

# 01204211 Discrete Mathematics

## Lecture 10a: Polynomials (1)<sup>1</sup>

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<sup>1</sup>This section is from Berkeley CS70 lecture notes.

## Quick exercise

For any integer  $a \neq 1$ ,  $a - 1 | a^2 - 1$ .

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For any integer  $a \neq 1$  and  $n \geq 1$ ,  $a - 1 | a^n - 1$ .

# Polynomials

A **single-variable polynomial** is a function  $p(x)$  of the form

$$p(x) = \underbrace{a_d x^d}_{\text{GF(P)}} + \underbrace{a_{d-1} x^{d-1}}_{\text{GF(P)}} + \cdots + a_1 x + a_0.$$

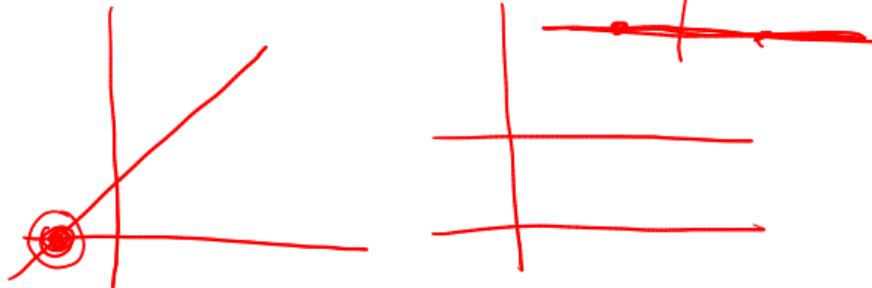
We call  $a_i$ 's coefficients. Usually, variable  $x$  and coefficients  $a_i$ 's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

## Examples

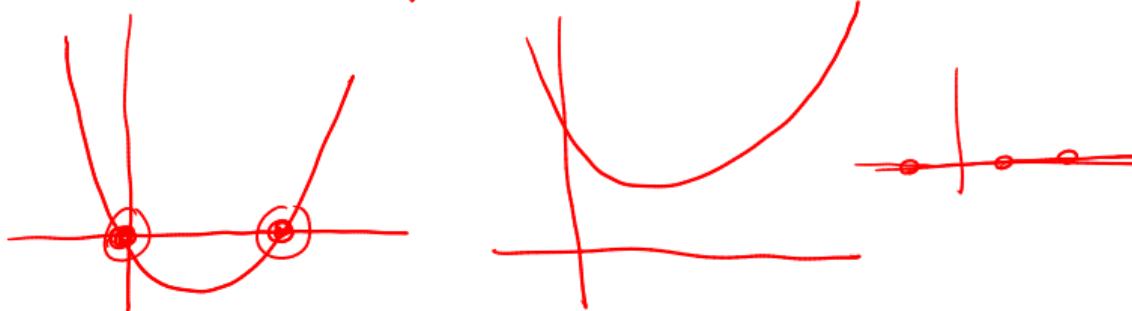
- $x^3 - 3x + 1$       3
- $x + 10$                 1
- $10$                       0
- $0$                         0

## Folklore

degree  $\leq 1$



degree  $\leq 2$



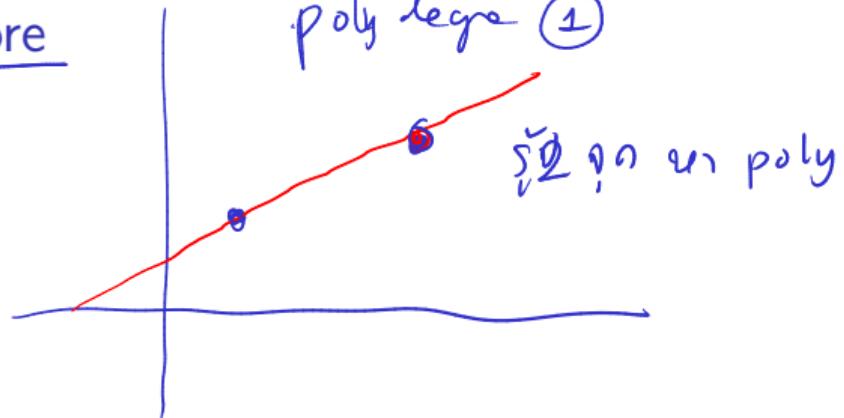
- polynomial degree  $d$   $\exists s \in \mathbb{R}^d$  non-zero

