System of linear equations

A system of m linear equations in n unknowns (or variables) $x_1, x_2, \dots x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (1)$$

The system is called linear because each variable x_j, $j = 1, 2, \dots m$ appears in the first power only.

$a_{11}, a_{12}, \dots a_{mn}$ are called the coefficients of the system.

$b_1, b_2, \dots b_m$ are the given numbers.

If all the b_j's are zero then the system of equations (1) is called a homogeneous system.

If atleast one b_j is not zero, then the system of equations (1) is called a non-homogeneous system.

Matrix form

We can express the system of linear equations in the matrix form as

$$\underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

This system of m linear equations in n variables $x_1, x_2, \ldots x_n$ is of the form

$$\boxed{AX = B}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Here A is called coefficient matrix

X is a column matrix of unknowns

B is a column matrix of constants

Definition

The matrix $[A:B]$ obtained by placing B to the right of the coefficient matrix A is called the augmented matrix.

$$[A:B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Q: Write down the augmented matrix for the following system

$$2x + y - z = -1$$
$$x + 3y + z = 10$$
$$4y + z = 11$$

Matrix form is $AX = B$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 10 \\ 11 \end{bmatrix}$$

Augmented matrix is given by

$$[A : B] = \left[\begin{array}{ccc:c} 2 & 1 & -1 & -1 \\ 1 & 3 & 1 & 10 \\ 0 & 4 & 1 & 11 \end{array}\right]$$

Solution of system of equations

The set of values of the variables x_i, $i=1,2,\dots,n$ satisfying $AX=B$ is called a solution.

The solution $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ for which $x_i = 0$, for all i is called the trivial solution of the system of equations

<u>Note</u>:

The trivial solution, $X=0$, is always a solution of the homogeneous system of equations $AX=0$.

Definition

A system of equations is said to be consistent if it has a solution (atleast one) and it is said to be inconsistent if it has no solution.

Row Echelon form (REF)

A non-zero matrix A is said to be in row echelon form if

1. The rows of zeros, if present, are the last rows
2. All elements below the leading element (pivot in each column are zeros
3. The number of zeros before the leading element in each row is greater than the number of zeros in the corresponding rows above it.

Examples

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and, } \begin{bmatrix} 4 & 3 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$ are in Row Echelon form.

1. Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Ans: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{bmatrix}$ $R_3 \rightarrow R_3 + R_2$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix}$

Here no: of non-zero rows = 3

$\therefore$ rank(A) = 3

2 Find the rank of $\begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 5 & 0 & 10 \end{bmatrix} \quad R_3 \rightarrow 3R_3 - 5R_1$$

$$\sim \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & -25 & 30 \end{bmatrix} \quad R_3 \rightarrow 3R_3 + 25R_2$$

$$\sim \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 215 \end{bmatrix}$$

$\therefore$ rank(A) $= \underline{3}$

3. Find the rank of the matrix $\begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix}$

$$A = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \quad R_2 \to 6R_2 + 4R_1$$

$$\sim \begin{bmatrix} 6 & -4 & 0 \\ 0 & -16 & 12 \\ 0 & 2 & 6 \end{bmatrix} \quad R_3 \to 8R_3 + R_2$$

$$\sim \begin{bmatrix} 6 & -4 & 0 \\ 0 & -16 & 12 \\ 0 & 0 & 60 \end{bmatrix}$$

Rank (A) $= 3$

4. Find the rank of $\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix} \quad R_3 \to 2R_3 - R_1$$

$$\sim \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\text{rank}(A) = 2}}$$

5. Find the rank of $\begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \to 9R_3 - R_1$$

$$\sim \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 9 & 8 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\text{rank}(A) = 3}$

Solve the system of equations by Gauss elimination method. $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$ and $2x-3y-z=5$

Ans: Matrix form is $AX=B$

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix is $[A:B] = \left[\begin{array}{ccc:c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right] \; R_2 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 3R_1 \\ \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to 3R_3 + 10R_2 \\ R_4 \to 3R_4 - 7R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & -17 & 68 \end{array}\right] \; R_4 \to R_4 + \frac{17}{29} R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-17 + \frac{17}{29} \times 29 = 0$

$68 + \frac{17}{\cancel{29}} \times \frac{\overset{4}{\cancel{-116}}}{} = 0$

$$AX=B \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -116 \\ 0 \end{bmatrix}$$

rank $(A:B) = 3$ = rank (A) = no: of unknowns

$\therefore$ The system has a unique solution

The solution is given by the equations

$x + 2y = 4$

$-3y + 2z = -11$

$29z = -116 \Rightarrow z = \frac{-116}{29} = \underline{\underline{-4}}$

$-3y + 2(-4) = -11 \Rightarrow y = \frac{-11+8}{-3} = \frac{-3}{-3} =$

$x + 2(1) = 4 \Rightarrow x = 4 - 2 = \underline{\underline{2}}$

$x = 2, \quad y = 1, \quad z = -4$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

1. Solve the system of equations by Gauss elimination method.

$$4x - 6y = -11$$
$$-3x + 8y = 10$$

Ans: matrix form is $AX = B$

$$\begin{bmatrix} 4 & -6 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 10 \end{bmatrix}$$

Augmented matrix is

$$[A:B] = \left[\begin{array}{cc:c} 4 & -6 & -11 \\ -3 & 8 & 10 \end{array}\right] \quad R_2 \to 4R_2 + 3R_1$$

$$\sim \left[\begin{array}{cc:c} 4 & -6 & -11 \\ 0 & 14 & 7 \end{array}\right]$$

rank $[A:B] = 2 =$ rank (A)

no: of unknowns $= 2$

$\therefore$ rank $(A:B) =$ rank $(A) =$ no: of unknowns

$\Rightarrow$ given system has unique solution

$$\begin{bmatrix} 4 & -6 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$$

Corresponding equations are $4x - 6y = -11$

$$14y = 7 \Rightarrow y = \frac{1}{2}$$

$$4x - 6\left(\frac{1}{2}\right) = -11$$

$$x = \frac{-11 + 3}{4}$$
$$= -2$$

$x = -2, \; y = \frac{1}{2}$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1}{2} \end{bmatrix}$$

Fundamental theorem for linear systems

For a system of non-homogeneous linear equations
$AX = B$

1 If rank $[A:B] \neq$ rank(A), then the system is inconsistent

2. If rank $[A:B]$ = rank(A) = no: of unknowns, then the system has a unique solution.

3. If rank $[A:B]$ = rank(A) < no: of unknowns, then the system has an infinite no: of solutions

If solutions exist, they can all be obtained by the Gauss elimination method.

Gauss elimination method

In this method the solution to the system of linear equations ① is obtained by the following steps

Step 1 Write the given system of equations into the matrix form $AX = B$

Step 2 Using elementary row operations, the augmented matrix $[A:B]$ is reduced to the row echelon form

Step 3 We get the solution in the order $x_n, x_{n-1}, \dots, x_1$ by back substitution

1. Solve the system of equations by Gauss elimination method.

$$4x - 6y = -11$$
$$-3x + 8y = 10$$

Ans: matrix form is $AX = B$

$$\begin{bmatrix} 4 & -6 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 10 \end{bmatrix}$$

Augmented matrix is

$$[A:B] = \begin{bmatrix} 4 & -6 & : & -11 \\ -3 & 8 & : & 10 \end{bmatrix} \quad R_2 \to 4R_2 + 3R_1$$

$$\sim \begin{bmatrix} 4 & -6 & : & -11 \\ 0 & 14 & : & 7 \end{bmatrix}$$

rank $(A:B) = 2 = $ rank (A)

no: of unknowns $= 2$

$\therefore$ rank $(A:B) =$ rank $(A) =$ no: of unknowns

$\Rightarrow$ given system has unique solution

$$\begin{bmatrix} 4 & -6 \\ 0 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$$

Corresponding equations are $4x - 6y = -11$

$14y = 7 \Rightarrow y = \frac{1}{2}$

$$4x - 6\left(\tfrac{1}{2}\right) = -11$$

$$x = \frac{-11+3}{4} = -2$$

$x = -2,\ y = \frac{1}{2}$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1}{2} \end{bmatrix}$$

2 Solve the system of equations by Gauss elimination method. $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$ and $2x-3y-z=5$

Ans: Matrix form is $AX=B$

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix is $[A:B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \quad R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \quad \begin{matrix} R_3 \to 3R_3 + 10R_2 \\ R_4 \to 3R_4 - 7R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 29 & : & -116 \\ 0 & 0 & -17 & : & 68 \end{bmatrix} \quad R_4 \to R_4 + \frac{17}{29} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 0 & 29 & : & -116 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$-17 + \frac{17}{29} \times 29$

