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A Primer on Derivatives and Maximization Problems

While calculus is not required for the class, and derivatives will not need to be used on exams, some parts of the class will seem clearer with some understanding of basic calculus. Here is brief description of what derivatives do and how to take simple derivatives.

What is a derivative?

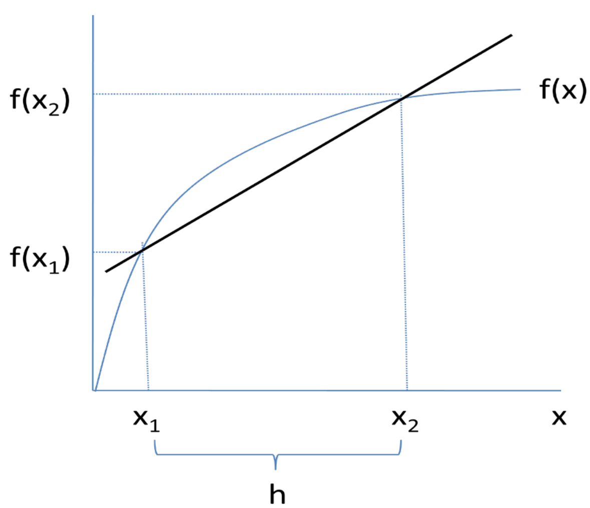
Imagine that we have some function of x: $y = f(x)$. This says that if I give you a value for x, you can plug it into the function and give me a value of y. The derivative is all about finding the slope of the function at any given x. Recall the basic equation for the slope of a line given two points on the line: (x_1,y_1) and (x_2,y_2):

$$Slope = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

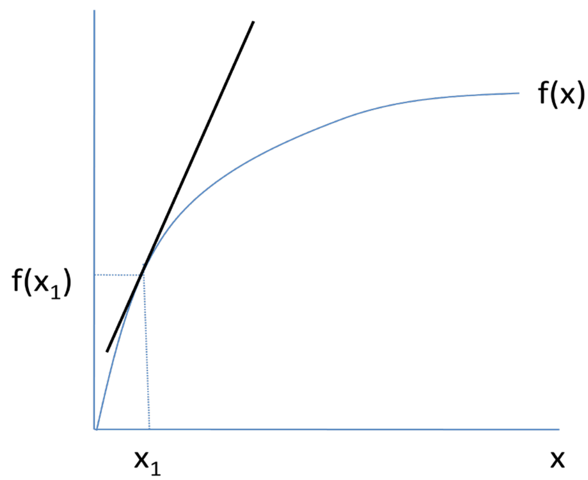
Now, mechanically, the difference between x_2 and x_1 is just some number h: $h=x_2-x_1$. So the slope of the line between (x_1,y_1) and any other point can be expressed in terms of how far the second point is from the first point.

$$Slope = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1+h) - f(x_1)}{x_1 + h - x_1} = \frac{f(x_1+h) - f(x_1)}{h}$$

These are purely mechanical manipulations. We've now expressed the slope of the line between any two points on function *f(x)* in terms of the difference between the *x*-values. This is illustrated in the graph below:



What our equation for the slope gives us is the slope of the line between the two points in the graph. Now imagine if we make h get smaller and smaller: i.e. we're moving x_2 closer and closer to x_1. As h becomes arbitrarily small, what we end getting is the slope of the function at the point x_1.



That's the derivative: it's the slope of the function at one point. Formally, it's the limit of the slope between two points as the horizontal distance between the two points shrinks to zero.

<u>*Definition 1*</u>*: If f is a function,* ***the derivative of the function f****, denoted by f', is defined by the formula*

$$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

This is the formal definition of the derivative. One can use this formula to find the slope of a function.

<u>*Example 1:*</u> Suppose $f(x)=3x^2$. Then

$$f(x+h) = 3(x+h)^2 = 3(x^2+2xh+h^2) = 3x^2+6xh+3h^2$$

So
$$f(x+h)-f(x) = (3x^2+6xh+3h^2)-3x^2 = 6xh+3h^2$$

Then divide by h:

$$\frac{f(x_1+h)-f(x_1)}{h} = \frac{6xh+3h^2}{h} = 6x+3h$$

Finally, let's take the limit as h goes to zero (just set $h=0$) to find f'

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 6x+3h = 6x$$

Thus, the derivative of the function $f(x)=3x^2$ at any x is given by $6x$. For example, when $x=1$, the slope of the function is 6(1)=6. When $x=2$, the slope of the function is 6(2)=12.

