MODELS

FRED ESPEN BENTH* and RODWELL KUFAKUNESU†

*Centre of Mathematics for Applications (CMA)
University of Oslo, P.O. Box 1053, Blindern
N0316 Oslo, Norway
fredb@math.uio.no

†University of Pretoria
Department of Mathematics and Applied Mathematics
Pretoria 0002, South Africa
rodwell.kufakunesu@up.ac.za

Received 4 September 2007
Revised 6 October 2008

Based on a non-Gaussian Ornstein–Uhlenbeck model for energy spot, we derive prices for Asian and spread options using Fourier techniques. The option prices are expressed in terms of the Fourier transform of the payoff function and the characteristic functions of the driving noises, being independent increment processes. In many relevant situations, these functions are explicitly available, and fast Fourier transform can be used for efficient numerical valuation. The arithmetic nature of our model implies that only a one-dimensional Fourier transform is required in the computation of the price, contrary to geometric models where transformation along both underlying variables is necessary.

Keywords: Energy markets; spread options; Asian options; fast Fourier transform; non-Gaussian Ornstein–Uhlenbeck processes; independent increment processes.

1. Introduction

The energy markets are trading exotic derivatives in large volumes. Except for some products, like crack spreads on NYMEX (spread between two refined oil products) and an Asian option on the spot of electricity traded at Nord Pool for a short period in the 90s, the market for these derivatives are mainly OTC. The class of spread contracts is huge, including important examples like the spark and dark spread. The spark spread is the difference between spot electricity and gas, whereas "dark" refers to the difference between electricity and coal. In this paper, we analyze the pricing of Asian and spread options for a class of arithmetic spot price models.

The price dynamics in typical energy markets possesses several specific features that must be accounted for in modeling, and which complicate the valuation of derivatives. Prices may show a strong seasonal influence, say due to a high demand for electricity or gas for heating in the winter. The prices are mean-reverting, but with significant spikes occurring possibly seasonally. These spikes may, in electricity markets say, occur due to a sudden drop in temperature not matched by sufficient production volume, or an unexpected shut down of a power plant. To get reliable prices on derivatives, one needs to have accessible spot price models which incorporate the stylized facts of energy prices. Furthermore, it is desirable to have models which are analytically tractable for derivatives pricing.

We consider a non-Gaussian Ornstein–Uhlenbeck (OU) model for the spot price dynamics of energy. The model was proposed by Benth *et al.* [2], and includes seasonality of prices, mean-reversion at different speeds and seasonal spikes. The model is arithmetic, however, defined to ensure the positivity of spot prices, a desirable feature in the market. In [2], analytical prices for energy forwards are derived, along with expressions for the price of call and put options on these forwards based on the characteristic functions of the stochastic processes driving the spot dynamics.

Here the analysis in [2] is generalized to cover average-type options and spread contracts of various kinds (including baskets). Issues of pricing using Fast Fourier transform (FFT) are discussed. It turns out that the spot price model is very flexible, and easily computable expressions come out for these exotic derivatives. This is a direct consequence of the arithmetic nature of the underlying spot process. Interestingly, for spread options, we can model bivariately the two energies in question, but still have a one-dimensional FFT problem to solve. Using geometric models, spread options inevitably lead to two-dimensional pricing problems, which in general are rather hard to valuate using FFT methods.

Margrabe's formula [15] gives an explicit expression for the price of a call option with zero strike on the spread between two financial assets following geometric Brownian motions. On the other hand, if the strike is different than zero, no formula is known. Recent and past methods to value spread options in energy and commodity markets have been thoroughly discussed by Carmona and Durrleman [6]. They present a detailed study of pricing of spreads based on diffusion models, and in particular approximation formulas are provided. Further, this article is also an excellent introduction to the zoology of spread options traded in the market. Dempster and Hong [8] propose and analyze a FFT method for spread options based on a two-dimensional geometric Brownian motion and stochastic volatility. Their method involves some ingenious approximations leading to expressions computable by the FFT. The approximations give tight bounds for the exact price. Benth and Šaltytė-Benth [5] have proposed to model the spark spread directly using an arithmetic jump-diffusion model. This idea is motivated by the findings in [6], and gives rise to option pricing formulas feasible for FFT methods. Recently, Benth and Kettler [4] have used copula theory to model the spark spread, and Monte Carlo simulated the price of put and call options.

It is known that Asian options written on a spot modeled by geometric Brownian motion does not allow for an explicit pricing formula. Many authors have investigated possible approaches to the pricing problem of Asian options, and we refer to the analysis of these in Geman [11], where the relevance and use of such options for energy and commodity markets are discussed. In particular, Geman and Yor [13] derive explicit representations for the Laplace transform being suitable for numerical valuation. Weron [18] considers the pricing of Asian options in the electricity market, more specifically, he numerically prices the options traded for some years at the Nord Pool, the Nordic Power Exchange. With an arithmetic jump-diffusion model, where the spot may become negative, he simulates prices for such options and investigates the implied market price of risk. Recently, Fusai *et al.* [10] derived a formula for the price of discretely monitored Asian options, with an underlying spot price process described by square-root dynamics. The formula is explicitly suitable for numerical valuation through Laplace transformation techniques.

Our results are presented as follows. In the next section, we introduce the spot price model which will be the basis for our analysis, together with some assumptions and properties. Next, in Sec. 3, Asian options are analyzed, and expressions feasible for FFT valuation are derived. The following section focuses on spread options, where we also discuss modeling issues connected to spreads of energy spots. Finally, in the last section, we conclude.

2. An Arithmetic Price Model for the Spot Price

Benth *et al.* [2] proposed to model the electricity spot price by a non-Gaussian OU model, which we introduce in this section. In connection with spread options, we will generalize to a bivariate version of the arithmetic model (see Sec. 4).

