01204211 Discrete Mathematics
Lecture 12a: Undecidability (1)

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September 19, 2023

Non-context-free languages

While

$$\{0^n 1^n \mid n \geq 0\}$$

non-regular

$S \to \varepsilon \mid 0S1$

is context free, the language

$$\{0^n 1^n 0^n \mid n \geq 0\}$$

is not.

Non-context-free languages

While

$$\{0^n 1^n \mid n \geq 0\}$$

is context free, the language

$$\{0^n 1^n 0^n \mid n \geq 0\}$$

is not.
Can we write a python program to check if a string w belongs to the language $\{0^n 1^n 0^n \mid n \geq 0\}$?

Big question

Is there a python program that "solves" any possible problem?

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Can a computer solve any problem?

language

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Is there a python program that "solves" any possible problem?
Can a computer solve any problem?
Is there an algorithm that solves every problem?
What is the limit of computation?

Answer by a counting argument

If there are “more” problems than any possible algorithms, then there should be some problem that algorithms cannot solve.
(Think of an algorithm as “a python program.”)

Answer by a counting argument

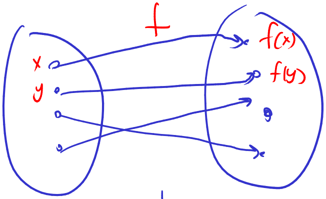
If there are “more” problems than any possible algorithms, then there should be some problem that algorithms cannot solve.
(Think of an algorithm as “a python program.)
However, there are infinitely many python programs and there are infinitely many problems. It is not obvious how to much such an argument formally.

Bijections

Definition

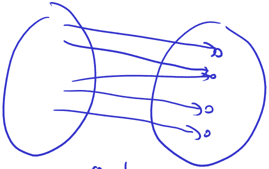
- A function $f : A \longrightarrow B$ from domain A to range B is **one-to-one** if for any $x \neq y \in A$, $f(x) \neq f(y)$.
- A function $f : A \longrightarrow B$ from domain A to range B is **onto** if for any $x' \in B$, there exists $x \in A$ such that $f(x) = x'$.
- A function $f : A \longrightarrow B$ is a **bijection** (or bijective) if it is one-to-one and onto.

Bijection: examples



one-to-one

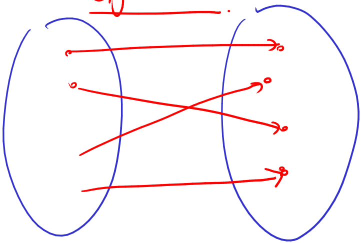
(ไม่ชน)



onto

(ใช้ครบ)

bijection.



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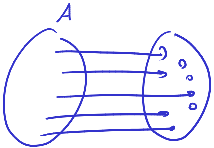

power set .

Lemma 1

For any set A, there is no bijective function $f : A \longrightarrow 2^A$.

Proof.

We prove by contradiction. Assume that there exists a bijective function f from A to 2^A. We construct a set $B \subseteq A$ such that there is no $x \in A$ such that $f(x) = B$.

B เป็นหลักฐาน ว่า f ไม่ onto .

$B \in 2^A$

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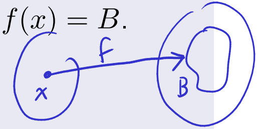
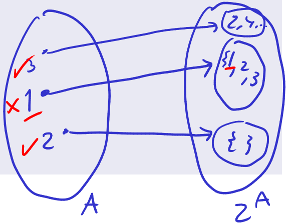


$$B = \{x \in A \mid x \notin f(x)\}.$$

Now suppose that there exists $\boxed{x} \in A$ such that $\boxed{f(x) = B}$. There are two cases to consider:

$= f(x)$

Case 1: $x \in B$. $\Rightarrow$ $x \in f(x)$ ✗ ขัดแย้งนิยาม B



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Now suppose that there exists $x \in A$ such that $f(x) = B$. There are two cases to consider:
Case 1: $x \in B$.
Case 2: $\underline{x \notin B} = f(x)$. $\Rightarrow$ $\underline{x \in B}$ จากนิยามของ B $*$

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We define B as follows.

$$B = \{x \in A \mid x \notin f(x)\}.$$

Now suppose that there exists $x \in A$ such that $f(x) = B$. There are two cases to consider:

Case 1: $x \in B$.

Case 2: $x \notin B$.

