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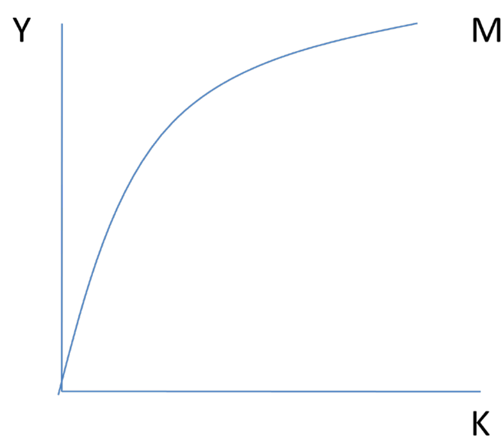
Econ 101B, Spring 2025

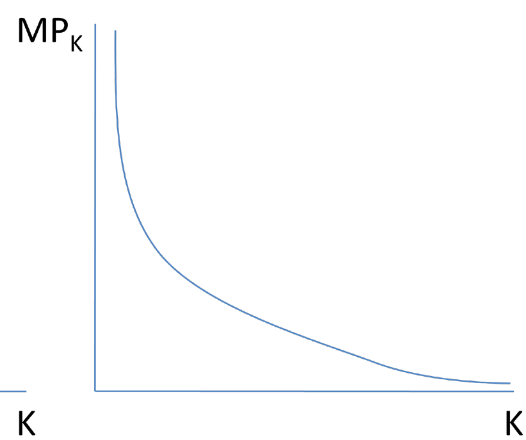
Notes on Investment

Investment, as defined by economists, is the purchase of durable items (capital) to be used in the process of production. This includes machinery, structures, trucks, computers, equipment, software, … but does not include financial variables like stocks or bonds. We will refer to an economy's capital stock at time t by K_t and to investment by I_t. Investment goods are typically durable, in the sense that they can be used many times. The depreciation rate δ is the rate at which capital gets used up in the production process.

The Marginal Product of Capital

When firms consider buying another unit of capital, a key consideration is by how much that extra unit of capital will expand production: the ***marginal product of capital***. As with utility, we will make some assumptions about how production is affected by capital. The first assumption is that more capital is always better in the sense that getting more capital never lowers production. The second assumption is capital is subject to a diminishing marginal product. This means that each additional unit of capital increases production by less than the last unit of capital purchased. Graphically, we can see these assumptions as:





The graph on the left shows that as the capital stock increases, production increases. However, the increase is of a smaller amount as the capital stock increases. This is better illustrated in the graph on the right, which shows the marginal product of capital (always positive) falling as the amount of capital increases.

The Optimal Level of the Capital Stock

Suppose the price of capital is one, for simplicity. A firm is trying to decide whether to purchase an additional unit of capital or not. Suppose that if it buys a unit of capital today, that capital becomes productive next period. We'll solve for the optimal level of capital using a perturbation argument.

Possibility 1: The firm buys a unit of capital (invests 1 unit). Next period, it earns the marginal product of capital from that unit MP_K. It can then sell the un-depreciated part of the capital ($1-\delta$) back. Since the price is one, it earns $1-\delta$ from the sale. Thus, with this option, the firm spends one unit on capital and earns a total of $MP_K+1-\delta$.

Possibility 2: The firm does not buy the unit of capital and instead saves that amount. In return it earns interest, so that next period it has $1+r$.

In equilibrium, the firm should be indifferent between the two options. This means that the equilibrium condition is $MP_K+1-\delta=1+r$, or just

$$\boldsymbol{MP_K=r+\delta}$$

The right hand side of this expression is known as the ***user cost of capital***. This is because it captures the costs associated with a unit of capital: δ is the depreciation on the unit of capital and r is the interest foregone by not saving, i.e. the opportunity cost of the capital. This expression determines the optimal level of the capital stock.

Example: Suppose the production function is of the Cobb-Douglas[1] type $Y=K^{1/3}L^{2/3}$ where L is the labor used in the production process. The marginal product of capital[2] is given by $MP_K=(1/3)K^{-2/3}L^{2/3}$. Plugging this into the user cost equation and solving for the capital stock K^* yields

$$K^* = [3(r+\delta)]^{-3/2}L$$

Notice that the more labor you have in production (L), the higher the level of the optimal capital stock (because the MP_K rises with labor). Also, if either r or δ goes up, the optimal level of the capital stock falls, since these represent the costs associated with holding capital.

