01204211 Discrete Mathematics
Lecture 10a: Polynomials (1)[1]

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October 15, 2024

[1] This section is from Berkeley CS70 lecture notes.

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

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For any integer $a \neq 1$, $a - 1 | a^2 - 1$.
For any integer $a \neq 1$ and $n \geq 1$, $a - 1 | a^n - 1$.

Polynomials

ถ้า f เป็น polynomial degree ≤ d
· ถ้าทราบจุดบน f ≥ d+1 จุด
จะสามารถสร้าง f กลับมาได้

มีแค่ polynomial degree ≤ d 1 poly ที่ผ่าน d+1 จุด

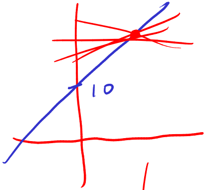
A **single-variable polynomial** is a function $p(x)$ of the form

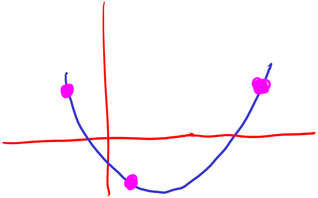
$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

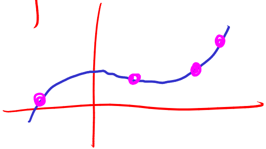
We call a_i's *coefficients*. Usually, variable x and coefficients a_i's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Examples

- $x^3 - 3x + 1$ 1   -3   1
- $x + 10$
- 10
- 0





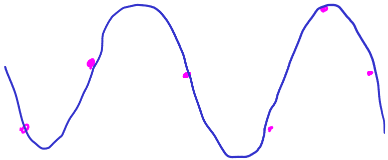


Folklore

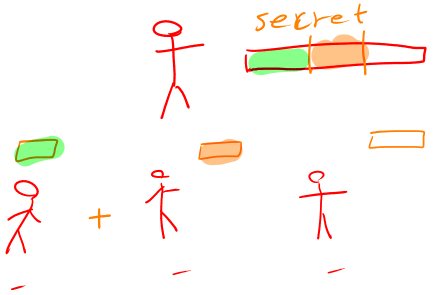
ถ้า f เป็น polynomial degree $\leq d$

- ถ้าทราบจุดบน f $\geq d+1$ จุด จะสามารถสร้าง f กลับมาได้

เวย polynomial degree $\leq d$ 1 poly ที่ผ่าน $d+1$ จุด



Applications



- Secret sharing

→ polynomial.

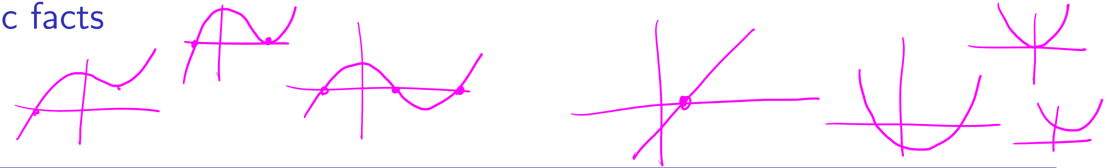
→ GF(p)
(จำนวนเฉพาะ)

secret → แบ่งคน n คน
จำนวนคน k ที่
① ถ้ามี $\boxed{k}$ คนขึ้นไป → หา secret ได้ ✓
② ถ้า ≤ $\boxed{k-1}$ คน จะไม่ทราบอะไรเกี่ยวกับ secret เลย
0%.

Applications

- Secret sharing
- Error-correcting codes

Basic facts



Definition

a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

Properties

Property 1: A non-zero polynomial of degree d has at most $\boxed{d}$ roots.
Property 2: Given $d + 1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

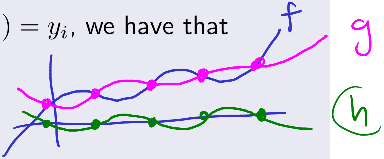
Lemma 1

If two polynomials $f(x)$ and $g(x)$ of degree at most d that share $d+1$ points $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$, where all x_i's are distinct, i.e., $f(x_i) = g(x_i) = y_i$, then $f(x) = g(x)$.

