01204211 Discrete Mathematics
Lecture 9a: Spans and Vector Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$\boldsymbol{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

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Write down the nutritions for a mixed drink that consists of 50ml of $\boldsymbol{v}$, 200ml of $\boldsymbol{c}$ and 10ml of $\boldsymbol{w}$.

Exercise

$$0{,}5\begin{bmatrix}100\\50\\0\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\0\\300\\0\end{bmatrix} + 0{,}1\begin{bmatrix}50\\0\\50\\10\end{bmatrix}$$

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

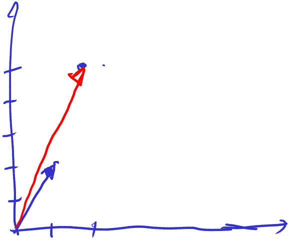
$$\boldsymbol{v} = \begin{bmatrix}100\\50\\0\\0\end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix}0\\0\\300\\0\end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix}50\\0\\50\\10\end{bmatrix}$$

1. Write down the nutritions for a mixed drink that consists of 50ml of $\boldsymbol{v}$, 200ml of $\boldsymbol{c}$ and 10ml of $\boldsymbol{w}$. (as a linear combination of v, c, & w)
2. Write that result as a matrix-vector product. (The matrix should be a 4×3 matrix.)

$$\begin{bmatrix}100 & 0 & 50\\50 & 0 & 0\\0 & 300 & 50\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix} = \begin{bmatrix}100 & 0 & 50\\50 & 0 & 0\\0 & 300 & 50\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix} = \begin{bmatrix}100 & 0 & 50\\50 & 0 & 0\\0 & 300 & 50\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix}$$

Example 1

Is $\text{Span}\,\{[1,2],[2,5]\} = \mathbb{R}^2$?



Example 2

Is $\mathrm{Span}\,\{[1,0,1],[1,1,0],[2,3,4]\} = \mathbb{R}^3$?

Example 3

Is $\mathrm{Span}\,\{[1,0,1],[1,1,0],[4,2,2]\} = \mathbb{R}^3$?

Elements in a vector

- We see examples of vectors over $\mathbb{R}$.
- However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- What do we want from an element in a vector?
  - We should be able to perform addition, subtraction, multiplication, and division.
  - Operations should be commutative and associative. $(ab)c = a(bc)$
  - Additive and multiplicative identity should exist.
  - Addition and multiplication should have inverses.
- We refer to a set with these properties as a **field**.

A field

Definition

A set $\mathbb{F}$ with two operations $+$ and $\times$ (or $\cdot$) is a **field** iff these operations satisfy the following properties:

- (Associativity): $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (Commutativity): $a+b = b+a$ and $a \cdot b = b \cdot a$
- (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a+0 = a$ and $a \cdot 1 = a$
- (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$
- (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an alement a^{-1} such that $a \cdot a^{-1} = 1$
- (Distributive): $a \cdot (b+c) = a \cdot b + a \cdot c$

Another useful field: $\boxed{GF(2)}$

$GF(2) = \{0, 1\}$. I.e., it is a "bit" field.
What are $+$ and $\cdot$ in $GF(2)$?

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- We define $b_1 + b_2$ to be XOR.

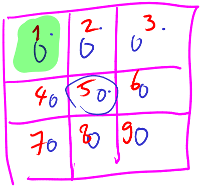
$$\begin{gathered} 0 + 0 = 0 \\ 0 + 1 = 1 + 0 = 1 \\ 1 + 1 = 0 \end{gathered}$$

- We define $b_1 \cdot b_2$ to be standard multiplication.

$$\begin{gathered} 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \\ 1 \cdot 1 = 1 \end{gathered}$$

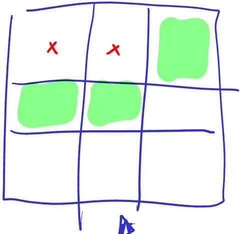
You can check that $GF(2)$ satisfies the axioms of fields.

