

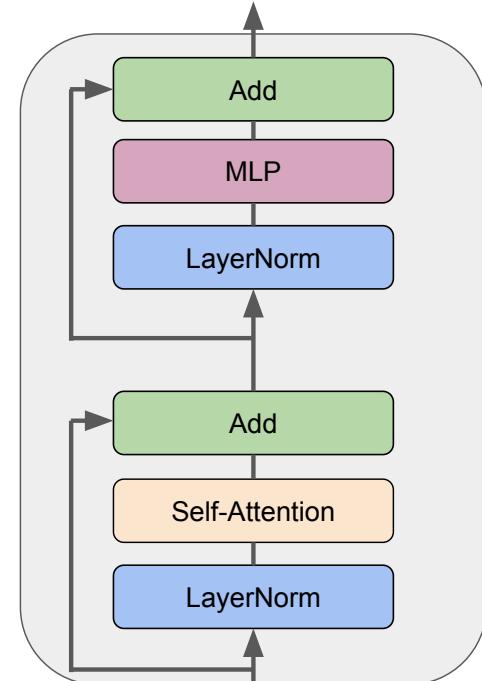
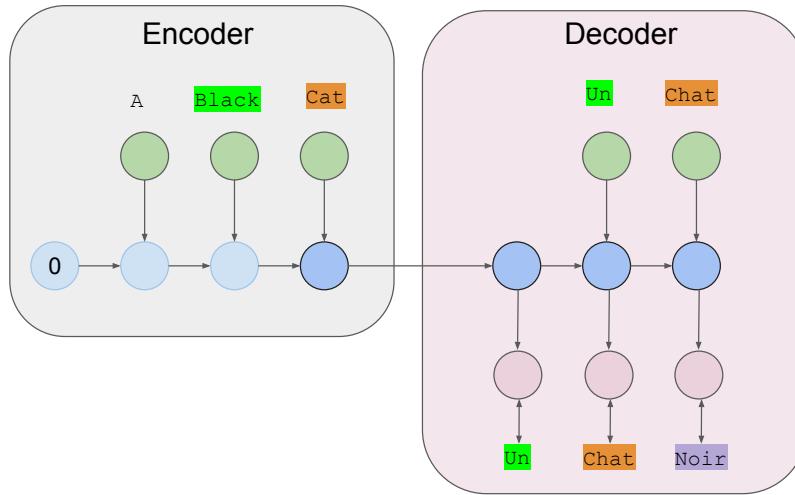
NYU CS-GY 6923

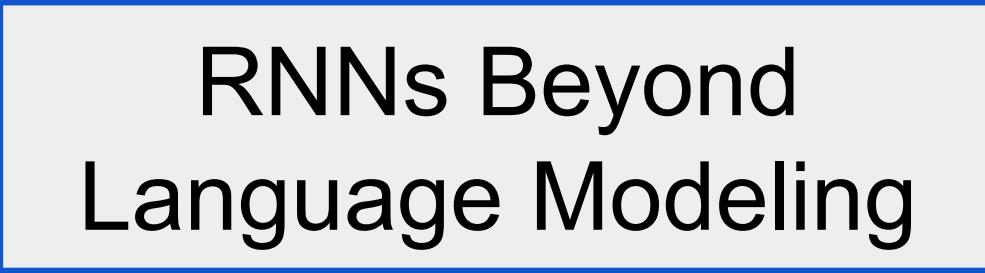
Machine Learning

Prof. Pavel Izmailov

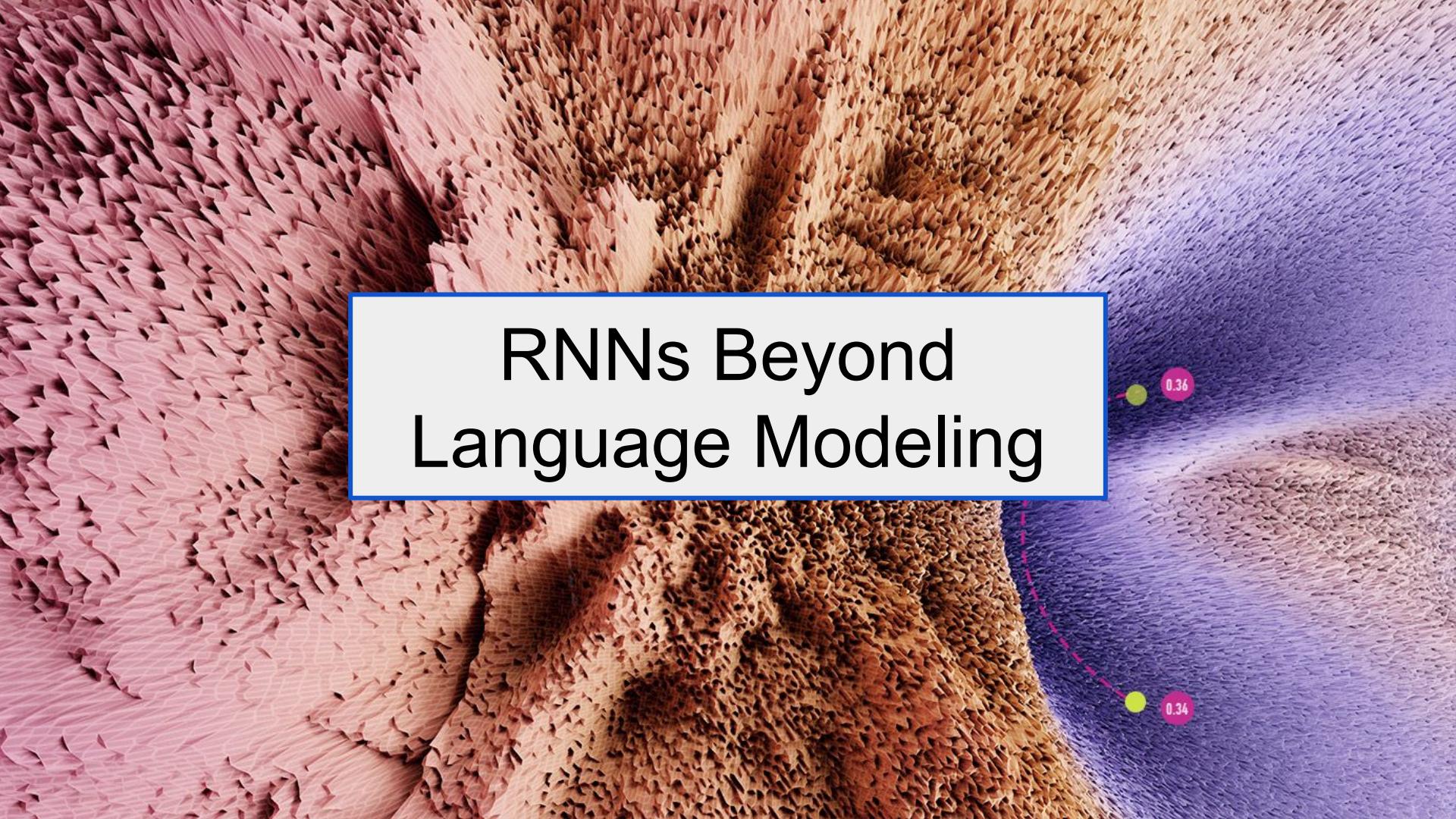
Today

- RNNs beyond language modeling
- Transformers
- Post-Training





RNNs Beyond Language Modeling

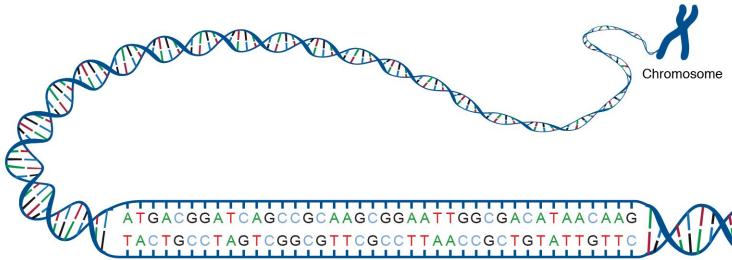


0.36

0.34

Sequence Modeling

Last week we talked about language modeling



Code Blame Raw ▾

```

82 static long long microseconds_since(ktime_t start)
83 {
84     ktime_t now = ktime_get();
85     return ktime_to_ns(ktime_sub(now, start)) >> 10;
86 }
87
88 static async_cookie_t lowest_in_progress(struct async_domain *domain)
89 {
90     struct async_entry *first = NULL;
91     async_cookie_t ret = ASYNC_COOKIE_MAX;
92     unsigned long flags;
93
94     spin_lock_irqsave(&async_lock, flags);
95
96     if (domain) {
97         if (!list_empty(&domain->pending))
98             first = list_first_entry(domain->pending,
99                                     struct async_entry, domain_list);
100    } else {
101        if (!list_empty(&async_global_pending))
102            first = list_first_entry(async_global_pending,
103                                     struct async_entry, global_list);
104    }
105 }
```



4

TERENCE TAO

The random set A we will propose for Theorem 1.3 will then be defined as the truncated random parabola

$$A := \{(x, y) \in \{0, \dots, n-1\}^2 : (ax+by)^2 = cx+dy+e \bmod p\}. \quad (1.2)$$

The standard truncated parabola $S := \{(x, y) \in \{0, \dots, n-1\}^2 : y = x^2 \bmod p\}$ is essentially the example considered in [16], [4], and is the special case of (1.2) when $a = d = 1$ and $b = c = 0$. Geometrically, S is formed by applying a random invertible affine transformation² to the parabola $\{(x, x^2) : x \in \mathbb{F}_p\}$ and then restricting to the grid $(0, \dots, n-1)^2$. While the construction (1.2) is not nearly as random as a completely random subset of \mathbb{F}_p^2 of density $\approx 1/p$, we shall see that there will (barely) still be enough “entropy” in the five random parameters a, b, c, d, e for the probabilistic method to be effective³. In particular, we avoid the difficult number-theoretic questions of trying to count the number of occurrences of the patterns x_1, x_2, \dots, x_5 in the standard truncated parabola S (cf. [5, Problem 2]).

A routine application of the second moment method reveals that A usually has the “right” cardinality (up to acceptable errors):

Lemma 1.5. *With probability at least 0.9 (say), the set A has cardinality $n^2/p + O(\sqrt{n})$. In particular, for n large enough, the cardinality of A is $\approx n$ with probability at least 0.9.*

Proof. Let us temporarily remove the non-degeneracy condition (1.1), so that a, b, c, d, e now become independent random variables. It is clear that any point $(x, y) \in \{0, \dots, n-1\}^2$ will now lie in A with probability $1/p$, just from the randomness of e alone. In fact, any two distinct points $(x, y), (x', y') \in \{0, \dots, n-1\}^2$ will both lie in A with a joint probability of $1/p^2$, from the randomness of c, d, e (since $(x, y, 1)$ and $(x', y', 1)$ are linearly independent in \mathbb{F}_p^3); thus the events $(x, y) \in A$ and $(x', y') \in A$ are pairwise independent. This implies that the cardinality of A has mean $n^2/p \times n$ and variance $O(n^2/p) = O(n)$, and hence from Chebyshev’s inequality, A will have cardinality $n^2/p + O(\sqrt{n})$ with probability at least 0.95 (say). Conditioning to the event (1.1), we obtain the claim. \square

Sequence Modeling

How do we model a time series?

+49.77 (35.98%) ↑ year to date

Closed: Nov 6, 7:59 PM EST • [Disclaimer](#)

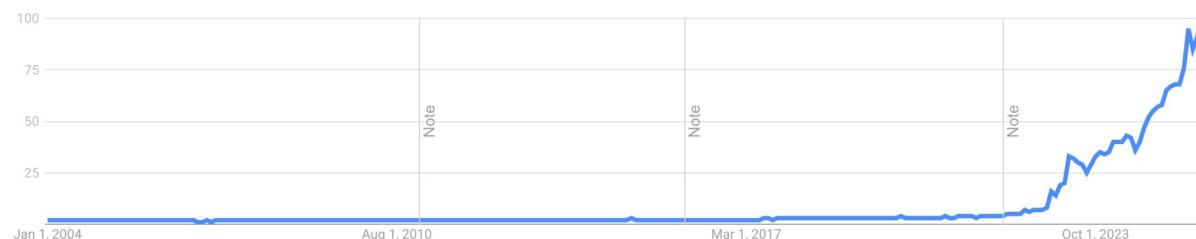
After hours 189.68 +1.60 (0.85%)



Interest over time ⓘ

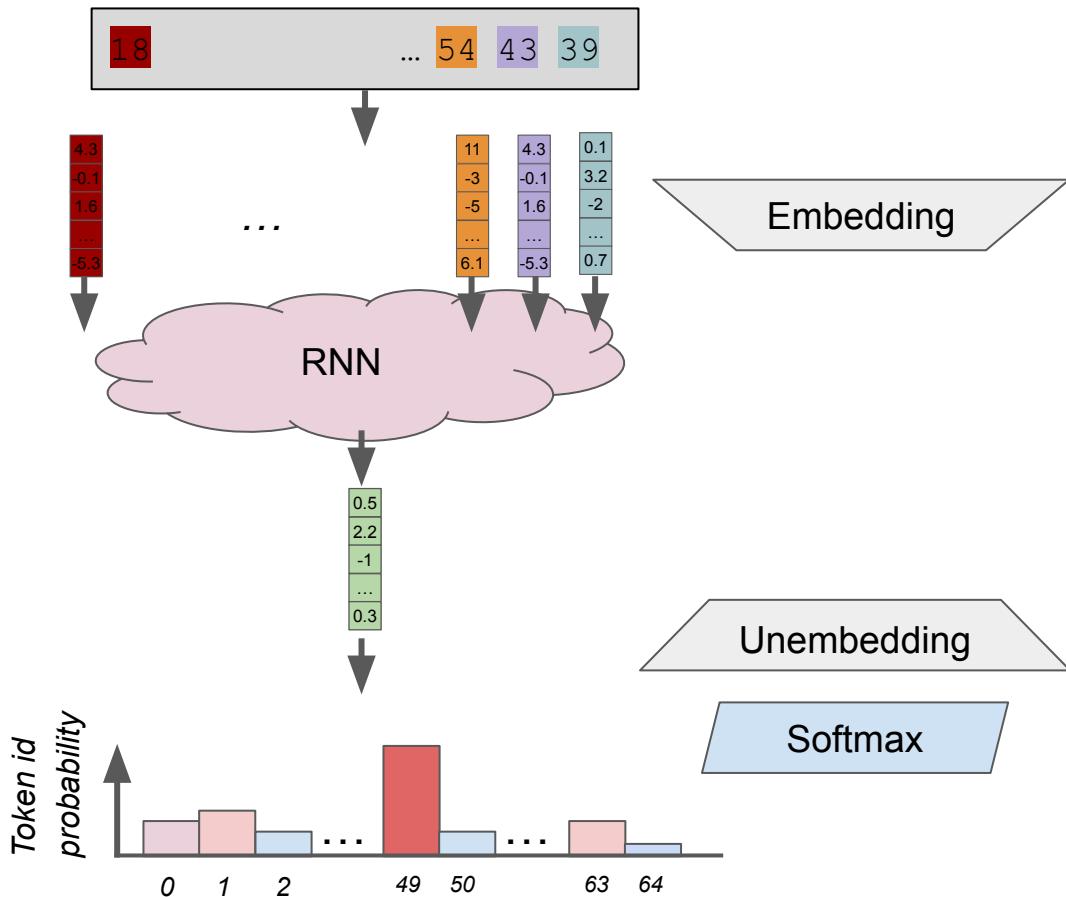
GoogleTrends

Searches for “AI”



Sequence Modeling

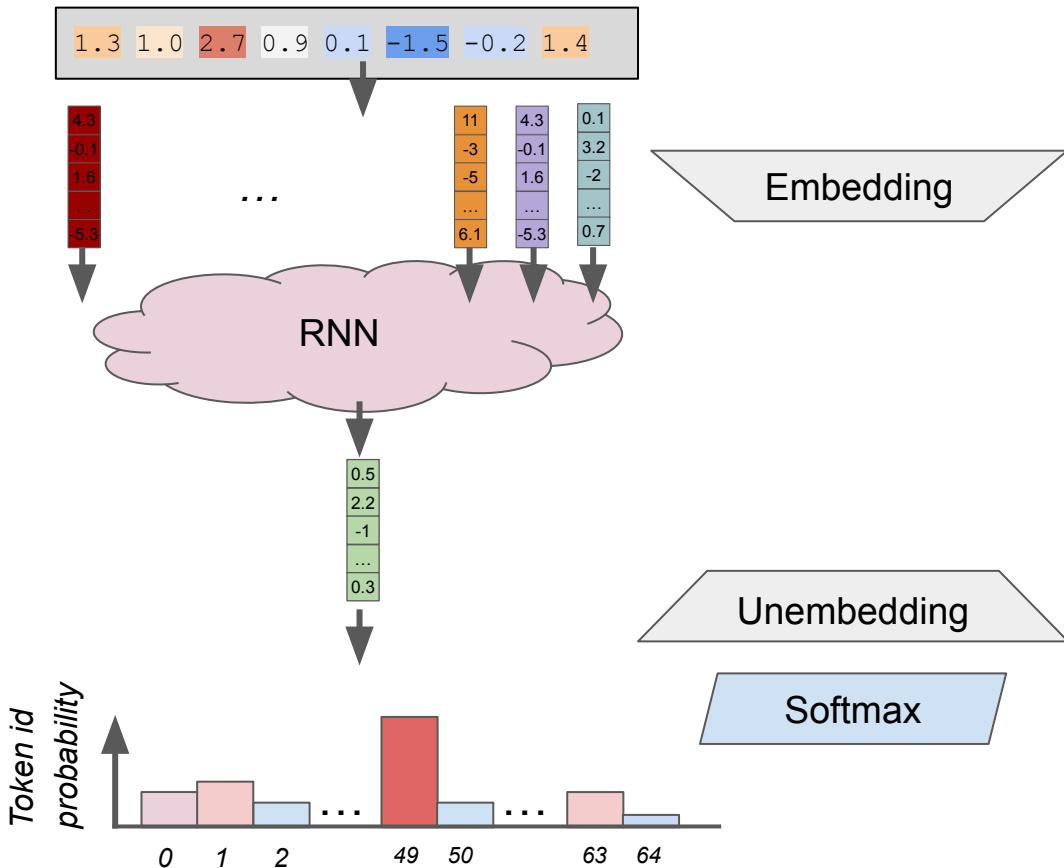
Question: What do we need to change?



Sequence Modeling

Tokens → numbers

Question: What do we need to change?

