01204211 Discrete Mathematics
Lecture 11b: Context-free languages and grammars (2)[1]

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[1] Based on lecture notes of *Models of Computation* course by Jeff Erickson.

Review: Definition

A **context-free grammer** consists of the following components:

- a finite set Σ, a set of *symbols* (or *terminals*),
- a finite set Γ disjoint from Σ, a set of *non-terminals* (you can think of them as variables),
- a finite set R of *production rules* of the form $A \to w$ where $A \in \Gamma$ and $w \in (\Sigma \cup \Gamma)^*$ is a string of symbols and variable, and
- a *starting* non-terminal (usually the non-terminal of the first production rule).

$\boxed{S} \to \_\ \_\ \_$

Review: Applying the rules

If you have strings $x, y, z \in (\Sigma \cup \Gamma)^*$ and the production rule

$$A \to y,$$

You can apply the rule to the string xAz. This yields the string

$$xyz.$$

We use the notation

$$xAz \rightsquigarrow xyz$$

to describe this application.

Review: Derivation

We say that z derives from x if we can obtain z from x by production rule applications, denoted by $x \rightsquigarrow^* z$.
Formally, for any string $x, z \in (\Sigma \cup \Gamma)^*$, we say that $x \rightsquigarrow^* z$ if either

- $x = z$, or
- $x \rightsquigarrow y$ and $y \rightsquigarrow^* z$ for some string $y \in (\Sigma \cup \Gamma)^*$.

Review: $L(w)$

G $\quad \boxed{S} \to$ $\quad L(G) = L(S)$

The *language* $L(w)$ of string $w \in (\Sigma \cup \Gamma)^*$ is the set of all strings in Σ^* that derive from w, i.e.,

$$L(w) = \{x \in \Sigma^* \mid w \rightsquigarrow^* x\}.$$

The language **generated by** a context-free grammar G, denoted by $L(G)$ is the language of its starting non-terminal.

A language L is **context-free** if there exists some context-free grammar G such that $L(G) = L$.

Review: Parse tree

- `00011`

$$
\begin{aligned}
S &\rightarrow A \mid B \\
A &\rightarrow 0A \mid 0C \\
B &\rightarrow B1 \mid C1 \\
C &\rightarrow \varepsilon \mid 0C1
\end{aligned}
$$

Ambiguity

- $1 + 1 + 1 + 1 + 1$

$$S \quad \rightarrow \quad 1 \mid S + S \mid S * S$$

- A string w is **ambiguous** with respect to a grammar G if more than one parse tree for w exists.
- A grammar G is **ambiguous** if some string is ambiguous with respect to G.

More example

Palindrome in $\{0, 1\}^*$

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Palindrome in $\{0, 1\}^*$

$$S \rightarrow 0S0 \mid 1S1 \mid 1 \mid 0 \mid \varepsilon$$

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

To show that

$$L(S) = \{0^n 1^n \mid n \geq 0\},$$

we have to prove

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$$S \longrightarrow 0S1 \mid \varepsilon$$

To show that

$$L(S) = \{0^n 1^n \mid n \geq 0\}, \quad L$$

we have to prove

1. ▶ $L(S) \supseteq \{0^n 1^n \mid n \geq 0\}$, and — ทุก string ใน L จะต้องสร้างได้จาก S ได้
2. ▶ $L(S) \subseteq \{0^n 1^n \mid n \geq 0\}$. ทุก string ที่ generated จาก S อยู่ใน L

$\boxed{S \leadsto^* 0^n 1^n \text{ สำหรับ } n \geq 0 \text{ ใดๆ}}$

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ for every non-negative integer n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ *for every non-negative integer* n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$. $0^n 1^n = \varepsilon$ [illegible] $S \rightsquigarrow^* \varepsilon$

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ for every non-negative integer n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$.
- Case 2: $n > 0$.

$S \rightsquigarrow^* 0^n 1^n$

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ for every non-negative integer n. $\{0^n 1^n \mid n \geq 0\} \subseteq L(S)$

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$.
- Case 2: $n > 0$. From I.H., we know that

$$S \rightsquigarrow^* 0^{n-1} 1^{n-1}.$$

We can apply rule $S \longrightarrow 0S1$ to obtain $0^n 1^n$, i.e.,

$$S \longrightarrow 0S1 \rightsquigarrow^* 0\boxed{0^{n-1} 1^{n-1}}1 = 0^n 1^n.$$

In both cases, we conclude that $S \rightsquigarrow^* 0^n 1^n$, as required.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n1^n \mid n \geq 0\}$

Proof. จาก lemma 1 ทราบว่า $L(S) \supseteq \{0^n1^n \mid n \geq 0\}$

เหลือต้องแสดงว่า $L(S) \subseteq \{0^n1^n \mid n \geq 0\}$ นั่นคือ แสดงว่า

ทุก string ที่ออกจาก Grammar จะอยู่ในรูปแบบ 0^n1^n

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(C)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(C)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(C)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.
Case 2: $w = 0x1$ for some $x \in L(S)$.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(C)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(C)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.
Case 2: $w = 0x1$ for some $x \in L(C)$. Since $|x| = |w| - 2 < |w|$, we can apply I.H., and get that $x = 0^k 1^k$;thus $w = 00^k 1^k 1$, i.e., $w = 0^n 1^n$ where $n = k + 1$, as required. □

Careful

$\#(a,w)$ = จำนวน a ใน w

$L = \{ w \in \{0,1\}^n \mid \#(0,w) = \#(1,w) \}$

- When using inductive proof, you have to ensure that each part of the string w is shorter than w.
- Consider this grammar

$$S \longrightarrow \varepsilon \mid \boxed{SS} \mid 0S1 \mid 1S0.$$

- When w is created by rule $S \to SS$, we know that $w = xy$ for $x, y \in L(S)$.
- Do we know that $|x| < |w|$ and $|y| < |w|$?