Let $(\Omega, P, \mathcal{F}, \{\mathcal{F}\}_{t\in[0,T]})$ be a complete filtered probability space, with $T < \infty$ a fixed finite time horizon. Let $S(t)$ denote the spot price of electricity at time t, and assume that

$$S(t) = \Lambda(t) + \sum_{j=1}^{n} Y_j(t), \tag{2.1}$$

where Λ is a deterministic seasonal floor function, and

$$dY_j(t) = -\lambda_j(t)\, Y_j(t)\, dt + \sigma_j(t)\, dL_j(t).$$

Here, L_j is an inhomogeneous subordinator, that is, an independent increment process with only positive jumps and no continuous martingale part. We refer to Jacod and Shiryaev [14] for a deep analysis of such processes and their properties. Furthermore, λ_j and σ_j, $j = 1, \ldots, n$, are positive, continuous functions on $[0, T]$. In [2], the speed of mean-reversion parameters λ_j are supposed to be constant, however, we let them be time-dependent here for the sake of generality.

An explicit representation of $Y_j(u)$ for $u \geq t \geq 0$ is

$$Y_j(u) = Y_j(t)\mathrm{e}^{-\int_t^u \lambda_j(s)\, ds} + \int_t^u \sigma_j(s)\mathrm{e}^{-\int_s^u \lambda_j(v)\, dv}\, dL_j(s). \tag{2.2}$$

All the OU processes will mean-revert toward zero, implying that the spot price mean-reverts to the seasonal floor function Λ. The *expected* spot price (which we may interpret as the seasonal function) will be a sum of the expectations of Y_j and the floor. Since Y_j will be positive, the spot price will become positive by definition if the floor function is positive. We suppose this to be the case. The idea now is to let some of the Y_j's account for the "normal" variations in the market, while one or two other represent the spikes. These have large jumps possibly with seasonally dependent intensity followed by a strong mean-reversion, whereas the normal variations are typically stationary OU-processes with more moderate jump sizes and mean-reversion. We refer to [2] for more issues around modeling and estimation of this spot price process.

The main advantage with this model, as shown by [2], is that one can analytically derive forward prices for contracts with delivery of the spot over a period rather than at a fixed future time point. The expressions involve the characteristic functions of the subordinators L_j. These are the typical forward contracts of the electricity and gas markets, where also plain vanilla call and put options on these are traded.

The independent increment processes L_j have a deterministic compensator measure $\ell_j(dz, ds)$ on the Borel sets of $\mathbb{R}\backslash\{0\} \times [0, T]$. Note that in the case of a Lévy process, it reduces to $\ell_j(dz, ds) = \ell_j(dz)\, ds$. Further, the cumulant function for a real-valued continuous function ϕ on $[0, T]$ is

$$
\begin{aligned}
\psi_j(t, T; \phi(\cdot)) &\triangleq \ln \mathbb{E}\left[\exp\left(\mathrm{i}\int_t^T \phi(s)\, dL_j(s)\right)\right] \\
&= \mathrm{i}\int_t^T \phi(s)\, d\gamma_j(s) + \int_t^T \int_{[0,\infty)} \{\mathrm{e}^{\mathrm{i}\phi(s)z} - 1\}\, \ell_j(dz, ds),
\end{aligned}
$$

with $\gamma_j(s)$ being the drift of the process L_j. When pricing derivatives, one must introduce a risk-neutral probability. We define an equivalent measure Q^θ using the Esscher transform as follows. Let $\theta \triangleq (\theta_1, \ldots, \theta_n)$ be an $\mathbb{R}^n$-valued continuous function on $[0, T]$, such that the density process of the Radon–Nikodym derivative of Q^θ with respect to P is

$$
\mathbf{Z}^\theta(t) \triangleq Z_1^\theta(t) \times \cdots \times Z_n^\theta(t),
$$

with,

$$
Z_j^\theta(t) = \exp\left(\int_0^t \theta_j(s)\, dL_j(s) - \psi_j(0, t; -\mathrm{i}\theta_j(\cdot))\right). \tag{2.3}
$$

The function θ is often referred to as the market price of risk, since it essentially introduces an additional drift in the dynamics of the Y_j's. This is translated into the spot price as a change of the floor function, or equivalently, the level to which the spot price is mean-reverting. Frequently, one chooses θ to be constant with respect to time. The Esscher transform is only well-defined under certain exponential integrability conditions on the compensator measures $\ell_j(dz, ds)$.

Condition (**I**): There exists a constant $\kappa_j > 0$ such that the compensator measure $\ell_j(dz, ds)$ satisfies the integrability condition

$$\int_0^T \int_1^\infty \mathrm{e}^{z\kappa_j}\, \ell_j(dz, ds) < \infty.$$

The process $Z_j^\theta(t)$ is well-defined for $t \in [0, T]$ as long as Condition (**I**) holds for $\kappa_j \triangleq \sup_{s\in[0,T]} |\theta_j(s)|$ for each $j = 1, \ldots, n$. We restrict our attention to functions θ for which this holds. In the rest of this article, we suppose that a fixed Q^θ is given, and shall understand with the notation $\mathbb{E}_\theta[\cdot]$ the expectation operator with respect to Q^θ.

We note in passing that contrary to most other financial markets, it is not necessary to require that the discounted energy spot price is a (local) martingale under the risk-neutral measure. The reason is that energies in general are far from tradable in the classical sense, since in these markets frictions like storage costs and transportation play a significant role. The most extreme example is the electricity market, where the spot is not storable, except indirectly in hydro reservoirs (for hydro power plants).

