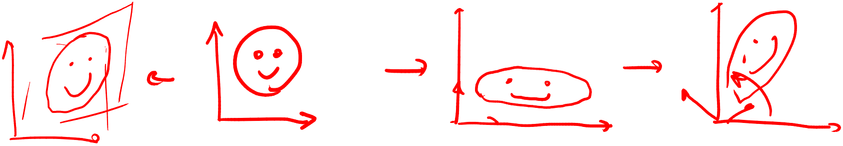
01204211 Discrete Mathematics
Lecture 13a: Linear functions (II)

Jittat Fakcharoenphol

September 20, 2022

Review: Linear functions



Linear functions

Consider vector spaces $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{R}$. A function $f : \mathcal{V} \to \mathcal{W}$ is **linear** if

1. for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, $f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$ and
2. for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x} \in \mathcal{V}$, $f(\alpha \boldsymbol{x}) = \alpha f(\boldsymbol{x})$.

Matrix-vector multiplication — ℑ a linear function

Given an $m \times n$ matrix M over $\mathbb{R}$, consider a product

$$M\boldsymbol{x}.$$

Note that for the multiplication to work, $\boldsymbol{x}$ must be in $\mathbb{R}^n$ and the result vector is in $\mathbb{R}^m$. Therefore, we can define a function $f : \mathbb{R}^n \to \mathbb{R}^m$ as

$$f(\boldsymbol{x}) = M\boldsymbol{x}.$$

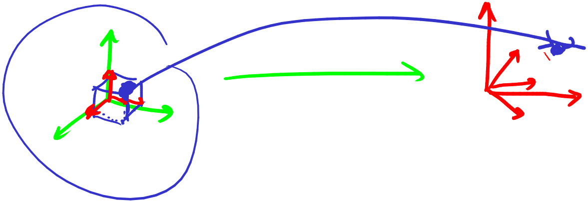
Note that f is linear because:

$$f(\boldsymbol{x}+\boldsymbol{y}) = M(\boldsymbol{x}+\boldsymbol{y}) = M\boldsymbol{x} + M\boldsymbol{y} = f(\boldsymbol{x}) + f(\boldsymbol{y}),$$

and

$$f(\alpha\boldsymbol{x}) = M(\alpha\boldsymbol{x}) = \alpha M\boldsymbol{x} = \alpha f(\boldsymbol{x}).$$

The converse



Lemma 1

For any linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, *there exists an* $m \times n$ *matrix* M *such that*

$$f(\boldsymbol{x}) = M\boldsymbol{x}.$$

Example: a system of linear equations

Consider the following homogeneous system $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 3 & 4 & 7 & 5 & 3 \\ 6 & 7 & 13 & 8 & 6 \\ 2 & 4 & 6 & 14 & 6 \\ 4 & 6 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: a system of linear equations

Consider the following homogeneous system $A\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 3 & 4 & 7 & 5 & 3 \\ 6 & 7 & 13 & 8 & 6 \\ 2 & 4 & 6 & 14 & 6 \\ 4 & 6 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's try to solve it on Colab.

Example: a system of linear equations

Let's look at what we've got so far (after row permutation)

rank = 3

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: a system of linear equations

Let's look at what we've got so far (after row permutation)

$$A' = \begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the rank of A? $= \text{rank}\, A' = 3$

Example: a system of linear equations

Let's look at what we've got so far (after row permutation)

5×5

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the rank of A? $= 3$

What does nullspace of A look like? $\in 0$

set [illegible]

dim nullspace $= 2$

Example: mullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & \boxed{2 \quad 1} & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

00 0 2 3 1

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & 0 & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

matrix $n \times n$

rank r

dim nullspace $n-r$

nullity

$$\begin{bmatrix} 10 & & & & \\ & 4 \cancel{4} & & & \\ & & 1 & \boxed{1 \quad 2} & \end{bmatrix}$$

