01204211 Discrete Mathematics
Lecture 8: Mathematical Induction 3

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Review: Mathematical Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:
Base case: $P(1)$
Inductive step: For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$

The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

The Induction Hypothesis

Theorem 1

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

The Induction Hypothesis

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Proof.

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The induction hypothesis is: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$.
We want to show $P(k+1)$, i.e.,
$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2$.
Then... $\square$

Strengtening the Induction Hypothesis (1)

- Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

"strong" enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?

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Why?

- To prove $P(k+1)$, we need a "gap" between the LHS and 2, so that we can add $1/(k+1)$ without blowing up the RHS.

Strengtening the Induction Hypothesis (2)

- Let's see a few values of the sum:
  - $1/1 = 1$.

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  Yes, there is a gap. But how large?
- We need the gap to be large enough to insert $1/(k+1)^2$.
- After a "mysterious" moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Strengtening the Induction Hypothesis (3)

Theorem 2

For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

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Inductive step: For $k \geq 1$, assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.
Adding $1/(k+1)^2$ on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right).$$

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Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

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Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \leq 2 - \frac{1}{k+1},$$

as required. $\square$

A Lesson learned

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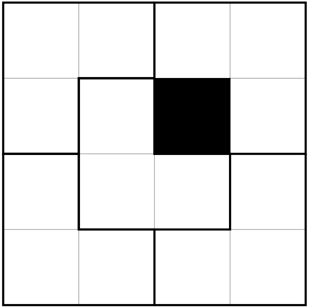
A Lesson learned

- Is a stronger statement easier to prove?
- In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.

L-shaped tiles (1)[1]

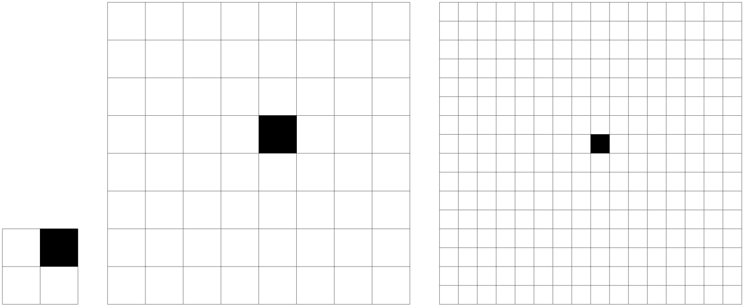
A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.



[1]This section is from Berkeley CS70 lecture notes.

L-shaped tiles (2)

This is true for 2x2 area, 8x8 area, even 16x16 area.



This motivates us to try to prove that it is possible to use L-shaped tiles to tile a $2^n \times 2^n$ area.

Proving the fact?

Theorem 3

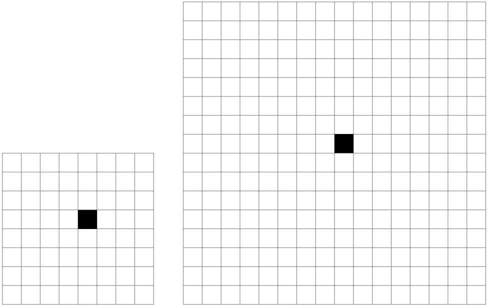
For integer $n \geq 1$, an area of size $2^n \times 2^n$ with one hole in the middle can be tiled with L-shaped tiles.

Proof: We prove by induction on n.
Base case: For $n = 1$, $2^1 \times 2^1$ area with a hole in the middle can be tiled.
Inductive step: Assume that for $k \geq 1$, an $2^k \times 2^k$ area with a hole in the middle can be tiled. We shall prove the statement for $n = k + 1$, i.e., that an $2^{k+1} \times 2^{k+1}$ area with one hole in the middle can be tiled.
(cont. on the next page)

Proving the fact?

Proof: (cont.)

