01204211 Discrete Mathematics
Lecture 8a: Integers and GCD

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Number theory: integers and divisibility

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We will cover:

- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.

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- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.
- Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b.
On the other hand if a does not divide b, we write $a \not| b$.

$a|b$

Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

Proof: เนื่องจาก $a|b$ มี k_1 ที่ $b = k_1 a$

เช่นกัน $a|c$ มี k_2 ที่ $c = k_2 a$

ดังนั้น $b + c = k_1 a + k_2 c$

$= (k_1 + k_2)\, a$

แต่ว่า k_1, k_2 เป็น จน.เต็ม ดังนั้นมี

จนเต็มที่ $k'a = b + c$,

k'

$\therefore\ a \mid (b+c)$.

Examples

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If $a|b$ and $b|c$, prove that $a|c$.

$\exists k_1$ [illegible] $b = k_1 a$ $\qquad$ $\exists k_2$ [illegible] [illegible] $k_2 b = c$

$$c = k_2 b = k_1 k_2 a$$

Remainder

Defintion (remainder)

The **remainder** of the division of b with a is an integer r when there exists an integer q such that

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where $0 \leq r < a$.

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Examples:

$5 \div 3$ rem $= 2$

$-5 \div 3$ rem $= 1$ $\quad -5 = (-2)3 + 1$

$-10 \div -7$ remainder $= 4$

$-10 = (2)(-7) + 4$

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Examples:

python / c

10 % 7

We use operator `mod` to denote an operation for finding the remainder of a division. I.e., $a \bmod b$ is the remainder of dividing a with b.

Examples

remainder r : $\exists q \in \mathbb{Z}$ $b = qa + r$

$0 \le r < a$

- Let r be the remainder of the division of b by a. Assume that $c|a$ and $c|b$. Prove that $c|r$.

$\exists k_1 \in \mathbb{Z}$

~~$c = k_1 a$~~

$a = k_1 c$

$b = k_2 c$

$r = b - qa$

More examples

$\neq 1$

For every integer a, $a - 1 | a^2 - 1$.

Primes

Definition (primes)

- An integer $p > 1$ is a **prime** if its divisors are only p, $-p$, 1, and -1.
- If an integer $n > 1$ is not a prime, it is called a **composite**.
- Note: 1 is not a prime and also not a composite.

Algorithm for testing primes

Recall our `CheckPrime2` algorithm

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run?

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```

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

Efficient algorithms

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n	number of bits of n	$\sqrt{n}$
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
1,267,650,600,228,229,401,496,703,205,376	100	1,125,899,906,842,624

Efficient algorithms

$\sqrt{n} = \left(2^{\log n}\right)^{1/2}$

$= 2^{\frac{\log n}{2}}$

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Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

$\boxed{O(\log n)}$

Greatest Common Divisors (GCD)

Definition (GCD) গসাগু

For integers x and y, the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $gcd(x, y)$.

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g = min(x,y)
while (x mod g != 0) or (y mod g != 0):
  g -= 1
return g
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return g
```

What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

$\log x + \log y$

Euclid's algorithm

112	44
44	24
24	20
20	4

```
Algorithm Euclid(x,y):
  if x mod y == 0:
    return y
  else:
    return Euclid(y, x mod y)
```

Euclid's algorithm

gcd(x,y) gcd(y, x mod y)

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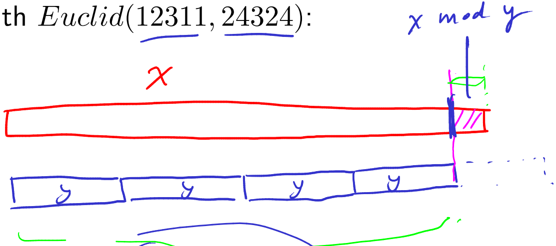
```

Let's see how it works with $Euclid(12311, 24324)$:

Euclid(12311, 24324)
Euclid(24324, 12311)
Euclid(12311, 12013)
Euclid(12013, 298)
Euclid(298, 93)
Euclid(93, 19)
Euclid(19, 17)
Euclid(17, 2)
Euclid(2, 1)



$\forall g$ $\quad g|x \ \& \ g|y \iff g|y \ \& \ g|x \bmod y$

Proofs

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid.

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We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid. on $x < y$, $x \bmod y = x$

Note that when $x < y$, $\mathrm{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\mathrm{Euclid}(y, x)$. After that, in each call, x is always larger than y. For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

Own technique

For any integers x and y such that $x > y$, $\text{Euclid}(x, y) = gcd(x, y)$.

$x \geq 0$, $y \geq 0$

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $gcd(x, y) = y$; therefore, the base case of the algorithm is correct. Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\text{Euclid}(x', y') = gcd(x', y')$.