Example: mullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's look at row 3:

$$2x_4 + x_5 = 0.$$

Example: mullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's look at row 3:

$$2x_4 + x_5 = 0.$$

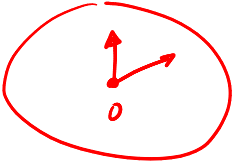
Let's look at row 2:

$$x_2 + \cancel{x_3} + 2x_4 = 0.$$

Example: nullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

nullspace



Let's look at row 3:

$$2x_4 + x_5 = 0.$$

Let's look at row 2:

$$x_2 + x_3 + 2x_4 = 0.$$

dimension of null space. = 2

Finally, let's look at row 1:

$$x_1 + 2x_2 + 3x_3 + 3x_4 + x_5 = 0.$$

Example: mullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's look at row 3:

$$2x_4 + x_5 = 0.$$

Let's look at row 2:

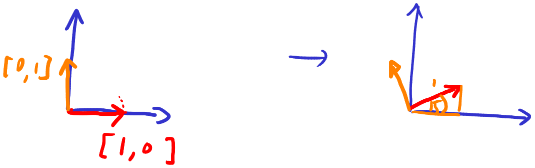
$$x_2 + x_3 + 2x_4 = 0.$$

Finally, let's look at row 1:

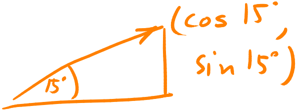
$$x_1 + 2x_2 + 3x_3 + 3x_4 + x_5 = 0.$$

How many "free" variable that you can set? 2

Ranks and nullities



Mx



Viewing matrix-vector multiplication as linear mapping

$x_1[\ ] + x_2[\ ] \cdots x_5[\ ]$

Mx

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 3 & 4 & 7 & 5 & 3 \\ 6 & 7 & 13 & 8 & 6 \\ 2 & 4 & 6 & 14 & 6 \\ 4 & 6 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

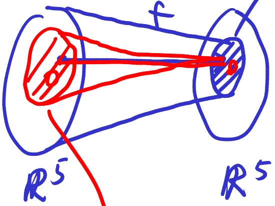
$\{f(x) \mid x \in \mathbb{R}^5\}$ range

$\{Mx \mid x \in \mathbb{R}^5\}$ = column space

$f(x)$

Image of f

$\{x \mid Mx = 0\}$ = nullspace



Kernel of f

Structures of linear functions

Zero

Lemma 2

Consider any linear function $f : \mathcal{V} \to \mathcal{W}$. *Let* $0_\mathcal{V}$ *denote the zero vector in* $\mathcal{V}$ *and* $0_\mathcal{W}$ *denote the zero vector in* $\mathcal{W}$. *We have that linear function* f *always maps zero to zero, i.e.,* $f(0_\mathcal{V}) = 0_\mathcal{W}$.

Proof.

First note that $0_\mathcal{V} = 0_\mathcal{V} + 0_\mathcal{V}$. Since f is linear, we have that

$$f(0_\mathcal{V}) = f(0_\mathcal{V} + 0_\mathcal{V}) = f(0_\mathcal{V}) + f(0_\mathcal{V}).$$

Subtracting $f(0_\mathcal{V})$ from both sides, we conclude that

$$0_\mathcal{W} = f(0_\mathcal{V}).$$

□

One-to-one linear functions and Onto linear functions

One-to-one and onto functions

Consider a function $f : D \to R$ (i.e., the domain of f is D and its range is R).

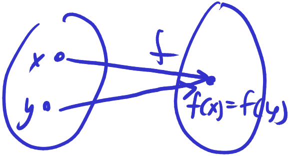
- Function f is **one-to-one** (or **injective**) if for all $x, y \in D$, $f(x) = f(y)$ implies that $x = y$.
- Function f is **onto** (or **surjective**) if for all $x \in R$, there exists $y \in D$ such that $f(y) = x$.

For this course, we consider only linear functions; therefore, we consider $f : \mathcal{V} \to \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are vector spaces.

One-to-one linear functions

Suppose that f is not one-to-one,

One-to-one linear functions



Suppose that f is not one-to-one, i.e., there exists a pair $x, y \in \mathcal{V}$ such that $x \neq y$ and $f(x) = f(y)$.

