

01204211 Discrete Mathematics
Lecture 9a: Spans and Vector Spaces

linear⁺ combination

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Exercise

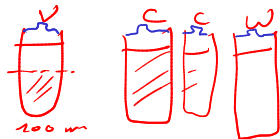
The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$\mathbf{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

Handwritten red annotations:

- ← Vitamin C (pointing to the first element of \mathbf{w})
- ← sugar (pointing to the second element of \mathbf{w})
- ← Vitamin B (pointing to the third element of \mathbf{w})
- ← salt (pointing to the fourth element of \mathbf{w})

Exercise



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Write down the nutritions for a mixed drink that consists of 50ml of v , 200ml of c and 10ml of w .

0,5 $\begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} + 0,1 \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 55 \\ 25 \\ 605 \\ 1 \end{bmatrix}$

Exercise

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$$\mathbf{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

Write down the nutritions for a mixed drink that consists of 50ml of \mathbf{v} , 200ml of \mathbf{c} and 10ml of \mathbf{w} .

The handwritten calculation shows the matrix multiplication of the ingredients matrix and the volume vector. The ingredients matrix is a 4x3 matrix with columns corresponding to ingredients v, c, and w. The first column (v) has values 100, 50, 0, 0. The second column (c) has values 0, 0, 300, 0. The third column (w) has values 50, 0, 50, 10. The volume vector is a 3x1 column vector with values 0.5, 2, and 0.1. The result is a 4x1 column vector with values 55, 25, 605, and 1. The calculation is as follows:

$$\begin{bmatrix} 100 & 0 & 50 \\ 50 & 0 & 0 \\ 0 & 300 & 50 \\ 0 & 0 & 10 \end{bmatrix} \times \begin{bmatrix} 0,5 \\ 2 \\ 0,1 \end{bmatrix} = \begin{bmatrix} 55 \\ 25 \\ 605 \\ 1 \end{bmatrix}$$

Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

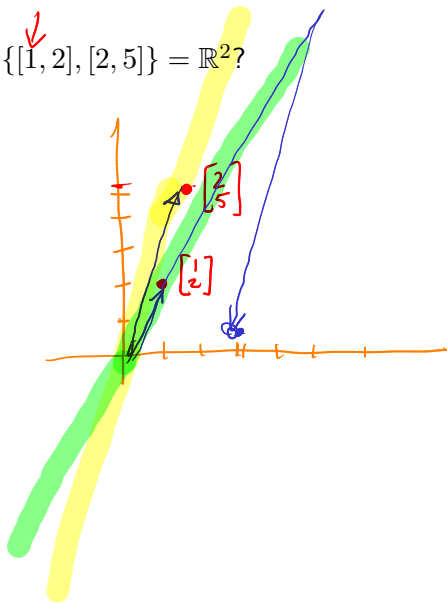
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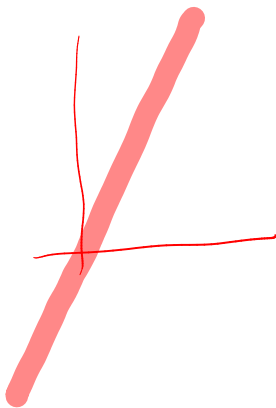
Write that result as a matrix-vector product. (The matrix should be a 4×3 matrix.)

Example 1

Is $\text{Span} \{[1, 2], [2, 5]\} = \mathbb{R}^2$?



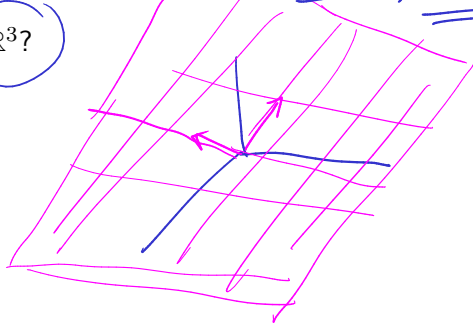
$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ ✓
 $\text{Span} \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$



Example 2

Is $\text{Span} \{[1, 0, 1], [1, 1, 0], [2, 3, 4]\} = \mathbb{R}^3$?

$\text{Span} \{[1, 0, 1], [1, 1, 0], [2, 1, 1]\}$



Can we write

$$[2, 3, 4]$$

as a linear combination of
 $[1, 0, 1]$ & $[1, 1, 0]$

$$4 \cdot [1, 0, 1] + 3 \cdot [1, 1, 0] = [7, 3, 4]$$

Example 3

Is $\text{Span} \{ \underline{[1, 0, 1]}, \underline{[1, 1, 0]}, \underline{[4, 2, 2]} \} \neq \mathbb{R}^3$?

