01204211 Discrete Mathematics
Lecture 8a: Linear systems of equations

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Linear algebra

Linear algebra studies
- ▶ matrices and operations with matrices
- ▶ systems of linear equations
- ▶ linear transformations
- ▶ linear spaces (and their structures)

Why?

- Lots of applications.
- Interesting perspectives.

A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?

Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

Then you can conclude that $y = -2$. Substitute it to one of the equation, you can find out the value of x.

Gaussian elimination (1)

Let's consider a system with 3 variables:

$$
\begin{aligned}
2x_1 &+& 4x_2 &+& 3x_3 &=& 7 \\
x_1 &+& && 5x_3 &=& 12 \\
4x_1 &+& 2x_2 &+& 3x_3 &=& 10
\end{aligned}
$$

Gaussian elimination (2)

Let's consider another system with 3 variables:

$$
\begin{array}{rcrcrcr}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
3x_1 & + & 8x_2 & + & x_3 & = & 10
\end{array}
$$

A closer look: 1st perspective

Consider

$$5x + 10y = 5$$
$$x - 3y = 11$$

Each equation (row) constraints certain values of x and y.

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$
\begin{aligned}
(\quad 5, \quad 10 \quad) &= \boldsymbol{u}_1 \\
(\quad 1, \quad -3 \quad) &= \boldsymbol{u}_2 \\
(\quad 0, \quad 25 \quad) &= \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2
\end{aligned}
$$

The third equation is a "combination" of the other two rows. In fact, it is a **linear combination** of the first two.

Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

Yes,

$$0.2 \cdot \boldsymbol{u}_1 - \boldsymbol{u}_2 = (0, 1).$$

It turns out that you can obtain any (a, b) from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A closer look: 1st perspective (more example)

Consider

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

What are the row vectors?

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.

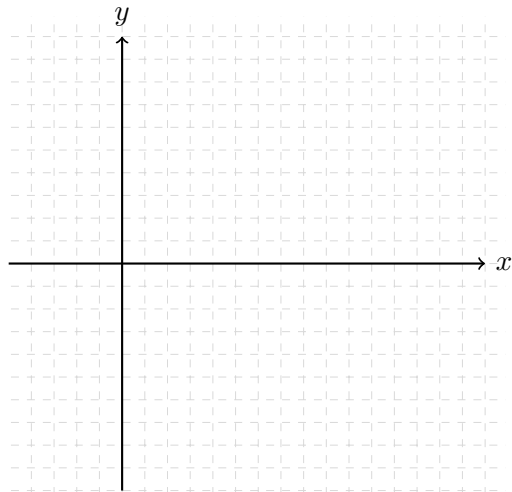But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and with x and y, we now see that $\boldsymbol{b}$ is a **linear combination** of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Finding x and y is essentially checking if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
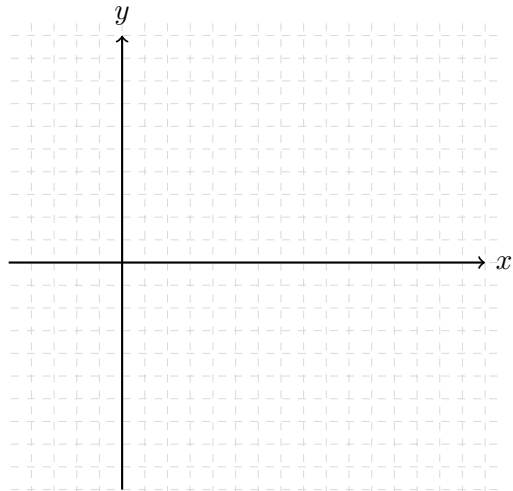
Example 2: a linear system with 2 variables

$$\begin{array}{ccccc} x & + & y & = & 7 \\ x & - & 2y & = & 13 \end{array}$$
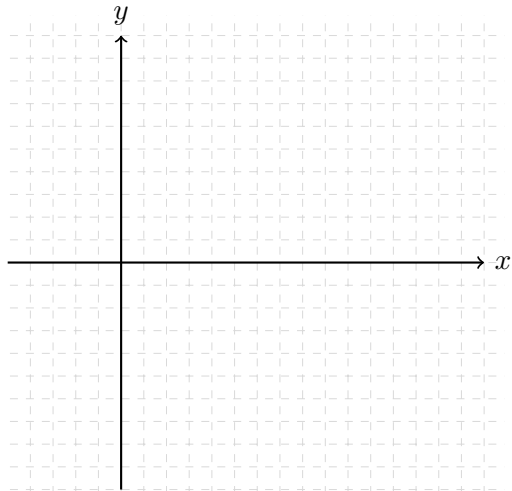
Example 3: a linear system with 2 variables

$$
\begin{array}{ccccc}
2x & + & y & = & 5 \\
4x & + & 2y & = & 10
\end{array}
$$

Example 4: a linear system with 2 variables

$$\begin{array}{rcrcr} x & + & 3y & = & 6 \\ 0.5 \cdot x & + & 1.5 \cdot y & = & 9 \end{array}$$

A linear system with 3 variables

Let's consider a system with 3 variables:

$$\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}$$

Row perspective

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &= 7 \\ x_1 + 5x_3 &= 12 \\ 4x_1 + 2x_2 + 3x_3 &= 10 \end{aligned}$$

Each equation becomes a **plane** in 3 dimensional space.

Row perspective: the goal of Gaussian Elimination

From vectors:
$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain
$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

In other words, what are the possible linear combinations of
$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

Column perspective

From

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array},
$$

we rewrite the system as

$$
\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.
$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$
\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.
$$

In other words, the vector b, for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$
\begin{array}{rcrcrcr}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
3x_1 & + & 8x_2 & + & x_3 & = & 10
\end{array}
$$

More example 2

Let's consider another system with 3 variables:

$$
\begin{aligned}
2x_1 &+ 4x_2 &+ 3x_3 &= 7 \\
x_1 &+ &5x_3 &= 12 \\
4x_1 &+ 2x_2 &+ 3x_3 &= 10 \\
5x_1 &+ 2x_2 &+ 8x_3 &= 22
\end{aligned}
$$

More failed example 3

Let's consider the last system with 3 variables:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & & & + & 5x_3 & = & 12 \\ 2x_1 & & & + & 10x_3 & = & 24 \end{array}$$

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.

What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions? It means that, for $\boldsymbol{u}$, you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that u and v are different solutions to the system:

$$
\begin{array}{rrrrrrl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12
\end{array}
$$

I.e.,

$$
\begin{array}{rrrrrrl}
2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\
u_1 & + & & & 5u_3 & = & 12
\end{array}
\qquad
\begin{array}{rrrrrrl}
2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\
v_1 & + & & & 5v_3 & = & 12
\end{array}
$$

Consider $u - v$. We see that

$$
\begin{array}{l}
(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\
\quad 2(u_1 - v_1) \;+\; 4(u_2 - v_2) \;+\; 3(u_3 - v_3) \;=\; (7 - 7) = 0 \\
(u_1 + 5u_3) - (v_1 + 5v_3) = \\
\quad (u_1 - v_1) \;+\; 5(u_1 - v_3) \;=\; (12 - 12) = 0
\end{array}
$$

Suppose that u and v are different solutions to the system:

$$\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12
\end{array}$$

It turns out that $u - v$ is a solution to the following system:

$$\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\
x_1 & + & & & 5x_3 & = & 0
\end{array}$$

It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.
It would play a central role when dealing with linear systems with many solutions.

Key take away

▶ There are 2 ways to look at how we solve linear systems: row perspective and column perspective.

▶ **Linear combination** is the main operation.