Let's see the Induction Hypothesis and the goal:



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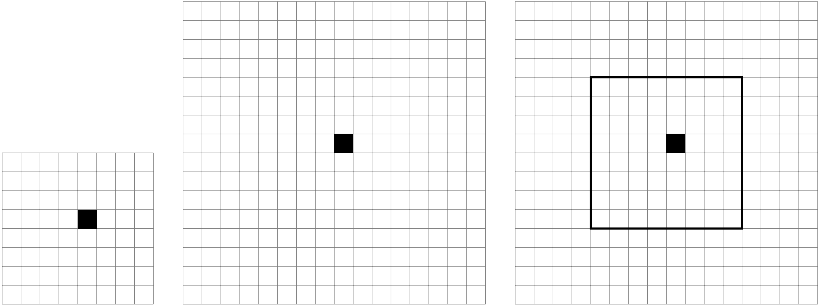

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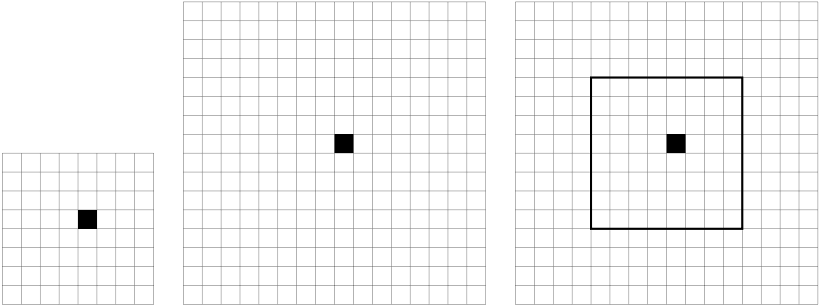
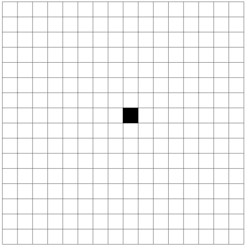


With the current form of the Induction Hypothesis, this is probably the way to use it. But it seems hard to go further with this approach....

Let's try a different approach



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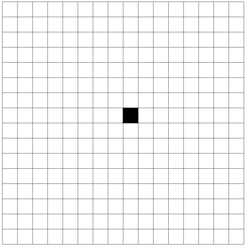

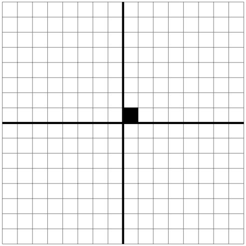

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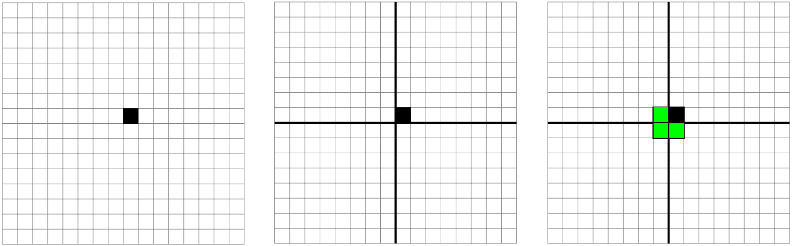

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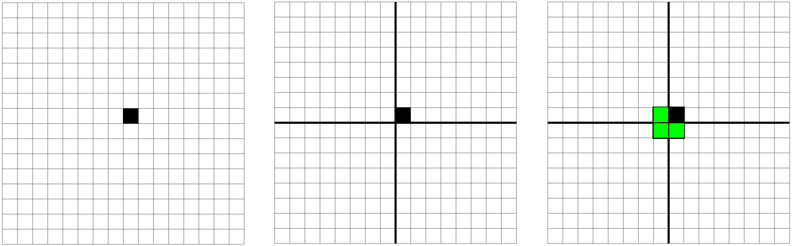

The last step seems nice, because it shows how we can solve the problem in the $2^{k+1} \times 2^{k+1}$ area with 4 problems in the $2^k \times 2^k$ areas.

