# 01204211 Discrete Mathematics Lecture 9c: Linear Independence and Bases

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## Review: Linear combinations

### Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a linear combination of  $u_1, \ldots, u_m$ .

Review: Span

## Definition

A set of all linear combination of vectors  $u_1, u_2, \dots, u_m$  is called the **span** of that set of vectors.

It is denoted by  $\mathrm{Span}\{u_1,u_2,\ldots,u_m\}$ .

## **Previous Lemmas**

## Lemma 1

Consider vectors  $u_1, u_2, \dots, u_n$ . If  $v_1, v_2, \dots, v_k$  are generators for  $\mathcal{V}$ , and for each i,

$$v_i \in \operatorname{Span} \{u_1, u_2, \dots, u_n\},\$$

we have that  $\mathcal{V} \subseteq \mathrm{Span} \ \{ oldsymbol{u}_1, oldsymbol{u}_2, \ldots, oldsymbol{u}_n \}.$ 

# Lemma 2

Consider vectors  $u_1, u_2, \ldots, u_n$ . If  $v \in \text{Span } \{u_1, u_2, \ldots, u_n\}$ , then

Span  $\{u_1, u_2, \dots, u_n, v\}$  = Span  $\{u_1, u_2, \dots, u_n\}$ 

Span  $\{u_1, u_2, \dots, u_{n-1}\} = \text{Span } \{u_1, u_2, \dots, u_n\}$ 

Consider vectors  $u_1, u_2, \ldots, u_n$ . If  $u_n \in \text{Span } \{u_1, u_2, \ldots, u_{n-1}\}$ , then

Lemma 3

### Proof of Lemma 2.

Since v can be written as a linear combination of other vectors, there exist  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n.$$

Consider any vector  $w \in \text{Span } \{u_1, u_2, \dots, u_n, v\}$ ; thus, we can write

$$\boldsymbol{w} = \beta_0 \boldsymbol{v} + \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \dots + \beta_n \boldsymbol{u}_n.$$

Plugging in v, we get that

$$\mathbf{w} = \beta_0 (\alpha_1 \mathbf{u}_1 + \dots + \alpha_n \mathbf{u}_n) + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k$$
  
=  $(\beta_0 \alpha_1 + \beta_1) \mathbf{u}_1 + (\beta_0 \alpha_2 + \beta_2) \mathbf{u}_2 + \dots + (\beta_0 \alpha_n + \beta_n) \mathbf{u}_n$ ,

implying that  $\boldsymbol{w} \in \operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}.$ 

# Linearly independence

### Definition

Vectors  $u_1, u_2, \dots, u_n$  are **linearly independent** if no vector  $u_i$  can be written as a linear combination of other vectors.

## (Another) Definition

Vectors  $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_n$  are linearly independent if the only solution of equation

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_n \boldsymbol{u}_n = \boldsymbol{0}$$

is

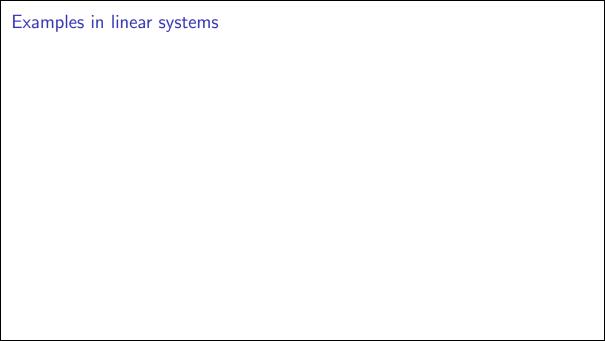
$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0.$$





Examples in  $\mathbb{R}^3$ 





# Subset of linearly independent vectors

### Lemma 4

If  $A = \{u_1, u_2, \dots, u_n\}$  be a set of linearly independent vectors, then any  $B \subseteq A$  is also a set of linearly independent vectors.

### Proof.

We prove by contradiction. Assume that B is **not** linearly independent. Without loss of generality, assume that  $B = \{u_1, u_2, \dots, u_k\}$  where  $k \leq n$ . This means that there exists  $\alpha_1, \alpha_2, \dots, \alpha_k$  such that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_k \boldsymbol{u}_k = \boldsymbol{0},$$

and some  $\alpha_i$ 's is nonzero. If we let  $\alpha_{k+1}=\alpha_{k+2}=\cdots=\alpha_n=0$ , we have that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_n \boldsymbol{u}_n = \boldsymbol{0},$$

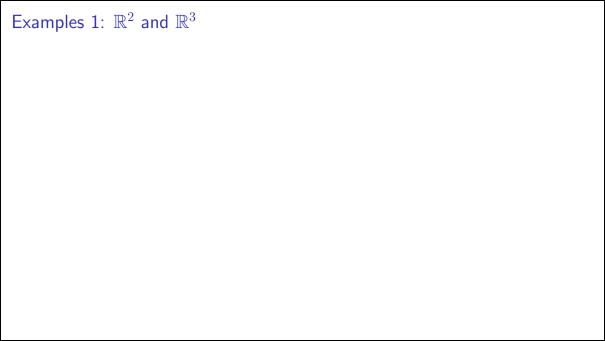
with some  $\alpha_i$ 's being nonzero as well. This implies that vectors in A are not linearly independent; leading to a contradiction.

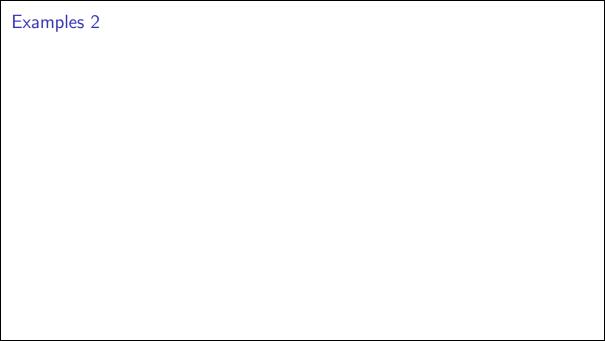
## **Bases**

## Definition

A set of vectors  $\{oldsymbol{u}_1,oldsymbol{u}_2,\dots,oldsymbol{u}_k\}$  is a <code>basis</code> for vector space  $\mathcal V$  if

- $ightharpoonup \operatorname{Span} \left\{ oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_k \right\} = \mathcal{V}$ , and
- $ightharpoonup u_1, u_2, \dots, u_k$  are linearly independent.





# Lemma 5 (Unique representation)

Let  $u_1, u_2, \ldots, u_k$  be a basis for vector space V. For any  $v \in V$ , there is a unique way to write v as a linear combination of  $u_1, \ldots, u_k$ .

## Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector  $v \in \mathcal{V}$  with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

and

$$\beta_1, \beta_2, \ldots, \beta_k,$$

that are not equal (i.e., there exists i where  $\alpha_i \neq \beta_i$ ) such that  $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_k \mathbf{u}_k$  and  $\mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_k \mathbf{u}_k$ . This implies that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k = \mathbf{v} = \beta_1 \mathbf{u}_1 + \beta \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k,$$

and

$$(\alpha_1 - \beta_1)\boldsymbol{u}_1 + (\alpha_2 - \beta_2)\boldsymbol{u}_2 + \dots + (\alpha_k - \beta_k)\boldsymbol{u}_k = 0.$$

Since  $\alpha_i \neq \beta_i$ , we have that at least one of the coefficients is non-zero, implying that  $u_1, \ldots, u_k$  are not linearly independent. This contradicts the assumption that  $u_1, \ldots, u_k$  form a basis.