01204211 Discrete Mathematics Lecture 3b: Proof techniques 2

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July 10, 2021

Proof techniques¹

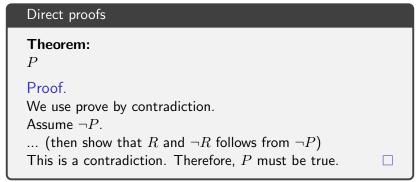
In this lecture, we will focus on two other proof techniques.

- ▶ Proofs by contradiction
- Proofs by cases

¹This lecture mostly follows Berkeley CS70 lecture notes. → ⟨≥⟩ ⟨≥⟩ ⟨≥⟩ ⟨≥⟩ ⟨≥⟩

Proofs by contradiction

We want to prove that proposition P is true. To do so, we first assume that P is false, and show that this logically leads to a contradiction. This means that it is impossible for P to be false; hence, P has to be true. This is called a proof by contradiction or reductio ad absurdum.



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 $\sqrt{2}$ is irrational.

Proof.

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Let's square both terms. We get $2 = a^2/b^2$, or

$$a^2 = 2b^2.$$

(cont. in next slide)



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By definition, we know that a^2 is an even number. From a theorem from last time, we know that a must also be an even number.

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By definition, we know that a^2 is an even number. From a theorem from last time, we know that a must also be an even number. Again by definition, there exists integer k such that a=2k. We then obtain

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i.e., $b^2=2k^2.$ This implies that b^2 is an even number. Again, this means that b must be an even number.

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This contradicts the fact that we choose the pair a and b that share no common factor.

Therefore, $\sqrt{2}$ must be irrational.

Proofs by cases

- ► The last proof technique that we shall discuss is closely related to proofs by exhaustion we tried before.
- Sometimes when we want to prove a statement, there are many possible cases. Also, we might not know which cases are true.
- ▶ We might still be able to prove the statement if we can show that the statement is true in every case.

Theorem 2

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Proof.

Let's split the process of picking 4 socks into 2 steps. First, pick 3 socks, then pick the last sock.

After we pick the first 3 socks. There are 2 possible cases: either I have a pair of socks with the same color, or I do not have such a pair. We shall consider each case separately.

(cont. in the next slide)

Proof. (cont.)

Case 1: I have a pair of socks with the same color.

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- **Case 2:** I do not have a pair of socks with the same color.

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- ▶ Case 2: I do not have a pair of socks with the same color. In this case, since I have 3 colors and 3 socks, I must have one sock for each color. Now, after we pick the last sock, whatever color the last one is, we have a color-matching sock in our first 3 socks. Therefore, the theorem is also true in this case.

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- ▶ Case 2: I do not have a pair of socks with the same color. In this case, since I have 3 colors and 3 socks, I must have one sock for each color. Now, after we pick the last sock, whatever color the last one is, we have a color-matching sock in our first 3 socks. Therefore, the theorem is also true in this case.

Since these two cases cover all possibilities, we conclude that the theorem is true.

Proofs by cases in propositional logic

In propositional logic, the following describe a proof by cases.

```
P \lor Q \lor R
P \Rightarrow S
Q \Rightarrow S
R \Rightarrow S
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Sometimes, when we have 2 cases, we also see:

$$P \vee \neg P$$

$$P \Rightarrow S$$

$$\neg P \Rightarrow S$$

$$S$$

Note that we can leave $P \vee \neg P$ out, because it is always true.