Now assume that $y \not| x$. The Euclid algorithm returns $\text{Euclid}(\underset{\underset{x'}{\shortparallel}}{y}, \underset{\underset{y'}{\shortparallel}}{x \bmod y})$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\text{Euclid}(y, x \bmod y) = gcd(y, x \bmod y).$$

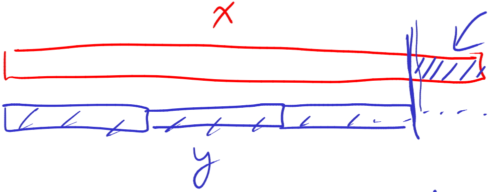
Thus, we are left to show that

$$\Rightarrow \quad gcd(x, y) = gcd(y, x \bmod y).$$

What is $x \bmod y$?

3.127

$x - 3 \cdot y$



$$x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

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Let $\lfloor a \rfloor$ be the largest integer a' such that $a' \leq \lfloor a \rfloor$.

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$$x \bmod y = x - \underbrace{\left\lfloor \frac{x}{y} \right\rfloor}_{} \cdot y$$

Lemma 2

If $a|x$ *and* $a|y$*, then* $a|x \bmod y$.

$$x \bmod y = \underline{x} - \left\lfloor \frac{x}{y} \right\rfloor \cdot \underline{y}$$

Lemma 2

If $a|x$ and $a|y$, then $a|x \bmod y$.

Lemma 3

$gcd(x, y) = gcd(y, x \bmod y)$

ต้องการว่า ทุกๆ common divisor ของ x และ y
จะเป็น common divisor ของ y และ $x \bmod y$.

และในทางกลับกัน c.d ของ y และ $x \bmod y$
จะเป็น c.d. ของ x และ y

ดังนั้น set ของ common divisor ของ x & y
และ y กับ $x \bmod y$ เท่ากัน $\Rightarrow$ gcd เท่ากัน

How many recursive calls does Euclid's algorithm makes?

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Consider Euclid(x, y):

- If we start with $x < y$, the next calls will always have that $\boxed{x > y;}$ so we have at most one call with $x < y$.

+1

(x, y) $\quad x > y$

$(y, x \bmod y)$

$x > y$
$y > x \bmod y$ } argument ของ gcd ลดลง ทั้งคู่

$\Downarrow$

$\boxed{\text{terminate}}$

→ เมื่อไหร่ argument ขนาด ลดลง ครึ่งหนึ่ง? ✓

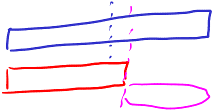
ถ้า $y \le x/2$

→ ถ้า $y > x/2$

$x \bmod y = x - y$

$< x/2$

x, y
$\downarrow$
$y, x \bmod y$
$x \bmod y, y \bmod (x \bmod y)$



$x \bmod y$
$= x - y$

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- When can we decrease the value of x or y in the calls?

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- When $y \leq x/2$, when Euclid(x, y) calls Euclid$(y, x \bmod y)$ the first argument decreases by half.

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  Euclid$(x \bmod y, y \bmod (x \bmod y))$ Note that in this case, $x \bmod y = x - y \leq x/2$. Thus, after two recursive calls, the first argument decreases by half.
- How many times can that happen?
- The first argument can decrease by a factor of two for at most $\log x$ times. Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$. $\longleftarrow$ poly time ใน

Computing power

$m^e \bmod n$

How fast can we compute x^y?

Computing power

$x^4 = (x \cdot x) \cdot (x \cdot x)$

$= (x^2)^2$

x^8

How fast can we compute x^y?

```
Algorithm power(x,y):
  a = 1
  for i = 1,2,...,y:
    a *= x
  return a
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What is the running time?

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    a *= x
  return a
```

What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

จน. ครั้งการคูณ

$$x^{16} = (x^8)^2 = ((x^4)^2)^2 = (((x^2)^2)^2)^2.$$

$\log y$

Repeated squaring

$x^5 = x \cdot x^4$

$x^{13} = x(x^6)^2$

$= x((x^3)^2)^2$

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```
Algorithm power(x,y):  // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

$a = x^{y/2}$

return $a^2 = (x^{y/2})^2 = x^y$

Repeated squaring (general y)

13

$(x^6)^2 \cdot x$

```
Algorithm power(x,y):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2))
    if y mod 2 == 0:
      return a*a
    else
      return a*a*x
```

Repeated squaring (general y)

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What is the number of recursive calls? $O(\log y)$

Repeated squaring (general y)

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What is the number of recursive calls?

What is the running time? ระวัง เพราะ a ขนาดใหญ่มาก!

Repeated squaring (general y)

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Algorithm power(x,y):
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    if y mod 2 == 0:
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    else
      return a*a*x
```

What is the number of recursive calls?
What is the running time?
While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y, mod n)

เลขไม่เกิน n
$O(\log n)$

Computing $x^y \bmod n$:

```
Algorithm power(x,y,n):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2)) mod n
    if y mod 2 == 0:
      return a*a mod n
    else
      return a*a*x mod n
```

$$\left.\begin{array}{ll} \gcd(x,y) & O(\log x + \log y) \\ \mathrm{power}(x,y,n) & O(\log y \cdot \log^2 n) \end{array}\right\}$$ เร็วดี.