$$\exists s \in \mathbb{R}^d \text{ non-zero}$$

- polynomial degree  $\leq d$   $\exists s \in \mathbb{R}^d$  non-zero  $\Rightarrow$   $\text{poly}_{\text{fun}} = 0$

## Folklore

poly degree ①

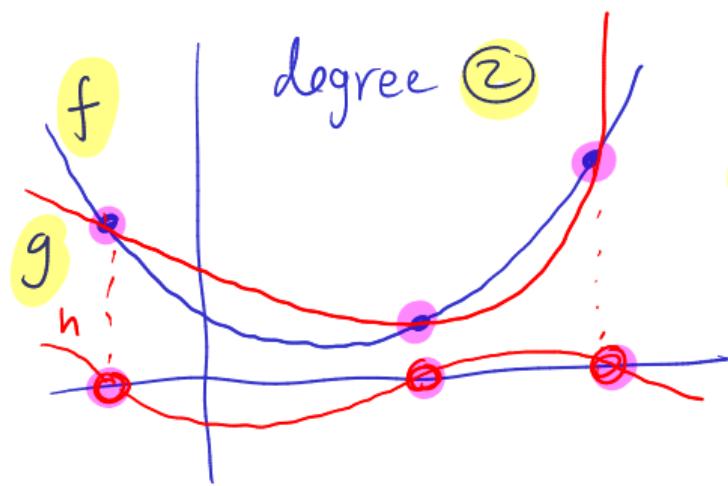


- degree ③

- ♡ 4 00



degree ②



$$h(x) = f(x) - g(x)$$

$h(x)$  degree 2

$$\Rightarrow h(x) = 0$$

$$\Rightarrow f(x) = g(x) \quad \checkmark$$

# Applications

- ▶ Secret sharing

# Applications

- ▶ Secret sharing
- ▶ Error-correcting codes

# Basic facts



hw

## Definition

$a$  is a root of polynomial  $f(x)$  if  $f(a) = 0$ .

## Properties

→ **Property 1:** A non-zero polynomial of degree  $d$  has at most  $d$  roots. \*

→ **Property 2:** Given  $d + 1$  pairs  $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$  with distinct  $x_i$ 's, there is a unique polynomial  $p(x)$  of degree at most  $d$  such that  $p(x_i) = y_i$  for  $1 \leq i \leq d + 1$ .



## Lemma 1

If two polynomials  $f(x)$  and  $g(x)$  of degree at most  $d$  that share  $d + 1$  points  $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$ , where all  $x_i$ 's are distinct, i.e.,  $f(x_i) = g(x_i) = y_i$ , then  $\underline{f(x) = g(x)}$ .

### Proof.

Suppose that  $f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0$  and  $g(x) = b_d x^d + b_{d-1} x^{d-1} + \dots + b_0$ . Let  $h(x) = f(x) - g(x)$ , i.e., let  $h(x) = c_d x^d + c_{d-1} x^{d-1} + \dots + c_0$ , where  $c_i = a_i - b_i$ . Note that  $h(x)$  is also a polynomial of degree (at most)  $d$ .