When $y=f(x)$, the derivative of y with respect to x ($f'(x)$) is also often denoted by dy/dx. Note that this looks a lot like $\Delta y/\Delta x$. The little d also means "a change in" but refers to very small change.

Shortcuts for taking derivatives

In theory, one could continue to take derivatives using the definition, but this is a very roundabout way. Instead, there are some great shortcuts that can be used to take derivatives very quickly and efficiently. Let's run through them:

Derivative of a constant: If $f(x)=c$, where c is a constant, then $f'(x)=0$.

Derivative of a line: If $f(x)=ax+b$, where a and b are constants, then $f'(x)=a$.

Power Rule: If $f(x)=x^n$ where n is a constant (not equal to 0), then $f'(x)=nx^{n-1}$.

Constants in front of functions: Suppose $g(x)=cf(x)$. Then $g'(x)=cf'(x)$.

Some examples:

a) $f(x)=x^4$, then $f'(x)=4x^3$ by the power rule.

b) $g(x)=2x^4$. By the rule about constants in front, combined with power rule, we have $g'(x)=2(4x^3)=8x^3$.
c) $f(x)=x^{-4}$. Then $f'(x)=-4x^{-5}$.

Derivative of a sum: Suppose $F(x)=f(x)+g(x)$, then $F'(x)=f'(x)+g'(x)$.

d) $F(x)=3x^2+5x$, then $F'(x)=3(2x)+5=6x+5$.

Derivative of a product: Suppose $F(x)=f(x)*g(x)$, then $F'(x)=f'(x)*g(x)+f(x)*g'(x)$.

e) $F(x)=(2x+5)(x^2)$ then $F'(x)=(2)(x^2)+(2x+5)(2x)=2x^2+(4x^2+10x)=6x^2+10x$.

<u>*The Chain Rule*</u>:

Suppose we have a function inside a function: $f(x)=g[u(x)]$. Then the derivative of f is given by

$$f'(x) = g'\big(u(x)\big)u'(x)$$

Example: Suppose $f(x)=[2x+5]^3$, then we can think of $2x+5$ as $u(x)$. First, treat $u(x)$ as constant and take the outside derivative, then multiply it by the derivative of the inside ($u(x)$).

$$f'(x) = 3(2x+5)^2(2) = 6(2x+5)^2$$

In general, you don't need to use the chain rule. You can always multiply things out first, then differentiate:
$f(x)=[2x+5]^3=(2x+5)(2x+5)(2x+5)=(4x^2+20x+25)(2x+5)=(8x^3+20x^2+40x^2+100x+50x+125)$. So $f(x)=8x^3+60x^2+150x+125$. Thus, $f'(x)=24x^2+120x+150$. This is the same answer as that above (check this by multiplying out the top expression). However, the latter approach can become tedious: imagine if instead of $f(x)=[2x+5]^3$ we had $f(x)=[2x+5]^{100}$. Multiplying this out would be infeasible. However, with the chain rule we would just have

$$f'(x) = 100(2x+5)^{99}(2) = 200(2x+5)^{99}$$

and we're done!

<u>*Partial Derivatives*:</u>

Suppose we have a function $y=f(x,z)$ that depends on two variables x and z. We want to know how y changes when we increase x by a tiny amount. What we can do is take the derivative of f with respect to x. What this means is that we simply treat z as a constant. We will be using this in the class.

Example: Suppose an individual receives utility from consuming two goods: x and y. His utility function is given by

$$U(x,y)=2x+3y^2.$$

By how much would his utility rise if we gave him just a little bit more *x*? In other words, what is the marginal utility of *x* (*MU(x)*)? Take the derivative with respect to *x*, i.e. treat *y* as constant. Then

$$MU(x) = \frac{dU(x,y)}{dx} = 2$$

Similarly, we could ask about the marginal utility of *y*, i.e. take the derivative of utility function with respect to *y*, holding *x* constant.

