

01204211 Discrete Mathematics
Lecture 9a: Spans and Vector Spaces
linear⁺ combination

Jittat Fakcharoenphol

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

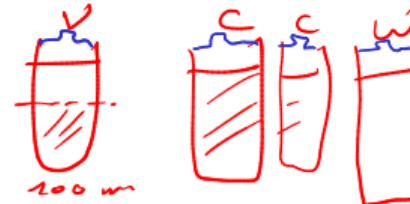
Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$\mathbf{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

*← vitamin C
← sugar
← vitamin B
← salt*

Exercise



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$$v = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad c = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

Write down the nutritions for a mixed drink that consists of 50ml of v , 200ml of c and 10ml of w .



$$0,5 \cdot \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} + 0,1 \cdot \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 55 \\ 25 \\ 605 \\ 1 \end{bmatrix}$$

Exercise

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Write down the nutritions for a mixed drink that consists of 50ml of \mathbf{v} , 200ml of \mathbf{c} and 10ml of \mathbf{w} .

Handwritten diagram illustrating the calculation of a mixed drink's nutrition values:

The diagram shows three vectors v , c , and w represented as columns of numbers, each multiplied by a scalar value above it:

- v (green box): $\begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \times 50$ (blue bracket)
- c (yellow box): $\begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \times 200$ (red bracket)
- w (orange box): $\begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix} \times 10$ (orange bracket)

The resulting vector is calculated as follows:

$$\begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \times 50 + \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \times 200 + \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix} \times 10 = \begin{bmatrix} 550 \\ 1000 \\ 600 \\ 100 \end{bmatrix}$$

Final result:

$$\begin{bmatrix} 550 \\ 1000 \\ 600 \\ 100 \end{bmatrix}$$

Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

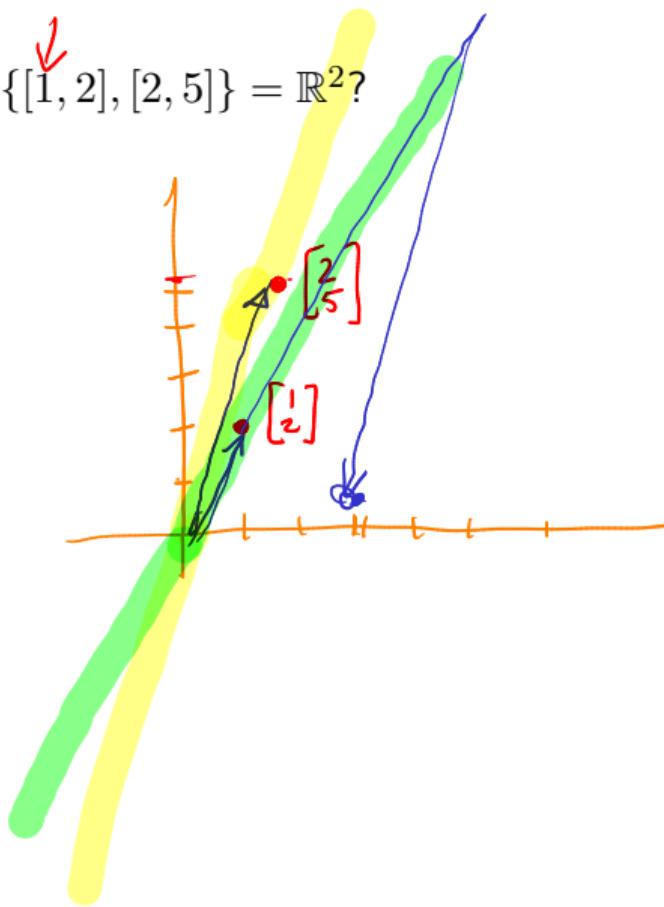
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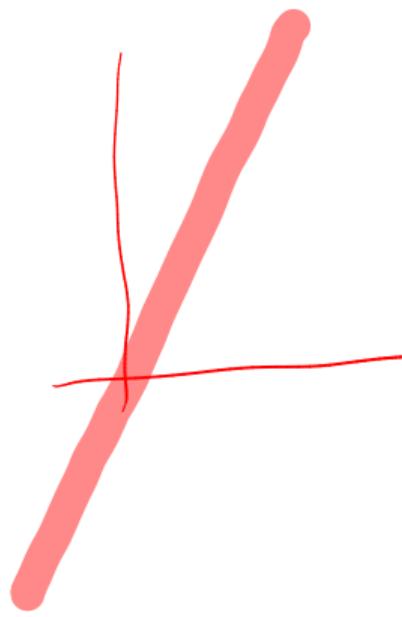
Write that result as a matrix-vector product. (The matrix should be a 4×3 matrix.)

