01204211 Discrete Mathematics
Lecture 6a: Counting 3

Jittat Fakcharoenphol

August 9, 2022

Quick recap

We have proved many useful facts.

- The number of subsets of a set with n elements is 2^n.

Quick recap

We have proved many useful facts.

- The number of subsets of a set with n elements is 2^n. In fact, we know 3 proofs of this fact:
  - We count the number of ways one can choose a subset.
  - We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.

Quick recap

We have proved many useful facts.

- The number of subsets of a set with n elements is 2^n. In fact, we know 3 proofs of this fact:
  - We count the number of ways one can choose a subset.
  - We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.
- For a set with n elements, the number of its permutations is $n!$.

This lecture's goals[1]

- Consider set $\{1, 2, 3, 4, 5\}$. How many subsets with 3 elements does this set have?

[1] This lecture mostly follows [LPV].

This lecture's goals[1]

- Consider set $\{1,2,3,4,5\}$. How many subsets with 3 elements does this set have?
- There are 10 subsets with 3 elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.
- In this lecture, we shall find out how to count these subsets.

[1] This lecture mostly follows [LPV].

This lecture's goals[1]

- Consider set $\{1,2,3,4,5\}$. How many subsets with 3 elements does this set have?
- There are 10 subsets with 3 elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.
- In this lecture, we shall find out how to count these subsets.

Abbreviations: We shall call a set with n elements as an n-**set**. We shall call a subset with k elements as a k-**subset**.

[1] This lecture mostly follows [LPV].

This lecture's goals[1]

- Consider set $\{1,2,3,4,5\}$. How many subsets with 3 elements does this set have?
- There are 10 subsets with 3 elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.
- In this lecture, we shall find out how to count these subsets.

Abbreviations: We shall call a set with n elements as an n-**set**. We shall call a subset with k elements as a k-**subset**.

- We will also discuss the inclusion-exclusion priciples.

[1]This lecture mostly follows [LPV].

Ordered subsets

In general, elements in a given set is unordered. I.e., sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same set.

Ordered subsets

In general, elements in a given set is unordered. I.e., sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same set.
However, sometimes, it is useful to treat sets as ordered.

Ordered subsets

In general, elements in a given set is unordered. I.e., sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same set.
However, sometimes, it is useful to treat sets as ordered.
For example, for set $\{1, 2, 3\}$, there are 6 ordered subsets with 2 elements: $\{1, 2\}$, $\{1, 3\}$, $\{2, 1\}$, $\{2, 3\}$, $\{3, 1\}$, $\{3, 2\}$.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

- First, we pick the 1st price winner: there are 10 choices.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

- First, we pick the 1st price winner: there are 10 choices.
- For any 1st price winner, there are 9 choices to choose the 2nd price winner.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

- First, we pick the 1st price winner: there are 10 choices.
- For any 1st price winner, there are 9 choices to choose the 2nd price winner.
- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.

Example: runners

> **Question:** There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

- First, we pick the 1st price winner: there are 10 choices.
- For any 1st price winner, there are 9 choices to choose the 2nd price winner.
- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- Therefore, we conclude that the number of ways is $10 \cdot 9 \cdot 8$.

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)
  - $10!$ is too many for our answer. Why?

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)
  - $10!$ is too many for our answer. Why?
- For a particular selection of 3 top winners, how many possible running results have exactly these 3 top winners?

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)
  - $10!$ is too many for our answer. Why?
- For a particular selection of 3 top winners, how many possible running results have exactly these 3 top winners?
  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are $7!$ of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)
  - $10!$ is too many for our answer. Why?
- For a particular selection of 3 top winners, how many possible running results have exactly these 3 top winners?
  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are $7!$ of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.
- Let X be the set of ordered subsets with 3 elements of an 10-set. We then have $|X| \times 7! = 10!$, because they count the same objects.

Example: runners (another look)

We can arrive at the same answer by a different way of counting.

- Let's count all possible running results: there are $10!$ results. (I.e., each running result is a permutation.)
  - $10!$ is too many for our answer. Why?
- For a particular selection of 3 top winners, how many possible running results have exactly these 3 top winners?
  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are $7!$ of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.
- Let X be the set of ordered subsets with 3 elements of an 10-set. We then have $|X| \times 7! = 10!$, because they count the same objects. Solving this yields

$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

General answers: numbers of ordered subsets

Using the same arguments (either one), we have this theorem.

Theorem 1

The number of ordered subsets with k elements of an n-set is

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start?

[2] This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity.

[2] This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a “quick” estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).

[2]This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).
- Let's think about $n!$.

[2]This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).
- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.

[2] This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a “quick” estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).
- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.
  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.)

---

[2] This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).
- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.
  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.) How about 2^n? Is it a lower bound?

[2] This section on estimation follows section 1.4 of [LPV].

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start? When we want to get an estimate, we usually start by finding an **upper bound** and a **lower bound** for the quantity. As the names suggest, the upper bound for x is a quantity that is not smaller than x, and the lower bound for x is a quantity that is not larger than x (maybe under some condition).
- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.
  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.) How about 2^n? Is it a lower bound? How about 3^n or 5^n? Are they lower bounds of $n!$?

