01204211 Discrete Mathematics
Lecture 2a: Quantifiers $\forall, \exists$

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ประโยค $\longrightarrow$ Proposition / Propositional form

ภาษาธรรมชาติ

"predicate"

Review (1)

$\underbrace{5+7}_{} = \underbrace{12}_{}$

$\neg(\neg P) \equiv P$

$\cancel{P \Leftrightarrow P} \equiv P$?

- A *proposition* is a statement which is either **true** or **false**.
- We can use variables to stand for propositions, e.g., $P =$ "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
  - **Conjunction:** $P \wedge Q$ ("P and Q"),
  - **Disjunction:** $P \vee Q$ ("P or Q"), and
  - **Negation:** $\neg P$ ("not P")
  - **Implication:** $P \Rightarrow Q$ ("P implies Q", "if P, then Q", "P, only if Q")
  - **Equivalence:** $P \Leftrightarrow Q$ ("P if and only if Q")

$\boxed{P \Leftrightarrow Q} \equiv \boxed{(P \Rightarrow Q) \wedge (Q \Rightarrow P)}$

ห้าม เฉพาะ

$\equiv$ equivalence.

นิพจน์ สมมูลกัน

Review (2): Testing primes

floor

$\lfloor x \rfloor$

ปดลง

Consider the following code.

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

$2, 3, \ldots, \lfloor \sqrt{n} \rfloor$

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time approximately proportional to $\sqrt{n}$ to run.

Ok, it should be faster. **But is it correct?**

The goals

- Let's recall what we are trying to do.

Original goal: To show that "Algorithm `CheckPrime2` is correct."

proposition

Current (sub) goal: Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.

$2 \leq b \leq \sqrt{n}$

The (sub) goal

- **Current (sub) goal:** Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
- We can be more specific about what values of n and b that we want to consider.

Revised statement

proposition

For all positive composite integer n, and for every divisor $\boxed{a}$ of n such that $\sqrt{n} < a < n$,

$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- Note that this revised statement is now "quantified," that is, every variable in the statement has specific scope. Now the statement is either true or false.

Predicates

Activity 01-3

- In many cases, the statement we are interested in contains variables.
- For example, "x is even," "p is prime," or "s is a student."

Predicates

- In many cases, the statement we are interested in contains variables.
- For example, "x is even," "p is prime," or "s is a student."
- As we previously did with propositions, we can use variables to represent these statements. E.g.,
  - let $E(x) \equiv$ "x is even",
  - let $P(y) \equiv$ "y is prime, and
  - let $S(w) \equiv$ "w is a student.

  friend(x)

  We call $E(x)$, $P(y)$, and $S(w)$ *predicates*. (You can think of predicates as statements that may be true of false depending on the values of its variables.)

Quantifiers (1)

"5 เป็นเลขคู่" – false

- As we note before, these predicates are not propositions. But if we know the values of their variables, then they becomes propositions. For example, if we let $x = 5$, then $E(5)$ is a proposition which is false. Also, $P(7)$ is true.

"7 เป็นจน. เฉพาะ" – true

- Since the truth values of predicates depend on the assignments of their variables, we can put $\boxed{\textit{quantifiers}}$ to specify the scopes of these variables and how to interprete the truth values of the predicates over these values.

$P(7)$ – proposition

Quantifiers (2): universal quantifiers $\forall$

- Let $A = \{2, 4, 6, 8\}$.
- Note that $E(2), E(4), E(6)$, and $E(8)$ are true, i.e., $E(x)$ is true $\boxed{\text{for every } x \in A.}$

Quantifiers (2): universal quantifiers

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  In this case, we say that the following proposition is true:

"ทุก $x \in A$, x เป็น จน.คู่"

$$\underbrace{(\forall x \in A)}\,\underbrace{E(x)}.$$

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$$(\forall x \in A) E(x).$$

- The quantifier $\forall$ is called a universal quantifier. (We usually pronounce "for all x", or "for every x.")

Quantifiers (3): existential quantifiers $\exists$

- Again, let $A = \{2, 4, 6, 8\}$.
- Note that $P(2)$ is true. This means that $P(y)$ is true for some $y \in A$.

"There exists y such that $P(y)$"

Quantifiers (3): existential quantifiers

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- The quantifier $\exists$ is called an existential quantifier. (We usually pronounce "for some x", or "there exists x.")

> When the universe A is clear, we can leave it out and just write $\forall x E(x)$ or $\exists y P(y)$.

