01204211 Discrete Mathematics Lecture 9a: Spans and Vector Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a linear combination of u_1, \ldots, u_m .

Review: Span

Definition

A set of all linear combination of vectors u_1, u_2, \dots, u_m is called the **span** of that set of vectors.

It is denoted by $\mathrm{Span}\{u_1,u_2,\ldots,u_m\}$.

Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$oldsymbol{v} = egin{bmatrix} 100 \ 50 \ 0 \ 0 \end{bmatrix} \quad oldsymbol{c} = egin{bmatrix} 0 \ 0 \ 300 \ 0 \end{bmatrix} \quad oldsymbol{w} = egin{bmatrix} 50 \ 0 \ 50 \ 10 \end{bmatrix}$$

Write down the nutritions for a mixed drink that consists of 50ml of v, 200ml of c and 10ml of w.

Write that result as a matrix-vector product. (The matrix should be a 4×3 matrix.)

Example 1

Is Span $\{[1,2],[2,5]\} = \mathbb{R}^2$?

Example 2

Is Span $\{[1,0,1],[1,1,0],[2,3,4]\} = \mathbb{R}^3$?

Example 3

Is Span $\{[1,0,1],[1,1,0],[4,2,2]\} = \mathbb{R}^3$?

Elements in a vector

- ightharpoonup We see examples of vectors over \mathbb{R} .
- ► However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
 - ▶ We should be able to perform addition, subtraction, multiplication, and division.
 - Operations should be commutative and associative.
 - Additive and multiplicative identity should exist.
 - Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a **field**.

A field

Definition

A set $\mathbb F$ with two operations + and \times (or \cdot) is a **field** iff these operations satisfy the following properties:

- (Associativity): (a+b)+c=a+(b+c) and $(a\cdot b)\cdot c=a\cdot (b\cdot c)$
- ▶ (Commutativity): a + b = b + a and $a \cdot b = b \cdot a$
- ▶ (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that a+0=a and $a\cdot 1=a$
- ▶ (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that a + (-a) = 0
- ▶ (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an alement a^{-1} such that $a \cdot a^{-1} = 1$
- ▶ (Distributive): $a \cdot (b+c) = a \cdot b + a \cdot c$

Another useful field: GF(2)

 $GF(2) = \{0, 1\}$. I.e., it is a "bit" field.

What are + and \cdot in GF(2)?

• We define $b_1 + b_2$ to be XOR.

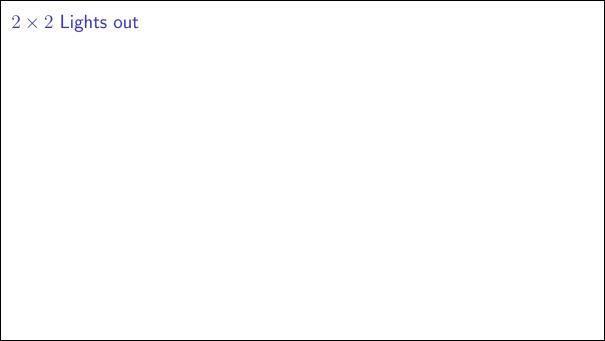
$$0+0=0 \\ 0+1=1+0=1 \\ 1+1=0$$

▶ We define $b_1 \cdot b_2$ to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$

 $1 \cdot 1 = 1$

You can check that GF(2) satisfies the axioms of fields.



Parity-check code

From message $a = [a_1, a_2, a_3, a_4]$, we compute (in GF(2)) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

Now our encoded message becomes

$$[a_1, a_2, a_3, a_4, a_5],$$

where $a_5=b=a_1+a_2+a_3+a_4$. It can detects a single-bit error.

What can we say about the condition on a_5 ? It is in fact a homogeneous linear equation (in GF(2)):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

Now, what is the set of all possible codewords?

