01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (II)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

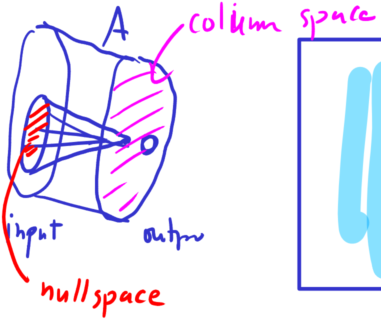
- The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$)
- The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
- The nullspace of A

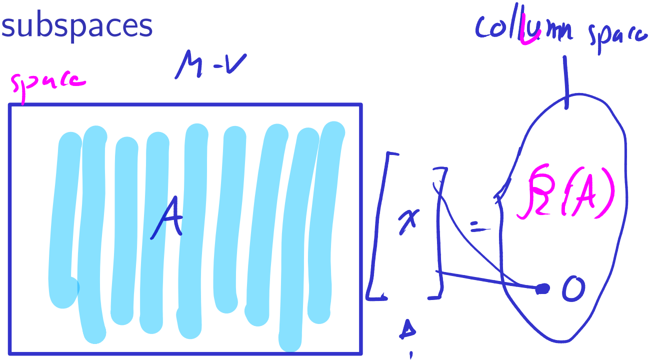
$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

- The left nullspace of A

$$\mathcal{N}(A^T) = \{\boldsymbol{y} \mid A^T\boldsymbol{y} = \boldsymbol{0}\} \subseteq \mathbb{R}^m$$

Four fundamental subspaces

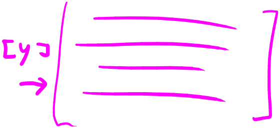




N(A)

บรรดา x ที่โดน map → 0

nullspace



Ranks

column sp.

Definition

Consider an m-by-n matrix A.

- The **row rank** of A is the maximum number of linearly independent rows of A.
- The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

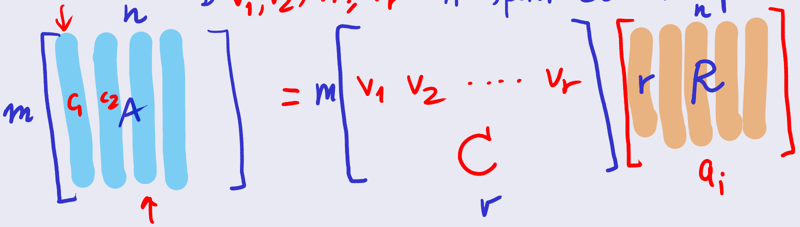
Theorem 1

For any matrix A, its row rank equals its column rank.

Proof. row rank $\leq$ column rank

Let r be the column rank. We will show that there are r n-vectors that span its row space. This implies that the row rank is at most r. We can use the same argument again on A^T to obtain that the column rank is at most the row rank; thus, they must be equal.

มี $v_1, v_2, \ldots, v_r$ ที่ span column space



$c_i = C \cdot a_i$

$\square$

Proof (cont.)

$$\underset{m}{} \underbrace{\begin{bmatrix} \quad A \quad \end{bmatrix}}_{n} = \underset{m}{} \underbrace{\begin{bmatrix} \quad C \quad \end{bmatrix}}_{r} \underset{r}{} \underbrace{\begin{bmatrix} \quad R \quad \end{bmatrix}}_{n} \Big\} \, r \text{ vectors}$$

Vector-matrix mult.

row space is span by $\underbrace{\text{rows of } R}_{r \text{ rows}}$

row rank $=$ dim row space $\leq r =$ column rank.

ทำแบบเดิมกับ A^T : $\underset{\text{ของ } A}{\text{col rank}} \leq \underset{\text{ของ } A}{\text{row rank}}$.

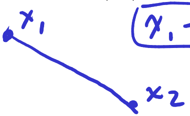
$\square$

Rank and nullity

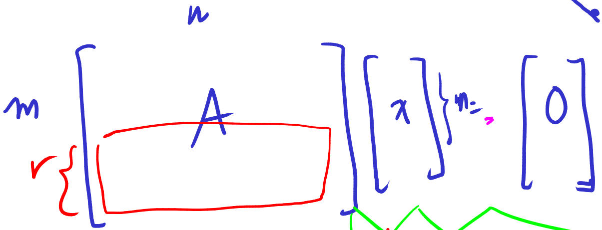
$m \times n$ ah rank $= r$

Given an m-by-n matrix A, the rank of A is $\dim\ \mathcal{R}(A)$. Let r be the rank of A.
What is $\dim\ \mathcal{N}(A)$?