By an extension of the arguments in Theorems 1 and 2 on page 685 in Shiryaev [17], it follows that L_j is still an independent increment process under Q^θ and the cumulant function becomes (for ϕ being a continuous function on $[0, T]$),

$$\ln \mathbb{E}_\theta\left[\exp\left(\mathrm{i}\int_t^T \phi(s)\, dL_j(s)\right)\right] = \psi_j(t, T; \phi(\cdot) - \mathrm{i}\theta_j(\cdot)) - \psi_j(t, T; -\mathrm{i}\theta_j(\cdot)). \quad (2.4)$$

We shall frequently use this formula in the derivations below.

Our analysis is based on Fourier transforms. For a real-valued function $f \in L^1(\mathbb{R})$, the space of Lebesgue integrable functions on $\mathbb{R}$, we define the Fourier transform as

$$\widehat{f}(y) \triangleq \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}xy}\, dx, \quad (2.5)$$

while

$$f(x) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(y)\mathrm{e}^{\mathrm{i}xy}\, dy, \quad (2.6)$$

is the inverse Fourier transform (the definitions are taken from Chap. 8 in Folland [9]).

3. Asian Options

We consider average-type options, or Asian options, written on an energy spot price $S(t)$ with the dynamics (2.1). Suppose the option pays $g\big(\int_\tau^T S(u)\, du\big)$, $\tau < T$, at maturity T, with $g \in L^1(\mathbb{R}_+)$, the space of integrable functions on $\mathbb{R}_+$. By recalling that the spot price is positive, only payoff functions on the positive real axis are relevant. We extend the function g to an integrable function on $\mathbb{R}$ by letting $g(x) = 0$

for $x < 0$. The risk-free interest rate is denoted $r > 0$, and assumed to be a constant for simplicity.

The price $C(t)$ of the option at time t is defined from the arbitrage theory to be

$$C(t) = \mathrm{e}^{-r(T-t)}\mathbb{E}_\theta \left[g\left(\int_\tau^T S(u)\,du \right) \middle| \mathcal{F}_t \right]. \tag{3.1}$$

For simplicity, we restrict our attention to Asian options which are traded up to the start of the averaging period, that is, we derive a price process $C(t)$ for $t \leq \tau$. Note that this corresponds to contracts which do not trade within the averaging period, but terminate trading when this starts. The price can be expressed in terms of characteristic functions, as the following Proposition shows:

Proposition 3.1. *The price* $C(t)$ *in Eq.* (3.1) *is given as*

$$C(t) = \frac{\mathrm{e}^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \widehat{g}(y)\Psi(t,\tau,T,y,\theta)\,dy,$$

where Ψ *is defined by*

$$\begin{aligned}
\ln \Psi(t,\tau,T,y,\theta) = \mathrm{i}y \int_\tau^T \Lambda(u)\,du &+ \mathrm{i}y \sum_{j=1}^n Y_j(t) \int_\tau^T \mathrm{e}^{-\int_t^u \lambda_j(v)\,dv}\,du \\
&+ \sum_{j=1}^n \psi_j\left(t, T; y\sigma_j(\cdot) \int_{\max(\cdot,\tau)}^T \mathrm{e}^{-\int_\cdot^u \lambda_j(v)\,dv}\,du \right. \\
&\left. - \mathrm{i}\theta_j(\cdot)\right) - \psi_j(t,T;-\mathrm{i}\theta_j(\cdot)).
\end{aligned}$$

Proof. Using the Fourier transform, we find

$$\mathbb{E}_\theta \left[g\left(\int_\tau^T S(u)\,du \right) \middle| \mathcal{F}_t \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(y) \mathbb{E}_\theta \left[\exp\left(\mathrm{i}y \int_\tau^T S(u)\,du \right) \middle| \mathcal{F}_t \right] dy.$$

We now calculate the conditional expectation in the expression for the inverse Fourier transform. Recall that for $u \geq t$

$$Y_j(u) = Y_j(t)\mathrm{e}^{-\int_t^u \lambda_j(v)\,dv} + \int_t^u \sigma_j(s)\mathrm{e}^{-\int_s^u \lambda_j(v)\,dv}\,dL_j(s).$$

Thus, from the stochastic Fubini theorem,

$$\begin{aligned}
\int_\tau^T Y_j(u)\,du &= Y_j(t) \int_\tau^T \mathrm{e}^{-\int_t^u \lambda_j(v)\,dv}\,du \\
&\quad + \int_\tau^T \int_t^T 1_{[t,u]}(s)\sigma_j(s)\mathrm{e}^{-\int_s^u \lambda_j(v)\,dv}\,dL_j(s)\,du \\
&= Y_j(t) \int_\tau^T \mathrm{e}^{-\int_t^u \lambda_j(v)\,dv}\,du + \int_t^T \sigma_j(s) \int_{\max(s,\tau)}^T \mathrm{e}^{-\int_s^u \lambda_j(v)\,dv}\,du\,dL_j(s).
\end{aligned}$$

From the $\mathcal{F}_t$-measurability of $Y_j(t)$, and the independent increment property of L_j preserved under Q^θ, we find

$$\ln \mathbb{E}_\theta\left[\exp\left(\mathrm{i}y\int_\tau^T S(u)\,du\right)\middle|\,\mathcal{F}_t\right]$$

$$= \mathrm{i}y\int_\tau^T \Lambda(u)\,du + \mathrm{i}y\sum_{j=1}^n Y_j(t)\int_\tau^T \mathrm{e}^{-\int_t^u \lambda_j(v)\,dv}\,du$$

$$+\sum_{j=1}^n \ln\mathbb{E}_\theta\left[\exp\left(\mathrm{i}y\int_t^T \sigma_j(s)\int_{\max(s,\tau)}^T \mathrm{e}^{-\int_s^u \lambda_j(v)\,dv}\,du\,dL_j(s)\right)\right].$$

