01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (I)

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September 13, 2022

What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

Vector spaces related to a matrix

Consider an m-by-n matrix A over $\mathbb{R}$.

Vector spaces related to a matrix

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} c_1 & c_2 & c_n \end{bmatrix} \end{array}$$

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We can view A as

- n columns of m-vectors: $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$

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$$\begin{bmatrix} \boxed{r_1} \\ \boxed{r_2} \\ \vdots \\ r_m \end{bmatrix}$$

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Subspaces

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that $\mathcal{V}$ is a **subspace** of $\mathcal{W}$.

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Examples:

- $\mathrm{Span}\,\{[1,1]\}$ is a subspace of $\mathbb{R}^2$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1]\}$ is a subspace of $\mathbb{R}^3$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of $\mathbb{R}^3$.

Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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$$A^T = \left[\begin{pmatrix}1\\2\\4\end{pmatrix} \begin{pmatrix}0\\1\\3\end{pmatrix}\right]$$

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- Row space:

vector space

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Example 1 (cont.)

vector

span

solution vec

num

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We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$A\boldsymbol{x} = 0.$$

In this case, we have

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The set of solutions $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product Ax. Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

ผลลัพธ์ของ การคูณ
เป็นอะไรได้บ้าง

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Column space

hom eq.

nullspace ของ A

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ได้บ้าง ที่

$\begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{R}(A)$)
- The row space of A (denoted by $\mathcal{R}(A^T)$)
- The nullspace of A

$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$$

- The left nullspace of A

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$y^T A = 0$

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Linearly independent rows

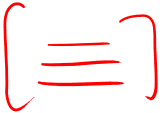
$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

rank of matrix = $\#$ $\underline{\text{lin. indep.}}$ $\boxed{\text{rows}}$

may

basis

Ranks



Definition

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- The **row rank** of A is the maximum number of linearly independent rows of A.
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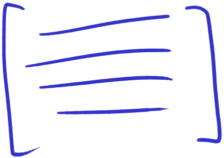
Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

Row rank = Column rank



Theorem 1

For any matrix A, its row rank equals its column rank.

We will prove this theorem next time.