3×3 Lights out



$$b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$



$$b_1 + b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$b_1, b_2 \, \ldots \, b_9$

Parity-check code

Parity-check code

From message $\boldsymbol{a} = [a_1, a_2, a_3, a_4]$, we compute (in $GF(2)$) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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$\boxed{1\,0\,1\,1\,1}$

$1+0+1+1 = 1$

Now our encoded message becomes

$$[a_1, a_2, a_3, a_4, a_5],$$

where $a_5 = b = a_1 + a_2 + a_3 + a_4$. It can detects a single-bit error.

0

0 0 1 1 1

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What can we say about the condition on a_5?
It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

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It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

Now, what is the set of all possible codewords?

Hamming code

You can detect and correct more errors with Hamming codes. In this version called a $[7,4]$ Hamming code, you encode 4-bit data $[a_1, a_2, a_3, a_4]$ into a 7-bit codeword $[p_1, p_2, a_1, p_3, a_2, a_3, a_4]$. Using the formula:

$$\begin{aligned} p_1 &= a_1 + a_2 + a_4 &= 0 \\ p_2 &= a_1 + a_3 + a_4 &= 0 \\ p_3 &= a_2 + a_3 + a_4 &= 0 \end{aligned}$$

1 1 1 0

a_1 a_2 a_3 a_4

P_1	P_2	a_1	P_3	a_2	a_3	a_4
1	0	1	0	1	1	1
1	1	1	0	1	1	1

$\longrightarrow$

S_3	S_2	S_1
0	0	1
0	1	1
1	1	1

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$$\begin{aligned} p_1 &= a_1 + a_2 + a_4 \\ p_2 &= a_1 + a_3 + a_4 \\ p_3 &= a_2 + a_3 + a_4 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

Let's see how this works.

7x4

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{m} = \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ a_1 \\ p_3 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{w}$$

Hamming code (encoding as matrix multiplication)

Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$. Let

$$
\begin{array}{ccc}
1 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}
$$

$$
\begin{aligned}
s_1 &= d_1 + d_3 + d_5 + d_7 && = 1 \\
s_2 &= d_2 + d_3 + d_6 + d_7 && = 0+1+1+1 = 1 \\
s_3 &= d_4 + d_5 + d_6 + d_7 && = 1+0+1+1 = 1
\end{aligned}
$$

Given a codewords $w = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero's.

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix}
=
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}
$$

7 3

s_3	s_2	s_1
1	1	1

Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$\begin{aligned} 0 = s_1 &= d_1 + d_3 + d_5 + d_7 \\ 0 = s_2 &= d_2 + d_3 + d_6 + d_7 \\ 0 = s_3 &= d_4 + d_5 + d_6 + d_7 \end{aligned}$$

Given a codewords $\boldsymbol{w} = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero's.

What if there is an error? Let's try.

Hamming code (parity check as matrix multiplication)

codeword ← 7 bits

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \boxed{a_3} \\ a_4 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ a_1 \\ p_3 \\ \boxed{a_2} \\ a_3 \\ a_4 \end{bmatrix}$$

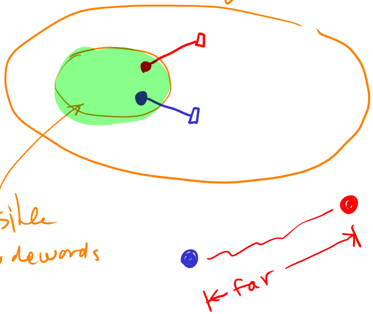
$2^4 = \boxed{16}$ possible messages

16 possible codewords

2^7 possible strings

$\parallel$

128



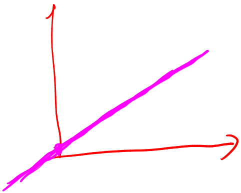
Codewords from Hamming code

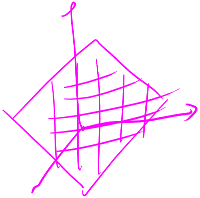
$[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$

