01204211 Discrete Mathematics
Lecture 9a: Fermat's Little Theorem

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Quick recap

For any integer x and y, there exist a pair of integers a and b such that

$$a \cdot x + b \cdot y = gcd(x, y).$$

How to find a and b? Use the extended GCD algorithm.

Finding a and b: Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $gcd(x, y)$.

```
Algorithm Euclid(x,y):
  if x mod y == 0:

    return y, 0, 1
  else:
    g, a', b' = Euclid(y, x mod y)

    a = b'

    b = a' - b'*floor(x / y)

    return g, a, b
```

Recap: Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

$$a \bmod m = b \bmod m,$$

we write

$$a \equiv b \pmod{m}.$$

We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m | (a - b)$$

Recap: Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $gcd(a, m) = 1$.

How to test if an integer n is prime

- Try to find factors of n. (Takes time $\sqrt{n}$)
- If there is a property that holds **iff** n is prime, we can check that property. If we can check that quickly, we can test if n is prime.
- If there is a property that holds **if** n is prime, how can we make use of that property?

Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

How can we use Fermat's Little Theorem to check if integer n is prime?

Fermat test

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
        return False
    else:
        return True
```

How good is the Fermat test?

When you call `CheckPrime(n)`:

- If n is prime, `CheckPrime` always return True.
- If n is composite, you want `CheckPrime` to return False, but **Fermat's Little Theorem does not guarantee that.**

Fermat test - when n is composite

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
        return False
    else:
        return True
```

If n is composite, the algorithm returns False when

- $gcd(a, n) \neq 1$, i.e., when you pick a with common factor with n.
- $a^{n-1} \bmod n \neq 1$, i.e., when you find a that violates the property. We want to be in this case. How likely?

Proof of Fermat's Little Thm: Idea

Let $p = 7$ and $a = 5$. Consider set

$$B = \{1, 2, 3, \ldots, p-1\} = \{1, 2, 3, 4, 5, 6\}$$

Also consider set

$$C = \{1 \cdot 5 \bmod 7,\ 2 \cdot 5 \bmod 7,\ 3 \cdot 5 \bmod 7, \ldots, 6 \cdot 5 \bmod 7\},$$

which is

$$C = \{5, 3, 1, 6, 4, 2\} = B.$$

Is this coincidental? No. (We will prove that. But can you quickly tell why.)
Since $B = C$, the following terms are equal:

$$\left(\prod_{i \in B} i\right) \bmod 7 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \bmod 7,$$

and

$$\begin{aligned} \left(\prod_{i \in C} i\right) \bmod 7 &= 5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \bmod 7 \\ &= (1a) \cdot (2a) \cdot (3a) \cdot (4a) \cdot (5a) \cdot (6a) \bmod 7 \\ &= (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot a^6 \bmod 7. \end{aligned}$$

Proof of Fermat's Little Thm.

Recall that $gcd(a,p) = 1$, i.e., there exists a multiplicative inverse a^{-1} of a modulo p. This implies that for $i \not\equiv j \pmod{p}$, $ai \not\equiv aj \pmod{p}$. Also note that $a \cdot 0 \equiv 0 \pmod{p}$.
Let $B = \{1, 2, \ldots, p-1\}$. Let

$$C = \{a \cdot i \bmod p | i \in B\}.$$

Since for different $i, j \in B$, we have different $ai \bmod p, aj \bmod p$, we know that $|C| = p - 1$. Also, $C \subseteq B$ because $0 \leq ai \bmod p \leq p - 1$ and $0 \notin C$. Thus, we can conclude that $C = B$. Since $B = C$, we have that $\prod_{i \in B} i \equiv \prod_{i \in C} i \pmod{p}$, i.e.

$$\begin{aligned} 1 \cdot 2 \cdots (p-1) &\equiv (a1) \cdot (a2) \cdot (a3) \cdots (a(p-1)) \pmod{p} \\ &\equiv (1 \cdot 2 \cdots (p-1)) \cdot a^{p-1} \pmod{p}. \end{aligned}$$

Since each of $1, 2, \ldots, p-1$ has an inverse modulo p, we can multiply both sides with $1^{-1}, 2^{-1}, \ldots, (p-1)^{-1}$ to obtain

$$1 \equiv a^{p-1} \pmod{p},$$

as required.

$\square$

Exercise

Prove that for any integer a and prime p,

$$a^p \equiv a \pmod{p}.$$

How good is the Fermat test when n is composite?

To answer correctly, we want a to be such that $gcd(a, n) \neq 1$ or

$$a^{n-1} \not\equiv 1 \pmod{n}.$$

We only consider the case where $gcd(a, n) = 1$ because when $gcd(a, n) \neq 1$, the test works perfectly.

We refer to $a \in \{1, 2 \ldots, p-1\}$ such that $gcd(a, n) = 1$ and $a^{n-1} \not\equiv 1 \pmod{n}$ as a **witness**. The other element b such that $b^{n-1} \equiv 1 \pmod{n}$ is called a **non-witness**. How likely that we randomly choose an element and get a witness?

Number of witnesses

Suppose that there exists a witness a; we know that $a^{n-1} \not\equiv 1 \pmod{n}$. How can we find other witnesses?

Consider a non-witness b such that $b^{n-1} \equiv 1 \pmod{n}$.

Carmichael Number

A **Carmicheal number** is a composite number n where

$$b^{n-1} \equiv 1 \pmod{n},$$

for every b which are relatively primve to n.

Carmicheal numbers are rare. The smallest is $561 = 3 \cdot 11 \cdot 17$. The next ones are $1105, 1729$, and 2465. There are $20, 138, 200$ Carmicheal numbers between 1 and 10^{21}.
So, if we ignore Carmicheal numbers, the Fermat test is very good. There are other probabilistic tests (e.g, Miller-Rabin test) that uses other properties that works for all numbers and there are deterministic algorithms for testing primes.

Lemma 3

If n is not a Carmicheal number, the Fermat test returns that n is a composite with probability at least $1/2$.

Note that if you repeat the test for k times, the probability that it gives the wrong answer is at most $1/2^k$.

Running time

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
        return False
    else:
        return True
```

Special case of Euler's theorem

Theorem 4 (Euler's theorem)

If p and q are different primes, for a such that $gcd(a, pq) = 1$, we have

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

Is this useful? Yes! In the RSA algorithm.

RSA

- Private key: (d, n), Public key: (e, n)
- Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

- Pick two primes p and q. Let $n = pq$.
- Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or $ed = k \cdot (p-1)(q-1) + 1$, for some integer k.
- What is $m^{ed} \bmod n$?

$$\begin{aligned}
m^{ed} &\equiv m^{k(p-1)(q-1)+1} \pmod{n} \\
&\equiv (m^{(p-1)(q-1)})^k \cdot m \pmod{n} \\
&\equiv 1^k \cdot m \pmod{n} \\
&\equiv m \pmod{n}
\end{aligned}$$

What is the requirement for m? $gcd(m, n) = 1$, otherwise you can use the message to factor n.