01204211 Discrete Mathematics
Lecture 8a: Integers and GCD

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Number theory: integers and divisibility

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We will cover:

- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.

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- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.
- Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

3|7 เพราะ

$7 = 3 \cdot \cancel{\tfrac{7}{3}}$

$a|b$ "มี จำนวนเต็ม k ที่ $b = ak$"

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b. On the other hand if a does not divide b, we write $a \nmid b$.

$a|b$ $\quad$ $\cancel{\tfrac{a}{b}}$ $\quad$ $\cancel{\tfrac{a}{b}}$ "$\frac{b}{a}$ เป็นจำนวนเต็ม"

(2), 4(a), 4(b)

Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

- จาก $a|b$ จะมี จน.เต็ม k_1 ที่ $b = ak_1$ (1)
- จาก $a|c$ จะมี จน.เต็ม k_2 ที่ $c = ak_2$ (2)
- พิจารณา

$$b+c = ak_1 + ak_2 \quad \text{(จาก (1) และ (2))}$$

$$= a(k_1+k_2)$$

แต่จาก k_1 และ k_2 เป็น จน.เต็ม จะได้ว่า k_1+k_2 เป็น จน.เต็มด้วย

ดังนั้น จะมี จน.เต็ม $k' = k_1+k_2$ ที่ $b+c = k'a$

นั่นคือ $a | b+c$

Examples

If $a|b$ and $a|c$, prove that $a|(b + c)$.

If $a|b$ and $b|c$, prove that $a|c$.

$b = ka$ และ $c = k'b$ โดย k, k' เป็น จน.เต็ม

จะได้ $c = k'b = \underbrace{k'k} \cdot a$

แต่ k & k' เป็น จน.เต็ม $k'k$ เป็น จนเต็ม ด้วย $a|c$.

Remainder

Defintion (remainder)

The **remainder** of the division of b with a is an integer r when there exists an integer q such that

$$b = qa + r,$$

where $0 \leq r < a$.

เศษของหาร 7 ด้วย 3 = 1

เศษของหาร 7 ด้วย -3 = 1

เศษของหาร -7 ด้วย 3 = 2

$-7 = (-3) \cdot 3 + 2$

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Examples:

We use operator `mod` to denote an operation for finding the remainder of a division.
I.e., $a \bmod b$ is the remainder of dividing a with b.

%

Examples

Let r be the remainder of the division of b by a. Assume that $c|a$ and $c|b$. Prove that $c|r$.

จากบทนิยาม มีจำนวนเต็ม q ที่

$$b = q \cdot a + r$$

โดย $0 \leq r < a$.

จัดพจน์ จะได้

$$r = b - q \cdot a$$

จาก $c|a$ และ $c|b$ จะมี k_1 ที่ $ck_1 = a$ และ k_2 ที่ $ck_2 = b$

แทนค่า k_1, k_2

$$r = ck_2 - q \cdot ck_1$$
$$= c \cdot (k_2 - qk_1)$$

More examples

For every integer a, $a - 1 | a^2 - 1$.

Primes

1

Definition (primes)

- An integer $p > 1$ is a **prime** if its divisors are only p, $-p$, 1, and -1.
- If an integer $n > 1$ is not a prime, it is called a **composite**.
- Note: 1 is not a prime and also not a composite.

Thm: There are infinitely many primes.

Proof: by contradiction. สมมติว่า จน.เฉพาะมีจำกัด. มี k ตัว

ให้ $p_1, p_2, \ldots, p_k$ เป็น จน.เฉพาะทั้งหมด.

พิจารณา $X = \underline{p_1 \cdot p_2 \cdot p_3 \cdots p_k} + 1$.

$\Rightarrow X \bmod p_i = 1$

Case 1: X เป็น จน.เฉพาะ: – contradiction.

