

01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (I)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array} \right]$$

Vector spaces related to a matrix

Consider an m -by- n matrix A over \mathbb{R} .

We can view A as

- ▶ n columns of m -vectors: $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$
- ▶ m rows of n -vectors: $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$

When we have a set of vectors, recall that its span forms a vector space.

We have

- ▶ Column space: $\text{Span} \{ \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n \} \subseteq \mathbb{R}^m$
- ▶ Row space: $\text{Span} \{ \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m \} \subseteq \mathbb{R}^n$

Subspaces

Definition

Let \mathcal{V} and \mathcal{W} be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that \mathcal{V} is a **subspace** of \mathcal{W} .

Examples:

- ▶ $\text{Span} \{[1, 1]\}$ is a subspace of \mathbb{R}^2 .
- ▶ $\text{Span} \{[1, 0, 0], [0, 1, 1]\}$ is a subspace of \mathbb{R}^3 .
- ▶ $\text{Span} \{[1, 0, 0], [0, 1, 1], [1, 1, 2]\}$ is a subspace of \mathbb{R}^3 .

Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

- Column space:

$$\mathcal{R}(A) = \{\alpha_1[1, 0] + \alpha_2[2, 1] + \alpha_3[4, 3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} = \mathbb{R}^2.$$

Note that: $\dim \mathcal{R}(A) = 2$

- Row space:

$$\mathcal{R}(A^T) = \{\alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

Note that: $\dim \mathcal{R}(A^T) = 2$

Example 1 (cont.)

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Is there any other way to obtain vector spaces from A ?

We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$A\mathbf{x} = \mathbf{0}.$$

In this case, we have

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The set of solutions $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A , we can look at the matrix-vector product Ax .
Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Four fundamental subspaces

Four fundamental subspaces

Given an m -by- n matrix A , we have the following subspaces

- ▶ The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$)
- ▶ The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
- ▶ The nullspace of A

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- ▶ The left nullspace of A

$$\mathcal{N}(A^T) = \{\mathbf{y} \mid A^T\mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

Linearly independent rows

Ranks

Definition

Consider an m -by- n matrix A .

- ▶ The **row rank** of A is the maximum number of linearly independent rows of A .
- ▶ The **column rank** of A is the maximum number of linearly independent columns of A .

Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

Row rank = Column rank

Theorem 1

For any matrix A , its row rank equals its column rank.

We will prove this theorem next time.