$68 + \frac{17}{\cancel{29}} \times \cancel{-116}^{\,4}$

$$AX = B \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 29 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -116 \\ 0 \end{bmatrix}$$

rank $(A:B) = 3 = $ rank $(A) =$ no: of unknowns

$\therefore$ The system has a unique solution

The solution is given by the equations

$x + 2y = 4$

$-3y + 2z = -11$

$29z = -116 \Rightarrow z = \frac{-116}{29} = \underline{\underline{-4}}$

$-3y + 2(-4) = -11 \Rightarrow y = \frac{-11+8}{-3} = \frac{-3}{-3} = \underline{\underline{1}}$

$x + 2(1) = 4 \Rightarrow x = 4 - 2 = \underline{\underline{2}}$

$x = 2, \quad y = 1, \quad z = -4$

$$\underline{\underline{X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}}}$$

$rank\ (A:B) = 3$ and $rank\ (A) = 2$

$rank\ (A:B) \neq rank\ (A)$

Hence the system is inconsistent

3 Solve the system of equations $2x-2z=6$, $y+z=1$, $2x+y-z=7$, and $3y+3z=0$

Ans

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 7 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right] \quad R_4 \leftrightarrow R_3 \quad \sim \left[\begin{array}{ccc:c} 2 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(A:B) = 3$, $\text{rank}(A) = 2$

4. Solve $2x+3y=9$, $x-2y=-13$ and $x+y=7$

Ans: $$[A:B] = \left[\begin{array}{cc:c} 2 & 3 & 9 \\ 1 & -2 & -13 \\ 1 & 1 & 7 \end{array}\right] \quad \begin{array}{l} \\ R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{cc:c} 2 & 3 & 9 \\ 0 & -7 & -35 \\ 0 & -1 & 5 \end{array}\right] \quad R_3 \to 7R_3 - R_2$$

$$\sim \left[\begin{array}{cc:c} 2 & 3 & 9 \\ 0 & -7 & -35 \\ 0 & 0 & 70 \end{array}\right]$$

rank $(A:B) = 3$ and rank $(A) = 2$

rank $(A:B) \neq$ rank(A)

$\therefore$ The system has no solution.

5. Show that the system of equations are inconsistent.

$2x + 6y = -11$, $6x + 20y - 6z = -3$ and $6y - 18z = -1$

Ans:

$$(A:B) = \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{array}\right]$$

6. Solve the system of equations by Gauss elimination method $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$ and $x-y+z=-1$

Ans: $(A:B) = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$ $\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$

$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right]$ $\begin{array}{l} R_3 \to 7R_3 - 6R_2 \\ R_4 \to 7R_4 - 3R_2 \end{array}$

$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{array}\right]$ $R_4 \to 5R_4 + R_3$

$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 0 & -7 & 5 & : & -8 \\ 0 & 0 & 5 & : & 20 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

rank $(A:B) = 3$ = rank(A) $\Rightarrow$ system has unique solution.

no: of unknowns $= 3$

Given system of equations is equivalent to

$$x + 2y - z = 3$$

$$-7y + 5z = -8$$

$$5z = 20 \Rightarrow z = \frac{20}{5} = \underline{\underline{4}}$$

$$-7y + 5(4) = -8 \Rightarrow y = \frac{-8-20}{-7} = \underline{\underline{4}}$$

$$\begin{aligned} x &= 3 - 2y + z \\ &= 3 - 2(4) + 4 \\ &= 3 - 8 + 4 \\ &= \underline{\underline{-1}} \end{aligned}$$

$$\underline{\underline{X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}}}$$

7 Solve $-2y-2z=8$ & $3x+4y-5z=8$ by Gauss elimination method

Ans: $(A:B) = \begin{bmatrix} 0 & -2 & -2 & : & 8 \\ 3 & 4 & -5 & : & 8 \end{bmatrix}$ $R_2 \leftrightarrow R_1$

$\sim \begin{bmatrix} 3 & 4 & -5 & : & 8 \\ 0 & -2 & -2 & : & 8 \end{bmatrix}$

rank $(A:B) = 2 =$ rank$(A) \Rightarrow$ System is consistent

no: of unknowns $= 3 = n$

rank $(A:B) =$ rank $(A) = r < n$

$\Rightarrow$ System has infinitely many solutions

Corresponding equations are $3x+4y-5z=8$

$-2y-2z=8$

rank $(A:B) = 3$ and rank $(A) = 2$

rank $(A:B) \neq$ rank (A)

Hence the system is inconsistent

Corresponding equations are $3x + 4y - 5z = 8$

$$-2y - 2z = 8$$

no: of arbitrary variables $= n - r$

$= 3 - 2$

$= 1$

Choose $z = t$

$$-2y - 2(t) = 8$$

$$y = \frac{8 + 2t}{-2}$$

$$= -4 - t$$

$$3x + 4(-4 - t) - 5t = 8$$

$$x = \frac{8 + 16 + 4t + 5t}{3}$$

$$= 8 + 3t$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8+3t \\ -4-t \\ t \end{bmatrix}$$

8. Solve $y + z - 2w = 0$

$2x - 3y - 3z + 6w = 2$

$4x + y + z - 2w = 4$

Ans: Matrix form is $AX = B$

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ 2 & -3 & -3 & 6 \\ 4 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{cccc:c} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right] R_2 \leftrightarrow R_1$$

$$\sim \left[\begin{array}{cccc:c} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 4 & 1 & 1 & -2 & 4 \end{array}\right] R_3 \to R_3 - 2R_1$$

$$\sim \left[\begin{array}{cccc:c} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 7 & 7 & -14 & 0 \end{array}\right] R_3 \to R_3 - 7R_2$$

$$\sim \left[\begin{array}{cccc:c} 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$[A:B] \sim \begin{bmatrix} 2 & -3 & -3 & 6 & : & 2 \\ 0 & 1 & 1 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

rank $(A:B)$ = rank$(A) = 2 <$ no: of unknowns

$\Rightarrow$ System has infinitely many solutions

$r < n \Rightarrow$ no: of arbitrary variables $= n - r$

$= 4 - 2$

$= 2$

$$2x - 3y - 3z + 6w = 2$$

$$y + z - 2w = 0$$

Choose $z = t$, $w = s$

$$y + t - 2s = 0 \Rightarrow y = \underline{\underline{2s - t}}$$

$$2x - 3(2s - t) - 3t + 6s = 2$$

$$x = \frac{2 + 6s - 3t + 3t - 6s}{2}$$

$$= \underline{\underline{1}}$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2s - t \\ t \\ s \end{bmatrix}$$

9. Solve $2x + z = 3$, $x - y - z = 1$ and $3x - y = 4$

Ans: Matrix form is $AX = B$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & -1 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 0 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ 3 & -1 & 0 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & 1 & 3 \\ 0 & -2 & -3 & -1 \\ 0 & -2 & -3 & -1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & 1 & 3 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(A:B) = \text{rank}(A) = 2 = r$

no: of unknowns, $n = 3$

$\operatorname{rank}(A:B) = \operatorname{rank}(A) = 2 = r$

no: of unknowns, $n = 3$

$r < n \Rightarrow$ system has infinitely many solutions

no: of arbitrary variables $= n - r$

$= 3 - 2$

$= 1$

corresponding equations are $2x + z = 3$

$-2y - 3z = -1$

choose $z = t$

$-2y - 3t = -1$

$y = \dfrac{-1+3t}{-2}$

$= \dfrac{1-3t}{2}$

$2x + t = 3$

$x = \dfrac{3-t}{2}$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3-t}{2} \\ \frac{1-3t}{2} \\ t \end{bmatrix}$

10. Show that the equations are consistent and Solve $y-3z=-1$, $x+z=1$, $3x+y=2$, and $x+y-2z=0$

Ans: Matrix form is $AX=B$

$$\begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 0 & 1 & -3 & : & -1 \\ 1 & 0 & 1 & : & 1 \\ 3 & 1 & 0 & : & 2 \\ 1 & 1 & -2 & : & 0 \end{bmatrix} R_2 \leftrightarrow R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank}(A:B) = \text{rank}(A) = 2 = r \Rightarrow$ System is consistent

no: of unknowns, $n = 3$

$r < n \Rightarrow$ System has Infinitely many solutions

no: of arbitrary variables $= n - r$

$= 3 - 2 = 1$

Corresponding equations are $x+z=1$

$y-3z=-1$

Choose $z=t$

$x+t=1 \Rightarrow x=1-t$

$y-3t=-1 \Rightarrow y=-1+3t$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1-t \\ -1+3t \\ t \end{bmatrix}$$

