01204211 Discrete Mathematics
Lecture 8a: Integers and GCD

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Number theory: integers and divisibility

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We will cover:

- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.

$\boxed{GF(2)}$ GF(p)

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- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.
- Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b. On the other hand if a does not divide b, we write $a \not| b$.

Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

Planning:

HAVE $\left\{ \begin{array}{l} a|b \to \exists k \in \mathbb{Z} \text{ s.t } b = a\cdot k \\ a|c \to \exists \ell \in \mathbb{Z} \ c = a \cdot \ell \end{array} \right.$

Goal: $a|(b+c)$

WANT $\exists x \in \mathbb{Z}$ s.t. $a \cdot x = b+c$

Proof: Because $a|b$, we know that there exists an integer k such that $b = k \cdot a$.

Also, since $a|c$, there exists an integer ℓ s.t. $c = \ell \cdot a$.

Therefore, we have that

$$b + c = k\cdot a + \ell \cdot a = a(\underbrace{k+\ell}_{\in \mathbb{Z}}).$$

However, k & ℓ are integers; thus $k+\ell$ is also an integer.

By the definition of divisibility, we can conclude that

$$a | b+c.$$

because $\exists$ an integer $x = k+\ell$, such that $ax = b+c$.

Examples

1. If $a|b$ and $a|c$, prove that $a|(b+c)$.

2. If $a|b$ and $b|c$, prove that $a|c$.

Proof: Since $a|b$, $\exists$ an integer k such that $b = k \cdot a$.

Also since $b|c$, $\exists$ an integer ℓ, s.t. $c = b \cdot \ell$. Thus

$$c = b \cdot \ell = k \cdot a \cdot \ell = (k \cdot \ell) \cdot a.$$

Since k & ℓ are integer $k\ell$ is also an integer, therefore, we know that

$$a | c.$$

$\blacksquare$

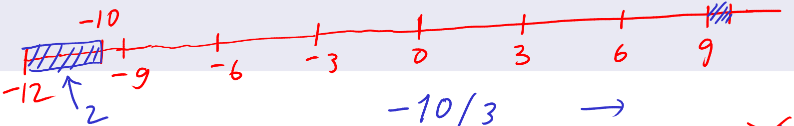
Remainder

Defintion (remainder) — positive

The **remainder** of the division of b with a is an integer r when there exists an integer q such that

$$b = qa + r,$$

where $0 \leq r < a$.



10 mod 3 = 1

25 mod 7 = 4

$-10/3$ → remainder ~~1~~ 2

$-10 \bmod 3 = 2$

$-25 \bmod 7 = 3$

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Examples:

We use operator `mod` to denote an operation for finding the remainder of a division.
I.e., $a \bmod b$ is the remainder of dividing a with b.

Examples

Let r be the remainder of the division of b by a. Assume that $c|a$ and $c|b$. Prove that $c|r$.

More examples

For every integer a, $a - 1 | a^2 - 1$.

Primes

Definition (primes)

- An integer $p > 1$ is a **prime** if its divisors are only p, $-p$, 1, and -1.
- If an integer $n > 1$ is not a prime, it is called a **composite**.
- Note: 1 is not a prime and also not a composite.

Fundamental theorem of arithmetic

Unique factorization

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

Algorithm for testing primes

Recall our CheckPrime2 algorithm

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run?

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How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

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n	number of bits of n	$\sqrt{n}$
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
1,267,650,600,228,229,401,496,703,205,376	100	1,125,899,906,842,624

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Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

Greatest Common Divisors (GCD)

Definition (GCD)

For integers x and y, the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $gcd(x, y)$.

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What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

Euclid's algorithm

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Let's see how it works with $Euclid(12311, 24324)$:

Euclid(12311, 24324)
Euclid(24324, 12311)
Euclid(12311, 12013)
Euclid(12013, 298)
Euclid(298, 93)
Euclid(93, 19)
Euclid(19, 17)
Euclid(17, 2)
Euclid(2, 1)

Proofs

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid.

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- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
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Note that when $x < y$, $\mathrm{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\mathrm{Euclid}(y, x)$. After that, in each call, x is always larger than y. For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

For any integers x and y such that $x > y$, $\mathrm{Euclid}(x, y) = gcd(x, y)$.

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $gcd(x, y) = y$; therefore, the base case of the algorithm is correct.
Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\mathrm{Euclid}(x', y') = gcd(x', y')$.
Now assume that $y \not| x$. The Euclid algorithm returns $\mathrm{Euclid}(y, x \bmod y)$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\mathrm{Euclid}(y, x \bmod y) = gcd(y, x \bmod y).$$

Thus, we are left to show that

$$gcd(x, y) = gcd(y, x \bmod y).$$

□

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$$x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

Lemma 2

If $a|x$ *and* $a|y$*, then* $a|x \bmod y$*.*

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Lemma 3

$gcd(x, y) = gcd(y, x \bmod y)$

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- How many times can that happen?
- The first argument can decrease by a factor of two for at most $\log x$ times. Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$.

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What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

$$x^{16} = (x^8)^2 = ((x^4)^2)^2 = (((x^2)^2)^2)^2.$$

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```
Algorithm power(x,y):  // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

Repeated squaring (general y)

```
Algorithm power(x,y):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2))
    if y mod 2 == 0:
      return a*a
    else
      return a*a*x
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```

What is the number of recursive calls?
What is the running time?
While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y, mod n)

Computing $x^y \bmod n$:

```
Algorithm power(x,y,n):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2)) mod n
    if y mod 2 == 0:
      return a*a mod n
    else
      return a*a*x mod n
```