After appealing to Eq. (2.4), the proof is complete. □

Typical examples of Asian options are plain vanilla put and call options written on the average of the spot. Consider a put option which has a payoff function $g(x) = \max(K - x, 0)$ for $x \geq 0$. We observe that by extending g with $g(x) = 0$ for $x < 0$ defines an integrable function on $\mathbb{R}$, and we can invoke the result in Prop. 3.1 to price options. Normally, the put options (and calls as well) are written on the average of the spot, which means that the above function g must be modified slightly in its definition. A straightforward calculation shows that the Fourier transform of g in the put option case is

$$\widehat{g}(y) = \frac{\mathrm{i}yK - \mathrm{e}^{-\mathrm{i}yK} + 1}{y^2}.$$

An FFT algorithm is thus readily available to calculate expressions like $C(t)$ efficiently on a computer for put options, as long as we know the cumulant functions ψ_j. We note that an appropriate numerical integration must be invoked in the calculations of the terms.

A call option will have a payoff function $g(x) = \max(x - K, 0)$ for $x \geq 0$, with the extension $g(x) = 0$ on the negative half of the real line. This is obviously not integrable on $\mathbb{R}$ due to the linear growth. Hence, we cannot use the Prop. 3.1 directly to price calls. In Carr and Madan [7], it is suggested to dampen the payoff function with an exponential function in order to obtain something integrable. We shall look at this in the next section, where we consider spread options. Alternatively, one may appeal to the put–call parity to price call options.

3.1. *A specific model for spot electricity*

We look at an example of a spot model, which has been considered in [2] in the context of electricity markets. Let the spot price be given by Eq. (2.1) with $n = 2$ and a seasonal floor

$$\Lambda(t) = a + b \times t + c\sin(2\pi(t - d)/365),$$

for constants a, b, c and d. As can be seen from the seasonal floor function, we suppose 365 trading days yearly in the market. Furthermore, we assume that the process Y_1 accounts for the "normal" variations in the market, while Y_2 models the spiky behavior of spot prices. The processes L_1 and L_2 are supposed to be a subordinator and a time-inhomogeneous compound Poisson process, respectively. To get spikes, Y_2 has a fast speed of mean-reversion. On the other hand, the process Y_1 reverts to zero at a much slower rate. Parameters derived in connection with a study of Nord Pool data are,[1]

$$\Lambda(t) = 100 + 0.025 \times t + 30 \sin(2\pi t/365),$$

whereas the speeds of mean-reversions are $\lambda_1 = 0.0846$ and $\lambda_2 = 1.12$. The stationary distribution of Y_1 is $\Gamma(\nu, \alpha)$, with $\nu = 8.055$ and $\alpha = 0.132$. The seasonal intensity function of the inhomogeneous compound Poisson process L_2 is chosen as

$$\lambda(t) = \frac{0.14}{\left|\sin\left(\frac{\pi(t-90)}{365}\right)\right| + 1} - 1. \tag{3.2}$$

The idea to use this seasonal intensity function is taken from Geman and Roncoroni [12], who apply it in an empirical study of several different electricity markets worldwide. They propose a parametric family of intensity functions, for which Eq. (3.2) is a special case. The jump sizes are exponentially distributed with mean set to 180.

A sketch of the estimation procedure which leads to these estimates goes as follows. First, we specify a seasonal floor Λ, to which the spot price is mean-reverting. The floor can be found by fitting the deterministic function to data, and next moving the whole function downward to assure that the price observations less the seasonal function are all positive. These are called "deseasonalized" spot prices. Next, one may identify the spikes by some filtering procedure. The data cleaned of spikes should be stationary, and by comparing the empirical autocorrelation function with the theoretical for Y_1, we can estimate the speed of mean-reversion. Furthermore, again due to stationarity of Y_1, we can identify and estimate the distribution of L_1. From the filtered spikes, one can reconstruct a time series which Y_2 models, and thus estimate the speed of mean-reversion and the jump characteristics of L_2.

Observe that for Y_1, we determine the stationary distribution rather than the distribution of the subordinator L_1. Since the Gamma-distribution is self-decomposable, there exists a so-called *background driving Lévy process* L_1 such that Y_1 has the desired $\Gamma(\nu, \alpha)$ marginal distribution (see for e.g., Barndorff–Nielsen and Shephard [1]). The Lévy measure of the subordinator $L_1(t)$ becomes

$$\ell_1(dz) = \nu\alpha \exp(-\alpha z)\, dz.$$

[1]The study was done by Thilo Meyer–Brandis, to whom we are indebted for providing the estimates.

Knowing the Lévy measure is sufficient for simulating paths of L_1 and Y_1. We refer to [1] for a discussion of Monte Carlo simulation of the paths. Furthermore, relevant for our analysis is that as long as we have the Lévy measure explicitly accessible, we can calculate the cumulant function associated to L_1 through the Lévy–Kintchine formula. Moreover, we have that

$$\psi_1(t,T;\phi(\cdot)) = \int_t^T \int_0^\infty (\mathrm{e}^{\mathrm{i}\phi(s)z} - 1)\,\ell(z)\,dz\,ds = \int_t^T \frac{\nu\alpha}{\alpha - \mathrm{i}\phi(s)} - \nu\,ds.$$

The cumulant function of L_2 is given by

$$\psi_2(t,T;\phi(\cdot)) = \int_t^T \lambda(s)(\mathrm{e}^{\psi(\phi(s))} - 1)\,ds,$$

where $\psi(\phi)$ is the cumulant function of the exponential distribution assumed for the jump size. A numerical integration routine is required for evaluating these cumulant functions.