One-to-one linear functions

Suppose that f is not one-to-one, i.e., there exists a pair $x, y \in \mathcal{V}$ such that $x \neq y$ and $f(x) = f(y)$. Since f is linear, we know that

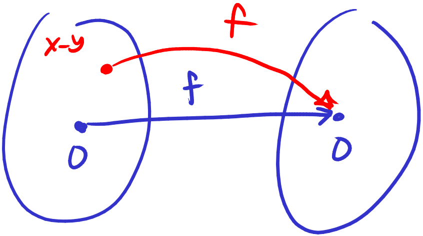
$$\underbrace{f(x-y)} = f(x) - f(y) = 0.$$

One-to-one linear functions

Suppose that f is not one-to-one, i.e., there exists a pair $x, y \in \mathcal{V}$ such that $x \neq y$ and $f(x) = f(y)$. Since f is linear, we know that

$$f(x - y) = f(x) - f(y) = 0.$$

Since $x \neq y$, $\boxed{x - y \neq 0}$ and we have that there exists a non-zero element $z = x - y$ that f maps to 0.

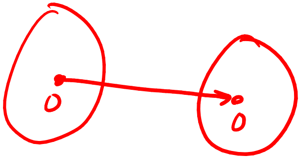


One-to-one linear functions

Suppose that f is not one-to-one, i.e., there exists a pair $x, y \in \mathcal{V}$ such that $x \neq y$ and $f(x) = f(y)$. Since f is linear, we know that

$$f(x - y) = f(x) - f(y) = 0.$$

Since $x \neq y$, $\boxed{x - y} \neq 0$ and we have that there exists a non-zero element $z = x - y$ that f maps to 0. The contraposition of this fact is as follows.
If the only element in $\mathcal{V}$ that f maps to $0_{\mathcal{W}}$ is $0_{\mathcal{V}}$, f is one-to-one (or injective).



Because the set of elements that f maps to zero is very important, we have a name for it.

Definition (Kernel)

The **kernel** of f, denoted by Ker f, is the set of all elements that f maps to zero, i.e.,

$$\text{Ker } f = \{\boldsymbol{v} \in \mathcal{V} : f(\boldsymbol{v}) = 0_{\mathcal{V}}\}.$$

We can now restate the condition for f to be one-to-one using this concept.

Lemma 3

A linear function f is one-to-one, if and only if $\boxed{\text{Ker } f = \{0\}}$.

The kernel is also a vector space

Lemma 4

Ker f *is a vector space.*

Proof.

The kernel is also a vector space

Lemma 4

Ker f *is a vector space.*

Proof.

First note that $f(0) = 0$; thus $0 \in \text{Ker } f$.

The kernel is also a vector space

Lemma 4

Ker f *is a vector space.*

Proof.

First note that $f(0) = 0$; thus $0 \in \text{Ker } f$.
Suppose that $x \in \text{Ker } f$, i.e., $f(x) = 0$. Note that for any scalar α,

$$f(\alpha x) = \alpha f(x) = \alpha 0 = 0.$$

The kernel is also a vector space

Lemma 4

Ker f *is a vector space.*

Proof.

V_1 First note that $f(0) = 0$; thus $0 \in$ Ker f.

V_2 Suppose that $x \in$ Ker f, i.e., $f(x) = 0$. Note that for any scalar α,

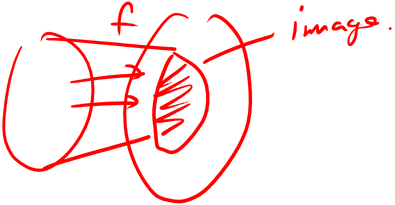
$$f(\alpha x) = \alpha f(x) = \alpha 0 = 0.$$

V_3 Also suppose $y \in$ Ker f. We have that

$$f(x+y) = f(x) + f(y) = 0 + 0 = 0.$$

□

Onto linear functions



Definition (Image)

For any function g, its **image**, denoted by $\operatorname{Im} g$, is the set of all elements that g maps to, i.e.,

$$\operatorname{Im} g = \{y : \text{there exists } x \text{ such that } g(x) = y\}.$$

The image is also a vector space

Lemma 5

The image of linear function f, $\operatorname{Im} f$, is a vector space.