Case 2: X ไม่เป็น จน.เฉพาะ: มี จน.เฉพาะ p' บางตัว ที่หาร X ลงตัว, p' ไม่อยู่ในนั้น, – contr…

Fundamental theorem of arithmetic

$100 = 20 \cdot 5 = 4 \cdot 5 \cdot 5 = \underline{2 \cdot 2 \cdot 5 \cdot 5}$

$15 = 5 \cdot 3 = 3 \cdot 5$

Unique factorization *

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

3

Algorithm for testing primes

$10^{50} \leq (2^4)^{50} = 2^{200}$

Recall our CheckPrime2 algorithm

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run? $O(\sqrt{n})$

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```

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

Efficient algorithms

Is $O(\sqrt{n})$ for checking a prime number efficient?

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When working with input consisting only a few numbers, we typically use the number of bits. For integer n, the number of bits of n is $\lceil \log_2 n \rceil$.

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n	number of bits of n	$\sqrt{n}$
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
1,267,650,600,228,229,401,496,703,205,376	100	1,125,899,906,842,624

Efficient algorithms

m $2^{m/2}$

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$\sqrt{n} = \sqrt{2^{\log n}}$

$= 2^{\log n / 2}$

Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

Greatest Common Divisors (GCD)

Definition (GCD)

For integers x and y, the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $gcd(x, y)$.

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100, ~~30~~ 15

A simple way to find $gcd(x, y)$:

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g = min(x,y)
while (x mod g != 0) or (y mod g != 0):
  g -= 1
return g
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return g
```

$\underline{O(min(x,y))}$

What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

Euclid's algorithm

```
Algorithm Euclid(x,y):
  if x mod y == 0:
    return y
  else:
    return Euclid(y, x mod y)
```

x	y
100 ,	15
15	10
10	5

15	100
100	15

Euclid's algorithm floor

$\lfloor x \rfloor$ = จำนวนเต็มที่มากที่สุดที่ $\leq x$