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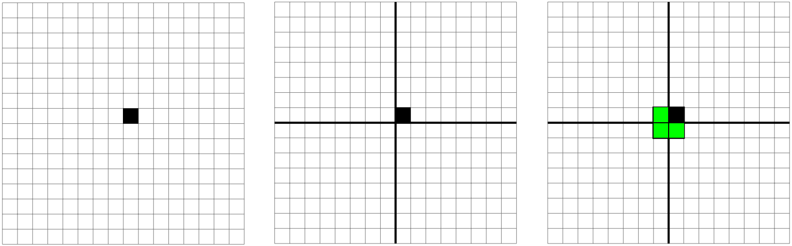

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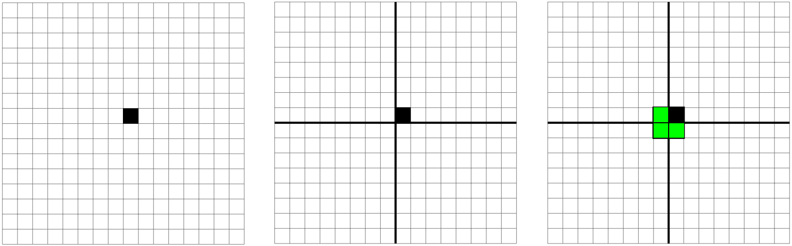

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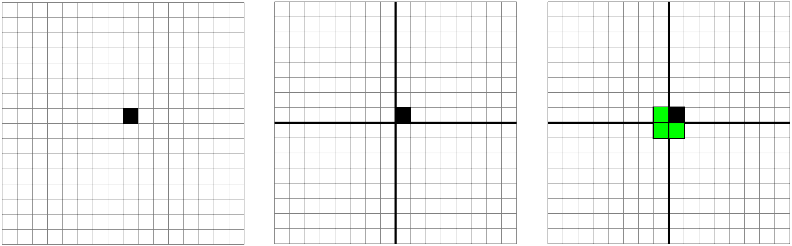

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A Stronger Inductive Hypothesis: Assume that for $k \geq 1$, an $2^k \times 2^k$ area with one hole can be tiled.

A stronger statement

Theorem: For integer $n \geq 1$, an area of size $2^n \times 2^n$ with one hole can be tiled with L-shaped tiles.
Proof: We prove by induction on n.
Base case: For $n = 1$, $2^1 \times 2^1$ area with one hole can be tiled;

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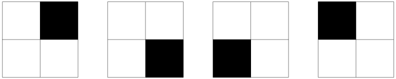

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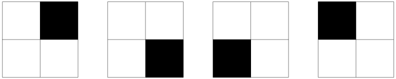

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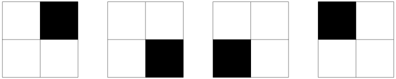

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(Try to finish it in homework.)

Proof of the Principle of Mathematical Induction[2]

Theorem 4

If $P(1)$ *and for any integer* $k \geq 1$, $P(k) \Rightarrow P(k+1)$, *then* $P(n)$ *for all natural number* n.

Proof.

[2] This section is from Berkeley CS70 lecture notes.

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Since m is smallest and $m > 1$, then $P(m-1)$ must be true.

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Since m is smallest and $m > 1$, then $P(m-1)$ must be true. However, because for any integer $k \geq 1$, $P(k) \Rightarrow P(k+1)$, we can conclude that $P(m)$ must be true. Again, we reach a contradiction.

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Therefore, $P(n)$ is true for every positive integer n. $\square$

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Therefore, $P(n)$ is true for every positive integer n. □

Is this proof correct?

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The Well-Ordering Property

- The proof of the Principle of Mathematical Induction depends on the following axiom of natural numbers $\mathbb{N}$:

The Well-Ordering Property: Any nonempty subset $S \subseteq \mathbb{N}$ contains the smallest element.

The Well-Ordering Property

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- Previously, we use the well-ordering property of natural numbers to prove the Principle of Mathematical Induction, but it turns out that we can use the induction to prove the well-ordering property as well. Therefore, we can take one as an axiom, and use it to prove the other.