01204211 Discrete Mathematics
Lecture 8b: Modular arithmetic

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Quick check 1

If $a|m$ and $b|m$, can we say that $ab|m$? Prove this fact or provide a counter example.

Quick check 2

If $a|m$, $b|m$, and $a \neq b$ are both prime, can we say that $ab|m$? Prove this fact or provide a counter example.

Prime factorization

One useful fact that we use over and over again is the following.

Unique Factorization (or Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be written *uniquely* as a product of prime numbers (up to the order of factors).

Examples:

- $10 = 2 \cdot 5$
- $13 = 13$
- $112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^4 \cdot 7$

Problem size

- Sort n numbers

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- Check if vertex u is reachable from vertex v in a graph with n vertices and m edges

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- Find $gcd(a, b)$ for inputs a and b

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- Check if vertex u is reachable from vertex v in a graph with n vertices and m edges
- Multiply two $n \times n$ matrices
- Add two integers $O(\log x + \log y)$
- Check if an integer n is prime
- Find $gcd(a, b)$ for inputs a and b

When inputs contain a few numbers

GCD and Power

Days

What day is it today?

Days

What day is it today? Thursday.

Days

What day is it today? ~~Thursday.~~ Monday
What day is 3 days after today?

Days

What day is it today? ~~Thursday.~~ Mon

What day is 3 days after today? ~~Sunday.~~ Thu

Days

What day is it today? ~~Thursday.~~ Mon

What day is 3 days after today? ~~Sunday.~~ Thu

What day is 20 days after today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today? Monday.

Clocks

Suppose that it is 1 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours? 9 o'clock.

Modular arithmetic

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$$4 + 5 = 9 \bmod m = 2.$$

Or

$$3 \cdot 4 =$$

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Or

$$2 - 6 =$$

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$4 \bmod 7 =$

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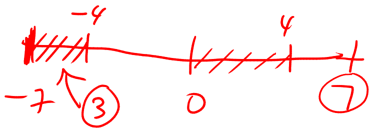

Suppose that $m = 7$. We would like to say that

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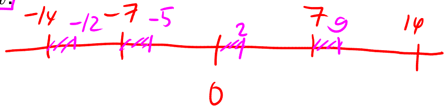


Suppose that $m = 7$. We would like to say that

$$4 + 5 = 9 \bmod m = 2.$$

Or

$$3 \cdot 4 = 12 \bmod m = 5.$$

Or

$$2 - 6 = -4 \bmod 7 = 3 \bmod 7 = 3.$$

Note that when you view integers under the lense of modulus 7, these numbers

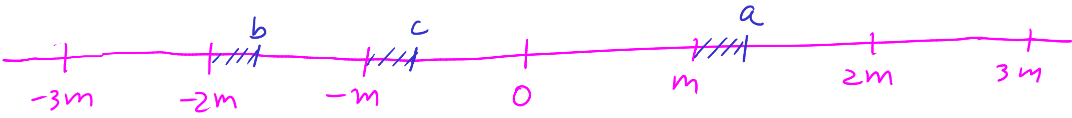
$$\ldots, -19, -12, -5, 2, 9, 16, 23, \ldots$$

are essentially **the same**.

Properties (1)

c mod m = a m

$a \bmod m = b \bmod m$, if and only if $m | a - b$.



Properties (1)

$a \bmod m = b \bmod m$, if and only if $m | a - b$.

Proof.

($\Rightarrow$) Let $r = a \bmod m$. We can write

$$a = qm + r,$$

and

$$b = pm + r,$$

for some integers q and p. Thus, we have

$$a - b = qm + r - pm - r = (q - p)m.$$

Therefore $m | a - b$.

($\Leftarrow$) Exercise. $\square$

Properties (2)

- $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- $(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$
- $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$

Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

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we write

$$a \equiv b \pmod{m}.$$

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We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m | (a - b)$$

Congruences: properties (1)

$x + 7 \equiv 5 \pmod{13}$

$x + 7 - 7 \equiv 5 - 7 \equiv -2 \equiv 11 \pmod{13}$

$x \equiv 11 \pmod{13}$

- (reflexivity)
  $a \equiv a \pmod{m}$.
- (symmetry)
  $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$.
- (transitiviey)
  $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ implies $a \equiv c \pmod{m}$.

Congruences: properties (2) – operations

If we have that

$$a \equiv b \pmod{m},$$

and

$$c \equiv d \pmod{m},$$

then

- $a + c \equiv b + d \pmod{m}$
- $a - c \equiv b - d \pmod{m}$
- $ac \equiv bd \pmod{m}$

$2x \equiv 7 \pmod{13}$

$7 \equiv 7 \pmod{13}$

$x \equiv 1 \cdot x \equiv 14x \equiv 7 \cdot 2x \equiv 7 \cdot 7 \equiv 49$

$\equiv 10 \pmod{13}$

$7 \cdot 2 \equiv 1 \pmod{13}$

13
26
39

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We can pretty much think of this "congruence" as a normal equation.

What is missing here?
Division!

