01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (I)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

Vector spaces related to a matrix

Consider an m-by-n matrix A over $\mathbb{R}$.
We can view A as

- n columns of m-vectors: $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$
- m rows of n-vectors: $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m$

When we have a set of vectors, recall that its span forms a vector space.

We have

- Column space: $\mathrm{Span}\,\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\} \subseteq \mathbb{R}^m$
- Row space: $\mathrm{Span}\,\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\} \subseteq \mathbb{R}^n$

Subspaces

Definition

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that $\mathcal{V}$ is a **subspace** of $\mathcal{W}$.

Examples:

- $\text{Span}\,\{[1,1]\}$ is a subspace of $\mathbb{R}^2$.
- $\text{Span}\,\{[1,0,0],[0,1,1]\}$ is a subspace of $\mathbb{R}^3$.
- $\text{Span}\,\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of $\mathbb{R}^3$.

Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

- Column space:

$$\mathcal{C}(A) = \{\alpha_1[1,0] + \alpha_2[2,1] + \alpha_3[4,3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} = \mathbb{R}^2.$$

  Note that: $\dim \mathcal{C}(A) = 2$

- Row space:

$$\mathcal{C}(A^T) = \{\alpha_1[1,2,4] + \alpha_2[0,1,3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

  Note that: $\dim \mathcal{C}(A^T) = 2$

Example 1 (cont.)

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Is there any other way to obtain vector spaces from A?

We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$A\boldsymbol{x} = 0.$$

In this case, we have

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The set of solutions $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product $A\boldsymbol{x}$.
Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{C}(A) \subseteq \mathbb{R}^m$)
- The row space of A (denoted by $\mathcal{C}(A^T) \subseteq \mathbb{R}^n$)
- The nullspace of A

$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- The left nullspace of A

$$\mathcal{N}(A^T) = \{\boldsymbol{y} \mid A^T\boldsymbol{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

Linearly independent rows

Ranks

Definition

Consider an m-by-n matrix A.

- The **row rank** of A is the maximum number of linearly independent rows of A.
- The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{C}(A)$. The row rank of A is $\dim \mathcal{C}(A^T)$.

Row rank = Column rank

Theorem 1

For any matrix A, its row rank equals its column rank.

We will prove this theorem next time.