? $\in \text{Span} \{ \begin{matrix} [1, 0, 1] \\ [1, 1, 0] \end{matrix} \}$

\boxed{No}

Elements in a vector

Vectors
- scalar $\langle \lambda, \vec{u} \rangle$
(scalar multiplication)
- $\vec{u} + \vec{v}$

elements are in \mathbb{R}

- ▶ We see examples of vectors over \mathbb{R} .
- ▶ However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
 - ▶ We should be able to perform addition, subtraction, multiplication and division.
 - ▶ Operations should be commutative and associative.
 - ▶ Additive and multiplicative identity should exist.
 - ▶ Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a field.

$$\underline{(a + b) + c = a + (b + c)}$$

\mathbb{R} $\text{GF}(2)$

A field

\mathbb{R}, \mathbb{C}

$\text{GF}(2) \dots$

$\text{GF}(p)$

Definition

A set \mathbb{F} with two operations $+$ and \times (or \cdot) is a **field** iff these operations satisfy the following properties:

- ▶ (Associativity): $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ▶ (Commutativity): $a + b = b + a$ and $a \cdot b = b \cdot a$
- ▶ (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$
- ▶ (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$
- ▶ (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an element a^{-1} such that $a \cdot a^{-1} = 1$
- ▶ (Distributive): $a \cdot (b + c) = a \cdot b + a \cdot c$

Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$. I.e., it is a “bit” field.

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What are $+$ and \cdot in $GF(2)$?

- We define $b_1 + b_2$ to be XOR.

$$0 + 0 = 0$$

$$0 + 1 = 1 + 0 = 1$$

$$1 + 1 = 0$$

- We define $b_1 \cdot b_2$ to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$

$$1 \cdot 1 = 1$$

You can check that $GF(2)$ satisfies the axioms of fields.

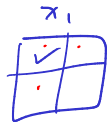
2 x 2 Lights out



state:
 $[1, 0, 0, 0]$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



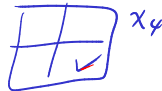
$[1, 1, 1, 0]$



$[1, 1, 0, 1]$



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Parity-check code

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From message $\mathbf{a} = [a_1, a_2, a_3, a_4]$, we compute (in $GF(2)$) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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Now our encoded message becomes

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where $a_5 = b = a_1 + a_2 + a_3 + a_4$. It can detect a single-bit error.

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It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

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Now, what is the set of all possible codewords?

Hamming code

You can detect and correct more errors with Hamming codes. In this version called a $[7, 4]$ Hamming code, you encode 4-bit data $[a_1, a_2, a_3, a_4]$ into a 7-bit codeword $[p_1, p_2, a_1, p_3, a_2, a_3, a_4]$. Using the formula:

$$p_1 = a_1 + a_2 + a_4$$

$$p_2 = a_1 + a_3 + a_4$$

$$p_3 = a_2 + a_3 + a_4$$

Hamming code

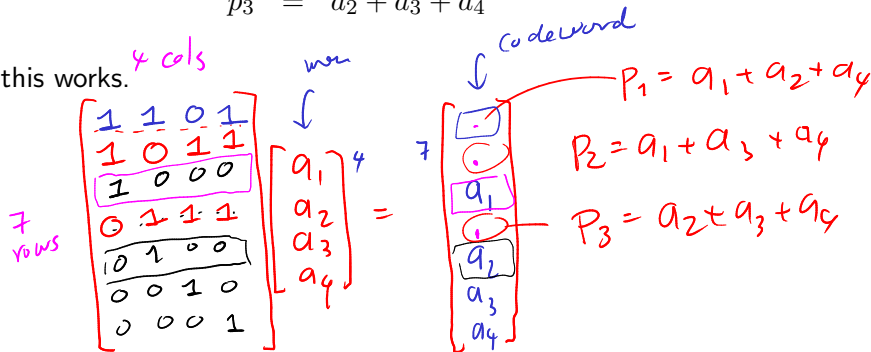
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$$p_3 = a_2 + a_3 + a_4$$

Let's see how this works.