Investment

We've solved for the optimal capital stock, but what does this tell us about investment? In general, new capital is not immediately available for production. There are numerous costs and delays that are associated with the purchase of capital: decision delays (time-to-plan), building delays (time-to-build), … Picture a firm trying to decide whether to build a new factory or not. The decision time will tend to be lengthy, as will the building process. A typical assumption is that there is a one-quarter lag between the time when investment is done and when capital is installed. Thus, one would write an ***equation of motion*** for capital as

$$\boldsymbol{K_{t+1} = K_t(1-\delta) + I_t}$$

which says that next quarter's capital stock (K_{t+1}) will be equal to the un-depreciated component of this quarter's capital stock ($K_t(1-\delta)$) plus the investment undertaken this quarter (I_t). The appropriate measure of time will depend on which type of capital one is referring to.

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[1] We'll talk a lot more about the Cobb-Douglas production function in the growth theory section of the course.

[2] This is found by taking the partial derivative of the production function with respect to K: $\frac{\partial Y}{\partial K}$.

This timing distinction is important because the relevant marginal product of capital in the user-cost equation depends on the timing assumption. If there is a one-quarter delay between when the investment is done and when it is operational, then the relevant left-hand side variable is the marginal product of capital next quarter. If it is a year-long delay, then it is the marginal product of capital next year, and so on… The key implication of this is that investment today depends on the user cost and the *expected* marginal product of capital in the future. Thus, just like consumption, investment decisions depend on expectations of future variables, in this case future marginal products of capital.

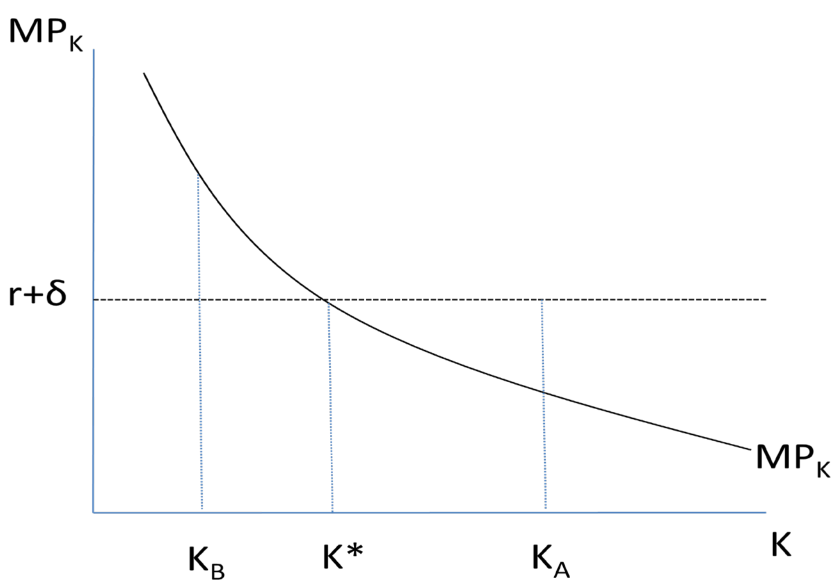
Thus, we will write the ***investment function*** as ***I=I(SP,r)*** where *SP* stands for "*animal spirits*," aka the optimism of firms about future profits.[3] When *SP* goes up, we'll assume that means firms are more optimistic, they expect higher marginal products of capital in the future, and will therefore want to invest more today. Secondly, when the interest rate *r* rises, the opportunity cost of saving rises for firms, and they will choose to invest less. A final thing to notice is that investment decisions should be independent of current output since *Y* does not appear anywhere in the investment function. Of course, output and firm optimism should tend to move together, so investment and output should tend to move together as well (holding the interest rate constant).

<u>A Graphical Interpretation</u>

We can represent the user cost of capital equation graphically and use this to understand how investment will change in response to changes in various variables. Recall that the user cost of capital equation is

$$MP_K = r + \delta$$

and that the marginal product of capital is a decreasing function of the capital stock. Thus, the graph below helps us identify the optimal level of the capital stock.



[3] This term was originally coined by Keynes.

The capital stock is on the x-axis, and the graph shows the MP_K line, which is decreasing with K. The user cost, r+δ, is shown as the horizontal line. Where the user cost and the MP_K line intersect determines the optimal level of the capital stock, K*. This means that if the economy begins at a level of capital given by K_B, less than K*, then the return to buying capital (MP_K) exceeds the marginal cost (r+δ) so firms will buy capital, raising K. As they do this, the marginal product will decrease (moving along the MP_K) line, until they reach K*. Thus, going from K_B to K*, investment will be high. The reverse is true starting from K_A.

One can also use this graph to figure out how the capital stock and investment will change when the interest rate, depreciation, or animal spirits change.

<u>Investment decisions with calculus</u>

When firms make investment decisions they solve complicated optimization problems. Typically, the cornerstone of this optimization problem is maximization of net present value of profits (sometimes, stream of cash flow). Again, using calculus can help us to formalize these problems and understand the general solution strategy.