Example 1

Is $\text{Span} \{[1, 2], [2, 5]\} = \mathbb{R}^2$?



$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ ✓
 $\text{Span} \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$



Example 2

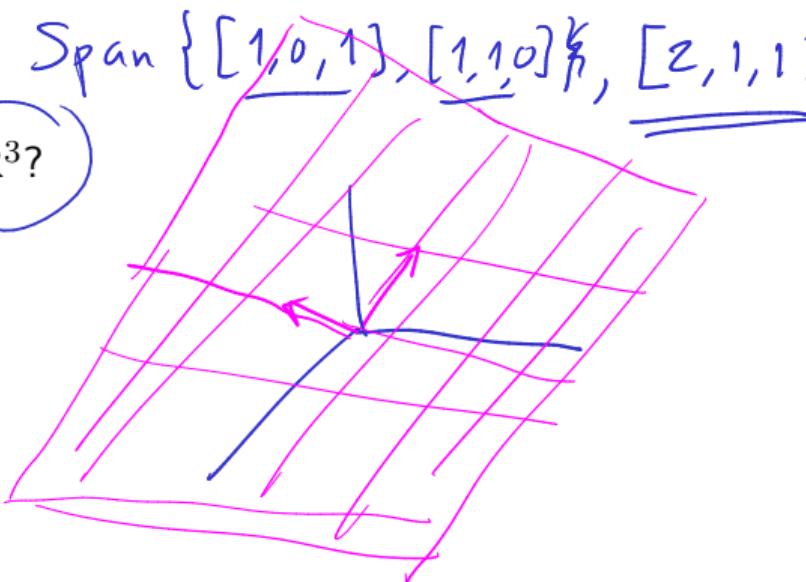
Is $\text{Span} \{[1, 0, 1], [1, 1, 0], [2, 3, 4]\} = \mathbb{R}^3$?

$\text{Span} \{[1, 0, 1], [1, 1, 0], [2, 3, 4]\}$

Can we write

$$[2, 1, 3, 4] = \underline{\quad}$$

as a linear combination of
 $[1, 0, 1]$ & $[1, 1, 0]$



$$4[1, 0, 1] + 3[1, 1, 0] = [7, 3, \underline{4}]$$

Example 3

? $\in \text{Span}\left\{\begin{bmatrix}1, 0, 1\\1, 1, 0\end{bmatrix}\right\}$

Is $\text{Span} \{[1, 0, 1], [1, 1, 0], [4, 2, 2]\} \neq \mathbb{R}^3?$ No

Elements in a vector

Vectors
- scale (scalar multiplication)
- $\vec{v} + \vec{r}$

elements are in \mathbb{R}

- ▶ We see examples of vectors over \mathbb{R} .
- ▶ However, elements in a vector can be from other sets with appropriate properties.
(I.e., they should behave like real numbers.)
- ▶ What do we want from an element in a vector?
 - ▶ We should be able to perform addition, subtraction, multiplication, and division.
 - ▶ Operations should be commutative and associative.
 - ▶ Additive and multiplicative identity should exist.
 - ▶ Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a field.

$(a+b)+c = a+(b+c)$

\mathbb{R} $GF(2)$

A field

\mathbb{R}, \mathbb{C}

$GF(2)$...

$(GF(p))$

Definition

A set \mathbb{F} with two operations $+$ and \times (or \cdot) is a **field** iff these operations satisfy the following properties:

- ▶ (Associativity): $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ▶ (Commutativity): $a + b = b + a$ and $a \cdot b = b \cdot a$
- ▶ (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$
- ▶ (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$
- ▶ (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an element a^{-1} such that $a \cdot a^{-1} = 1$
- ▶ (Distributive): $a \cdot (b + c) = a \cdot b + a \cdot c$

Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$. I.e., it is a “bit” field.