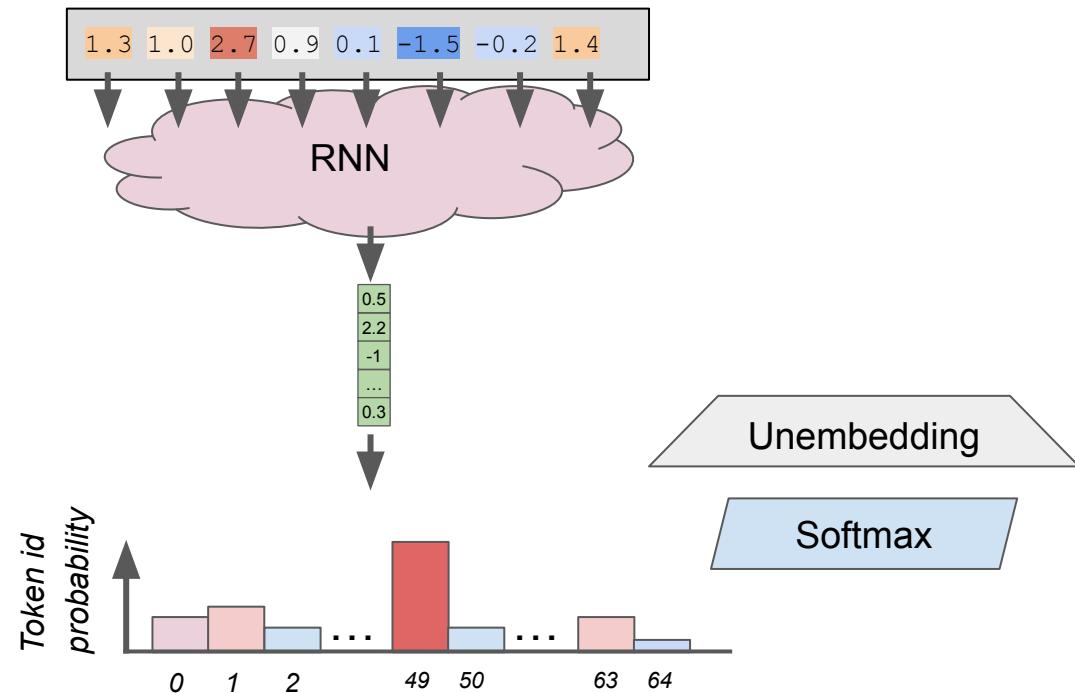


Sequence Modeling

Tokens → numbers

We can remove the embedding layer

*Question: What do we need
to change?*



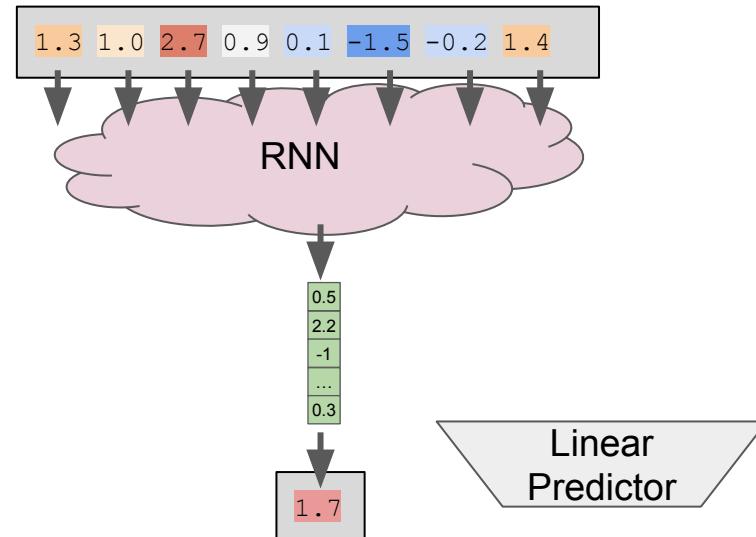
Sequence Modeling

Tokens → numbers

We can remove the embedding layer

*Question: What do we need
to change?*

Unembedding is now just a linear
layer with one output, and we use
regression loss

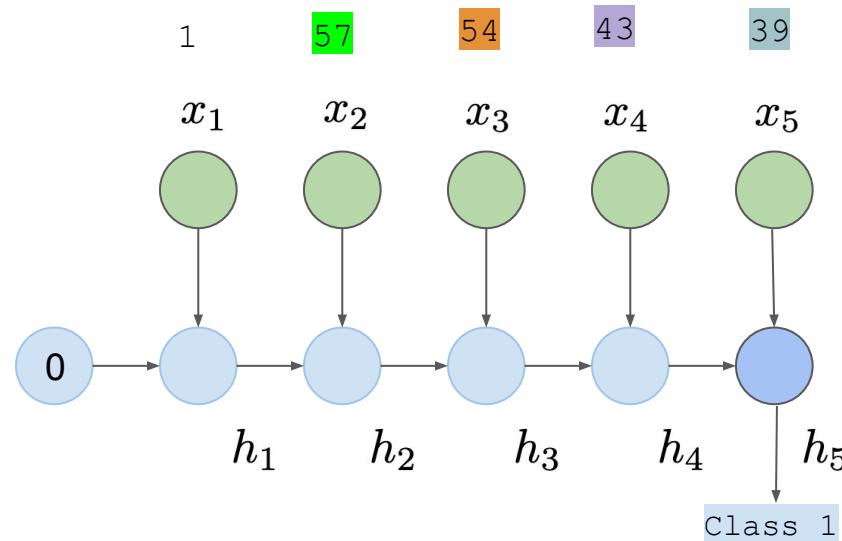


Sequence Classification

- Sentiment analysis: is this a positive review?
- Language detection: which language is this?
- Does this python program compile?
- ...

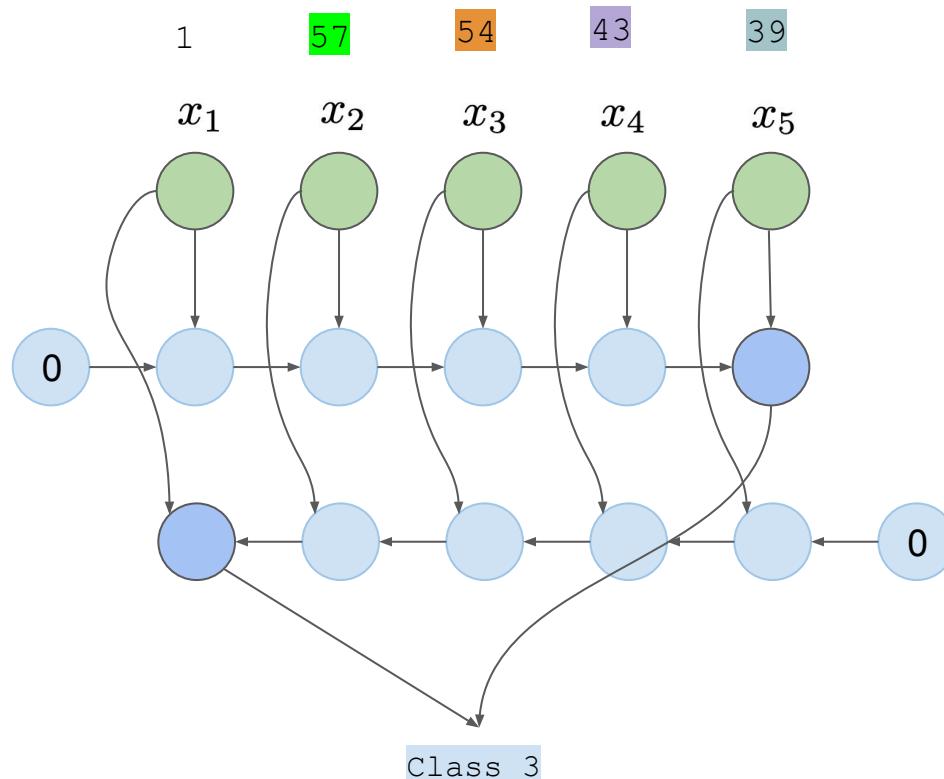


Sequence Classification



We can just use the hidden representation for the last token to make predictions

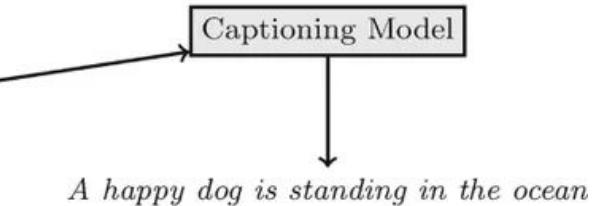
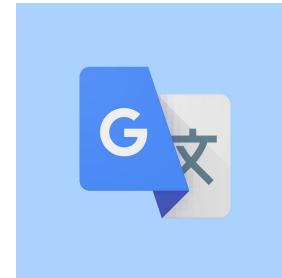
Sequence Classification



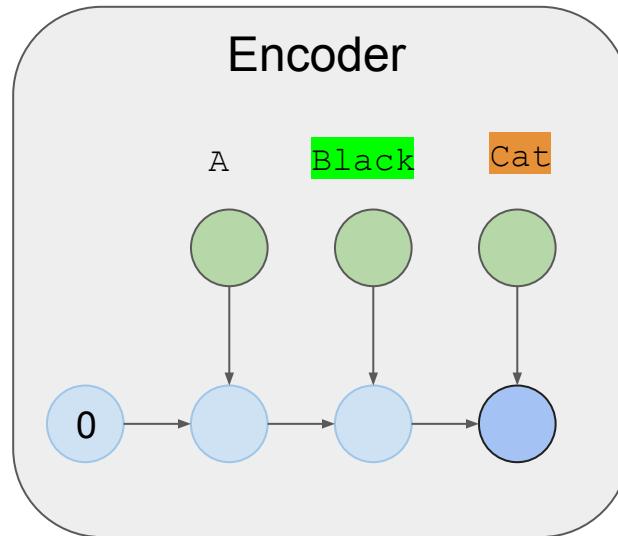
We can generalize the idea to bidirectional RNNs: use the two nodes with full context

Sequence to Sequence

- Machine Translation: Translating text from one language to another
- Text Summarization
- Speech Recognition
- Image Captioning

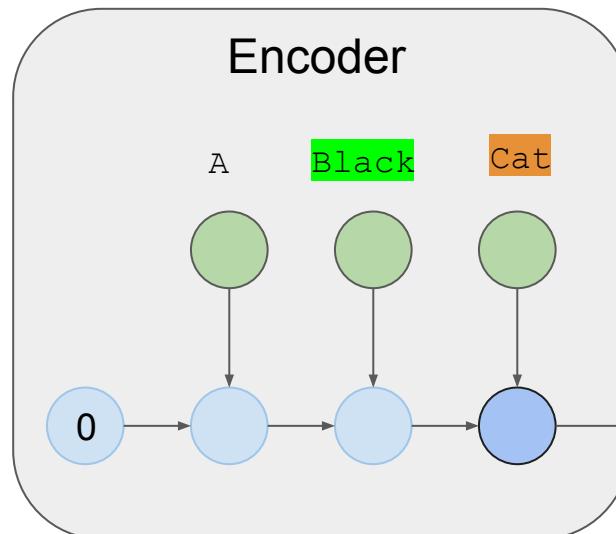


Sequence to Sequence

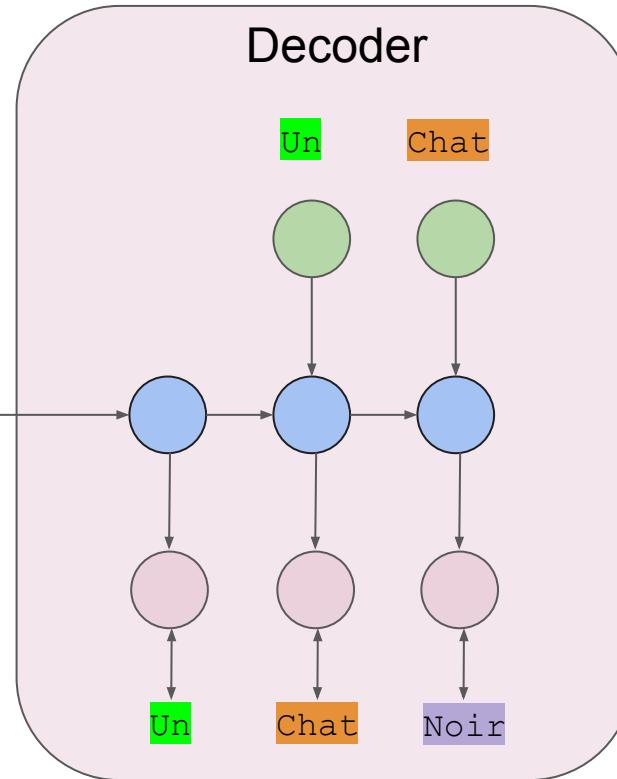


Encoder produces a hidden representation that will be used to decode the sequence

Sequence to Sequence

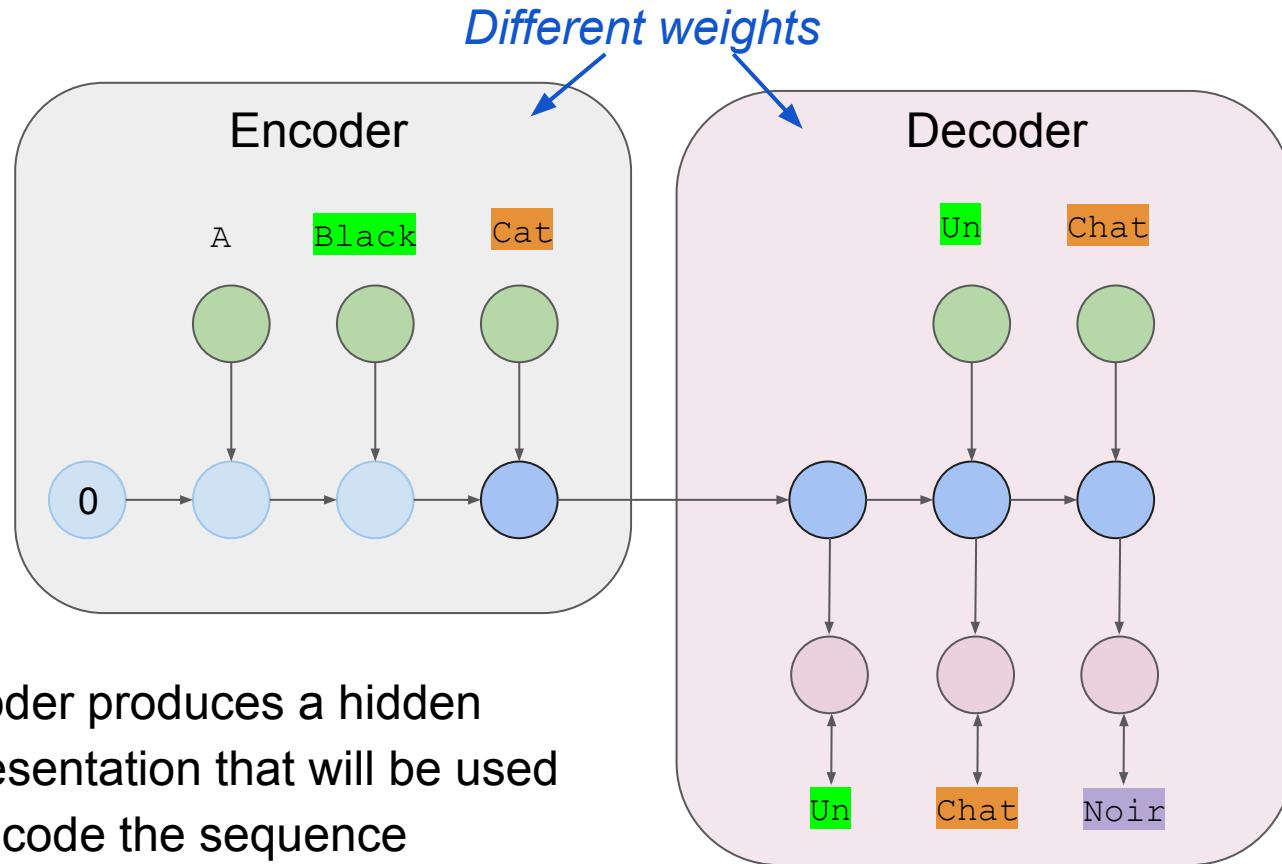


Encoder produces a hidden representation that will be used to decode the sequence



Decoder is doing language modeling

Sequence to Sequence

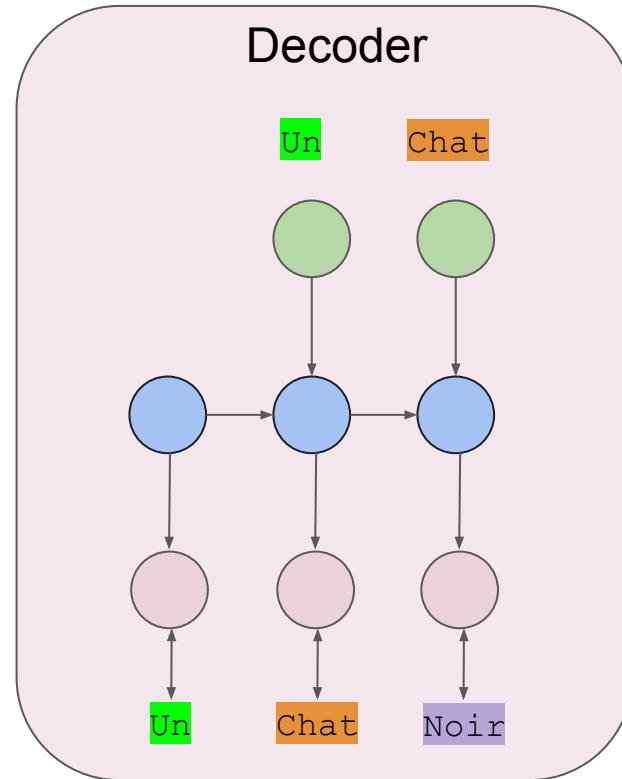


Encoder produces a hidden representation that will be used to decode the sequence