We claim that  $h(x)$  has  $d + 1$  roots. Note that since  $f(x_i) = g(x_i) = y_i$ , we have that

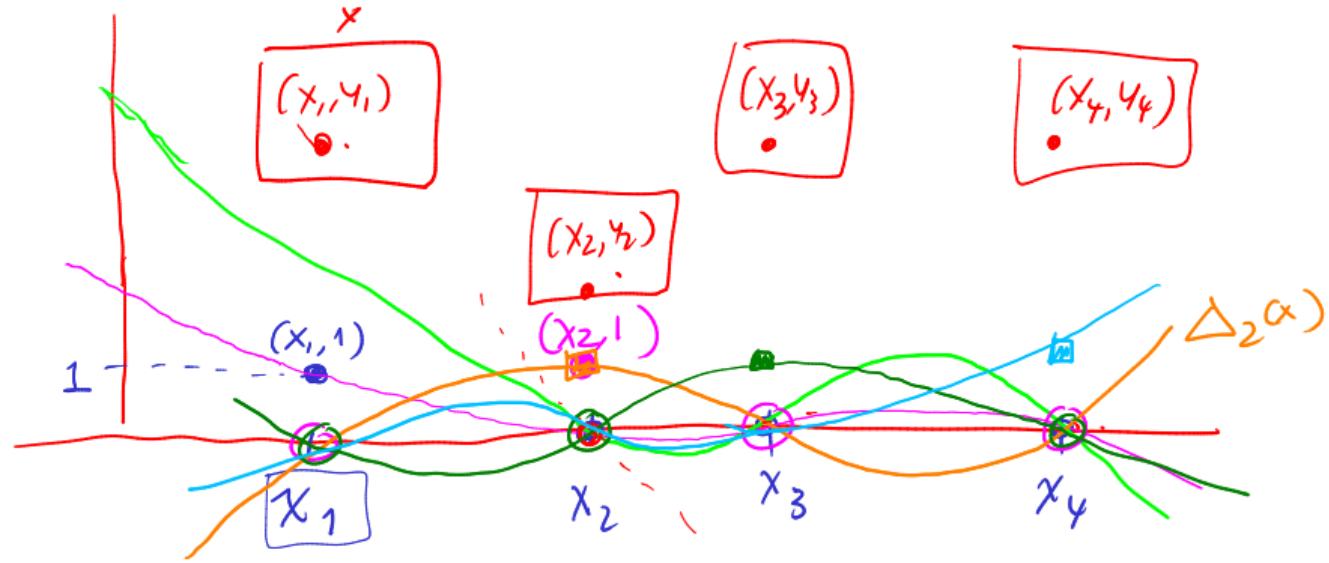
$$h(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0,$$

i.e., every  $x_i$  is a root of  $h(x)$ .

From **Property 1** if  $h(x)$  is non-zero it has at most  $d$  roots; therefore,  $h(x)$  must be zero, i.e.,  $f(x) - g(x) = 0$  or  $f(x) = g(x)$  as required.  $\square$