11. Solve the system of equations
$$x+y-z=0$$
$$2x-y-z=3$$
$$4x+2y-2z=2$$

Ans: matrix form is $AX=B$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 4 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 2 & -1 & -1 & : & 3 \\ 4 & 2 & -2 & : & 2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & -3 & 1 & : & 3 \\ 0 & -2 & 2 & : & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to 3R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & : & 0 \\ 0 & -3 & 1 & : & 3 \\ 0 & 0 & 4 & : & 0 \end{bmatrix}$$

rank $(A:B)$ = rank(A) = 3 = no: of unknowns

$\Rightarrow$ system has unique solution

$$x + y - z = 0$$

$$-3y + z = 3$$

$$4z = 0 \Rightarrow z = 0$$

$$-3y = 3 \Rightarrow y = -1$$

$$x + (-1) - 0 = 0 \Rightarrow x = 1$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

12. Find the values of λ & μ for which system of equatio

$x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

has (i) no Solution
(ii) a unique solution and
(iii) infinite solution

Ans: Matrix form is $AX=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$(A:B) = \left[\begin{array}{ccc:c} \boxed{1} & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 1 & \lambda_{-1} & \mu-6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(i) System has no solution

If rank $(A:B) \neq$ rank (A),

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

$\lambda - 3 = 0$, $\mu - 10 \neq 0$

$\underline{\underline{\lambda = 3, \ \mu \neq 10}}$

(ii) System has a unique solution

If rank $(A:B)$ = rank $(A) = 3$

$\lambda - 3 \neq 0$, μ may have any value

$\underline{\underline{\lambda \neq 3}}$, μ may have any value

(iii) System has infinite no: of solutions

If rank $(A:B)$ = rank(A) < 3

$\lambda - 3 = 0$, $\mu - 10 = 0$

$\underline{\underline{\lambda = 3, \ \mu = 10}}$

13. Find the values of λ & μ for which the System of equations

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

has (i) no solution (ii) a unique solution and (iii) infinite solution

Ans: Matrix form is $AX = B$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -15 & -39 & : & -47 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

(i) no solution

If $\text{rank}(A:B) \neq \text{rank}(A)$

$\underline{\underline{\lambda = 5, \quad \mu \neq 9}}$

(ii) a unique solution

If $\text{rank}(A:B) = \text{rank}(A) = n$

$\lambda \neq 5$, μ may have any value

(iii) infinite no: of solutions

If $\text{rank}(A:B) = \text{rank}(A) < n$

$\underline{\underline{\lambda = 5, \quad \mu = 9}}$

14 Find the values of a & b for which the system of equations

$$x+2y+2z=2$$
$$2x-y+3z=10$$
$$5x-y+az=b$$

has (i) no solution (ii) a unique solution and (iii) infinitely many solutions

Ans: Matrix form is $AX=B$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 10 \\ b \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & -1 & 3 & 10 \\ 5 & -1 & a & b \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -5 & -1 & 6 \\ 0 & -11 & a-10 & b-10 \end{array}\right] \quad R_3 \to 5R_3 - 11R_2$$

$$5a-50+11 = 5a-39$$

$$5b-50-66 = 5b-116$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -5 & -1 & 6 \\ 0 & 0 & 5a-39 & 5b-116 \end{array}\right]$$

(i) no solution

If $\operatorname{rank}(A:B) \neq \operatorname{rank}(A)$

$5a-39=0, \quad 5b-116 \neq 0$

$a=\frac{39}{5}, \quad b \neq \frac{116}{5}$

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 2 \\ 0 & -5 & -1 & 6 \\ 0 & 0 & 5a-39 & 5b-1 \end{array}\right.$$

(ii) a unique solution

If $\operatorname{rank}(A:B) = \operatorname{rank}(A) = n$

$5a-39 \neq 0$, $5b-116$ = any value

$a \neq \frac{39}{5}$, b may have any value

(iii) infinite no: of solution

If $\operatorname{rank}(A:B) = \operatorname{rank}(A) < n$

$5a-39=0, \quad 5b-116=0$

$a=\frac{39}{5}, \quad b=\frac{116}{5}$

15. Find the values of μ for which the system of equations

$$x+y+z=1$$
$$x+2y+3z=\mu$$
$$x+5y+9z=\mu^2$$

will be consistent

Ans: $[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & \mu \\ 1 & 5 & 9 & \mu^2 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 4 & 8 & \mu^2-1 \end{array}\right] \quad R_3 \to R_3 - 4R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \mu-1 \\ 0 & 0 & 0 & \mu^2-4\mu+3 \end{array}\right]$$

$$\mu^2-1-4(\mu-1)$$
$$=\mu^2-4\mu+4-1$$
$$=\mu^2-4\mu+3$$

System is consistent

If rank $(A:B)$ = rank (A)

$$\mu^2-4\mu+3=0$$
$$(\mu-1)(\mu-3)=0$$
$$\underline{\mu=1,\ 3}$$

Fundamental theorem for homogeneous linear system

A system of homogeneous linear equations ($AX = 0$) has either trivial solution or infinite no: of solutions

i If rank(A) = no: of unknowns, then the system has only trivial solution $X = 0$

ii If rank (A) < no: of unknowns, then the system has an infinite no: of non-trivial solutions

1. Solve the system of equations $3x + 2y + z = 0$
$2x + 3z = 0$
$x + 2y + 3z = 0$

Ans: $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ $R_2 \rightarrow 3R_2 - 2R_1$
$R_3 \rightarrow 3R_3 - R_1$

$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & -4 & 7 \\ 0 & 4 & 8 \end{bmatrix}$ $R_3 \rightarrow R_3 + R_2$

$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & -4 & 7 \\ 0 & 0 & 15 \end{bmatrix}$

rank (A) = 3 = no: of unknowns

∴ The system has only trivial solution $x = 0, y = 0, z = 0$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. Solve by Gauss elimination method

$$x - 3y - 8z = 0$$

$$3x + y = 0$$

$$2x + 5y + 6z = 0$$

Ans: $A = \begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & 0 \\ 2 & 5 & 6 \end{bmatrix}$ $\quad R_2 \to R_2 - 3R_1$, $\quad R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 11 & 22 \end{bmatrix} \quad R_3 \to 10R_3 - 11R_2$$

$$\sim \begin{bmatrix} 1 & -3 & -8 \\ 0 & 10 & 24 \\ 0 & 0 & -44 \end{bmatrix}$$

$$\begin{array}{r} 24 \\ 11 \\ \hline 24 \\ 24 \\ \hline 264 \\ 220 \\ \hline 44 \end{array}$$

rank (A) = 3 = no: of unknowns

$\therefore$ System has only trivial solution

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3 Solve $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A) = 2 <$ no: of unknowns

$\therefore$ System has infinite no: of non-trivial solutions

$$x + 3y - 2z = 0$$

$$-7y + 8z = 0$$

$$x + 3y - 2z = 0$$

$$-7y + 8z = 0$$

Choose $z = t$

$$-7y + 8t = 0 \Rightarrow y = \frac{-8t}{-7} = \frac{8t}{7}$$

$$x + 3\left(\frac{8}{7}t\right) - 2t = 0 \Rightarrow x = -\frac{24t}{7} + 2t$$

$$= -\frac{10t}{7}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{10}{7}t \\ \frac{8}{7}t \\ t \end{bmatrix}$$

4. Show that the equations have non-trivial solution and find them

$$x + 2y - z = 0$$

$$3x + y - z = 0$$

$$2x - y = 0$$

Ans:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & -5 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2 < n$

$\Rightarrow$ System has non-trivial solution

$x + 2y - z = 0$

$-5y + 2z = 0$

Choose $z = t$

$-5y + 2t = 0 \implies y = \frac{2}{5}t$

$x + 2\left(\frac{2}{5}t\right) - t = 0 \implies x = t - \frac{4}{5}t = \underline{\underline{\frac{1}{5}t}}$

$= \frac{3}{5}t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{5}t \\ \frac{2}{5}t \\ t \end{bmatrix}$$