Hamming code

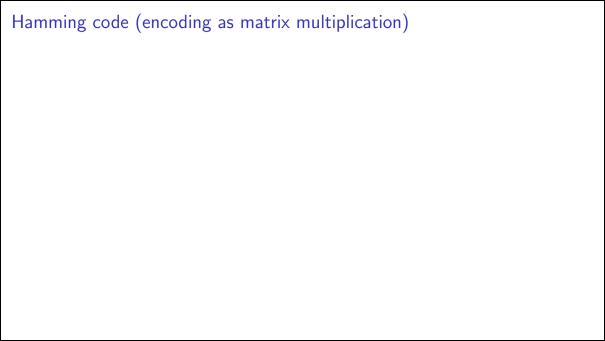
You can detect and correct more errors with Hamming codes. In this version called a [7,4] Hamming code, you encode 4-bit data $[a_1,a_2,a_3,a_4]$ into a 7-bit codeword $[p_1,p_2,a_1,p_3,a_2,a_3,a_4]$. Using the formula:

$$p_1 = a_1 + a_2 + a_4$$

$$p_2 = a_1 + a_3 + a_4$$

$$p_3 = a_2 + a_3 + a_4$$

Let's see how this works.



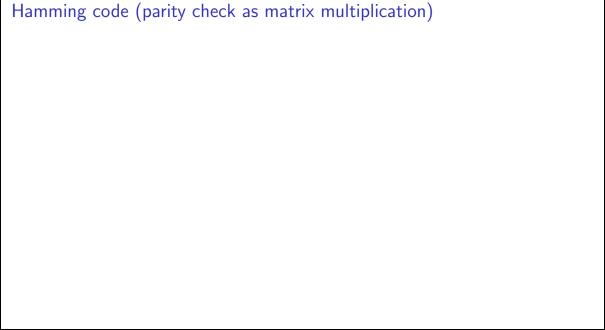
Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$s_1 = d_1 + d_3 + d_5 + d_7
 s_2 = d_2 + d_3 + d_6 + d_7
 s_3 = d_4 + d_5 + d_6 + d_7$$

Given a codewords ${m w}=[c_1,c_2,\ldots,c_7]$, if we compute s_1,s_2,s_3 , we would get all zero's.

What if there is an error? Let's try.



Codewords from Hamming code

Turning the formula for p_1, p_2, p_3 around, we have 3 homogeneous linear equations:

$$d_1 + d_3 + d_5 + d_7 = 0$$

$$d_2 + d_3 + d_6 + d_7 = 0$$

$$d_4 + d_5 + d_6 + d_7 = 0$$

and again the set of all possible codewords $\mathcal W$ forms a vector space over GF(2).

Can you solve 2×2 Lights out?

Let $u_1 = [1, 1, 1, 0]$, $u_2 = [1, 1, 0, 1]$, $u_3 = [1, 0, 1, 1]$, and $u_4 = [0, 1, 1, 1]$.

Given ${\pmb b}=[b_1,b_2,b_3,b_4]$, can you always find $a_1,a_2,a_3,a_4\in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Same question: Is Span $\{u_1, u_2, u_3, u_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\mathbf{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot u_1 + a_2 \cdot u_2 + a_3 \cdot u_3 + a_4 \cdot u_4 = b$$
?

Can you solve 2×2 Lights out?

 $[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1] \in \text{Span } \{u_1,u_2,u_3,u_4\},\$

and

Span $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\} = GF(2)^4$, what can we say about Span $\{u_1, u_2, u_3, u_4\}$?

Generators

Definition

Let $\mathcal V$ be a set of vectors. Consider vectors $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_n.$

If $\mathrm{Span}\ \{oldsymbol{u}_1,oldsymbol{u}_2,\ldots,oldsymbol{u}_n\}=\mathcal{V}$, we say that

- $lackbox{ } \{oldsymbol{u}_1,oldsymbol{u}_2,\ldots,oldsymbol{u}_n\}$ is a **generating set** for $\mathcal V$
- lacktriangle vectors $oldsymbol{u}_1, oldsymbol{u}_2 \dots, oldsymbol{u}_n$ are **generators** for $\mathcal V$

Examples

Standard generators

Note that $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ are generators for $GF(2)^4$. Why?

They are called standard generators for $GF(2)^4$, written as e_1, e_2, e_3, e_4 .

