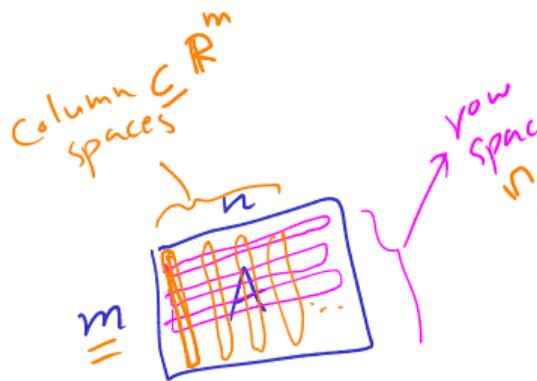


vectors  
↓  
spans  
↓  
vector spaces  
solutions  
homogen  
line  $\text{eqn}$

01204211 Discrete Mathematics

Lecture 11b: Four fundamental subspaces (preview)



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# What is a matrix?

basis.

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\text{Span } \{ [1, 4, 7, 10], [2, 5, 8, 11], [3, 6, 9, 12] \} = \text{column space}$$

$\subseteq \mathbb{R}^4$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{bmatrix}$$

span of rows of a matrix  
row space

$$\text{Span } \{ [1, 2, 4], [4, 5, 6], [7, 8, 9], [10, 11, 12] \} \subseteq \mathbb{R}^3$$

# Matrices and elimination

## Matrices and elimination

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

## Matrices and elimination



$\textcircled{A} \leftrightarrow \textcircled{B}$

$r = k$

vow sp

$$\left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{array} \right] \Rightarrow \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\textcircled{k}$   
linear  
in  
vows

A



$k \geq r$

B



$\textcircled{r}$   
linear  
in  
var

## Matrices and elimination

Row space of A

Span {  $\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 5 & 1 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 2 & 10 & 2 & 4 \end{bmatrix}$ ,  $\begin{bmatrix} 2 & 7 & 1 & 10 \end{bmatrix}$  }

$$\left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{array} \right] \Rightarrow \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

B

dim row space of B = 3

Span {  $\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 3 & 1 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 0 & 0 & 2 \end{bmatrix}$  }

$$\{ [1, 2, 0, 1], [0, 3, 1, 0], [0, 0, 0, 2] \}$$

B a basis for the row space of B

# Matrices and elimination

Row space of A  $C(A^T)$

$$= \text{Span} \{ \bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4 \}$$

dimension of  $C(A^T)$

$$\dim C(A^T) = 3$$

row rank of A (= 3)

= max number  
of linearly  
independent  
rows of A.

$$C(A^T) = C(B^T)$$

V is a vector space

- ①  $0 \in V$
- ②  $\forall u \in V$ ,
- $\alpha u \in V$
- ③  $\forall u, v \in V$ ,
- $u + v \in V$ .

$\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \bar{r}'_1 \\ \bar{r}'_2 \\ \bar{r}'_3 \\ \bar{r}'_4 \end{matrix}$$

$$(1) C(A^T) \supseteq C(B^T)$$

$$(2) C(A^T) \subseteq C(B^T)$$

Row Space of B

$$C(B^T) = \text{Span} \{ \bar{r}_1, \bar{r}_2, \bar{r}_3 \}$$

$\{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$  is a basis of  $C(B^T)$

$$\dim C(B^T) = 3$$

row rank of B

$$= 3$$

## Matrices and elimination

column space of  $A$

$C(A)$

$= \text{Span} \{ \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4 \}$

$$\boxed{\dim C(A)} = 3$$

||

$$\boxed{\dim C(A^T)} = 3$$

$A$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4$

## Matrices and elimination

Column space

$$\text{Span} \left\{ \begin{bmatrix} 1, 0, 0, 0 \end{bmatrix}, \begin{bmatrix} 2, 1, 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 1, 0, 0 \end{bmatrix}, \begin{bmatrix} 1, 0, 2, 0 \end{bmatrix} \right\}$$

B

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Matrices and elimination

$$N(A) = \{\bar{x} \mid A\bar{x}=0\} = \{\bar{x} \mid B\bar{x}=0\} = \left\{ \alpha \begin{bmatrix} 2/3, -1/3, 1, 0 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$$

$$\dim N(A) = 1$$

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{array} \right] \Rightarrow \begin{array}{c} B \\ \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array} \end{array}$$

