

# 01204211 Discrete Mathematics

## Lecture 9a: Fermat's Little Theorem

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## Quick recap

For any integer  $x$  and  $y$ , there exist a pair of integers  $a$  and  $b$  such that

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How to find  $a$  and  $b$ ? Use the extended GCD algorithm.

## Finding $a$ and $b$ : Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns  $a$  and  $b$  together with  $\gcd(x, y)$ .

```
Algorithm Euclid(x,y):  
    if x mod y == 0:  
  
        return y, 0, 1  
    else:  
        g, a', b' = Euclid(y, x mod y)  
  
        a = b'  
  
        b = a' - b'*floor(x / y)  
  
    return g, a, b
```

## Recap: Congruences

### Definition (congruences)

For an integer  $m > 0$ , if integers  $a$  and  $b$  are such that

$$a \bmod m = b \bmod m,$$

we write

$$a \equiv b \pmod{m}.$$

We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a - b)$$

## Recap: Multiplicative inverse modulo $m$

### Definition

The multiplicative inverse modulo  $m$  of  $a$ , denoted by  $a^{-1}$ , is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

### Theorem 1

*An integer  $a$  has a multiplicative inverse modulo  $m$  iff  $\gcd(a, m) = 1$ .*

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- ▶ If there is a property that holds **iff**  $n$  is prime, we can check that property. If we can check that quickly, we can test if  $n$  is prime.
- ▶ If there is a property that holds **if**  $n$  is prime, how can we make use of that property?

## Theorem 2 (Fermat's Little Theorem)

*If  $p$  is prime and  $a$  is an integer such that  $\gcd(a, p) = 1$ ,*

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*How can we use Fermat's Little Theorem to check if integer  $n$  is prime?*

## Fermat test

```
Algorithm CheckPrime(n):  
    pick integer a from 2,...,n-1  
  
    if gcd(a,n) != 1:  
        return False  
  
    if power(a,n-1,n) != 1:  
        return False  
    else:  
        return True
```

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When you call `CheckPrime(n)`:

- ▶ If  $n$  is prime, `CheckPrime` always return `True`.
- ▶ If  $n$  is composite, you want `CheckPrime` to return `False`, but **Fermat's Little Theorem does not guarantee that.**

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- ▶  $\gcd(a, n) \neq 1$ , i.e., when you pick  $a$  with common factor with  $n$ .
- ▶  $a^{n-1} \bmod n \neq 1$ , i.e., when you find  $a$  that violates the property. We want to be in this case. How likely?

# Proof of Fermat's Little Thm: Idea

Let  $p = 7$  and  $a = 5$ . Consider set

$$B = \{1, 2, 3, \dots, p-1\} = \{1, 2, 3, 4, 5, 6\}$$

Also consider set

$$C = \{1 \cdot 5 \bmod 7, 2 \cdot 5 \bmod 7, 3 \cdot 5 \bmod 7, \dots, 6 \cdot 5 \bmod 7\},$$

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Is this coincidental? No. (We will prove that. But can you quickly tell why.)

Since  $B = C$ , the following terms are equal:

$$\left( \prod_{i \in B} i \right) \bmod 7 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \bmod 7,$$

and

$$\begin{aligned} \left( \prod_{i \in C} i \right) \bmod 7 &= 5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \bmod 7 \\ &= (1a) \cdot (2a) \cdot (3a) \cdot (4a) \cdot (5a) \cdot (6a) \bmod 7 \\ &= (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot a^6 \bmod 7. \end{aligned}$$

## Proof of Fermat's Little Thm.

Recall that  $\gcd(a, p) = 1$ , i.e., there exists a multiplicative inverse  $a^{-1}$  of  $a$  modulo  $p$ . This implies that for  $i \not\equiv j \pmod{p}$ ,  $ai \not\equiv aj \pmod{p}$ . Also note that  $a \cdot 0 \equiv 0 \pmod{p}$ .