Proof.

Suppose that $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $g(x) = b_d x^d + b_{d-1}x^{d-1} + \cdots + b_0$.
Let $h(x) = f(x) - g(x)$, i.e., let $h(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_0$, where $c_i = a_i - b_i$. Note that $h(x)$ is also a polynomial of degree (at most) d.
We claim that $h(x)$ has $d+1$ roots. Note that since $f(x_i) = g(x_i) = y_i$, we have that

$$h(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0,$$



i.e., every x_i is a root of $h(x)$.
From **Property 1**, if $h(x)$ is non-zero it has at most d roots; therefore, $h(x)$ must be zero, i.e.,
$h(x) = \boxed{f(x) - g(x)} = 0$ or $f(x) = g(x)$ as required. □

Polynomial interpolation - ideas

evaluation

ให้ $p(x)$

ที่จุด $x_1, x_2, \dots$

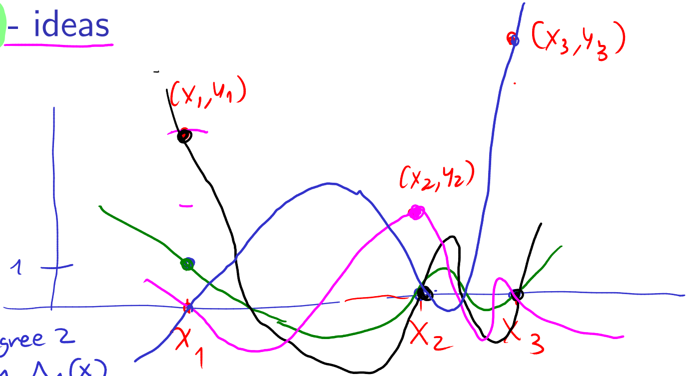
ได้ $p(x_1), p(x_2), \dots$

interpolation

ให้ $(x_1, y_1), (x_2, y_2) \dots$

(x_{d+1}, y_{d+1})

หา $p(x)$ ที่ deg



degree 2

ขอ $\Delta_1(x)$

- $\Delta_1(x_1) = 1$ *
- $\Delta_1(x_2) = \Delta_1(x_3) = 0$ ---- f_1

$$y_1 \left(\boxed{\Delta_1(x)} = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \right) y_1$$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d

$$\boxed{\Delta_i(x)} = \begin{cases} 0 & \text{ถ้า } \boxed{x = x_j} \text{ ที่ } j \neq i \\ 1 & \text{ถ้า } x = x_i \end{cases}$$

Lagrange polynomial

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Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) =$

Lagrange polynomial

For $d+1$ points $(x_1,y_1),(x_2,y_2),\ldots,(x_{d+1},y_{d+1})$ where all x_i's are distinct, let

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Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) = 1$.

We can use $\Delta_i(x)$ to construct a degree-d polynomial

$$\boxed{p(x)} = y_1 \cdot \Delta_1(x) + y_2 \cdot \Delta_2(x) + \cdots y_{d+1} \cdot \Delta_{d+1}(x).$$

What can you say about $p(x_i)$?

Property 2

Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.

Proof of Property 2.

Using Lagrange interpolation, we know that there exists a polynomial $\boxed{p(x)}$ of degree d such that $p(x_i) = y_i$ for all $1 \leq i \leq d+1$.
For uniqueness, assume that there exists another polynomial $g(x)$ of degree d also satifying the condition. Since $p(x)$ and $g(x)$ agrees on more than d points, $p(x)$ and $g(x)$ must be equal from Lemma 1. □

Polynomials over a finite field $GF(p)$

Examples - evaluation

Suppose that we work over $GF(m)$ where $m = 11$. Let $p(x) = 4 \cdot x^2 + 5 \cdot x + 3$. We have

x	$p(x)$	$p(x) \bmod m$
0	3	3
1	12	1
2	29	7
3	54	10
4	87	10
5	128	7
6	177	1
7	234	3
8	299	2
9	372	9
10	453	2
11	542	3

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2, 7), (4, 10)$, and $(7, 3)$. Find $p(x)$.