4. Spread Options

We move on to analyze prices of spread options in energy markets. Since spreads involve (usually) two energies, we first extend the spot model (2.1) to a bivariate case, where we include both unique and common risk. Some properties of this model are studied before we calculate the price of spread options, including the case of baskets.

Consider two energies A and B, with spot price dynamics defined by

$$S^A(t) = \Lambda^A(t) + \sum_{i=1}^m X_i^A(t) + \sum_{j=1}^n Y_j^A(t) \tag{4.1}$$

$$S^B(t) = \Lambda^B(t) + \sum_{i=1}^m X_i^B(t) + \sum_{j=1}^n Y_j^B(t). \tag{4.2}$$

We suppose that the first m factors X_i^A and X_i^B are common, in the sense that the OU processes are driven by the same jump processes. Hence, we suppose

$$dX_i^A(t) = -\alpha_i^A X_i^A(t)\,dt + \sigma_i^A\,dL_i(t)$$

$$dX_i^B(t) = -\alpha_i^B X_i^B(t)\,dt + \sigma_i^B\,dL_i(t),$$

where $\alpha_i^A, \alpha_i^B, \sigma_i^A$ and σ_i^B are positive constants, and the independent increment processes L_i, $i = 1, \ldots, m$ are independent. Note that in order to have slightly simpler expressions (without too many integrals involved), we have dispensed with the general time-dependent mean-reversion and variation coefficients α's and σ's and consider only constants. Further, we let

$$dY_j^A(t) = -\beta_j^A Y_j^A(t)\,dt + \eta_j^A\,dL_j^A(t)$$

$$dY_j^B(t) = -\beta_j^B Y_j^B(t)\,dt + \eta_j^B\,dL_j^B(t),$$

with all the parameters again being assumed positive constants and where L_j^A and L_j^B are independent increment processes being mutually independent. Moreover, all the independent increment processes involved have positive jumps only, being inhomogeneous subordinators. We suppose further that Λ^A and Λ^B are positive, so that the price process for each energy is positive. Condition (**I**) is naturally extended for all compensator measures $\ell_i(dz, ds)$ and $\ell_j^{A,B}(dz, ds)$, for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

The two spot price dynamics in Eq. (4.1) imply a rather explicit correlation structure. The exact form of this is stated in the following proposition.

Proposition 4.1. *Suppose that Condition* (**I**) *holds for constants κ bigger than or equal to 2 for all the compensator measures. Then the covariance between $S^A(t)$ and $S^B(t)$ is*

$$\mathrm{Cov}(S^A(t), S^B(t)) = \sum_{i=1}^m \sigma_i^A \sigma_i^B \int_0^t \int_{\mathbb{R}} z^2 \mathrm{e}^{-(\alpha_i^A + \alpha_i^B)(t-s)} \, \ell_i(dz, ds).$$

Proof. First, recall the explicit solutions of X and Y as

$$X_i(t) = X_i(0)\mathrm{e}^{-\alpha_i t} + \int_0^t \sigma_i \mathrm{e}^{-\alpha_i(t-s)} \, dL_i(s)$$

$$Y_j(t) = Y_j(0)\mathrm{e}^{-\beta_j t} + \int_0^t \eta_j \mathrm{e}^{-\beta_j(t-s)} \, dU_j(s),$$

where for a moment we have skipped the superscripts A and B and write U_j for the subordinators $L_j^{A,B}$ in order to avoid confusing notation. Imposing Condition (**I**), we compensate L_i and U_j and obtain

$$X_i(t) = X_i(0)\mathrm{e}^{-\alpha_i t} + \mu_i^X(t) + \int_0^t \sigma_i \mathrm{e}^{-\alpha_i(t-s)} \, d\widetilde{L}_i(s)$$

$$Y_j(t) = Y_j(0)\mathrm{e}^{-\beta_j t} + \mu_j^Y(t) + \int_0^t \eta_j \mathrm{e}^{-\beta_j(t-s)} \, d\widetilde{U}_j(s),$$

where $\widetilde{L}_i$ and $\widetilde{U}_j$ are martingales, and μ_i^X and μ_j^Y are (deterministic) drift terms induced by the compensation. By using the independence of the different jump terms, we find

$$\begin{aligned}\mathrm{Cov}(S^A(t), S^B(t)) &= \mathbb{E}\left[\left(\sum_{i=1}^m \int_0^t \sigma_i^A \mathrm{e}^{-\alpha_i^A(t-s)} \, d\widetilde{L}_i(s)\right)\right.\\ &\quad \left.\times \left(\sum_{i=1}^m \int_0^t \sigma_i^B \mathrm{e}^{-\alpha_i^B(t-s)} \, d\widetilde{L}_i(s)\right)\right]\\ &= \sum_{i=1}^m \mathbb{E}\left[\int_0^t \sigma_i^A \mathrm{e}^{-\alpha_i^A(t-s)} \, d\widetilde{L}_i(s) \int_0^t \sigma_i^B \mathrm{e}^{-\alpha_i^B(t-s)} \, d\widetilde{L}_i(s)\right].\end{aligned}$$

Hence, the proposition is proved. $\square$

If the common factors L_i, $i = 1, \ldots, m$ are subordinators, we have

$$\mathrm{Cov}(S^A(t), S^B(t)) = \sum_{i=1}^{m} \frac{\sigma_i^A \sigma_i^B}{\alpha_i^A + \alpha_i^B}(1 - \mathrm{e}^{-(\alpha_i^A + \alpha_i^B)t}) \int_{\mathbb{R}} z^2 \,\ell_i(dz),$$

since the compensator measures then are $\ell_i(dz, ds) = \ell_i(dz)\,ds$. Letting $t \to \infty$, we derive a stationary covariance function

$$\mathrm{Cov}(S^A(t), S^B(t)) \approx \sum_{i=1}^{m} \frac{\sigma_i^A \sigma_i^B}{\alpha_i^A + \alpha_i^B} \int_{\mathbb{R}} z^2 \,\ell_i(dz).$$

This can be exploited when calibrating the spot models to actual data.