For \mathbb{R}^n , we also have $[1, 0, 0, \dots, 0], [0, 1, 0, \dots, 0], [0, 0, 1, \dots, 0], \dots, [0, 0, 0, \dots, 1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors u_1, u_2, \ldots, u_n . If v_1, v_2, \ldots, v_k are generators for $\mathcal V$, and for each i,

$$v_i \in \mathrm{Span} \ \{u_1, u_2, \ldots, u_n\},$$

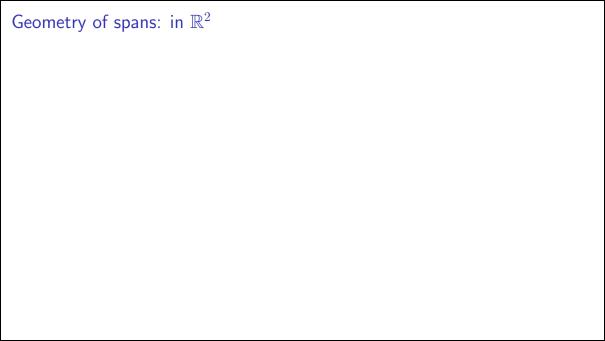
we have that $V \subseteq \operatorname{Span} \{u_1, u_2, \dots, u_n\}$.

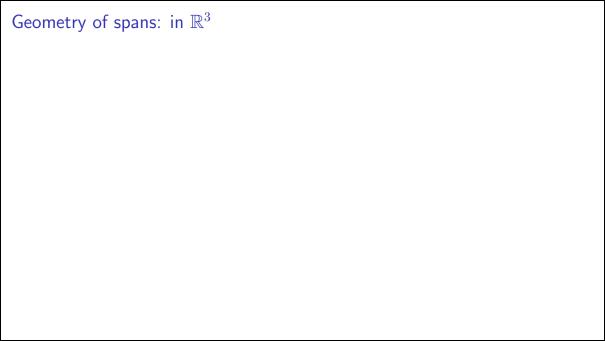
Adding a vector into a span

Lemma 2

Consider vectors u_1, u_2, \ldots, u_n . If $v \in \mathrm{Span}\ \{u_1, u_2, \ldots, u_n\}$, then

Span
$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n, \boldsymbol{v}\} = \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$$





Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- they pass through the origin,
- lacktriangle if vector $m{u}$ is in the objects, $lpha m{u}$ for any scalar lpha is also in the objects, and
- lacktriangle if $m{u}$ and $m{v}$ are in the objects, $m{u}+m{v}$ is also in the objects.

Vector spaces

Definition

A set $\mathcal V$ of vectors over $\mathbb F$ is a **vector space** iff

- ightharpoonup (V1) $\mathbf{0} \in \mathcal{V}$,
- ightharpoonup (V2) for any $u \in \mathcal{V}$,

$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

 $u + v \in \mathcal{V}$.

for any $\alpha \in \mathbb{F}$, and

ightharpoonup (V3) for any $oldsymbol{u},oldsymbol{v}\in\mathcal{V}$,

Span of vectors is a vector space

Consider *n*-vectors u_1, u_2, \ldots, u_m ,

Span
$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m\}$$

is a vector space.

Let's check if properties V1, V2, and V3 are satisfied.

Solutions to homogeneous linear equations is a vector space

Consider a set S of all n-vectors in the form $[x_1, x_2, \ldots, x_n]$ where

$$\begin{array}{rcl} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \dots + a_{1n} \cdot x_n & = & 0 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2n} \cdot x_n & = & 0 \\ & \vdots & = & \vdots \end{array}$$

$$a_{m1}x \cdot_1 + a_{m2} \cdot x_2 + \dots + a_{mn} \cdot x_n = 0$$

Let's check if properties V1, V2, and V3 are satisfied.

Dot product

Definition

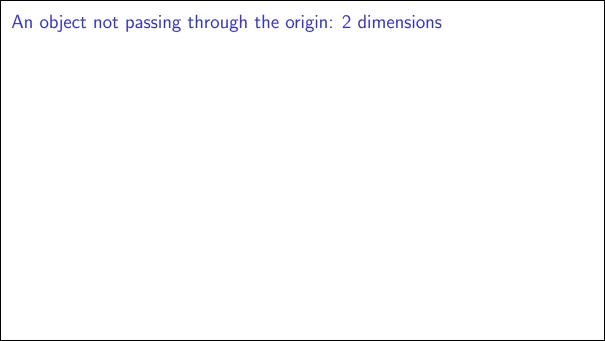
For n-vectors $u = [u_1, u_2, \dots, u_n]$ and $v = [v_1, v_2, \dots, v_n]$, the **dot product** of u and v, denoted by $u \cdot v$, is