$n - r$

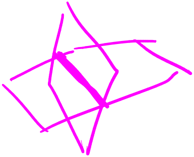


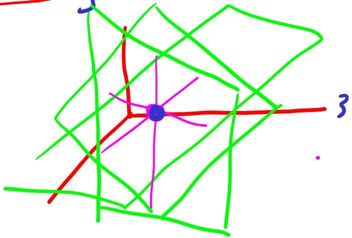
$\boxed{x_1 - x_2}$

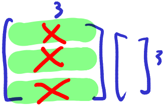


$x_1 + x_2 - x_3 = 7$

rank $A = r$







Dimensions

Four fundamental subspaces

Given an m-by-n matrix A of rank r, we have the following subspaces

- The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$)
  $\dim\ \mathcal{R}(A) = r.$
- The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
  $\dim\ \mathcal{R}(A^T) = r.$
- The nullspace of A (denoted by $\mathcal{N}(A) \subseteq \mathbb{R}^n$)
  $\dim\ \mathcal{N}(A) = \boxed{n - r.}$
- The left nullspace of A (denoted by $\mathcal{N}(A^T) \subseteq \mathbb{R}^m$)
  $\dim\ \mathcal{N}(A) = \boxed{m - r.}$

Application: Singular Value Decomposition (SVD)

Any n-by-d matrix A can be factored into the form of UDV^T, i.e.,

$$\begin{array}{c} n \\ \end{array}\begin{array}{c}\begin{bmatrix} & A & \end{bmatrix} \\ d\end{array} = \begin{array}{c} n \\ \end{array}\begin{array}{c}\begin{bmatrix} & U & \end{bmatrix} \\ r\end{array} \begin{array}{c} r \\ \end{array}\begin{array}{c}\begin{bmatrix} & & 0 \\ & D & \\ 0 & & \end{bmatrix} \\ r\end{array} \begin{array}{c} r \\ \end{array}\begin{array}{c}\begin{bmatrix} & V^T & \end{bmatrix} \\ d\end{array}$$

where

- U is an n-by-r matrix,
- D is a diagonal r-by-r matrix, and
- V is an d-by-r matrix (i.e., V^T is an r-by-d matrix)
- (Also, columns of U and D are "orthonormal.")

$u^T v = [x \cdot \quad] \begin{bmatrix} \cdot \\ \\ \end{bmatrix}$

$$\begin{bmatrix} \alpha_1 r \\ \alpha_2 r \\ \alpha_3 r \\ \alpha_4 r \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} & r & \end{bmatrix}$$

$u \cdot v^T$

See demo.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} \boxed{u_1} & \boldsymbol{u}_i & U & \end{bmatrix} \begin{bmatrix} d_{ii} & \\ & D \end{bmatrix} \begin{bmatrix} \boxed{v_1^T} \\ \boldsymbol{v}_i^T \\ V^T \\ \end{bmatrix}$$

$$d_1 u_1 v_1^T + \cdots$$

$700 \times 1000 = 700{,}000$

$170{,}000$

$$\left[\; A \;\right] = \left[\; \boldsymbol{u}_i \quad U \;\right] \underbrace{\left[\; d_{ii} \quad D \;\right]}_{r} \; \underbrace{\begin{bmatrix} \boldsymbol{v}_i^T \\ V^T \end{bmatrix}}_{n}$$

$m \times n$

$$
\begin{array}{c} \\ m \end{array}
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}
\overset{n}{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}} = m \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \\ a_4 b_1 & a_4 b_2 & a_4 b_3 \end{bmatrix}
$$

A

$$A = d_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + d_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + d_r \boldsymbol{u}_r \boldsymbol{v}_r^T .$$

$700 + 1000 + 1$

10

[illegible] $=$ d_1

$\boxed{1701}$ .... $\underline{17{,}010}$ 750