Suppose a firm decides whether to make an investment today or not. Denote the amount of investment the firm makes with K, which means that next period the firm will have capital stock K. The payoff, however, occurs tomorrow. We model this payoff as $A\times K^{1/2}$, which corresponds to a production function with one input (here capital K) and the level of technology given by parameter A. We will assume that the level of technology is uncertain at the time when the investment is made.

Note that the firm borrows today and returns the debt tomorrow. Similar to the consumption problem we considered before, we need to keep everything in today's dollars we need to discount the return tomorrow by 1/(1+r).

Formally, the firm's maximization is

$$\max\left\{-K+\frac{1}{(1+r)}A\times K^{\frac{1}{2}}\right\}$$

Recall that the necessary condition for optimal level of capital is that the derivative of the maximized function should be equal to zero (the first order condition). Hence, we need to find K such that

$$-1+\frac{1}{(1+r)}\left(\frac{1}{2}\times A\times K^{1/2-1}\right)=-1+\frac{1}{(1+r)}\left(\frac{1}{2}\times A\times K^{-1/2}\right)=0$$

Here, "-1" corresponds to the marginal cost of borrowing funds for finance the investment. Note that this cost does not vary with the level of capital. The term $(\frac{1}{2}A\times K^{-1/2})$ is the marginal product of capital. Observe that as we increase A we increase the marginal product. In contrast, adding more capital decreases the marginal product of capital. Finally, $\frac{1}{(1+r)}$ captures the fact that we need to discount future return to present dollars.

We can rewrite this equation as follows

$$1 = \frac{1}{(1+r)}\left(\frac{1}{2} \times A \times K^{-1/2}\right),$$

where on the left hand side we have the marginal cost of funds, and on the right hand side we have the marginal return on capital in today's dollars. The optimal level of investment equalizes the marginal cost and return of investment and this level is given by

$$K^* = \left(\frac{A}{2(1+r)}\right)^2$$

Consistent with our intuition, investment increases with expected level of productivity A and decreases with the interest rate *r*.

In many important cases, an investment becomes operational many periods after money is spent. How would this affect the results. Consider a simple modification of the model when the investment pays off after *N* periods. In this case, the objective function is

$$\max\left\{-K + \frac{1}{(1+r)^N} A \times K^{\frac{1}{2}}\right\}$$

Note the discount factor is $\frac{1}{(1+r)^N}$ instead of $\frac{1}{(1+r)}$. After we repeat the derivations, the optimal level of investment is

$$K^* = \left(\frac{A}{2(1+r)^N}\right)^2$$

Provided $r > 0$, we have that $(1+r)^N > (1+r)^{N-1} > \cdots > (1+r)$ and hence holding everything else constant more distant payoffs will be associated with smaller investments. At the same time, more investments with more distant payoffs are more sensitive to changes in the interest rate. Specially,

$$\frac{\partial K^*}{\partial r} = -\left(\frac{A}{2(1+r)^N}\right)^2 \times \frac{2N}{(1+r)} < 0$$

which suggests that as *N* increases the absolute value of the derivative increases. This explains why investment in structures, which typically takes long time to build, is typically more sensitive to the interest rate than investment in equipment or software, which typically takes a short time to install. Likewise, it explains why the prices of zero-coupon bonds with long maturities are more sensitive to changes in the interest rate than the prices of zero-coupon bonds with short maturities.

Extension 1: What determines the price of capital?

In the previous section, we derived the user cost of capital equation to determine the optimal level of the capital stock for a firm. However, in doing so, we implicitly assumed that the price of capital was constant (and normalized to be equal to 1). In principle, there is no reason to believe that the price of capital is constant. So what determines the price of capital?

Let the price of capital at time *t* be denoted by p_t. Consider the case of a firm who is deciding whether or not to spend another dollar on capital. Again, we can use the perturbation argument:

> *Possibility One:* We have one dollar available to buy capital. With that one dollar, the firm can buy $1/p_t$ units of capital. Again, each unit of capital purchased yields $MP_{K,t+1}$ in extra output and depreciates by δ in the process. The un-depreciated component can then be resold at price p_{t+1} per unit. Thus, this transaction earns $(1/p_t)[MP_K+(1-\delta)p_{t+1})$.
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> *Possibility Two:* The firm can save that one dollar and earn r in interest. Next period it has a total of $1+r$.

