

01204211 Discrete Mathematics

Lecture 2: Quantifiers

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July 10, 2019

This lecture covers:

- ▶ More connectives: implications and equivalences
- ▶ Quantifiers

Review (1)

- ▶ A *proposition* is a statement which is either **true** or **false**.
- ▶ We can use variables to stand for propositions, e.g., $P =$ “today is Tuesday”.
- ▶ We can use connectives to combine variables to get propositional forms.
 - ▶ **Conjunction:** $P \wedge Q$ (“ P and Q ”),
 - ▶ **Disjunction:** $P \vee Q$ (“ P or Q ”), and
 - ▶ **Negation:** $\neg P$ (“not P ”)

Review (2)

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1) $P \wedge \neg P$ and (2) $P \vee \neg P$.

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Note that $P \wedge \neg P$ is always false and $P \vee \neg P$ is always true. A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Implications¹

Given P and Q , an implication

$$P \Rightarrow Q$$

stands for “if P , then Q ”. This is a very important propositional form.

It states that “when P is true, Q must be true”. Let’s try to fill in its truth table:

Implications

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¹Materials in this lecture are mostly from Berkeley CS70’s lecture notes.

What?

- ▶ Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
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- ▶ Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
- ▶ We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true*.
- ▶ You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P , then Q we sometimes mean something more than that in the logical expression " $P \Rightarrow Q$."

One explanation

- ▶ But let's look closely at what it means when we say that:

if P is true, Q must be true.

- ▶ Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.

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- ▶ Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.
- ▶ Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.

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- ▶ Note that this statement does not say anything about the case when P is false, i.e., it only considers the case when P is true.
- ▶ Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.
- ▶ This is an example when mathematical language is “stricter” than natural language.

Noticing if-then

We can write “if P , then Q ” for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P , (2) P only if Q , or (3) when P , then Q .

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Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- ▶ If you do not have enough sleep, you will feel dizzy during class.
- ▶ If you eat a lot and you do not have enough exercise, you will get fat.
- ▶ You can get A from this course, only if you work fairly hard.

Only-if

Let P be “you get A from this course.”

Let Q be “you work fairly hard.”

Let R be “You can get A from this course, only if you work fairly hard.”

Let's think about the truth values of R .

Only if you work fairly hard.

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Thus, R should be logically equivalent to $P \Rightarrow Q$. (We write $R \equiv P \Rightarrow Q$ in this case.)

If and only if: (\Leftrightarrow)

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the statement “ P if and only if Q .” It is logically equivalent to

$$(P \Leftarrow Q) \wedge (P \Rightarrow Q),$$

i.e., $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \wedge (P \Rightarrow Q)$.

Let's fill in its truth table.

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

An implication and its friends

When you have two propositions

- ▶ $P =$ “I own a cell phone”, and
- ▶ $Q =$ “I bring a cell phone to class”.

We have

- ▶ an implication $P \Rightarrow Q \equiv$
“If I own a cell phone, I’ll bring it to class”,
- ▶ its **converse** $Q \Rightarrow P \equiv$
“If I bring a cell phone to class, I own it”, and
- ▶ its **contrapositive** $\neg Q \Rightarrow \neg P \equiv$
“If I do not bring a cell phone to class, I do not own one”.

Quick check 3

Let's consider the following truth table:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
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T	F			
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Do you notice any equivalence?

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Right, $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$.

How about our subgoal?

- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ x is even,” “ p is prime,” or “ s is a student.”

How about our subgoal?

- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ x is even,” “ p is prime,” or “ s is a student.”
- ▶ As we previously did with propositions, we can use variables to represent these statements. E.g.,
 - ▶ let $E(x) \equiv$ “ x is even”,
 - ▶ let $P(y) \equiv$ “ y is prime, and
 - ▶ let $S(w) \equiv$ “ w is a student.

We call $E(x)$, $P(y)$, and $S(w)$ *predicates*. (You can think of predicates as statements that may be true or false depending on the values of its variables.)

Quantifiers (1)

- ▶ As we note before, these predicates are not propositions. But if we know the values of their variables, then they become propositions. For example, if we let $x = 5$, then $E(5)$ is a proposition which is false. Also, $P(7)$ is true.
- ▶ Since the truth values of predicates depend on the assignments of their variables, we can put *quantifiers* to specify the scopes of these variables and how to interpret the truth values of the predicates over these values.

Quantifiers (2): universal quantifiers

- ▶ Let $A = \{2, 4, 6, 8\}$.
- ▶ Note that $E(2)$, $E(4)$, $E(6)$, and $E(8)$ are true, i.e., $E(x)$ is true for every $x \in A$.

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$$(\forall x \in A)E(x).$$

- ▶ The quantifier \forall is called a universal quantifier. (We usually pronounce “for all x ”, or “for every x .”)

Quantifiers (3): existential quantifiers

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- ▶ The quantifier \exists is called an existential quantifier. (We usually pronounce “for some x ”, or “there exists x .”)

When the universe A is clear, we can leave it out and just write $\forall x E(x)$ or $\exists y P(y)$.

The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input n is an integer. Thus, we are interested in every $n \in \mathbb{Z}$, where \mathbb{Z} denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where $C(n) \equiv$

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 $P(n) \equiv$

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where $C(n) \equiv$ "CheckPrime2(n) returns True", and
 $P(n) \equiv$ " n is a prime."

Quantified propositions with more than one variables

Let our universe be integers (\mathbb{Z}). Which of the following statements is true?

- ▶ $\forall x \forall y (x = y)$
- ▶ $\forall x \exists y (x = y)$
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When you have many quantifiers, we can interpret the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

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Also note that usually, $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$.

Quick check 4

We will consider the universe to be “everything”. Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.

Quick check 5

- ▶ Let's consider the current subgoal. (Note that in this version, variable b is replaced with n/a .)

Another revised statement

For all positive composite integer n , and for every divisor a of n such that $\sqrt{n} < a < n$,

$$2 \leq n/a \leq \sqrt{n}.$$

- ▶ Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

Negations of quantified propositions (1)

Let consider a set of positive integers \mathbb{Z}^+ as our universe. Let predicate $P(x) \equiv$ “ x is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

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When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since $P(4)$ is false, $\forall x P(x)$ is false.

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This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

Negations of quantified propositions (2)

Let consider a set of positive integers \mathbb{Z}^+ as our universe. Let predicate $Q(x) \equiv$ “if $x > 2$, then $x^2 \leq 2x$.”

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Consider this proposition

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When showing that an existential quantified proposition is false, we need to show that $Q(x)$ is false for every possible values of x . In this case, since $x^2 = x \cdot x > 2 \cdot x$ for every $x > 2$, we have that $(\exists x)Q(x)$ is false.

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Negations of quantified propositions (3)

Thus, the following equivalences:

► $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$

► $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.