NMTFD1 Task3 Report

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1 Introduction

Given the dimensionless Navier-Stokes equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{2}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial (v^2)}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{3}$$

It must advance in time $t \in [0, 0.1]$, and boundary conditions are:

$$u(x,1,t) = 1, \quad u(x,0,t) = u(0,y,t) = u(1,y,t) = 0,$$
$$v(x,0,t) = v(x,1,t) = v(0,y,t) = v(1,y,t) = 0$$

and initial velocity of the rest fluid is 0.

This task is to simulate the cavity flow with a driven lid. In this report, we are going to use fractional-step approach with collocated grids to solve the unsteady Navier-Stokes equation.

2 Fractional-Step Method[1]

Firstly we simplify the convective term with chain rule of partial derivative:

$$\frac{\partial (u^2)}{\partial x} + \frac{\partial uv}{\partial y} = \frac{\partial (u^2)}{\partial u}\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} \overset{Conti}{=} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \tag{4}$$

Similarily, we have the term of another direction: $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$.

To handel the pressure gradient in Navier-Stokes equation, we combine the two conservative equations to obtain an equation for p.

The basic idea of the method is to predict the velocity without pressure, and then calculate the pressure gradient using Poisson Pressure equation and correct the velocity with the pressure. Firstly we use CDS to discretize the terms in the Navier-Stokes equation[2][3]:

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \tag{5}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \tag{6}$$

$$\frac{\partial v}{\partial x} = \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} \tag{7}$$

$$\frac{\partial v}{\partial y} = \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \tag{9}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \tag{10}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2} \tag{11}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2} \tag{12}$$

$$\frac{\partial p}{\partial x} = \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} \tag{13}$$

$$\frac{\partial p}{\partial y} = \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y} \tag{14}$$

2.1 Prediction[5]

In this step, the momentum equation is solved by neglecting the pressure gradient. Here we introduce the intermediate velocity term u^*. Replace the derivative terms with discretized terms and the momentum equation becomes:

$$\begin{aligned}
\frac{u^*_{i,j} - u^n_{i,j}}{\Delta t} =& -(u^n_{i,j}\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2x} + v^n_{i,j}\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2y}) \\
&+ \frac{1}{Re}(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}) \\
\Longleftrightarrow& \\
u^*_{i,j} =& u^n_{i,j} - \Delta t(u^n_{i,j}\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2x} + v^n_{i,j}\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2y}) \\
&+ \frac{\Delta t}{Re}(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2})
\end{aligned} \tag{15}$$

Similarily, the prediction in y is:

$$\begin{aligned} v^*_{i,j} =& v^n_{i,j} - \Delta t(u^n_{i,j}\frac{v^n_{i+1,j}-v^n_{i-1,j}}{2x} + v^n_{i,j}\frac{v^n_{i,j+1}-v^n_{i,j-1}}{2y}) \\ &+ \frac{\Delta t}{Re}(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\Delta y^2}) \end{aligned} \tag{16}$$

To simplify the equation, let:

$$\begin{aligned} C_x &= u^n_{i,j}\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2x} + v^n_{i,j}\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2y} \\ D_x &= \frac{1}{Re}(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}) \\ C_y &= u^n_{i,j}\frac{v^n_{i+1,j}-v^n_{i-1,j}}{2x} + v^n_{i,j}\frac{v^n_{i,j+1}-v^n_{i,j-1}}{2y} \\ D_y &= \frac{1}{Re}(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{\Delta x^2} + \frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{\Delta y^2} \end{aligned}$$

So the eq.15 and eq.16 become:

$$u^*_{i,j} = u^n_{i,j} + \Delta t(-C_x + D_x) \tag{17}$$

$$v^*_{i,j} = v^n_{i,j} + \Delta t(-C_y + D_y) \tag{18}$$

Since the intermediate velocity u^* doesn't fulfill the continuity equation. We need to estimate the pressure for the new time step p^{n+1}.

2.2 Pressure Poisson Equation[5]

Firstly, we discretized the momentum equation at time step n+1 considering pressure gradient:

$$\begin{aligned} \frac{u^{n+1}_{i,j}-u^n_{i,j}}{\Delta t} =& -u^n_{i,j}\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2x} + v^n_{i,j}\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2y} \\ &+ \frac{1}{Re}(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta y^2}) - \frac{\partial p^{n+1}}{\partial x} \end{aligned} \tag{19}$$

Subtract eq.15 from eq.26 and lead to a relation between pressure and intermediate velocity

$$\frac{u^{n+1}_{i,j}-u^*_{i,j}}{\Delta t} = -\frac{\partial p^{n+1}}{\partial x} \tag{20}$$

