MAT2013: Probability and Statistics [1] [2] **Spring 2017**

Lecture 5: R.V. Transformation and MGF

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5.1 Random Variable Transformation

We discussed about the distributional approach to the transformation of random variables. Now, we shall discuss the calculus-based approach. First, we will deal with an univariate transformation and then extend the concept to a multivariate transformation.

5.1.1 Univariate Transformation

First of all, recall the basic calculus "Integration of Substitution", given by

$$\int_{\phi(a)}^{\phi(b)} f(x)\, dx = \int_a^b f(\phi(t))\, |\phi^i(t)|\, dt \tag{5.1}$$

where $\phi(t) : [a, b] \to 1$. This formula is used to transform one integral into another integral. Therefore, from this formula, we can find the continuous probability distribution $P[Y \in A]$ of which integrand is a density:

$$P[Y \in A] = \int_{\phi^{-1}(A)} f_X(x)\, dx = \int_A f_Y(\phi^{-1}(y))\, |(\phi^{-1})'(y)|\, dy \tag{5.2}$$

5.1.2 Multivariate Transformation

Let us begin the discussion about the case of multivariate transformation in some rigorous sense.

Let G be an open set in $\mathbb{R}^n$ and let $\phi : G \to \mathbb{R}^n$ be continuously differentiable. Suppose ϕ is injective on G and its Jacobian never vanishes then

$$\int_{\phi(G)} f(y)\, dy = \int_G f(\phi(x))\, ||J_\phi(x)||\, dx \tag{5.3}$$

where

$$J_\phi(x) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix}. \tag{5.4}$$

5.1.3 Bivariate transformation

A specific example is given by the $\mathbb{R}^2 \to \mathbb{R}^2$ transformation, which is called bivariate transformation.

Let X be a random vector and ϕ be a transformation from $\mathbb{R}^n \to \mathbb{R}^n$ we want to find the distribution of $Y = \phi(X)$. Suppose ϕ is good enough to satisfy any regularity needed to apply the theorem. Note that, from a random vector $Y = (Y_1, Y_2)$, given by

$$\begin{aligned} Y_1 &= \phi_1(X_1, X_2) \\ Y_2 &= \phi_2(X_1, X_2), \end{aligned} \tag{5.5}$$

we can find ϕ^{-1} such that

$$\begin{aligned} X_1 &= \phi_1^{-1}(Y_1, Y_2) \\ X_2 &= \phi_2^{-1}(Y_1, Y_2). \end{aligned} \tag{5.6}$$

Then, by the Change of Variables

$$\begin{aligned} P[Y \in A] &= \iint_{\phi^{-1}(A)} f_X(x_1, x_2)\, dx_1 dx_2 \\ &= \iint_A f_X(\phi^{-1}(y_1, y_2))\, ||J||\, dy_1 dy_2 \end{aligned} \tag{5.7}$$

where

$$J(x) = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix} \tag{5.8}$$

From this specific example, we can understand the following theorem of $\mathbb{R}^n \to \mathbb{R}^n$ generalization.

Theorem 5.1.1. Let $X = (x_1, x_2, \ldots, x_n)$ have joint distributions f. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and intjective with non-vaninsh Jacobian. Then $Y = \phi(x)$ has density:

$$f_Y(y) = \begin{cases} f_X(\phi^{-1}(y))\, ||(J_{\phi^{-1}}(y))||, & y \in \phi(\mathbb{R}^n) \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

□

Example 5.1.2. Let X and Y be i.i.d. normal R.V's with $\mu = 0$ and $\sigma^2 = 1$. What is the joint

distribution of $(U, V) = (X+Y, X-Y)$?
Sketch:

1. Define $\phi : (U, V) \to (X, Y)$ and find ϕ^{-1}
2. Evaluate Jacobian $J_{\phi^{-1}}(U, V)$ and it's determinant $||J||$
3. Use the theorem to find $f_{(U,V)}(u, v)$.