Decoder is doing language modeling

Sequence to Sequence

Last week, we were looking at decoder-only models



Decoder is
doing
language
modeling



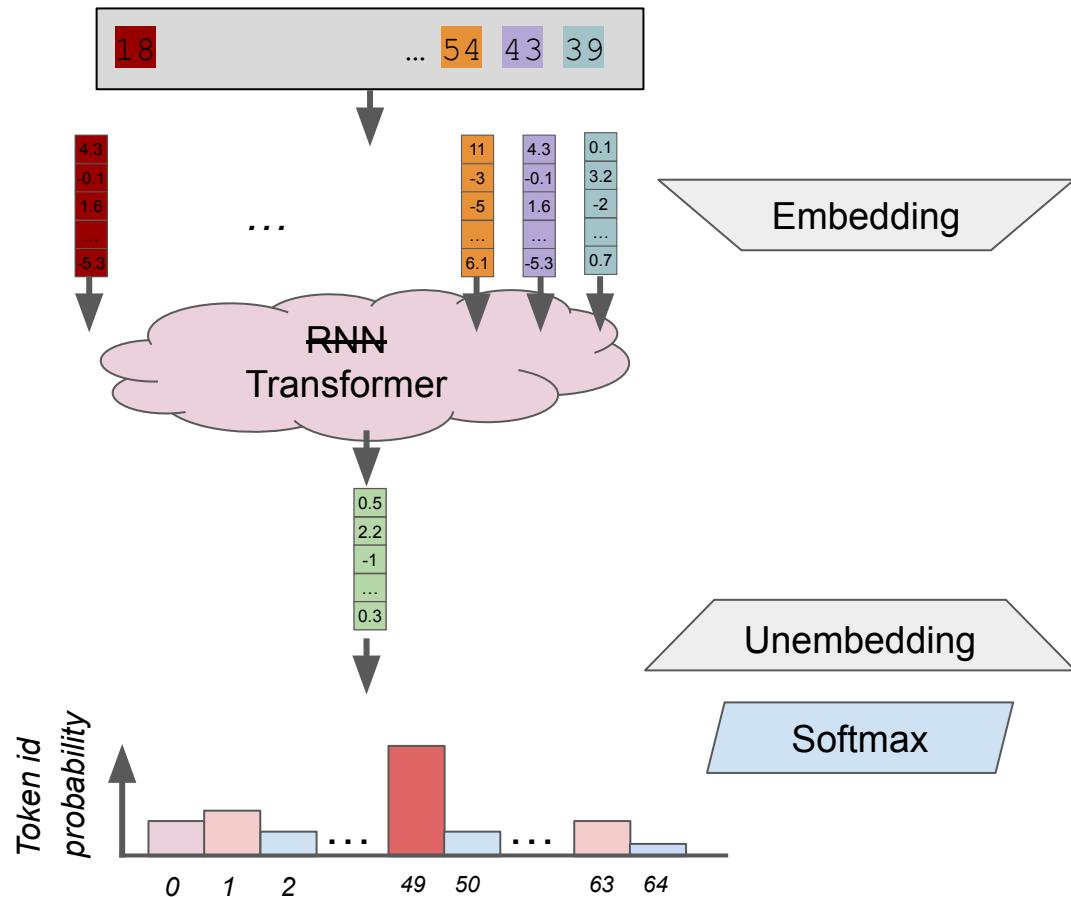
Transformers

0.36

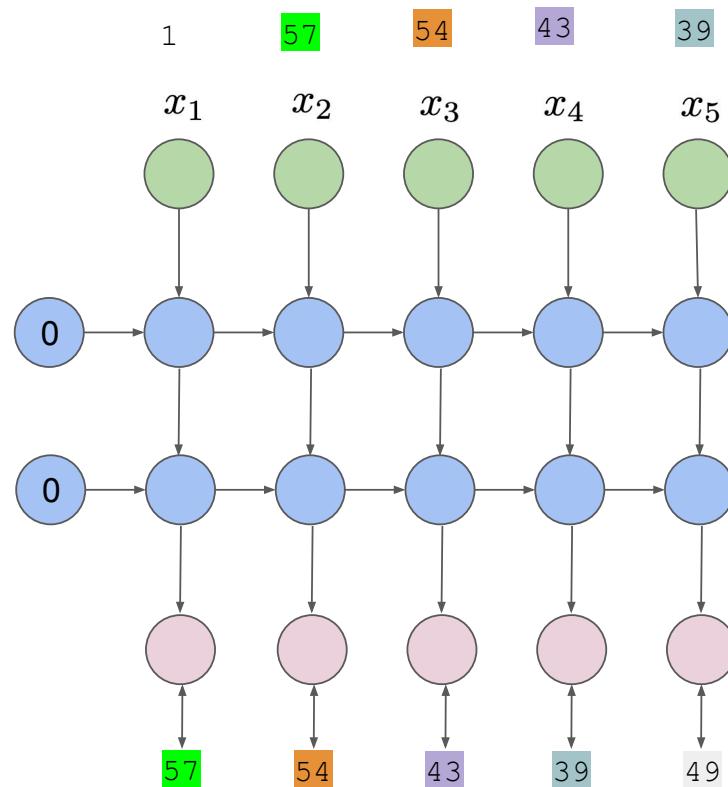
0.34

Transformers

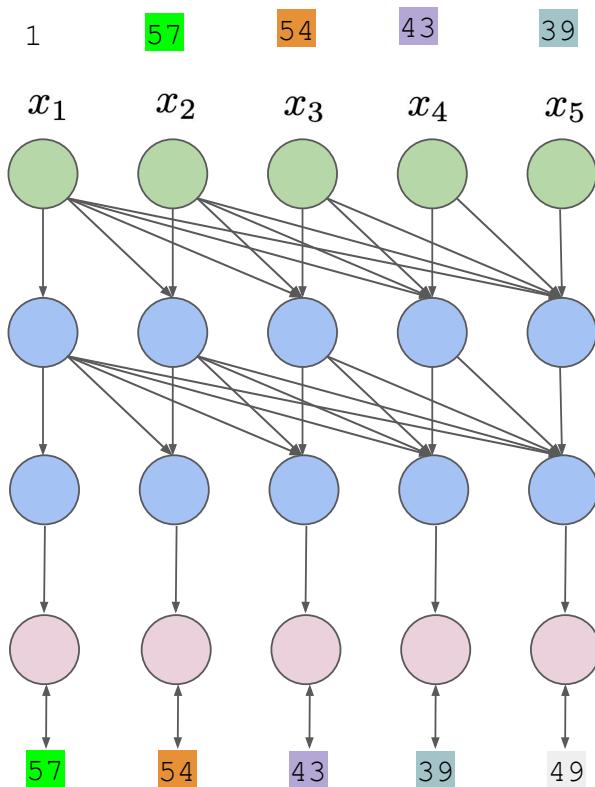
We will go back to our decoder-only language modeling setting.



Transformers

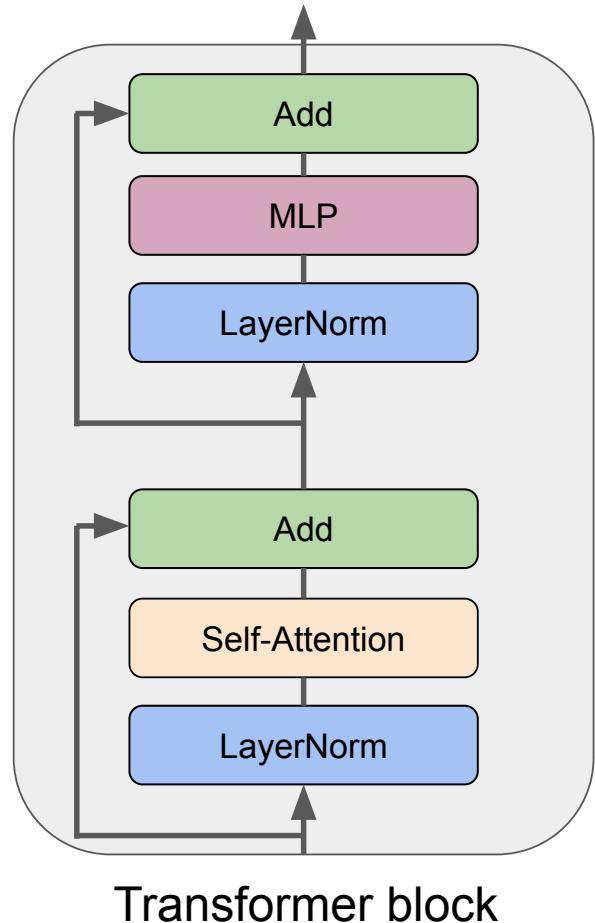


RNN



Transformer

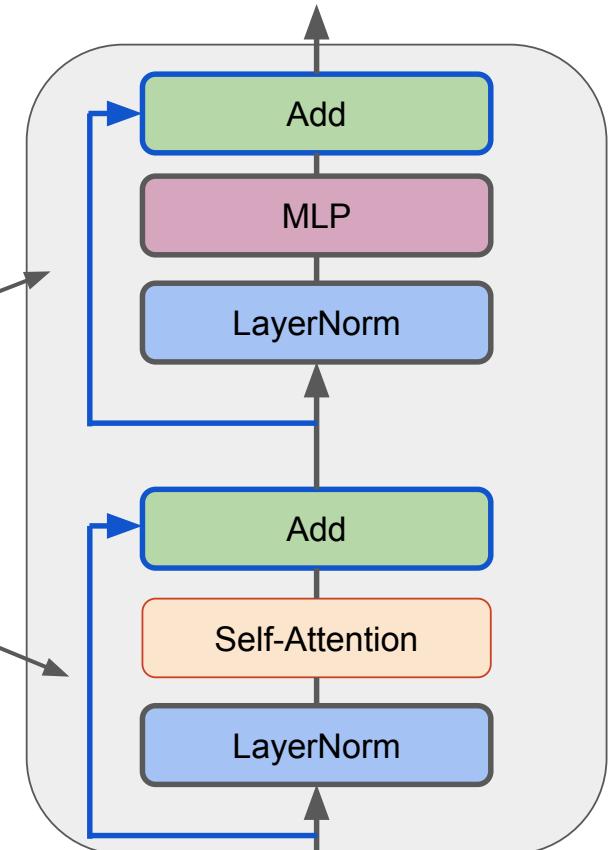
Transformers



Transformers

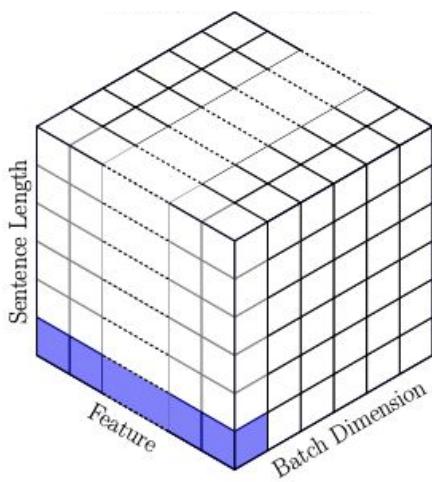
We know almost all of the components!

Skip Connections



Transformer block

Layernorm



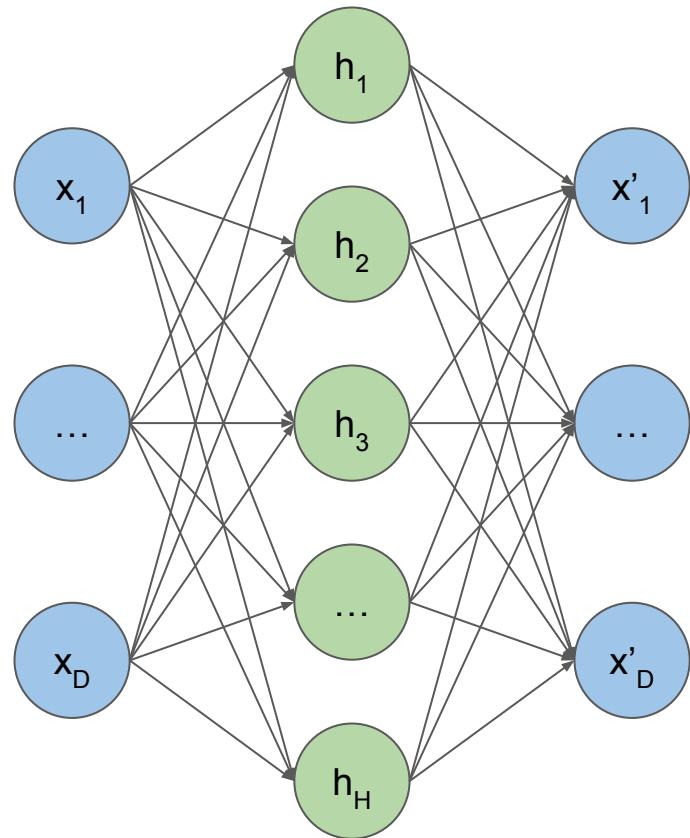
$$y = \frac{x - \mathbb{E}[x]}{\sqrt{\text{Var}[x] + \epsilon}} * \gamma + \beta$$

Annotations for the equation:

- "Layernorm output" points to the final result y .
- "Input" points to the term x .
- "Trainable scale" points to the term γ .
- "Trainable shift" points to the term β .
- " $\mathbb{E}[x]$ " and " $\text{Var}[x]$ " are highlighted in green boxes.

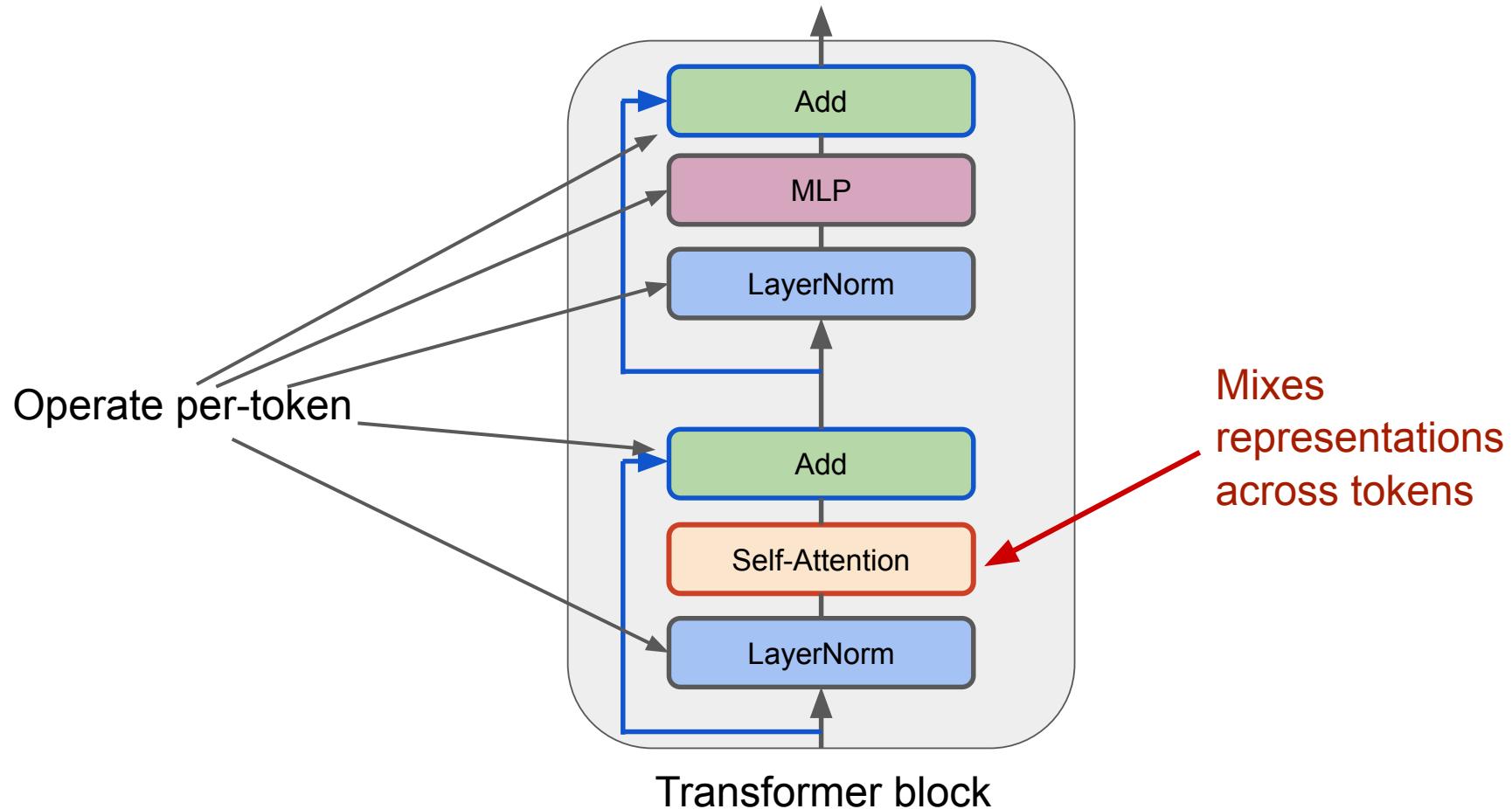
- We normalize the activations per token
- Mean and variance are computed over all dimensions of the input

MLP

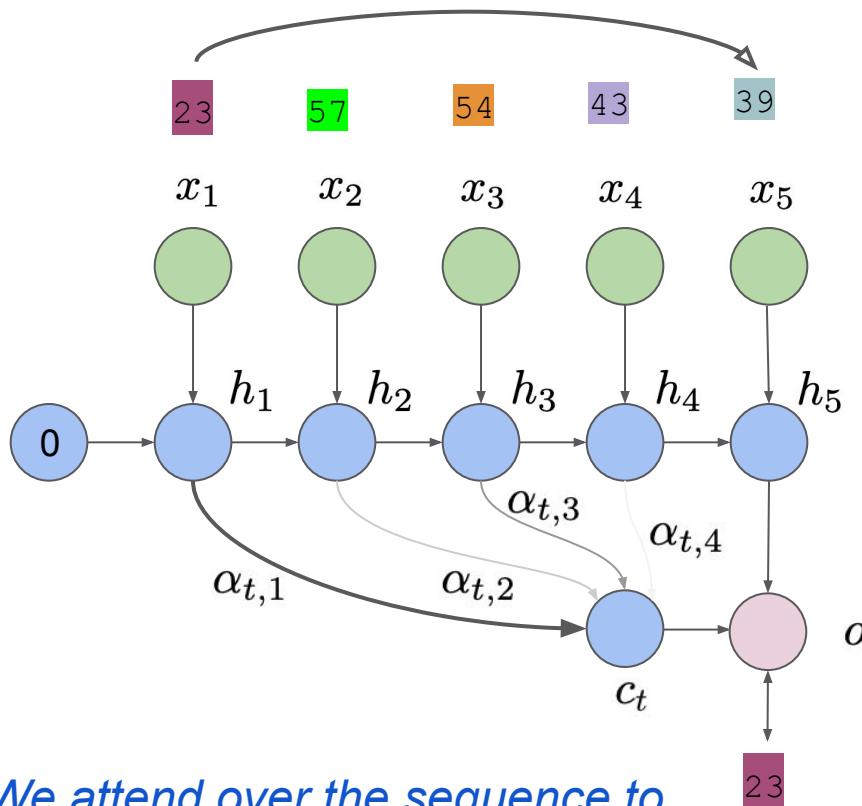


- Same MLP as we discussed before
- Transforms the features of each token separately

Transformers



RNN Variants: Attention



We attend over the sequence to look for relevant information!

