

01204211 Discrete Mathematics

Lecture 8a: Integers and GCD

Jittat Fakcharoenphol

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Number theory: integers and divisibility

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We will cover:

- ▶ Basic concepts of divisibility, prime numbers, and congruence.
- ▶ How to quickly check if a number is prime.
- ▶ How to essentially perform “division” with integers, allowing us to work with important and useful objects like polynomials using only integers.

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- ▶ How to quickly check if a number is prime.
- ▶ How to essentially perform “division” with integers, allowing us to work with important and useful objects like polynomials using only integers.
- ▶ Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b . On the other hand if a does not divide b , we write $a \nmid b$.

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If $a|b$ and $a|c$, prove that $a|(b+c)$.

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Examples:

We use operator `mod` to denote an operation for finding the remainder of a division.
I.e., $a \bmod b$ is the remainder of dividing a with b .

Examples

Let r be the remainder of the division of b by a . Assume that $c|a$ and $c|b$. Prove that $c|r$.

More examples

For every integer a , $a - 1 \mid a^2 - 1$.

Primes

Definition (primes)

- ▶ An integer $p > 1$ is a **prime** if its divisors are only p , $-p$, 1 , and -1 .
- ▶ If an integer $n > 1$ is not a prime, it is called a **composite**.
- ▶ Note: 1 is not a prime and also not a composite.

Fundamental theorem of arithmetic

Unique factorization

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

Algorithm for testing primes

Recall our CheckPrime2 algorithm

```
Algorithm CheckPrime2(n): // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
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How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

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n	number of bits of n	\sqrt{n}
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
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Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

Greatest Common Divisors (GCD)

Definition (GCD)

For integers x and y , the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $\gcd(x, y)$.

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What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

Euclid's algorithm

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Let's see how it works with *Euclid*(12311, 24324):

Euclid(12311, 24324)

Euclid(24324, 12311)

Euclid(12311, 12013)

Euclid(12013, 298)

Euclid(298, 93)

Euclid(93, 19)

Euclid(19, 17)

Euclid(17, 2)

Euclid(2, 1)

Proofs

We have to prove two properties:

- ▶ For any integers x and y , $\text{Euclid}(x, y) = \gcd(x, y)$.
- ▶ The running time of Euclid.

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- ▶ For any integers x and y , $\text{Euclid}(x, y) = \text{gcd}(x, y)$.
- ▶ The running time of Euclid.

Note that when $x < y$, $\text{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\text{Euclid}(y, x)$. After that, in each call, x is always larger than y . For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

For any integers x and y such that $x > y$, $\text{Euclid}(x, y) = \text{gcd}(x, y)$.

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $\text{gcd}(x, y) = y$; therefore, the base case of the algorithm is correct.

Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\text{Euclid}(x', y') = \text{gcd}(x', y')$.

Now assume that $y \nmid x$. The Euclid algorithm returns $\text{Euclid}(y, x \bmod y)$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\text{Euclid}(y, x \bmod y) = \text{gcd}(y, x \bmod y).$$

Thus, we are left to show that

$$\text{gcd}(x, y) = \text{gcd}(y, x \bmod y).$$



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$$x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

Lemma 2

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Lemma 3

$$\gcd(x, y) = \gcd(y, x \bmod y)$$

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- ▶ How many times can that happen?
- ▶ The first argument can decrease by a factor of two for at most $\log x$ times.
Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$.

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What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

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```
Algorithm power(x,y): // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

Repeated squaring (general y)

```
Algorithm power(x,y):  
  if y == 0:  
    return 1  
  else:  
    a = power(x, floor(y / 2))  
    if y mod 2 == 0:  
      return a*a  
    else  
      return a*a*x
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What is the number of recursive calls?

What is the running time?

While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y , mod n)

Computing $x^y \bmod n$:

```
Algorithm power(x,y,n):  
  if y == 0:  
    return 1  
  else:  
    a = power(x, floor(y / 2)) mod n  
    if y mod 2 == 0:  
      return a*a mod n  
    else  
      return a*a*x mod n
```