01204211 Discrete Mathematics
Lecture 6a: Counting 3

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Quick recap

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  - We prove the fact by induction.

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- The number of subsets of a set with n elements is 2^n. In fact, we know 3 proofs of this fact:
  - We count the number of ways one can choose a subset.
  - We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.
- For a set with n elements, the number of its permutations is $n!$.

This lecture's goals[1]

- Consider set $\{1, 2, 3, 4, 5\}$. How many subsets with 3 elements does this set have?

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- There are 10 subsets with 3 elements: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 4, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$.
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- We will also discuss the inclusion-exclusion priciples.

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However, sometimes, it is useful to treat sets as ordered.
For example, for set $\{1, 2, 3\}$, there are 6 ordered subsets with 2 elements: $\{1, 2\}$, $\{1, 3\}$, $\{2, 1\}$, $\{2, 3\}$, $\{3, 1\}$, $\{3, 2\}$.

Example: runners

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- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.

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- For any 1st price winner, there are 9 choices to choose the 2nd price winner.
- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- Therefore, we conclude that the number of ways is $10 \cdot 9 \cdot 8$.

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  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are $7!$ of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.

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- Let X be the set of ordered subsets with 3 elements of an 10-set. We then have $|X| \times 7! = 10!$, because they count the same objects. Solving this yields

$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

General answers: numbers of ordered subsets

Using the same arguments (either one), we have this theorem.

Theorem 1

The number of ordered subsets with k elements of an n-set is

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
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- Let's think about $n!$.
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Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

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Are they any good?

n	2^{n-1}	$n!$	n^{n-1}
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	3,628,800	1,000,000,000

A better bound?

Let's consider $n!$ again, but for simplicity, let's consider only the case when n is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2 - 1) \cdot (n/2) \cdot (n/2 + 1) \cdots n$$

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To get a better lower bound, we may move our cutting point from 2 to, say, $n/2$. Note that at least $n/2$ factors are at least $n/2$. Thus,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots n \\ &\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2} \\ &= (n/2)^{n/2} = \sqrt{(n/2)^n}. \end{aligned}$$

Better?

n	2^{n-1}	$\sqrt{(n/2)^n}$	$n!$	n^{n-1}
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	7,776
10	512	3,125	3,628,800	1,000,000,000
12	2,048	46,656	479,001,600	743,008,370,688

OK. A bit better.

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$.

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$.

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$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100\log(100/e) + \log(200\pi) \approx 157.9696.$$

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Note that the correct answer is 158 digits.

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- This upper bound of n^2 is very good as the gaps between the upper bounds and the actual values will not be larger than 2, as $\frac{n^2}{n(n+1)/2} < 2$.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

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$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of k-subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required. $\square$

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

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- $\binom{n}{0} = 1$ (why?), and,
- when $k > n$, $\binom{n}{k} = 0$.

Properties (1)

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Properties (2)

Theorem: When $n, k > 0$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Properties (3)

Theorem: When $n, k > 0$, then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$