$x_1 + 2x_2 + x_4 = 0$   
 $\boxed{3x_2 + x_3 = 0}$   
 $\underline{2x_4 = 0}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_3 \\ -\frac{1}{3}x_3 \\ x_3 \\ 0 \end{bmatrix}$$

$$\begin{array}{l} x_4 = 0 \\ x_2 = \frac{-x_3}{3} \end{array}$$

$$x_1 - \frac{2x_3}{3} = 0$$
$$x_1 = \frac{2}{3}x_3$$

## Matrices and elimination

Column rank  
" "  
Row rank

Column rank = 3  
of B

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Row echelon form

## Linearly independent rows

## Vector spaces related to a matrix

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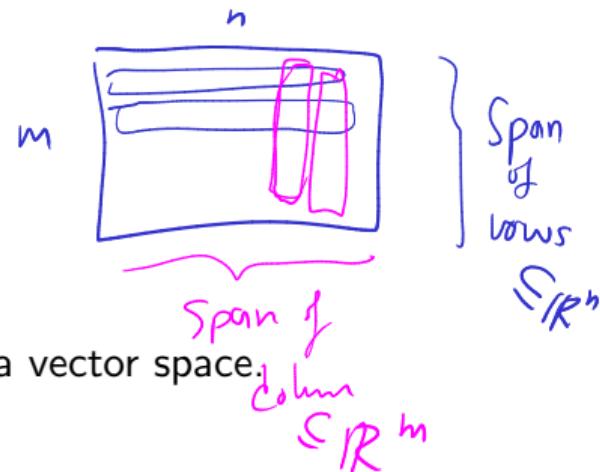
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$$\text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\} = \text{row space}$$

# Subspaces

## Definition

Let  $\mathcal{V}$  and  $\mathcal{W}$  be vector spaces such that  $\mathcal{V} \subseteq \mathcal{W}$ . We say that  $\mathcal{V}$  is a **subspace** of  $\mathcal{W}$ .

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## Examples:

- ▶ Span  $\{[1, 1]\}$  is a subspace of  $\mathbb{R}^2$ .
- ▶ Span  $\{[1, 0, 0], [0, 1, 1]\}$  is a subspace of  $\mathbb{R}^3$ .
- ▶ Span  $\{[1, 0, 0], [0, 1, 1], [1, 1, 2]\}$  is a subspace of  $\mathbb{R}^3$ .

## Example 1

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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$$\mathcal{C}(A) = \{\alpha_1[1, 0] + \alpha_2[2, 1] + \alpha_3[4, 3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

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- ▶ Row space:

$$\mathcal{C}(A^T) = \{\alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

## Example 1

Vector spaces

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

• Basis  
⊗ dimension

- ▶ Column space:

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- ▶ Row space:

$$\mathcal{C}(A^T) = \{\alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

## Example 1 (cont.)

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left\{ \begin{array}{l} 1(x_1) + 2(x_2) + 4(x_3) = 0 \\ 1x_2 + 3x_3 = 0 \end{array} \right. \quad \leftarrow$$

Let

$N(A)$

Is there any other way to obtain vector spaces from  $A$ ?

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

homogeneous system

$$\bar{x} = [x_1, x_2, x_3]$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \boxed{\{ \bar{x} \mid A\bar{x}=0 \}}$$

$$1\cancel{x_1} + 2\cancel{x_2} + 4x_3 = 0 \quad \text{--- (1)}$$

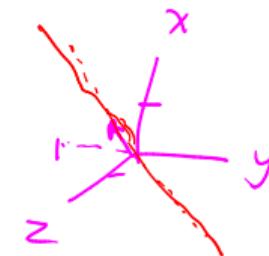
$$1x_2 + 3x_3 = 0 \quad \text{--- (2)}$$

from (2)

$$\boxed{x_2 = -3x_3}$$

plug into (1)  $x_1 + 6x_3 + 4x_3 = 0$

$$\boxed{x_1 = 2x_3}$$



$$\underline{\underline{N(A)}} = \{ \bar{x} \mid A\bar{x}=0 \} = \left\{ \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\} = \left\{ x_3 \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}$$

## Example 1 (cont.)

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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We can think of  $A$  as a coefficient matrix of a system of homogenous linear equations:

$$Ax = 0.$$

In this case, we have

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The set of solutions  $\{x \mid Ax = 0\}$  form a vector space.