Then apply the divergence on both hands leads to:

$$\frac{\partial}{\partial x}\frac{\partial p^{n+1}}{\partial x} = -\frac{1}{\Delta t}(\frac{\partial u^{n+1}_{i,j}}{\partial x} - \frac{\partial u^*_{i,j}}{\partial x}) \tag{21}$$

Neglect the derivative term of u^{n+1} from eq.21 since it fulfills the continuity equation.

$$\frac{\partial}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{p}^{n+1}}{\partial \boldsymbol{x}} = \frac{1}{\Delta t}\frac{\partial \boldsymbol{u}^*_{i,j}}{\partial \boldsymbol{x}} \tag{22}$$

$$\iff \frac{u^*_{i+1,j} - u^*_{i-1,j}}{2\Delta t\Delta x} + \frac{u^*_{i,j+1} - u^*_{i,j-1}}{2\Delta t\Delta y} = \frac{p^{n+1}_{i+1,j} - 2p^{n+1}_{i,j} + p^{n+1}_{i-1,j}}{\Delta x^2} + \frac{p^{n+1}_{i,j+1} - 2p^{n+1}_{i,j} + p^{n+1}_{i,j-1}}{\Delta y^2}$$

$$\iff p^{n+1}_{i,j} = \frac{p^{n+1}_{i+1,j} + p^{n+1}_{i-1,j}}{\Delta x^2} + \frac{p^{n+1}_{i,j+1} + p^{n+1}_{i,j-1}}{\Delta y^2} - \frac{\frac{u^*_{i+1,j} - u^*_{i-1,j}}{2\Delta t\Delta x} + \frac{u^*_{i,j+1} - u^*_{i,j-1}}{2\Delta t\Delta y}}{\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2}} \tag{23}$$

And eq.23 can be solved using successive over-relaxation (SOR) method. In the MATLAB code, we set $\omega = 1.4$ and 200 times iteration to have the pressure residual less than 0.001.

2.3 Correction[5]

After obtaining the pressure distribution in the whole domain at n+1 time step, the velocity at n+1 time step can be estimated from eq.20:

$$u^{n+1}_{i,j} = u^*_{i,j} - \frac{\Delta t}{2\Delta x}(p^{n+1}_{i+1,j} - p^{n+1}_{i-1,j}) \tag{24}$$

$$v^{n+1}_{i,j} = v^*_{i,j} - \frac{\Delta t}{2\Delta y}(p^{n+1}_{i,j+1} - p^{n}_{i-1,j+1}) \tag{25}$$

After this step, the pressure and velocity at time step n+1 are known and ready for the next loop.

3 Stability Analysis[3]

The discretized equation in the predictor step can be written as:

$$\begin{aligned} u^{n+1}_{i,j} =& u^n_{i,j} + (-u^n_{i,j}\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2\Delta x} - v^n_{i,j}\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y} \\ &+ \frac{1}{Re}\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} + \frac{1}{Re}\frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{\Delta y^2}) \end{aligned} \tag{26}$$

We can write $u^n_{x,y} = e^{zkx}e^{zly}$ and $u^{n+1}_{x,y} = Ge^{zkx}e^{zly}$. G is the growth factor and z is the imaginary number.

$$\begin{aligned} Ge^{zki\Delta x}e^{zlj\Delta y} =& e^{zki\Delta x}e^{zlj\Delta y} + \frac{\Delta t}{2\Delta x}[-u(e^{zk(i+1)\Delta x}e^{zlj\Delta y} - e^{zk(i-1)\Delta x}e^{zlj\Delta y})] \\ &+ \frac{\Delta t}{2\Delta y}[-v(e^{zki\Delta x}e^{zl(j+1)\Delta y} - e^{zki\Delta x}e^{zl(j-1)\Delta y})] \\ &+ \frac{\Delta t}{Re\Delta x^2}[e^{zk(i+1)\Delta x}e^{zlj\Delta y} - 2e^{zki\Delta x}e^{zlj\Delta y} + e^{zk(i-1)\Delta x}e^{zlj\Delta y}] \\ &+ \frac{\Delta t}{Re\Delta y^2}[e^{zki\Delta x}e^{zl(j+1)\Delta y} - 2e^{zki\Delta x}e^{zlj\Delta y} + e^{zki\Delta x}e^{zl(j-1)\Delta y}] \end{aligned} \tag{27}$$