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- We define $b_1 + b_2$ to be XOR.

$$0 + 0 = 0$$

$$0 + 1 = 1 + 0 = 1$$

$$1 + 1 = 0$$

- We define $b_1 \cdot b_2$ to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$

$$1 \cdot 1 = 1$$

You can check that $GF(2)$ satisfies the axioms of fields.

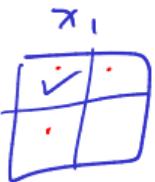
2×2 Lights out



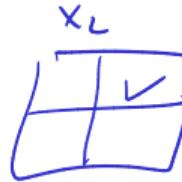
state:
 $[1, 0, 0, 0]$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



$[1, 1, 1, 0]$



$[1, 1, 0, 1]$

x_3
 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
 $[1, 0, 1, 1]$

x_4
 $\begin{bmatrix} & \\ & \checkmark \end{bmatrix}$
 $[0, 1, 1, 1]$

Parity-check code

Parity-check code

From message $\mathbf{a} = [a_1, a_2, a_3, a_4]$, we compute (in $GF(2)$) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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Now our encoded message becomes

$$[a_1, a_2, a_3, a_4, \underline{a_5}],$$

where $a_5 = b = a_1 + a_2 + a_3 + a_4$. It can detect a single-bit error.

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What can we say about the condition on a_5 ?

It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

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Now, what is the set of all possible codewords?

Hamming code

You can detect and correct more errors with Hamming codes. In this version called a [7, 4] Hamming code, you encode 4-bit data $[a_1, a_2, a_3, a_4]$ into a 7-bit codeword $[p_1, p_2, a_1, p_3, a_2, a_3, a_4]$. Using the formula:

$$p_1 = a_1 + a_2 + a_4$$

$$p_2 = a_1 + a_3 + a_4$$

$$p_3 = a_2 + a_3 + a_4$$

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Let's see how this works.

4 cols

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}^4 = \begin{bmatrix} p_1 \\ p_2 \\ a_1 \\ p_3 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}^7$$

rows

cols

row

codeword

$$P_1 = a_1 + a_2 + a_4$$
$$P_2 = a_1 + a_3 + a_4$$
$$P_3 = a_2 + a_3 + a_4$$

Hamming code (encoding as matrix multiplication)

Hx

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}^4$$

Parity check

(P₁) P₂ (a₁) P₃ (a₂) a₃ (a₄)

Suppose that we are given [d₁, d₂, d₃, d₄, d₅, d₆, d₇] Let

$$s_1 = d_1 + d_3 + d_5 + d_7$$

$$\boxed{s_2} = d_2 + d_3 + d_6 + d_7$$

$$s_3 = d_4 + d_5 + d_6 + d_7$$

$$a_1 + a_2 + a_4$$

$$\boxed{P_1 + a_1 + a_2 + a_4 = 0}$$

$$= \boxed{P_2 + a_1 + a_3 + a_4 = 0}$$

$$= \boxed{P_3 + a_2 + a_3 + a_4 = 0}$$

Given a codewords $w = [c_1, c_2, \dots, c_7]$, if we compute s_1, s_2, s_3 , we would get all zero's.

Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$\begin{aligned}s_1 &= \underline{d_1 + d_3 + d_5 + d_7} \\s_2 &= d_2 + d_3 + d_6 + d_7 \\s_3 &= d_4 + d_5 + d_6 + d_7\end{aligned}$$

Given a codewords $\mathbf{w} = [c_1, c_2, \dots, c_7]$, if we compute s_1, s_2, s_3 , we would get all zero's.

What if there is an error? Let's try.

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right] \left[\begin{array}{c} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{array} \right] = \left[\begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} \right]$$

Hamming code (parity check as matrix multiplication)

Codewords from Hamming code

Turning the formula for p_1, p_2, p_3 around, we have 3 homogeneous linear equations:

$$\begin{aligned}d_1 + d_3 + d_5 + d_7 &= 0 \\d_2 + d_3 + d_6 + d_7 &= 0 \\d_4 + d_5 + d_6 + d_7 &= 0\end{aligned}$$

and again the set of all possible codewords \mathcal{W} forms a vector space over $GF(2)$.