We will try to fix the long-range dependence issue.

$$e_{t,i} = \text{score}(h_t, h_i), \quad i < t$$

$$\alpha_{t,i} = \frac{\exp(e_{t,i})}{\sum_{j < t} \exp(e_{t,j})}$$

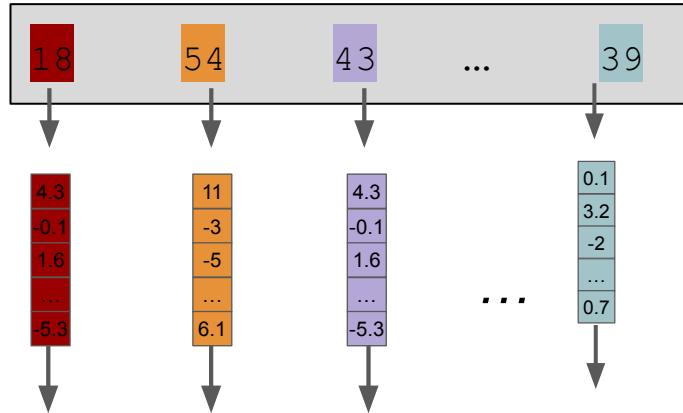
Compute *attention weights* for each previous token

$$c_t = \sum_{i < t} \alpha_{t,i} h_i$$

Use a weighted sum of all representations to make the prediction

Self-Attention

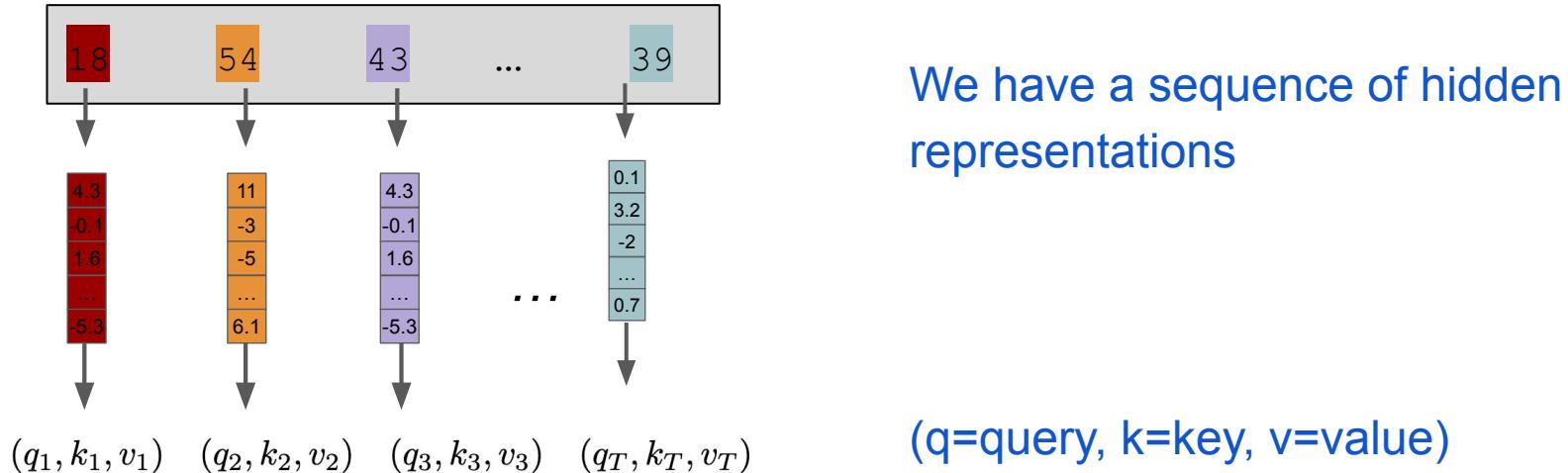
We want a transformation that can mix representations over multiple tokens.



We have a sequence of hidden representations

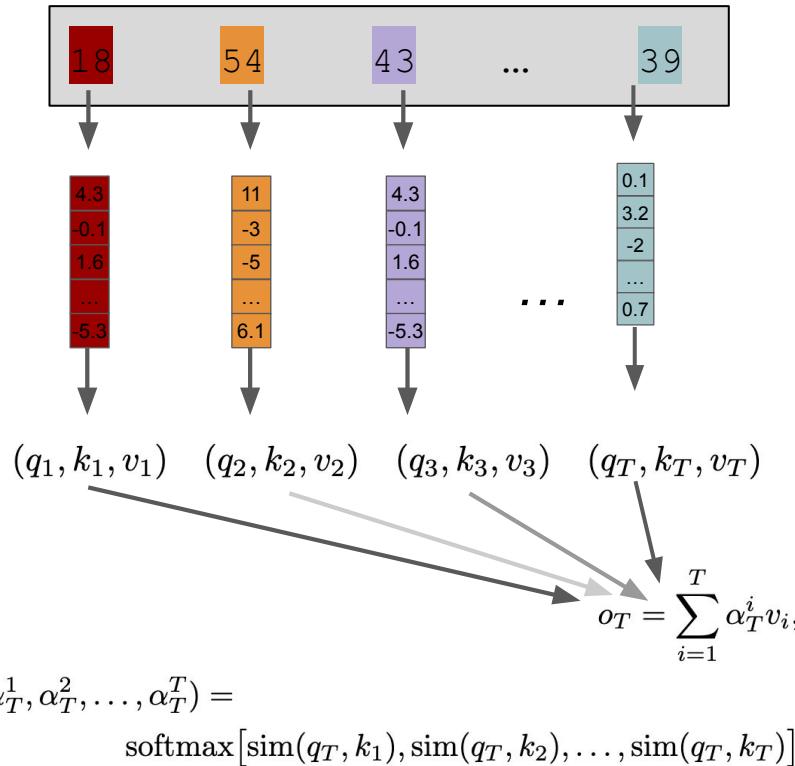
Self-Attention

We want a transformation that can mix representations over multiple tokens.



Self-Attention

We want a transformation that can mix representations over multiple tokens.



We have a sequence of hidden representations

(q =query, k =key, v =value)

Output is a mix of values with weights based on $\text{similarity(query_T, key_i)}$

Key-Query-Value example

Item	Key (car features)	Value (car mpg)
Honda Civic	HP: 158, Engine: 2.0L, Year: 2020	33
Ford Mustang	HP: 310, Engine: 5.0L, Year: 2019	18
Toyota Prius	HP: 121, Engine: 1.8L, Year: 2021	52
...
Chevy Silverado	HP: 285, Engine: 4.3L, Year: 2020	20

Item	Query
Mystery Sedan	HP: 150, Engine: 1.9L, Year: 2020

Key-Query-Value example

Item	Key (car features)	Value (car mpg)	Similarity(key, query)
Honda Civic	HP: 158, Engine: 2.0L, Year: 2020	33	0.92
Ford Mustang	HP: 310, Engine: 5.0L, Year: 2019	18	0.31
Toyota Prius	HP: 121, Engine: 1.8L, Year: 2021	52	0.78
...	
Chevy Silverado	HP: 285, Engine: 4.3L, Year: 2020	20	0.35

Item	Query
Mystery Sedan	HP: 150, Engine: 1.9L, Year: 2020

Key-Query-Value example

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...	
Chevy Silverado	HP: 285, Engine: 4.3L, Year: 2020	20	0.35

Item	Query
Mystery Sedan	HP: 150, Engine: 1.9L, Year: 2020

$$o = \sum_{i=1}^T \alpha^i v_i,$$

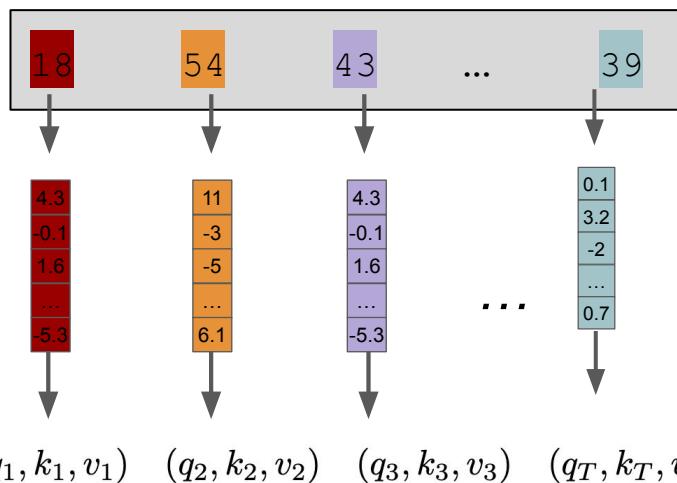
O = 34.2 MPG

$$(\alpha^1, \alpha^2, \dots, \alpha^T) = \text{softmax} [\text{sim}(q, k_1), \text{sim}(q, k_2), \dots, \text{sim}(q, k_T)]$$

Key-Query-Value example

For each token:

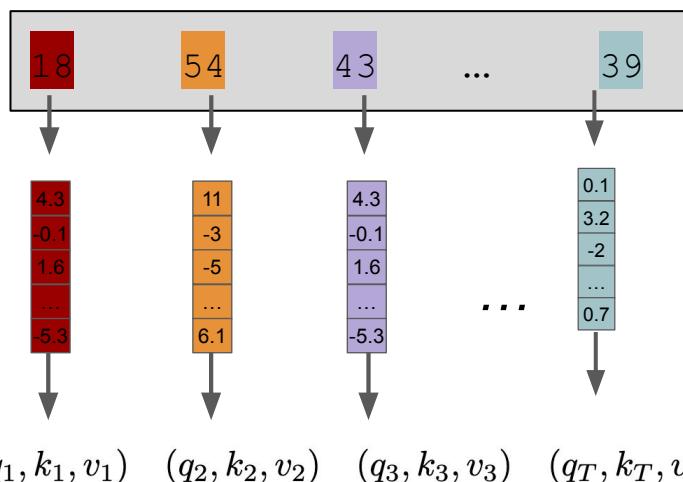
- k: key, a description of the information in this token, used to find this token
- v: value, output associated with the key
- q: query, what we want to find in other tokens



Key-Query-Value example

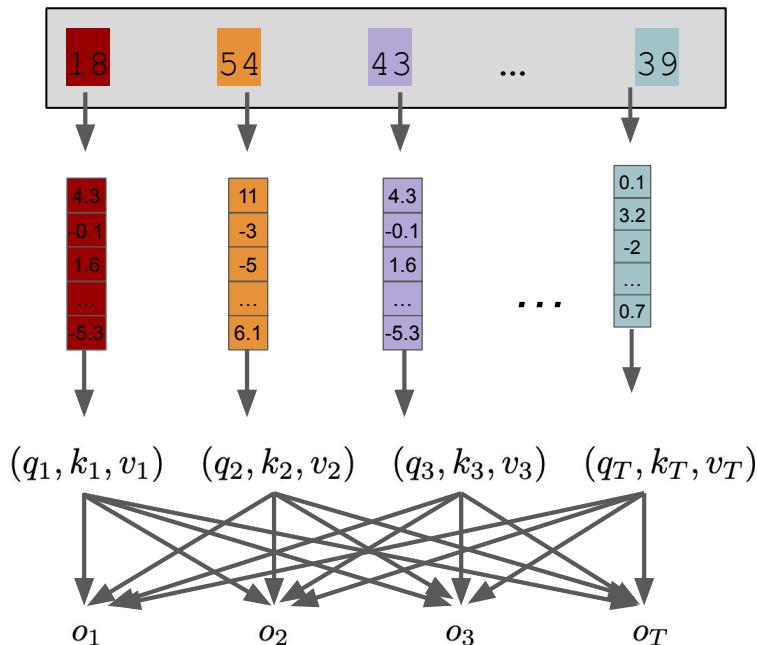
For each token:

- k: key, a description of the information in this token, used to find this token
- v: value, output associated with the key
- q: query, what we want to find in other tokens



Unlike our previous example, in transformers (q, k, v) are abstract and learned by the model; they will not have an interpretable meaning

Key-Query-Value attention



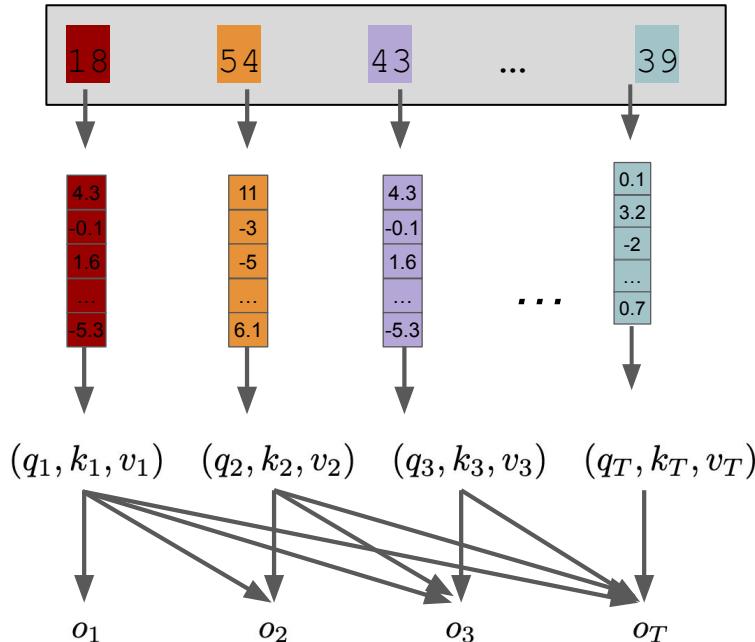
We get an output for each token, which depends on all other tokens

$$o_j = \sum_{i=1}^T \alpha_j^i v_i,$$

$$(\alpha_j^1, \alpha_j^2, \dots, \alpha_j^T) =$$

$$\text{softmax}[\text{sim}(q_j, k_1), \text{sim}(q_j, k_2), \dots, \text{sim}(q_j, k_T)]$$

Key-Query-Value attention



We get an output for each token, which depends on all other tokens

For language modeling we apply a causal mask: outputs can not depend on future tokens

$$o_j = \sum_{i=1}^T \alpha_j^i v_i,$$

$$(\alpha_j^1, \alpha_j^2, \dots, \alpha_j^T) =$$

$$\text{softmax}[\text{sim}(q_j, k_1), \text{sim}(q_j, k_2), \dots, \text{sim}(q_j, k_T)]$$

Dot-Product Self Attention

$$W_Q \quad x \quad q = \begin{matrix} 4.3 \\ -0.1 \\ 1.6 \\ \dots \\ -5.3 \end{matrix}$$

$$W_V \quad x \quad v = \begin{matrix} 4.3 \\ -0.1 \\ 1.6 \\ \dots \\ -5.3 \end{matrix}$$

$$W_K \quad x \quad k = \begin{matrix} 4.3 \\ -0.1 \\ 1.6 \\ \dots \\ -5.3 \end{matrix}$$

(q, k, v) are all produced as linear transformations of the hidden vector
 W_Q, W_K, W_V are learned, shared across all tokens

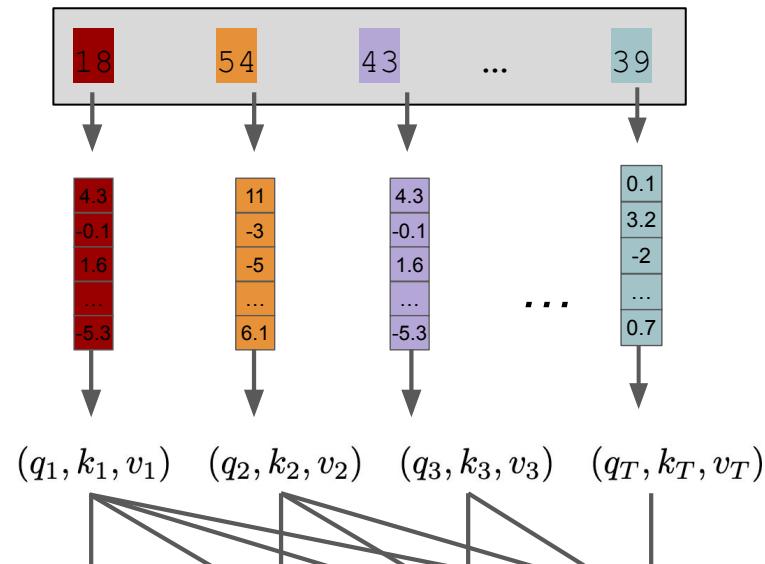
Dot-Product Self Attention

$$\text{sim}(q, k) = \frac{\overbrace{q^T \cdot k}^{\sqrt{d_k}}}{\sqrt{d_k}}$$

The diagram illustrates the dot-product self attention formula. On the left, the inputs q (green, 4 units high) and k (blue, 4 units high) are shown. An equals sign follows. To the right, the formula is expanded: q^T is represented by a row of four green squares, and k is represented by a column of four blue squares. A vertical double-headed arrow between them indicates their height is d_k .

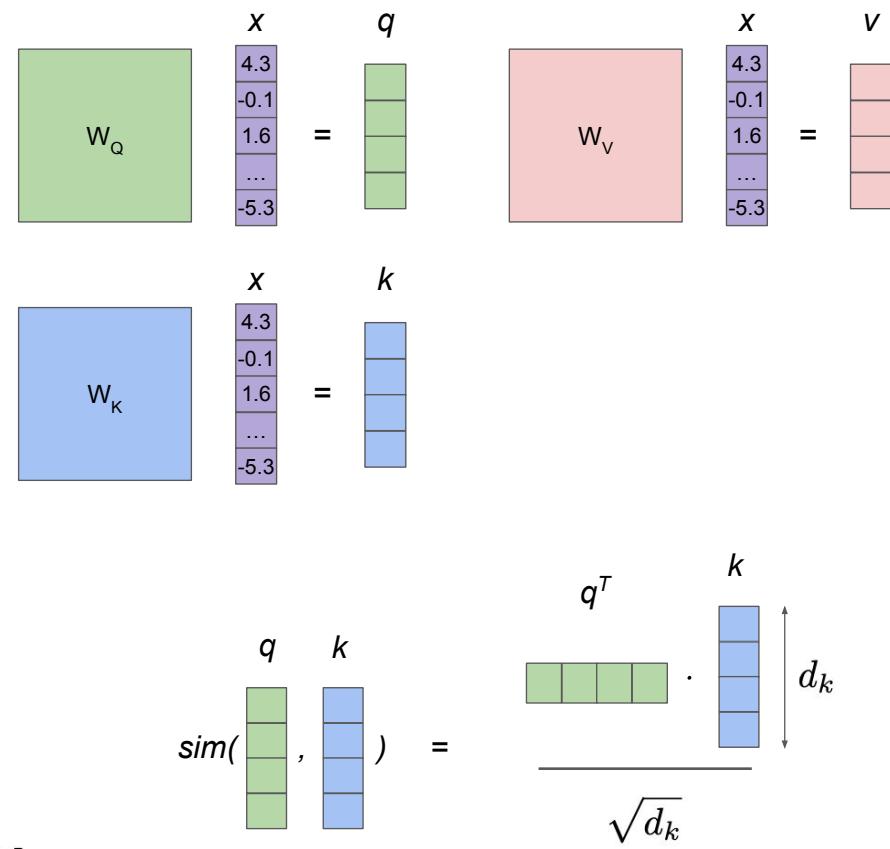
Our similarity score is a scaled dot product. We will explain the scale later.

