

# 01204211 Discrete Mathematics

## Lecture 10b: Dimensions

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## Review: Linear combinations

### Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Review: Span

### Definition

A set of all linear combination of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **span** of that set of vectors.

It is denoted by  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ .

# Review: Vector spaces

## Definition

A set  $\mathcal{V}$  of vectors over  $\mathbb{F}$  is a **vector space** iff

- ▶ (V1)  $\mathbf{0} \in \mathcal{V}$ ,
- ▶ (V2) for any  $\mathbf{u} \in \mathcal{V}$ ,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any  $\alpha \in \mathbb{F}$ , and

- ▶ (V3) for any  $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ ,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

# Review: Linearly independence

## Definition

Vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  are **linearly independent** if no vector  $\mathbf{u}_i$  can be written as a linear combination of other vectors.

## (Another) Definition

Vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  are **linearly independent** if the only solution of equation

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

## Review: Bases

### Definition

A set of vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  is a **basis** for vector space  $\mathcal{V}$  if

- ▶  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} = \mathcal{V}$ , and
- ▶  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent.

## Lemma 1 (Superfluous Vector Lemma)

*Consider vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . If  $\mathbf{v} \in \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ , then*

$$\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\} = \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

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## Lemma 2

Consider vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . If  $\mathbf{u}_n \in \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\}$ , then

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### Lemma 3 (Unique representation)

*Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  be a basis for vector space  $\mathcal{V}$ . For any  $\mathbf{v} \in \mathcal{V}$ , there is a unique way to write  $\mathbf{v}$  as a linear combination of  $\mathbf{u}_1, \dots, \mathbf{u}_k$ .*

# Examples in $\mathbb{R}^2$ and $\mathbb{R}^3$

## Examples in $GF(2)$ - Vector spaces from graphs

## Examples in $GF(2)$ - Cycles

## Examples in $GF(2)$ - Basis

## Number of vectors in bases

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- ▶ This is not a coincident.
- ▶ In this course, we will see two proofs.
- ▶ Remark: there are vector spaces whose basis has infinite size, but we are not dealing with those vector spaces in this course.



## Theorem 4 (Main Theorem)

*If  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_m$  are bases for vector space  $\mathcal{W}$ , then  $n = m$ .*

## Exchange Lemma

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### Lemma 5 (Simplified Exchange Lemma)

*Consider a set of vectors  $S$  and let  $z$  be a non-zero vector in  $\text{Span } S$ . There is a vector  $w \in S$  such that  $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$ .*

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### Lemma 6 (Exchange Lemma)

*Consider a set of vectors  $S$  and its subset  $A$ . Let  $z$  be a non-zero vector in  $\text{Span } S$  such that  $A \cup \{z\}$  is linearly independent. There is a vector  $w \in S - A$  such that  $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$ .*

## Lemma 7 (Morphing Lemma)

*If a set of vectors  $S$  spans a vector space  $\mathcal{W}$  and  $B$  is a linearly independent set of vectors in  $\mathcal{W}$ , then  $|B| \leq |S|$ .*

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### Proof.

Let  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ . We show how to construct  $S_1, \dots, S_n$  such that for each  $i$ ,  $|S_i| = |S|$ ,  $\text{Span } S_i = \text{Span } S$ , and

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We then let  $S_i = S_{i-1} \cup \{\mathbf{u}_i\} - \{\mathbf{w}\}$ . (You can check that  $S_i$  has the properties as claimed above.)

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## Morphing Lemma $\Rightarrow$ Main Theorem

### Theorem 8 (Main Theorem)

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Therefore,  $n = m$ . □

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How can we prove the full lemma?

# Dimensions

## Definition

The **dimension** of a vector space  $\mathcal{V}$  is the size of its basis.  
The dimension of  $\mathcal{V}$  is written as  $\dim \mathcal{V}$ .