01204211 Discrete Mathematics
Lecture 10b: Dimensions

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Review: Vector spaces

span

hom line

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,

$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

  for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,

$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

Review: Linearly independence

Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if no vector $\boldsymbol{u}_i$ can be written as a linear combination of other vectors.

(Another) Definition

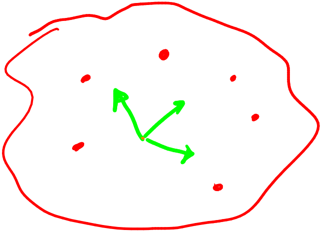
Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if the only solution of equation

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n = \mathbf{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Review: Bases



Definition

A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a **basis** for vector space $\mathcal{V}$ if

- $\text{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \mathcal{V}$, and
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

Lemma 1 (Superfluous Vector Lemma)

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

$$\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

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Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$. *If* $\underline{\boldsymbol{u}_n \in \operatorname{Span}}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{n-1}\}$, *then*

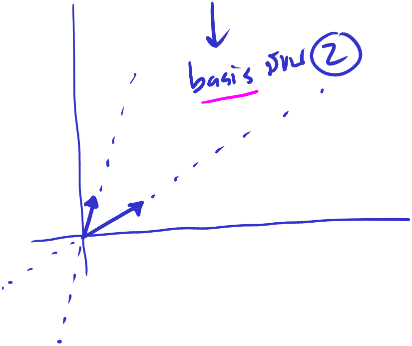
$$\operatorname{Span}\{\underline{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{n-1}}\} = \operatorname{Span}\{\underline{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n}\}$$

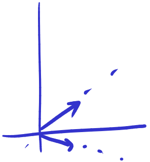
Lemma 3 (Unique representation)

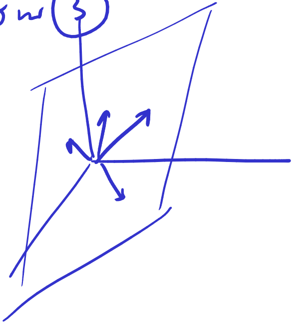
Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be a basis for vector space $\mathcal{V}$. For any $\boldsymbol{v} \in \mathcal{V}$, there is a unique way to write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$.

Examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ → basis ขนาด 3

basis ขนาด 2







Examples in $GF(2)$ - Vector spaces from graphs

Examples in $GF(2)$ - Cycles

Examples in $GF(2)$ - Basis

Number of vectors in bases

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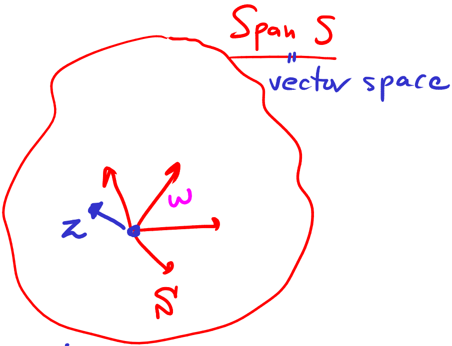
- This is not a coincident.
- In this course, we will see two proofs.
- Remark: there are vector spaces whose basis has infinite size, but we are not dealing with those vector spaces in this course.

Theorem 4 (Main Theorem)

If $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are bases for vector space $\mathcal{W}$, then $n = m$.

Exchange Lemma

We will prove the main theorem using the "exchange" lemma.

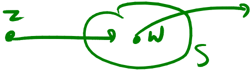


$z \in \operatorname{Span} S$

$$\operatorname{Span} S = \operatorname{Span}(S \cup \{z\})$$

หา $w \in S$ ที่ $\operatorname{Span}(S \cup \{z\} - \{w\}) = \operatorname{Span} S$

Exchange Lemma



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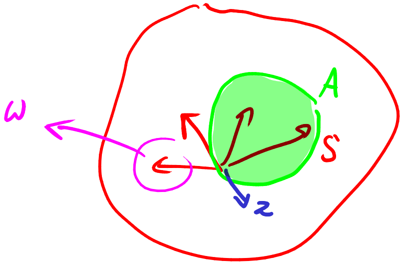
Lemma 5 (Simplified Exchange Lemma)

Consider a set of vectors S and let $\boldsymbol{z}$ be a non-zero vector in $\mathrm{Span}\ S$. *There is a vector $\boldsymbol{w} \in S$ such that* $\mathrm{Span}\ (S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\}) = \mathrm{Span}\ S$.



มี $w \in S - A$
ที่ $\mathrm{Span}\ (S \cup \{z\} - \{w\})$
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$z \in \mathrm{Span}\ S$, $A \cup \{z\}$ lin. indep.

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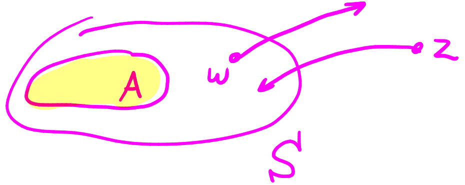
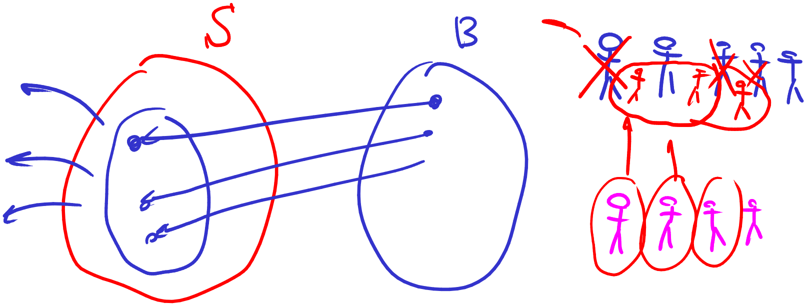
Lemma 6 (Exchange Lemma)

Consider a set of vectors S and its subset A. Let $\boldsymbol{z}$ be a non-zero vector in $\mathrm{Span}\ S$ *such that $\boxed{A \cup \{\boldsymbol{z}\}}$ is linearly independent. There is a vector $\boldsymbol{w} \in S - A$ such that* $\mathrm{Span}\ (S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\}) = \mathrm{Span}\ S$.



Lemma 7 (Morphing Lemma)

If a set of vectors S spans a vector space $\mathcal{W}$ and B is a linearly independent set of vectors in $\mathcal{W}$, then $|B| \le |S|$.



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Proof.

Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. We show how to construct $S_1, \ldots, S_n$ such that for each i, $|S_i| = |S|$, $\text{Span } S_i = \text{Span } S$, and

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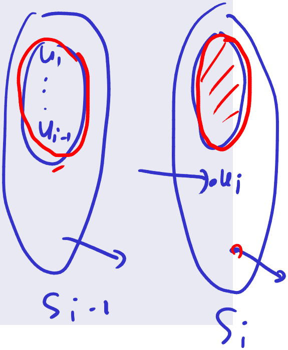

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Since $|S_n| = |S|$ and $B \subseteq S_n$, we have that $|B| \leq |S|$. □

Morphing Lemma $\Rightarrow$ Main Theorem

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$$\left.\begin{array}{l} m \le n \\ n \le m \end{array}\right\} \quad n = m$$

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Therefore, $n = m$. $\square$

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$\alpha_i \neq 0$

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How can we prove the full lemma? $u_i \notin A$

Dimensions

เป็นคุณสมบัติของ Vector space
(ไม่ขึ้นกับวิธีเลือก basis)

Definition

The **dimension** of a vector space $\mathcal{V}$ is the size of its basis.
The dimension of $\mathcal{V}$ is written as $\boxed{\dim \mathcal{V}}$.