Cancelling the relevant terms on both sides, we get

$$\begin{aligned}\implies G =&1 + \frac{\Delta t}{2\Delta x}\left[-u(e^{zk1\Delta x} - e^{-zk\Delta x})\right] + \frac{\Delta t}{2\Delta y}\left[-v(e^{zl\Delta y} - e^{-zl\Delta y})\right]\\ &+ \frac{\Delta t}{Re\Delta x^2}\left[e^{zk\Delta x} - 2 + e^{-zk\Delta x}\right] + \frac{\Delta t}{Re\Delta y^2}\left[e^{zl\Delta y} - 2 + e^{-zl\Delta y}\right]\end{aligned} \tag{28}$$

$$\begin{aligned}\implies G =&1 + \frac{\Delta t}{2\Delta x}\left[-u(2Cos(k\Delta x))\right] + \frac{\Delta t}{2\Delta y}\left[-v(Cos(l\Delta y))\right]\\ &+ \frac{\Delta t}{Re\Delta x^2}\left[2Sin(k\Delta x) - 2\right] + \frac{\Delta t}{Re\Delta y^2}\left[2Sin(l\Delta y) - 2\right]\end{aligned} \tag{29}$$

As $\Delta x \to 0$ and $\Delta y \to 0$

$$\implies G = 1 - \frac{u\Delta t}{\Delta x} - \frac{v\Delta t}{\Delta y} - \frac{2\Delta t}{Re\Delta x^2} - \frac{2\Delta t}{Re\Delta y^2} \tag{30}$$

$$\implies \frac{u\Delta t}{\Delta x} + \frac{v\Delta t}{\Delta y} + \frac{2\Delta t}{Re\Delta x^2} + \frac{2\Delta t}{Re\Delta y^2} = 1 - G \tag{31}$$

$$\implies \Delta t = \frac{1 - G}{\frac{u}{\Delta x} + \frac{v}{\Delta y} + \frac{2}{Re\Delta x^2} + \frac{2}{Re\Delta y^2}} \tag{32}$$

$$\implies \Delta t \leq \frac{1}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + \frac{2}{Re\Delta x^2} + \frac{2}{Re\Delta y^2}} \tag{33}$$

We have chosen a $\Delta x = \frac{1}{40}$ and $\Delta y = \frac{1}{40}$ for our domain. So, to conform to stability requirement derived previously we need at $\Delta t < 0.000015$ for $Re = 0.1$ and $\Delta t < 0.00015$ for $Re = 1$.

4 Results

In this problem, we have a square cavity with its upper "lid" drawn in the positive x-direction at a constant velocity and thereby inducing a circular motion, ie creating a vortex. We have chosen a $\Delta x = 0.025$ and $\Delta y = 0.025$ for space step and $\Delta t = 0.00001$ for time step considering the stability. The run-time for Re = 0.1 (dt = 0.00001) is 86 seconds on a 1.4Ghz Quad Core Intel Core i5 MacBook Pro. The run-time for Re = 1 (dt = 0.0001) is 8.6 seconds on a 1.4Ghz Quad Core Intel Core i5 MacBook Pro.

In figure(1), we can see the pressure contours of the domain at different times, $t = 0 : 0.02 : 0.1$, for $Re = 0.1$. We can notice 2 singularities in the top corners, this arises because our velocity changes from 0 to 1. So when the pressure-poisson equation is solved using the SOR method, the effect of the velocity change is clearly seen in the pressure field as time progresses. The pressure in the domain keeps dropping constantly inside the vortex but remains vastly constant throughout the domain.