The main goal

- Let's try to be more specific about our main goal:

> Algorithm `CheckPrime2` is correct.

- Can we re-write this statement so that the input/output of the algorithm are explicit?
- Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where $\mathbb{Z}$ denote the set of all integers.
- Let's rewrite the goal as:

$$\forall n \in \mathbb{Z},\ C(n) \Leftrightarrow P(n),$$

where $C(n) \equiv$

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where $C(n) \equiv$ "`CheckPrime2(n)` returns True", and
$P(n) \equiv$

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- Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where $\mathbb{Z}$ denote the set of all integers.
- Let's rewrite the goal as:

"quantify"

$$\forall n \in \mathbb{Z}, \; C(n) \Leftrightarrow P(n),$$

where $C(n) \equiv$ "`CheckPrime2(n)` returns True", and $P(n) \equiv$ "n is a prime."

$C(n) \Leftrightarrow P(n)$

Quantified propositions with more than one variables

Let our universe be integers $(\mathbb{Z})$ Which of the following statements is true?

⟨PAUSE⟩

- False ▶ $\forall x \forall y (x = y)$ לכל x לכל y $x=y$ –
- True ▶ $\forall x \exists y (x = y)$
- False ▶ $\exists x \forall y (x = y)$
- True ▶ $\exists x \exists y (x = y)$

$x=0,\ y=0$

$x=y$

ϕ

פשוט universe $\{0,1\}$?

-2 $\exists y=2$

-1 $\exists y=-1$

0 $\exists y=0$

1

2

$\vdots$

$\forall x$

-3

-2

-1

$0 \equiv y$

1

2

$\vdots$

$\exists x$

Quantified propositions with more than one variables

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- $\forall x \forall y (x = y)$
- $\forall x \exists y (x = y)$
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When you have many quantifiers, we can interprete the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

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Also note that usually, $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$.

Quick check 4

human(x) = "x is a human"
die(x) = "x dies"

We will consider the universe to be "everything". Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- Every human must die. $\forall x\,(human(x) \Rightarrow die(x))$
- Some animal eats other animals. $\exists x \exists y\,(A(x) \land A(y) \land E(x,y) \land O(x,y))$
  - $O(x,y)$: $[x \neq y]$
- If a student works hard, that student will be successful.
- Everyone has someone that care about him or her. them

$\forall x\,(student(x) \land wh(x) \Rightarrow successful(x))$

$\forall y \exists x\; care(x,y)$ care(x,y) = "x cares about y"

Quick check 5

- Let's consider the current subgoal. (Note that in this version, variable b is replaced with n/a.)

Another revised statement

For all positive composite integer n, and for every divisor a of n such that $\boxed{\sqrt{n} < a < n,}$

$$\boxed{2 \le n/a \le \sqrt{n}.}$$

C D(x,y) R b

$Q(x,y) = $ "$2 \le y/x \le \sqrt{y}$"

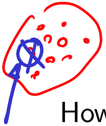
- Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

$\forall n \forall a [C(n) \wedge D(a,n) \wedge R(a,n) \Rightarrow Q(n/a, n)]$

Negations of quantified propositions (1)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $P(x) \equiv$ "x is a prime number."
Consider this proposition

ทุกๆ จน.เต็มบวก x
x เป็น จน.เฉพาะ



$$(\forall x \in \mathbb{Z}^+) P(x).$$

How can we show that this is false?

$$(\exists x \in \mathbb{Z}^+) \neg P(x)$$

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When showing that a universally quantified proposition is false, we need to show "one" counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.

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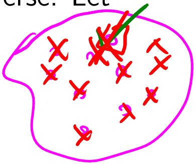
When showing that a universally quantified proposition is false, we need to show "one" counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.
This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions (2)

Let consider a set of positive integers $\mathbb{Z}^+$ as our universe. Let predicate $Q(x) \equiv$ "if $x > 2$, then $x^2 \leq 2x$."
Consider this proposition



$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

$$(\forall x)\neg Q(x)$$

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Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?
When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x. In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x)Q(x)$ is false.

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Negations of quantified propositions (3)

Thus, the following equivalences:

- $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$
- $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- Every human must die.
- Some animal eats other animals.
- If a student works hard, that student will be successful.
- Everyone has someone that care about him or her.

$$\neg[\forall y \exists x\, C(x,y)] \equiv \exists y [\neg [\exists x\, C(x,y)]]$$

$$\equiv \exists y \forall x [\neg C(x,y)]$$