MAT2013: Probability and Statistics **Spring 2017**

Problem Set 3

Name: Jisung Lim (2014147040)

Solution

1. Consider a random variable vector (X, Y) with joint pdf

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

   (a) Compute $P(X+Y \geq 1)$.
   Solution:

$$\begin{aligned} P(X+Y \geq 1) &= 1 - P(X+Y < 1) \\ &= 1 - \iint_{\mathbb{R}^2} f(x,y)\, dxdy \\ &= 1 - \int_0^{\frac{1}{2}} \int_x^{1-x} e^{-y}\, dydx \\ &= 1 - \int_0^{\frac{1}{2}} -e^{-y}\Big|_x^{1-x}\, dx \\ &= 1 - \int_0^{\frac{1}{2}} -e^{x-1} + e^{-x}\, dx \\ &= 1 - (-e^{x-1} - e^{-x})\Big|_0^{\frac{1}{2}} \\ &= 1 - \left(-2e^{-1/2} + e^{-1} + 1\right) \\ &= 2e^{-1/2} - e^{-1} \end{aligned}$$

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   (b) Find the marginal pdfs f_X and f_Y.
   Solution:
   If $x < 0$, then $f_X(x) = 0$. If $x \geq 0$, then

$$\begin{aligned} f_X(x) &= \int_x^{\infty} e^{-y}\, dy \\ &= \lim_{t\to 0} -e^{-y}\Big|_x^t \\ &= \lim_{t\to 0} e^{-x} - e^{-t} \\ &= e^{-x} \end{aligned}$$

   If $y < 0$, then $f_Y(y) = 0$. If $y \geq 0$, then

$$\begin{aligned} f_Y(y) &= \int_0^{y} e^{-y}\, dx \\ &= ye^{-y} \end{aligned}$$

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2. Let X_i, $i = 1, 2, \dots$ be independent exponential random variables with rate η_i. Let $Z = \min\{X_1, X_2, \dots, X_n\}$ and $Y = \max\{X_1, X_2, \dots, X_n\}$. Find the distributions of Z and Y.
Solution:
For the random variable X_i, the probability distribution function $f^{(i)}(x)$ is given by

$$f^{(i)}(x) = \eta_i e^{-\eta_i x} \quad \forall x \in (0, \infty),$$

and the distribution function $F^{(i)}(x)$ is given by

$$F^{(i)}(x) = \int_0^x \eta_i e^{-\eta_i t}\, dt = -e^{-\eta_i t}\Big|_0^x = 1 - e^{-\eta_i x}$$

Now, let the random variable Z is given by

$$Z = \min\{X_1, X_2, \dots, X_n\},$$

then distribution function of Z, says $F_Z(z)$ is given by

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = \mathbb{P}[\{\omega \in \Omega : Z(\omega) \leq z\}] \\
&= \mathbb{P}[\{Z \leq z\}] \\
&= \mathbb{P}[\{\min\{X_1, X_2, \dots, X_n\} \leq z\}] \\
&= \mathbb{P}[\{X_1 \leq z\} \cup \{X_2 \leq z\} \cup \cdots \cup \{X_n \leq z\}] \\
&= \mathbb{P}[\Omega \setminus (\{X_1 > z\} \cap \{X_2 > z\} \cap \cdots \cap \{X_n > z\})] \\
&= 1 - \mathbb{P}[\{X_1 > z\} \cap \{X_2 > z\} \cap \cdots \cap \{X_n > z\}]
\end{aligned}$$

Since the random variable $X_1, \dots, X_n$ are mutually independent, then for A_i in σ-algebra $\mathcal{A}$, every events $\{\omega \in \Omega : X_i(\omega) \in A_i\}$ are mutually independent. Therefore,

$$\begin{aligned}
F_Z(z) &= 1 - \mathbb{P}[\{X_1 > z\} \cap \{X_2 > z\} \cap \cdots \cap \{X_n > z\}] \\
&= 1 - \mathbb{P}[\{X_1 > z\}]\mathbb{P}[\{X_2 > z\}] \cdots \mathbb{P}[\{X_n > z\}] \quad \text{(by independence)} \\
&= 1 - P_{X_1}(X_1 > z) P_{X_2}(X_2 > z) \cdots P_{X_n}(X_n > z)
\end{aligned}$$

Since $P_{X_i}(X_i > z) = 1 - P_{X_i}(X_i \leq z) = 1 - F^{(i)}(z) = e^{-\eta_i z}$, then

$$\begin{aligned}
F_Z(z) &= 1 - P_{X_1}(X_1 > z) P_{X_2}(X_2 > z) \cdots P_{X_n}(X_n > z) \\
&= 1 - e^{-\eta_1 z} e^{-\eta_2 z} \cdots e^{-\eta_n z} \\
&= 1 - e^{-(\eta_1 + \eta_2 + \cdots + \eta_n) z}
\end{aligned}$$