[2] This section on estimation follows section 1.4 of [LPV].

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

Similarly, since all factors of $n!$ is at most n, we have that

$$n! \leq n^n.$$

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

Similarly, since all factors of $n!$ is at most n, we have that

$$n! \leq n^n.$$

A slightly better upper bound is n^{n-1} because we can, again, ignore 1.

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

Similarly, since all factors of $n!$ is at most n, we have that

$$n! \leq n^n.$$

A slightly better upper bound is n^{n-1} because we can, again, ignore 1.

Are they any good?

Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \leq n!.$$

Similarly, since all factors of $n!$ is at most n, we have that

$$n! \leq n^n.$$

A slightly better upper bound is n^{n-1} because we can, again, ignore 1.

Are they any good?

n	2^{n-1}	$n!$	n^{n-1}
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	$3,628,800$	$1,000,000,000$

A better bound?

Let's consider $n!$ again, but for simplicity, let's consider only the case when n is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2 - 1) \cdot (n/2) \cdot (n/2 + 1) \cdots n$$

A better bound?

Let's consider $n!$ again, but for simplicity, let's consider only the case when n is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2 - 1) \cdot (n/2) \cdot (n/2 + 1) \cdots n$$

To get a better lower bound, we may move our cutting point from 2 to, say, $n/2$. Note that at least $n/2$ factors are at least $n/2$. Thus,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots n \\ &\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2} \\ &= (n/2)^{n/2} = \sqrt{(n/2)^n}. \end{aligned}$$

Better?

n	2^{n-1}	$\sqrt{(n/2)^n}$	$n!$	n^{n-1}
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	$7,776$
10	512	$3,125$	$3,628,800$	$1,000,000,000$
12	$2,048$	$46,656$	$479,001,600$	$743,008,370,688$

OK. A bit better.

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$.

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$. With Stirling's formula, We can use a calculator to estimate the number of digits for $100!$.

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)
$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$.

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$. With Stirling's formula, We can use a calculator to estimate the number of digits for $100!$. The estimate for $100!$ is

$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100 \log(100/e) + \log(200\pi) \approx 157.9696.$$

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$. With Stirling's formula, We can use a calculator to estimate the number of digits for $100!$. The estimate for $100!$ is

$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100\log(100/e) + \log(200\pi) \approx 157.9696.$$

Note that the correct answer is 158 digits.

Another example

- Consider the sum $1 + 2 + \cdots + n$.

Another example

- Consider the sum $1 + 2 + \cdots + n$.
- While know that it is $n(n+1)/2$, we can get a very easy upper bound by noting that each term in the sum is at most n; thus,

$$1 + 2 + \cdot + n \leq \underbrace{n + n + \cdots + n}_{n \text{ terms}} = n \times n = n^2$$

Another example

- Consider the sum $1 + 2 + \cdots + n$.
- While know that it is $n(n+1)/2$, we can get a very easy upper bound by noting that each term in the sum is at most n; thus,

$$1 + 2 + \cdot + n \leq \underbrace{n + n + \cdots + n}_{n \text{ terms}} = n \times n = n^2$$

- This upper bound of n^2 is very good as the gaps between the upper bounds and the actual values will not be larger than 2, as $\frac{n^2}{n(n+1)/2} < 2$.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

Consider the following process for choosing an ordered subsets with k elements of an n-set.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

Consider the following process for choosing an ordered subsets with k elements of an n-set. First, we choose a k-subset, then we permute it. Let B be the number of k-subsets. For each subset that we choose in the first step, the second step has $k!$ choices.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

Consider the following process for choosing an ordered subsets with k elements of an n-set. First, we choose a k-subset, then we permute it. Let B be the number of k-subsets. For each subset that we choose in the first step, the second step has $k!$ choices. Therefore, we can choose an ordered subset in $B \cdot k!$ possible ways.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

Consider the following process for choosing an ordered subsets with k elements of an n-set. First, we choose a k-subset, then we permute it. Let B be the number of k-subsets. For each subset that we choose in the first step, the second step has $k!$ choices. Therefore, we can choose an ordered subset in $B \cdot k!$ possible ways. From the previous discussion, we know that

$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of k-subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required. $\square$

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

(which reads "n choose k"). These numbers are called **binomial coefficients**.

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

(which reads "n choose k"). These numbers are called **binomial coefficients**.

Note that

- $\binom{n}{n} = 1$ (why?),

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

(which reads "n choose k"). These numbers are called **binomial coefficients**.

Note that

- $\binom{n}{n} = 1$ (why?),
- $\binom{n}{0} = 1$ (why?),

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

(which reads "n choose k"). These numbers are called **binomial coefficients**.

Note that

- $\binom{n}{n} = 1$ (why?),
- $\binom{n}{0} = 1$ (why?), and,
- when $k > n$, $\binom{n}{k} = 0$.

Properties (1)

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Properties (2)

Theorem: When $n, k > 0$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Properties (3)

Theorem: When $n, k > 0$, then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$