นี่แสดงไม่มี x ที่ $f(x) = B$

In both case, we have a contradiction; therefore, our assumption is false. Thus, there is no bijection between A and 2^A. □

Example: finite set

Let $A = \{1, 2, 3, 4, 5, 6, 7.\}$ Consider function $f : A \longrightarrow 2^A$ defined as

$$
\begin{aligned}
f(1) &= \{\} \\
f(2) &= \{1, 2, 3\} \\
f(3) &= \{1, 2, 3, 4, 5, 6, 7\} \\
f(4) &= \{1, 3, 5, 7\} \\
f(5) &= \{2, 4, 6\} \\
f(6) &= \{7\} \\
f(7) &= \{1, 2, 3\}
\end{aligned}
$$

$B = \{1, 4, 5, 6, 7\}$

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$$

	1	2	3	4	5	6	7
1	×	×	×	×	×	×	×
2	✓	✓	✓				
3	✓	✓	✓	✓	✓	✓	✓
4	✓		✓		✓		✓
5		✓		✓		✓	
6							✓
7	✓	✓	✓				

× ✓ × × ✓ ✓ ✓ ✓

$B = \{1, 4, 5, 6, 7\}$

Example: infinite set

Let $A = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Consider function $f : A \longrightarrow 2^A$ defined as

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f(7) &= \{1, 2, 3, 11, 12, 13, 21, 22, 23, \ldots\} \\
&\vdots \qquad \vdots
\end{aligned}
$$

	1	2	3	4	5	6	7	⋯
1	×	×	×					
2	✓	✓	✓					
3	✓	✓	✓	✓	✓	✓	✓	…
4	✓		✓		✓		✓	
5		✓		✓		✓		✓ …
6							✓	
7	✓	✓	✓					
⋮								
x	✓	×	×	✓	✓	✓	✓	…

$B = \{1$

$\Sigma = \{0,1\}$

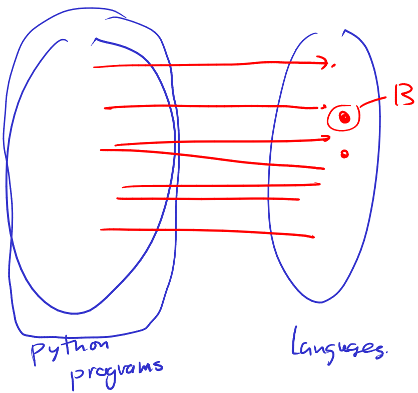
The previous lemma informally states that there are “more” subsets than the number of elements in the set.
Let’s think about:

- A set of all python programs, and
- A set of all languages.

$\subseteq \Sigma^* = A$

$2^{\Sigma^*} = 2^A$

1 language $L \subseteq \Sigma^*$



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But what exactly is a problem that cannot be “solved”?

Decision problems

- Given an integer x, is x odd?
- Given a string w, is w palindrome?
- Given a string w, is $w \in \{0^n 1^n \mid n \geq 0\}$?
- Given a map, a starting position s, a destination t, and an integer k, does there exist a path from s to t with distance at most k?
- Given a program P and input string w, when running P with w as an input, does P terminate?

Decision problems and languages

For this problem:

Given an integer x, is x ~~odd~~?
even

we can define a corresponding language

$$L_E = \{, \ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}.$$

To solve this problem, given x, we can ask if $x \in L_E$.

Programs and inputs

```
x = int(input())
if x % 2 == 0:
    print('yes')
else:
    print('no')
```

Programs and inputs

```
$ python le.py
10
yes
$ python le.py
7
no
```

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$ python le.py
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Traceback (most recent call last):
  File "le.py", line 1, in <module>
    x = int(input())
ValueError: invalid literal for int()
with base 10: 'fjdsklfjsdf'
```

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$ python le.py < le.py
```

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Nice programs

We can systematically modify any python program P so that

- P contains a main function that works with the input as a string.
- P never crashes. (If the original P crashes, the modified P outputs `no`.)