Hence,

$$F_Z(z) = 1 - e^{-(\eta_1 + \eta_2 + \cdots + \eta_n) z}$$

For Y, in the same manner,

$$\begin{aligned}
F_Y(y) &= P_Y(Y \leq y) = \mathbb{P}[\{Y \leq y\}] \\
&= \mathbb{P}[\{\max\{X_1, X_2, \dots, X_n\} \leq y\}] \\
&= \mathbb{P}[\{X_1 \leq y\} \cap \{X_2 \leq y\} \cap \cdots \cap \{X_n \leq z\}] \\
&= \mathbb{P}[\{X_1 \leq y\}]\mathbb{P}[\{X_2 \leq y\}] \cdots \mathbb{P}[\{X_n \leq z\}] \\
&= P_{X_1}(X_1 \leq y) P_{X_2}(X_2 \leq y) \cdots P_{X_n}(X_n \leq y) \\
&= (1 - e^{-\eta_1 y})(1 - e^{-\eta_2 y}) \cdots (1 - e^{-\eta_n y})
\end{aligned}$$

Hence,

$$F_Y(y) = (1 - e^{-\eta_1 y})(1 - e^{-\eta_2 y}) \cdots (1 - e^{-\eta_n y})$$

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3. Prove the following statements:

(a) $Cov(X,Y) = Cov(Y,X)$
Solution:

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathbb{E}[(Y-\mu_Y)(X-\mu_X)] \\ &= Cov(Y,X) \end{aligned}$$

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(b) $Cov(X,X) = Var(X)$
Solution:

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}[(X-\mu_X)(X-\mu_X)] \\ &= \mathbb{E}[(X-\mu_X)^2] \\ &= Var(X) \end{aligned}$$

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(c) $Cov(aX,Y) = aCov(X,Y)$
Solution:

$$\begin{aligned} Cov(aX,Y) &= \mathbb{E}[(aX-\mathbb{E}[aX])(Y-\mu_Y)] \\ &= \mathbb{E}[(aX-a\mathbb{E}[X])(Y-\mu_Y)] \\ &= a\mathbb{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= aCov(X,Y) \end{aligned}$$

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(d) $Cov(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i) = \sum_{i=1}^n \sum_{i=1}^n Cov(X_i, Y_i)$
Solution:

$$\begin{aligned} Cov\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right) &= \mathbb{E}[(\sum_{i=1}^n X_i - \mathbb{E}[\sum_{i=1}^n X_i])(\sum_{i=1}^n Y_i - \mathbb{E}[\sum_{i=1}^n Y_i])] \\ &= \mathbb{E}[(\sum_{i=1}^n X_i - \sum_{i=1}^n \mathbb{E}[X_i])(\sum_{i=1}^n Y_i - \sum_{i=1}^n \mathbb{E}[Y_i])] \\ &= \mathbb{E}[\sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \sum_{i=1}^n (Y_i - \mathbb{E}[Y_i])] \\ &= \mathbb{E}[\sum_{i=1}^n \sum_{i=1}^n (X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])] \\ &= \sum_{i=1}^n \sum_{i=1}^n \mathbb{E}[(X_i - \mathbb{E}[X_i])(Y_i - \mathbb{E}[Y_i])] \\ &= \sum_{i=1}^n \sum_{i=1}^n Cov(X_i, Y_i) \end{aligned}$$

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4. Suppose that X and Y are independent continuous random variables. Find the distribution of $X+Y$.
Solution:
Let Z be a random variable given by $Z = X+Y$. Then a distribution function of Z, says $F_Z(z)$, is given by

$$\begin{aligned}F_Z(z) &= P_Z(Z \geq z) = \mathbb{P}[\{Z \geq z\}] \\ &= \mathbb{P}[\{\omega \in \Omega : X(\omega) + Y(\omega) \geq z\}] \\ &= \int_{-\infty}^{\infty} \mathbb{P}[\{X \geq z-k\} \cap \{Y \geq k\}]\, dk\end{aligned}$$

Since the random variable X and Y are mutually independent, then two events $\{\omega \in \Omega : X(\omega) \in A\}$ and $\{\omega \in \Omega : Y(\omega) \in B\}$ are mutually independent where A and B in σ-algebra $\mathcal{A}$. Therefore,

$$\begin{aligned}F_Z(z) &= \int_{-\infty}^{\infty} \mathbb{P}[\{X \geq z-k\} \cap \{Y \geq k\}]\, dk \\ &= \int_{-\infty}^{\infty} \mathbb{P}[\{X \geq z-k\}]\mathbb{P}[\{Y \geq k\}]\, dk \\ &= \int_{-\infty}^{\infty} P_X(X \geq z-k)P_Y(Y \geq k)\, dk \\ &= \int_{-\infty}^{\infty} F_X(z-k)F_Y(k)\, dk\end{aligned}$$

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