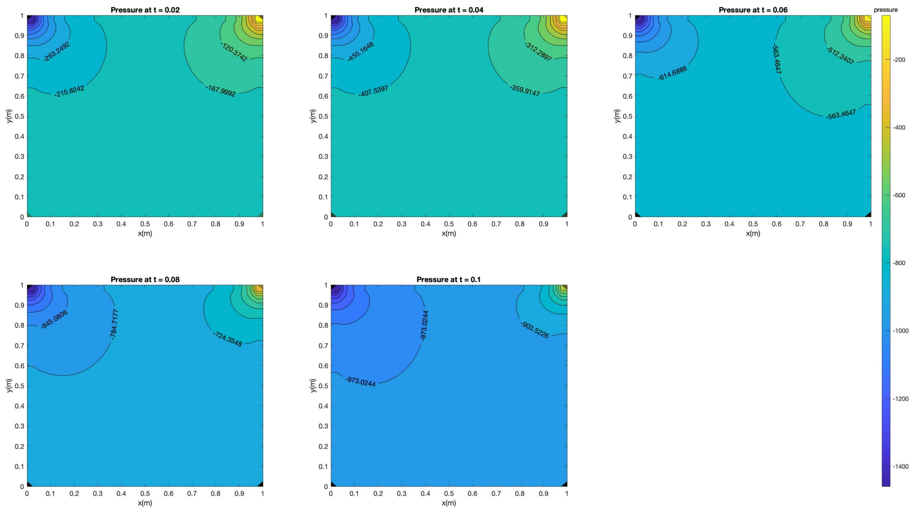


Figure 1: Pressure contours for Re = 0.1

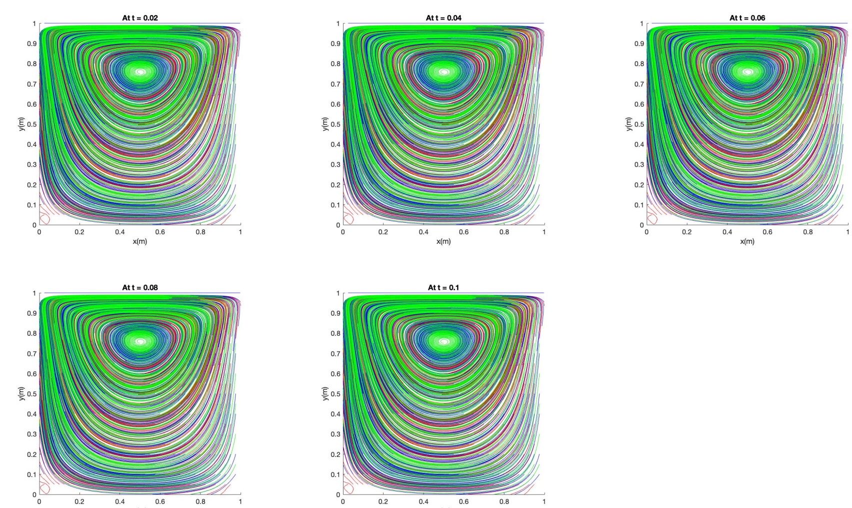
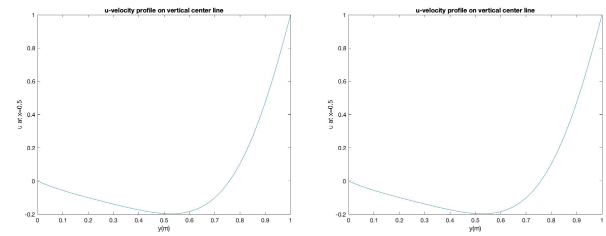


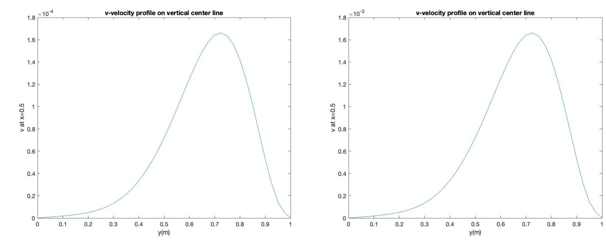
Figure 2: Pressure contours for Re = 0.1



(a) u, Re = 0.1

(b) u, Re = 1

Figure 3: u-velocity profile on vertical center line



(a) v, Re = 0.1

(b) v, Re = 1

Figure 4: v-velocity profile on vertical center line

In figure(2), we can see the streamline plots of the domain at different times, t = 0:0.02:0.1, for Re = 0.1. Since we are running a very low Reynolds number there is negligible change in the velocity fields after a certain time. The streamlines are also nearly symmetrical, due to the symmetry in the velocity field of a typical stokes flow, $Re \to 0$. One can also notice the smaller vortex forming in the lower left corner of the domain characterized by separating streamlines. And since the Reynolds number is very small, which means the viscous force dominant under this situation. Therefore it reaches the steady state quickly, and that's why the streamlines look alike.

In both cases of figure(3), $Re = 0.1$ and $Re = 1$, we can see that the velocity in x direction drops first and the picks as we progress up in the vertical direction right in the center of the domain at $x = 0.5$. This happens because of the presence of the vortex in the center of our 2D domain. Although not evident, the location of the vortex changes slightly as our Reynolds number increases. This difference can be more prominently visible if we further increase the reynolds number. As we move up vertically, the value of $u \to 1$ and eventually reaches our boundary condition on the top.

In both cases of figure(4), $Re = 0.1$ and $Re = 1$, we can see that the velocity in y direction increases first and then drops as we progress up in the vertical direction right in the center of the domain at $x = 0.5$. However, the magnitude of these values are apart by an order of 10^1 at least. Indicating that the velocity in y-direction is higher for higher Reynolds numbers

5 Conclusion

In this Report, we tried to use the Fractional Step Method which is given in the lecture to solve the typical CFD question, the driven lid cavity. Although there are more alternative methods like SIMPLEC and IFSM etc., different grid arrangements and boundary handling methods like ghost nodes, which seems more suitable for Finite Volume Method, limited by the laptop performance and our laziness, we basically only used the simplest way to solve NS-equation in our last report, but we do hope that in the following study we could learn and try more.

References

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