Also, we wish we can do "cancellation", i.e., if

$$xa \equiv xb \pmod{m},$$

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then $a \equiv b \pmod{m}$. **BUT THIS IS NOT ALWAYS TRUE.**
Let’s see the following example:

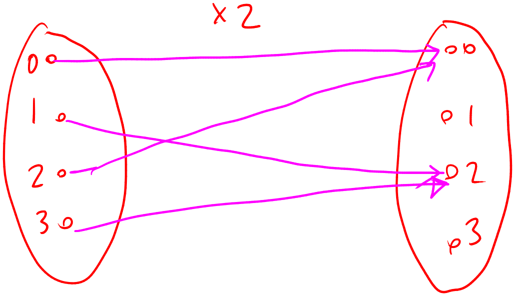
$$2 \cdot 1 \equiv 2 \cdot 3 \pmod{4},$$

but

$$1 \not\equiv 3 \pmod{4}.$$

Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \bmod 4$.

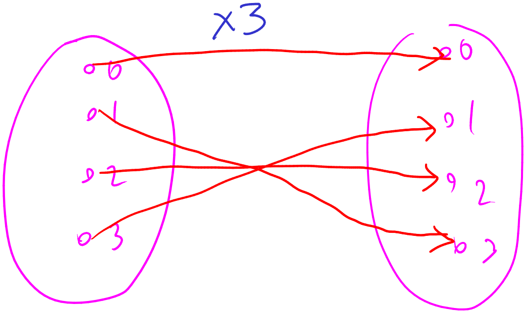


Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \bmod 4$.

Let's also see $g(x) = 3 \cdot x \bmod 4$.

$$3 \cdot x \equiv 2 \pmod 4$$



Multiplications as functions

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Which functions have inverses?

Multiplicative inverses (standard arithmetic)

In standard arithmetic, what is $\underline{2/5}$?

$x = 2/5$

Solving

$$\boxed{5x = 2}$$

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We are looking to a number x such that $2 = 5x$. How can we do that?

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By dividing on both sides with 5:

$$2/5 = 5x/5 = x,$$

Multiplicative inverses (standard arithmetic)

$$\boxed{\tfrac{2}{5}} = 2\left(\tfrac{1}{5}\right) = 5x\left(\tfrac{1}{5}\right) = \cancel{5}\left(\tfrac{1}{\cancel{5}}\right)x$$

In standard arithmetic, what is 2/5?
We are looking to a number x such that $2 = 5x$. How can we do that?
By dividing on both sides with 5:

$$2/5 = 5x/5 = x,$$

or equivalently, by multiplying with $(1/5) = 5^{-1}$:

$$2 \cdot 5^{-1} = 5x \cdot 5^{-1} = x \cdot (5) \cdot (5^{-1}) = x \cdot \underline{1} = x.$$

Here 5^{-1} is a multiplicative inverse of 5.

Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be $m = 7$. Note that

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod 7.$$

Therefore, $\boxed{5^{-1} \equiv 3 \pmod 7.}$

$$x \equiv 15x \equiv 3 \cdot 5x \equiv \underbrace{2 \cdot 3}_{6} \pmod 7$$

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Therefore, $5^{-1} \equiv 3 \pmod{7}$.

To find $2/5$, we can view our goal as to find the value of x such that

$$2 \equiv 5x \pmod{7}.$$

We can multiply both sides with $5^{-1} \equiv 3$ to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

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$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

Let's check:

$$5 \cdot 6 \equiv 30 \equiv 2 \pmod{7},$$

as requied.

Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Multiplicative inverse modulo ~~11~~ 13

45
39

Let's try to figure out multiplicative inverse of every integer modulo 1~~1~~3.

a	$a^{-1} \pmod{\cancel{11}}$ 13
1	1
2	7
3	9
4	10
5	8
6	11
7	2
8	5
9	3
10	4
11	6
12	12

27

(1) $\quad 2x + 5y \equiv 6 \pmod{13}$

(2) $\quad x + y \equiv 7 \pmod{13}$

(1) − 2(2) $\quad 3y \equiv 6 - 14$

$\equiv 6 - 1 \equiv 5 \pmod{13}$

$9 \cdot 3y \equiv 27y \equiv 1 \cdot y \equiv 9 \cdot 5$

$\equiv 45 \equiv \boxed{6} \pmod{13}$

$x + 6 \equiv 7$

$\boxed{x \equiv 1}$ $\quad \boxed{y \equiv 6}$ $\quad \pmod{13}$

Multiplicative inverse modulo 11

56

Let's try to figure out multiplicative inverse of every integer modulo 11.

a	$a^{-1} \pmod{11}$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

$$8(5x + 7y) \equiv 8(7) \pmod{13}$$

$$\boxed{x + 4y \equiv 4 \pmod{13}}$$

Examples: division in modular arithmetic

Suppose that we know that every non-zero integer a has an inverse modulo m. Can you solve this equation?

