

01204211 Discrete Mathematics
Lecture 8b: Vectors

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What is a vector?

You can think of a **vector** as an “ordered” list of elements (which are typically numbers). For example:

▶ $[1, 2, 5, 20]$

▶ $[0, 0, 1, 1, 0, 0, 0, 1]$

You can also view a vector as a **function**, e.g., you can view $\mathbf{u} = [1, 2, 5, 20]$ as a function \mathbf{u} that maps

$$0 \mapsto 1, \quad 1 \mapsto 2, \quad 2 \mapsto 5, \quad 3 \mapsto 20.$$

Each element in the vector is typically a real number (\mathbb{R}), but can be an element from other sets with appropriate property (more on this later).

Remark: Mathematically, a vector is an element of a vector space. We will understand this more later.

What can be represented as a vector?

Viewing vectors: vectors in \mathbb{R}^2

Viewing vectors: vectors in \mathbb{R}^3

n -vectors over \mathbb{R}

- ▶ We mostly deal with vectors with finite number of elements.
- ▶ This is a **4-vector**: $[10, 20, 500, 4]$.
- ▶ We sometimes also write it as a column vector:

$$\begin{bmatrix} 10 \\ 20 \\ 500 \\ 4 \end{bmatrix}$$

- ▶ When every element of a vector is from some set, we say that it is a vector **over** that set. For example, $[10, 20, 500, 4]$ is a 4-vector over \mathbb{R} .

Vector operations

- ▶ As discussed in the previous slides, when working with a system of linear equations, we mostly deal with **linear combinations** of vectors.
- ▶ We will look at the operations we do to vectors to obtain their linear combinations.
- ▶ The operations are:
 - ▶ Vector additions
 - ▶ Scalar multiplications
- ▶ These operations motivate the definition of vector spaces.

Vector additions

Given two n -vectors

$$\mathbf{u} = [u_1, u_2, \dots, u_n]$$

and

$$\mathbf{v} = [v_1, v_2, \dots, v_n],$$

we have that

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n].$$

Vector additions, in picture

Zero vectors

A zero n -vector $\mathbf{0} = [0, 0, \dots, 0]$ is an additive identity, i.e., for any vector \mathbf{u} ,

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}.$$

Scalar multiplications

For a vector over \mathbb{R} , we refer to an element α in \mathbb{R} as a scalar. For an n -vector

$$\mathbf{u} = [u_1, u_2, \dots, u_n],$$

we have that

$$\alpha \cdot \mathbf{u} = [\alpha \cdot u_1, \alpha \cdot u_2, \dots, \alpha \cdot u_n],$$

Scalar multiplications, in pictures

Linear combinations

For any scalar

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Examples:

A linear system with 3 variables

Give the following linear system.

$$\begin{array}{rrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ & x_1 & + & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

If we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

This becomes the problem of expressing a vector as linear combination of other vectors. I.e., given vectors

$$\mathbf{u}_1 = [2, 1, 4], \quad \mathbf{u}_2 = [4, 0, 2], \quad \mathbf{u}_3 = [3, 5, 3]$$

we would like to find coefficients x_1, x_2, x_3 such that

$$x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2 + x_3 \cdot \mathbf{u}_3 = [7, 12, 10].$$

Span

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denote by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Examples:

Convex combination

For any scalar

$$\alpha_1, \alpha_2, \dots, \alpha_m,$$

such that $\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$ and $\alpha_i \geq 0$ for all i , and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **convex combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Examples:

Elements in a vector

- ▶ We see examples of vectors over \mathbb{R} .
- ▶ However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
 - ▶ We should be able to perform addition, subtraction, multiplication, and division.
 - ▶ Operations should be commutative and associative.
 - ▶ Additive and multiplicative identity should exist.
 - ▶ Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a **field**.

A field

A set \mathbb{F} with two operations $+$ and \times (or \cdot) is a **field** iff these operations satisfy the following properties:

- ▶ (Associativity): $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ▶ (Commutativity): $a + b = b + a$ and $a \cdot b = b \cdot a$
- ▶ (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$
- ▶ (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$
- ▶ (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an element a^{-1} such that $a \cdot a^{-1} = 1$
- ▶ (Distributive): $a \cdot (b + c) = a \cdot b + a \cdot c$