5. Solve $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + w = 0$

Ans: $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$ $\begin{array}{l} R_2 \to 4R_2 - 6R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$

$$A \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad R_3 \rightarrow 10R_3 + R_2$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank (A) $= 2 <$ no: of unknowns

no: of arbitrary variables $= n - r$

$= 4 - 2$

$= 2$

$$4x + 2y + z + 3w = 0$$

$$10z + 10w = 0$$

Choose $w = t$

$$10z + 10t = 0 \Rightarrow z = -t$$

$$y = s$$

$$4x + 2s - t + 3t = 0 \Rightarrow x = \frac{-2s-2t}{4}$$

$$= \frac{-s-t}{2}$$

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{s-t}{2} \\ s \\ -t \\ t \end{bmatrix}$$

6. Solve

$$4x + y + 2z = 0$$
$$-3x + 2y + 4z = 0$$
$$8x - y - 2z = 0$$

Ans.

$$A = \begin{bmatrix} 4 & 1 & 2 \\ -3 & 2 & 4 \\ 8 & -1 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to 4R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 4 & 1 & 2 \\ 0 & 11 & 22 \\ 0 & -3 & -6 \end{bmatrix} \quad R_3 \to 11R_3 + 3R_2$$

$$\sim \begin{bmatrix} 4 & 1 & 2 \\ 0 & 11 & 22 \\ 0 & 0 & 0 \end{bmatrix}$$

rank (A) = 2 < no: of unknowns

$\therefore$ System has non-trivial solution

no: of arbitrary variables $= n - r$

$= 3 - 2$

$= 1$

$$4x + y + 2z = 0$$

$$11y + 22z = 0$$

Choose $z = t$

$$11y + 22t = 0$$

$$y = -2t$$

$$4x + (-2t) + 2t = 0$$

$$4x = 0$$

$$\underline{\underline{x = 0}}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix}$$

7. Solve

$$4x + 2y + z + 3u = 0$$
$$4x + 2y + 4z + 7u = 0$$
$$2x + y + u = 0$$

Ans:

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 4 & 2 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad R_3 \to 3R_3 + R_2$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A) = 3 <$ no: of unknowns

$$4x + 2y + z + 3u = 0$$

$$3z + 4u = 0$$

$$u = 0 \Rightarrow z = 0$$

Choose $y = t$

$$4x + 2t = 0$$

$$x = \frac{-2t}{4} = \underline{\underline{\frac{-1}{2}t}}$$

$$\underline{\underline{X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} \frac{-t}{2} \\ t \\ 0 \\ 0 \end{bmatrix}}}$$

8. Solve

$$x - y + z = 0$$
$$2x + y - z = 0$$
$$x + 5y - 5z = 0$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & -5 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 6 & -6 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A) = 2 <$ no: of unknowns

no: of arbitrary variables $= n - r$

$= 3 - 2$

$= 1$

$$x - y + z = 0$$

$$3y - 3z = 0$$

Choose $z = t$

$$3y - 3t = 0 \Rightarrow y = t$$

$$x - t + t = 0 \Rightarrow x = 0$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Eigenvalues for a 2x2 matrix

Let $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, then the characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - (a_1 + b_1)\lambda + |A| = 0$$

ie, The characteristic equation for a 2x2 matrix A is $\lambda^2 - (\text{trace } A)\lambda + \det A = 0$

Eigenvalues for a 3x3 matrix

The characteristic equation for a 3x3 matrix A is

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

trace $A = a_{11} + a_{22} + a_{33}$ (= sum of diagonal elements)

Sum of principal minors $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Note:

Sum of eigenvalues = trace A

Product of eigenvalues = det A

1. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Ans: Characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\text{trace } A = -5 + (-2) = \underline{\underline{-7}}$$

$$\det A = (-5)(-2) - (2)(2) = 10 - 4 = \underline{\underline{6}}$$

$$\therefore \quad \lambda^2 - (-7)\lambda + 6 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\underline{\underline{\lambda = -1, -6}}$$

2. Find the eigenvalues of $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Ans: Characteristic equation is

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$

trace $A = 3+3+5 = 11$

$\det A = 3\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} -2 & 3 \\ 0 & 0 \end{vmatrix}$

$= 3(15) + 2(-10) + 0$

$= 45 - 20$

$= 25$

Sum of principal minors $= \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}$

$= 15 + 15 + 9 - 4$

$= 35$

$$\therefore \lambda^3 - 11\lambda^2 + 35\lambda - 25 = 0$$

$$\lambda = 1, 5, 5$$

eigenvalues are 1, 5 and 5

3. Find the Spectrum of $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

Ans: trace $A = 4+5+3 = \underline{\underline{12}}$

$\det A = 4\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 5 \\ -2 & 0 \end{vmatrix}$

$= 4(15) - 2(6) - 2(0-(-10))$

$= 60 - 12 - 20$

$= \underline{\underline{28}}$

Sum of principal minors $= \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix}$

$= 15 + (12-4) + (20-4)$

$= 15 + 8 + 16$

$= 39$

Characteristic equation is

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$

$\lambda = 1, 7, 4$

Spectrum of $A = \underline{\underline{\{1, 7, 4\}}}$

Eigenvectors

1. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Ans: trace $A = -2+1+0 = \underline{\underline{-1}}$

$$\det A = -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$$= -2(-12) - 2(-6) - 3(-4+1)$$

$$= 24 + 12 + 9$$

$$= \underline{\underline{45}}$$

Sum of Principal Minors $= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$$= -12 - 3 + (-2-4)$$

$$= -21$$

Characteristic equation is

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$$

$$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = 5, -3, -3$$

Eigenvalues are 5, -3 and -3

Eigenvectors corresponding to $\lambda=5$ is given by

$$(A-\lambda I)X=0$$

$$(A-5I)X=0$$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$$A-5I=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}-\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \quad \begin{matrix} R_2 \to 7R_2+2R_1 \\ R_3 \to 7R_3-R_1 \end{matrix}$$

$$\sim\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24 & -48 \\ 0 & -16 & -32 \end{bmatrix} \quad \begin{matrix} R_2 \to -\frac{1}{24}R_2 \\ R_3 \to \frac{-1}{16}R_3 \end{matrix}$$

$$\sim\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3-R_2$$

$$\sim\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-5I) = 2 < n$

$n - r = 3 - 2$
$= 1$

$-7x + 2y - 3z = 0$
$y + 2z = 0$

choose $z = t$
$y + 2t = 0 \Rightarrow y = -2t$
$-7x + 2(-2t) - 3t = 0 \Rightarrow -7x = 7t$
$x = -t$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = -3$ is given by

$$(A - \lambda I) X = 0$$

$$(A - (-3) I) X = 0$$

$$(A + 3I) X = 0$$

$$A + 3I = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A+3I) = 1 < n$

$n - r = 3 - 1 = 2$ arbitrary variables

$(x + 2y - 3z = 0$

Choose $z = t$, $y = s$

$x = 3t - 2s$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t-2s \\ s \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ eigen vectors are $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

Eigenvector corresponding to $\lambda = -3$ is given by

$(A - \lambda I)X = 0$

$(A - (-3)I)X = 0$

$(A + 3I)X = 0$

$$A + 3I = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A+3I) = 1 < n$

$n - r = 3 - 1 = 2$ arbitrary variables

$x + 2y - 3z = 0$

Choose $z = t$, $y = s$

$x + 2s - 3t = 0 \Rightarrow x = 3t - 2s$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t - 2s \\ s \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ eigenvectors are $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Algebraic multiplicity

The order of an eigenvalue λ as a root of the characteristic equation is called the algebraic multiplicity of λ.

AM of $5 = 1$

AM of $-3 = 2$

Geometric multiplicity

The no: of Linearly independent eigenvectors corresponding to the eigenvalue λ is called the geometric multiplicity of λ.

