Yuriy Gorodnichenko
Econ 101B, Spring 2025

# A note on budget constraint in intertemporal optimization

In the Fisher model, a consumer wants to choose a combination of consumption levels in various times that provides the highest possible utility. When we have intertemporal optimization (i.e. optimal allocation of resources across time), we can treat consumption (income) at time $T$ as a separate good called "CONSUMPTION AT TIME $T$". We typically express the price of "CONSUMPTION AT TIME T" in terms of the first period consumption which is in CURRENT dollars. The interest rate measures the opportunity cost of consumption today relative to consumption in the future. Hence, the price of "CONSUMPTION AT TIME T" in terms of "CONSUMPTION AT TIME 1" is $1/(1+r)^{T-1}$, which shows that a consumer must sacrifice $1/(1+r)^{T-1}$ units of consumption today to get one unit of consumption in period T. (An alternative way to see the pricing of consumption at time T is to note that if one puts \$1 today in a bank, he or she would get $\$(1+r)^{T-1}$ in the beginning of period T.)

In fact, it is useful to think that every period people are paid in various goods and likewise consumption in various periods is consumption of various goods. For instance, consumption at the first period can be thought as consumption of apples, and consumption at period 2 as a consumption of oranges. Likewise, income at period 1 is paid in apples, and at period 2 in oranges. Note that in this interpretation, the price of oranges in terms of apples is $1/(1+r)$. The value of a given combination of apples and oranges in terms of apples will be:

**(*number of apples*) + (*number of oranges*) × (*price of oranges in terms of apples*)**

Hence, in general case, the value of a given path of consumption choices $\{C_1, C_2, C_3, ..., C_T\}$ will be

$$
\begin{aligned}
PV_C &= p_1 C_1 + p_2 C_2 + \ldots + p_i C_i + \ldots + p_{T-1} C_{T-1} + p_T C_T = \\
&= 1 \cdot C_1 + \frac{1}{1+r} C_2 + \ldots + \frac{1}{(1+r)^{i-1}} C_i + \ldots + \frac{1}{(1+r)^{T-2}} C_{T-1} + \frac{1}{(1+r)^{T-1}} C_T = \\
&= C_1 + \frac{1}{1+r} C_2 + \ldots + \frac{1}{(1+r)^{i-1}} C_i + \ldots + \frac{1}{(1+r)^{T-2}} C_{T-1} + \frac{1}{(1+r)^{T-1}} C_T
\end{aligned}
$$

Note that this is nothing else but the present value of consumption path $\{C_1, C_2, C_3, ..., C_T\}$, that is how much we need to spend in terms of CURRENT dollars to purchase $\{C_1, C_2, C_3, ..., C_T\}$.

If we continue to use the metaphor of fruits in the Fisher model, the value of our total income in terms of apples would be:

***(income we get in apples) plus (income we get in oranges)*(the price of oranges in terms of apples)***

Again the price oranges in terms of apples is $1/(1+r)$. If we generalize this to N periods (N fruits!) then the value of our incomes in various periods $\{W_1, W_2, W_3, ..., W_T\}$ (often called endowments) will be

$$PV_W = p_1 W_1 + p_2 W_2 + ... + p_i W_i + ... + p_{T-1} W_{T-1} + p_T W_T =$$
$$= 1 \cdot W_1 + \frac{1}{1+r} W_2 + ... + \frac{1}{(1+r)^{i-1}} W_i + ... + \frac{1}{(1+r)^{T-2}} W_{T-1} + \frac{1}{(1+r)^{T-2}} W_T =$$
$$= W_1 + \frac{1}{1+r} W_2 + ... + \frac{1}{(1+r)^{i-1}} W_i + ... + \frac{1}{(1+r)^{T-2}} W_{T-1} + \frac{1}{(1+r)^{T-1}} W_T$$

Again $PV_W$ is the value of our incomes in various times (endowment) in terms of CURRENT dollars.

Not surprisingly, $PV_C$ and $PV_W$ are respectively called present value of expenditures on consumption and present value of endowment since we measure everything in terms of current (or present, or period 1) dollars.

Like in standard consumption problem, total expenditures cannot exceed budget. In this problem, total expenditures on consumption in various periods are like total expenditures on all sorts of fruits and the budget is nothing else but the value of fruits in the fruits endowment. Provided that resources (fruits) are not wasted, $PV_C = PV_W$ must hold.

Moreover, if we know that consumption is the same across time, i.e. $C_1 = C_2 = ... = C_{T-1} = C_T = C$, then

$$PV_C = C_1 + \frac{1}{1+r} C_2 + \frac{1}{(1+r)^2} C_3 ... + \frac{1}{(1+r)^{i-1}} C_i + ... + \frac{1}{(1+r)^{T-2}} C_{T-1} + \frac{1}{(1+r)^{T-1}} C_T =$$
$$= C\left(1 + \frac{1}{1+r} + ... + \frac{1}{(1+r)^{i-1}} + ... + \frac{1}{(1+r)^{T-2}} + \frac{1}{(1+r)^{T-1}}\right)$$

And given $PV_C = PV_W$, we have

$$PV_W = PV_C = C\left(1 + \frac{1}{1+r} + ... + \frac{1}{(1+r)^{i-1}} + ... + \frac{1}{(1+r)^{T-2}} + \frac{1}{(1+r)^{T-1}}\right) \Rightarrow$$
$$C = \frac{W_1 + \frac{1}{1+r} W_2 + ... + \frac{1}{(1+r)^{i-1}} W_i + ... + \frac{1}{(1+r)^{T-2}} W_{T-1} + \frac{1}{(1+r)^{T-1}} W_T}{\left(1 + \frac{1}{1+r} + ... + \frac{1}{(1+r)^{i-1}} + ... + \frac{1}{(1+r)^{T-2}} + \frac{1}{(1+r)^{T-1}}\right)}$$

If interest rate is equal to zero, i.e. $r = 0$, then

$$C = \frac{W_1 + \frac{1}{1+0} W_2 + ... + \frac{1}{(1+0)^{i-1}} W_i + ... + \frac{1}{(1+0)^{T-2}} W_{T-1} + \frac{1}{(1+0)^{T-1}} W_T}{\left(1 + \frac{1}{1+0} + ... + \frac{1}{(1+0)^{i-1}} + ... + \frac{1}{(1+0)^{T-2}} + \frac{1}{(1+0)^{T-1}}\right)} =$$
$$= \frac{W_1 + W_2 + ... + W_i + ... + W_{T-1} + W_T}{\underbrace{(1 + 1 + ... + 1 + ... + 1 + 1)}_{T \quad times}} = \frac{1}{T}\left(W_1 + W_2 + ... + W_i + ... + W_{T-1} + W_T\right)$$