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Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7),(4,10)$, and $(7,3)$. Find $p(x)$.
Let

- $\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2-11x+28}{(-2)\cdot(-5)} = \frac{x^2+6}{10} = 10x^2 + 5$

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- $\Delta_2(x) = \frac{(x-2)(x-7)}{(4-2)(4-7)} = \frac{x^2-9x+14}{2\cdot(-3)} = \frac{x^2+2x+3}{5} = 9x^2 + 7x + 5$

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7),(4,10)$, and $(7,3)$. Find $p(x)$.

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- $\Delta_2(x) = \frac{(x-2)(x-4)}{(7-2)(7-4)} = \frac{x^2-6x+8}{5\cdot 3} = \frac{x^2+5x+8}{4} = 3x^2+4x+2$

Thus,

$$\begin{aligned} p(x) &= 7\Delta_1(x) + 10\Delta_2(x) + 3\Delta_3(x) \\ &= (70x^2+35) + (90x^2+70x+50) + (9x^2+12x+6) \\ &= 4x^2+5x+3 \end{aligned}$$

Secret sharing scheme - settings

Secret sharing scheme - settings

5 3

- There are $\overset{5}{n}$ people, a secret s, and an integer $\overset{3}{k}$.
- We want to "distribute" the secret in such a way that any set of $k-1$ people cannot know anything about s, but any set of k people can reconstruct s.

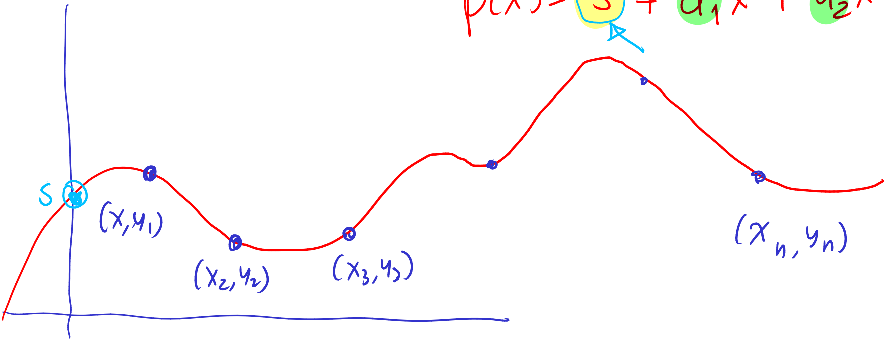
Secret sharing scheme

$S \longrightarrow p(x)$

▷ k คน สร้างความลับได้

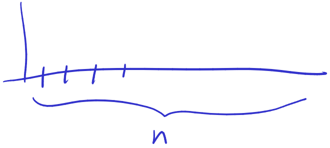
$\longrightarrow$ สร้าง $p(x)$ — $\boxed{p(x) \text{ ต้องมี degree } k-1}$

$$p(x) = S + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1}$$



ให้คนที่ $i \leftarrow (x_i, y_i)$

Secret sharing scheme



mod m

#an

secret

- Pick m to be larger than n and s. (Much larger than s, i.e., $m >>> s$.)
- Pick a random polynomial of degree $\boxed{k-1}$ such that $P(0) = s$.
- Give $\boxed{P(i)}$ to person i, for $1 \le i \le n$.
- Correctness: for any set of k people,

$a_1, a_2, \ldots, a_{k-1}$

$\boxed{(i, P(i))}$

get P → get $P(0)$ → the secret

interpolation

Secret sharing scheme

- Pick m to be larger than n and s. (Much larger than s, i.e., $m >>> s$.)
- Pick a random polynomial of degree $k-1$ such that $P(0) = s$.
- Give $P(i)$ to person i, for $1 \leq i \leq n$.
- Correctness: for any set of k people,
- Correctness: for any set of $k-1$ people, how many possible candidate secrets compatible with the information these people have?