01204211 Discrete Mathematics
Lecture 9b: Polynomials (1)[1]

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[1] This section is from Berkeley CS70 lecture notes.

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

$$a^2 - 1 = \underbrace{(a-1)}(a+1)$$

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

For any integer $a \neq 1$ and $n \geq 1$, $a - 1 | a^n - 1$.

$$\begin{array}{r l} & a^4 + a^3 + a^2 + a + 1 \\ a - 1 \,\big)\!\!\! & \overline{a^5 \qquad\qquad\qquad - 1} \\ & a^5 - a^4 \\ & \overline{a^4} \\ & a^4 - a^3 \\ & \overline{a^3} \\ & a^3 - a^2 \\ & \overline{a^2} \\ & a^2 - a \\ & \overline{a} \\ & a - 1 \end{array}$$

$$\boxed{a^n - 1 = \underbrace{(a-1)}\,\underbrace{(a^{n-1} + a^{n-2} + \cdots + a + 1)}}$$

$$(a^n + \cancel{a^{n-1}} + \cdots + \cancel{a}) - (\cancel{a^{n-1}} + a^{n-2} + \cdots \cancel{+} 1)$$

$$a^5 - 1 = (a-1)(a^4 + a^3 + a^2 + a + 1)$$

Polynomials พหุนาม

A **single-variable polynomial** is a function $p(x)$ of the form

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

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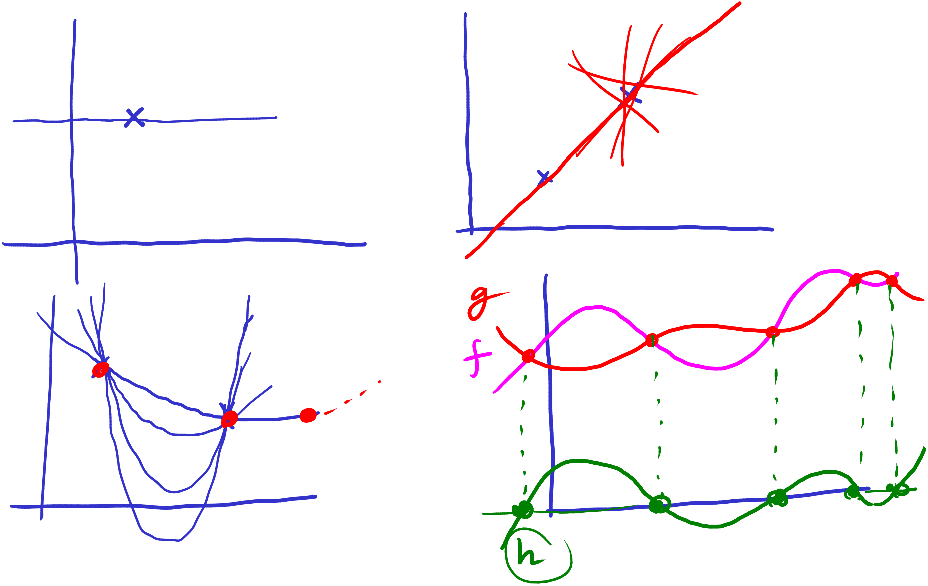
We call a_i's *coefficients*. Usually, variable x and coefficients a_i's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Examples

- $x^3 - 3x + 1$ — degree 3
- $x + 10$ — degree 1
- 10 — d 0
- $\boxed{0}$

Folklore

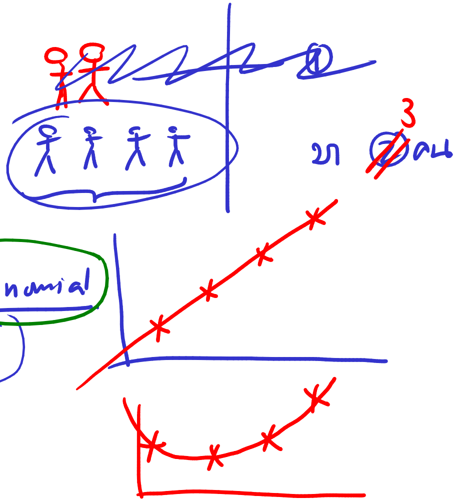


Applications

- Secret sharing

polynomial

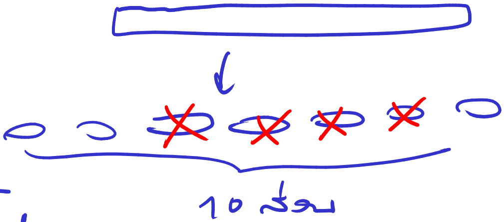
d

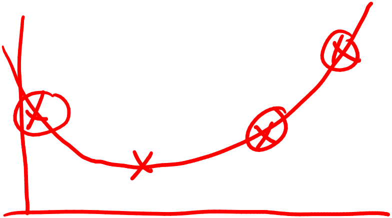


Applications

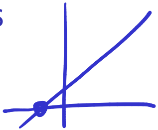
- Secret sharing
- Error-correcting codes

