01204211 Discrete Mathematics
Lecture 8a: Integers and GCD

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- polynomials
- secret sharing *
- error correction

applications

+ − × ÷ จำนวนเต็ม

modular arithmetic

$\boxed{GF(p)}$

Number theory

divisibility

gcd, lcm (หรม, ครน)

cryptography

application

public key crypto

RSA

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- How to quickly check if a number is prime.
- How to essentially perform "division" with integers, allowing us to work with important and useful objects like polynomials using only integers.

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- Applications like cryptography (RSA), secret sharing, erasure codes and error correcting codes.

Definitions

Definition (divisibility)

We say that an integer a **divides** b or b **is divisible by** a if there exist an integer k such that

$$b = ak.$$

If it is the case, we also write $a|b$. We also say that a is a **divisor** (or a **factor**) of b. On the other hand if a does not divide b, we write $a \nmid b$.

$a|b$

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Examples

If $a|b$ and $a|c$, prove that $a|(b+c)$.

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จากนิยาม ∃ k_1 และ k_2 ที่เป็น จน.เต็ม ที่

$b = k_1 a$, $c = k_2 a$

ดังนั้น $b + c = k_1 a + k_2 a = (k_1 + k_2) a$

→ เนื่องจาก k_1 & k_2 เป็น จน.เต็ม ดังนั้น $k_1 + k_2$ จึงเป็น จน.เต็มด้วย

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$b = k_1 a$, $c = k_2 b$

ดังนั้น $c = k_2 \cdot b = (k_2 \cdot k_1) \cdot a$, เพราะ k_1 & k_2 เป็น จน.เต็ม

จะได้ว่า $k_2 \cdot k_1$ เป็น จน.เต็ม ทำให้ สรุปได้ $a|c$

Remainder

12 หาร 7 → เหลือ 5

Defintion (remainder)

The **remainder** of the division of b with a is an integer r when there exists an integer q such that

$$b = qa + r,$$

where $0 \leq r < a$.

-12 หาร 7 เหลือ 2

$-12 = (-2) \cdot 7 + r$

$r = +2$

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Examples:

$-12 \ \underline{\text{mod}} \ 7 = 2$

We use operator `mod` to denote an operation for finding the remainder of a division.
I.e., $a \bmod b$ is the remainder of dividing a with b.

%

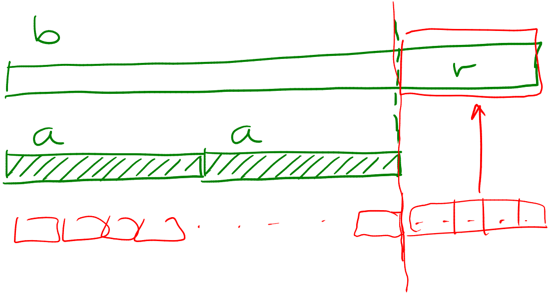
Examples

- Let r be the remainder of the division of b by a. Assume that $c|a$ and $c|b$. Prove that $c|r$.

① $r = b \bmod a$

② $c|a$, $c|b$

พิสูจน์ว่า $c|r$



จาก ① มีจำนวนเต็ม q ที่

$$b - q \cdot a = r$$

โดยที่ $0 \le r < a$

เขียน $r = b - q \cdot a$

⋮

More examples

For every integer a, $a-1|a^2-1$.

$$a^2 - 1 = a^2 - 1^2 = \underbrace{(a+1)}(a-1)$$

Primes

Definition (primes)

- An integer $p > 1$ is a **prime** if its divisors are only p, $-p$, 1, and -1.
- If an integer $n > 1$ is not a prime, it is called a **composite**.
- Note: 1 is not a prime and also not a composite.

Fundamental theorem of arithmetic

Unique factorization

Every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors.

Algorithm for testing primes

Recall our `CheckPrime2` algorithm

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run?

Algorithm for testing primes

$\boxed{\log n}$

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```

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time $O(\sqrt{n})$ to run.

$2^{\log n/2}$

Efficient algorithms

Is $O(\sqrt{n})$ for checking a prime number efficient?

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n	number of bits of n	$\sqrt{n}$
2	1	1.414
4	2	2
16	4	4
1,024	10	32
1,048,576	20	1,024
1,125,899,906,842,624	50	33,554,432
1,267,650,600,228,229,401,496,703,205,376	100	1,125,899,906,842,624

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Side note: Recall that the first step in RSA is to find a pair of large primes. Typically we want them to be of size in the *thousand* bits.

Greatest Common Divisors (GCD)

Definition (GCD)

For integers x and y, the **greatest common divisor** (or GCD) of x and y is the largest integer g such that $g|x$ and $g|y$. We refer to it as $gcd(x, y)$.

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g = min(x,y)
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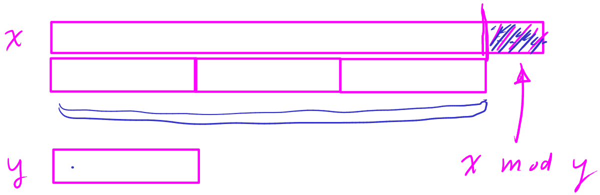
What is the running time of this algorithm? Does it run in polynomial time on the size of the input?

