

# 01204211 Discrete Mathematics

## Lecture 3: Inference rules

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# How to prove a mathematical statement?

This lecture covers two fundamental concepts in mathematical proofs:

- ▶ Proofs by exhaustion
- ▶ Inference rules<sup>1</sup>

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<sup>1</sup>The materials on inference rules are from [Rosen].

# De Morgan's Laws

Given propositions  $P$  and  $Q$ , these are a very useful logical equivalences (referred to as the De Morgan's Laws).

$$\blacktriangleright \neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

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(Note that  $\neg$  takes precedence over  $\vee$  or  $\wedge$ .)

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In this case, since there are not too many cases to consider, we can enumerate all the possibilities to show that the proposition is true.

# Proof by exhaustion

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$P$	$Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg Q \wedge \neg P$
$T$	$T$			
$T$	$F$			
$F$	$T$			
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Note that for all possible truth values of  $P$  and  $Q$ ,  $\neg(P \vee Q)$  equals  $\neg P \wedge \neg Q$ . Thus, the statement is true. □

## Quick check 1

Prove the following statement by exhaustion.

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I have 2 pairs of socks in 2 colors: black and white. If I pick any 3 socks, I will have at least a pair of socks of the same color.

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This is clearly a brute force method. Sometimes, even in small cases, proofs by exhaustion can be very tedious and error-prone.

# Logical deduction (1)

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- ▶ It rains.
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If we believe in these statements (i.e., if we believe that they are all true), is it OK to conclude that:

- ▶ It is dangerous to drive very fast.

## Quick check 3

Define propositional variables representing each proposition inside these statements and write proposition forms of them.

- ▶ It rains.
- ▶ If it rains, then the road will get wet.
- ▶ If the road is wet, it will be dangerous to drive very fast.
- ▶ It is dangerous to drive very fast.

## Logical deduction (2)

Using that proposition variables, our problem translate to the following.

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There are 3 variables. These are all possible cases.

$R$	$W$	$D$
$T$	$T$	$T$
$T$	$T$	$F$
$T$	$F$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$T$	$F$
$F$	$F$	$T$
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We believe that  $R$ ,  $R \Rightarrow W$ , and  $W \Rightarrow D$  are true, and we want to conclude that  $D$  must be true.



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Proofs by exhaustion can be exhausted...

# Valid arguments (1)

Very often, the statement we want to prove is in the form:

Given:

- ▶ Hypothesis 1,
- ▶ Hypothesis 2,
- ▶ ...
- ▶ Hypothesis  $n$

Then:

- ▶ Conclusion

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Then:

- ▶ Conclusion

We say that the statement is **valid** if when all hypotheses are true, the conclusion must be true as well. In that case, we say that the conclusion **logically follows** from the hypotheses.

## Valid arguments (2)

More precisely, to show that conclusion  $Q$  logically follows from hypotheses  $P_1, P_2, \dots, P_n$ , we need to show that

$$(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow Q,$$

is always true, i.e., is a tautology.

## An example

Consider the following argument:

- ▶ Hypotheses:  $P$  and  $P \Rightarrow Q$
- ▶ Conclusion:  $Q$

Is this a valid argument?

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Is this a valid argument?

It is. See the following truth table.

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

## $R/W/D$ again

Since we know that the previous argument is valid, maybe we can use that “small” step in our previous example.

Recall our hypotheses:

- ▶  $R$
- ▶  $R \Rightarrow W$
- ▶  $W \Rightarrow D$

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Using the same reasoning, we can say that from  $R$  and  $R \Rightarrow W$ ,  $W$  logically follows.



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Recall our hypotheses:

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- ▶  $R \Rightarrow W$
- ▶  $W \Rightarrow D$

Using the same reasoning, we can say that from  $R$  and  $R \Rightarrow W$ ,  $W$  logically follows.

Then, since we know that  $W$  is now true, and  $W \Rightarrow D$ , we can conclude that  $D$  must follow.

## A rule of inference

The previous “small” valid step that we can use in our argument is extremely useful when making arguments. It is called *Modus ponens*, and is one of many useful rules of inference.

Modus ponens

$$\frac{P \quad P \Rightarrow Q}{Q}$$

# Other rules of inference

## Addition

$$\frac{P}{P \vee Q}$$

## Simplification

$$\frac{P \wedge Q}{P}$$

## Modus tollens

$$\frac{\neg Q \quad P \Rightarrow Q}{\neg P}$$

## Hypothetical syllogism

$$\frac{P \Rightarrow Q \quad Q \Rightarrow R}{P \Rightarrow R}$$

## Conjunction

$$\frac{P \quad Q}{P \wedge Q}$$

## Disjunctive syllogism

$$\frac{P \vee Q \quad \neg P}{Q}$$

## Using inference rules

Argue that  $P \Rightarrow Q$ ,  $(P \vee R)$ , and  $\neg R$  logically leads to the conclusion  $Q$ .

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1. $P \vee R$	Hypothesis

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2. $\neg R$	Hypothesis

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1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. $P$	Disjunctive syllogism using Step 1 and 2



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1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. $P$	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis

# Using inference rules

Argue that  $P \Rightarrow Q$ ,  $(P \vee R)$ , and  $\neg R$  logically leads to the conclusion  $Q$ .

Steps	Reasons
1. $P \vee R$	Hypothesis
2. $\neg R$	Hypothesis
3. $P$	Disjunctive syllogism using Step 1 and 2
4. $P \Rightarrow Q$	Hypothesis
5. $Q$	Modus ponens using Step 3 and 4.

## Other useful logical equivalences

We have discussed De Morgan's Laws, which are logical equivalences. The following logical equivalences are also useful when making valid arguments. (Notes: do not get confused with operator  $\Leftrightarrow$  and notation  $P \equiv Q$ .)

Equivalences	Names
$\neg(\neg P) \equiv P$	Double negation law
$(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$	Distributive law
$(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$	Distributive law
$P \Rightarrow Q \equiv \neg P \vee Q$	

## Another example

Argue that  $P \Rightarrow R$  and  $Q \Rightarrow R$  logically leads to the conclusion  $(P \vee Q) \Rightarrow R$ .

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## Another example

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### Steps

1.  $P \Rightarrow R$
2.  $\neg P \vee R$

### Reasons

Hypothesis  
Equivalence of Step 1

## Another example

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Steps	Reasons
1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis



## Another example

Argue that  $P \Rightarrow R$  and  $Q \Rightarrow R$  logically leads to the conclusion  $(P \vee Q) \Rightarrow R$ .

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1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3

## Another example

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1. $P \Rightarrow R$	Hypothesis
2. $\neg P \vee R$	Equivalence of Step 1
3. $Q \Rightarrow R$	Hypothesis
4. $\neg Q \vee R$	Equivalence of Step 3
5. $(\neg P \vee R) \wedge (\neg Q \vee R)$	Conjunction of Steps 2 and 4.

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2. $\neg P \vee R$	Equivalence of Step 1
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4. $\neg Q \vee R$	Equivalence of Step 3
5. $(\neg P \vee R) \wedge (\neg Q \vee R)$	Conjunction of Steps 2 and 4.
6. ... (left as homework)	