GM of $5 = 1$

GM of $-3 = 2$

2. Find the eigenvalues & eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

Ans: trace $A = 2+1+4 = 7$

$$\det A = 2\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 0 & -1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= 2(4+2) - 1(0) + 0$$

$$= 12$$

Sum of principal minors $= \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$$= (4-(-2)) + (8-0) + (2-0)$$

$$= 6 + 8 + 2$$

$$= 16$$

characteristic equation is

$\lambda^3 - (\text{trace } A)\,\lambda^2 + (\text{Sum of principal minors})\,\lambda - \det A = 0$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

$\lambda = 3, 2, 2$

eigenvalues are 3 and 2

Eigenvector corresponding to $\lambda = 3$ is given by

$$(A - \lambda I) X = 0$$

$$(A - 3I) X = 0$$

$$A - 3I = \begin{bmatrix} 2-3 & 1 & 0 \\ 0 & 1-3 & -1 \\ 0 & 2 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-3I) = 2 < n$

$n-r = 3-2 = 1$ arbitrary variable

$-x + y = 0$

$-2y - z = 0$

Choose $y = t$

$-x + t = 0 \Rightarrow x = t$

$-2t - z = 0 \Rightarrow z = -2t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 2$ is given by

$(A - \lambda I) X = 0$

$(A - 2I) X = 0$

$$A - 2I = \begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 1-2 & -1 \\ 0 & 2 & 4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $(A - 2I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$y = 0$

$-z = 0$

choose $x = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Here $AM(3) = 1 = GM(1)$

$AM(2) = 2, \quad GM(2) = 1$

3. Find the eigenvalues & eigenvectors of

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Ans: trace $A = 8 + 2 = 10$

$$\det A = \begin{vmatrix} 8 & -4 \\ 2 & 2 \end{vmatrix} = 16 + 8 = 24$$

Characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$\lambda = 4, 6$$

eigenvalues are 4 and 6

Eigenvector corresponding to $\lambda = 4$ is given by

$(A - \lambda I) X = 0$

$(A - 4I) X = 0$

$$A - 4I = \begin{bmatrix} 8-4 & -4 \\ 2 & 2-4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \quad R_2 \to 2R_2 - R_1$$

$$\sim \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}$$

rank $(A-4I)=1$

$n-r = 2-1=1$

$4x-4y=0$

$x=y$

Choose $y=t=x$

$$X=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} t \\ t \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda=6$ is given by

$$(A-\lambda I)\,X=0$$

$$(A-6I)\,X=0$$

$$A-6I=\begin{bmatrix} 8-6 & -4 \\ 2 & 2-6 \end{bmatrix}$$

$$\sim\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \quad R_2 \to R_2-R_1$$

$$\sim\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}$$

$\text{rank}(A-6I) = 1 < n$

$n-r = 2-1 = 1$ arbitrary variable

$2x - 4y = 0$

Choose $y = t$

$2x = 4t$

$x = 2t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Properties of eigenvalues and eigenvectors

1. Eigenvalues of A and A^T are same

2. If λ is an eigenvalue of A, then

i λ^n is an eigenvalue of A^n, where n is a positive integer

ii $k\lambda$ is an eigenvalue of kA

iii $\lambda - k$ is an eigenvalue of $A - kI$

iv $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

v $\frac{|A|}{\lambda}$ is an eigenvalue of $\text{adj}A$

Q: Find the eigenvalues of A and A^T if $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Ans: Characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - (4+2)\lambda + (8-3) = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\underline{\underline{\lambda = 1, 5}}$$

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\det A^T = 8 - 3 = 5$$

$$\text{trace } A^T = 4 + 2 = 6$$

characteristic equation is

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\underline{\underline{\lambda = 1, 5}}$$

$\therefore$ eigenvalues of $\underline{A \;\&\; A^T}$ are 1, 5

Q: If 2 is an eigenvalue of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, without using its characteristic equation, find the other eigenvalues. Also find the eigenvalues of A^3, A^T, A^{-1}, $5A$, $A-3I$ and adj A

Ans: let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

and $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A

trace A = Sum of diagonal elements

$\lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3$

$2 + \lambda_2 + \lambda_3 = 11$

$\boxed{\lambda_2 + \lambda_3 = 9}$

$$\boxed{\lambda_2 + \lambda_3 = 9}$$

$$\det A = 3\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix}$$

$$\lambda_1 \lambda_2 \lambda_3 = 3(15-1) + 1(-3+1) + (1-5)$$

$$2\lambda_2\lambda_3 = 42 - 2 - 4$$

$$2\lambda_2\lambda_3 = 36$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\boxed{\lambda_2\lambda_3 = 18}$$

$\lambda_2 = 6$, $\lambda_3 = 3$

$\therefore$ eigenvalues are 2, 3 and 6

eigenvalues of $A = 2, 3, 6$

eigenvalues of $A^T = 2, 3, 6$

eigenvalues of $A^3 = 2^3, 3^3, 6^3$

$= 8, 27, 216$

eigenvalues of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

eigenvalues of $5A = 5(2), 5(3), 5(6)$

$= 10, 15, 30$

eigenvalues of $A - 3I = 2-3,\ 3-3,\ 6-3$

$= -1,\ 0,\ 3$

eigenvalues of $\operatorname{adj} A = \dfrac{\det A}{\lambda}$

$= \dfrac{36}{2},\ \dfrac{36}{3},\ \dfrac{36}{6}$

$= 18,\ 12,\ 6$

Properties of eigenvalues and eigenvectors

3. Eigenvalues of triangular matrices (upper or lower) and diagonal matrices are its diagonal elements

4. If an $n \times n$ matrix A has n distinct eigenvalues, then A has n distinct eigenvectors $x_1, x_2, \cdots x_n$

Q Find the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

s: Given an upper triangular matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

eigenvalues of A $\underline{\underline{= 1, 3}}$

Q: Find the eigenvalues of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Ans: Given a diagonal matrix.

$\therefore$ eigenvalues are $\underline{\underline{2, -1, 3}}$

Q: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

Ans: Given matrix is lower triangular

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$

eigenvalues of $A = 3, 4, 1$

Eigenvector corresponding to $\lambda = 3$ is given by

$(A - \lambda I)X = 0$

$(A - 3I)X = 0$

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 1 & 0 \\ 3 & 6 & -2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 6 & -2 \\ 5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to 3R_2 - 5R_1$$

$$\sim \begin{bmatrix} 3 & 6 & -2 \\ 0 & -27 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-3I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$$3x + 6y - 2z = 0$$
$$-27y + 10z = 0$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-2}{27}t \\ \frac{10}{27}t \\ t \end{bmatrix} = \frac{t}{27}\begin{bmatrix} -2 \\ 10 \\ 27 \end{bmatrix}$$

Choose $z = t$

$$-27y + 10t = 0 \Rightarrow y = \frac{10}{27}t$$

$$3x + 6\left(\frac{10}{27}t\right) - 2t = 0 \Rightarrow x = \frac{-2}{27}t$$

Eigenvector corresponding to $\lambda = 4$ is given by

$(A-4I)X = 0$

$$A-4I = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 5 & 0 & 0 \\ 3 & 6 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & -3 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 0 & 0 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A - 4I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$-x = 0$

$6y - 3z = 0$

choose $y = t$

$6t - 3z = 0 \Rightarrow z = \frac{6t}{3} = 2t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 2t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 1$ is given by

$(A - I)X = 0$

$$A - I = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{matrix} \\ R_2 \to 2R_2 - 5R_1 \\ R_3 \to 2R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 12 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$rank(A-I) = 2 < n$

$n-r = 3-2 = 1$ arbitrary variable

$2x = 0$

$6y = 0$

choose $z = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvalues are 3, 4 and 1

The eigenvectors are $\begin{bmatrix} -2 \\ 10 \\ 27 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Symmetric, Skew-Symmetric and orthogonal matrices:

A real square matrix $A = [a_{ij}]$ is called

* Symmetric if $A^T = A$

* Skew-Symmetric if $A^T = -A$

* Orthogonal if $A^T = A^{-1}$ or $AA^T = I$

1. Show that the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is symmetric. Find its spectrum.

Ans: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

$= A \Rightarrow A$ is symmetric.

Characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - (-1)\lambda + (-6) = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3$$

Spectrum of $\underline{\underline{A = \{-3, 2\}}}$

2. Show that the matrix $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ is skew symmetric. Also find the eigenvalues of A.

Ans: $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$

$= - \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$= -A$

$\therefore A^T = -A \Rightarrow$ A is skew symmetric.

Characteristic equation is

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$$

$$\text{trace } A = 0$$

$$\det A = 0 - (-9)\begin{vmatrix} -9 & 20 \\ 12 & 0 \end{vmatrix} + (-12)\begin{vmatrix} -9 & 0 \\ 12 & -20 \end{vmatrix}$$

$$= -9(-240) - 12(180)$$

$$= 0$$

$$\text{Sum of principal minors} = \begin{vmatrix} 0 & 20 \\ -20 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -12 \\ 12 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 9 \\ -9 & 0 \end{vmatrix}$$

$$= 400 + 144 + 81$$

$$= 625$$

$$\lambda^3 - 0\lambda^2 + 625\lambda - 0 = 0$$

$$\lambda^3 + 625\lambda = 0$$

$$\lambda(\lambda^2 + 625) = 0$$

$$\lambda = 0, \quad \lambda = -25i, \quad \lambda = 25i$$

$\therefore$ The eigenvalues of a skew symmetric matrix are purely imaginary or zero.