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Since for different  $i, j \in B$ , we have different  $ai \bmod p, aj \bmod p$ , we know that  $|C| = p-1$ . Also,  $C \subseteq B$  because  $0 \leq ai \bmod p \leq p-1$  and  $0 \notin C$ . Thus, we can conclude that  $C = B$ .

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$$\begin{aligned} 1 \cdot 2 \cdots (p-1) &\equiv (a1) \cdot (a2) \cdot (a3) \cdots (a(p-1)) \pmod{p} \\ &\equiv (1 \cdot 2 \cdots (p-1)) \cdot a^{p-1} \pmod{p}. \end{aligned}$$

Since each of  $1, 2, \dots, p-1$  has an inverse modulo  $p$ , we can multiply both sides with  $1^{-1}, 2^{-1}, \dots, (p-1)^{-1}$  to obtain

$$1 \equiv a^{p-1} \pmod{p},$$

as required.



## Exercise

Prove that for any integer  $a$  and prime  $p$ ,

$$a^p \equiv a \pmod{p}.$$

## How good is the Fermat test when $n$ is composite?

To answer correctly, we want  $a$  to be such that  $\gcd(a, n) \neq 1$  or

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We refer to  $a \in \{1, 2, \dots, p-1\}$  such that  $\gcd(a, n) = 1$  and  $a^{n-1} \not\equiv 1 \pmod{n}$  as a **witness**. The other element  $b$  such that  $b^{n-1} \equiv 1 \pmod{n}$  is called a **non-witness**. How likely that we randomly choose an element and get a witness?

## Number of witnesses

Suppose that there exists a witness  $a$ ; we know that  $a^{n-1} \not\equiv 1 \pmod{n}$ . How can we find other witnesses?



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Consider a non-witness  $b$  such that  $b^{n-1} \equiv 1 \pmod{n}$ .

# Carmichael Number

A **Carmicheal number** is a composite number  $n$  where

$$b^{n-1} \equiv 1 \pmod{n},$$

for every  $b$  which are relatively primve to  $n$ .

Carmicheal numbers are rare. The smallest is  $561 = 3 \cdot 11 \cdot 17$ . The next ones are 1105, 1729, and 2465. There are 20,138,200 Carmicheal numbers between 1 and  $10^{21}$ .

So, if we ignore Carmicheal numbers, the Fermat test is very good. There are other probabilistic tests (e.g, Miller-Rabin test) that uses other properties that works for all numbers and there are deterministic algorithms for testing primes.

## Lemma 3

*If  $n$  is not a Carmicheal number, the Fermat test returns that  $n$  is a composite with probability at least  $1/2$ .*

Note that if you repeat the test for  $k$  times, the probability that it gives the wrong answer is at most  $1/2^k$ .

# Running time

```
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```

## Special case of Euler's theorem

### Theorem 4 (Euler's theorem)

*If  $p$  and  $q$  are different primes, for  $a$  such that  $\gcd(a, pq) = 1$ , we have*

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

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*Is this useful?* Yes! In the RSA algorithm.

# RSA

- ▶ Private key:  $(e, n)$ , Public key:  $(d, n)$
- ▶ Encryption  $E(m) = m^e \bmod n$ , Decryption:  $D(w) = w^d \bmod n$ .
- ▶ Goal: Select  $e, d, n$  such that  $D(E(m)) = m^{ed} \bmod n = m$ .

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- ▶ Pick two primes  $p$  and  $q$ . Let  $n = pq$ .
- ▶ Pick  $e$  (usually a small number)
- ▶ Pick  $d$  such that  $d = e^{-1} \pmod{(p-1)(q-1)}$ , i.e.,  $ed \equiv 1 \pmod{(p-1)(q-1)}$ , or  $ed = k \cdot (p-1)(q-1) + 1$ , for some integer  $k$ .
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What is the requirement for  $m$ ?  $\gcd(m, n) = 1$ , otherwise you can use the message to factor  $n$ .