Proof.

The image is also a vector space

Lemma 5

The image of linear function f, $\mathrm{Im}\ f$, is a vector space.

Proof.

Since $f(0_{\mathcal{V}}) = 0_{\mathcal{W}}$, $0_{\mathcal{W}} \in \mathrm{Im}\ f$.

The image is also a vector space

Lemma 5

The image of linear function f, $\operatorname{Im} f$, is a vector space.

Proof.

Since $f(0_{\mathcal{V}}) = 0_{\mathcal{W}}$, $0_{\mathcal{W}} \in \operatorname{Im} f$.
Consider $y \in \operatorname{Im} f$. We have that there exists x such that $f(x) = y$. Consider any scalar α. We know that $\alpha y \in \operatorname{Im} f$ because $f(\alpha x) = \alpha f(x) = \alpha y$.

The image is also a vector space

Lemma 5

The image of linear function f, Im f, is a vector space.

Proof.

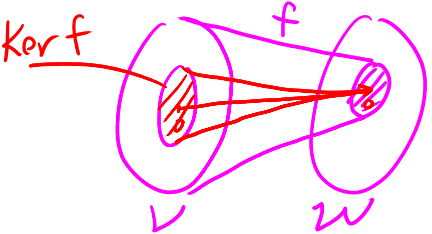
Since $f(0_{\mathcal{V}}) = 0_{\mathcal{W}}$, $0_{\mathcal{W}} \in \text{Im } f$.
Consider $y \in \text{Im } f$. We have that there exists x such that $f(x) = y$. Consider any scalar α. We know that $\alpha y \in \text{Im } f$ because $f(\alpha x) = \alpha f(x) = \alpha y$.
Consider, also, $y' \in \text{Im } f$. Let x' be such that $f(x') = y'$. Since $y' \in \text{Im } f$, we know that x' exists. We have that

$$f(x + x') = f(x) + f(x') = y + y'.$$

This implies that $y + y' \in \text{Im } f$. □

Kernels and images

$\mathbb{R}^2 \to \mathbb{R}^5$



Theorem 6 (Kernel-Image Theorem) ✱

Consider a linear function $f : \mathcal{V} \to \mathcal{W}$. We have that

$$\dim \mathcal{V} = \dim \mathsf{Ker}\ f + \dim \mathsf{Im}\ f.$$

Matrix $n \times n$

$$\boxed{n = \text{rank } A + \text{nullity } A}$$

Completing the basis

Lemma 7

For a set of linearly independent vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$$

in $\mathcal{V}$ with basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ (where $k \leq n$), there exists a set of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k} \in B$ such that

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}\}$$

is also a basis for $\mathcal{V}$.

Completing the basis

Lemma 7

For a set of linearly independent vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$$

in $\mathcal{V}$ with basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ (where $k \leq n$), there exists a set of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k} \in B$ such that

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}\}$$

is also a basis for $\mathcal{V}$.

Proof.

Use the morphing lemma. $\square$

Theorem 8 (Kernel-Image Theorem)

For a linear function $f : \mathcal{V} \to \mathcal{W}$, $\dim \mathcal{V} = \dim \mathsf{Ker}\ f + \dim \mathsf{Im}\ f$.

Proof of Kernel-Image Theorem (1).

Theorem 8 (Kernel-Image Theorem)

For a linear function $f : \mathcal{V} \to \mathcal{W}$, $\dim \mathcal{V} = \dim \mathsf{Ker}\ f + \dim \mathsf{Im}\ f$.

Proof of Kernel-Image Theorem (1).

Let $n = \dim \mathcal{V}$ and $k = \dim \mathsf{Ker}\ f$. Our goal is to show that $\dim \mathsf{Im}\ f = n - k$.

Theorem 8 (Kernel-Image Theorem)

For a linear function $f : \mathcal{V} \to \mathcal{W}$, $\dim \mathcal{V} = \dim \mathsf{Ker}\ f + \dim \mathsf{Im}\ f$.