3. Determine the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ is Symmetric, Skew-Symmetric or Orthogonal. Also find its determinant.

Ans: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta+\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ A is orthogonal.

$\det A = 1(\cos^2\theta + \sin^2\theta)$

$= \underline{\underline{1}}$

<u>Note</u>: Determinant of an orthogonal matrix has the value +1 or -1.

Q: Show that the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is orthogonal. Find the eigenvalues of A.

Ans: $$AA^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ A is orthogonal.

Characteristic equation is

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$$

$$\lambda^3 - 0\lambda^2 + 0\lambda - 1 = 0$$

$$\lambda^3 - 1 = 0$$

$$\lambda = 1, \quad -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Properties of eigenvalues and eigenvectors

5. The eigenvalues of a symmetric matrix are real

6. The eigenvalues of a skew-symmetric matrix are purely imaginary or zero.

7. The eigenvalues of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1.

Similar matrices

An $n \times n$ matrix B is called similar to an $n \times n$ matrix A if $B = P^{-1}AP$ for some non-singular $n \times n$ matrix P (ie, $|P| \neq 0$)

Note: If B is similar to A, then B has the same eigenvalues as A

Q: Write down the eigenvalues of $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

what are the eigenvalues of $P^{-1}AP$

where $P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

Ans: Here A is a diagonal matrix

eigenvalues of $A = 2, -1$

Given $P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

$$|P| = \begin{vmatrix} -4 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= 4 - 6$$

$$= -2 \neq 0$$

$\therefore$ A is similar to $P^{-1}AP$

The eigenvalues of $P^{-1}AP = 2, -1$

Diagonalization of a matrix

If an $n \times n$ matrix A is similar to a diagonal matrix D (ie, If $P^{-1}AP = D$) then the matrix A is diagonalizable.

Where D is a diagonal matrix with eigenvalues of A as the entries on the main diagonal.

and P is called the modal matrix with eigenvector as column vectors.

__Note:__

1. If $P^{-1}AP = D$ then $D^m = P^{-1}A^mP$ and $A^m = PD^mP^{-1}$

2. Suppose that A is an $n \times n$ matrix, then the following are equivalent

i A is diagonalizable

ii A has n linearly independent eigenvectors

1. Diagonalize the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

Ans: Characteristic equation is $|A - \lambda I| = 0$

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 0\lambda^2 + (-3)\lambda - (-2) = 0$

$\lambda^3 - 3\lambda + 2 = 0$

$\underline{\underline{\lambda = -2, 1, 1}}$

$\text{rank}(A+2I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$3x - 3y + 3z = 0$

$-3y + 6z = 0$

choose $z = t$

$-3y + 6t = 0 \Rightarrow y = 2t$

$3x - 3(2t) + 3t = 0 \Rightarrow x = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = -2$ is given by $(A - (-2)I)X = 0$

$$(A + 2I)X = 0$$

$$A+2I = \begin{bmatrix} 1+2 & -3 & 3 \\ 0 & -5+2 & 6 \\ 0 & -3 & 4+2 \end{bmatrix}$$

$$A+2I = \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda=1$ is given by

$(A-I)X = 0$

$$A-I = \begin{bmatrix} 1-1 & -3 & 3 \\ 0 & -5-1 & 6 \\ 0 & -3 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-I) = 1 < n$

$n - r = 3 - 1 = 2$ arbitrary variables

$-3y+3z=0$

Choose $z=t$, $x=s$

$$-3y+3t=0 \Rightarrow y=t$$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} s \\ t \\ t \end{bmatrix}=s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore X_2=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $X_3=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are two eigenvectors corresponding to $\lambda=1$

Eigenvalues are $-2, 1, 1$

Eigenvectors are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

So there are 3 linearly independent eigenvectors, then the matrix A is diagonalizable.

$$\therefore P^{-1}AP = D$$

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= D$$

2. Examine whether the matrix $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$ is diagonalizable.

Ans: Charasteristic equation is $|A - \lambda I| = 0$

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - 10\lambda + 25 = 0$$

$$\underline{\underline{\lambda = 5, 5}}$$

Eigenvectors corresponding to $\lambda = 5$ is given by

$(A - 5I)X = 0$

$$A - 5I = \begin{bmatrix} 7-5 & -1 \\ 4 & 3-5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$\text{rank}(A - 5I) = 1 < n$

$n - r = 2 - 1 = 1$, arbitrary Variable

$2x - y = 0$

choose $x = t$

$y = 2t$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Here there is only one eigenvector corresponding to $\lambda = 5$, then the matrix A is not diagonalizable.

3. Examine whether the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

is diagonalizable.

n: Characteristic equation is $|A - \lambda I| = 0$

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{Sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1, 2, 2.$

Eigenvector corresponding to $\lambda = 1$ is given by

$(A-I)X = 0$

$$A - I = \begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} R_1 \leftrightarrow R_3 \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$$-x + 2y + z = 0$$
$$y + z = 0$$

Choose $z = t$

$$y = -t$$
$$x = 2(-t) + t$$
$$= -t$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \therefore X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda=2$ is given by

$(A-2I)X=0$

$$A-2I = \begin{bmatrix} 1-2 & 2 & 2 \\ 0 & 2-2 & 1 \\ -1 & 2 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank }(A-2I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary Variable

$-x + 2y + 2z = 0$

$z = 0$

Choose $y = t$

$x = 2t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t\underline{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}, \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues are 1, 2, 2.

Eigenvecton are $X_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Here A has only two independent eigenvectors, A is <u>not</u> diagonalizable.

4. Diagonalize the matrix $A = \begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix}$

Ans: Given matrix is triangular,

the eigenvalues are 6, 2, 9.

Eigenvectors corresponding to $\lambda = 6$ is obtained by solving $(A - 6I)X = 0$

$$A - 6I = \begin{bmatrix} 6-6 & 0 & 0 \\ 12 & 2-6 & 0 \\ 21 & -6 & 9-6 \end{bmatrix}$$

$$A-6I = \begin{bmatrix} 0 & 0 & 0 \\ 12 & -4 & 0 \\ 21 & -6 & 3 \end{bmatrix} \begin{array}{l} R_2 \rightarrow \frac{R_2}{4} \\ \frac{R_3}{3} \leftrightarrow R_1 \end{array}$$

$$\sim \begin{bmatrix} 7 & -2 & 1 \\ 3 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \rightarrow 7R_2 - 3R_1$$

$$\sim \begin{bmatrix} 7 & -2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A-6I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable.

$$7x - 2y + z = 0$$
$$-y - 3z = 0$$

Choose $z = t$

$$y = -3t$$

$$7x - 2(-3t) + t = 0 \Rightarrow x = -t$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ -3t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \qquad \therefore X_1 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 2$ is given by

$(A - 2I)X = 0$

$$A - 2I = \begin{bmatrix} 6-2 & 0 & 0 \\ 12 & 2-2 & 0 \\ 21 & -6 & 9-2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 12 & 0 & 0 \\ 21 & -6 & 7 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 21 & -6 & 7 \\ 12 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to 21R_2 - 12R_1 \\ R_3 \to 21R_3 - 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 21 & -6 & 7 \\ 0 & 72 & -84 \\ 0 & 24 & -28 \end{bmatrix} \quad R_3 \to 3R_3 - R_2$$

$$\sim \begin{bmatrix} 21 & -6 & 7 \\ 0 & 72 & -84 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-2I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$21x - 6y + 7z = 0$

$72y - 84z = 0$

Choose $z = t$

$72y - 84t = 0 \Rightarrow y = \frac{84}{72}t = \frac{7}{6}t$

$21x - 6\left(\frac{7}{6}t\right) + 7t = 0$

$x = 0$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{7}{6}t \\ t \end{bmatrix} = \frac{t}{6}\begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix} \quad \therefore X_2 = \begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 9$ is given by

$(A - 9I)X = 0$

$$A - 9I = \begin{bmatrix} 6-9 & 0 & 0 \\ 12 & 2-9 & 0 \\ 21 & -6 & 9-9 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 12 & -7 & 0 \\ 21 & -6 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & -6 & 0 \end{bmatrix} \quad R_3 \to 7R_3 - 6R_2$$

$$\sim \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A-9I)=2<n$

$n-r=3-2=1$, arbitrary variable.

$-3x=0 \Rightarrow x=0$

$-7y=0 \Rightarrow y=0$

choose $z=t$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}=t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \therefore X_3=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues are 6, 2, 9

Eigenvectors are $X_1 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 7 \\ 6 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Here there are 3 linearly independent eigenvectors, A is diagonalizable.