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We can even perform gaussian elimination (*which is very useful later*):

$$\begin{aligned} 2x + y &\equiv 3 \pmod 7 \\ x + 3y &\equiv 5 \pmod 7 \end{aligned}$$

There are 3 clocks. At this moment, all three clocks ring at the same time. The first clock rings every 3 hours, the second clock rings every 4 hours, and the third clock rings every 10 hours. How long do you have to wait until you would hear all clocks ring a the same time again?

You have a large water container and two smaller buckets. The first bucket carries 3 litres of water and the second bucket carries 5 litres of water.
Can you put exactly 1 litre of water in the water container?

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.
What is the minimum volume of water you can exactly put in the water container?

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In general if you have two buckets of volumes x and y, the amount that you can exactly make must be in the form of

$$ax + by,$$

for some integers x and y. (Note that x and y may be negative.)

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for some integers x and y. (Note that x and y may be negative.)
Do you see why the sum must be divisible by any common divisor of x and y?

Useful fact

For any integer x and y, consider the term

$$a \cdot x + b \cdot y,$$

for some integer a and b.

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For any integer x and y, consider the term

$$a \cdot x + b \cdot y,$$

for some integer a and b.
When the term is non-zero, it must be divisible by $gcd(x, y)$, so it has to be at least $gcd(x, y)$.

It turns out that you can actually attain that value, i.e., there exist a pair of integer a and b such that

$$a \cdot x + b \cdot y = gcd(x, y).$$

Finding a and b: Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $gcd(x, y)$.

```
Algorithm Euclid(x,y):
  if x mod y == 0:

    return y,          ,
  else:
    g, a', b' = Euclid(y, x mod y)

    a =

    b =

    return g, a, b
```

Notes:

We have a' and b' such that

$$a' \cdot y + b' \cdot (x \bmod y) = g.$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $gcd(a, m) = 1$.

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($\Leftarrow$) Recall that there exist integers x and y such that

$$x \cdot a + y \cdot m = gcd(a, m) = 1.$$

Thus, $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$, i.e., $x \cdot a \equiv 1 \pmod{m}$. Therefore x is the inverse.

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($\Rightarrow$) Let $r = gcd(a, m)$. Suppose that b is the multiplicative inverse of a modulo m, i.e., we have that

$$b \cdot a \equiv 1 \pmod{m},$$

Thus, $ba \bmod m = 1 \bmod m = 1$, i.e., there exists an integer q such that

$$ba = qm + 1,$$

or $ba - qm = 1$. However, r since $r|a$ and $r|m$, r also divides $bd - qm$ and 1. But it $r \not| 1$ because $r > 1$ and we have the contradiction. $\square$

Examples: division in modular arithmetic

Since the requirement for an existance of a^{-1} modulo m is that $gcd(a, m) = 1$, if we let m be a prime number, every a which is not a multiple of m has an inverse.
Can you solve this equation?

$$4x + 9 \equiv 0 \pmod{11}.$$

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Public-key cryptography

public key k_1

private key k_2

Alice

key

Eve

Bob

key

Bob: 121134

Private key cryptography

$E(M, k_1)$

$$D(E(M,k_1),k_2) \Rightarrow M$$

$k_1 = 121134$

$\boxed{k_2}$

Bob

M

RSA

Rivest Shamir Adelman

Public key (e, n) big number

Private key (d, n)

Message: m

- $\text{encrypt}(m) = (m^e) \bmod n$ %
- $\text{decrypt}(r) = (r^d) \bmod n$

Pick two prime numbers: p, q

$\boxed{n = pq}$

pick $\boxed{e}$ 65535

Calculate d: $\boxed{e^{-1} \pmod{(p-1)(q-1)}}$

RSA

$$(m^e) \bmod n$$

$$(a+b) \bmod n$$

$$=$$

$$((a \bmod n)+(b \bmod n)) \bmod n$$

$$(a \cdot b) \bmod n$$

$$=$$

$$((a \bmod n)(b \bmod n)) \bmod n$$

RSA: steps

- Private key: (d, n), Public key: (e, n)
- Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

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- Pick two primes p and q. Let $n = pq$.
- Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or

  $$ed = k \cdot (p-1)(q-1) + 1,$$

  for some integer k.
- What is $m^{ed} \bmod n$?

Secret sharing

Secret sharing scheme based on straight lines

Example: secret sharing

- Think of a secret number $m \in \{0, 1, \ldots, 10\}$.
- Pick a random number $a \in \{1, 2, \ldots, 10\}$.
- Your straight line function $f(x) = (ax + m) \bmod 11$.
- We will generate 3 points from f and give them to 3 of your friends, each with only 1 point. Pick 3 numbers x_1, x_2, x_3 from $\{1, 2, \ldots, 10\}$.
- Let's compute
$$(x_1, f(x_1)), \quad (x_2, f(x_2)), \quad (x_3, f(x_3)).$$
- Give them to 3 of your friends and challenge them to form a group of 2 people and figure out your number m.

What's next?

- We will prove Fermat's Little Theorem and show how to efficiently test if a number is prime.
- We will also use Fermat's Little Theorem to prove the correctness of RSA.
- Modular arithmetic is also key to our usage of polynomials to perform secret sharing and error correcting codes, because now we can do Gaussian elimination using only integers.