

01204211 Discrete Mathematics

Lecture 2b: Quantifiers

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Review (1)

- ▶ A *proposition* is a statement which is either **true** or **false**.
- ▶ We can use variables to stand for propositions, e.g., $P =$ “today is Tuesday”.
- ▶ We can use connectives to combine variables to get propositional forms.
 - ▶ **Conjunction:** $P \wedge Q$ (“ P and Q ”),
 - ▶ **Disjunction:** $P \vee Q$ (“ P or Q ”), and
 - ▶ **Negation:** $\neg P$ (“not P ”)
 - ▶ **Implication:** $P \Rightarrow Q$ (“ P implies Q ”, “if P , then Q ”, “ P , only if Q ”)
 - ▶ **Equivalence:** $P \Leftrightarrow Q$ (“ P if and only if Q ”)

Review (2): Testing primes

Consider the following code.

```
Algorithm CheckPrime2(n): // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time approximately proportional to \sqrt{n} to run.

Ok, it should be faster. **But is it correct?**

The goals

- ▶ Let's recall what we are trying to do.

Original goal: To show that Algorithm CheckPrime2 is correct.

Current (sub) goal: Consider a positive composite n and its positive divisor a , where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.

The (sub) goal

- ▶ **Current (sub) goal:** Consider a positive composite n and its positive divisor a , where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
- ▶ We can be more specific about what values of n and b that we want to consider.

Revised statement

For all positive composite integer n , and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq b \leq \sqrt{n},$$

where $b = n/a$.

- ▶ Note that this revised statement is now “quantified,” that is, every variable in the statement has specific scope. Now the statement is either true or false.

Predicates

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- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ x is even,” “ p is prime,” or “ s is a student.”
- ▶ As we previously did with propositions, we can use variables to represent these statements. E.g.,
 - ▶ let $E(x) \equiv$ “ x is even”,
 - ▶ let $P(y) \equiv$ “ y is prime, and
 - ▶ let $S(w) \equiv$ “ w is a student.

We call $E(x)$, $P(y)$, and $S(w)$ *predicates*. (You can think of predicates as statements that may be true or false depending on the values of its variables.)

Quantifiers (1)

- ▶ As we note before, these predicates are not propositions. But if we know the values of their variables, then they become propositions. For example, if we let $x = 5$, then $E(5)$ is a proposition which is false. Also, $P(7)$ is true.
- ▶ Since the truth values of predicates depend on the assignments of their variables, we can put *quantifiers* to specify the scopes of these variables and how to interpret the truth values of the predicates over these values.

Quantifiers (2): universal quantifiers

- ▶ Let $A = \{2, 4, 6, 8\}$.
- ▶ Note that $E(2)$, $E(4)$, $E(6)$, and $E(8)$ are true, i.e., $E(x)$ is true for every $x \in A$.

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- ▶ The quantifier \forall is called a universal quantifier. (We usually pronounce “for all x ”, or “for every x .”)

Quantifiers (3): existential quantifiers

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- ▶ The quantifier \exists is called an existential quantifier. (We usually pronounce “for some x ”, or “there exists x .”)

When the universe A is clear, we can leave it out and just write $\forall x E(x)$ or $\exists y P(y)$.

The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where \mathbb{Z} denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where $C(n) \equiv$

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 $P(n) \equiv$

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$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where $C(n) \equiv$ "CheckPrime2(n) returns True", and
 $P(n) \equiv$ " n is a prime."

Quantified propositions with more than one variables

Let our universe be integers (\mathbb{Z}). Which of the following statements is true?

- ▶ $\forall x \forall y (x = y)$
- ▶ $\forall x \exists y (x = y)$
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When you have many quantifiers, we can interpret the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

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Also note that usually, $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$.

Quick check 4

We will consider the universe to be “everything”. Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.

Quick check 5

- ▶ Let's consider the current subgoal. (Note that in this version, variable b is replaced with n/a .)

Another revised statement

For all positive composite integer n , and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq n/a \leq \sqrt{n}.$$

- ▶ Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

Negations of quantified propositions (1)

Let consider a set of positive integers \mathbb{Z}^+ as our universe. Let predicate $P(x) \equiv$ “ x is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

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When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.

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When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.

This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions (2)

Let consider a set of positive integers \mathbb{Z}^+ as our universe. Let predicate $Q(x) \equiv$ “if $x > 2$, then $x^2 \leq 2x$.”

Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

Negations of quantified propositions (2)

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Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x . In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x)Q(x)$ is false.

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$$(\forall x)(\neg Q(x)).$$

Negations of quantified propositions (3)

Thus, the following equivalences:

► $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$

► $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.