$$MU(y) = \frac{dU(x,y)}{dy} = 3(2y) = 6y$$

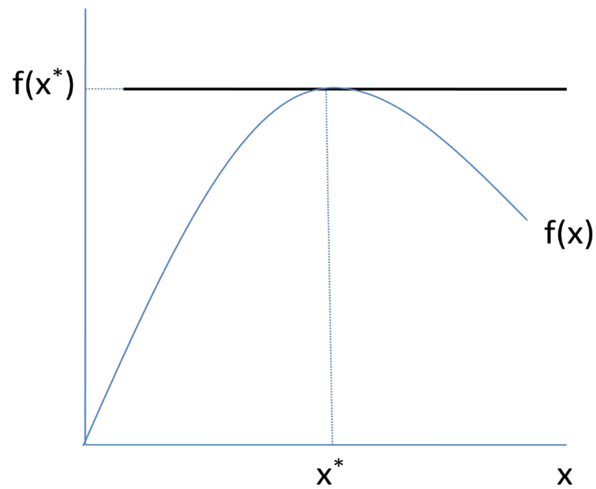
Suppose instead the utility function is given by *U(x,y)=6xy*. Then, by the same logic, the marginal utilities are given by:

$$MU(x) = \frac{dU(x,y)}{dx} = 6y$$

$$MU(y) = \frac{dU(x,y)}{dy} = 6x$$

<u>*Maximization Problems*</u>:

In economic problems, most derivatives are taken as part of a maximization problem, and not just for the fun of taking derivatives. So what does the derivative have to do with maximization? Suppose we have a function *f(x)* as shown in the graph below. We want to know what value of *x* gives us the maximum value of *f(x)*.



In the graph, it's easy to tell that the maximum is reached when *x=x**. The key feature is that the slope of the function at *x** is zero. Why does this have to be the case? When the slope of the function is positive, it means that if I give you a little more *x*, then *f(x)* will increase. Clearly, you cannot have reached the maximum value if a little more *x* will give you a higher *f(x)*. By the same logic, if the slope is negative, then raising *x* will lower *f(x)*. This also means that lowering *x* will raise *f(x)*. Again, this cannot be a maximum. Thus, *what has to be true at the maximum is that the slope of the function is zero*.

This is where derivatives come in. The derivative of the function gives the slope of the function at each point. The condition for a maximum is that the slope is zero. So all we need to do to find conditions for a maximum is to *"take the derivative and set it equal to zero"* (sometimes this is called the first order condition).[1]

x* is a maximum of f(x) if f'(x*)=0.

Example 1:

Suppose we want to find the maximum of $f(x)=-2x^2+5$.

Let's take the derivative: $f'(x)=-4x$.

Now let's set it equal to zero and solve for x: $-4x=0 \rightarrow x^*=0$.

The maximum value of f(x) occurs at x=0.

[1] In general, there are some other conditions one needs to worry about. For the purposes of this class, we don't need to worry about them. So the definition provided here is sufficient for our purposes.

Example 2:

Consider a firm that wants to maximize profits. It uses only labor N in production and must pay its workers a fixed wage W. It sells its goods at a price p determined by the market: the firm's decisions have no effect on this price (it's called a price-taker). Production for the firm Y is determined by how much labor it uses according to the function $Y=N^{\alpha}$ where α is some constant between zero and 1 . The firm's total profit is given by total revenues (P*Y) minus total costs (W*N). The goal is to maximize profits by choosing N. We would write this problem as

$$\max_{N} PN^{\alpha} - WN$$

where I've plugged in N^{α} for Y. How do find the value of N that maximizes profits? Let's take the derivative with respect to N and set it equal to zero:

$$\alpha PN^{\alpha-1} - W = 0$$

which we can rearrange as

$$N = \left[\alpha \frac{P}{W}\right]^{\frac{1}{1-\alpha}}$$

This implies that the higher the wage, the less labor the firm will hire. The higher the price, the more labor the firm will hire. So given these variables that are determined by forces outside of the firm's control, we've solved the firm's maximization problem and now know how the firm will change its employment and production decisions as these other variables change.

Note that derivatives also help us solve for key economic concepts:

Marginal product of labor: additional output from hiring more labor is $dY/dN=\alpha N^{\alpha-1}$.

Marginal cost: additional cost from hiring more labor is $d(WN)/dN=W$.