Solution: $\frac{1}{\sqrt{4\pi}} \exp\{-u^2/4\} \frac{1}{\sqrt{4\pi}} \exp\{-v^2/4\}$ ■

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5.2 Moment Generating Function

Definition 5.2.1. Let X be a random variable, then

$$M_X(t) = \begin{cases} \sum_i e^{tx_i} f(x_i) & x\text{: distrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & x\text{: continuous} \end{cases} \tag{5.10}$$

when it exists and call it the *moment generating function* as a function fo t. □

By the definition of mathematical expectation $\mathbb{E}[\cdot]$, MGF can be denoted by

$$M_X(t) = \mathbb{E}_X[e^{tX}] \tag{5.11}$$

and by differentiating the function with respect to t, we obtain,

$$\begin{aligned} M'_X(t) &= \frac{d}{dt}\mathbb{E}[e^{tX}] \\ &= \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dt} e^{tx} f(x)\, dx \quad \text{(Not always commutable)} \\ &= \mathbb{E}[\frac{d}{dt}(e^{tX})] \\ &= \mathbb{E}[Xe^{tX}] \end{aligned} \tag{5.12}$$

and let take $t = 0$ then,

$$M'_X(0) = \mathbb{E}[X] \tag{5.13}$$

which is called the first moment. Similarly, the second derivative is given by

$$\begin{aligned}
M_X''(t) &= \frac{d}{dt} M_X'(t) \\
&= \frac{d}{dt} \mathbb{E}[Xe^{tX}] \\
&= \frac{d}{dt} \int_{-\infty}^{\infty} xe^{tx} f(x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{d}{dt} xe^{tx} f(x)\, dx \quad \text{(Not always commutable)} \\
&= \mathbb{E}[\frac{d}{dt}(Xe^{tX})] \\
&= \mathbb{E}[X^2 e^{tX}]
\end{aligned} \tag{5.14}$$

and when $t = 0$, the second moment is given by $M_X''(0) = \mathbb{E}[X^2]$. In general, the kth derivative is given by

$$M_X^{(k)}(t) = \mathbb{E}[X^k e^{tX}] \tag{5.15}$$

and take $t = 0$, then

$$M_X^{(k)}(0) = \mathbb{E}[X^k] \tag{5.16}$$

which is called the k-th moment of random variable X.

Example 5.2.1. Let X be a random variable, the PDF of which is given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad \text{where} \quad 0 < x < \infty,\ \alpha > 0,\ \beta > 0$$

Find MGF of X.
Solution:

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(t)\, dx \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\, dx \\
&= \frac{(\beta/(1-t\beta))^\alpha}{\beta^\alpha} \int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)(\beta/(1-t\beta))^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta/(1-t\beta)}}\, dx \\
&= \frac{(\beta/(1-t\beta))^\alpha}{\beta^\alpha} \\
&= \left\{ \frac{1}{1-t\beta} \right\}^\alpha
\end{aligned}$$

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Properties 5.2.2. *MFG has following properties:*

- **Uniqueness:**
  If two random variables have the same MGF, than they have the same distribution.

- **Linear Transformation:**
  If $Y = aX + b$, then

$$M_Y(t) = e^{bt} M_X(at).$$

- **Independent sums:**
  For independent random variable, says X and Y, with $M_X(t), M_Y(t)$, respectively, the MGF of $Z = X + Y$ is given by

$$M_Z(t) = M_{X+Y}(t) = M_X(t)M_Y(t) = \iint e^{t(x+y)} f_X(x) f_Y(y) \, dxdy.$$

References

[1] Youngstown State University G. Jay Kerns. *Introduction to Probability and Statistics Using R.* `http://ipsur.org/index.html`. G. Jay Kerns, 2010.

[2] Babatunde A. Ogunnaike. *Random Phenomena: Fundamentals of Probability and Statistics for Engineers.* CRC Press, 2009.