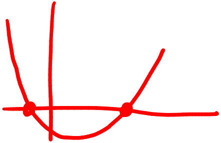
erasure codes

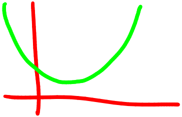




Basic facts







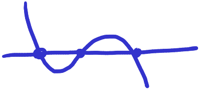
Definition

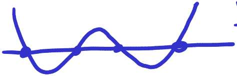
a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

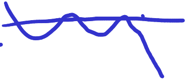
Properties

Property 1: A non-zero polynomial of degree d has at most d roots.

Property 2: Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.







Lemma 1

(Uniqueness) via Prop 2.

If two polynomials $f(x)$ *and* $g(x)$ *of degree at most* d *that share* $d+1$ *points* $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$*, where all* x_i*'s are distinct, i.e.,* $f(x_i) = g(x_i) = y_i$*, then* $f(x) = g(x)$*.*

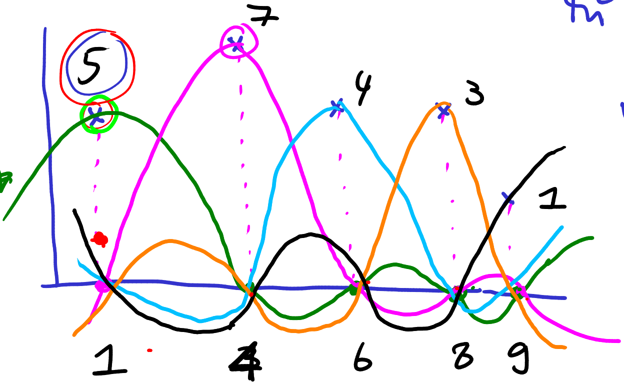
Proof.

Suppose that $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $g(x) = b_d x^d + b_{d-1}x^{d-1} + \cdots + b_0$.
Let $h(x) = f(x) - g(x)$, i.e., let $h(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_0$, where $c_i = a_i - b_i$. Note that $h(x)$ is also a polynomial of degree (at most) d.
We claim that $h(x)$ has $d+1$ roots. Note that since $f(x_i) = g(x_i) = y_i$, we have that

$$h(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0,$$

i.e., every x_i is a root of $h(x)$.
From **Property 1**, if $h(x)$ is non-zero it has at most d roots; therefore, $h(x)$ must be zero, i.e., $f(x) - g(x) = 0$ or $f(x) = g(x)$ as required. □

Polynomial interpolation - ideas



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$\Downarrow$

ได้ polynomial degree 4

$f(2)=1$

$f(4)=0$

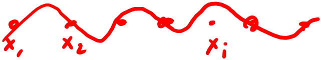
$f(-1)=0$

$f(-3)=0$

$$\frac{(x-4)(x+1)(x+3)}{(2-4)(2+1)(2+3)}$$

$$\boxed{\Delta_1(x)} = \frac{(x-4)(x-6)(x-8)(x-9)}{(1-4)(1-6)(1-8)(1-9)}$$

Lagrange polynomial



For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

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Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\underline{\Delta_i(x_j)} =$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

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Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) =$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

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- $\Delta_i(x_i) = 1$.

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{d+1})}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) = 1$.

We can use $\Delta_i(x)$ to construct a degree-d polynomial

$$\boxed{p(x)} = \underline{y_1 \cdot \Delta_1(x)} + y_2 \cdot \Delta_2(x) + \cdots y_{d+1} \cdot \Delta_{d+1}(x).$$

What can you say about $p(x_i)$? $= y_i \underbrace{\cancel{\Delta_i(x_i)}}_{1} + \overbrace{\cancel{\Delta_j(x_i)}}^{0}$

$= y_i$

Property 2

Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.

Proof of Property 2.

Using Lagrange interpolation, we know that there exists a polynomial $p(x)$ of degree d such that $p(x_i) = y_i$ for all $1 \leq i \leq d+1$.
For uniqueness, assume that there exists another polynomial $g(x)$ of degree d also satifying the condition. Since $p(x)$ and $g(x)$ agrees on more than d points, $p(x)$ and $g(x)$ must be equal from Lemma 1. $\square$