We now move on to analyse the price of European options written on the spread. It is assumed that we have chosen a risk-neutral measure Q^θ, where

$$\theta \triangleq (\theta_1, \ldots, \theta_m, \theta_1^A, \ldots, \theta_n^A, \theta_1^B, \ldots, \theta_n^B),$$

and the definition of the density process $\mathbf{Z}^\theta(t)$ is extended in the natural way. Condition (**I**) is supposed to hold for

$$\kappa_i = \sup_{s \in [0,T]} |\theta_i(s)|\,, \quad i = 1, \ldots, m,$$

and

$$\kappa_j^{A,B} = \sup_{s \in [0,T]} |\theta_j^{A,B}(s)|\,, \quad j = 1, \ldots, n.$$

This specifies the risk-neutral meausure to be used for pricing.

We consider contracts written on the linear combination

$$S(t) \triangleq aS^A(t) + bS^B(t), \tag{4.3}$$

with a and b being two constants. Hence, we in fact consider options written on a basket of two energies, being slightly more general than merely restricting our attention to spreads. Typical examples include spark and dark spreads, where $a = 1$ and $b = -c$, with c being a constant converting energy B into the units of A. For the spark spread, A is electricity and B is gas, whereas B is coal for the dark spread. The constant c is the *heat rate*. If we let $a = b = 0.5$, we have the average price of two energies.

Consider a European option with exercise time T and payoff $g(S(T))$. Contrary to the Asian option case studied in the previous section, we can have that $S(T)$ is negative (specifically for spreads this is the case). Hence, we suppose that g is defined on the whole real line, with the condition $g \in L^1(\mathbb{R})$. The price $C(t)$ at time t of the option is defined by

$$C(t) = \mathrm{e}^{-r(T-t)}\mathbb{E}_\theta[g(S(T)) \,|\, \mathcal{F}_t]. \tag{4.4}$$

The price may be expressed in terms of the cumulant functions of the jump processes, as seen in the following proposition:

Proposition 4.2. *The price $C(t)$ of a European option with payoff $g(S(T))$ at exercise time $T \geq t$ is*

$$C(t) = \frac{e^{-r(T-t)}}{2\pi} \int_{\mathbb{R}} \widehat{g}(y)\Psi(t, T, y, \theta)\, dy,$$

where

$$\begin{aligned}
\ln \Psi(t, T, y, \theta) = {} & \mathrm{i}y(a\Lambda^A(T) + b\Lambda^B(T)) \\
& + \mathrm{i}y \sum_{i=1}^{m} aX_i^A(t)\mathrm{e}^{-\alpha_i^A(T-t)} + bX_i^B(t)\mathrm{e}^{-\alpha_i^B(T-t)} \\
& + \mathrm{i}y \sum_{j=1}^{n} aY_j^A(t)\mathrm{e}^{-\beta_j^A(T-t)} + bY_j^B(t)\mathrm{e}^{-\beta_j^B(T-t)} \\
& + \sum_{i=1}^{m} \psi_i(t, T; y(a\sigma_i^A \mathrm{e}^{-\alpha_i^A(T-\cdot)} + b\sigma_i^B \mathrm{e}^{-\alpha_i^B(T-\cdot)} \\
& - \mathrm{i}\theta_i(\cdot))) - \psi_i(t, T; -\mathrm{i}\theta_i(\cdot)) \\
& + \sum_{j=1}^{n} \psi_j^A(t, T; y(a\eta_j^A \mathrm{e}^{-\beta_j^A(T-\cdot)} - \mathrm{i}\theta_j^A(\cdot))) - \psi_j^A(t, T; -\mathrm{i}\theta_j^A(\cdot)) \\
& + \sum_{j=1}^{n} \psi_j^B(t, T; y(a\eta_j^B \mathrm{e}^{-\beta_j^B(T-\cdot)} - \mathrm{i}\theta_j^B(\cdot))) - \psi_j^B(t, T; -\mathrm{i}\theta_j^B(\cdot)).
\end{aligned}$$

Here, ψ_i, $i = 1, \ldots, m$ are the cumulant functions of L_i, while ψ_j^A and ψ_j^B are the cumulant functions of L_j^A and L_j^B, respectively, for $j = 1, \ldots, n$.

Proof. By definition of the inverse Fourier transform, we have

$$\mathbb{E}_\theta[g(S(T)) \,|\, \mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(y)\mathbb{E}_\theta[\mathrm{e}^{\mathrm{i}yS(T)} \,|\, \mathcal{F}_t]\, dy,$$

where $\widehat{g}$ is the Fourier transform of the payoff function. We must calculate the characteristic function of $S(T)$, and show that this coincides with $\Psi(t, T, y, \theta)$ defined in the proposition.

