01204211 Discrete Mathematics
Lecture 12b: Undecidability (2)

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September 26, 2024

Decision problems

- Given an integer x, is x odd?
- Given a string w, is w palindrome?
- Given a string w, is $w \in \{0^n 1^n \mid n \geq 0\}$?
- Given a map, a starting position s, a destination t, and an integer k, does there exist a path from s to t with distance at most k?
- Given a program P and input string w, when running P with w as an input, does P terminate?

Decision problems and languages

For this problem:

Given an integer x, is x odd?

we can define a corresponding language

$$L_E = \{, \ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}.$$

To solve this problem, given x, we can ask if $x \in L_E$.

Deciders

We say that a python program P **decides** the language L if for any input string x, P when running with x as an input,

- P always terminates,
- P outputs **yes**, if $x \in L$, and
- P outputs **no**, if $x \notin L$.

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- P always terminates,
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If we believe that anything that a computer can do can be written as a python program, and there is no python program that decides a language A, then we say that

A is undecidable.

Language $\textsc{HaltA}$ and $\textsc{Halt}$

Let $\mathbb{P}$ be the set of all python programs. Let the language $\textsc{HaltA}$ be

$$\{P \in \mathbb{P} \mid \text{when running } P \text{ with } P \text{ as an input, } P \text{ terminates}\}$$

Or with a more concise notation:

$$\textsc{HaltA} = \{P \in \mathbb{P} \mid P(P) \text{ terminates}\}.$$

We also have another related language

$$\textsc{Halt} = \{(P, w) \mid P \text{ is a python program such that } P(w) \text{ terminates}\}$$

Lemma 1

HALTA *and* HALT *are undecidable.*

Lemma 2

There is no python program that decides A? = HALTA.

$Q \to \boxed{P} \xrightarrow{Y/N}$

Y - Q(Q) term

N ·

Proof.

We prove by contradiction. Assume that there is a python program P that decides A.

Lemma 2

There is no python program that decides A.

decides $HALT_A$

Proof.

We prove by contradiction. Assume that there is a python program P that decides A. We describe a python program B that reads a string Q as an input as follows:

prgm

```
Program B
Input Q
1.    Load P as module Pmod
2.    if Pmod.main(Q) == 'yes':    # when Pmod outputs yes
3.       while True: pass          #    loop forever
4.    else:                        # when Pmod outputs no
5.       quit()                    #    halt
```

$P(Q) == \text{'yes'}$

จาก ที่ว่า P decides HALT

Given program Q as an input, B loops forever when $P(Q) = \text{yes} \Rightarrow Q(Q)$ halts

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Program B
Input Q
      Load P as module Pmod
      if Pmod.main(Q) == 'yes':      # when Pmod outputs yes
         while True: pass            #    loop forever
      else:                          # when Pmod outputs no
         quit()                      #    halt
```

Given program Q as an input, B loops forever when

It terminates when P(Q) == 'no' , ⇒ Q(Q) loops

□

Proof.

We know that

- $B(Q)$ loops when $Q(Q)$ terminates, and
- $B(Q)$ terminates when $Q(Q)$ loops.

Does running B using B as an input terminate?

$B(B)$ loops $B(B)$ termin... ✗

$B(B)$ terminates $B(B)$ loops ✗

contradiction

وجود ندارد B

Proof.

We know that

- $B(Q)$ loops when $Q(Q)$ terminates, and
- $B(Q)$ terminates when $Q(Q)$ loops.

Does running B using B as an input terminate?
Let's try to plug in $Q = B$. We have

- $B(B)$ loops when

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Does running B using B as an input terminate?
Let's try to plug in $Q = B$. We have

- $B(B)$ loops when $B(B)$ terminates, and
- $B(B)$ terminates when $B(B)$ loops.

Since either $B(B)$ loops or terminates, and we cannot be in any of the cases, we obtain a contradiction. $\Rightarrow$ ไม่มี B

Therefore, we conclude that program P does not exist. $\square$

Reduction: proving undecidability of HALT

$\S$ HALTA undecidable (Q)

Q

$\boxed{P \Rightarrow \neg Q} \Rightarrow \neg P$

(P)

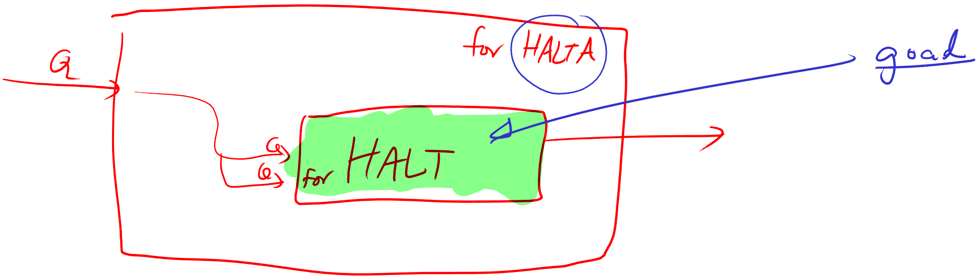
- We show that if HALT is decidable, then HALTA is also decidable.

(¬Q)

- However, HALTA IS UNDECIDABLE.
- We conclude that HALT is also undecidable.

proof by contradiction.

Reduction in picture



reduction of HALTA to HALT

source target

Let $\textsc{Accept} = \{(P, w) \mid P \in \mathbb{P}$ and $P(w)$ terminates with acceptance$\}$.

Lemma 3

$\textsc{Accept}$ *is undecidable.*

Proof.

We prove the lemma by contradiction. Assume that there is a python program Q that decides $\textsc{Accept}$. We construct a program C that decides $\textsc{Halt}$ as follows