Dot-Product Self Attention

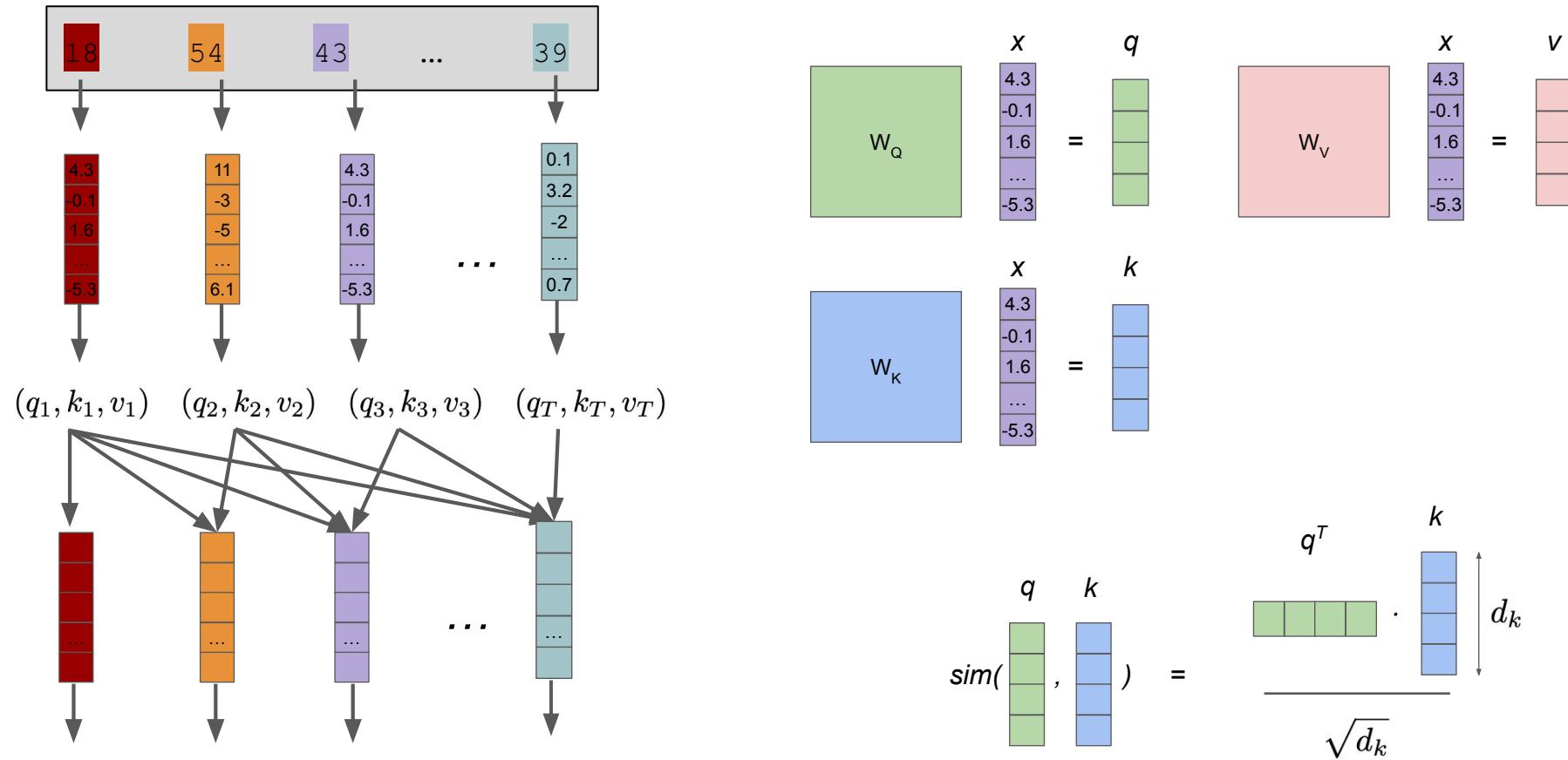


$$o_j = \sum_{i=1}^T \alpha_j^i v_i,$$

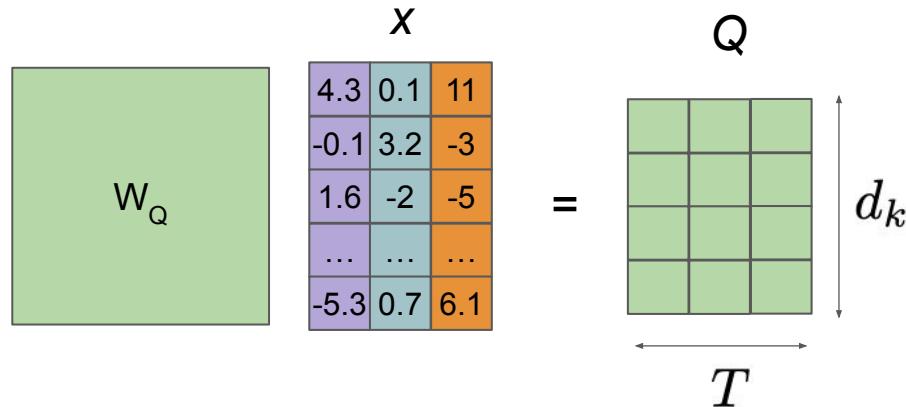
$$(\alpha_j^1, \alpha_j^2, \dots, \alpha_j^T) = \text{softmax}[\text{sim}(q_j, k_1), \text{sim}(q_j, k_2), \dots, \text{sim}(q_j, k_T)]$$



Dot-Product Self Attention

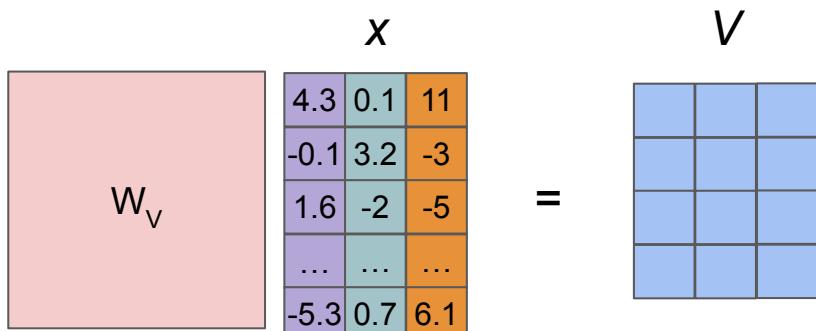
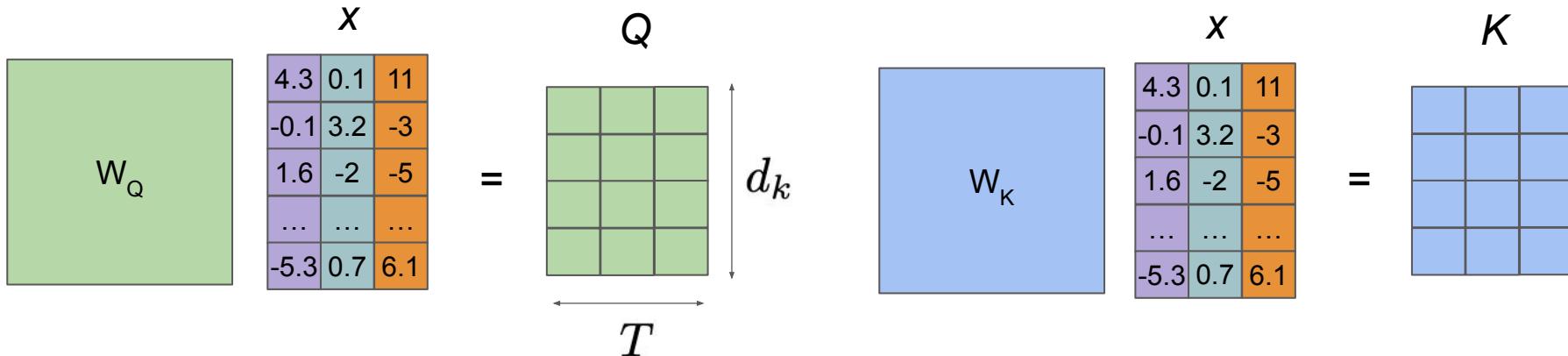


Dot-Product Self Attention



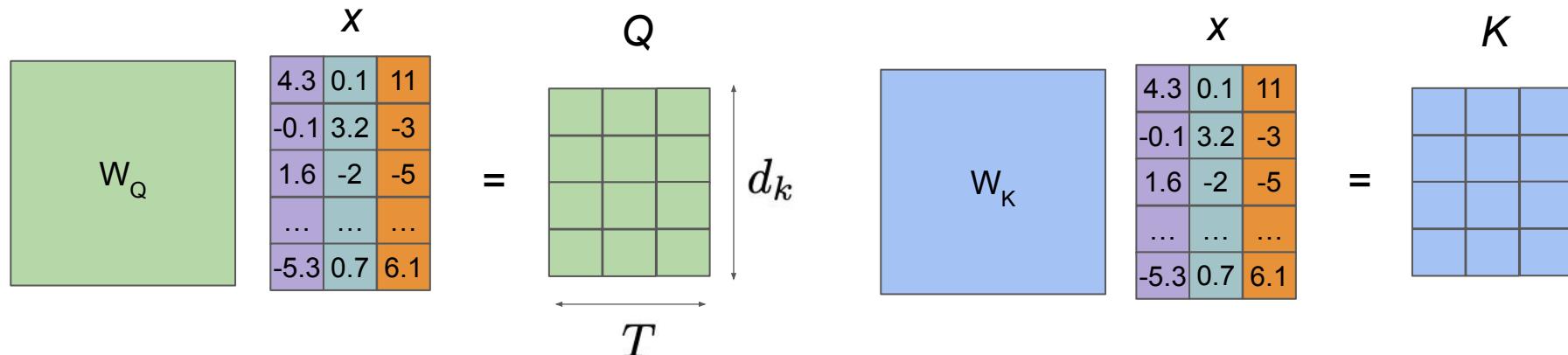
Let's process all T tokens in parallel.

Dot-Product Self Attention



Let's process all T tokens in parallel.

Dot-Product Self Attention

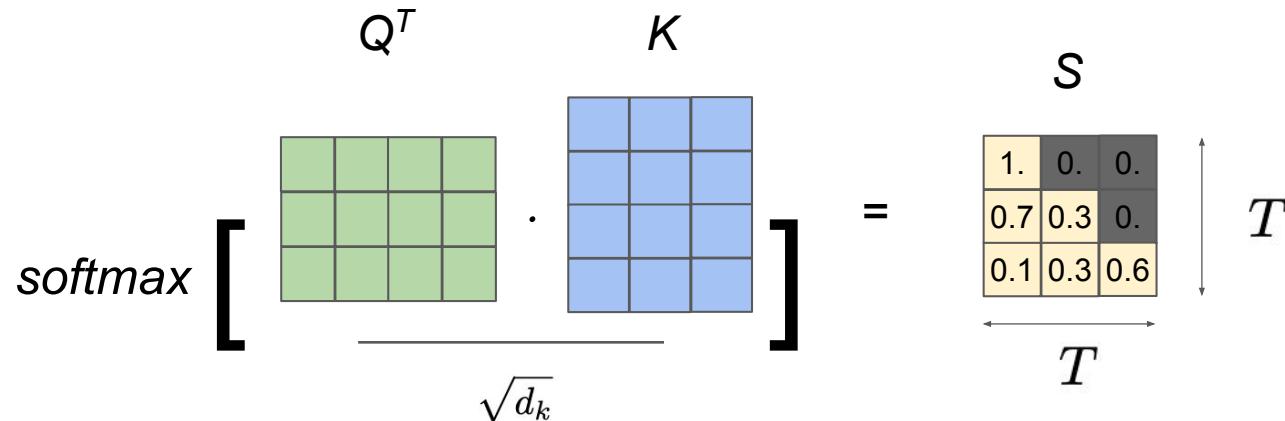
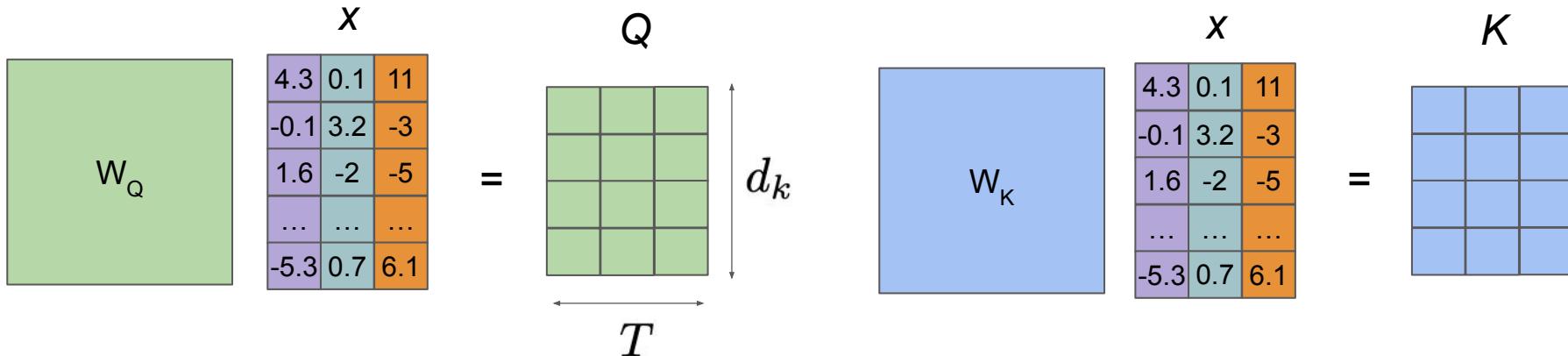


The diagram illustrates the computation of the similarity matrix S from the Query matrix Q^T and the Key matrix K . The Query matrix Q^T is shown as a 3x5 grid of green squares. The Key matrix K is shown as a 3x3 grid of blue squares. The resulting similarity matrix S is a $T \times T$ matrix, where T is the sequence length. The similarity matrix S is shown as a 3x3 grid of orange squares with values: $\begin{matrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{matrix}$. The formula for the similarity matrix is given as:

$$\text{softmax} \left[\frac{Q^T \cdot K}{\sqrt{d_k}} \right] = S$$

Then, construct a T by T similarity matrix.

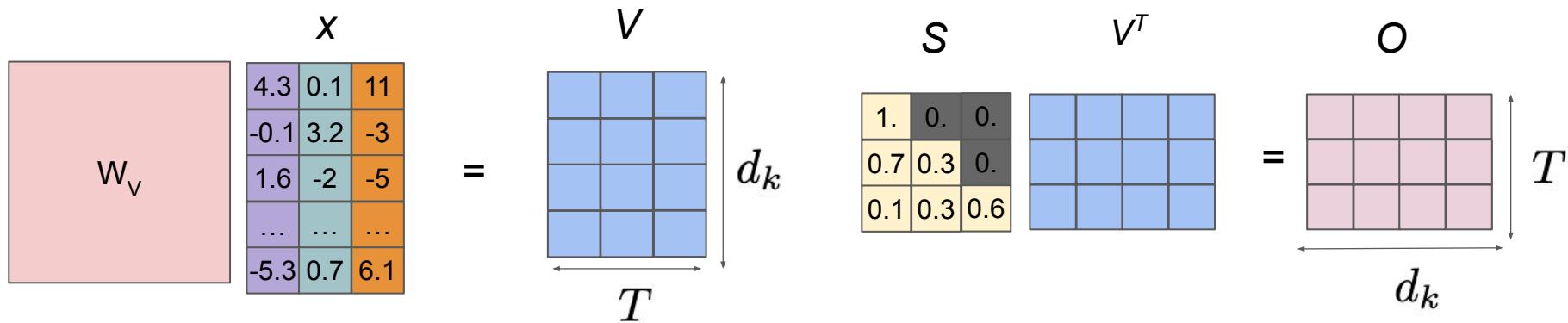
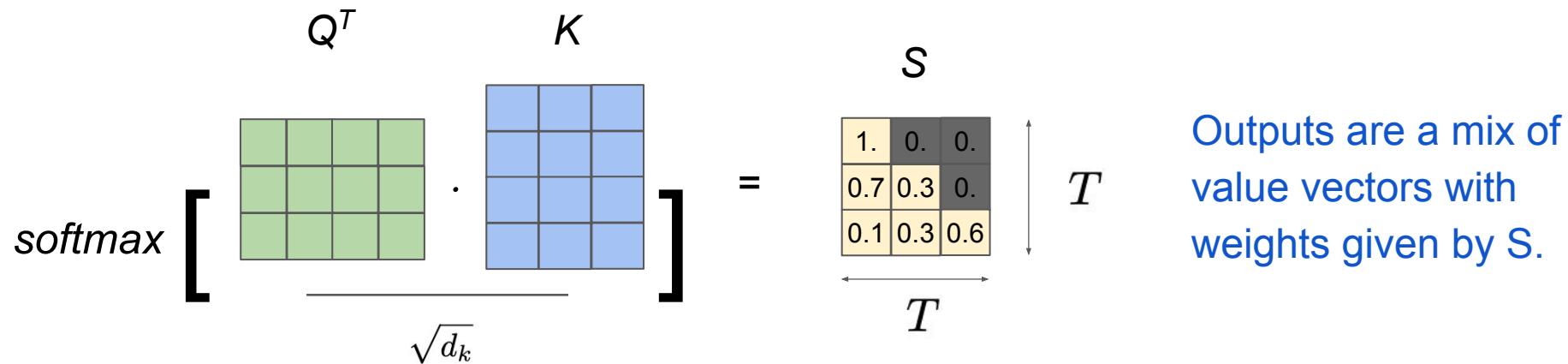
Dot-Product Self Attention



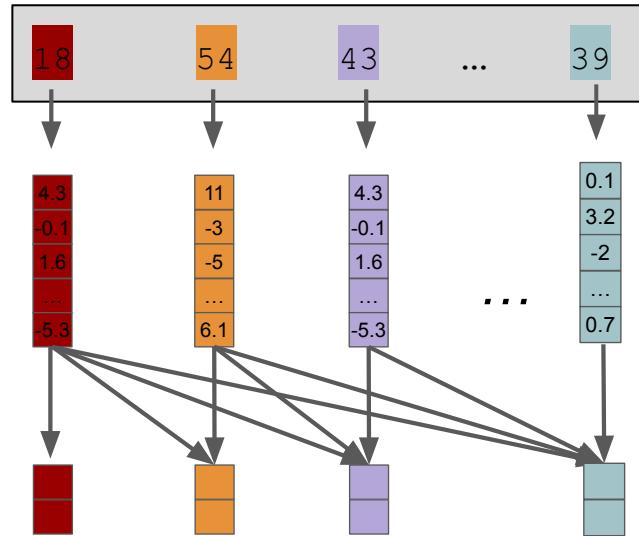
Then, construct a T by T similarity matrix.