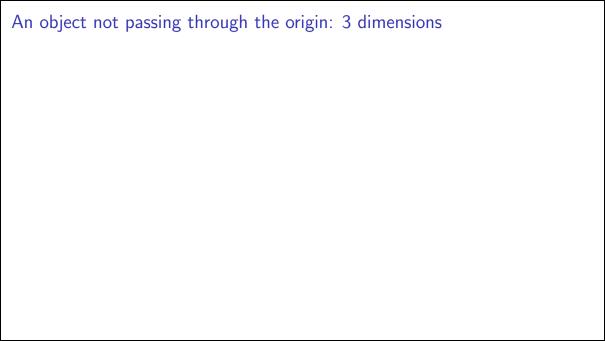
$$u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

Using dot products, the previous set ${\mathcal S}$ can be written as

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} = 0, \boldsymbol{a}_2 \cdot \boldsymbol{x} = 0, \dots, \boldsymbol{a}_m \cdot \boldsymbol{x} = 0\}$$

and we know that ${\cal S}$ is a vector space.





Translation

If we have a line or a plane passing through a vector a, but not through the origin, how can we represent it?

- ► Translate the object so that it passes through the origin.
- ightharpoonup We obtain a vector space \mathcal{V} .
- ightharpoonup Then we translate it back so that it passes through a.
- ► We get the set

$$\mathcal{A} = \{ \boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V} \}$$

- ► Question: Is A a vector space?
- ightharpoonup We also write it as a + V.

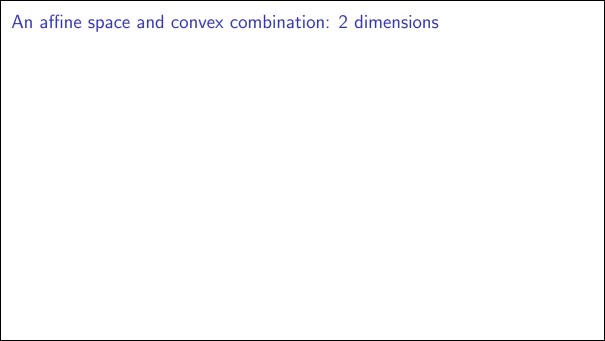
Affine spaces

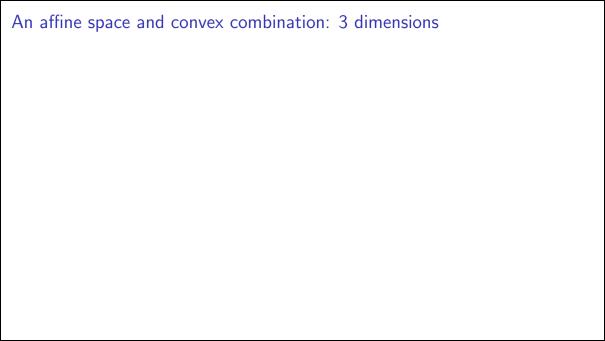
Definition

If a is a vector and $\mathcal V$ is a vector space, then

$$a + \mathcal{V}$$

is an affine space.





Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of u_1, \ldots, u_m .

Definition

The set of all affine combinations of vectors u_1, u_2, \ldots, u_m is called the **affine hull** of u_1, u_2, \ldots, u_m .

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors u_1, u_2, \ldots, u_m , we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of u_1, \ldots, u_m .

Definition

The set of all convex combinations of vectors u_1, u_2, \ldots, u_m is called the **convex hull** of u_1, u_2, \ldots, u_m .

Writing an affine space using a span

An affine space

An affine space passing through $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_n$ is

$$u_1 + \text{Span } \{u_2 - u_1, u_3 - u_1, \dots, u_n - u_1\}.$$

Non-homogeneous linear system

Two linear systems:

What can you say about the solution sets of these two related linear systems? $\mathbf{0}$ is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $m{u}_1$ and $m{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$a_i u_1 - a_i u_2 = b_i - b_i = 0 = a_i (u_1 - u_2).$$

This implies that u_1-u_2 is a solution to the homogeneous linear system.

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{ \boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m \},$$

and let $u \in \mathcal{W}$ be one of the solutions, we have that

$$\{v - u : v \in \mathcal{W}\}$$

is a vector space, because

$$\{v - u : v \in W\} = \{x : a_i x = 0, \text{ for } 1 < i < m\}$$

In other words.

In other words,
$${\cal W}=m{u}+\{m{v}-m{u}:m{v}\in{\cal W}\}\ =m{u}+\{m{x}:m{a}_im{x}=0, \ ext{for}\ 1\leq i\leq m\},$$

i.e., \mathcal{W} is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.