```
Program C
Input P,w
1.   Replace every "print('no')" statement in P with "print('yes')"
1.   if Q(P,w) == 'yes':
2.      print('yes')
3.   else
4.      print('no')
```

Proof (cont.)

```
Program C
Input P,w
1.    Replace every "print('no')" statement in P with "print('yes')"
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3.    else
4.        print('no')
```

เปลี่ยน P ที่แก้แล้วเป็น P'

P'(w) จะ accept

ต้องการให้ C decides HALT

We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.

✓ P(w) halts ⇒ C ตอบ yes

✓ P(w) ไม่ halt ⇒ C ตอบ no

- P'(w) จะ accept *โดยไม่ได้ run

พอเรียก Q(P,w) ⇒ ตอบ yes ⇒ C ตอบ yes

Proof.

We prove the lemma by contradiction. Assume that there is a python program Q that decides ACCEPT. We construct a program C that decides HALT as follows

```
Program C
Input P,w
1.    Replace every "print('no')" statement in P with "print('yes')"
1.    if Q(P,w) == 'yes':
2.        print('yes')
3.    else
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```

We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.

Case 1: when $P(w)$ halts.

- P'(w) จะ accept *โดยไม่ให้ run

พอเรียก Q(P,w) ⇒ ตอบ yes ⇒ C ตอบ yes

P'(w) ไม่ halt, Q(P',w) ตอบ no ⇒ C ตอบ no

- Let Reject = {(P, w) | P ∈ 𝒫, P(w) reject}

จะพิสูจน์ว่า Reject is undecidable

Since in both cases, C answers correctly, we know that given program Q deciding ACCEPT, we can construct a program C that decides HALT. However, we know that HALT is undecidable; thus, we reach a contradiction. We conclude that ACCEPT is also undecidable. □