Causal mask: cannot look at future tokens.

Dot-Product Self Attention

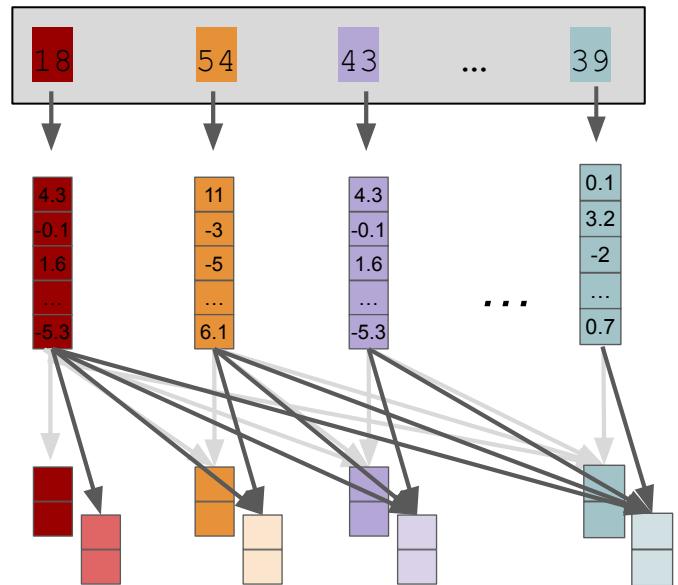


Multi-Head Attention



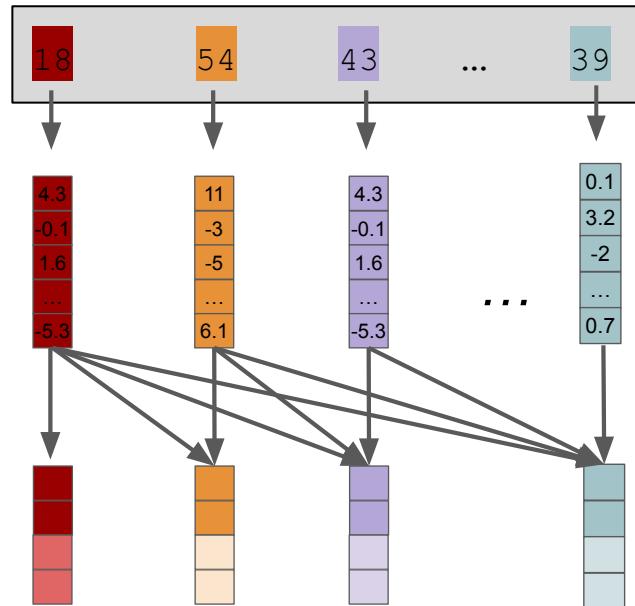
Let's just do attention twice with different projection matrices W_Q , W_K , W_V

Multi-Head Attention



Let's just do attention twice with different projection matrices W_Q , W_K , W_V

Multi-Head Attention



Let's just do attention twice with different projection matrices W_Q , W_K , W_V

We can just concatenate the outputs.

We now have two *attention heads*.

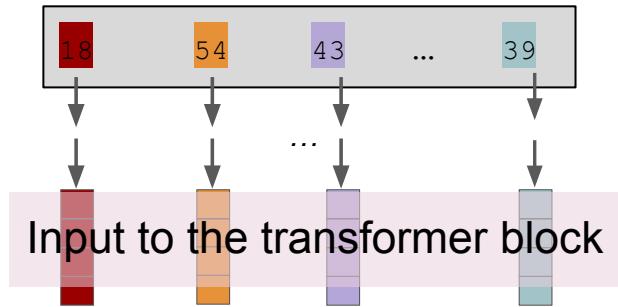
In particular, attention heads will use different attention weights.

$$S_1 \quad S_2$$

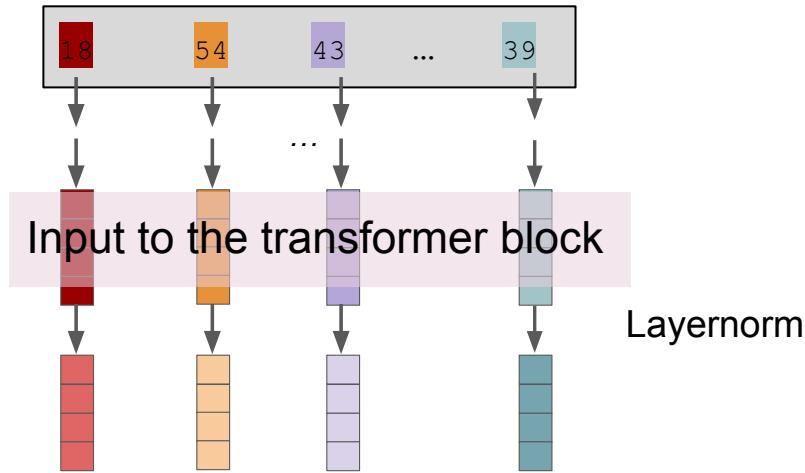
1.	0.	0.
0.7	0.3	0.
0.1	0.3	0.6

1.	0.	0.
0.4	0.6	0.
0.8	0.1	0.1

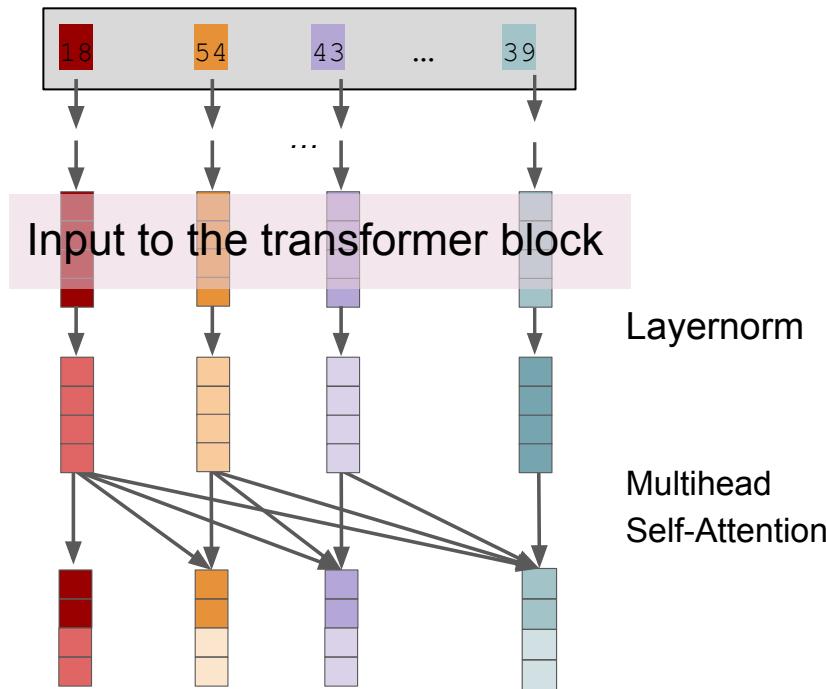
Transformer: Putting it all together



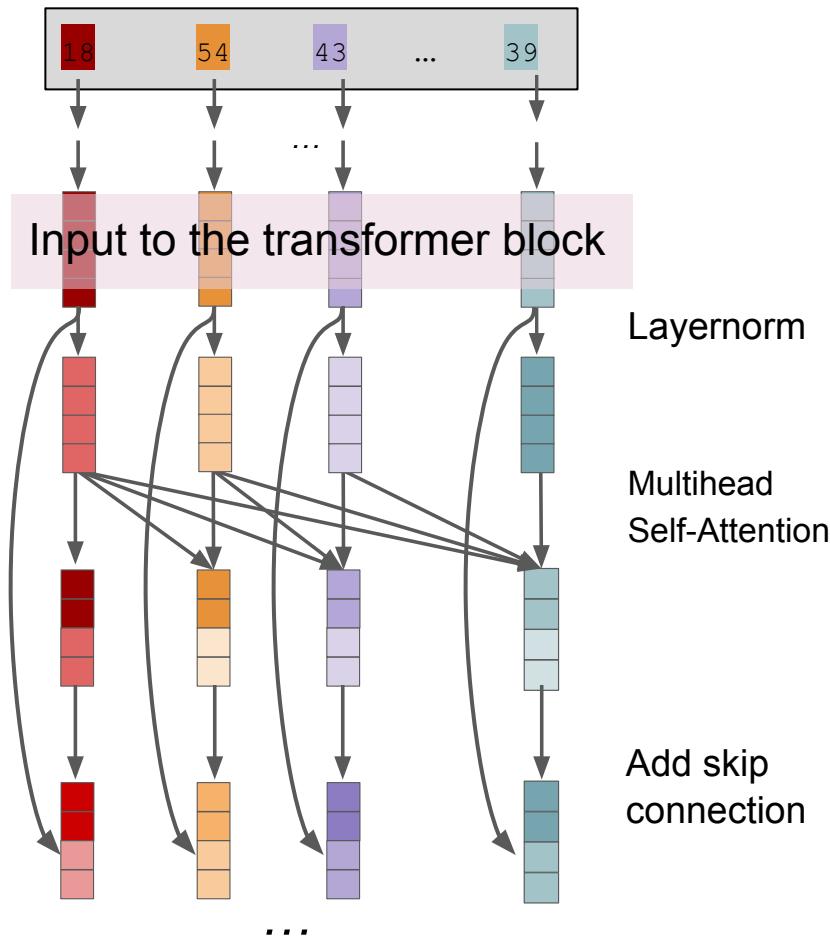
Transformer: Putting it all together



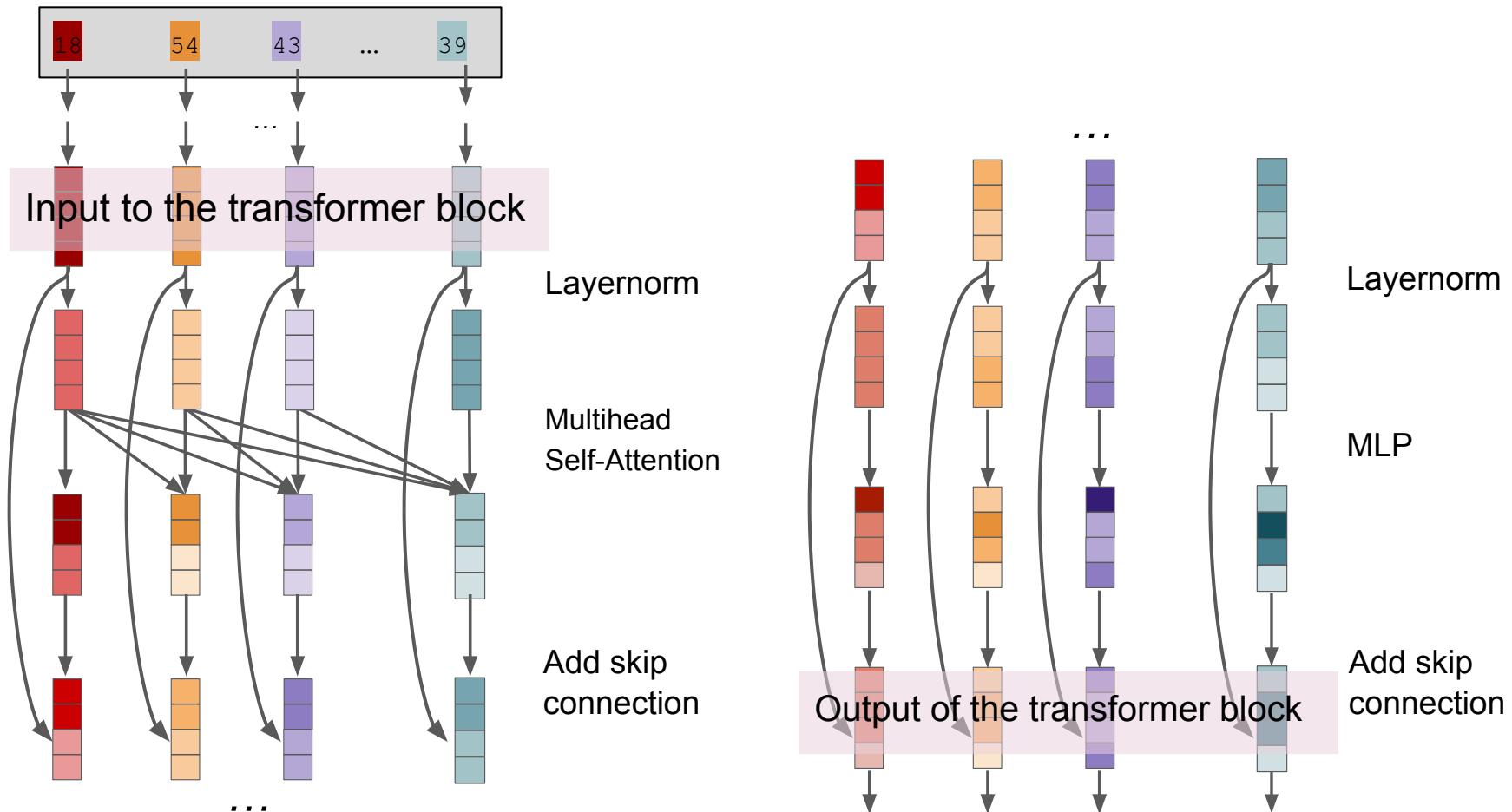
Transformer: Putting it all together



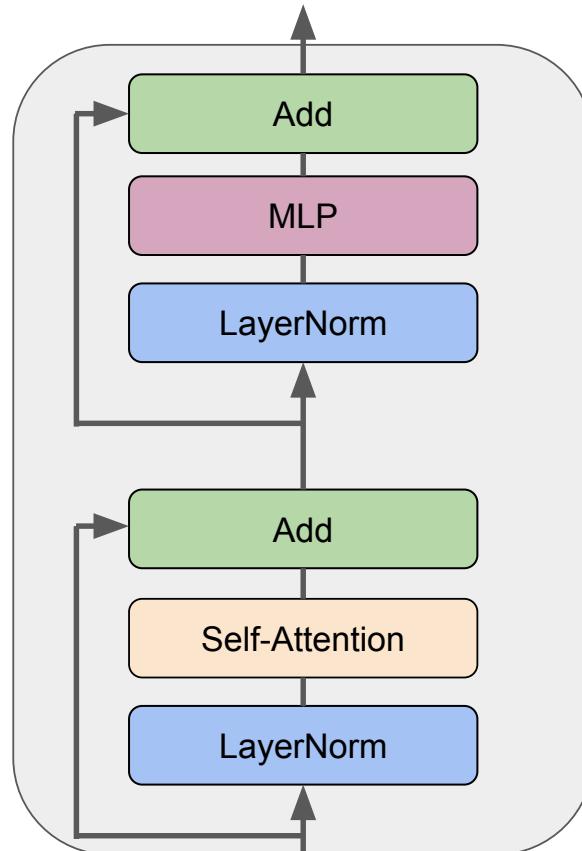
Transformer: Putting it all together



Transformer: Putting it all together



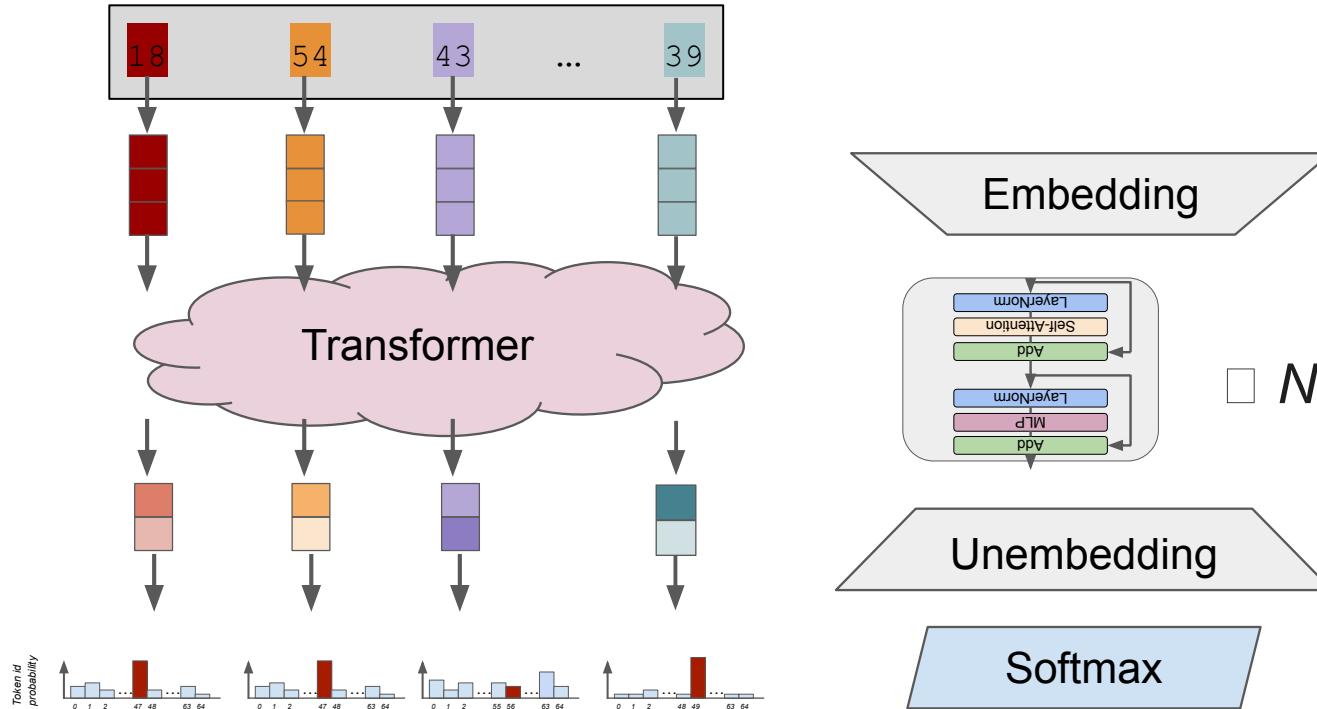
Transformer: Putting it all together



This is our new layer, can repeat!