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```
x = int(input())
if x % 2 == 0:
    print('yes')
else:
    print('no')
```

```
import sys
def main(w):
    try:
        x = int(w)
        if x % 2 == 0:
            print('yes')
        else:
            print('no')
    except:
        print('no')

if __name__ == '__main__':
    w = sys.stdin.read()
    main(w)
```

main("hello")

main("2") = ...

When running a program

When you run a program P with input x, there are three possible outcomes:

- P terminates and outputs **yes**,
- P terminates and outputs **no**, and
- P does not terminate. (It runs forever.)

When running a program

When you run a program P with input x, there are three possible outcomes:

- P terminates and outputs **yes**,
- P terminates and outputs **no**, and
- P does not terminate. (It runs forever.)

Remarks: if P crashes (even after modification), we treat it as if it terminates and outputs **no**.

Deciders

We say that a python program P **decides** the language L if for any input string x, P when running with x as an input,

- P always terminates,
- P outputs **yes**, if $x \in L$, and
- P outputs **no**, if $x \notin L$.

Deciders: more examples

Language A

Let $\mathbb{P}$ be the set of all python programs. Let the language A be

$$\{P \in \mathbb{P} \mid \text{when running } P \text{ with } P \text{ as an input, } P \text{ terminates}\}$$

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We use a function call notation $P(x)$ when refering to the execution of program P with input x.
We restate the definition of A as

$$\{\underline{P} \in \mathbb{P} \mid \underline{P}(\underline{P}) \text{ terminates}\}.$$

python p.py < p.py $\Rightarrow$ $\underline{p(p)}$

Not a decider for A

```
Input: python program P (as a string)

1. Load module P as Pmod
2. Call Pmod.main(P)
3. print('yes')                  # we reach this line,
                                 #  only if M.main(P) terminates
```

ไม่ terminate ถ้า P(P) ไม่ terminate.

Lemma 2

There is no python program that decides A.

Proof.

We prove by contradiction. Assume that there is a python program P that decides A.

python p.py < p.py

Lemma 2

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$A = \{ P' \mid P'(P') \text{ terminates} \}$

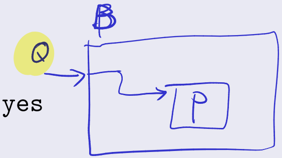
Proof.

We prove by contradiction. Assume that there is a python program P that decides A. We describe a python program B that reads a string Q as an input as follows:

program

```
Program B   สร้างจาก P
Input Q
    Load P as module Pmod
    if Pmod.main(Q) == 'yes':      # when Pmod outputs yes
        while True: pass           #   loop forever
    else:                          # when Pmod outputs no
        quit()                     #   halt
```

$Q \in A$, $Q(Q)$ termnate



Given program Q as an input, B loops forever when $Q(Q)$ terminates

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4.    else:                         # when Pmod outputs no
5.       quit()                     #   halt
```

ถ้า Q(Q) terminate.

Given program Q as an input, B loops forever when

It terminates when P ตอบ 'no' , Q(Q) ไม่ termint

∎

Proof.

We know that

p mn yes

- $B(Q)$ loops when $Q(Q)$ terminates, and

p nw no

- $B(Q)$ terminates when $Q(Q)$ loops.

Does running B using B as an input terminate? — $B(B)$????

Proof.

We know that

- $B(Q)$ loops when $Q(Q)$ terminates, and
- $B(Q)$ terminates when $Q(Q)$ loops.

Does running B using B as an input terminate?

Let's try to plug in $Q = B$. We have

- $B(B)$ loops when B(B) terminates ✗

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ไม่มีทางมี B

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- $B(B)$ loops when $B(B)$ terminates, and ✗
- $B(B)$ terminates when $B(B)$ loops. ✗

มี B ไม่ได้
มี P ไม่ได้

Since either $\boxed{B(B)}$ loops or terminates, and we cannot be in any of the cases, we obtain a contradiction.
Therefore, we conclude that program P does not exist. □

Undecidability

If we believe that anything that a computer can do can be written as a python program,

Undecidability

If we believe that anything that a computer can do can be written as a python program, and there is no python program that decides A, ~~w~~then we say that

> A is undecidable.

Language A will be very important later on, we give it a proper name as $\textsc{HaltA}$.

The proof as a table

List all python programs in $\mathbb{P}$ as $P_1, P_2, P_3, \ldots$.

input

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	$\ldots$
P_1	✓	✗	✓	✗	✓	✗	✓	✓
P_2	✗	✗	✗	✗	✗	✗	✗	
P_3	✓	✓	✗	✓	✓	✗	✓	
P_4	✓	✓	✓	✓	✓	✓	✓	
P_5	✗	✓	✗	✓	✗	✓	✓	
P_6	✗	✓	✓	✓	✗	✓	✗	
$\vdots$								
(B)	✗	✓	✓	✗	✓	✗		

What does B do on each input program P_i?