Proof (cont.)

```
Program C
Input P,w
1.   Replace every "print('no')" statement in P with "print('yes')"
1.   if Q(P,w) == 'yes':
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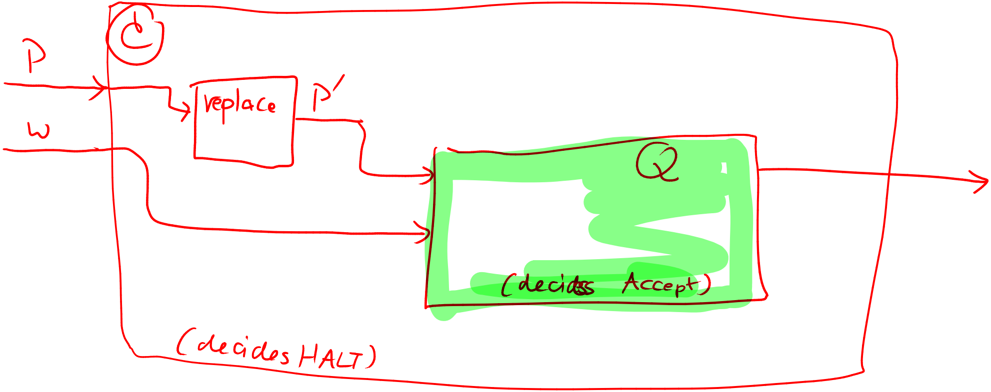
We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.
Since in both cases, C answers correctly, we know that given program Q deciding ACCEPT, we can construct a program C that decides HALT. However, we know that HALT is undecidable; thus, we reach a contradiction. We conclude that ACCEPT is also undecidable. □

Reduction from HALT to ACCEPT in picture

target



How about $\textsc{Reject}$?

Let

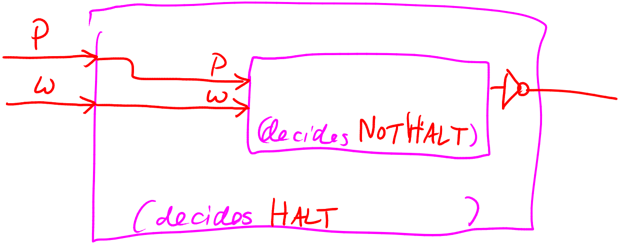
$$\textsc{Reject} = \{(P, w) \mid P \in \mathbb{P} \text{ and } P \text{ rejects } w\}.$$

P terminates & rejects w

How about NotHalt?

Let

$$\textsc{NotHalt} = \{(P, w) \mid P \in \mathbb{P} \text{ and } P(w) \text{ does not terminate}\}.$$



Language of program P

For a python program P, let $L(P)$ be the set of all strings that P accepts, i.e.,

$$L(P) = \{w \in \Sigma^* \mid P(w) = yes\}.$$

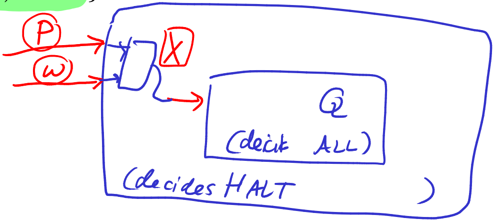
Let

$$\textsc{All} = \{P \in \mathbb{P} \mid L(P) = \Sigma^*\}.$$

สร้าง $\boxed{X}$ ที่

X accepts ทุก string

ถ้า $P(w)$ halts



Lemma 4

ALL is undecidable.

Proof.

We prove by reduction from HALT. Assume that ALL is decidable, i.e., there is a python program Q that decides ALL. We construct program C that decides HALT as follows

```
Program C (input: P,w)
1. Construct another program R from P and w:
     | Program R (input: x)
     | 1.  Run program P on input w, suppressing any output from P
     | 2.  Accept x
2. if Q(R) == 'yes':
3.     return 'yes'
4. else: return 'no'
```

Proof (cont.)

```
Program C (input: P,w)
1. Construct another program R from P and w:
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```

To ensure the correctness, we have to consider two cases.

Case 1: when $P(w)$ halts. : $R(x)$ $\forall x$: accepts [illegible] $\Rightarrow$ $R \in ALL \Rightarrow Q(R) = yes$

C rtn yes

Proof (cont.)

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Program C (input: P,w)
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```

To ensure the correctness, we have to consider two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.

$R(x)$ 永不 halt, $R \notin ALL$, $Q(R) =$ 'no', C ans no

Proof (cont.)

```
Program C (input: P,w)
1. Construct another program R from P and w:
     | Program R (input: x)
     | 1.  Run program P on input w, suppressing any output from P
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```

To ensure the correctness, we have to consider two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.
Since we can construct a program for $\textsc{Halt}$ using a program that decides $\textsc{All}$, but $\textsc{Halt}$ is undecidable; therefore, we conclude that $\textsc{All}$ is undecidable. □

Empty

Let

$$\textsc{Empty} = \{P \in \mathbb{P} \mid L(P) = \emptyset\}.$$

Lemma 5

EMPTY *is undecidable.*

Proof.

We prove by reduction from HALT. Assume that EMPTY is decidable, i.e., there is a python program Q that decides EMPTY. We construct program C that decides HALT as follows