จะพิสูจน์ว่า $L(S) = L$

Careful

- When using inductive proof, you have to ensure that each part of the string w is shorter than w.
- Consider this grammar
$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$
- When w is created by rule $S \rightarrow SS$, we know that $w = xy$ for $x, y \in L(S)$.
- Do we know that $|x| < |w|$ and $|y| < |w|$?
- We can consider a **minimum-length derivation** in the proof to avoid this problem.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.

$\boxed{L(S) \subseteq L}$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$.

$w = 0x1$, ที่ $x \in L(S)$, $|x| < |w|$

จาก I.H. $\#(0,x) = \#(1,x)$

เพราะ $\#(0,w) = \#(0,x)+1$
$\#(1,w) = \#(1,x)+1$

ดัง $\#(0,w) = \#(1,w)$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow \underline{S}\,\underline{S}$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

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  From I.H.,

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
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  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\#(0, w) \quad = \quad \#(0, x) + \#(0, y)$$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
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  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\begin{aligned} \#(0, w) &= \#(0, x) + \#(0, y) \\ &= \#(1, x) + \#(1, y) \end{aligned}$$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof. $L(S) \subseteq \{ w \mid \#(0,w) = \#(1,w) \}$

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

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  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\begin{aligned} \#(0, w) &= \#(0, x) + \#(0, y) \\ &= \#(1, x) + \#(1, y) = \#(1, w) \end{aligned}$$

In all cases, we conclude that $\#(0, w) = \#(1, w)$.

Examples: Not palindromes

Strings in $(0+1)^*$ that are not palindromes.

$$\begin{aligned} S &\longrightarrow 0S0 \mid 1S1 \mid 0Z1 \mid 1Z0 \\ Z &\longrightarrow \varepsilon \mid 0Z \mid 1Z \end{aligned}$$

Why does this work?

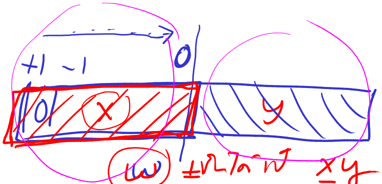
Strings with the same number of 0s and 1s

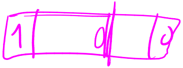
$L(S) \supseteq \{ w \mid \#(0,w) = \#(1,w) \}$

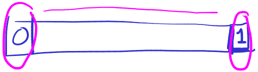
$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

We already show that every string in $L(S)$ contains the same number of 0s and 1s. Why does it contain all possible required strings?

▶ พิจารณา string w ใดๆ ที่ $\#(0,w) = \#(1,w)$



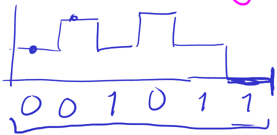




w เขียนได้ xy

ที่ $\#(0,x) = \#(1,x)$

$x \neq \varepsilon$



Strings in which the number of 0s is greater than or equal to the number of 1s

We can start with the previous grammar

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

And try to add more rules.

Strings in which the number of 0s is greater than or equal to the number of 1s

We can start with the previous grammar

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

And try to add more rules.

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0 \mid \underline{0S} \mid \underline{S0}.$$

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid 0E1 \mid 1E0.$$

There are two cases.

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid \mathtt{0}E\mathtt{1} \mid \mathtt{1}E\mathtt{0}.$$

There are two cases.

$$S \longrightarrow O \mid I$$

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid \mathtt{0}E\mathtt{1} \mid \mathtt{1}E\mathtt{0}.$$

There are two cases.

$$S \longrightarrow O \mid I$$

$$O \longrightarrow E\mathtt{0}O \mid E\mathtt{0}E$$

How about I?

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid 0E1 \mid 1E0.$$

There are two cases.

$$S \longrightarrow O \mid I$$

$$O \longrightarrow E0O \mid E0E$$

How about I?

$$I \longrightarrow E1I \mid E1E$$

Balanced parentheses

$$S \longrightarrow (S) \mid SS \mid \varepsilon$$

Balanced parentheses

$$S \longrightarrow (S) \mid SS \mid \varepsilon$$

$$S \longrightarrow (S)S \mid \varepsilon$$

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?
From inspection, we may guess that $L(S) = (\mathtt{01})^*$. But how can we prove that?

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?
From inspection, we may guess that $L(S) = (\mathtt{01})^*$. But how can we prove that?
To prove $L(S) = (\mathtt{01})^*$, we must also prove $L(A) = (\mathtt{10})^*$ *at the same time*.