MAT2013: Probability and Statistics **Spring 2017**

Problem Set 3

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Solution

1. Exisiting setting

There exists n users which is denoted by

$$u_1, u_2, \dots, u_n, \tag{3.1}$$

and a list of enemies of u_i, $\mathcal{E}(u_i)$, briefly $\mathcal{E}_i$, is given. Let $\mathcal{F}(u_i)$, briefly $\mathcal{F}_i$, is a list of friends of u_i, then we have three conditions as follows

- Condition 1.

$$\forall i, j(\neq i) \in \{1, \dots, n\}, \quad u_i \in \{\mathcal{E}_j\} \quad \textit{iff} \quad u_j \in \{\mathcal{E}_i\} \tag{3.2}$$

- Condition 2.

$$\forall i \in \{1, \dots, n\}, \quad u_i \notin \{\mathcal{E}_i\} \tag{3.3}$$

- Condition 3.
  For all $i, j(\neq i) \in \{1, \dots, n\}$, u_i and u_j are friend *iff*

$$\exists u_k \in \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} \tag{3.4}$$

2. Define a matrix

For all $i, j \in \{1, \dots, n\}$ let E and F be $n \times n$ matrices which is given by

$$(E)_{ij} = \begin{cases} 1, & \text{if } u_i \text{ and } u_j \text{ are enemy} \\ 0, & \text{otherwise} \end{cases}, \tag{3.5}$$

and

$$(F)_{ij} = \begin{cases} 1, & \text{if } u_i \text{ and } u_j \text{ are friend} \\ 0, & \text{otherwise} \end{cases}. \tag{3.6}$$

By condition 1 and 2, the matrix E is a symmetry matrix (i.e., $(E)_{ij} = (E)_{ji}\ \forall i, j$) and has zero diagonal entries (i.e. $E_{ii} = 0\ \forall i$), hence column vectors $\mathbf{c}_j(E)$ and row vectors $\mathbf{r}_i(E)$, briefly $\mathbf{c}_j$ and $\mathbf{r}_i$, satisfies

$$i = j \Rightarrow \mathbf{c}_j = \mathbf{r}_i. \tag{3.7}$$

Now, let us consider the inner product of $\mathbf{r}_i$ and $\mathbf{c}_j$, which is given by

$$\mathbf{r}_i \cdot \mathbf{c}_j = \sum_{k=i}^{n} (\mathbf{r}_i)_k (\mathbf{c}_j)_k = \sum_{k=i}^{n} (E)_{ik}(E)_{kj} = \sum_{k=i}^{n} \mathbb{I}_k(i, j) \tag{3.8}$$

where indicator function $\mathbb{I}_k(i,j)$ is

$$\mathbb{I}_k(i,j) = \begin{cases} 1, & \exists u_k \in \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} \\ 0, & \text{otherwise} \end{cases} , \tag{3.9}$$

then we can say that

$$\mathbf{r}_i \cdot \mathbf{c}_j = \begin{cases} l_{ij}, & \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} \neq \emptyset \\ 0, & \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} = \emptyset \end{cases} \tag{3.10}$$

where $0 < l_{ij} < n$.

3. Find the matrix F from E

Let a $n \times n$ matrix $M = EE$, then, by (3.10),

$$(M)_{ij} = \mathbf{r}_i \cdot \mathbf{c}_j = \begin{cases} l_{ij}, & \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} \neq \emptyset \\ 0, & \{\mathcal{E}_i\} \cap \{\mathcal{E}_j\} = \emptyset \end{cases} \tag{3.11}$$

where $0 < l_{ij} < n$. By condition 3,

$$(M)_{ij} = \begin{cases} l_i j, & \text{if } u_i \text{ and } u_j \text{ are friend} \\ 0, & \text{if } u_i \text{ and } u_j \text{ are not friend} \end{cases} , \quad \forall i, j (\neq i). \tag{3.12}$$

Be careful that if $i = j$, there may be inconsistent result such that

$$(M)_{ii} = \begin{cases} 0, & \text{if } \mathcal{E}_i = \emptyset \\ l_{ii}, & \text{otherwise} \end{cases} \tag{3.13}$$

Hence, if we assume that every user u_i is friend of itself, then we can find the matrix F as follows

$$(F)_{ij} = \begin{cases} 1, & i = j \\ 0, & (M)_{ij} = 0 \\ (M)_{ij}/l_{ij}, & (M)_{ij} = l_{ij} \end{cases} \tag{3.14}$$