## Polynomial interpolation - ideas

$(x_1, y_1), (x_2, y_2), (x_3, y_3)$   
 $(x_4, y_4)$

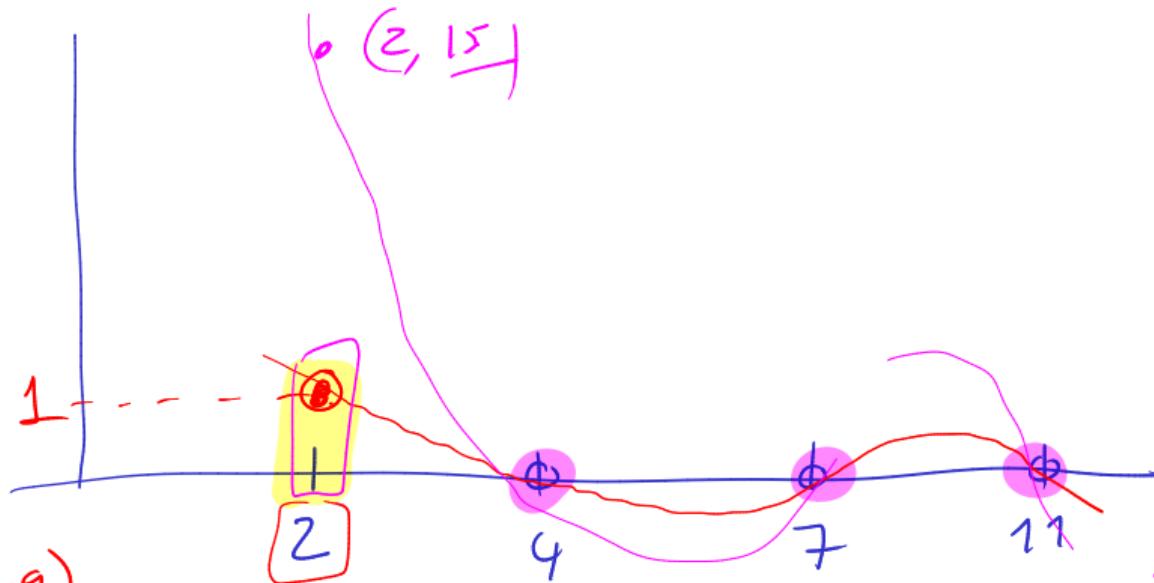


$$\Delta_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$\Delta_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$\Delta_3(x)$$
  
$$\Delta_4(x)$$

## Polynomial interpolation - ideas



$$(-2)(-5)(-9)$$

-90

$$\boxed{A_1(x)}$$

$$= \frac{(x-4)(x-7)(x-11)}{(2-4)(2-7)(2-11)}$$

} divisor  
 $x=2$

## Lagrange polynomial

For  $d + 1$  points  $(x_1, y_1), (x_2, y_2), \dots, (x_{d+1}, y_{d+1})$  where all  $x_i$ 's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_{d+1})}{(x_i - x_1)(x_i - x_2) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_{d+1})}.$$

Note that  $\Delta_i(x)$  is a polynomial of degree  $d$

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- ▶ For  $j \neq i$ ,  $\Delta_i(x_j) =$

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Note that  $\Delta_i(x)$  is a polynomial of degree  $d$ . Also we have that

- ▶ For  $j \neq i$ ,  $\Delta_i(x_j) = 0$ , and
- ▶  $\Delta_i(x_i) = 1$ .

We can use  $\Delta_i(x)$  to construct a degree- $d$  polynomial

$$p(x) = \underbrace{y_1 \cdot \Delta_1(x)} + \underbrace{y_2 \cdot \Delta_2(x)} + \cdots + \underbrace{y_{d+1} \cdot \Delta_{d+1}(x)}.$$

What can you say about  $p(x_i)$ ?

## → Property 2

Given  $d + 1$  pairs  $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$  with distinct  $x_i$ 's there is a unique polynomial  $p(x)$  of degree at most  $d$  such that  $p(x_i) = y_i$  for  $1 \leq i \leq d + 1$ .

## Proof of Property 2.

✓ Using Lagrange interpolation, we know that there exists a polynomial  $p(x)$  of degree  $d$  such that  $p(x_i) = y_i$  for all  $1 \leq i \leq d + 1$ .

For uniqueness, assume that there exists another polynomial  $g(x)$  of degree  $d$  also satisfying the condition. Since  $p(x)$  and  $g(x)$  agrees on more than  $d$  points,  $p(x)$  and  $g(x)$  must be equal from Lemma 1. □

## Polynomials over a finite field $GF(p)$

- known mod p
- division  $\rightarrow$  requires multiplicative inverse.

## Examples - evaluation

Suppose that we work over  $GF(m)$  where  $m = 11$ . Let  $p(x) = 4 \cdot x^2 + 5 \cdot x + 3$ . We have

$x$	$p(x)$	$p(x) \bmod m$
0	3	3
1	12	1
2	29	7
3	54	10
4	87	10
5	128	7
6	177	1
7	234	3
8	299	2
9	372	9
10	453	2
11	542	3

## Examples - interpolation

Let  $m = 11$ . Suppose that  $p(x)$  is a polynomial over  $GF(m)$  of degree 2 passing through  $(2, 7), (4, 10)$ , and  $(7, 3)$ . Find  $p(x)$ .

