01204211 Discrete Mathematics
Lecture 4a: Mathematical Induction 1

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July 17, 2021

Mathematical Induction

$P \Rightarrow Q$

$\neg Q \Rightarrow \neg P$

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$$\sum_{i=1}^{n} i = n(n+1)/2,$$

$1+2+\cdots+n$

or for any integer $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1),$$

$1^2+2^2+\cdots+n^2$

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quantifier

$$P(n) = \text{"}\sum_{i=1}^{n} i = n(n+1)/2,\text{"}$$

or for any integer $n \geq 1$,

$$Q(n) = \text{"}\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1),\text{"}$$

or "We can pay any integer amount $x \geq 4$ baht with 2-baht coins and 5-baht coins."

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- The range of the index may be sets. For example, let $A = \{1, 2, 4, 15\}$, we have that $\sum_{i \in A} i^2 = 1^2 + 2^2 + 4^2 + 15^2.$
- What is $\sum_{i=5}^{2} i$? Note that in this case, the range is empty. This sum is called an **empty sum**. By convention, we define it to be zero.

Informal arguments (1)

- Let's try to check that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$, by experimentation.
- Try $n = 1$:

[1] LHS = left hand side
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$\underbrace{xxx}=\underbrace{4444}$

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- Try $n = 2$: LHS: $1 + 2 = 3$,

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- Try $n = 3$: LHS: $1 + 2 + 3 = 6$, RHS: $3(3+1)/2 = 6$, OK
- Try ...

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- Try $n = 3$: LHS: $1 + 2 + 3 = 6$, RHS: $3(3+1)/2 = 6$, OK
- Try ...
- With this trying-all approach, we can't actually prove this statement.

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Informal arguments (2)

- Our goal is to show that $\sum_{i=1}^{n} i = n(n+1)/2$, for any integer $n \geq 1$.
- Try $n = 2$: LHS: $1 + 2 = 3$, RHS: $2(2+1)/2 = 3$.
- Try $n = 3$: LHS: $\boxed{1 + 2} + 3$, RHS: $3(3+1)/2$

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- Try $n = 3$: LHS: $1 + 2 + 3$, RHS: $3(3+1)/2$
- If we compare these two lines, we can see that

$$1 + 2 + 3 \quad = \quad (1+2) + 3$$

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$$\begin{aligned} 1 + 2 + 3 &= (1+2) + 3 \\ &= 2(2+1)/2 + 3 \qquad (*) \end{aligned}$$

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จริงที่ n=3

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&= 2(2+1)/2 + 2 \cdot (2+1)/2 \\
&= (2+2)(2+1)/2 = (3+1)(3)/2,
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which is equal to $3(3+1)/2$.

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\end{aligned}
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  which is equal to $3(3+1)/2$.
- Line (*) is important here. That is because we use the fact that the statement is true when $n = 2$ there.

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- Assume that the statement is true for $n = k$. I.e.,

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$$\sum_{i=1}^{k} i = k(k+1)/2.$$

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Informal arguments (3)

- Goal: show that $\boxed{\sum_{i=1}^{n} i = n(n+1)/2,}$ for any integer $n \geq 1$.
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assumption.

$$\sum_{i=1}^{k} i = k(k+1)/2.$$

- Can we show that, with this assumption, the statement is true for $n = k+1$? I.e., can we show that

goal.

$$\sum_{i=1}^{k+1} i = \underbrace{(k+1)}((\underbrace{k+1}_{\text{" } k+2}) + 1)/2?$$

Informal arguments (4)

(direct proof)

Let's try...

Assumption: $\boxed{\sum_{i=1}^{k} i} = k(k+1)/2$.

Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

เขียน $\displaystyle\sum_{i=1}^{k+1} i = (1+2+\cdots+k) + (k+1)$

$= k(k+1)/2 + (k+1)$ (* ใช้ assumption)

$= k(k+1)/2 + 2(k+1)/2$

$= (k+1)/2 \times (k+2)$

$= \underline{(k+1)((k+1)+1)/2}$ ตามต้องการ ∎

Informal arguments (4)

Let's try...
Assumption: $\sum_{i=1}^{k} i = k(k+1)/2$.
Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

$$\sum_{i=1}^{k+1} i \quad = \quad \left(\sum_{i=1}^{k} i\right) + (k+1)$$

Informal arguments (4)

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Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\ &= k(k+1)/2 + (k+1)\end{aligned}$$

Informal arguments (4)

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Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

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\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
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Goal: $\sum_{i=1}^{k+1} i = (k+1)((k+1)+1)/2$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= \boxed{\left(\sum_{i=1}^{k} i\right)} + (k+1) \\
&= k(k+1)/2 + (k+1) \quad \leftarrow \text{จาก assumption} \\
&= k(k+1)/2 + 2\cdot(k+1)/2 \\
&= (k+2)(k+1)/2 \\
&= (k+1)((k+1)+1)/2,
\end{aligned}
$$

as required.

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We have all the ingredients required to prove this statement:

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The statement we want to prove becomes:

> For any natural number n, $P(n)$.

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We have shown:

1. $P(1)$ (by experimentation)
2. $P(k) \Rightarrow P(k+1)$ for any integer $k \geq 1$.

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What do these two statements imply?

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$P(1)$ (1st statement itself)

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2. $P(k) \Rightarrow P(k+1)$ for any integer $k \geq 1$.

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$P(1)$ (1st statement itself)
$\Rightarrow P(2)$ (from 2nd statement, let $k = 1$)

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$\Rightarrow P(2)$ (from 2nd statement, let $k = 1$)
$\Rightarrow P(3)$ (from 2nd statement, let $k = 2$)

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$P(1)$ (1st statement itself)
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$\Rightarrow P(3)$ (from 2nd statement, let $k = 2$)
$\Rightarrow P(4)$ (from 2nd statement, let $k = 3$)

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$\Rightarrow P(4)$ (from 2nd statement, let $k = 3$)
$\Rightarrow P(5)$

$\vdots$

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$\Rightarrow P(5) \Rightarrow P(6)$

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$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7)$

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$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7) \ldots$

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$\Rightarrow P(4)$ (from 2nd statement, let $k = 3$)
$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7) \ldots$

Informally, these chain of reasoning will eventually reach any natural number n. Therefore, we can conclude that $P(n)$ for any natural number n.

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What do these two statements imply?

$P(1)$ (1st statement itself)
$\Rightarrow P(2)$ (from 2nd statement, let $k = 1$)
$\Rightarrow P(3)$ (from 2nd statement, let $k = 2$)
$\Rightarrow P(4)$ (from 2nd statement, let $k = 3$)
$\Rightarrow P(5) \Rightarrow P(6) \Rightarrow P(7) \ldots$

Informally, these chain of reasoning will eventually reach any natural number n. Therefore, we can conclude that $P(n)$ for any natural number n.
We have just shown the statement with mathematical induction.

Mathematical Induction

basis

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:

Base case: $\boxed{P(1)}$

Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

I.H.

Let's re-write the proof again

$\forall n \in \mathbb{N}, P(n)$

Theorem 1

For every natural number n, $\sum_{i=1}^{n} i = n(n+1)/2$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i = n(n+1)/2$."

$\forall k \geq 1,$ $P(k) \Downarrow P(k+1)$

Base case: We can plug in $n = 1$ to check that $P(1)$ is true: $1 = 1(1+1)/2$.

Inductive step: We assume that $P(k)$ is true for $k \geq 1$ and show that $P(k+1)$ is true.

Let's state the Induction Hypothesis $P(k)$: $\sum_{i=1}^{k} i = k(k+1)/2$.

Let's show $P(k+1)$. We write $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$. Using the Induction Hypothesis, we know that this is equal to

$$
\begin{aligned}
k(k+1)/2 + (k+1) &= k(k+1)/2 + 2 \cdot (k+1) \\
&= (k+2)(k+1)/2,
\end{aligned}
$$

which implies $P(k+1)$ as required.

From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every natural number n. (quantifier)