

01204211 Discrete Mathematics  
Lecture 9b: Affine Spaces

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# Review: Linear combinations

## Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Review: Span

### Definition

A set of all linear combination of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **span** of that set of vectors.

It is denoted by  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ .

# Review: Vector spaces

## Definition

A set  $\mathcal{V}$  of vectors over  $\mathbb{F}$  is a **vector space** iff

- ▶ (V1)  $\mathbf{0} \in \mathcal{V}$ ,
- ▶ (V2) for any  $\mathbf{u} \in \mathcal{V}$ ,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any  $\alpha \in \mathbb{F}$ , and

- ▶ (V3) for any  $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ ,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Examples of vector spaces:

- ▶ A span of vectors is a vector space.
- ▶ A solution set to homogeneous linear equations is a vector space.

# Translation

If we have a line or a plane passing through a vector  $\mathbf{a}$ , but not through the origin, how can we represent it?

- ▶ Translate the object so that it passes through the origin.
- ▶ We obtain a vector space  $\mathcal{V}$ .
- ▶ Then we translate it back so that it passes through  $\mathbf{a}$ .
- ▶ We get the set

$$\mathcal{A} = \{\mathbf{a} + \mathbf{u} : \mathbf{u} \in \mathcal{V}\}$$

- ▶ *Question:* Is  $\mathcal{A}$  a vector space?
- ▶ We also write it as  $\mathbf{a} + \mathcal{V}$ .

# Affine spaces

## Definition

If  $\mathbf{a}$  is a vector and  $\mathcal{V}$  is a vector space, then

$$\mathbf{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions



# Affine combination

## Definition

For any scalars  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ , we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Definition

The set of all affine combinations of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **affine hull** of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ .

# Convex combination: review

## Definition

For any scalars  $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$  such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ , we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **convex combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Definition

The set of all convex combinations of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **convex hull** of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ .

## Writing an affine space using a span

### An affine space

An affine space passing through  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

## Non-homogeneous linear system

Two linear systems:

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} & = & b_2 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & b_m \end{array} \qquad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ \mathbf{a}_2 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

What can you say about the solution sets of these two related linear systems?

$\mathbf{0}$  is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

## Solutions of the two systems

Recall that if  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are both solutions to the non-homogeneous linear system, we have that for any  $i$

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

This implies that  $\mathbf{u}_1 - \mathbf{u}_2$  is a solution to the homogeneous linear system.

Suppose that  $\mathcal{W}$  is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let  $\mathbf{u} \in \mathcal{W}$  be one of the solutions, we have that

$$\{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\}$$

is a vector space, because

$$\{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\}$$

In other words,

$$\begin{aligned}\mathcal{W} &= \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} \\ &= \mathbf{u} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\},\end{aligned}$$

i.e.,  $\mathcal{W}$  is an affine space.

# Solutions to a non-homogeneous linear system

## Lemma 1

*If the solution set of a linear system is not empty, it is an affine space.*