In equilibrium, the firm must be indifferent between the two options. Hence, we can set the two expressions equal to each other and solve for p_t to get

$$p_t = \left(\frac{1}{1+r}\right)\left[MP_{K,t+1} + (1-\delta)p_{t+1}\right]$$

which says that the current price of capital depends on the *expected marginal product of capital and the expected resale price of the un-depreciated capital.*

Now suppose we write the same equation at time *t+1:* $p_{t+1} = \left(\frac{1}{1+r}\right)\left[MP_{K,t+2} + (1-\delta)p_{t+2}\right]$ and plug that in for p_{t+1} in the previous equation, we get

$$p_t = \left(\frac{1}{1+r}\right)\left[MP_{K,t+1} + \left(\frac{1-\delta}{1+r}\right)MP_{K,t+2} + \left(\frac{(1-\delta)^2}{1+r}\right)p_{t+2}\right]$$

Then we could plug in the same way for p_{t+2} and then p_{t+3} and so on until we got

$$p_t = \left(\frac{1}{1+r}\right)\left[MP_{K,t+1} + \left(\frac{1-\delta}{1+r}\right)MP_{K,t+2} + \left(\frac{1-\delta}{1+r}\right)^2 MP_{K,t+3} + \cdots\right]$$

or equivalently

$$\boldsymbol{p_t = \left(\frac{1}{1+r}\right)\sum_{j=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^j MP_{K,t+1+j}}$$

This expression tells us that the price of capital is equal to the present discounted value of current and future marginal products of capital. This price is therefore a very forward-looking variable. In particular, the lower the depreciation rate, the more important are marginal products of capital far in the future.

Extension 2: Fundamental Determinants of Stock Prices

While the previous section focused on investment in capital goods, one can use a very similar approach to understand the determinants of stock prices, a financial asset. A stock is a claim to ownership of a corporation. The benefit of owning of a stock is the earning of dividends (corporate profits returned to shareholders) and capital gains (reselling the stock at a (potentially) higher price).

Let P_t be the price of a share of stock at time *t* and D_t be dividends earned at time *t* from each share of stock. As with the price of capital goods, we can use an arbitrage argument to describe an equilibrium relationship for stock prices. Consider an investor that has $1 to allocate in one of two ways:

1. stock purchase: the investor can buy some stock today. Given a price of P_t per share, the investor can buy $1/P_t$ shares of stock. Each share of stock expects to earn dividends of D_{t+1}^e the following period and can then be resold at an expected price of P_{t+1}^e. Thus, the return from pursuing this option is
$$\left(\frac{1}{P_t}\right)[D_{t+1}^e + P_{t+1}^e]$$
2. saving: alternatively, the investor can save that dollar and would receive *1+r* next period.

In equilibrium, these two options must yield the same return, else one could earn riskless profits by either borrowing funds and buying stocks, or short-selling stocks and saving money now. Equality in returns implies

$$1 + r = \left(\frac{1}{P_t}\right)[D_{t+1}^e + P_{t+1}^e]$$

which can be rewritten as

$$\boldsymbol{P_t = \left(\frac{1}{1+r}\right)[D_{t+1}^e + P_{t+1}^e]}$$

The expression above is crucially important. It says that the current price of a stock should equal the discounted value of expected profits and the resale price. Thus, stock prices (like capital goods prices) are forward-looking; they depend on expectations of future values.

We can go further with this expression, just as we did with capital goods prices. Specifically, what has to be true about the current expectation of the price at time t+1? The same arbitrage argument applied at time t+1 implies that the following must also be true:

$$P_{t+1}^e = \left(\frac{1}{1+r}\right)[D_{t+2}^e + P_{t+2}^e]$$

so that next period's price should equal the discounted value of dividends at time *t+2* and expected resale price at time *t+2*. Plugging this into our expression for the current stock price yields

$$P_t = \left(\frac{1}{1+r}\right)\left[D_{t+1}^e + \frac{D_{t+2}^e}{1+r} + \frac{P_{t+2}^e}{1+r}\right]$$

Applying this same approach ad nauseam yields:

$$\boldsymbol{P_t = \left(\frac{1}{1+r}\right)\left[D_{t+1}^e + \frac{D_{t+2}^e}{1+r} + \frac{D_{t+3}^e}{(1+r)^2} + \frac{D_{t+4}^e}{(1+r)^3} + \cdots\right] = \left(\frac{1}{1+r}\right)\sum_{j=0}^{\infty}\frac{D_{t+1+j}^e}{(1+r)^j}}$$

This final expression identifies the fundamental stock price: *the current price of a stock should be equal to the present discounted sum of current and future dividend payments*. Thus, current stock prices should change whenever either current or expected future dividends change, or when interest rates change or are expected to change in the future.

Now consider the implications of this relationship for how stock prices respond to news. If good news is reported about a company's current or expected future profits, then future dividends are expected to be higher and the firm's stock price will rise immediately. However, if good news is reported about the economy, this does not necessarily mean that stock prices will rise. It does imply that expected dividends will be higher, so that tends to push stock prices higher. But with higher economic growth, central banks typically raise interest rates. The higher the interest rate, the lower stock prices will tend to be. Thus, good macroeconomic news can have positive or negative effects on current stock prices depending on how much people revise their expectations of dividends relative to their expectations of future interest rates.

Finally, because current stock prices depend on current expectations of future dividend payments (and hence of future profits), stock market prices can be interpreted as indicators of "animal spirits", i.e. expectations of firms' future profits.