```
Algorithm Euclid(x,y):
  if x mod y == 0:
    return y
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    return Euclid(y, x mod y)
```

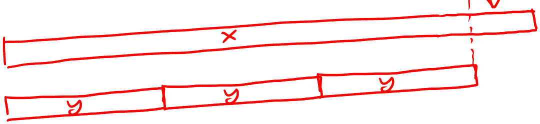
$gcd(x,y) = gcd(y, x \bmod y)$

Let's see how it works with $Euclid(12311, 24324)$:

Euclid(12311, 24324)
Euclid(24324, 12311)
Euclid(12311, 12013)
Euclid(12013, 298)
Euclid(298, 93)
Euclid(93, 19)
Euclid(19, 17)
Euclid(17, 2)
Euclid(2, 1)

$gcd(12311, 24324) = 1$

x mod y



Proofs

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid.

ถูกมั้ย?
เร็วมั้ย

Proofs

ถ้า $x < y$

if $\cancel{x}$ mod $y == 0$

else

euclid(y, x mod y)

$= x$

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid.

Note that when $x < y$, $\mathrm{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\mathrm{Euclid}(y, x)$. After that, in each call, x is always larger than y. For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

For any integers x and y such that $x > y$, $\text{Euclid}(x, y) = gcd(x, y)$.

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $gcd(x, y) = y$; therefore, the base case of the algorithm is correct.
Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\text{Euclid}(x', y') = gcd(x', y')$.
Now assume that $y \not| x$. The Euclid algorithm returns $\text{Euclid}(y, x \bmod y)$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\text{Euclid}(y, x \bmod y) = gcd(y, x \bmod y).$$

Euclid(x,y)
=
return Euclid(y, x mod y)

Thus, we are left to show that

$$gcd(x, y) = gcd(y, x \bmod y). \quad *$$

$\square$

What is $x \bmod y$?

$$x - \left(\left\lfloor \frac{x}{y} \right\rfloor \right) \cdot y$$

What is $x \mod y$?

Let $\lfloor a \rfloor$ be the largest integer a' such that $a' \leq \lfloor a \rfloor$.

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Lemma 2

If $\underbrace{a|x \text{ and } a|y}$, *then* $\underbrace{a|x \bmod y}$.

แสดงว่า $x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$

ถ้า $a|x$, $a|y$ จะได้ว่า

$$a \,\Big|\, \left(x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y\right)$$

4(a)

$$\Rightarrow \gcd(x, y) \le \gcd(\underbrace{y}, \underbrace{x \bmod y})$$

Lemma 2

If $a|x$ *and* $a|y$, *then* $a|x \bmod y$. $\Rightarrow \; gcd(x,y) \le gcd(y, x \bmod y)$ ×

Lemma 3

$gcd(x, y) = gcd(y, x \bmod y)$

... ถ้า $\underline{a \mid y}$ และ $a \mid \underline{x \bmod y}$ แล้ว $a \mid x$ ด้วย.

4(b)

$$\Rightarrow \; gcd(y, x \bmod y) \le gcd(x,y).$$ ×

How many recursive calls does Euclid's algorithm makes?

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- When can we decrease the value of x or y in the calls?

Euclid(1000, 7)
↓
Euclid(7, ...)

Eu(1000, 999)
↓
Eu(999, 1)
↓
Eu(1, —)

Eu(x, y)
|
Eu(y, x mod y)
|
Eu(x mod y, y mod (x mod y))

Eucl(x, y)
↓
Eu(y, x mod y)

- $y \leq x/2$ ✓
- $y > x/2$

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- When $y \leq x/2$, when Euclid(x, y) calls Euclid$(y, x \bmod y)$ the first argument decreases by half.

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  ไม่ลด

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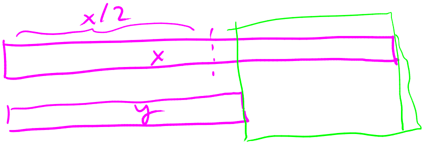
  Euclid$(x, y) \Rightarrow$ Euclid$(y, x \bmod y) \Rightarrow$ Euclid$(\underline{x \bmod y}, y \bmod (x \bmod y))$

ถ้า $y > x/2$

$x \bmod y = x - y$

$< x/2$



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$x \bmod y \leq x/2$

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- If we start with $x < y$, the next calls will always have that $x > y$; so we have at most one call with $x < y$.
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  Euclid$(x, y) \Rightarrow$ Euclid$(y, x \bmod y) \Rightarrow$ Euclid$(x \bmod y, y \bmod (x \bmod y))$ Note that in this case, $x \bmod y = x - y \leq x/2$. Thus, after two recursive calls, the first argument decreases by half.
- How many times can that happen?
- The first argument can decrease by a factor of two for at most $\log x$ times. Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$.

Computing power

$x^m \bmod n$

How fast can we compute x^y?

$x^2 = x \cdot x$

$x^3 = x \cdot x^2$

$x^4 = (x^2)^2 = (x^2) \times (x^2)$

$x^8 = x^4 \cdot x^4$

$x^{2^k} = (x^{2^{k-1}}) \cdot (x^{2^{k-1}})$ → ทำได้โดยการคูณ k ครั้ง

Computing power

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```
Algorithm power(x,y):
  a = 1
  for i = 1,2,...,y:
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Computing power

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What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

$$x^{16} = (x^8)^2 = ((x^4)^2)^2 = (((x^2)^2)^2)^2.$$

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```
Algorithm power(x,y):  // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

Repeated squaring (general y)

```
Algorithm power(x,y):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2))
    if y mod 2 == 0:
      return a*a
    else
      return a*a*x
```

$x^{y/2} \cdot x^{y/2} = x^{y}$

y เป็นคี่

$\underbrace{x^{(y-1)/2}} \cdot \underbrace{x^{(y-1)/2}} = x^{y-1} \cdot x$

จน. ครั้งในการคูณ $O(\log y)$

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```

What is the number of recursive calls?
What is the running time?
While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y, mod n)

$m^e \bmod n$

$m^d \bmod n$

Computing $x^y \bmod n$:

```
Algorithm power(x,y,n):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2)) mod n
    if y mod 2 == 0:
      return a*a mod n
    else
      return a*a*x mod n
```