

01204211 Discrete Mathematics  
Lecture 11b: Four fundamental subspaces (II)

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## What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[ \begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array} \right]$$

# Four fundamental subspaces

## Four fundamental subspaces

Given an  $m$ -by- $n$  matrix  $A$ , we have the following subspaces

- ▶ The column space of  $A$  (denoted by  $\mathcal{R}(A) \subseteq \mathbb{R}^m$  )
- ▶ The row space of  $A$  (denoted by  $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$  )
- ▶ The nullspace of  $A$

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- ▶ The left nullspace of  $A$

$$\mathcal{N}(A^T) = \{\mathbf{y} \mid A^T\mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

## Four fundamental subspaces

# Ranks

## Definition

Consider an  $m$ -by- $n$  matrix  $A$ .

- ▶ The **row rank** of  $A$  is the maximum number of linearly independent rows of  $A$ .
- ▶ The **column rank** of  $A$  is the maximum number of linearly independent columns of  $A$ .

**Remark:** The column rank of  $A$  is  $\dim \mathcal{R}(A)$ . The row rank of  $A$  is  $\dim \mathcal{R}(A^T)$ .

## Theorem 1

*For any matrix  $A$ , its row rank equals its column rank.*

### Proof.

Let  $r$  be the column rank. We will show that there are  $r$   $n$ -vectors that span its row space. This implies that the row rank is at most  $r$ . We can use the same argument again on  $A^T$  to obtain that the column rank is at most the row rank; thus, they must be equal.



## Proof (cont.)



## Rank and nullity

Given an  $m$ -by- $n$  matrix  $A$ , the rank of  $A$  is  $\dim \mathcal{R}(A)$ . Let  $r$  be the rank of  $A$ . What is  $\dim \mathcal{N}(A)$ ?



# Dimensions

## Four fundamental subspaces

Given an  $m$ -by- $n$  matrix  $A$  of rank  $r$ , we have the following subspaces

- ▶ The column space of  $A$  (denoted by  $\mathcal{R}(A) \subseteq \mathbb{R}^m$ )  
 $\dim \mathcal{R}(A) = r$ .
- ▶ The row space of  $A$  (denoted by  $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$ )  
 $\dim \mathcal{R}(A^T) = r$ .
- ▶ The nullspace of  $A$  (denoted by  $\mathcal{N}(A) \subseteq \mathbb{R}^n$ )  
 $\dim \mathcal{N}(A) = n - r$ .
- ▶ The left nullspace of  $A$  (denoted by  $\mathcal{N}(A^T) \subseteq \mathbb{R}^m$ )  
 $\dim \mathcal{N}(A^T) = m - r$ .

## Application: Singular Value Decomposition (SVD)

Any  $n$ -by- $d$  matrix  $A$  can be factored into the form of  $UDV^T$ , i.e.,

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

where

- ▶  $U$  is an  $n$ -by- $r$  matrix,
- ▶  $D$  is a diagonal  $r$ -by- $r$  matrix, and
- ▶  $V$  is an  $d$ -by- $r$  matrix (i.e.,  $V^T$  is an  $r$ -by- $d$  matrix)
- ▶ (Also, columns of  $U$  and  $D$  are “orthonormal.”)

See demo.

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \mathbf{u}_i & U \end{bmatrix} \begin{bmatrix} d_{ii} & \\ & D \end{bmatrix} \begin{bmatrix} \mathbf{v}_i^T \\ V^T \end{bmatrix}$$

$$A = d_1 \mathbf{u}_1 \mathbf{v}_1^T + d_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + d_r \mathbf{u}_r \mathbf{v}_r^T.$$