01204211 Discrete Mathematics
Lecture 8a: Integers and GCD

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Number theory: integers and divisibility

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$a | b$

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a can ~~divides~~ divide b

$\frac{b}{a}$

b is divided by a

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We will cover:

- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.

$a | b \rightarrow$ b is a multiple of a

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- Basic concepts of divisibility, prime numbers, and congruence.
- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.
- Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

$a|b \neq a/b = \frac{a}{b}$

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b. On the other hand if a does not divide b, we write $a \nmid b$.

Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

① Because $a|b$, there exists integer k such that
$$b = ka.$$

② Because $a|c$, $\exists$ integer m, s.t. $c = ma$.

③ from ① & ②,
$$b+c = ka + ma$$
$$= (k+m)a.$$

④ Since k & m are integers, $k+m$ is an integer.

⑤ $\exists$ integer x, s.t. $b+c = ax$ $\Rightarrow$ $a|(b+c)$.
$\underset{(k+m)}{\overset{\shortparallel}{x}}$

Definition: for any integer x & y, we say that
$$x|y$$
if there exists an integer k such that
$$y = kx$$

$$\forall x, \forall y\, [\, x|y \Leftrightarrow \exists k,\ y = kx \,]$$
(universe = set of all integers.)

ex: $3 | 12$ because $\exists k = 4$
$12 = 4 \cdot 3$

$3 \nmid 10$ because for k
$10 \neq 3 \cdot k$

Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

Because $a|b$ & $a|c$, there exist integers k & m such that

$$b = k \cdot a \quad \& \quad c = m \cdot a$$

Thus, $b + c = k \cdot a + m \cdot a = (k+m) \cdot a$.

Since k & m are integers, $k+m$ is also an integer.

Therefore there exists an integer $x = k+m$, s.t.

$$b + c = a \cdot x.$$

This implies that $a|(b+c)$.

Examples

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If $a|b$ and $b|c$, prove that $a|c$.

Then exist integers k & m s.t.

$$b = ka, \quad c = mb.$$

Therefore

$$c = mb = m(ka) = (mk) \cdot a.$$

Since m & k are int, mk is an integer.

Thus there [illegible] $x = mk$ s.t. $c = x \cdot a = (mk)a$

$\Rightarrow a|c.$ as required.

Remainder

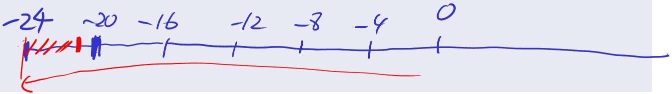
non negative

Defintion (remainder)

The **remainder** of the division of b with a is an integer r when there exists an integer q such that

$$b = qa + r,$$

where $\boxed{0 \leq r < a}$.



divide 21 with 4

remainder = 1

divide -21 with 4

remainder = 3

Remainder

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$b \bmod a$.

Examples:

We use operator `mod` to denote an operation for finding the remainder of a division.
I.e., $b \bmod a$ is the remainder of dividing b with a.

Examples

$r = b \bmod a.$

If $c|a$ & $c|b \Rightarrow c|r$

Let r be the remainder of the division of b by a. Assume that $c|a$ and $c|b$. Prove that $c|r$.

Examples

$r = b \bmod a$

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$r = b \bmod a$

Since $c|a$ & $c|b$, there exist integers k & m s.t. $a = k\cdot c$ & $b = m\cdot c$.

Since $r = b \bmod a$, $\exists$ integer q s.t. $b = q\cdot a + r$

$$\Rightarrow r = b - q\times a.$$

(Your work: prove that $c|r$) Write $r = b - q\cdot a = m\cdot c - q\cdot k\cdot c$

$$= (m - qk)c.$$

*Because q, k, m are int, $m - qk$ is an int, ...

More examples

For every integer a, $a - 1 | a^2 - 1$.

Primes

Definition (primes)

- An integer $p > 1$ is a **prime** if its divisors are only p, $-p$, 1, and -1.
- If an integer $n > 1$ is not a prime, it is called a **composite**.
- Note: 1 is not a prime and also not a composite.

Fundamental theorem of arithmetic

Unique factorization

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

Algorithm for testing primes

Recall our `CheckPrime2` algorithm

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run?

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How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

Efficient algorithms

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n	number of bits of n	$\sqrt{n}$
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
1,267,650,600,228,229,401,496,703,205,376	100	1,125,899,906,842,624

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Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

Greatest Common Divisors (GCD)

Definition (GCD)

For integers x and y, the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $gcd(x, y)$.

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while (x mod g != 0) or (y mod g != 0):
  g -= 1
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What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

Euclid's algorithm

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Let's see how it works with $Euclid(12311, 24324)$:

Euclid(12311, 24324)
Euclid(24324, 12311)
Euclid(12311, 12013)
Euclid(12013, 298)
Euclid(298, 93)
Euclid(93, 19)
Euclid(19, 17)
Euclid(17, 2)
Euclid(2, 1)

Proofs

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
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- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
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Note that when $x < y$, $\mathrm{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\mathrm{Euclid}(y, x)$. After that, in each call, x is always larger than y. For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

For any integers x and y such that $x > y$, $\mathrm{Euclid}(x, y) = gcd(x, y)$.

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $gcd(x, y) = y$; therefore, the base case of the algorithm is correct.
Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\mathrm{Euclid}(x', y') = gcd(x', y')$.
Now assume that $y \not| x$. The Euclid algorithm returns $\mathrm{Euclid}(y, x \bmod y)$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\mathrm{Euclid}(y, x \bmod y) = gcd(y, x \bmod y).$$

Thus, we are left to show that

$$gcd(x, y) = gcd(y, x \bmod y).$$

$\square$

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$$x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

Lemma 2

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Lemma 3

$gcd(x, y) = gcd(y, x \bmod y)$

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- How many times can that happen?
- The first argument can decrease by a factor of two for at most $\log x$ times. Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$.

Computing power

How fast can we compute x^y?

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What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

$$x^{16} = (x^8)^2 = ((x^4)^2)^2 = (((x^2)^2)^2)^2.$$

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```
Algorithm power(x,y):  // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

Repeated squaring (general y)

```
Algorithm power(x,y):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2))
    if y mod 2 == 0:
      return a*a
    else
      return a*a*x
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What is the number of recursive calls?
What is the running time?
While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y, mod n)

Computing $x^y \bmod n$:

```
Algorithm power(x,y,n):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2)) mod n
    if y mod 2 == 0:
      return a*a mod n
    else
      return a*a*x mod n
```