01204211 Discrete Mathematics
Lecture 10b: Polynomials (2)[1]

Jittat Fakcharoenphol

October 15, 2024

[1]This section is from Berkeley CS70 lecture notes.

Fun fact: Check digit for Thai National ID

Review: Polynomials

A **single-variable polynomial** is a function $p(x)$ of the form

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

We call a_i's *coefficients*. Usually, variable x and coefficients a_i's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Review: Basic facts

Definition

a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

Properties

Property 1: A non-zero polynomial of degree d has at most d roots.
Property 2: Given $d + 1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

Polynomial division

If you have a polynomial $p(x)$ of degree d, you can divide it with a polynomial $q(x)$ of degree $\leq d$. You have that there exists a pair of polynomial $q'(x)$ and $r(x)$ such that

$$p(x) = q'(x)q(x) + r(x),$$

and $r(x)$ is of degree **less** than $q(x)$'s degree.

Lemma 1

If a is a root of polynomial $p(x)$ with degree $d \geq 1$, then $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ with degree at most $d - 1$

Proof.

Dividing $p(x)$ with $(x - a)$, we get that

$$p(x) = q'(x)(x - a) + r(x),$$

where $r(x)$ is of degree at most $1 - 1 = 0$, i.e., $r(x)$ must be a constant; thus, we assume that $r(x) = c$. Let's evaluate $p(a)$; note that $p(a) = c$, since

$$p(a) = q'(a)(a - a) + c = 0 + c = c.$$

However we know that a is a root of $p(x)$, i.e., $p(a) = 0$. Therefore $c = 0$, or $r(x) = 0$. Thus, the lemma follows. □

Lemma 2

If $p(x)$ is a polynomial of degree d with d distinct roots $a_1, a_2, \ldots, a_d$, $p(x)$ can be written as $c(x - a_1)(x - a_2) \cdots (x - a_d)$.

Proof.

We prove by induction on d.

Base case:

Inductive step: Assume that $p(x)$ is a polynomial of degree $d + 1$ with distinct roots $a_1, \ldots, a_d, a_{d+1}$. Since a_{d+1} is $p(x)$'s root, we can divide $p(x)$ with $(x - a_{d+1})$ and get that

$$p(x) = (x - a_{d+1})q(x),$$

where $q(x)$ is a polynomial of degree d with d distinct roots $a_1, \ldots, a_d$. $\square$

Property 1

Polynomials over a finite field $GF(p)$

Examples - evaluation

Suppose that we work over $GF(m)$ where $m = 11$. Let $p(x) = 4 \cdot x^2 + 5 \cdot x + 3$. We have

x	$p(x)$	$p(x) \bmod m$
0	3	3
1	12	1
2	29	7
3	54	10
4	87	10
5	128	7
6	177	1
7	234	3
8	299	2
9	372	9
10	453	2
11	542	3

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7),(4,10)$, and $(7,3)$. Find $p(x)$.

Let

- $\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2-11x+28}{(-2)\cdot(-5)} = \frac{x^2+6}{10} = 10x^2+5$
- $\Delta_2(x) = \frac{(x-2)(x-7)}{(4-2)(4-7)} = \frac{x^2-9x+14}{2\cdot(-3)} = \frac{x^2+2x+3}{5} = 9x^2+7x+5$
- $\Delta_2(x) = \frac{(x-2)(x-4)}{(7-2)(7-4)} = \frac{x^2-6x+8}{5\cdot 3} = \frac{x^2+5x+8}{4} = 3x^2+4x+2$

Thus,

$$\begin{aligned} p(x) &= 7\Delta_1(x) + 10\Delta_2(x) + 3\Delta_3(x) \\ &= (70x^2+35) + (90x^2+70x+50) + (9x^2+12x+6) \\ &= 4x^2+5x+3 \end{aligned}$$

How many?

Two ways of specifying a polynomial $p(x)$ of degree d:

- Specify its coefficients $a_0, a_1, \ldots, a_d$, i.e., the polynomial is

$$p(x) = a_d x^d + \ldots a_1 x + a_0.$$

- Specify $d+1$ points, i.e., $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$, where all x_i are distinct. There is a *unique* polynomial $p(x)$ of degree at most d that passes through these points (from Property 2).

For polynomials of degree at most d over $GF(m)$, if you specify q points, there are:

q	numbers of polynomials
$d+1$	1
d	m
$d-1$	m^2
$d-2$	m^3
$\vdots$	$\vdots$
1	m^d
0	m^{d+1}

Secret sharing scheme - settings

- There are n people, a secret s, and an integer k.
- We want to "distribute" the secret in such a way that any set of $k-1$ people cannot know anything about s, but any set of k people can reconstruct s.

Secret sharing scheme

- Pick m to be larger than n and s. (Much larger than s, i.e., $m >>> s$.)
- Pick a random polynomial of degree $k-1$ such that $P(0) = s$.
- Give $P(i)$ to person i, for $1 \leq i \leq n$.
- Correctness: for any set of k people,
- Correctness: for any set of $k-1$ people, how many possible candidate secrets compatible with the information these people have?

A more complex secret sharing scheme

Suppose that a company has 3 VPs and 5 senior members. You want to distribute a secret such that (1) any 2 VPs can obtain the secret or (2) a single VP with 3 senior members can also obtain the secret. How can you do that?

Sending a message

Suppose that you want to send a message `1,2,1,1,3,4,4,10` over the internet.
Since the internet does not maintain the ordering (if you send with UDP), you have to maintain the "ordering" youself, e.g., you can add the message indices, i.e.,
Lossy internet:

Erasure codes

Suppose that we want to send a message $m_1, m_2, \ldots, m_n$ where $m_i \leq p - 1$ for some prime p.

However, we know that our communication channel is lossy, i.e., some messages can be *dropped*. How can we send this message?

Two ways of encoding

Suppose that we want to send a message $m_1, m_2, \ldots, m_n$ where $m_i \leq p-1$ for some prime p. We want to tolerate up to k missing messages.
We use a polynomial of degree $n-1$ and generate $n+k$ points.
How can we obtain the polynomial $P(x)$?

- We can let the message be the coefficients, i.e., let

$$P(x) = m_n \cdot x^{n-1} + m_{n-1} \cdot x^{n-2} + \cdots + m_2 \cdot x + m_1.$$

- We can try to obtain a degree-$(n-1)$ polynomial $P(x)$ such that

$$P(0) = m_1,\ P(1) = m_2,\ \ldots P(n-2) = m_{n-1},\ P(n-1) = m_n.$$