Hamming code (encoding as matrix multiplication)

$$Hx$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}^4$$

Parity check

Suppose that we are given $\underbrace{[d_1, d_2, d_3, d_4, d_5, d_6, d_7]}$ Let

$$s_1 = d_1 + d_3 + d_5 + d_7$$

$$s_2 = d_2 + d_3 + d_6 + d_7$$

$$s_3 = d_4 + d_5 + d_6 + d_7$$

$$\begin{aligned} & d_1 + a_2 + a_4 \\ & \boxed{p_1 + a_1 + a_2 + a_4 = 0} \\ & p_2 + a_1 + a_3 + a_4 = 0 \\ & p_3 + a_2 + a_1 + a_4 = 0 \end{aligned}$$

Given a codeword $w = [c_1, c_2, \dots, c_7]$, if we compute s_1, s_2, s_3 , we would get all zero's.

Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$s_1 = \frac{d_1 + d_3 + d_5 + d_7}{2}$$

$$s_2 = \frac{d_2 + d_3 + d_6 + d_7}{2}$$

$$s_3 = \frac{d_4 + d_5 + d_6 + d_7}{2}$$

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What if there is an error? Let's try.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Hamming code (parity check as matrix multiplication)

Codewords from Hamming code

Turning the formula for p_1, p_2, p_3 around, we have 3 homogeneous linear equations:

$$d_1 + d_3 + d_5 + d_7 = 0$$

$$d_2 + d_3 + d_6 + d_7 = 0$$

$$d_4 + d_5 + d_6 + d_7 = 0$$

and again the set of all possible codewords \mathcal{W} forms a vector space over $GF(2)$.

Can you solve 2×2 Lights out?

Let $\mathbf{u}_1 = [1, 1, 1, 0]$, $\mathbf{u}_2 = [1, 1, 0, 1]$, $\mathbf{u}_3 = [1, 0, 1, 1]$, and $\mathbf{u}_4 = [0, 1, 1, 1]$.

Given $\mathbf{b} = [b_1, b_2, b_3, b_4]$, can you always find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

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$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

Same question: Is $\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\mathbf{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

Can you solve 2×2 Lights out?

Since

$$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \},$$

and

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and

$$\text{Span} \{ [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \} = GF(2)^4,$$

Can you solve 2×2 Lights out?

Since

$$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\},$$

and

$$\text{Span } \{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\} = GF(2)^4,$$

what can we say about $\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$?

Generators

Definition

Let \mathcal{V} be a set of vectors. Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

If $\text{Span} \{ \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \} = \mathcal{V}$, we say that

- ▶ $\{ \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \}$ is a **generating set** for \mathcal{V}
- ▶ vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **generators** for \mathcal{V}

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Examples

Standard generators

Note that $\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$ are generators for $GF(2)^4$.
Why?

Standard generators

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Why?

They are called **standard generators** for $GF(2)^4$, written as e_1, e_2, e_3, e_4 .

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Why?

They are called **standard generators** for $GF(2)^4$, written as e_1, e_2, e_3, e_4 .

For \mathbb{R}^n , we also have $[1, 0, 0, \dots, 0], [0, 1, 0, \dots, 0], [0, 0, 1, \dots, 0], \dots, [0, 0, 0, \dots, 1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors u_1, u_2, \dots, u_n . If v_1, v_2, \dots, v_k are generators for \mathcal{V} , and for each i ,

$$v_i \in \text{Span} \{u_1, u_2, \dots, u_n\},$$

we have that $\mathcal{V} \subseteq \text{Span} \{u_1, u_2, \dots, u_n\}$.

Adding a vector into a span

Lemma 2

Consider vectors u_1, u_2, \dots, u_n . If $v \in \text{Span} \{u_1, u_2, \dots, u_n\}$, then

$$\text{Span} \{u_1, u_2, \dots, u_n, v\} = \text{Span} \{u_1, u_2, \dots, u_n\}$$

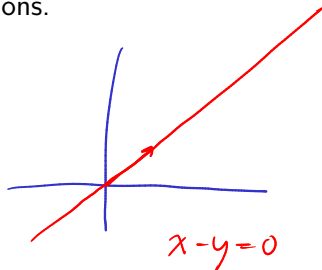
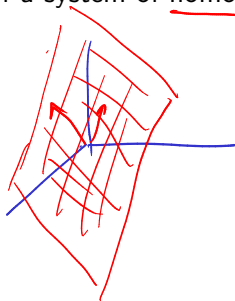
Geometry of spans: in \mathbb{R}^2

Geometry of spans: in \mathbb{R}^3

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.



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What are common properties of these geometric objects?

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- ▶ they pass through the origin,
- ▶ if vector \mathbf{u} is in the objects, $\alpha\mathbf{u}$ for any scalar α is also in the objects, and
- ▶ if \mathbf{u} and \mathbf{v} are in the objects, $\mathbf{u} + \mathbf{v}$ is also in the objects.