Null space of  $A$

## Example 1 (cont.)

Given a matrix  $A$ , we can look at the matrix-vector product  $A\mathbf{x}$ .

Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

# Four fundamental subspaces

## Four fundamental subspaces

Given an  $m$ -by- $n$  matrix  $A$ , we have the following subspaces

- ▶ The column space of  $A$  (denoted by  $\mathcal{C}(A)$ )
- ▶ The row space of  $A$  (denoted by  $\mathcal{C}(A^T)$ )
- ▶ The nullspace of  $A$

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$$

- ▶ The left nullspace of  $A$

$$\mathcal{N}(A^T) = \{\mathbf{y} \mid A^T\mathbf{y} = \mathbf{0}\}$$

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## Four fundamental subspaces



$$\text{rank}(A) = r$$

## Four fundamental subspaces

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$$\dim \mathcal{C}(A) = r$$

► The row space of  $A$  (denoted by  $\mathcal{C}(A^T) \subseteq \mathbb{R}^n$ )

$$\dim \mathcal{C}(A^T) = \boxed{r}$$

→ ► The nullspace of  $A$

$$\mathcal{N}(A) = \{x \mid Ax = \mathbf{0}\} \subseteq \mathbb{R}^n$$

$$\dim \mathcal{N}(A) = n - r$$

→ ► The left nullspace of  $A$

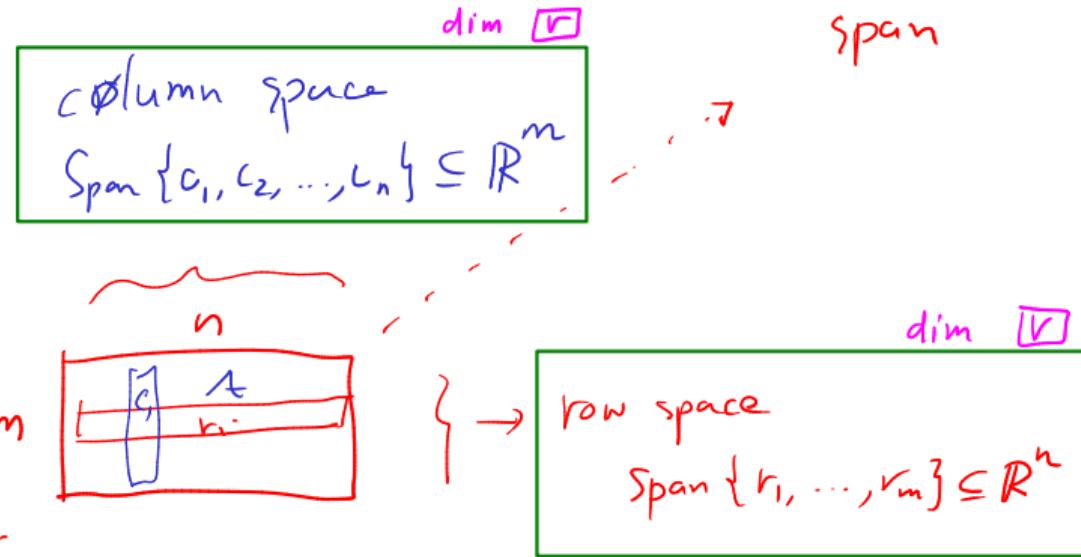
$$\mathcal{N}(A^T) = \{y \mid A^T y = \mathbf{0}\} \subseteq \mathbb{R}^m$$

$$\dim \mathcal{N}(A^T) = m - r$$

dim  $n-r$

$N(A)$   
 $\{x \mid Ax=0\} \subseteq \mathbb{R}^n$

Homogen  
sysh  
lik  
q.



$N(A^T)$   
 $\{y \mid A^T y = 0\} \subseteq \mathbb{R}^m$

dim  $m-r$