$(\Omega, \mathcal{A}, P)$ Conditioning

$B \in \mathcal{B}$, $P_X(B) = P[\{X \in B\}]$

$\{\{X \in B\} : B \in \mathcal{B}\}$ $\mathcal{A}$

$\|$

$\{w \in \Omega : X(w) \in B\}$

A collection of subsets of {w \in \Omega : w \in X^{-1} (B) }

subset of the sample space

(Consider you tossed three coins 10 원, 50 원, 100 원)

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

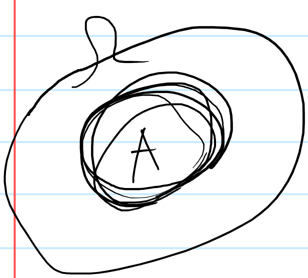
Define X by $X(w)$: the values of those coins that land heads up are added up.

Q: What is the expected value of X given that two coins have landed heads up?

$\|\|$

B {w : X(w) in B}

$B = \{HHT, HTH, THH\}$



$X(HHT) = 60$

$X(HTH) = 110$

$X(THH) = 150$

$P(\{X=60\} \cap B)$

$X(HTH)$

$X(THH) = 150$

$$\frac{P(\{X=60\} \cap B)}{P(B)}$$

$$E[X|B] := X(HHT)\, P[\{X=60\} | B] + X(HTH)\, P[\{X=110\} | B] + X(THH)\, P[\{X=150\} | B]$$

$$= \frac{1}{P(B)} \Big(X(HHT)\, P(\{HHT\}) + X(HTH)\, P(HTH) + X(THH)\, P(THH) \Big)$$

$$= \frac{1}{P(B)} \sum_{x \in X(B)} x f_X(x)$$

$$\frac{1}{P(B)} \int_{X(B)} x f_X(x)\, dx.$$

Now, consider another random variable Y defined by $Y(\omega)$

: the sum of the values of 10 and 50 when they show heads

heads

Then, $Y(\omega) = \begin{cases} 0 & \omega \in \{TTH, TTT\} \\ 10 \\ 50 \\ 60 \end{cases}$

$\{Y=0\} = \{TTH, TTT\}$

$\{Y=10\} = \{HTT, HTH\}$

$\{Y=50\} =$

$\{Y=60\} =$

$E[X \mid \{Y=0\}] = a$

$E[X \mid \{Y=10\}] = b$

$E[X \mid \{Y=50\}] = c$

$E[X \mid \{Y=60\}] = d$

$$E[X \mid Y](\omega) = \begin{cases} a & \text{if } Y(\omega)=0 \\ b & \text{if } Y(\omega)=10 \\ c & \text{if } Y(\omega)=50 \\ d & \text{if } Y(\omega)=60 \end{cases}$$

" Conditional expectation of X given Y.

$E[X \mid Y] = \phi(Y)$

$$E[X|Y] = \phi(Y)$$

$$E[X|Y=y] := \phi(y)$$

if X and Y are discrete random variables

$$E[X|Y=y] := \phi(y) = \sum_x x \, f_{X|Y}(x|y)$$

where

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{f_Y(x,y)}{f_Y(y)}, & f_Y(y)>0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \int x \, f_{X|Y}(x|y) \, dy$$

if X and Y are absolutely continuous

$B \subset \Omega$.

$X(\omega) = \mathbb{1}_B(\omega)$ indicator function

$$E[X|Y] = E[\mathbb{1}_B | Y]$$

$$:= P[B|Y]$$ conditional probability of B given Y

$$P(B) = E[\mathbb{1}_B]$$

$$= 1 \times P(B) + 0 \times P(B^c)$$

If X and Y are discrete,

$$P[\{X=x\} | Y=y] = f_{X|Y}(x|y).$$

If X and Y are absolutely continuous

$$P[\{Y \in C\} \mid X = x] = \int_C f_{Y|X}(y|x)\,dy$$

In practice conditioning is useful in computing probability and expectation

① Computing expectation by conditioning

<u>theorem</u> $E[X] = E[E[X|Y]]$

② Computing probability by conditioning

$$P(E) = \sum_y P(E \mid Y = y) \quad \text{if } Y \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} P(E \mid Y = y)\, f_Y(y)\,dy \quad \text{if } Y \text{ is continuous}$$

$$P[X_1 > X_2] = \int_{-\infty}^{\infty} P[\{X_1 > x_2\} \mid X_2 = x_2]\, f_{X_2}(x_2)\,dx_2$$

$$(=) \int_{-\infty}^{\infty} P[\{X_1 > x_2 \mid X_2 = x_2] \, f_{X_2}(x_2) \, dx_2$$