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Let

$$\blacktriangleright \Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2 - 11x + 28}{(-2) \cdot (-5)} = \frac{x^2 + 6}{10} = 10x^2 + 5$$

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- ▶  $\Delta_2(x) = \frac{(x-2)(x-7)}{(4-2)(4-7)} = \frac{x^2-9x+14}{2\cdot(-3)} = \frac{x^2+2x+3}{5} = 9x^2 + 7x + 5$

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- ▶  $\Delta_3(x) = \frac{(x-2)(x-4)}{(7-2)(7-4)} = \frac{x^2-6x+8}{5\cdot3} = \frac{x^2+5x+8}{4} = 3x^2 + 4x + 2$

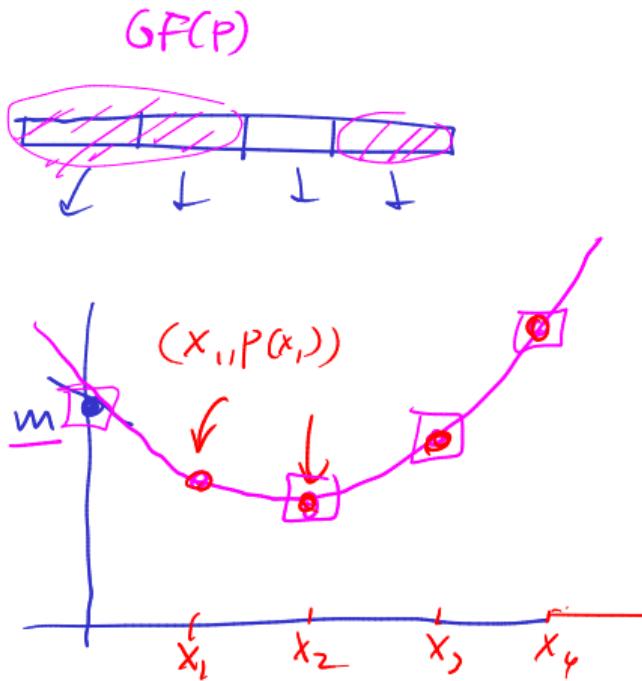
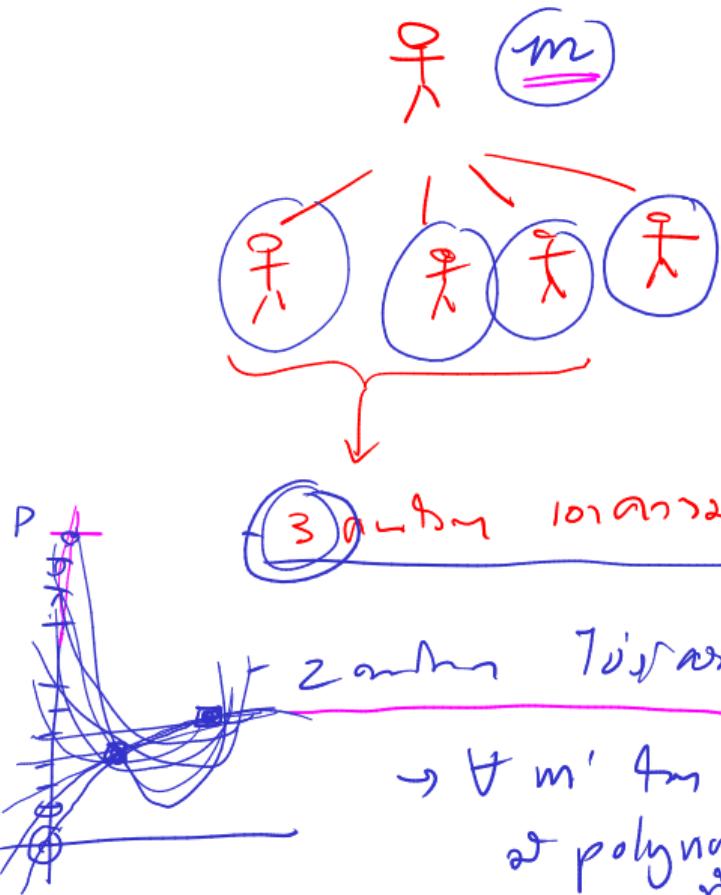
Thus,

$$\begin{aligned} p(x) &= 7\Delta_1(x) + 10\Delta_2(x) + 3\Delta_3(x) \\ &= (70x^2 + 35) + (90x^2 + 70x + 50) + (9x^2 + 12x + 6) \\ &= 4x^2 + 5x + 3 \end{aligned}$$