From the explicit dynamics of the OU processes, we have

$$\begin{aligned}
S(T) = {} & a\Lambda^A(T) + b\Lambda^B(T) + \sum_{i=1}^{m} aX_i^A(t)\mathrm{e}^{-\alpha_i^A(T-t)} + bX_i^B \mathrm{e}^{-\alpha_i^B(T-t)} \\
& + \sum_{j=1}^{n} aY_j^A(t)\mathrm{e}^{-\beta_j^A(T-t)} + bY_j^B(t)\mathrm{e}^{-\beta_j^B(T-t)}
\end{aligned}$$

$$+\sum_{i=1}^{m}\int_t^T (a\sigma_i^A \mathrm{e}^{-\alpha_i^A(T-s)} + b\sigma_i^B \mathrm{e}^{-\alpha_i^B(T-s)})\, dL_i(s)$$

$$+\sum_{j=1}^{n}\int_t^T a\eta_j^A \mathrm{e}^{-\beta_j^A(T-s)}\, dL_j^A(s) + \int_t^T b\eta_j^B \mathrm{e}^{-\beta_j^B(T-s)}\, dL_j^B(s).$$

Recall that all the involved independent increment processes will remain independent increment processes under Q^θ, and moreover, they are mutually independent. Further, $X_i^A(t), X_i^B(t), Y_j^A(t)$ and $Y_j^B(t)$ are all $\mathcal{F}_t$-measurable. Hence,

$$\begin{aligned}
&\ln \mathbb{E}_\theta[\mathrm{e}^{\mathrm{i}yS(T)} \mid \mathcal{F}_t] \\
&\quad = \mathrm{i}y(a\Lambda^A(T) + b\Lambda^B(T)) \\
&\qquad + \mathrm{i}y\sum_{i=1}^{m} aX_i^A(t)\mathrm{e}^{-\alpha_i^A(T-t)} + bX_i^B \mathrm{e}^{-\alpha_i^B(T-t)} \\
&\qquad + \mathrm{i}y\sum_{j=1}^{n} aY_j^A(t)\mathrm{e}^{-\beta_j^A(T-t)} + bY_j^B(t)\mathrm{e}^{-\beta_j^B(T-t)} \\
&\qquad + \sum_{i=1}^{m}\ln \mathbb{E}_\theta\left[\exp\left(\mathrm{i}y\int_t^T (a\sigma_i^A \mathrm{e}^{-\alpha_i^A(T-s)} + b\sigma_i^B \mathrm{e}^{-\alpha_i^B(T-s)})\, dL_i(s)\right)\right] \\
&\qquad + \sum_{j=1}^{n}\ln \mathbb{E}_\theta\left[\exp\left(\mathrm{i}y\int_t^T a\eta_j^A \mathrm{e}^{-\beta_j^A(T-s)}\, dL_j^A(s)\right)\right] \\
&\qquad + \sum_{j=1}^{n}\ln \mathbb{E}_\theta\left[\exp\left(\mathrm{i}y\int_t^T b\eta_j^B \mathrm{e}^{-\beta_j^B(T-s)}\, dL_j^B(s)\right)\right].
\end{aligned}$$

Using Eq. (2.4), the proof is complete. □

The expression for the price of the option is suitable for the FFT method, as long as we know the cumulant functions and the Fourier transform of g. The calculation involves only a one-dimensional inversion of the Fourier transform. Note that if we base our spot price dynamics on geometric models, for instance like the mean-reverting model proposed by Schwartz [16], we will end up with the problem of calculating a two-dimensional inverse Fourier transform. The expressions become considerably more intricate, as shown by Dempster and Hong [8].

We remark that the standard contracts such as European call options are not included in the class of payoff-functions $g(x)$ that we can treat directly. Note further that put options are not covered in general either, which is due to the fact that $S(t)$ may attain arbitrary negative values (think of a spark spread, where electricity becomes very cheap and gas prices rocket to the sky), yielding an unbounded payoff for the holder of the option. Of course, if both a and b are positive, then the payoff from a put option is indeed in $L^1(\mathbb{R})$ since $g(x) = \max(K - x, 0)$, and $aS^A(T) + bS^B(T)$ is the weighted sum of two positive processes. Call options may now be priced from the put–call parity. Standard knock-out structures on calls and puts are, on the other hand, included.

To allow for pricing of plain vanilla calls and puts in the general case, we can use the damping of the payoff function by an exponential function, as suggested by [7]. We now consider this for call options. Let, for $x \in \mathbb{R}$,

$$g(x) = \max(x - K, 0).$$

For $\delta > 0$, introduce the exponentially dampened function

$$g_\delta(x) \triangleq \mathrm{e}^{-\delta x} g(x)$$

It is easily seen that $g_\delta(x) \in L^1(\mathbb{R})$, and from the inverse Fourier transform it holds

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\delta(y) \mathrm{e}^{\mathrm{i}x(y - \mathrm{i}\delta)}\, dy.$$

A straightforward calculation shows that the Fourier transform of g_δ is

$$\widehat{g}_\delta(y) = \frac{\mathrm{e}^{K(\mathrm{i}y - \delta)}}{(\mathrm{i}y - \delta)^2}.$$

We can now represent the price of a call option on $S(T)$ as

$$\mathbb{E}_\theta[g(S(T)) \,|\, \mathcal{F}_t] = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_\delta(y) \mathbb{E}_\theta[\mathrm{e}^{\mathrm{i}(y - \mathrm{i}\delta) S(T)} \,|\, \mathcal{F}_t]\, dy.$$

First, for this to be valid, it is sufficient that Condition (**I**) holds for

$$\kappa_i^{A,B} = \delta \sigma_i^{A,B} T + \sup_{s \in [0,T]} |\theta_i(s)|, \quad i = 1, \ldots, m,$$

and

$$\kappa_j^{A,B} = \delta \eta_j^{A,B} T + \sup_{s \in [0,T]} \left|\theta_j^{A,B}(s)\right|, \quad j = 1, \ldots, n.$$

We can now calculate as before, obtaining the price of a call in terms of the damping factor δ, the Fourier transform of g_δ, and the cumulant functions of L_i and $L_j^{A,B}$. In fact, the price will be the same as in Prop. 4.2, except for two modifications. Firstly, $\widehat{g}$ is substituted with $\widehat{g}_\delta$, and secondly, instead of the argument y in the function $\ln \Psi(t, T, y, \theta)$, we must use $y - \mathrm{i}\delta$. A put option can be calculated using a similar damping idea, or by appealing to the put–call parity.