Euclid's algorithm

```
Algorithm Euclid(x,y):
  if x mod y == 0:
    return y
  else:
    return Euclid(y, x mod y)
```

$gcd(x,y) \longrightarrow$



① ถ้า $b|x, b|y$ แสดงว่า
$b | x \bmod y$

$gcd(y, x \bmod y)$

② ถ้า $b|y, b|(x \bmod y)$
$b|x$ ด้วย

Euclid's algorithm

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```

Let's see how it works with $Euclid(12311, 24324)$:

Euclid(12311, 24324)
Euclid(24324, 12311)
Euclid(12311, 12013)
Euclid(12013, 298)
Euclid(298, 93)
Euclid(93, 19)
Euclid(19, 17)
Euclid(17, 2)
Euclid(2, 1)

Proofs

We have to prove two properties:

- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
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- For any integers x and y, $\mathrm{Euclid}(x, y) = gcd(x, y)$.
- The running time of Euclid.

Note that when $x < y$, $\mathrm{Euclid}(x, y)$ just calls itself with both arguments swapped, i.e., $\mathrm{Euclid}(y, x)$. After that, in each call, x is always larger than y. For simplicity of the analysis, we shall work only with the case that $x > y$.

Theorem 1

For any integers x and y such that $x > y$, $\mathrm{Euclid}(x, y) = gcd(x, y)$.

Proof.

We prove using strong induction. For the base case, note that when $y|x$, $gcd(x, y) = y$; therefore, the base case of the algorithm is correct.
Our induction hypothesis is: for any $x' < x$ and $y' < y$, $\mathrm{Euclid}(x', y') = gcd(x', y')$.
Now assume that $y \nmid x$. The Euclid algorithm returns $\mathrm{Euclid}(y, x \bmod y)$ as the gcd. Note that $y < x$ and $x \bmod y < y$. Therefore, we can use the I.H. to claim that

$$\mathrm{Euclid}(y, x \bmod y) = gcd(y, x \bmod y).$$

Thus, we are left to show that

$$\boxed{gcd(x, y) = gcd(y, x \bmod y).}$$

What is $x \bmod y$?

$$x \bmod y = x - \overbrace{\left\lfloor \frac{x}{y} \right\rfloor}^{\text{integer}} \cdot y$$

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Lemma 2

If $a|x$ and $a|y$, then $a|x \bmod y$.

רק מחוכל $\quad x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$

Lemma 2

If $a|x$ *and* $a|y$*, then* $a|x \bmod y$*.*

Lemma 3

$gcd(x, y) = gcd(y, x \bmod y)$

① ถ้า $g|x$, $g|y$,

$\Rightarrow g | x \bmod y$

② ถ้า $g|y$, $g | x \bmod y$,

$\Rightarrow g|x$

How many recursive calls does Euclid's algorithm makes?

$x > y$

Euclid(x, y)

$\downarrow$

Euclid$(y, x \bmod y)$

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- When can we decrease the value of x or y in the calls?

$\gcd(x, y)$

$\downarrow$

$\gcd(y, x \bmod y)$

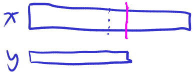
$\downarrow$

$\gcd(x \bmod y, \; y \bmod (x \bmod y))$

Case 1: if $y \le x/2$,

เรียกตัวเอง 1 ครั้ง

argument แรก ลดลง ครึ่งหนึ่ง.



Case 2: if $y > x/2$

เรียกตัวเอง 2 ครั้ง

argument แรก = $\boxed{x \bmod y}$

$= x - y \le x/2$

ดังนั้น argument แรก ลดลง ครึ่ง...

How many recursive calls does Euclid's algorithm makes?

Consider Euclid(x, y):

- If we start with $x < y$, the next calls will always have that $x > y$; so we have at most one call with $x < y$.
- When can we decrease the value of x or y in the calls?
- When $y \leq x/2$, when Euclid(x, y) calls Euclid$(y, x \bmod y)$ the first argument decreases by half.

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- How many times can that happen?
- The first argument can decrease by a factor of two for at most $\log x$ times. Therefore, the Euclid algorithm runs in time $O(\log \max\{x, y\}) = O(\log x + \log y)$.

Computing power

8214652700.121778

121124412577

How fast can we compute x^y?

Computing power

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```
Algorithm power(x,y):
  a = 1
  for i = 1,2,...,y:
    a *= x
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What is the running time?

Computing power

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What is the running time? Is it efficient?

Repeated squaring

If y is a power of two, we can find x^y using small number of multiplications using repeated squaring. E.g.,

$$x^{16} = (x^8)^2 = ((x^4)^2)^2 = (((x^2)^2)^2)^2.$$

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```
Algorithm power(x,y):  // for y=2^k
  if y == 0:
    return 1
  else:
    a = power(x, y / 2)
    return a*a
```

Repeated squaring (general y)

x^{13} $(x^6 \cdot x^6) \cdot x$

```
Algorithm power(x,y):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2))
    if y mod 2 == 0:
      return a*a
    else
      return a*a*x
```

$x^y \bmod n$

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```

What is the number of recursive calls?
What is the running time?
While the number of multiplication is small, the numbers involved is huge as x^y has $y \log x$ bits. Computing x^y exactly definitely takes a long time.

Repeated squaring (general y, mod n)

recursive log y low

Computing x^y mod n:

```
Algorithm power(x,y,n):
  if y == 0:
    return 1
  else:
    a = power(x, floor(y / 2)) mod n
    if y mod 2 == 0:
      return a*a mod n
    else
      return a*a*x mod n
```