$\therefore P^{-1} A P = D$

Where $P = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 7 & 0 \\ 1 & 6 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -\frac{3}{7} & \frac{1}{7} & 0 \\ \frac{25}{7} & -\frac{6}{7} & 1 \end{bmatrix}$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ -\frac{3}{7} & \frac{1}{7} & 0 \\ \frac{25}{7} & -\frac{6}{7} & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 12 & 2 & 0 \\ 21 & -6 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ -3 & 7 & 0 \\ 1 & 6 & 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}}}$$

Orthogonal transformations

Orthogonal transformations are transformations (or mapping) $y = Ax$ where A is an orthogonal matrix.

Length (or norm) of any vector $\vec{a}$ is given by

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$$

Orthonormality of Vectors

A real square matrix is orthogonal iff its column vectors $a_1, a_2, \cdots a_n$ (also its row vectors) form an orthonormal system.

ie, $a_i \cdot a_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Diagonalization of Symmetric matrix

A real square matrix 'A' is symmetric if $\boxed{A^T = A}$

The eigenvalues of A are real and the eigenvectors of A form an orthonormal system. We have to normalize the vectors in the matrix P. To normalize the vectors we have to divide each vectors by its length or norm. ie, $P = \left[\frac{X_1}{\|X_1\|} \; \frac{X_2}{\|X_2\|} \; \frac{X_3}{\|X_3\|} \right]$. Such normalized matrix P is orthogonal. So $\boxed{P^{-1} = P^T}$

1\. Diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Ans: Characteristic eqn is $|A - \lambda I| = 0$

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$$

$$\lambda^3 - 11\lambda^2 + 35\lambda - 25 = 0$$

$$\underline{\underline{\lambda = 1, 5, 5}}$$

Eigenvector corresponding to $\lambda = 1$ is given by

$(A - I)X = 0$

$$A - I = \begin{bmatrix} 3-1 & -2 & 0 \\ -2 & 3-1 & 0 \\ 0 & 0 & 5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad R_3 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A - I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$2x - 2y = 0 \Rightarrow x = y$

$4z = 0 \Rightarrow z = 0$

Choose $x = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 5$ is given by

$(A - 5I)X = 0$

$$A - 5I = \begin{bmatrix} 3-5 & -2 & 0 \\ -2 & 3-5 & 0 \\ 0 & 0 & 5-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-5I) = 1 < n$

$n-r = 3-1 = 2$, arbitrary variables

$-2x - 2y = 0 \Rightarrow x = -y$

Choose $x = t$, $z = s$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 5 \qquad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 5 \qquad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 \cdot x_2 = (1\ 1\ 0)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 - 1 + 0 = 0$$

$$x_1 \cdot x_3 = (1\ 1\ 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$x_2 \cdot x_3 = (1\ -1\ 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\|x_1\| = \sqrt{x_1 \cdot x_1} = \sqrt{(1)^2 + (1)^2 + 0^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{x_2 \cdot x_2} = \sqrt{(1)^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\|x_3\| = \sqrt{x_3 \cdot x_3} = \sqrt{0^2 + 0^2 + 1^2} = 1$$

$$P = \left[\frac{X_1}{\|X_1\|} \quad \frac{X_2}{\|X_2\|} \quad \frac{X_3}{\|X_3\|} \right]$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \; P^{-1}AP = D \Rightarrow P^{T}AP = D$$

$$P^{T}AP = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

2. Diagonalize $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and hence find A^4.

Ans: Characteristic equation is $|A - \lambda I| = 0$

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1, 2, 3$

Eigenvector corresponding to $\lambda = 1$ is given by

$(A - I) X = 0$

$$A - I = \begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$rank(A-I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$x + z = 0 \implies x = -z$

$y = 0$

choose $z = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\underline{\lambda=2}$:

$$A-2I = \begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_1,$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-2I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$x = 0$

$z = 0$

Choose $y = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda = 3}$:

$$A - 3I = \begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $(A-3I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$-x + z = 0 \Rightarrow x = z$

$-y = 0 \Rightarrow y = 0$

Choose $z = t$

$$X = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \underline{\underline{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}}$$

Eigenvalues : $\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$

Eigenvectors : $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X_1 \cdot X_2 = 0 \qquad \|X_1\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$$X_1 \cdot X_3 = 0 \qquad \|X_2\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$

$$X_2 \cdot X_3 = 0 \qquad \|X_3\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \underline{\underline{D}}$$

Since A is diagonalizable,

$$P^T A P = D \Rightarrow P^T A^4 P = D^4$$

$$\therefore \; A^4 = P D^4 P^T$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 41 & 0 & 40 \\ 0 & 16 & 0 \\ 40 & 0 & 41 \end{bmatrix}}}$$

3. Diagonalize the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Ans: Characteristic eqn is $|A - \lambda I| = 0$

$$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$$

$$\lambda^3 - 9\lambda^2 + (8 + (8) + 8)\lambda - 20 = 0$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

$$\lambda = \underline{\underline{5, 2, 2}}$$

Eigenvector corresponding to $\lambda = 2$ is given by

$$(A - \lambda I) X = 0$$

$$(A - 2I) X = 0$$

$$A - 2I = \begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 3-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}\,(A - 2I) = 1 < n$

$n - r = 3 - 1 = 2$ arbitrary variables

$x - y + z = 0$

Choose $x = t$, $z = s$

$y = t + s$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to $\lambda = 5$ is given by

$(A-5I)X = 0$

$$A-5I = \begin{bmatrix} 3-5 & -1 & 1 \\ -1 & 3-5 & -1 \\ 1 & -1 & 3-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} -2 & -1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A-5I) = 2 < n$

$n - r = 3 - 2 = 1$ arbitrary variable

$-2x - y + z = 0$

$-3y - 3z = 0$

choose $z = t$

$y = -t$

$-2x - (-t) + t = 0 \Rightarrow x = t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \therefore X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvalues : $\lambda_1 = 2 \quad \lambda_2 = 2 \quad \lambda_3 = 5$

eigenvectors : $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$X_1 \cdot X_2 = 1 \neq 0$$

$$X_2 \cdot X_3 = 0$$

$$X_1 \cdot X_3 = 0$$

Let $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ∋ $X_1 \cdot X_2 = 0$ and $X_2 \cdot X_3 = 0$

$$X_1 \cdot X_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow a + b + 0c = 0$$

$$X_2 \cdot X_3 = 0 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0 \Rightarrow a - b + c = 0$$

$$\begin{array}{cccc} & a = & b = & c \\ \hline 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{a}{1-0} = \frac{b}{0-1} = \frac{c}{-1-1}$$

$$\frac{a}{1} = \frac{b}{-1} = \frac{c}{-2}$$

$$\therefore X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\|x_1\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{1^2+(-1)^2+(-2)^2} = \sqrt{6}$$

$$\|x_3\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = 2 \quad \lambda_3 = 5$$

$$P = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

and P is orthogonal

$$P^T A P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \underline{\underline{D}}$$

Quadratic form

A quadratic form Q in the components x_1, x_2, x_3 of a vector x is a sum

$$Q = x^T A x$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 +$$
$$a_{21} x_1 x_2 + a_{22} x_2^2 + a_{23} x_2 x_3 +$$
$$a_{31} x_1 x_3 + a_{32} x_2 x_3 + a_{33} x_3^2$$

Where $A = [a_{ij}]$ is called the coefficient matrix of the quadratic form.

Here we assume that A is symmetric

$\therefore$ We get the matrix P as orthogonal

then $P^{-1}AP = D \Rightarrow P^{T}AP = D$

$\Rightarrow A = PDP^{T}$

$$\boxed{\therefore Q = X^{T} PDP^{T} X}$$

Canonical form or principal axes form

If we set the orthogonal transformation

$$Y = P^T X \Rightarrow \boxed{X = PY}$$

then $Y^T = (P^T X)^T = X^T P$ and

$$Q = X^T P D P^T X \Rightarrow Q = Y^T D Y$$

i.e., $\boxed{Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2}$

which is the principal axes form or Canonical form.

Where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the symmetric matrix A and P is an orthogonal matrix with corresponding eigenvectors X_1, X_2, X_3 respectively, as column vectors.