Can you solve 2×2 Lights out?

Let $\mathbf{u}_1 = [1, 1, 1, 0]$, $\mathbf{u}_2 = [1, 1, 0, 1]$, $\mathbf{u}_3 = [1, 0, 1, 1]$, and $\mathbf{u}_4 = [0, 1, 1, 1]$.

Given $\mathbf{b} = [b_1, b_2, b_3, b_4]$, can you always find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}$$

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$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}$$

Same question: Is $\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\mathbf{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}$$

Can you solve 2×2 Lights out?

Since

$$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\},$$

and

Can you solve 2×2 Lights out?

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Since

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and

$$\text{Span } \{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\} = GF(2)^4,$$

what can we say about $\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$?

Generators

Definition

Let \mathcal{V} be a set of vectors. Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

If $\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} = \mathcal{V}$, we say that

- ▶ $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a **generating set** for \mathcal{V}
- ▶ vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **generators** for \mathcal{V}

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Examples

Standard generators

Note that $\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$ are generators for $GF(2)^4$.
Why?

Standard generators

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Why?

They are called **standard generators** for $GF(2)^4$, written as e_1, e_2, e_3, e_4 .

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They are called **standard generators** for $GF(2)^4$, written as e_1, e_2, e_3, e_4 .

For \mathbb{R}^n , we also have $[1, 0, 0, \dots, 0], [0, 1, 0, \dots, 0], [0, 0, 1, \dots, 0], \dots, [0, 0, 0, \dots, 1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are generators for \mathcal{V} , and for each i ,

$$\mathbf{v}_i \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\},$$

we have that $\mathcal{V} \subseteq \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$.

Adding a vector into a span

Lemma 2

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{v} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$, then

$$\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

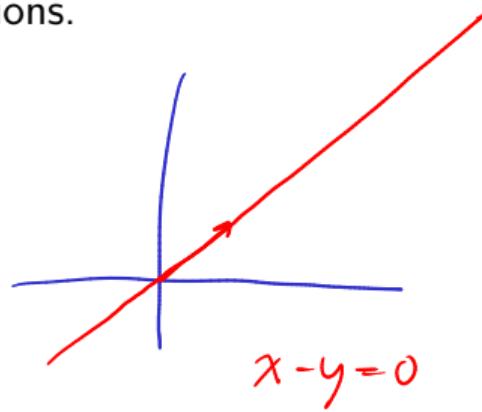
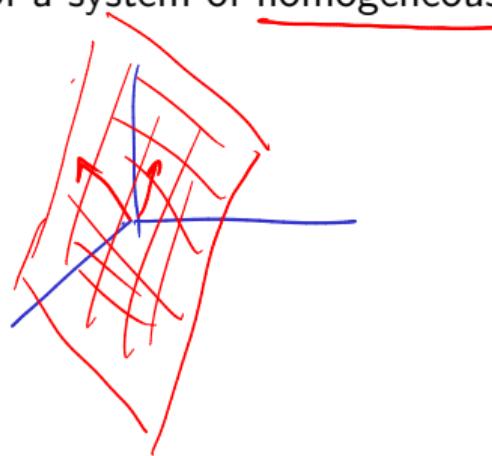
Geometry of spans: in \mathbb{R}^2

Geometry of spans: in \mathbb{R}^3

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.



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What are common properties of these geometric objects?

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- ▶ they pass through the origin,
- ▶ if vector \mathbf{u} is in the objects, $\alpha\mathbf{u}$ for any scalar α is also in the objects, and
- ▶ if \mathbf{u} and \mathbf{v} are in the objects, $\mathbf{u} + \mathbf{v}$ is also in the objects.

Vector spaces

\mathbb{R}

Definition

A set \mathcal{V} of vectors over \mathbb{F} is a vector space iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\mathbf{u} \in \mathcal{V}$,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- (V3) for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Span of vectors is a vector space

Consider n -vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$,

$$\text{Span} \{ \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m \}$$

is a vector space.

is a vector space.