## Secret sharing scheme - settings

$$\boxed{p > m}$$

$GFP(p)$



$$p(x) = \boxed{a_2} x^2 + \boxed{a_1} x + \boxed{m}$$

## Secret sharing scheme - settings

- There are  $n$  people, a secret  $s$ , and an integer  $k$ .
- We want to “distribute” the secret in such a way that any set of  $k-1$  people cannot know anything about  $s$ , but any set of  $k$  people can reconstruct  $s$ .

- degree  $k-1$

mod  $p$

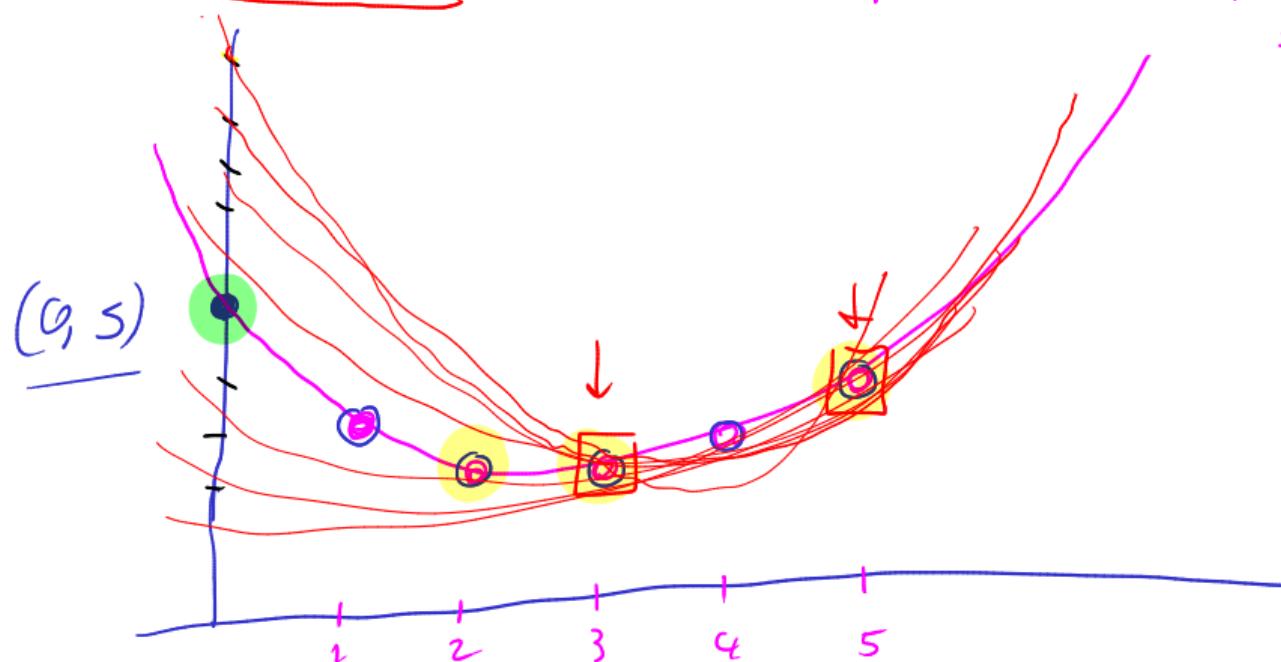
- $P(x) = a_{k-1} \underline{x^{k-1}} + a_{k-2} \underline{x^{k-2}} + \dots + a_1 x + \boxed{s}$

