

# 01204211 Discrete Mathematics

## Lecture 11b: Four fundamental subspaces (preview)

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# What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[ \begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array} \right]$$

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$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row echelon form

## Linearly independent rows

## Vector spaces related to a matrix

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# Subspaces

## Definition

Let  $\mathcal{V}$  and  $\mathcal{W}$  be vector spaces such that  $\mathcal{V} \subseteq \mathcal{W}$ . We say that  $\mathcal{V}$  is a **subspace** of  $\mathcal{W}$ .

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## Examples:

- ▶  $\text{Span} \{[1, 1]\}$  is a subspace of  $\mathbb{R}^2$ .
- ▶  $\text{Span} \{[1, 0, 0], [0, 1, 1]\}$  is a subspace of  $\mathbb{R}^3$ .
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The set of solutions  $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$  form a vector space.

## Example 1 (cont.)

Given a matrix  $A$ , we can look at the matrix-vector product  $A\mathbf{x}$ .

Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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- ▶ The column space of  $A$  (denoted by  $\mathcal{C}(A)$  )
- ▶ The row space of  $A$  (denoted by  $\mathcal{C}(A^T)$  )
- ▶ The nullspace of  $A$

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$$

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