01204211 Discrete Mathematics
Lecture 13b: Eigenvalues and Eigenvectors

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Review: Hamming codes (1)

The code is defined by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Consider the encoding function $e : GF(2)^4 \to GF(2)^7$. Let $e(\boldsymbol{x}) = G\boldsymbol{x}$. What is Ker e?

What is $\dim$ Im e?

Review: Hamming codes (2)

The code is defined by the generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

What can you say about the minimum “distance”?

Examples: random walks

Examples: differential equations (1)

Let's start with a simple system with one variable.

$$\frac{du}{dt} = au,$$

with $u = u(0)$ when $t = 0$.

Examples: differential equations (2)

Now consider a system with two variables v and w:

$$\begin{aligned} \frac{dv}{dt} &= 4v - 5w \\ \frac{dw}{dt} &= 2v - 3w \end{aligned}$$

with $v = 5$ and $w = 4$ when $t = 0$, or if we let $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ and

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix},$$

we have

$$\frac{du}{dt} = Au,$$

with $u(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Examples: differential equations (3)

Eigenvalues and eigenvectors

Definition

For an n-by-n matrix A, a vector $\boldsymbol{v}$ is an **eigenvector** of A if

$$A\boldsymbol{v} = \lambda\boldsymbol{v},$$

and $\boldsymbol{v} \neq \mathbf{0}$. The scalar λ is called an **eigenvalue** associated with $\boldsymbol{v}$.

Example

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$.

If we let $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$A\boldsymbol{v}_1 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5+7 \\ -5+3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-2) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

If we let $\boldsymbol{v}_2 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, we have

$$A\boldsymbol{v}_2 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35+35 \\ 35+15 \end{bmatrix} = \begin{bmatrix} 70 \\ 50 \end{bmatrix} = 10 \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

See demo in colab.

Invariant subspace

Definition (invariant subspace)

For an n-by-n matrix A, subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called an **invariant subspace** under linear map $f(\boldsymbol{x}) = A\boldsymbol{x}$ if for all $\boldsymbol{u} \in \mathcal{V}$, $f(\boldsymbol{u}) = A\boldsymbol{u} \in \mathcal{V}$.

Eigenvector

If $\boldsymbol{v}$ is an eigenvector of matrix A, then

$$\mathrm{Span}\,\{\boldsymbol{v}\}$$

is a 1-dimensional invariant subspace under linear map defined by A.

Finding eigenvalues and eigenvectors

Given A, we want to find an eigenvalue λ and a vector $\boldsymbol{u} \neq \mathbf{0}$ such that

$$A\boldsymbol{u} = \lambda \boldsymbol{u}.$$

After some writing, we want to solve this equation

$$(A - \lambda I)\boldsymbol{u} = 0,$$

where $\boldsymbol{u} \neq 0$.

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

$$A\boldsymbol{x} = \mathbf{0}.$$

Suppose that there exists $\boldsymbol{x} \neq \mathbf{0}$ that satisfies the equation, what can you say about A?

Clearly, A cannot have an inverse because no matrix B can bring $\boldsymbol{x}$ back from $A\boldsymbol{x} = 0$. In this case, we say that A is **singular**.

Equavilent conditions:

- The rank of A is less than n.
- Rows of A are not linearly independent.
- The linear function $f(\boldsymbol{x}) = A\boldsymbol{x}$ is not injective.
- Ker $f \neq \{\mathbf{0}\}$.
- $\dim \text{Ker } f \neq 0$.

Finding λ

From this equation

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

Since we want it to have nonzero solution $\boldsymbol{x}$. Our goal is to find λ so that $A - \lambda I$ becomes singular.

Typically, the tool to use is the **determinant**. However, we do not cover this topic in this class. We will look at small examples and consider an iterative method instead.

Example: 2×2 matrix

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We want to find λ such that

$$\begin{bmatrix} 5-\lambda & 7 \\ 5 & 3-\lambda \end{bmatrix}$$

is singular. This amounts to solving

$$\frac{5-\lambda}{5} = \frac{7}{3-\lambda},$$

i.e.,

$$\lambda^2 - 8\lambda - 20 = 0.$$

The equation can be re-written as $(\lambda - 10)(\lambda + 2) = 0$; thus, it has 2 roots: 10 and -2.
You can find associated eigenvectors by solving corresponding $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ equations.

Matrix multiplication (again)

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We know that A has two eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

with corresponding eigenvalues $\lambda_1 = 10$ and $\lambda_2 = -2$.
What can we say about

$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7-2 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \left(\begin{bmatrix} 7 \\ 5 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

Matrix multiplication (again and again)

Fact: An n-by-n matrix A has n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. (They might not be real vectors.)
Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis, for any vector $\boldsymbol{x}$ there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n.$$

Let's multiply $\boldsymbol{x}$ with A:

$$\begin{aligned} A\boldsymbol{x} &= A(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n) \\ &= A\alpha_1 \boldsymbol{v}_1 + A\alpha_2 \boldsymbol{v}_2 + \cdots + A\alpha_n \boldsymbol{v}_n \\ &= \lambda_1 \alpha_1 \boldsymbol{v}_1 + \lambda_2 \alpha_2 \boldsymbol{v}_2 + \cdots + \lambda_n \alpha_n \boldsymbol{v}_n. \end{aligned}$$

We can keep multiplying with A many times:

$$A^k \boldsymbol{x} = \lambda_1^k \alpha_1 \boldsymbol{v}_1 + \lambda_2^k \alpha_2 \boldsymbol{v}_2 + \cdots + \lambda_n^k \alpha_n \boldsymbol{v}_n.$$

The power method

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$|\lambda_1| > |\lambda_i|,$$

for $i \neq 1$. We call λ_1 the **dominant eigenvalue**. We also call the eigenvectors corresponding to λ_1 **dominant eigenvectors**.

The power method (or power iteration)

- Start with a random vector $\boldsymbol{x_0}$.
- For $i = 0, 1, \ldots, k$,
  Let $\boldsymbol{x}_{i+1} = A\boldsymbol{x}_i$, with probably some scaling.