

01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (preview)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array} \right]$$

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$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

Linearly independent rows

Vector spaces related to a matrix

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Examples:

- ▶ Span $\{[1, 1]\}$ is a subspace of \mathbb{R}^2 .
- ▶ Span $\{[1, 0, 0], [0, 1, 1]\}$ is a subspace of \mathbb{R}^3 .
- ▶ Span $\{[1, 0, 0], [0, 1, 1], [1, 1, 2]\}$ is a subspace of \mathbb{R}^3 .

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$Ax = 0.$$

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The set of solutions $\{x \mid Ax = 0\}$ form a vector space.

Example 1 (cont.)

Given a matrix A , we can look at the matrix-vector product $A\mathbf{x}$.

Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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- ▶ The column space of A (denoted by $\mathcal{C}(A)$)
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$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$$

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