

# 01204211 Discrete Mathematics

## Lecture 11: Counting 3

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## Quick recap

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  - ▶ We count the number of ways one can choose a subset.
  - ▶ We provide a bijection between subsets and binary strings.
  - ▶ We prove the fact by induction.
- ▶ For a set with  $n$  elements, the number of its permutations is  $n!$ .

# This lecture's goals<sup>1</sup>

- ▶ Consider set  $\{1, 2, 3, 4, 5\}$ . How many subsets with 3 elements does this set have?

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- ▶ We will also discuss the inclusion-exclusion principles.

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However, sometimes, it is useful to treat sets as ordered.

For example, for set  $\{1, 2, 3\}$ , there are 6 ordered subsets with 2 elements:  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 1\}$ ,  $\{2, 3\}$ ,  $\{3, 1\}$ ,  $\{3, 2\}$ .

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- ▶ For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- ▶ Therefore, we conclude that the number of ways is  $10 \cdot 9 \cdot 8$ .

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- ▶ Let  $X$  be the set of ordered subsets with 3 elements of an 10-set. We then have  $|X| \times 7! = 10!$ , because they count the same objects. Solving this yields

$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

## General answers: numbers of ordered subsets

Using the same arguments (either one), we have this theorem.

### Theorem 1

*The number of ordered subsets with  $k$  elements of an  $n$ -set is*


$$n \cdot (n - 1) \cdots (n - k + 1) = \frac{n!}{(n - k)!}.$$



# How big is $100!$ ?

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## Bounds for $n!$

Recall that  $n! = 1 \cdot 2 \cdot 3 \cdots n$ . Since all its factor, except the first one is at least 2, we have that

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$n$	$2^{n-1}$	$n!$	$n^{n-1}$
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	3,628,800	1,000,000,000

## A better bound?

Let's consider  $n!$  again, but for simplicity, let's consider only the case when  $n$  is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2 - 1) \cdot (n/2) \cdot (n/2 + 1) \cdots n$$

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To get a better lower bound, we may move our cutting point from 2 to, say,  $n/2$ . Note that at least  $n/2$  factors are at least  $n/2$ . Thus,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots n \\ &\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2} \\ &= (n/2)^{n/2} = \sqrt{(n/2)^n}. \end{aligned}$$

## Better?

$n$	$2^{n-1}$	$\sqrt{(n/2)^n}$	$n!$	$n^{n-1}$
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	7,776
10	512	3,125	3,628,800	1,000,000,000
12	2,048	46,656	479,001,600	743,008,370,688

OK. A bit better.



# Stirling's formula

An even better estimate for  $n!$  exists.

## Theorem 2 (Stirling's formula)

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

When we write  $a(n) \sim b(n)$ , we mean that  $\frac{a(n)}{b(n)} \rightarrow 1$  as  $n \rightarrow \infty$ .

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With Stirling's formula, We can use a calculator to estimate the number of digits for  $100!$ . The estimate for  $100!$  is

$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log \left( (100/e)^{100} \cdot \sqrt{200\pi} \right) = 100 \log(100/e) + \log(200\pi) \approx 157.9696.$$

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Note that the correct answer is 158 digits.

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- ▶ This upper bound of  $n^2$  is very good as the gaps between the upper bounds and the actual values will not be larger than 2, as  $\frac{n^2}{n(n+1)/2} < 2$ .

# The number of subsets

**Theorem:** The number of  $k$ -subsets of an  $n$ -set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.



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$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of  $k$ -subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required.

# Binomial coefficients

The number of  $k$ -subsets of an  $n$ -set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

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- ▶  $\binom{n}{0} = 1$  (why?), and,
- ▶ when  $k > n$ ,  $\binom{n}{k} = 0$ .



# Properties (1)

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}.$$

## Properties (2)

**Theorem:** When  $n, k > 0$ , then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

## Properties (3)

**Theorem:** When  $n, k > 0$ , then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

## Quick questions (1)

There are 40 students in the classroom. There are 35 students who like Naruto, 10 students who like Bleach, and 7 students who like both of them. How many students in this classroom who do not like either Bleach or Naruto?

## Quick questions (2)

There are 35 students in the classroom. There are 25 students who like Naruto, 15 students who like Bleach, 12 students who like One Piece. There are 10 students who like both Naruto and Bleach, 7 students who like both Bleach and One Piece, and 9 students who like both Naruto and One Piece. There are 5 students who like all of them.

How many students in this classroom who do not like any of Bleach, Naruto, or One Piece?

# Is this correct?

The answer from the previous quick question is

$$35 - (25 + 15 + 12 - 10 - 7 - 9 + 5) = 4.$$

Is this correct? Why?

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Is this correct? Why?

Let's try to argue that this answer is, in fact, correct and try to find general answers to this kind of counting questions.

## Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	+10	+7	+9	-5	4
Alfred	N,O									



## Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	+10	+7	+9	-5	4
Alfred	N,O	*	*		*			*		
Bobby	B									

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Bobby	B	*		*						
Cathy	B,O									

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Alfred	N,O	*	*		*			*		
Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O									

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Alfred	N,O	*	*		*			*		
Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-									

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Alfred	N,O	*	*		*			*		
Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-	*								
⋮	⋮									

## Let's look at an individual student (2)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	+10	+7	+9	-5	4
Alfred	N,O	1	-1		-1			+1		0
Bobby	B	1		-1						0
Cathy	B,O	1		-1	-1		+1			0
Dave	N,B,O	1	-1	-1	-1	+1	+1	+1	-1	0
Eddy	-	1								1
⋮	⋮									

# Let's see how each one is counted

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Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} =$$

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Do you see any patterns here?

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Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0$$

Do you see any patterns here? How about

$$1 - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} \quad ?$$

# Underlying structures

Let's write 1 as  $\binom{5}{0}$ . Also, let's separate plus terms and minus terms:

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} \quad \heartsuit \quad \binom{5}{1} + \binom{5}{3} + \binom{5}{5}$$



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Note that the left terms are the number of even subsets and the right terms are the number of odd subsets. Do you recall one of the homework?

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Note that the left terms are the number of even subsets and the right terms are the number of odd subsets. Do you recall one of the homework? We have proved this:

**Theorem:** The number of even subsets is equal to the number of odd subsets.

This theorem also shows that our calculation technique is correct. This technique is usually called the **Inclusion-Exclusion principle**.