Proof of Kernel-Image Theorem (1).

Let $n = \dim \mathcal{V}$ and $k = \dim \mathsf{Ker}\ f$. Our goal is to show that $\dim \mathsf{Im}\ f = n - k$.
Since Ker f is a vector space, there is a basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

Theorem 8 (Kernel-Image Theorem)

For a linear function $f : \mathcal{V} \to \mathcal{W}$, $\dim \mathcal{V} = \dim \mathsf{Ker}\ f + \dim \mathsf{Im}\ f$.

Proof of Kernel-Image Theorem (1).

Let $n = \dim \mathcal{V}$ and $k = \dim \mathsf{Ker}\ f$. Our goal is to show that $\dim \mathsf{Im}\ f = n - k$.

Since Ker f is a vector space, there is a basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$. From the previous slide, we can find other $n - k$ vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}$ to extend B to be a basis S for $\mathcal{V}$, i.e., we have that

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-k}\}$$

is a basis for $\mathcal{V}$. □

Proof of Kernel-Image Theorem (2).

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis.

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis. Consider $f(\boldsymbol{u})$. We have that

$$f(\boldsymbol{u}) = f(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k})$$

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis. Consider $f(\boldsymbol{u})$. We have that

$$\begin{aligned} f(\boldsymbol{u}) &= f(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}) \\ &= f(\alpha_1 \boldsymbol{v}_1) + \cdots + f(\alpha_k \boldsymbol{v}_k) + f(\beta_1 \boldsymbol{w}_1) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \end{aligned}$$

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis. Consider $f(\boldsymbol{u})$. We have that

$$\begin{aligned} f(\boldsymbol{u}) &= f(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}) \\ &= f(\alpha_1 \boldsymbol{v}_1) + \cdots + f(\alpha_k \boldsymbol{v}_k) + f(\beta_1 \boldsymbol{w}_1) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\ &= f(\beta_1 \boldsymbol{w}_1) + f(\beta_2 \boldsymbol{w}_2) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \end{aligned}$$

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis. Consider $f(\boldsymbol{u})$. We have that

$$\begin{aligned}
f(\boldsymbol{u}) &= f(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= f(\alpha_1 \boldsymbol{v}_1) + \cdots + f(\alpha_k \boldsymbol{v}_k) + f(\beta_1 \boldsymbol{w}_1) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= f(\beta_1 \boldsymbol{w}_1) + f(\beta_2 \boldsymbol{w}_2) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= \beta_1 f(\boldsymbol{w}_1) + \beta_2 f(\boldsymbol{w}_2) + \cdots + \beta_{n-k} f(\boldsymbol{w}_{n-k})
\end{aligned}$$

(Note that the second step follows because $\boldsymbol{v}_i \in \text{Ker } f$. Other steps use the fact that f is linear.)

This calculation shows that

Proof of Kernel-Image Theorem (2).

Consider any $\boldsymbol{u} \in \mathcal{V}$. We can write $\boldsymbol{u}$ as

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k},$$

because S is a basis. Consider $f(\boldsymbol{u})$. We have that

$$\begin{aligned}
f(\boldsymbol{u}) &= f(\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_k \boldsymbol{v}_k + \beta_1 \boldsymbol{w}_1 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= f(\alpha_1 \boldsymbol{v}_1) + \cdots + f(\alpha_k \boldsymbol{v}_k) + f(\beta_1 \boldsymbol{w}_1) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= f(\beta_1 \boldsymbol{w}_1) + f(\beta_2 \boldsymbol{w}_2) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= \beta_1 f(\boldsymbol{w}_1) + \beta_2 f(\boldsymbol{w}_2) + \cdots + \beta_{n-k} f(\boldsymbol{w}_{n-k})
\end{aligned}$$

(Note that the second step follows because $\boldsymbol{v}_i \in \text{Ker } f$. Other steps use the fact that f is linear.)

This calculation shows that an image of f can be written as a linear combination of $f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})$. That is

$$\text{Im } f = \text{Span}\,\{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}.$$

□

Proof of Kernel-Image Theorem (3).

Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$. If we can show that S' is a basis for Im f, we are done because that would imply that $\dim \text{Im } f = n - k$ as required.

Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$. If we can show that S' is a basis for Im f, we are done because that would imply that $\dim \text{Im } f = n - k$ as required.

We already know that S' spans Im f.

Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$. If we can show that S' is a basis for Im f, we are done because that would imply that $\dim$ Im $f = n - k$ as required.

We already know that S' spans Im f. To show that S' is a basis we still need to show that S' is linearly independent.

Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$. If we can show that S' is a basis for Im f, we are done because that would imply that $\dim \text{Im } f = n - k$ as required.

We already know that S' spans Im f. To show that S' is a basis we still need to show that S' is linearly independent.

Suppose that there exist $\beta_1, \ldots, \beta_{n-k}$ such that

$$\beta_1 f(\boldsymbol{w}_1) + \beta_2 f(\boldsymbol{w}_2) + \cdots + \beta_{n-k} f(\boldsymbol{w}_{n-k}) = 0_{\mathcal{W}}.$$

Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$. If we can show that S' is a basis for Im f, we are done because that would imply that $\dim \text{Im } f = n - k$ as required.

We already know that S' spans Im f. To show that S' is a basis we still need to show that S' is linearly independent.

Suppose that there exist $\beta_1, \ldots, \beta_{n-k}$ such that

$$\beta_1 f(\boldsymbol{w}_1) + \beta_2 f(\boldsymbol{w}_2) + \cdots + \beta_{n-k} f(\boldsymbol{w}_{n-k}) = 0_{\mathcal{W}}.$$

Since f is linear we know that

$$\begin{aligned}
0_{\mathcal{W}} &= \beta_1 f(\boldsymbol{w}_1) + \beta_2 f(\boldsymbol{w}_2) + \cdots + \beta_{n-k} f(\boldsymbol{w}_{n-k}) \\
&= f(\beta_1 \boldsymbol{w}_1) + f(\beta_2 \boldsymbol{w}_2) + \cdots + f(\beta_{n-k} \boldsymbol{w}_{n-k}) \\
&= f(\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}),
\end{aligned}$$

i.e., $\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k}$ is in Ker f. $\square$

Proof of Kernel-Image Theorem (4).

Suppose that some $\beta_i \neq 0$.
Since

$$\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k} \in \text{Ker } f,$$

we know that it is a linear combination of vectors from B, as B is a basis for vector space Ker f.

Proof of Kernel-Image Theorem (4).

Suppose that some $\beta_i \neq 0$.
Since

$$\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k} \in \mathrm{Ker}\ f,$$

we know that it is a linear combination of vectors from B, as B is a basis for vector space Ker f.
From here, we can reach a contradiction using the fact that vectors in S are linearly independent.

Proof of Kernel-Image Theorem (4).

Suppose that some $\beta_i \neq 0$.
Since

$$\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-k} \boldsymbol{w}_{n-k} \in \text{Ker } f,$$

we know that it is a linear combination of vectors from B, as B is a basis for vector space Ker f.
From here, we can reach a contradiction using the fact that vectors in S are linearly independent.

Therefore, we conclude that all $\beta_1, \ldots, \beta_{n-k}$ must be 0. Hence, $S' = \{f(\boldsymbol{w}_1), \ldots, f(\boldsymbol{w}_{n-k})\}$ is linearly independent as needed. □

Direct sum (optional)

Consider two subspaces $\mathcal{V}$ and $\mathcal{W}$ of a vector space $\mathcal{Z}$. If $\mathcal{V} \cap \mathcal{W} = \{0\}$, we can define their *direct sum* to be another vector space $\mathcal{V} \oplus \mathcal{W}$ as

$$\mathcal{V} \oplus \mathcal{W} = \{\boldsymbol{v} + \boldsymbol{u} : \boldsymbol{v} \in \mathcal{V}, \boldsymbol{u} \in \mathcal{W}\}.$$

Note, again, that $\mathcal{V} \oplus \mathcal{W}$ is defined only when $\mathcal{V} \cap \mathcal{W} = \{0\}$.