As mentioned earlier, Dempster and Hong [8] consider the problem of applying FFT techniques to spread options. They base the analysis on bivariate geometric models of geometric Brownian motion type, including possibly stochastic volatility. Due to the form of the exercise region of the spread call option, it seems impossible to dampen the payoff to obtain an integrable function which is Fourier transformable. To remedy this problem, Dempster and Hong [8] suggest to approximate the exercise region from above and below with polygons, defining regions where one can define integrable payoff functions after possibly dampening. In this way, they obtain upper and lower bounds for the price which can be computed by FFT methods. Although the bounds are tight, the method involves much preparations in terms of defining the various domains, being very specific for the praticular option in question. Many FFT problems must be solved, giving only an approximation of the price. Our method yields straighforwardly the application of a one-dimensional FFT problem for the exact price, rather than several two-dimensional ones for the approximative price. Further, it is much more versatile since the change of payoff function involves only a recalculation of the Fourier transform, and not an additional domain splitting.

In the marketplace, there are contracts written on the spread between the futures prices of two energies, rather than their spot. For instance, this is particularly relevant for gas and electricity. With the models above, it is possible to derive an explicit dynamics for the futures prices, which become arithmetic as well (see [2]). Hence, similar calculations as above will yield expressions for the option price for such spreads being suitable for FFT methods.

5. Conclusions and Open Problems

For the positive arithmetic spot price model proposed by Benth *et al.* [2], Asian and spread options can be easily calculated by fast Fourier transform methods. By the simple form of the spot price process, expressions for the option prices are easily derived based on the characteristic functions of the driving stochastic innovations, being independent increment processes. For relevant examples, these are easily implemented on a computer, for which prices can be efficiently computed.

In particular, spread options can be represented by pricing formulas which have a one-dimensional nature in the sense that the payoff function does not need to be transformed along both marginals defining the spread. The spread between the two energy spots can in our case be represented again by an arithmetic model, much in contrast with more frequently used geometric models. For the latter, the geometric form of the price processes enforces a two-dimensional Fourier transform of the payoff when pricing the option. This is considerably more difficult to handle in practice for relevant payoff functions like for instance call and put options.

A key to obtaining market relevant option prices is the determination of the market price of risk (denoted by θ in our context). With our approach, this could be estimated from observed option prices in the market. One must keep in mind that the options we have considered here are mostly traded OTC, making it hard to obtain good and reliable data. This complicates such an approach. Furthermore, we have supposed a simple deterministic form of the market price of risk. There are reasons to believe that it may be stochastic, making the problem of pricing even harder. We refer to Benth *et al.* [3] for a thorough discussion of the market price of risk and the risk premium in energy markets, and a possible approach to explain it using the certainty equivalence principle. How this, and other approaches may work in our context, we leave for future research.

Acknowledgments

We are grateful to Thilo Meyer–Brandis for parameter estimates of Nord Pool electricity spot prices. An anonymous referee is acknowledged for carefully reading the manuscript.

References

[1] O. E. Barndorff-Nielsen and N. Shephard, Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in economics, *J. R. Statist. Soc. B* **63**(2) (2001) 167–241.

[2] F. E. Benth, J. Kallsen and T. Meyer-Brandis, A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price and derivates pricing, *Applied Mathematic Finance* **14**(2) (2007) 153–169.

[3] F. E. Benth, A. Cartea and R. Kiesel, Pricing forward contracts in power markets by the certainty equivalence principle: Explaining the sign of the market risk premium, *Journal of Banking Finance* **32**(10) (2008) 2006–2021.

[4] F. E. Benth and P. C. Kettler, Dynamic copula models for the spark spread, *E-print*, 14, September 2006, Department of Mathematics, University of Oslo (2006).

[5] F. E. Benth and J. Šaltytė-Benth, Analytical approximation for the price dynamics of spark spread options, *Stud. Nonlinear Dyn. Econom.* **10**(3) (2006), article 8 (electronic publication).

[6] R. Carmona and V. Durrleman, Pricing and hedging spread options, *SIAM Reviews* **45** (2003) 627–685.

[7] P. Carr and D. P. Madan, Option valuation using the fast Fourier transform, *Journal of Computing Finance* **2** (1998) 61–73.

[8] M. A. H. Dempster and S. S. G. Hong, Spread option valuation and the fast Fourier transform, *WP 26/2000* (Judge Institute of Management Studies, University of Cambridge, 2000).

[9] G. B. Folland, *Real Analysis* (John Wiley and Sons, 1984).

[10] G. Fusai, M. Marena and A. Roncoroni, Analytical pricing of discretely monitored Asian-style options: Theory and application to commodity markets, *Journal of Banking Finance* **32**(10) (2008) 2033–2045.

[11] H. Geman, *Commodities and Commodity Derivatives* (John Wiley & Sons, 2005).

[12] H. Geman and A. Roncoroni, Understanding the fine structure of electricity prices, *Journal of Business* **79**(3) (2006) 1225–1261.

[13] H. Geman and M. Yor, Bessel processes, Asian options and perpetuities, *Math. Finance* **4**(3) (1993) 349–375.

[14] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes* (Springer-Verlag, 1987).

[15] W. Margrabe, The value of an option to exchange one asset for another, *Journal of Finance* **33** (1978) 177–187.

[16] E. S. Schwartz, The stochastic behaviour of commodity prices: Implications for valuation and hedging, *Journal of Finance* **LII**(3) (1997) 923–973.

[17] A. N. Shiryaev, *Essentials of Stochastic Finance* (World Scientific, 1999).

[18] R. Weron, *Modeling and Forecasting Electricity Loads and Prices* (John Wiley & Sons, 2006).