01204211 Discrete Mathematics
Lecture 4c: Mathematical Induction 3

Jittat Fakcharoenphol

July 17, 2021

Review: Mathematical Induction

> Suppose that you want to prove that property $P(n)$ is true for every natural number n.
>
> Suppose that we can prove the following two facts:
> **Base case:** $P(1)$
> **Inductive step:** For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$
>
> The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

The Induction Hypothesis

Theorem 1
For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

The Induction Hypothesis

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Proof.
The statement $P(n)$ that we want to prove is
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Inductive step: For $k \geq 1$, let's assume $P(k)$ and we prove that $P(k+1)$ is true.
The induction hypothesis is: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$.
We want to show $P(k+1)$, i.e.,
$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2$.
Then... $\qquad \square$

Strengtening the Induction Hypothesis (1)

▶ Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

"strong" enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?

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Why?

▶ To prove $P(k+1)$, we need a "gap" between the LHS and 2, so that we can add $1/(k+1)$ without blowing up the RHS.

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  - $1/1 = 1$.

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  - ▶ $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236$.

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  Yes, there is a gap. But how large?

- We need the gap to be large enough to insert $1/(k+1)^2$.

- After a "mysterious" moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Strengtening the Induction Hypothesis (3)

Theorem 2
For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

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Proof.
(... the beginning is left out ...)

Inductive step: For $k \geq 1$, assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.

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Inductive step: For $k \geq 1$, assume that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.
Adding $1/(k+1)^2$ on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right).$$

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$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right).$$

Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

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Since $1/k - 1/(k+1) = 1/(k(k+1))$, we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right) \leq 2 - \frac{1}{k+1},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A Lesson learned

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- ▶ Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.