01204211 Discrete Mathematics Lecture 9b: Affine Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a linear combination of u_1, \ldots, u_m .

Review: Span

Definition

A set of all linear combination of vectors $m{u}_1, m{u}_2, \dots, m{u}_m$ is called the span of that set of vectors.

It is denoted by $Span\{u_1, u_2, \dots, u_m\}$.

Review: Vector spaces

Definition

A set $\mathcal V$ of vectors over $\mathbb F$ is a **vector space** iff

- ightharpoonup (V1) $0 \in \mathcal{V}$,
- ightharpoonup (V2) for any $u\in\mathcal{V}$,

$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

ightharpoonup (V3) for any $oldsymbol{u},oldsymbol{v}\in\mathcal{V}$,

$$u + v \in \mathcal{V}$$
.

Examples of vector spaces:

- A span of vectors is a vector space.
- A solution set to homogeneous linear equations is a vector space.

Translation

If we have a line or a plane passing through a vector a, but not through the origin, how can we represent it?

- ▶ Translate the object so that it passes through the origin.
- ightharpoonup We obtain a vector space \mathcal{V} .
- lacktriangle Then we translate it back so that it passes through a.
- We get the set

$$\mathcal{A} = \{ \boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V} \}$$

- Question: Is A a vector space?
- ightharpoonup We also write it as a + V.

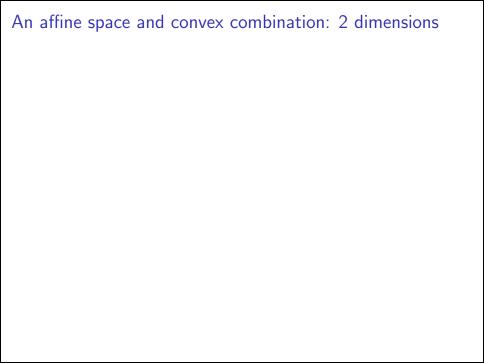
Affine spaces

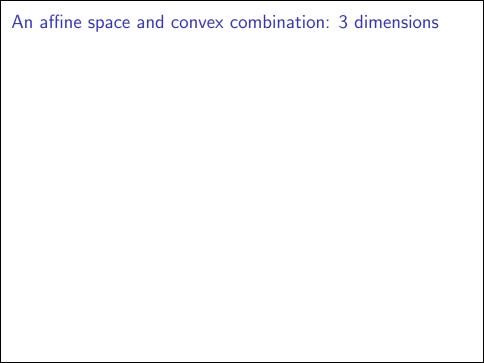
Definition

If a is a vector and ${\mathcal V}$ is a vector space, then

$$a + V$$

is an affine space.





Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of u_1, \ldots, u_m .

Definition

The set of all affine combinations of vectors u_1, u_2, \ldots, u_m is called the affine hull of u_1, u_2, \ldots, u_m .

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of u_1, \ldots, u_m .

Definition

The set of all convex combinations of vectors u_1, u_2, \dots, u_m is called the **convex hull** of u_1, u_2, \dots, u_m .

Writing an affine space using a span

An affine space

An affine space passing through $oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_n$ is

$$u_1 + \text{Span } \{u_2 - u_1, u_3 - u_1, \dots, u_n - u_1\}.$$

Non-homogeneous linear system

Two linear systems:

What can you say about the solution sets of these two related linear systems?

0 is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if u_1 and u_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$a_i u_1 - a_i u_2 = b_i - b_i = 0 = a_i (u_1 - u_2).$$

This implies that u_1-u_2 is a solution to the homogeneous linear system.

Suppose that $\ensuremath{\mathcal{W}}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{ \boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m \},$$

and let $u \in \mathcal{W}$ be one of the solutions, we have that

$$\{v - u : v \in \mathcal{W}\}$$

is a vector space, because

$$\{v - u : v \in \mathcal{W}\} = \{x : a_i x = 0, \text{ for } 1 \le i \le m\}$$

In other words.

$$\mathcal{W} = \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\}\$$

= $\mathbf{u} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\},$

i.e., ${\mathcal W}$ is an affine space.

Solutions to a non-homogeneous linear system

Lemma 1

If the solution set of a linear system is not empty, it is an affine space.