- On  $(1, P(1)), (2, P(2)), (3, P(3)), \dots$

$P > s$   
 $P > k$

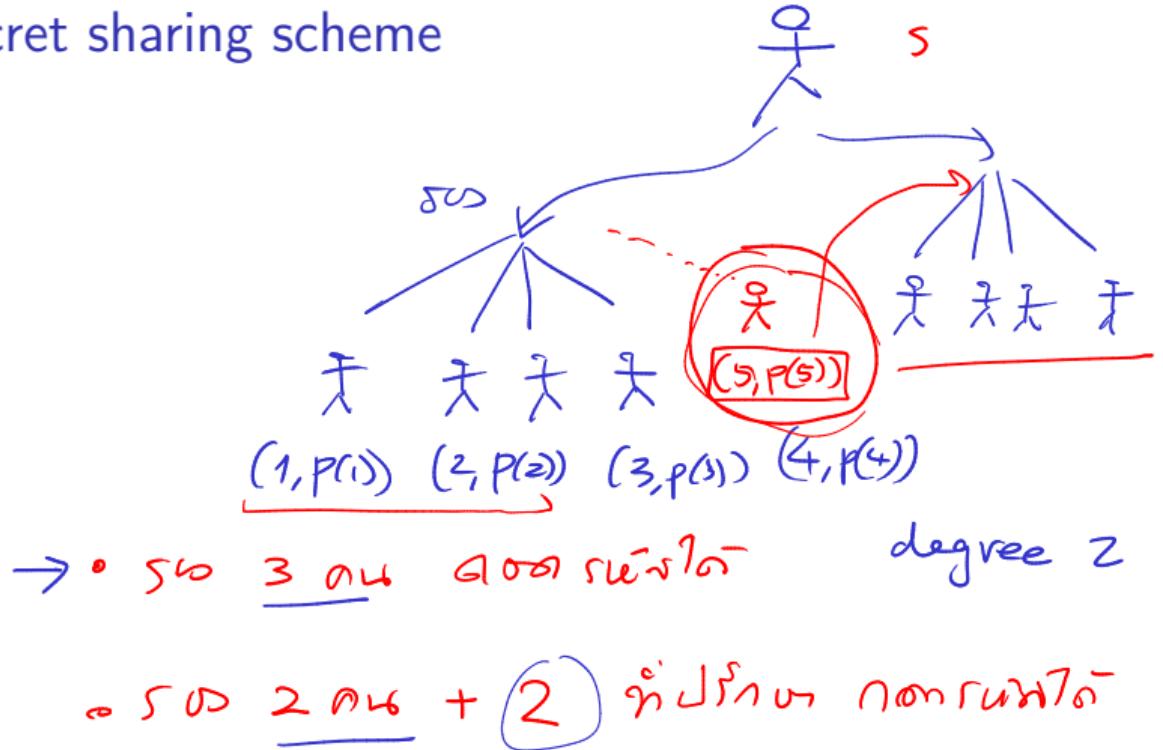
## Secret sharing scheme

$\tilde{s} \in \mathbb{F}_q, n = 3 \Rightarrow \tilde{s}$  poly  
 $\Rightarrow$  degree 2



$$P(x) = a_2 x^2 + a_1 x + s$$

## Secret sharing scheme



The diagram shows a graph of a polynomial function  $P(x)$  plotted against  $x$ . The x-axis has tick marks from 1 to 5. The y-axis has tick marks from 0 to 10. Five points are plotted:  $(1, P(1))$ ,  $(2, P(2))$ ,  $(3, P(3))$ ,  $(4, P(4))$ , and  $(5, P(5))$ . A smooth curve is drawn through these points. The equation  $P(x) = a_2x^2 + a_1x + a_0$  is written next to the graph.

$$P(x) = a_2x^2 + a_1x + a_0$$

## Secret sharing scheme

- ▶ Pick  $m$  to be larger than  $n$  and  $s$ . (Much larger than  $s$ , i.e.,  $m \gg s$ .)
- ▶ Pick a random polynomial of degree  $k - 1$  such that  $P(0) = s$ .
- ▶ Give  $P(i)$  to person  $i$ , for  $1 \leq i \leq n$ .
- ▶ Correctness: for any set of  $k$  people,

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- ▶ Correctness: for any set of  $k - 1$  people, how many possible candidate secrets compatible with the information these people have?