Vector spaces

Definition

A set \mathcal{V} of vectors over \mathbb{F} is a vector space iff

- ▶ (V1) $\mathbf{0} \in \mathcal{V}$,
- ▶ (V2) for any $\mathbf{u} \in \mathcal{V}$,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- ▶ (V3) for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Span of vectors is a vector space

Consider n -vectors u_1, u_2, \dots, u_m ,

$\text{Span } \overset{A}{\{u_1, u_2, \dots, u_m\}}$

is a vector space.

is a vector space.

(V1) $\overset{\text{check}}{0 \in A}$

(V2) $\overset{\text{check}}{\forall u \in A, \forall \alpha, \alpha u \in A}$

(V3) $\overset{\text{check}}{\forall u, v \in A, u+v \in A}$

Span of vectors is a vector space

✓ (V3) ...

Consider n -vectors u_1, u_2, \dots, u_m ,

$$\text{Span} \{u_1, u_2, \dots, u_m\}$$

is a vector space.

Let's check if properties V1, V2, and V3 are satisfied.

✓ (V1) $0 \in \text{Span}$ ^{ben} 0 is a linear $0 = 0 \cdot \bar{u}_1 + 0 \cdot \bar{u}_2 + \dots + 0 \cdot \bar{u}_m$

✓ (V2) let $\bar{v} \in \text{Span} \{u_1, \dots, u_m\}$. there exists $\alpha_1, \alpha_2, \dots, \alpha_m$ s.t.
$$\alpha_1 \cdot u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m = \bar{v}$$

Consider $\beta \cdot \bar{v} = \beta \alpha_1 (u_1) + \beta \alpha_2 (u_2) + \dots + \beta \alpha_m (u_m) \leftarrow$ is also
a linear comb of $u_1, \dots, u_m \Rightarrow \beta \bar{v} \in \text{Span} \{u_1, \dots, u_m\}$

Solutions to homogeneous linear equations is a vector space

Consider a set \mathcal{S} of all n -vectors in the form $[x_1, x_2, \dots, x_n]$ where

$$\begin{aligned} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= 0 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n &= 0 \\ \vdots &= \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n &= 0 \end{aligned}$$

Let's check if properties V1, V2, and V3 are satisfied.

✓ $\vec{x} = [x_1, x_2, \dots, x_n]$

$$\vec{a}_1 = [a_{11}, a_{12}, a_{13}, \dots, a_{1n}]$$

$$\vec{a}_1 \cdot \vec{x} = 0$$

$$\cancel{x+y} + 2z = 0$$



Dot product

Definition

For n -vectors $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$, the **dot product** of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} \cdot \mathbf{v}$, is

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Using dot products, the previous set \mathcal{S} can be written as

$$\underline{y}, \underline{x} \in \mathcal{S} \\ \propto x$$

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_1 \cdot \mathbf{x} = 0, \mathbf{a}_2 \cdot \mathbf{x} = 0, \dots, \mathbf{a}_m \cdot \mathbf{x} = 0\}$$

and we know that \mathcal{S} is a vector space.

$$\begin{aligned} \textcircled{1} \quad \bar{\mathbf{a}}_1 \cdot \alpha \bar{\mathbf{x}} &= \alpha (\bar{\mathbf{a}}_1 \cdot \bar{\mathbf{x}}) = \alpha \cdot 0 = 0 \\ \textcircled{2} \quad \bar{\mathbf{a}}_1 \cdot (\bar{\mathbf{x}} + \bar{\mathbf{y}}) &= \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{x}} + \bar{\mathbf{a}}_1 \cdot \bar{\mathbf{y}} = 0 + 0 = 0 \end{aligned}$$

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

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- ▶ *Question:* Is \mathcal{A} a vector space?
- ▶ We also write it as $\mathbf{a} + \mathcal{V}$.

Affine spaces

Definition

If \mathbf{a} is a vector and \mathcal{V} is a vector space, then

$$\mathbf{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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Definition

The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **affine hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

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Writing an affine space using a span

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An affine space

An affine space passing through $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= b_2 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= b_m \end{aligned}$$

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

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What can you say about the solution sets of these two related linear systems?

$\mathbf{0}$ is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

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This implies that $\mathbf{u}_1 - \mathbf{u}_2$ is a solution to the homogeneous linear system.

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

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i.e., \mathcal{W} is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.