```
Program C (input: P,w)
1. Construct another program R from P and w:
     | Program R (input: x)
     | 1.  Run program P on input w, suppressing any output from P
     | 2.  Accept x
2. if Q(R) == 'yes':
3.     return 'no'
4. else: return 'yes'
```

Since we can construct a program for HALT using a program that decides EMPTY, but HALT is undecidable; therefore, we conclude that EMPTY is undecidable. □

Lemma 6

Let $\textsc{Eq} = \{(P_1, P_2) \mid P_1, P_2 \in \mathbb{P}$ *and* $L(P_1) = \cancel{L(P_2)}\}$. $\textsc{Eq}$ *is undecidable.*

Σ^*

$L(P_2) = \Sigma^*$

Proof.

We prove by reduction from $\textsc{All}$. Assume that $\textsc{Eq}$ is decidable, i.e., there is a python program Q that decides $\textsc{Eq}$. We construct program C that decides $\textsc{All}$ as follows

```
Program C (input: P)
1. Construct another program R:
      | Program R (input: x)
      | 1.   Accept x
2. if Q(P,R) == 'yes':
3.       return 'yes'
4. else: return 'no'
```

$L(R) = \Sigma^*$

in $L(P) = L(R) = \Sigma^* \Rightarrow P \in \textsc{All}$

Since we can construct a program for $\textsc{All}$ using a program that decides $\textsc{Eq}$, but $\textsc{All}$ is undecidable; therefore, we conclude that $\textsc{Eq}$ is undecidable. □

Another proof for EQ

Proof.

We prove by reduction from EMPTY. Assume that EQ is decidable, i.e., there is a python program Q that decides EQ. We construct program C that decides EMPTY as follows

```
Program C (input: P)
1. Construct another program R:
     | Program R (input: x)
     | 1.  Reject x
2. if Q(P,R) == 'yes':
3.     return 'yes'
4. else: return 'no'
```

$L(R) = \emptyset$

$// L(P) = L(R) = \emptyset \Rightarrow P \in \text{EMPTY}$

Since we can construct a program for EMPTY using a program that decides EQ, but EMPTY is undecidable; therefore, we conclude that EQ is undecidable. $\square$

Lemma 7

Let HELLO $= \{P \in \mathbb{P} \mid L(P) = \{\texttt{hello}\}\}$. HELLO *is undecidable.*

INCORRECT PROOF.

We prove by reduction from EQ. Assume that EQ is decidable, i.e., there is a python program Q that decides EQ. We construct program C that decides HELLO as follows

```
Program C (input: P)
1. Construct another program R:
     | Program R (input: x)
     | 1.  if x == 'hello':
     | 2.     print('yes')      # accept x
     | 3.  else
     | 4.     print('no')       # reject x
2. if Q(P,R) == 'yes':
3.     return 'yes'
4. else: return 'no'
```

$L(R) = \{hello\}$

$L(P) = L(R) = \{hello\}$ $P \in$ HELL

$\square$

Lemma 8

Let $\textsc{Hello} = \{P \in \mathbb{P} \mid L(P) = \{\mathtt{hello}\}\}$. $\textsc{Hello}$ *is undecidable.*

Proof

We prove by reduction from $\textsc{Halt}$. Assume that $\textsc{Hello}$ is decidable, i.e., there is a python program Q that decides $\textsc{Hello}$. We construct program C that decides $\textsc{Halt}$ as follows

$$L(R) = \begin{cases} \{\text{hello}\} & \text{אם } P(w) \text{ halts} \\ \emptyset & \text{אם } P(w) \text{ לא halts} \end{cases}$$

```
Program C (input: P,w)
1. Construct another program R:
     | Program R (input: x)
     | 1.  if x != 'hello':
     | 2.    print('no')      # reject x
     | 3.  Replace any output statements from P
     | 4.  Run the modified P on w
     | 5.  print('yes')       # accept x
2. if Q(R) == 'yes':
3.      return 'yes'
4. else: return 'no'
```

Proof (cont.)

```
Program C (input: P,w)
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```

We consider two cases:

Case 1: $P(w)$ halts.

