# 01204211 Discrete Mathematics Lecture 8b: Vectors

Jittat Fakcharoenphol

August 25, 2022

#### What is a vector?

You can think of a **vector** as an "ordered" list of elements (which are typically numbers). For example:

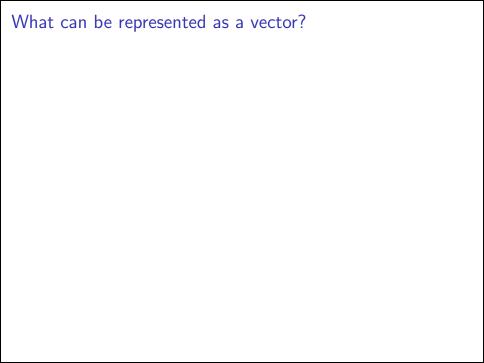
- ightharpoonup [1, 2, 5, 20]
- $\triangleright$  [0, 0, 1, 1, 0, 0, 0, 1]

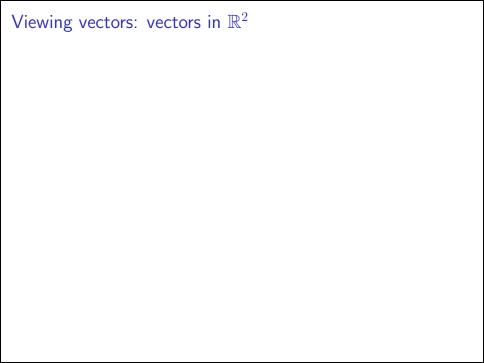
You can also view a vector as a **function**, e.g., you can view  ${\pmb u}=[1,2,5,20]$  as a function  ${\pmb u}$  that maps

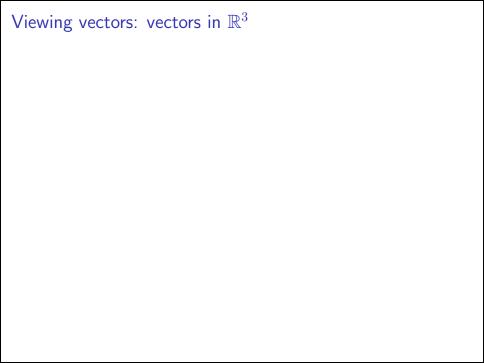
$$0 \mapsto 1$$
,  $1 \mapsto 2$ ,  $2 \mapsto 5$ ,  $3 \mapsto 20$ .

Each element in the vector is typically a real number  $(\mathbb{R})$ , but can be an element from other sets with appropriate property (more on this later).

Remark: Mathematically, a vector is an element of a vector space. We will understand this more later.







#### n-vectors over $\mathbb R$

- We mostly deal with vectors with finite number of elements.
- ► This is a 4-vector: [10, 20, 500, 4].
- ▶ We sometimes also write it as a column vector:

$$\begin{bmatrix} 10\\20\\500\\4 \end{bmatrix}$$

When every element of a vector is from some set, we say that it is a vector **over** that set. For example, [10, 20, 500, 4] is a 4-vector over  $\mathbb{R}$ .

### Vector operations

- As discussed in the previous slides, when working with a system of linear equations, we mostly deals with linear combinations of vectors.
- We will look at the operations we do to vectors to obtain their linear combinations.
- ► The operations are:
  - Vector additions
  - Scalar multiplications
- ▶ These operations motivate the definition of vector spaces.

### Vector additions

Given two n-vectors

$$\boldsymbol{u} = [u_1, u_2, \dots, u_n]$$

and

$$\boldsymbol{v} = [v_1, v_2, \dots, v_n],$$

we have that

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n].$$



#### Zero vectors

A zero  $n\text{-vector }\mathbf{0}=[0,0,\dots,0]$  is an additive identity, i.e., for any vector  $\boldsymbol{u},$ 

$$0 + u = u + 0 = u$$
.

## Scalar multiplications

For a vector over  $\mathbb R$ , we refer to an element  $\alpha$  in  $\mathbb R$  as a scalar. For an n-vector

$$\boldsymbol{u} = [u_1, u_2, \dots, u_n],$$

we have that

$$\alpha \cdot \boldsymbol{u} = [\alpha \cdot u_1, \alpha \cdot u_2, \dots, \alpha \cdot u_n],$$



### Linear combinations

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of  ${m u}_1,\dots,{m u}_m.$ 

Examples:

# A linear system with 3 variables

Give the following linear system.

If we rewrite the system as

$$\begin{bmatrix} 2\\1\\4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4\\0\\2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3\\5\\3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7\\12\\10 \end{bmatrix}.$$

This becomes the problem of expressing a vector as linear combination of other vectors. I.e., given vectors

$$u_1 = [2, 1, 4], \quad u_2 = [4, 0, 2], \quad u_3 = [3, 5, 3]$$

we would like to find coefficients  $x_1, x_2, x_3$  such that

$$x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2 + x_3 \cdot \mathbf{u}_3 = [7, 12, 10].$$

### Span

A set of all linear combination of vectors  $u_1, u_2, \ldots, u_m$  is called the **span** of that set of vectors. It is denote by  $\mathrm{Span}\{u_1, u_2, \ldots, u_m\}$ .

Examples:

### Convex combination

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m,$$

such that  $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$  and  $\alpha_i \ge 0$  for all i, and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of  $oldsymbol{u}_1,\ldots,oldsymbol{u}_m.$ 

Examples:

#### Elements in a vector

- $\blacktriangleright$  We see examples of vectors over  $\mathbb{R}$ .
- However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
  - We should be able to perform addition, subtraction, multiplication, and division.
  - Operations should be commutative and associative.
  - Additive and multiplicative identity should exist.
  - Addition and multiplication should have inverses.
- We refer to a set with these properties as a **field**.

#### A field

A set  $\mathbb{F}$  with two operations + and  $\times$  (or  $\cdot$ ) is a **field** iff these operations satisfy the following properties:

- (Associativity): (a+b)+c=a+(b+c) and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ► (Commutativity): a + b = b + a and  $a \cdot b = b \cdot a$
- (Identities): There exist two elements  $0\in\mathbb{F}$  and  $1\in\mathbb{F}$  such that a+0=a and  $a\cdot 1=a$
- ▶ (Additive inverse): For every element  $a \in \mathbb{F}$ , there is an element  $-a \in \mathbb{F}$  such that a + (-a) = 0
- ▶ (Multiplicative inverse): For every element  $a \in \mathbb{F} \setminus \{0\}$ , there is an alement  $a^{-1}$  such that  $a \cdot a^{-1} = 1$
- ▶ (Distributive):  $a \cdot (b+c) = a \cdot b + a \cdot c$

# Another useful field: GF(2)

$$GF(2)=\{0,1\}.$$
 I.e., it is a "bit" field. What are  $+$  and  $\cdot$  in  $GF(2)$ ?

▶ We define  $b_1 + b_2$  to be XOR.

$$0+0=0 \\ 0+1=1+0=1 \\ 1+1=0$$

 $\blacktriangleright$  We define  $b_1 \cdot b_2$  to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$
  
 $1 \cdot 1 = 1$ 

You can check that GF(2) satisfies the axioms of fields.



