01204211 Discrete Mathematics
Lecture 8b: Vectors

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What is a vector?

You can think of a **vector** as an "ordered" list of elements (which are typically numbers). For example:

- $[1, 2, 5, 20]$
- $[0, 0, 1, 1, 0, 0, 0, 1]$

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You can also view a vector as a **function**, e.g., you can view $\boldsymbol{u} = [1, 2, 5, 20]$ as a function $\boldsymbol{u}$ that maps

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Remark: Mathematically, a vector is an element of a vector space. We will understand this more later.

What can be represented as a vector?

Viewing vectors: vectors in $\mathbb{R}^2$

Viewing vectors: vectors in $\mathbb{R}^3$

n-vectors over $\mathbb{R}$

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- When every element of a vector is from some set, we say that it is a vector **over** that set. For example, $[10, 20, 500, 4]$ is a 4-vector over $\mathbb{R}$.

Vector operations

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- We will look at the operations we do to vectors to obtain their linear combinations.

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- We will look at the operations we do to vectors to obtain their linear combinations.
- The operations are:
  - Vector additions
  - Scalar multiplications
- These operations motivate the definition of vector spaces.

Vector additions

Given two n-vectors

$$\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$$

and

$$\boldsymbol{v} = [v_1, v_2, \ldots, v_n],$$

we have that

$$\boldsymbol{u} + \boldsymbol{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

Vector additions, in picture

Zero vectors

A zero n-vector $\mathbf{0} = [0, 0, \ldots, 0]$ is an additive identity, i.e., for any vector $\boldsymbol{u}$,

$$\mathbf{0} + \boldsymbol{u} = \boldsymbol{u} + \mathbf{0} = \boldsymbol{u}.$$

Scalar multiplications

For a vector over $\mathbb{R}$, we refer to an element α in $\mathbb{R}$ as a scalar. For an n-vector

$$\boldsymbol{u} = [u_1, u_2, \ldots, u_n],$$

we have that

$$\alpha \cdot \boldsymbol{u} = [\alpha \cdot u_1, \alpha \cdot u_2, \ldots, \alpha \cdot u_n],$$

Scalar multiplications, in pictures

Linear combinations

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

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Examples:

A linear system with 3 variables

Give the following linear system.

$$
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

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If we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

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This becomes the problem of expressing a vector as linear combination of other vectors. I.e., given vectors

$$\boldsymbol{u}_1 = [2, 1, 4], \quad \boldsymbol{u}_2 = [4, 0, 2], \quad \boldsymbol{u}_3 = [3, 5, 3]$$

we would like to find coefficients x_1, x_2, x_3 such that

$$x_1 \cdot \boldsymbol{u}_1 + x_2 \cdot \boldsymbol{u}_2 + x_3 \cdot \boldsymbol{u}_3 = [7, 12, 10].$$

Span

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denote by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Examples:

Convex combination

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m,$$

such that $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$ and $\alpha_i \geq 0$ for all i, and vectors

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- We refer to a set with these properties as a **field**.

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- (Distributive): $a \cdot (b+c) = a \cdot b + a \cdot c$

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You can check that $GF(2)$ satisfies the axioms of fields.