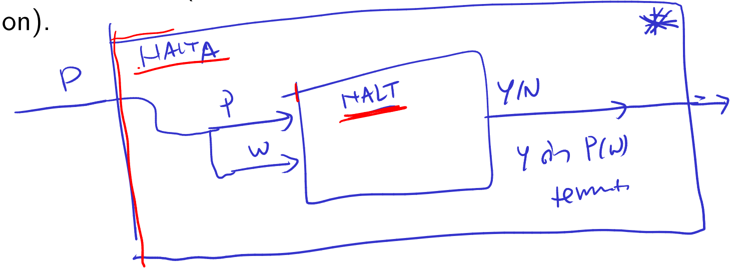
Another language HALT

HALTA = { P | P(P) terminates }

Let

$$\boxed{\text{HALT}} = \{(P, w) \mid P \text{ is a python program such that } P(w) \text{ terminates}\}$$

We shall prove that HALT is also undecidable (if we believe that python programs represent all possible computation).



Lemma 3

HALT *is undecidable.*

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides HALT.

Lemma 3

HALT *is undecidable.*

Proof.

We prove the lemma by contradiction. Assume that there is a python program $\boxed{P}$ that decides HALT.
We construct a program C as follows

Q: decide $\overline{\text{HALT}}_A$

```
Program C
Input Q
1.    Load P as module Pmod
2.    if Pmod.main(Q,Q) == 'yes':
3.        print('yes')
4.    else
5.        print('no')
```

□

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides HALT.
We construct a program C as follows

```
Program C
Input Q
1.   if P(Q,Q) == 'yes':
2.       print('yes')
3.   else
4.       print('no')
```

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides HALT.
We construct a program C as follows

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Program C
Input Q
1.    if P(Q,Q) == 'yes':
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3.    else
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```

Given program P, we can construct a program C that decides HALTA.

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides $\textsc{Halt}$.
We construct a program C as follows

```
Program C
Input Q
1.   if P(Q,Q) == 'yes':
2.       print('yes')
3.   else
4.       print('no')
```

Given program P, we can construct a program C that decides $\textsc{HaltA}$. However, we know that $\textsc{HaltA}$ is undecidable; thus, we reach a contradiction.
We conclude that there does not exist a python program P that decides $\textsc{Halt}$. $\square$

Reduction

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- We show that if HALT is decidable, then HALTA is also decidable.

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- However, HALTA IS UNDECIDABLE.

Reduction

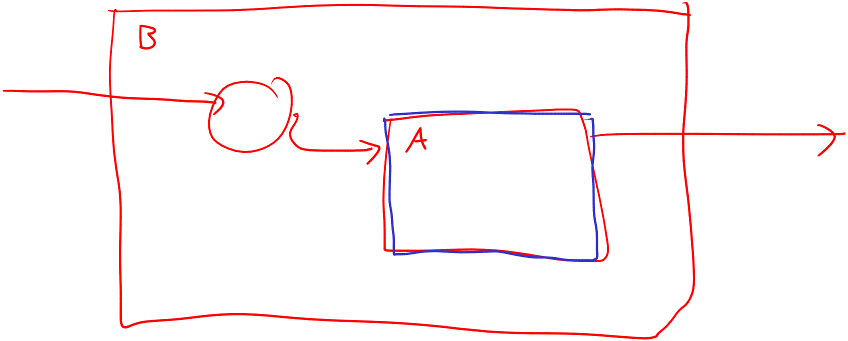
จะแสดงว่า HALT undecidable

- We show that if HALT is decidable (P), then HALTA is also decidable (Q). $\boxed{P \Rightarrow Q}$
- However, HALTA IS UNDECIDABLE. $\neg Q$
- We conclude that HALT is also undecidable. $\neg P$

Reduction in picture

ถ้า มี python prog สำหรับ A

⇒ มี python prog สำหรับ B



ถ้าทราบว่า ไม่มี python prog สำหรับ B

⇒ ไม่มี prog สำหรับ A ด้วย

Python as computation

Do you believe in this assumption:

Anything that a computer can do can be written as a python program.

Turing machines

Anything that a computer can do can be carried out using Turing machines.

Turing machines

Anything that a computer can do can be carried out using Turing machines.

Any possible computation can be performed by Turing machines.