<u>*Total Derivatives:*</u>

The final aspect of differentiation that we may use is what's known as a total derivative. First recall how we did a partial derivative when there were two variables in a function y=f(x,z):

$$\frac{dy}{dx} = \frac{df(x,z)}{dx}$$

For example: if $f(x,z)=x^2z^2$, then $dy/dx=2xz^2$. We could rewrite this as

$$dy = 2xz^2 dx$$

which says that the change in y (dy) is $2xz^2$ times the change in x, where we held z constant. We could also do the same with respect to z to get

$$dy = 2x^2 z dz$$

which tells us by how much *y* changes for a given change in *z*.

Sometime, we want to know how *y* changes when both *x* and *z* change at the same time. In that case, we would use the total derivative:

$$dy = \frac{df(x,z)}{dx} dx + \frac{df(x,z)}{dz} dz$$

which decomposes changes in *y* in terms of both changes in *x* and *z*. The key is that the coefficients in front of *dx* and *dz* are the partial derivatives of *f(x,z)* with respect to *x* and *z* respectively.

Using the same example: if $f(x,z)=x^2z^2$, then we can decompose changes in *y* using the total derivative:

$$dy = \frac{df(x,z)}{dx} dx + \frac{df(x,z)}{dz} dz = 2xz^2 dx + 2x^2 z dz$$

When would this be useful? Suppose we have a consumer with utility function over two goods:

$$U(x,y) = x^2 + y^2$$

Then we can decompose changes in utility that come changes in the consumption of *x* and *y* using the total derivative:

$$dU(x,y) = 2xdx + 2ydy$$

Then we could ask, suppose that our agent gets a little more *y* to consume. By how much can he lower his consumption of *x* and still keep his utility unchanged? Keeping utility unchanged means dU=0. Plugging this in gives

$$0 = 2xdx + 2ydy$$

or equivalently

$$dx = -\frac{y}{x} dy$$

which would tell us that for a change *dy*, the consumer could reduce his consumption of *x* by *y/x* times the change in *y* and keep the same level of utility. This process allows us to define *indifference curves* for consumers: combinations of *x* and *y* that all yield the same level of utility for the consumer.

Practice Problems:

A) Take derivatives of the following functions:
1) $F(x)=2x^4$
2) $F(x)=12x+x^5$
3) $F(x)=(4x-3)(1-2x)$
4) $F(x)=(2x-1)/(3x+5)$ *hint*: $1/x=x^{-1}$ and $1/x^2=x^{-2}$

B) I give you a utility function U(x,y), you solve for the marginal utility of x: MU(x)=dU/dx.
1) $U(x,y)=2x+y$
2) $U(x,y)=2y$
3) $U(x,y)=4xy^2$
4) $U(x,y)=(3x^2-1)^2(2y-5)$

C) Take total derivatives to decompose changes in Y into changes in N, K, and A.
1) $Y=AKN$
2) $Y=AK^{\alpha}N^{1-\alpha}$
3) $Y=A+2K+3N$

D) Find the value of N that maximizes profits, where profits are given by PY-WN
1) P is exogenous, $Y=N^{\alpha}$, W is exogenous.
2) $P=Y^{-\tau}$, $Y=N$, and W is exogenous.
3) $P=Y^{-\tau}$, $Y=AN^{\alpha}$ and W is exogenous.
4) $P=Y^{-\tau}$, $Y=AN^{\alpha}$ and $W=PN^{\eta}$.

Selected Solutions

Part A: 1) $8x^3$ 2) $12+5x^4$ 3) $4(1-2x)+(4x-3)(-2)$ 4) $2/(3x+5) – 3(2x-1)/(3x+5)^2$

Part B: 1) $MU(x)=2$ 2) $MU(x)=0$ 3) $MU(x)=4y^2$ 4) $MU(x)=2(3x^2-1)(6x)(2y-5)$

Part C: 1) $dY=KNdA+ANdK+AkdN$ 2) $dY=K^{\alpha}N^{\alpha}dA+\alpha AK^{\alpha-1}N^{\alpha}dK+(1-\alpha) AK^{\alpha}N^{-\alpha}dN$

Part D: 1) $N=(W/\alpha P)^{1/(\alpha-1)}$ 2) $N=(W/\alpha(1-\tau)P)^{1/(\alpha-1)}$