(V1) $0 \in A$

check

(V2) $\boxed{\forall} u \in A, t\alpha, \alpha u \in A$.

check

(V3) $\boxed{\forall} u, v \in A, u+v \in A$

Span of vectors is a vector space

V((V3)) . . .

Consider n -vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$,

Span $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$

is a vector space.

Let's check if properties V1, V2, and V3 are satisfied.

✓(V1) $0 \in \text{Span}$ $\overset{\text{Def}}{0}$ is a linear $0 = 0 \cdot \bar{\mathbf{u}}_1 + 0 \cdot \bar{\mathbf{u}}_2 + \dots + 0 \cdot \bar{\mathbf{u}}_m$

✓(V2) Let $\bar{v} \in \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$. There exists $\alpha_1, \alpha_2, \dots, \alpha_m$ s.t
 $\alpha_1 \cdot \mathbf{u}_1 + \alpha_2 \cdot \mathbf{u}_2 + \dots + \alpha_m \cdot \mathbf{u}_m = v$

Then $\beta \cdot \bar{v} = \beta \alpha_1(\mathbf{u}_1) + \beta \alpha_2(\mathbf{u}_2) + \dots + \beta \alpha_m(\mathbf{u}_m)$ \leftarrow is also
a linear comb of $\mathbf{u}_1, \dots, \mathbf{u}_m \Rightarrow \beta \bar{v} \in \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$

Solutions to homogeneous linear equations is a vector space $\frac{x+y}{2} + 2z = 0$

Consider a set \mathcal{S} of all n -vectors in the form $[x_1, x_2, \dots, x_n]$ where

$$\begin{aligned} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= 0 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n &= 0 \\ \vdots &= \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n &= 0 \end{aligned}$$

Let's check if properties V1, V2, and V3 are satisfied.

✓
 $\bar{x} = [x_1, x_2, \dots, x_n]$

$$\boxed{\bar{a}_1 \cdot \bar{x} = 0}$$

$$\bar{a}_1 = [a_{11}, a_{12}, a_{13}, \dots, a_{1n}]$$

Dot product

Definition

For n -vectors $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$, the **dot product** of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} \cdot \mathbf{v}$, is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

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Using dot products, the previous set \mathcal{S} can be written as

$$\underline{y}, \underline{x} \in \mathcal{S}$$
$$\alpha x$$

$$\{x \in \mathbb{R}^n : \underline{a_1 \cdot x} = 0, \underline{a_2 \cdot x} = 0, \dots, \underline{a_m \cdot x} = 0\}$$

and we know that \mathcal{S} is a vector space.

$$(V_2) \quad \bar{a}_1 \cdot \alpha \bar{x} = \alpha (\bar{a}_1 \cdot \bar{x}) = \alpha 0$$

$$(V_3) \quad \bar{a}_1 (\bar{x} + \bar{y}) = \bar{a}_1 \bar{x} + \bar{a}_1 \cdot \bar{y} = 0 + 0 = 0$$

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

Translation

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- ▶ *Question:* Is \mathcal{A} a vector space?
- ▶ We also write it as $\mathbf{a} + \mathcal{V}$.

Affine spaces

Definition

If a is a vector and \mathcal{V} is a vector space, then

$$a + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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Definition

The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **affine hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

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Writing an affine space using a span

Writing an affine space using a span

An affine space

An affine space passing through $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

Non-homogeneous linear system

Two linear systems:

$$a_1 \cdot x = b_1$$

$$a_2 \cdot x = b_2$$

$$\vdots$$

$$a_m \cdot x = b_m$$

$$a_1 \cdot x = 0$$

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What can you say about the solution sets of these two related linear systems?

$\mathbf{0}$ is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

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This implies that $\mathbf{u}_1 - \mathbf{u}_2$ is a solution to the homogeneous linear system.

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\mathbf{u} \in \mathcal{W}$ be one of the solutions, we have that

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In other words,

$$\begin{aligned}\mathcal{W} &= \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} \\ &= \mathbf{u} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\},\end{aligned}$$

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i.e., \mathcal{W} is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.