Transformer block

Transformer: Putting it all together



Attention: Temperature

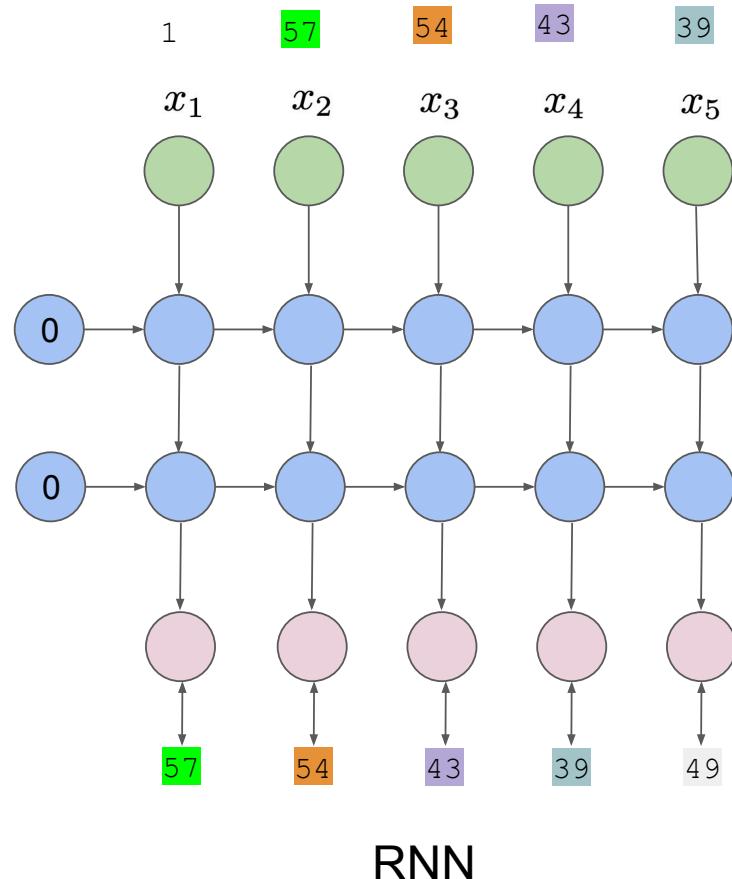
Positional Encoding

Transformers vs RNNs

0.36

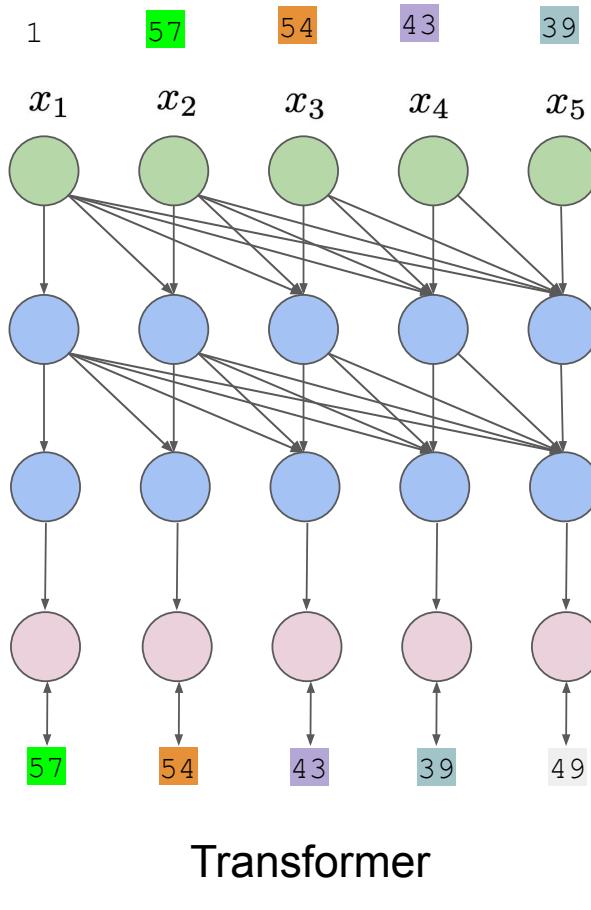
0.34

Transformers vs RNNs

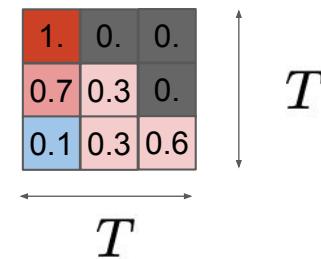


- Process inputs **sequentially**
- Memory
 - Linear in seq len T
- Compute
 - Linear in seq len T
- Time
 - Linear in seq len T
- Context compressed into a fixed-size hidden vector
- Bad at long-range dependencies

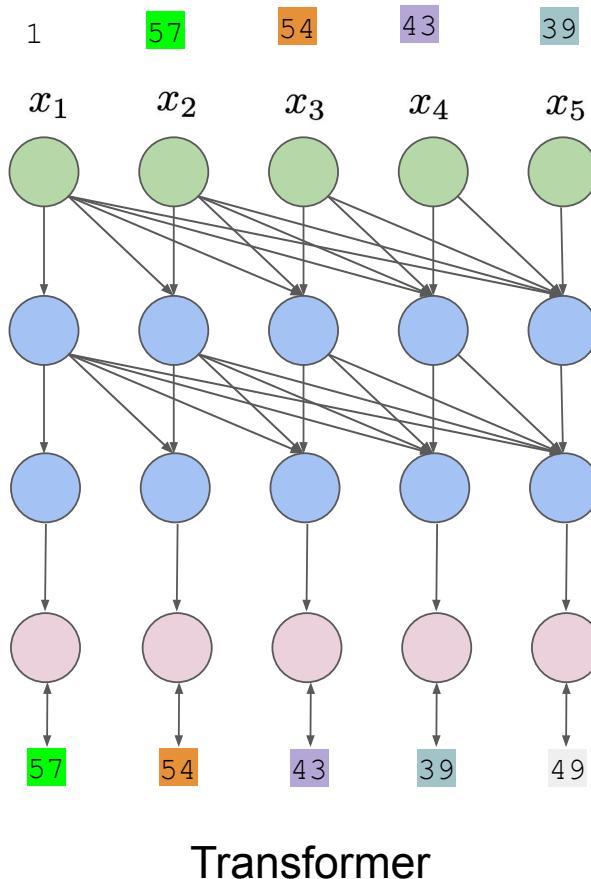
Transformers vs RNNs



- Process inputs in **parallel**
- Memory
 - Quadratic in seq len T
- Compute
 - Quadratic in seq len T
- Time
 - Linear in seq len T
- Good at long-range dependencies

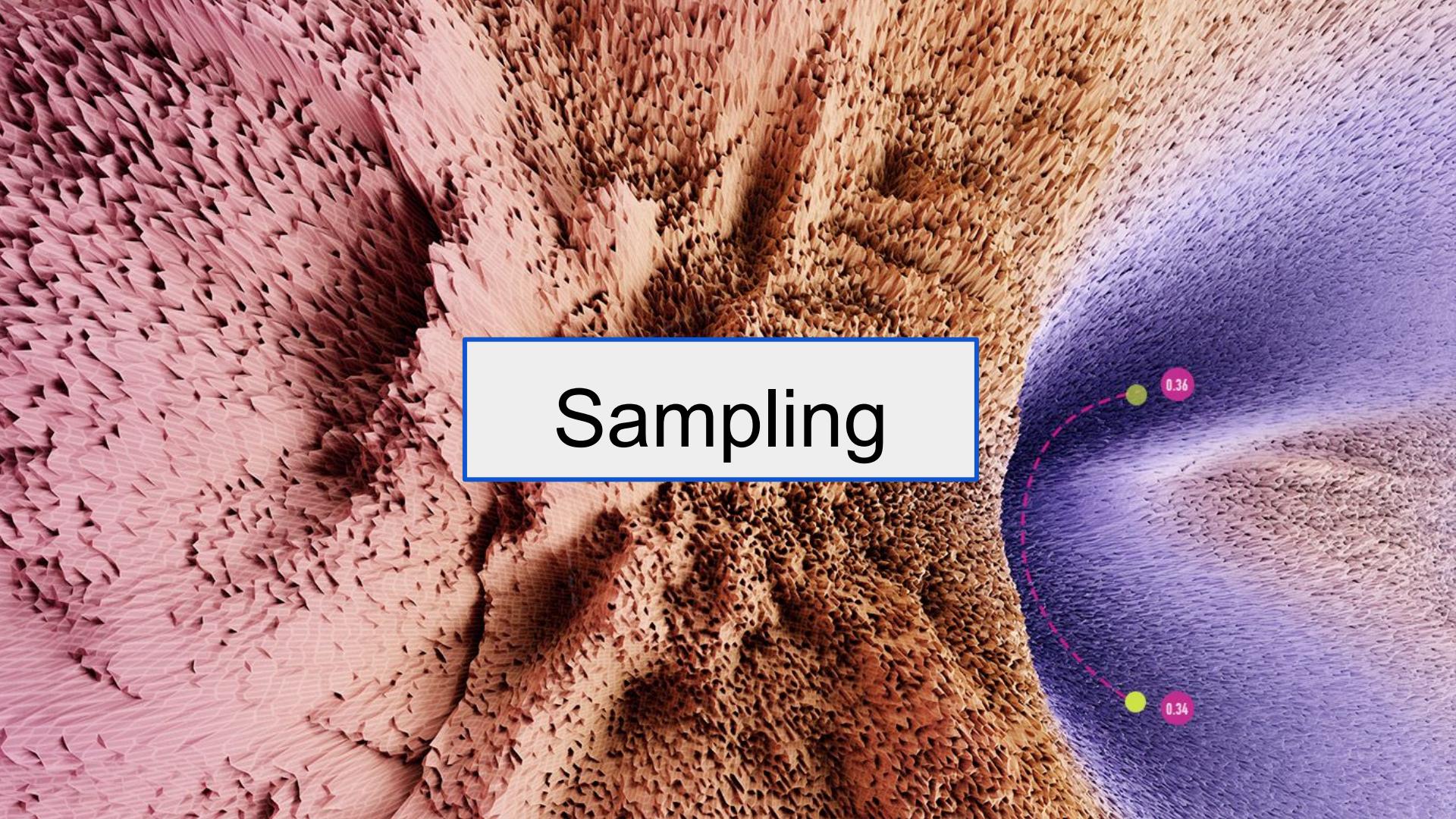


Transformers vs RNNs



- Process inputs in **parallel**
- Can be scaled very efficiently on large GPU clusters



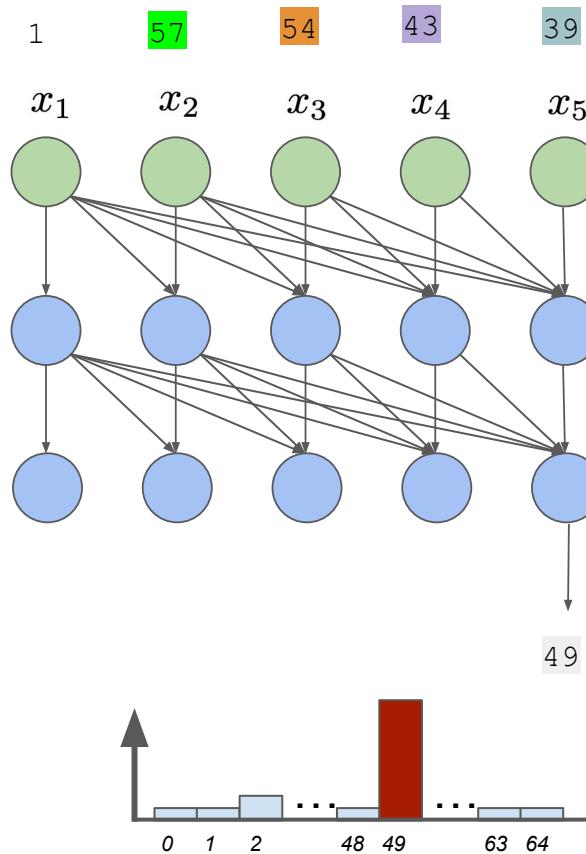


Sampling

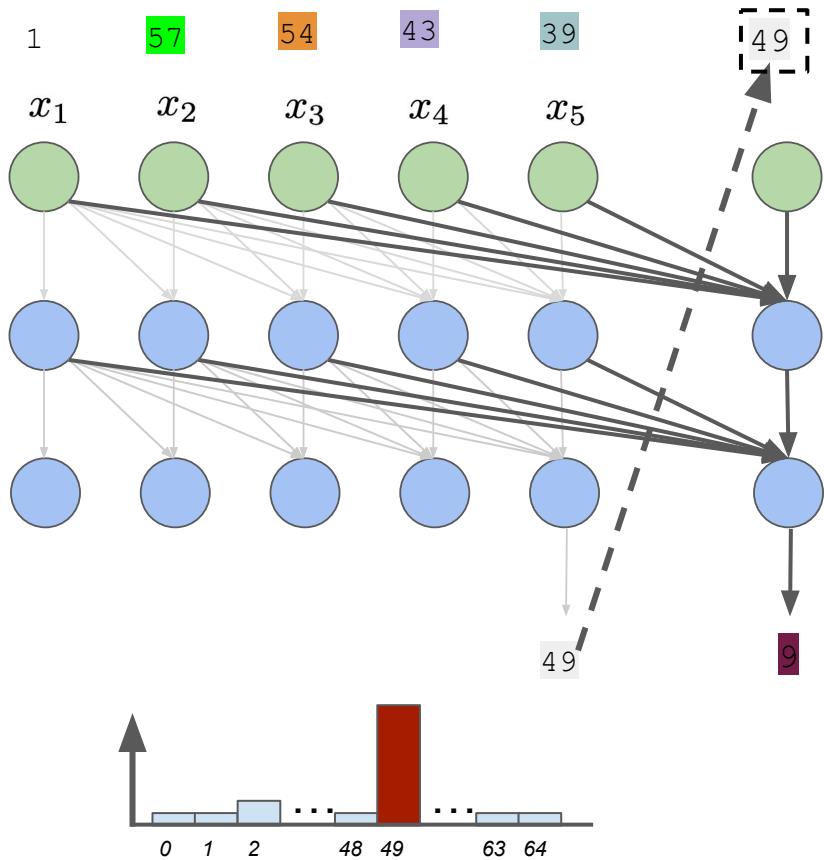
0.36

0.34

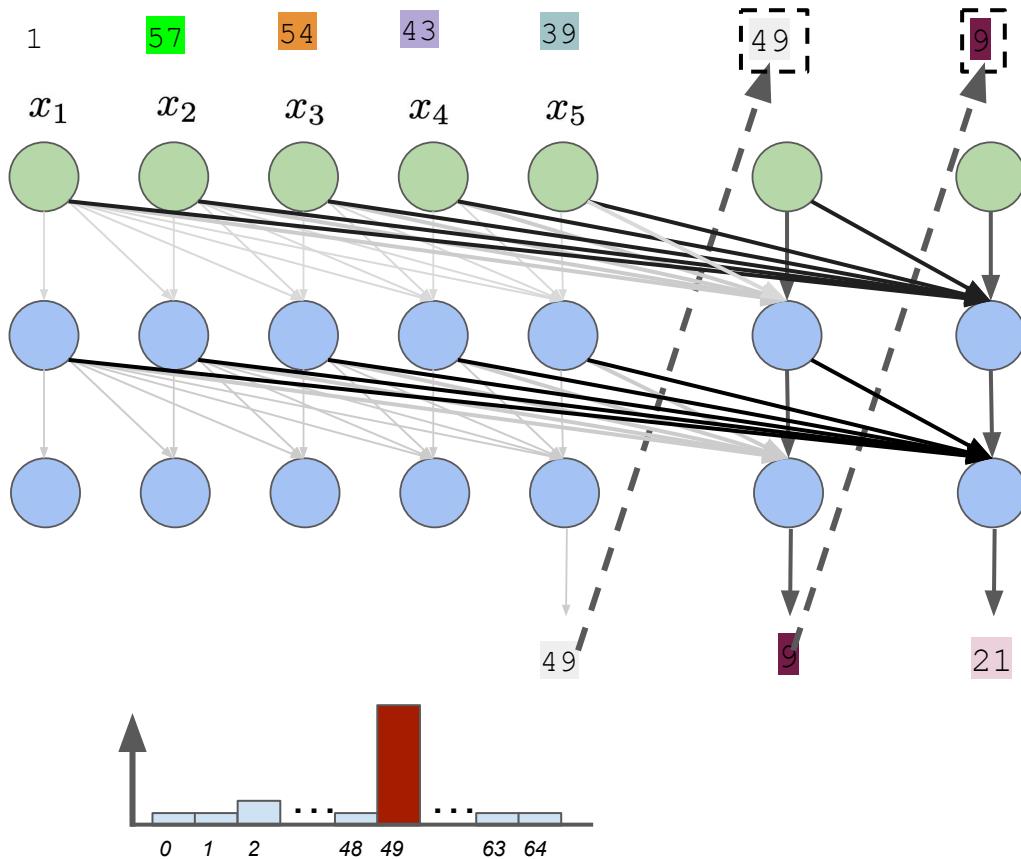
Transformers vs RNNs



Transformer Sampling



Transformer Sampling



- Same as for RNNs, we can sample many tokens *autoregressively*
- Sampling is sequential

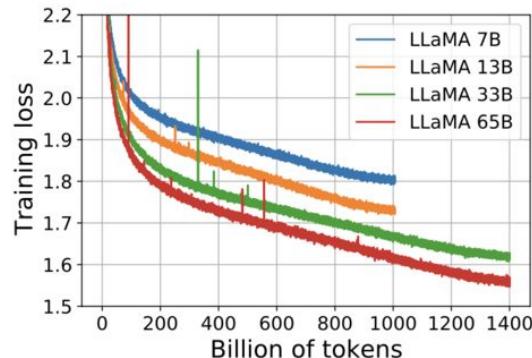
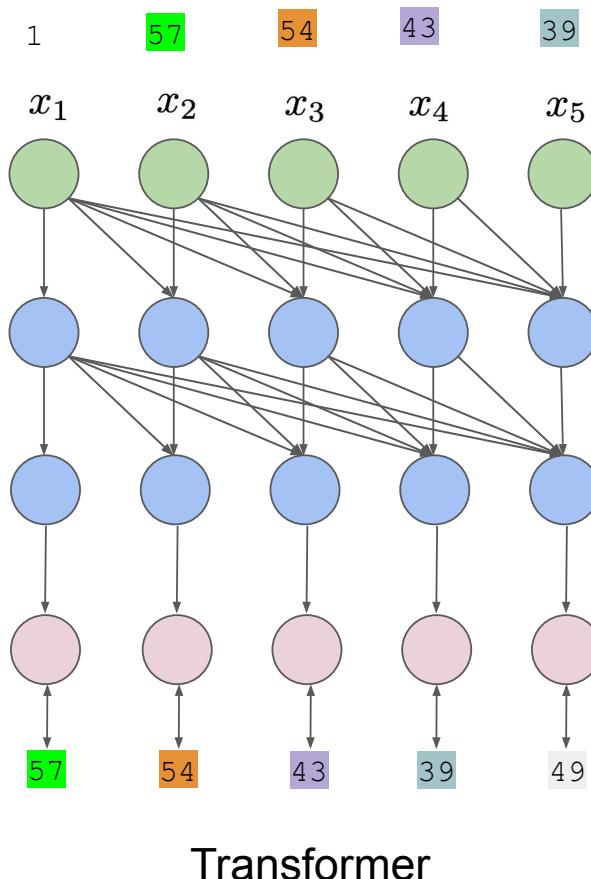


Post-Training

0.36

0.34

LLM Pretraining



Base Language Models

- Next token prediction on the internet does not give us a chatbot
- They continue the text, not respond to your questions
- These are hard to work with

Prompt

Explain the moon landing to a 6 year old in a few sentences.

