01204211 Discrete Mathematics
Lecture 2a: Inference rules

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How to prove a mathematical statement?

This lecture covers two fundamental concepts in mathematical proofs:

- Proofs by exhaustion
- Inference rules[1]

[1] The materials on inference rules are from [Rosen].

De Morgan's Laws

Given propositions P and Q, these are a very useful logical equivalences (referred to as the De Morgan's Laws).

- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

(Note that $\neg$ takes precedence over $\vee$ or $\wedge$.)

How can we prove that the first statement is true?

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(Note that $\neg$ takes precedence over $\vee$ or $\wedge$.)

How can we prove that the first statement is true?
In this case, since there are not too many cases to consider, we can enumerate all the possibilities to show that the proposition is true.

Proof by exhaustion

For any proposition P and Q, $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

Proof.

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P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q \wedge \neg P$
T	T			
T	F			
F	T			
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Note that for all possible truth values of P and Q, $\neg(P \vee Q)$ equals $\neg P \wedge \neg Q$. Thus, the statement is true. $\square$

Quick check 1

Prove the following statement by exhaustion.

For any proposition P and Q, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

Quick check 2

Prove the following statement by exhaustion.

> I have 2 pairs of socks in 2 colors: black and white. If I pick any 3 socks, I will have at least a pair of socks of the same color.

Quick check 2

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This is clearly a brute force method. Sometimes, even in small cases, proofs by exhaustion can be very tedious and error-prone.

Logical deduction (1)

Consider the following statements:

- It rains.
- If it rains, then the road will get wet.
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If we believe in these statements (i.e., if we believe that they are all true), is it OK to conclude that:

- It is dangerous to drive very fast.

Quick check 3

Define propositional variables representing each proposition inside these statements and write proposition forms of them.

- It rains.
- If it rains, then the road will get wet.
- If the road is wet, it will be dangerous to drive very fast.
- It is dangerous to drive very fast.

Logical deduction (2)

Using that proposition variables, our problem translate to the following.

Let's try to prove by exhaustion

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There are 3 variables. These are all possible cases.

R	W	D
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

We believe that R, $R \Rightarrow W$, and $W \Rightarrow D$ are true, and we want to conclude that D must be true.

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We believe that R, $R \Rightarrow W$, and $W \Rightarrow D$ are true, and we want to conclude that D must be true.
Proofs by exhaustion can be exhausted...

Valid arguments (1)

Very often, the statement we want to prove is in the form:

Given:

- Hypothesis 1,
- Hypothesis 2,
- ...
- Hypothesis n

Then:

- Conclusion

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Given:

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- ...
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Then:

- Conclusion

We say that the statement is **valid** if when all hypotheses are true, the conclusion must be true as well. In that case, we say that the conclusion **logically follows** from the hypotheses.

Valid arguments (2)

More precisely, to show that conclusion Q logically follows from hypotheses $P_1, P_2, \ldots, P_n$, we need to show that

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Rightarrow Q,$$

is always true, i.e., is a tautology.

An example

Consider the following argument:

- Hypotheses: P and $P \Rightarrow Q$
- Conclusion: Q

Is this a valid argument?

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- Conclusion: Q

Is this a valid argument?

It is. See the following truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$R/W/D$ again

Since we know that the previous argument is valid, maybe we can use that "small" step in our previous example.
Recall our hypotheses:

- R
- $R \Rightarrow W$
- $W \Rightarrow D$

$R/W/D$ again

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Using the same reasoning, we can say that from R and $R \Rightarrow W$, W logically follows.

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Since we know that the previous argument is valid, maybe we can use that "small" step in our previous example.
Recall our hypotheses:

- R
- $R \Rightarrow W$
- $W \Rightarrow D$

Using the same reasoning, we can say that from R and $R \Rightarrow W$, W logically follows.
Then, since we know that W is now true, and $W \Rightarrow D$, we can conclude that D must follow.

A rule of inference

The previous “small” valid step that we can use in our argument is extremely useful when making arguments. It is called *Modus ponens*, and is one of many useful rules of inference.

Modus ponens

$$\begin{array}{l} P \\ P \Rightarrow Q \\ \hline Q \end{array}$$

Other rules of inference

Addition

$$\frac{P}{P \lor Q}$$

Simplification

$$\frac{P \land Q}{P}$$

Modus tollens

$$\frac{\begin{array}{l}\neg Q \\ P \Rightarrow Q\end{array}}{\neg P}$$

Hypothetical syllogism

$$\frac{\begin{array}{l}P \Rightarrow Q \\ Q \Rightarrow R\end{array}}{P \Rightarrow R}$$

Conjuction

$$\frac{\begin{array}{l}P \\ Q\end{array}}{P \land Q}$$

Disjunctive syllogism

$$\frac{\begin{array}{l}P \lor Q \\ \neg P\end{array}}{Q}$$

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q.

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1. $P \vee R$	Hypothesis

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2. $\neg R$	Hypothesis

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Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q.

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1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q.

Steps	**Reasons**
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis

Using inference rules

Argue that $P \Rightarrow Q$, $(P \vee R)$, and $\neg R$ logically leads to the conclusion Q.

Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. P	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis
5. Q	Modus ponens using Step 3 and 4.

Other useful logical equivalences

We have discussed De Morgan's Laws, which are logical equivalences. The following logical equivalences are also useful when making valid arguments. (Notes: do not get confused with operator $\Leftrightarrow$ and notation $P \equiv Q$.)

Equivalences	Names
$\neg(\neg P) \equiv P$	Double negation law
$(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$	Distributive law
$(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$	Distributive law
$P \Rightarrow Q \equiv \neg P \vee Q$	

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

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1. $P \Rightarrow R$	Hypothesis

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Steps	**Reasons**
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	**Reasons**
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	**Reasons**
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \vee Q) \Rightarrow R$.

Steps	**Reasons**
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3
5. $(\neg P \vee R) \wedge (\neg Q \vee R)$	Conjuction of Steps 2 and 4.

Another example

Argue that $P \Rightarrow R$ and $Q \Rightarrow R$ logically leads to the conclusion $(P \lor Q) \Rightarrow R$.

Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \lor R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \lor R$	Equivalence of Step 3
5. $(\neg P \lor R) \land (\neg Q \lor R)$	Conjuction of Steps 2 and 4.
6. ... (left as homework)	