01204211 Discrete Mathematics
Lecture 7a: Binomial Coefficients (1)

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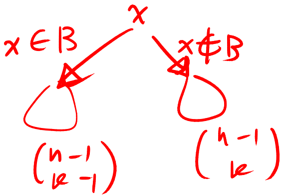
The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem

[1] This lecture mostly follows Chapter 3 of [LPV].

The equation



Last time we proved that, for $n, k > 0$,

เลือกของ k ชิ้น จาก n ชิ้น $\leftarrow$

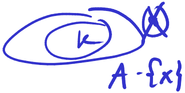
$$\boxed{\binom{n}{k}} = \underbrace{\binom{n-1}{k-1} + \binom{n-1}{k}}.$$

Proof: พิจารณา set A ที่ $|A| = n$ ให้ $\boxed{x} \in A$

k-subset ทั้งหมดของ A แบ่งได้ 2 แบบ

- case 1: มี x, มีกี่ subsets? $\binom{n-1}{k-1}$.

- case 2: ไม่มี x
  มี $\binom{n-1}{k}$.



ดังนั้น จำนวน k-subset ของ A = $\binom{n-1}{k-1} + \binom{n-1}{k}$

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Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k-subsets reveals interesting insights. This equation also hints us how to compute the value of $\boxed{\binom{n}{k}}$ using values of $\binom{n-1}{\cdot}$'s.

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The table

We shall use the fact that $\binom{n}{0} = 1$ and $\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				

$\binom{2}{1}$ $\binom{2}{2}$

$=$

$$\binom{2}{1} = \binom{1}{0} + \binom{1}{1}$$

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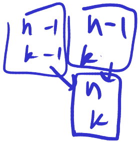

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$$\binom{4}{1} = \binom{3}{0} + \binom{3}{1}$$

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You can note that the table is left-right symmetric. This is true because of the fact that $\boxed{\binom{n}{k} = \binom{n}{n-k}.}$

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & \vdots & & \vdots & & & & & & & & \vdots & &
\end{array}
$$

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x + y$
- $(x+y)^2 = x^2+2xy+y^2$

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- $(x+y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?*

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$(x+y)^n = x^n$

$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $\boxed{x^{n-1}y.}$ Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x+y)^4$ again. It is

$$(x+y)(x+y)(x+y)(x+y).$$

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- How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3} = \binom{4}{1}$ ways to do so.

The binomial theorem

$\overbrace{(x+y)(x+y)\cdots(x+y)}^{n}$

Theorem: If you expand $(x+y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \cdots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients. For example, if we let $x = 1$ and $y = 1$, we get that

$$\underline{(1+1)^n} = 2^n = \underbrace{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}}.$$

$$\underline{x}^{\cancel{k}} \cdot \underline{y}^{n-k} = 1$$

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Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.