Turning the formula for p_1, p_2, p_3 around, we have 3 non homogeneous linear equations:

$$
\begin{aligned}
d_1 + d_3 + d_5 + d_7 &= 0 \\
d_2 + d_3 + d_6 + d_7 &= 0 \\
d_4 + d_5 + d_6 + d_7 &= 0
\end{aligned}
$$

and ~~again the set of all possible codewords $\mathcal{W}$ forms a vector space over $GF(2)$~~.





Can you solve 2×2 Lights out?

(Skipped)

Let $\boldsymbol{u}_1 = [1, 1, 1, 0]$, $\boldsymbol{u}_2 = [1, 1, 0, 1]$, $\boldsymbol{u}_3 = [1, 0, 1, 1]$, and $\boldsymbol{u}_4 = [0, 1, 1, 1]$.

Given $\boldsymbol{b} = [b_1, b_2, b_3, b_4]$, can you always find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Can you solve 2×2 Lights out?

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$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Same question: Is $\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\boldsymbol{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Can you solve 2×2 Lights out?

Since

$$[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1] \in \mathrm{Span}\,\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3,\boldsymbol{u}_4\},$$

and

Can you solve 2×2 Lights out?

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and

$$\text{Span}\,\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\} = GF(2)^4,$$

Can you solve 2×2 Lights out?

Since

$$[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1] \in \text{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$$

and

$$\text{Span}\ \{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\} = GF(2)^4,$$

what can we say about $\text{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$?

Generators

Definition

Let $\mathcal{V}$ be a set of vectors. Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$.
If $\operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$, we say that

- $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a **generating set** for $\mathcal{V}$
- vectors $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_n$ are **generators** for $\mathcal{V}$

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Examples

Standard generators

Note that $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ are generators for $GF(2)^4$. Why?

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They are called **standard generators** for $GF(2)^4$, written as $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$.

Standard generators

Note that $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ are generators for $GF(2)^4$. Why?

They are called **standard generators** for $GF(2)^4$, written as $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$.

For $\mathbb{R}^n$, we also have $[1,0,0,\ldots,0],[0,1,0,\ldots,0],[0,0,1,\ldots,0],\ldots,[0,0,0,\ldots,1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ *are generators for* $\mathcal{V}$, *and for each* i,

$$\boldsymbol{v}_i \in \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\},$$

we have that $\mathcal{V} \subseteq \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$.

Adding a vector into a span

Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

$$\mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

Geometry of spans: in $\mathbb{R}^2$

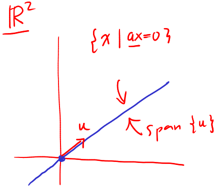
Geometry of spans: in $\mathbb{R}^3$

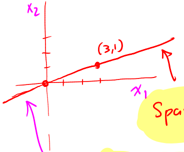
Two representations

Vector space

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.

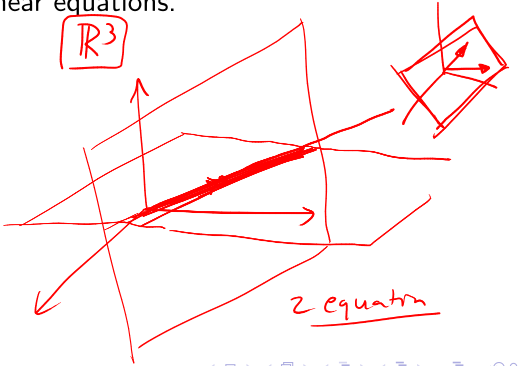




$\frac{x_2}{x_1} = 3 \Rightarrow \boxed{x_2 - 3x_1 = 0}$

Span $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

$\{[x_1, x_2] \mid x_2 - 3x_1 = 0\}$



Two representations

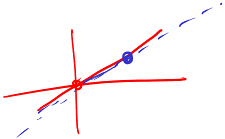
$Ax = b$ 0

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

Two representations



There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- they pass through the origin,
- if vector $\boldsymbol{u}$ is in the objects, $\alpha \boldsymbol{u}$ for any scalar α is also in the objects, and
- if $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the objects, $\boldsymbol{u} + \boldsymbol{v}$ is also in the objects.

closed under scalar multiplication

closed under addition.

