01204211 Discrete Mathematics
Lecture 5c: Counting 2

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Listing all subsets[1]

- From the previous lecture, we know that a set with n elements has 2^n subsets.
- Let's try to enumerate them.
- As an example, consider set $\{a, b, c\}$ and its subsets.
- There are many ways of listing all 8 subsets.
  - $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$, $\{b\}$, $\{b, c\}$, $\{c\}$
    Note that we treat each subset as a word in a dictionary and use the dictionary order.
  - $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$
    In this case, we order by their cardinalities, then use dictionary ordering for subsets with the same numbers of elements.

[1] This section follows section 1.3 from [LPV].

A different representation (1)

- There is a different representation for subsets which is particularly useful when listing subsets.
- To represent a subset of $A = \{a, b, c\}$, we consider each element of A one-by-one in some fixed order. If that element is in the subset, we write down 1, if it is not we write down 0.
- For examples:
  - $\{a, c\}$ is represented as: 101
  - $\{a\}$ is represented as: 100
  - $\{b, c\}$ is represented as: 011
  - $\{\}$ is represented as: 000

A different representation (2)

- Note that we represent a subset as a string with 0's and 1's. You may recall that these strings can be considered as binary numbers.
- Thus, we can associate the numerical values of the representations with the subsets:
  - $\{a, c\}$ is rep. as: $101_2 = 5$, $\{a\}$ is rep. as: $100_2 = 4$
  - $\{b, c\}$ is rep. as: $011_2 = 3$, $\{\}$ is rep. as: $000_2 = 0$
- Also, this representation can be considered backwards, i.e., if we start with an integer 6, we can write down its binary representation: 110_2 and turns it into a subset $\{a, b\}$.

A correspondence

Let's see a full list of correspondence between $\{0, 1, 2, \ldots, 7\}$ and subsets of $\{a, b, c\}$.

- $0 \leftrightarrow 000_2 \leftrightarrow \{\}$
- $1 \leftrightarrow 001_2 \leftrightarrow \{c\}$
- $2 \leftrightarrow 010_2 \leftrightarrow \{b\}$
- $3 \leftrightarrow 011_2 \leftrightarrow \{c, b\}$
- $4 \leftrightarrow 100_2 \leftrightarrow \{a\}$
- $5 \leftrightarrow 101_2 \leftrightarrow \{a, c\}$
- $6 \leftrightarrow 110_2 \leftrightarrow \{a, b\}$
- $7 \leftrightarrow 111_2 \leftrightarrow \{a, b, c\}$

Do you notice anything interesting?

A general case

Similarly, we can describe a representation for each subset of a set A with n elements. As we consider each element a of A, we put 1 if $a \in A$ and put 0 if $a \notin A$.
Each subset is represented uniquely as a string of 0 and 1 of length n. Also, each string corresponds to only one subset. Then, we can conclude that the number of subsets equal the number of bit strings of length n.
How many bit strings of length n are there?

There are 2^n bit strings; hence, the number of subsets is also 2^n. This is another proof of the following theorem:

Theorem: The number of subsets of a set with n elements is 2^n.

Two proofs

Why do we need two proofs of the same statement?
Really, it does not make a statement stronger, truer, "more" correct. But each proof usually reveals additional facts related to the statement.

- The first proof considers a procedure for constructing subsets.
- The second proof introduces a nice technique for counting. I.e., instead of counting subsets directly, we show that we have a "special" correspondence between subsets and binary numbers, and then just count the numbers.

A bijection

What is so special about this correspondence?

- For each number, there is exactly **one** subset that corresponds to it.
- For each subset, there is exactly **one** number that it corresponds to.

With these two properties, we can conclude that both sets have the same cardinality.

> This type of correspondence is called a **one-to-one correspondence** or **bijection**.

Sequences of choices

Previously, when we want to count the number of bit strings of length n, we use this argument:

> Suppose that to select an object, you have to make k decisions. The first decision has n_1 choices, the second decision has n_2 choices, and so on. More precisely, for $1 \leq i \leq k$, the i-th decision has n_i choices. Then the number of ways you can select an object is $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$.

Example 1

A car license number consists of two English letters and one number from 1 to 9999. How many possible license numbers are there?

Example 2

10 students stand in a line. You want to give them ice cream. There are 4 flavours, but you don't want to give the same flavour to any consecutive students. In how many ways can you give out the ice cream to these students?

Permutations

Counting permutations: an example

We want to count the number of permutations. Let's try with a small example: permutations of set $\{a, b, c\}$.

Counting permutations

Number of permutations

We have proved this theorem.

Theorem: The number of permutations of a set with n elements is $n!$.