1. Find out what type of conic section the following quadratic form represents and transform it to principal axes.

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$$

Ans: We have $Q = X^T A X$

where $A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Characteristic equation is $|A-\lambda I| = 0$

$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$

$\lambda^2 - 34\lambda + 64 = 0$

$\lambda = 2, 32$

Thus $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$= \underline{2y_1^2 + 32y_2^2}$; Principal axes form

Given $Q = 128 \Rightarrow 2y_1^2 + 32y_2^2 = 128$

$$\frac{2y_1^2}{128} + \frac{32y_2^2}{128} = \frac{128}{128}$$

$$\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$ which is an ellipse

Conic Section: ellipse.

$\underline{\lambda_1 = 2}$: $A - 2I = \begin{bmatrix} 17-2 & -15 \\ -15 & 17-2 \end{bmatrix}$

$= \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$ $R_2 \to R_2 + R_1$

$= \begin{bmatrix} 15 & -15 \\ 0 & 0 \end{bmatrix}$

$\text{rank}(A - 2I) = 1 < n$

$n - r = 2 - 1 = 1$ arbitrary variable

$15x - 15y = 0 \Rightarrow x = y$

Choose $y = t$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \underline{\underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$

$\underline{\lambda_2 = 32}$: $A - 32I = \begin{bmatrix} 17-32 & -15 \\ -15 & 17-32 \end{bmatrix}$

$= \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \quad R_2 \to R_2 - R_1$

$\approx \begin{bmatrix} -15 & -15 \\ 0 & 0 \end{bmatrix}$

$-15x - 15y = 0 \Rightarrow x = -y$

choose $y = t$

$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \underline{\underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}}$

Eigenvalues : $\lambda_1 = 2 \quad \lambda_2 = 32$

Eigenvectors : $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$X_1 \cdot X_2 = 0 \qquad \|X_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\|X_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$$P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The orthogonal transformation is

$$X = PY$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

ie,
$$x_1 = \frac{1}{\sqrt{2}} y_1 - \frac{1}{\sqrt{2}} y_2$$

$$x_2 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2$$

2. What kind of conic section is given by the quadratic form $x_1^2 - 12x_1x_2 + x_2^2 = 70$. Also find its equation.

Ans: We have $Q = X^T A X$

$$A = \begin{bmatrix} 1 & -6 \\ -6 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - 2\lambda + (-35) = 0$$

$$\lambda^2 - 2\lambda - 35 = 0$$

$$\lambda = 7, -5$$

Thus $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$$= 7y_1^2 - 5y_2^2$$

Given $Q = 70 \Rightarrow 7y_1^2 - 5y_2^2 = 70$

$$\frac{7y_1^2}{70} - \frac{5y_2^2}{70} = 1$$

$$\frac{y_1^2}{10} - \frac{y_2^2}{14} = 1$$

$$\frac{y_1^2}{(\sqrt{10})^2} - \frac{y_2^2}{(\sqrt{14})^2} = 1$$

Conic Section is <u>hyperbola</u>

Index

The no: of positive eigenvalues of a matrix is called the index of the quadratic form.

Rank

The no: of non-zero eigenvalues of a matrix is called the rank of the quadratic form

Signature

(The no: of positive eigenvalues) − (The no: of negative eigenvalues)

Nature of a quadratic form

A real quadratic form $Q = X^T A X$ is said to be

i positive definite if all eigenvalues of A are positive.

ii negative definite if all eigenvalues of A are negative iii positive semi definite if all eigenvalues of A are positive with atleast one of them being zero.

iv negative semi definite if all eigenvalues of A are negative with at least one of them being zero.

v indefinite if A has both positive and negative eigenvalues.

Q: Find the nature, rank, index and signature of the quadratic form

$$3x^2 - 2y^2 - z^2 - 4xy + 8xz + 12yz$$

Ans: $Q = X^T A X$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}$$

Characteristic equation is

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 0\lambda^2 + (-34 - 19 - 10)\lambda - (-162) = 0$

$\lambda^3 - 63\lambda + 162 = 0$

$\lambda = -9, 6, 3$

nature : indefinite

rank = no: of non-zero eigenvalues = 3

index = no: of positive eigenvalues = 2

Signature = no: of positive eigenvalues − no: of negative eigenvalues

= 2 − 1

= $\underline{\underline{1}}$

Q. Find the nature, rank, index and Signature of the quadratic form

$$-5x_1^2 - 2x_2^2 + 4x_1x_2$$

Ans: $Q = X^T A X$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

characteristic equation is

$$\lambda^2 - (\text{trace } A)\lambda + \det A = 0$$

$$\lambda^2 - (-7)\lambda + 6 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\lambda = -1, -6$$

nature : negative definite

rank = no: of non-zero eigenvalues

= 2

index = no: of positive eigenvalues

= 0

Signature = no: of +ve eigenvalues $-$ no: of negative eigenvalues

$= 0 - 2$

$= \underline{\underline{-2}}$

4 Find the orthogonal transformation which will transform $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into canonical form.

Ans:

$$Q = X^T A X$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Characteristic equation is $|A-\lambda I| = 0$

$\lambda^3 - (\text{trace } A)\lambda^2 + (\text{sum of principal minors})\lambda - \det A = 0$

$\lambda^3 - 12\lambda^2 + (8+14+14)\lambda - 32 = 0$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 8, 2, 2$

Thus $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$= \underline{\underline{2y_1^2 + 2y_2^2 + 8y_3^2}}$; canonical form

$\underline{\lambda = 2}$: $A - 2I = \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}$

$= \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ $\begin{array}{l} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$

$\sim \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A-2I) = 1 < n$

$n - r = 3 - 1 = 2$, arbitrary variables

$4x - 4y + 2z = 0$

choose $x = t$, $y = s$

$$z = \frac{4y - 4x}{2}$$

$$= 2s - 2t$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ s \\ 2s-2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$\underline{\lambda=8}$:

$$A-8I=\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} R_3 \to R_3 - R_2 \sim \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A - 8I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$-2x - 2y + 2z = 0$

$-3y - 3z = 0$

Choose $z = t$

$y = -z = -t$

$-2x - 2(-t) + 2t = 0 \Rightarrow x = \frac{-4t}{-2} = 2t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\text{rank}(A - 8I) = 2 < n$

$n - r = 3 - 2 = 1$, arbitrary variable

$-2x - 2y + 2z = 0$

$-3y - 3z = 0$

Choose $z = t$

$y = -z = -t$, $x = 2t$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

rank $(A-2I) = 1 < n$

$n - r = 3 - 1 = 2$, arbitrary variables

$4x - 2y + 2z = 0$

Choose $y = t$

$z = s$

$4x - 2t + 2s = 0 \Rightarrow x = \frac{2t - 2s}{4}$

$= \frac{t-s}{2}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{t-s}{2} \\ t \\ s \end{bmatrix} = \frac{t}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{s}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

eigenvectors are $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$X_1 \cdot X_2 = -1 \neq 0$ $\quad X_2 \cdot X_3 = -2 + 0 + 2 = 0$

$X_1 \cdot X_3 = 2 - 2 + 0 = 0$

Let $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $X_2 \cdot X_3 = 0$

$X_1 \cdot X_2 = 0$

$$X_2 \cdot X_3 = 0 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 0 \Rightarrow 2a - 1b + 1c = 0$$

$$X_1 \cdot X_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow 1a + 2b + 0c = 0$$

$$\begin{array}{cccccccc} & a & & & b & & & c \\ \hline -1 & & 1 & & 2 & & -1 & \\ & \times & & \times & & \times & & \\ 2 & & 0 & & 1 & & 2 & \end{array}$$

$$\frac{a}{0-2} = \frac{b}{1-0} = \frac{c}{4+1}$$

$$\frac{a}{-2} = \frac{b}{1} = \frac{c}{5}$$

$$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$$

Eigenvectors are $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$\|X_1\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$\|X_2\| = \sqrt{(-2)^2 + 1^2 + 5^2} = \sqrt{30}$$

$$\|X_3\| = \sqrt{(2)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$P = \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The orthogonal transformation is

$$X = P Y$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\therefore \quad x = \frac{1}{\sqrt{5}} y_1 - \frac{2}{\sqrt{30}} y_2 + \frac{2}{\sqrt{6}} y_3$$

$$y = \frac{2}{\sqrt{5}} y_1 + \frac{1}{\sqrt{30}} y_2 - \frac{1}{\sqrt{6}} y_3$$

$$z = \frac{5}{\sqrt{30}} y_2 + \frac{1}{\sqrt{6}} y_3$$

Quadratic form $Q = X^T A X$ in two variables

$$ax^2 + by^2 + 2hxy = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underset{1\times 2}{\begin{bmatrix} x & y \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} a & h \\ h & b \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

Quadratic form $Q = X^T A X$ in three variables :-

$$ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fxz$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$