Vector space

Vector spaces

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,
$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$
for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,
$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

$\boxed{\text{Span of vectors}}$ is a vector space

V1 0

V2 scale

V3 addition

Consider n-vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$,

$$\operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$$

is a vector space.

Span of vectors is a vector space

Consider n-vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$,

$$\boxed{\text{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}}$$

is a vector space.

Let's check if properties V1, V2, and V3 are satisfied.

(V1) $0 \in \text{Span}\{u_1, u_2, \ldots, u_m\}$ ✓

(V2) If $\underline{u} \in \text{Span}\underline{\{u_1, \ldots, u_m\}}$, any β

Proof: there exist $\alpha_1, \alpha_2, \ldots, \alpha_m$ s.t.

$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$

$\beta u = \beta(\qquad) = \beta\alpha_1 u_1 + \beta\alpha_2 u_2 + \cdots \in \text{Span}\{-\}$

(V3) if $x \in \text{Span}$ ___

$y \in$ ___

Solutions to homogeneous linear equations is a vector space

Consider a set $\mathcal{S}$ of all n-vectors in the form $[x_1, x_2, \dots, x_n]$ where

homogeneous

m equation

$$\left\{\begin{array}{rcl} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n & = & 0 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n & = & 0 \\ \vdots & = & \vdots \\ a_{m1} x \cdot_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n & = & 0 \end{array}\right.$$

Let's check if properties V1, V2, and V3 are satisfied.

V1

Dot product

Definition

For n-vectors $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$ and $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$, the **dot product** of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

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Using dot products, the previous set $\mathcal{S}$ can be written as

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} = 0, \boldsymbol{a}_2 \cdot \boldsymbol{x} = 0, \ldots, \boldsymbol{a}_m \cdot \boldsymbol{x} = 0\}$$

and we know that $\mathcal{S}$ is a vector space.

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

Translation

If we have a line or a plane passing through a vector $\boldsymbol{a}$, but not through the origin, how can we represent it?

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$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

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- *Question:* Is $\mathcal{A}$ a vector space?
- We also write it as $\boldsymbol{a} + \mathcal{V}$.

Affine spaces

Definition

If $\boldsymbol{a}$ is a vector and $\mathcal{V}$ is a vector space, then

$$\boldsymbol{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

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Definition

The set of all affine combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **affine hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

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The set of all convex combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **convex hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Writing an affine space using a span

Writing an affine space using a span

An affine space

An affine space passing through $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is

$$\boldsymbol{u}_1 + \operatorname{Span}\{\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{u}_3 - \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n - \boldsymbol{u}_1\}.$$

Non-homogeneous linear system

Two linear systems:

$$
\begin{aligned}
\boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\
\boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\
&\vdots \\
\boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m
\end{aligned}
\qquad\qquad
\begin{aligned}
\boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\
\boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\
&\vdots \\
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\end{aligned}
$$

What can you say about the solution sets of these two related linear systems?

Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?
$\mathbf{0}$ is always a solution to the linear system on the right.
Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

This implies that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution to the homogeneous linear system.

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \le i \le m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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In other words,

$$\begin{aligned} \mathcal{W} &= \boldsymbol{u} + \{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\} \\ &= \boldsymbol{u} + \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = 0, \text{ for } 1 \leq i \leq m\}, \end{aligned}$$

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i.e., $\mathcal{W}$ is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.