Completion

GPT-3

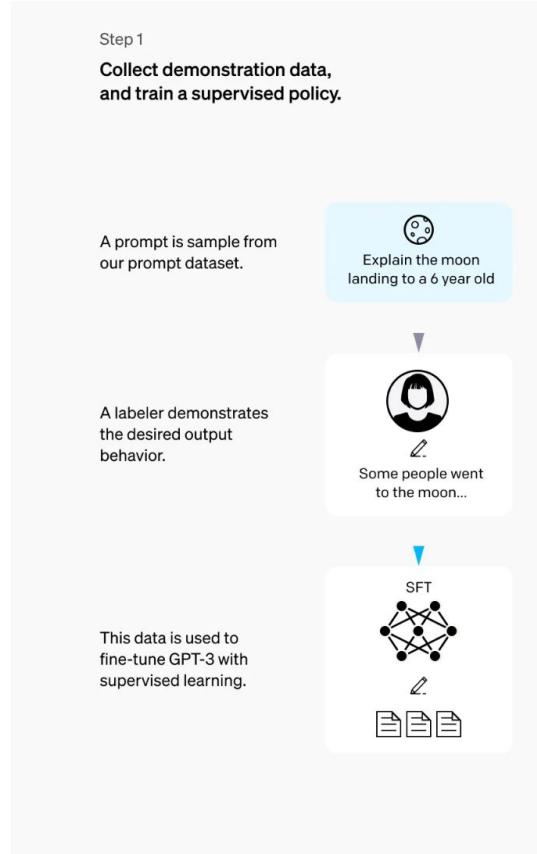
Explain the theory of gravity to a 6 year old.

Explain the theory of relativity to a 6 year old in a few sentences.

Explain the big bang theory to a 6 year old.

Explain evolution to a 6 year old.

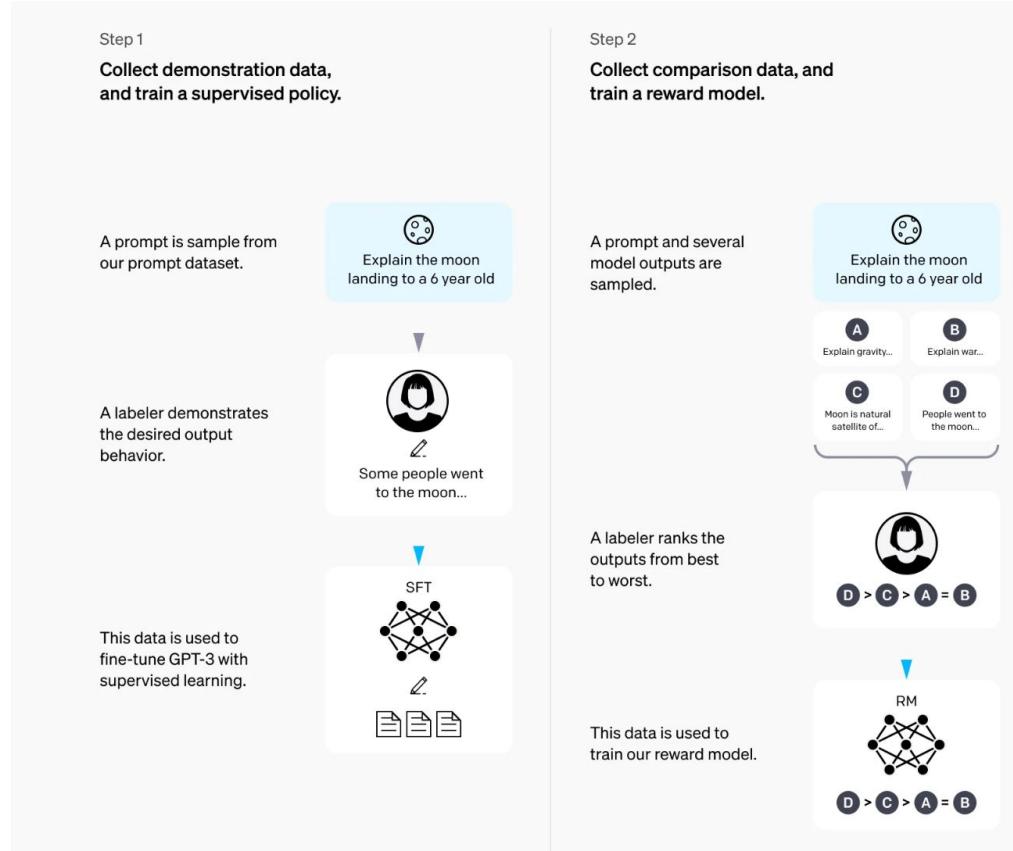
Instruction Tuning



We can just collect examples of responses we like and finetune on those

Issue: these examples might be narrow and hard for the model to imitate and generalize

Instruction Tuning

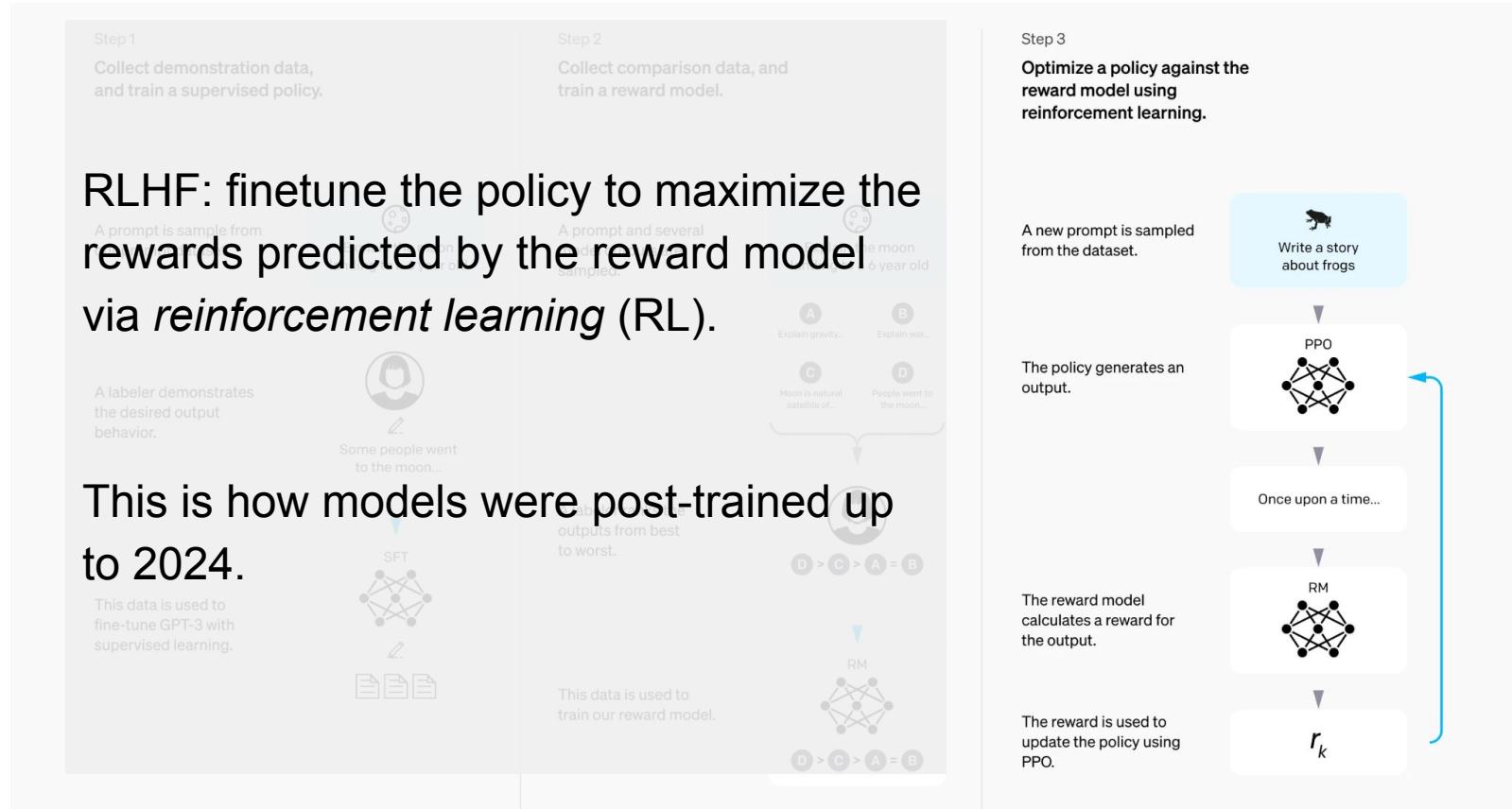


We will train a model to assign rewards (scores) to model-generated responses

Humans give preferences to model-generated responses

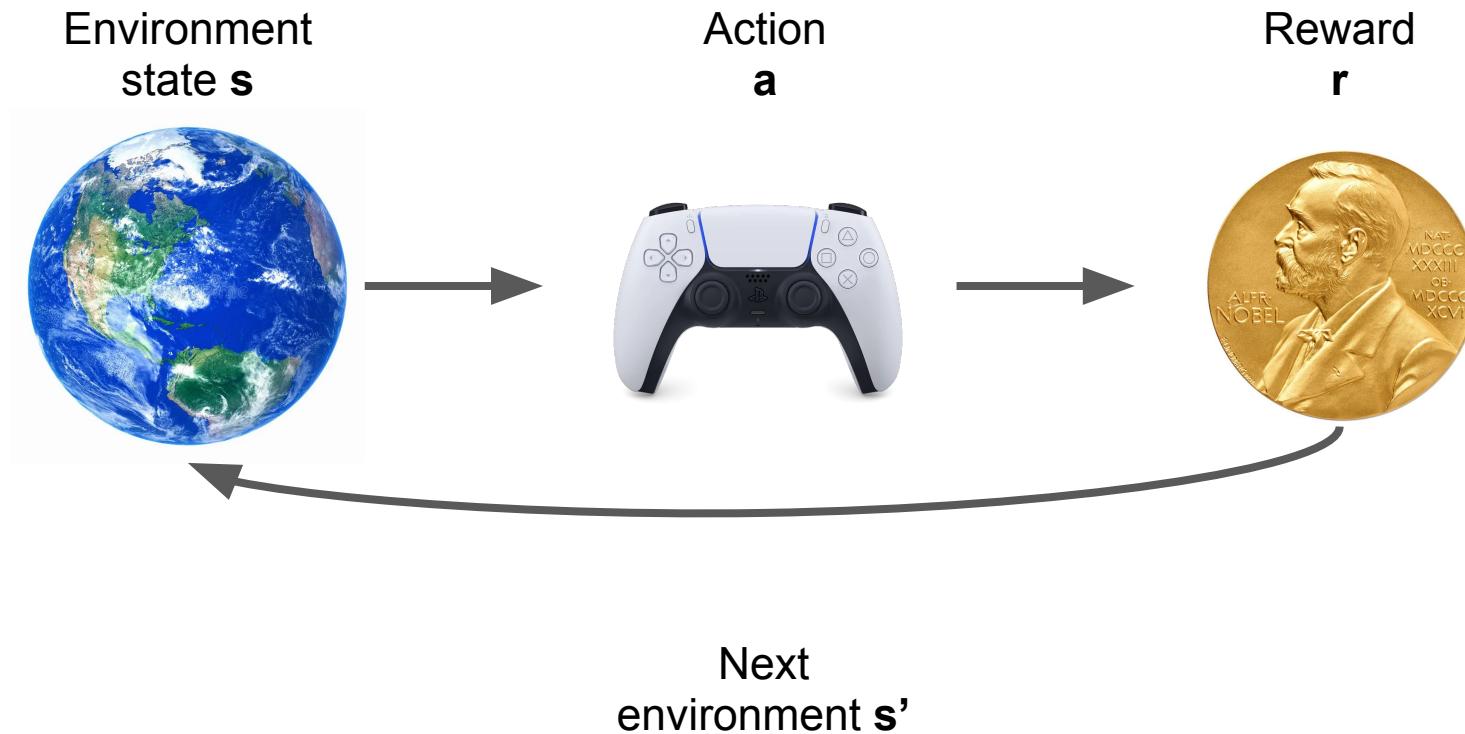
Reward model learns to predict those preferences

Instruction Tuning



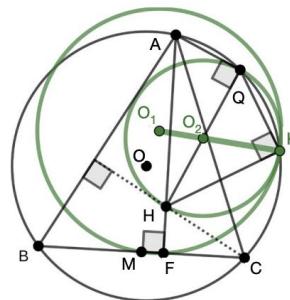
<https://openai.com/index/instruction-following/>

RL for Reasoning



RL for Reasoning

Environment
state s



Math Problem

$$a \sim \pi_\theta$$

Action
 a

...

Construct D: midpoint BH [a]
[a], O_2 midpoint HQ $\Rightarrow BQ \parallel O_2D$ [20]

...

Construct G: midpoint HC [b] ...
 $\angle GMD = \angle G0_2D \Rightarrow M, O_2, G, D$ cyclic [26]

...

[a], [b] $\Rightarrow BC \parallel DG$ [30]

...

Construct E: midpoint MK [c]
..., [c] $\Rightarrow \angle KFC = \angle K0_1E$ [104]

...

*Text: Solution
Attempt*

Reward
 r

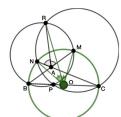
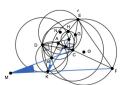
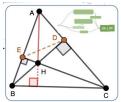
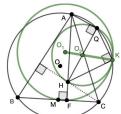


1 if correct else 0

$$\mathbb{E}_{s \sim \mathcal{D}, a \sim \pi_\theta} r$$

RL for Reasoning

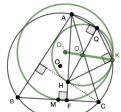
$s \sim \mathcal{D}$



RL for Reasoning

$s \sim \mathcal{D}$

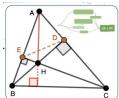
$a \sim \pi_\theta$



Step 1. $AO = A_2O$, $AO = BO$ and $BO = CO \Rightarrow A, A_2, B, C$ are cyclic.

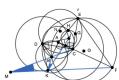
Step 2. A, A_2, B, C are cyclic $\Rightarrow \angle AA_2C = \angle ABC$.

Step 3. A, A_1, A_2 are collinear, A_1, Q_1, Q are collinear, $\angle AA_2C = \angle ABC$ and $\angle ABC = \angle QQ_1C \Rightarrow \angle A_1A_2C = \angle A_1Q_1C$.



Let PB_1 and QA_1 meet line AB at X and Y .

Since $\overline{XY} \parallel \overline{PQ}$ it is equivalent to show P_1XYQ_1 is cyclic (Reim's theorem).
Note the angle condition implies P_1CXA and Q_1CYB are cyclic.

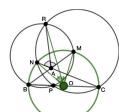


By the external version of Pitot theorem, the existence of ω implies that

$$BA + AD = CB + CD.$$

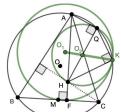
Let \overline{PQ} and \overline{ST} be diameters of ω_1 and ω_2 with $P, T \in \overline{AC}$. Then the length relation on $ABCD$ implies that P and T are reflections about the midpoint of \overline{AC} .

Now orient AC horizontally and let K be the “uppermost” point of ω , as shown.

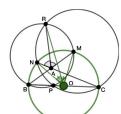
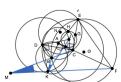
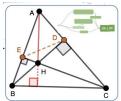


RL for Reasoning

$s \sim \mathcal{D}$



$a \sim \pi_\theta$



$r(s, a)$



Step 1. $AO = A_2O$, $AO = BO$ and $BO = CO \Rightarrow A, A_2, B, C$ are cyclic.

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$$\det \begin{bmatrix} -rs & ru & st \\ ru & -rt & st \\ -(u+s) & u+t & 0 \end{bmatrix} = rst [[u(u+t) - t(u+s)] + [s(u+t) - u(u+s)]] \\ = rst [(u^2 - st) + (st - u^2)] = 0.$$

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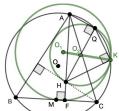
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RL for Reasoning

$s \sim \mathcal{D}$

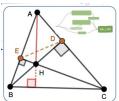
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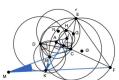
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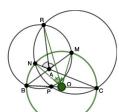


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$r(s, a)$

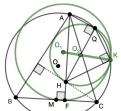


Reinforce: SFT policy on successful solutions

RL for Reasoning

$s \sim \mathcal{D}$

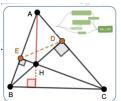
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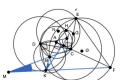
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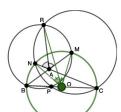


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$r(s, a)$



Reinforce: SFT policy on successful solutions

Summary: How to Train a ChatGPT

- Pretrain a large transformer language model on all of internet
- Post-train it to make it useful:
 - Instruction tuning via RLHF
 - Reasoning RL with verifiable rewards

What are you working on?