Proof (cont.)

```
Program C (input: P,w)
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Case 1: $P(w)$ halts.

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We consider two cases:
Case 1: $P(w)$ halts.
Case 2: $P(w)$ does not halt.
Since we can construct a program that decides HALT given a program that decides HELLO, but HALT is undecidable. We conclude that HELLO is also undecidable. □

Python as computation

Do you believe in this assumption:

Anything that a computer can do can be written as a python program.

Turing machines

Anything that a computer can do can be carried out using Turing machines.

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Anything that a computer can do can be carried out using Turing machines.

Any possible computation can be performed by Turing machines.

Turing Machines

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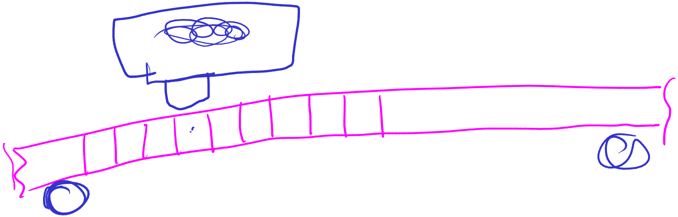
Turing Machines

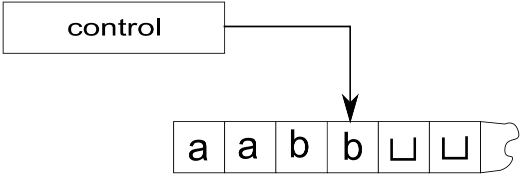
- Proposed by Alan Turing in 1936.
- A finite automaton with an unlimited memory with unrestricted access.
- Can perform any tasks that a computer can. (we'll see)
- However, there are problems that TM can't solve. These problems are beyond the limit of computation.

Components



- An infinite **tape**.
- A tape head that can
  - **read and write** to the tape, and
  - **move** around the tape.

Schematic



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- The machine reads a symbol from of the tape where its head is at.
- It can write a symbol back and move left or right.
- At the end of the computation, the machine outputs accept or reject, by entering accept state of reject state. (After changing, it halts.)
- It can go on forever (not entering any accept or reject states).

Examples

Formal definition of a Turing Machine

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- So, δ is in the form: $Q \times \Gamma \rightarrow Q \times \Gamma \times \{\texttt{LEFT}, \texttt{RIGHT}\}$
- E.g., if $\delta(q, a) = (r, b, \texttt{LEFT})$, then if the machine is in state q and reads a, it will change its state to r, write b to the tape and move to the left.

Definition

Definition (Turing Machine)

A **Turing machine** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the **blank symbol** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$,
4. $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathtt{LEFT}, \mathtt{RIGHT}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.

Variants of Turing Machines

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- They all have the same power. This demonstrates the robustness in the definition of TM's.

Variants of Turing Machines

- There are many alternative definitions of TM's.
- They are called variants of TM's.
- They all have the same power. This demonstrates the robustness in the definition of TM's. Also, this is an evidence that TM's "capture" the idea of computation (because whatever computing machine we can think of they are all equivalent to TM's).

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  - For any "stay put" transition, we replace with two transitions: "right" and "left".

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- E.g., if $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, \texttt{LEFT}, \texttt{RIGHT}, \ldots, \texttt{LEFT})$ then if the machine is at state q_i and each head on tape i reads symbol a_i, it'll write b_i on each tape i, change state to q_j and move each head accordingly.

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- Can nondeterminism help?

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Since you can write a C interpreter in Pascal and Pascal interpreter in C, what you can do in C, you can do in Pascal.

Turing machine

If you believe that Turing machines are ultimate model of computing, all those programming languages are equivalent because they all can simulate Turing machines (and they runs on Turing machines).

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Hilbert's problems

Mathematician David Hilbert asked:
"Find a process according to which it can be determined by a finite number of operations if a given polynomial has intergral root"

To say NO

We need an argument (a mathematical proof) that covers all possible “processes” or all “computations”.

Possible definitions

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- Turing's machines

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They both turned out to be **equivalent**.

Church-Turing thesis

Turing machine algorithms $=$ intuitive notion of algorithms

Final answer to Hilbert

No, there doesn't exist any algorithm for determining if a polynomial has integral root.