

# 01204211 Discrete Mathematics

## Lecture 9c: Linear Independence and Bases

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# Review: Linear combinations

## Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Review: Span

### Definition

A set of all linear combination of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **span** of that set of vectors.

It is denoted by  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ .

## Previous Lemmas

### Lemma 1

*Consider vectors  $u_1, u_2, \dots, u_n$ . If  $v_1, v_2, \dots, v_k$  are generators for  $\mathcal{V}$ , and for each  $i$ ,*

$$v_i \in \text{Span} \{u_1, u_2, \dots, u_n\},$$

*we have that  $\mathcal{V} \subseteq \text{Span} \{u_1, u_2, \dots, u_n\}$ .*

## Lemma 2

Consider vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . If  $\mathbf{v} \in \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ , then

$$\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\} = \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

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## Lemma 3

Consider vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ . If  $\mathbf{u}_n \in \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\}$ , then

$$\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\} = \text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

## Proof of Lemma 2.

Since  $v$  can be written as a linear combination of other vectors, there exist  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

Consider any vector  $w \in \text{Span} \{u_1, u_2, \dots, u_n, v\}$ ; thus, we can write

$$w = \beta_0 v + \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n.$$

Plugging in  $v$ , we get that

$$\begin{aligned} w &= \beta_0 (\alpha_1 u_1 + \dots + \alpha_n u_n) + \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_n u_n \\ &= (\beta_0 \alpha_1 + \beta_1) u_1 + (\beta_0 \alpha_2 + \beta_2) u_2 + \dots + (\beta_0 \alpha_n + \beta_n) u_n, \end{aligned}$$

implying that  $w \in \text{Span} \{u_1, u_2, \dots, u_n\}$ .



# Linearly independence

## Definition

Vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  are **linearly independent** if no vector  $\mathbf{u}_i$  can be written as a linear combination of other vectors.

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## (Another) Definition

Vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  are **linearly independent** if the only solution of equation

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

# Examples in $\mathbb{R}^2$

# Examples in $\mathbb{R}^3$

## Examples in $GF(2)$

# Examples in linear systems

# Subset of linearly independent vectors

## Lemma 4

*If  $A = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be a set of linearly independent vectors, then any  $B \subseteq A$  is also a set of linearly independent vectors.*

## Proof.

We prove by contradiction. Assume that  $B$  is **not** linearly independent. Without loss of generality, assume that  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  where  $k \leq n$ .

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$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0},$$

and some  $\alpha_i$ 's is nonzero.

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$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k = \mathbf{0},$$

and some  $\alpha_i$ 's is nonzero. If we let  $\alpha_{k+1} = \alpha_{k+2} = \dots = \alpha_n = 0$ , we have that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0},$$

with some  $\alpha_i$ 's being nonzero as well.

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with some  $\alpha_i$ 's being nonzero as well. This implies that vectors in  $A$  are not linearly independent; leading to a contradiction. □

# Bases

## Definition

A set of vectors  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  is a **basis** for vector space  $\mathcal{V}$  if

- ▶  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\} = \mathcal{V}$ , and
- ▶  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent.

## Examples 1: $\mathbb{R}^2$ and $\mathbb{R}^3$

## Examples 2

## Lemma 5 (Unique representation)

*Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  be a basis for vector space  $\mathcal{V}$ . For any  $\mathbf{v} \in \mathcal{V}$ , there is a unique way to write  $\mathbf{v}$  as a linear combination of  $\mathbf{u}_1, \dots, \mathbf{u}_k$ .*

## Proof of unique representation lemma.

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We prove by contradiction. Assume that there exists a vector  $\mathbf{v} \in \mathcal{V}$  with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \dots, \alpha_k,$$

and

$$\beta_1, \beta_2, \dots, \beta_k,$$

that are not equal (i.e., there exists  $i$  where  $\alpha_i \neq \beta_i$ ) such that  $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k$  and  $\mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k$ .

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$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k = \mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k,$$

and

$$(\alpha_1 - \beta_1) \mathbf{u}_1 + (\alpha_2 - \beta_2) \mathbf{u}_2 + \dots + (\alpha_k - \beta_k) \mathbf{u}_k = \mathbf{0}.$$

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and

$$(\alpha_1 - \beta_1) \mathbf{u}_1 + (\alpha_2 - \beta_2) \mathbf{u}_2 + \dots + (\alpha_k - \beta_k) \mathbf{u}_k = \mathbf{0}.$$

Since  $\alpha_i \neq \beta_i$ , we have that at least one of the coefficients is non-zero, implying that  $\mathbf{u}_1, \dots, \mathbf{u}_k$  are not linearly independent. This contradicts the assumption that  $\mathbf{u}_1, \dots, \mathbf{u}_k$  form a basis. □