01204211 Discrete Mathematics
Lecture 4b: Mathematical Induction 2

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Review: Mathematical Induction

$P(1) \downarrow P(2) \downarrow P(3) \downarrow \vdots \downarrow P(x) \vdots \downarrow$

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:

Base case: $\boxed{P(1)}$

Inductive step: For any $k \geq 1$, $\underbrace{P(k) \Rightarrow P(k+1)}_{\text{induction}}$

The **Principle of Mathematical Induction** states that "$P(n)$ is true for every natural number n."

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

Example 1

Base case จะแสดง $P(1)$

$1^2 = \frac{1}{6}(1+1)(2+1)$

* $P(2)$: $1^2+2^2 = 5$ $\quad \frac{2}{6}(3)(5)$

Theorem: For every natural number n, $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$."

Inductive step Assume ว่า $P(k)$ พิสูจน์ $P(k+1)$ สำหรับ $k \geq 1$

Induction hypothesis: $P(k)$: "$\sum_{i=1}^{k} i^2 = \frac{k}{6}(k+1)(2k+1)$"

จะพิสูจน์ $P(k+1)$: "$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)}{6}(k+2)(2k+3)$"

พิจารณา $\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 =$ จาก I.H. $\frac{k}{6}(k+1)(2k+1) + (k+1)^2$

$= \left(\frac{k+1}{6}\right)\left(k(2k+1) + 6(k+1)\right) = \left(\frac{k+1}{6}\right)\left(2k^2+k+6k+6\right)$

$= \left(\frac{k+1}{6}\right)(k+2)(2k+3)$ ตามต้องการ

จาก Principle of M.I. จะได้ว่า $P(n)$ สำหรับ $n \geq 1$ ■

Example 1

Theorem: For every natural number n,
$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$

Proof: We prove by induction. The property that we want to prove $P(n)$ is "$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$."

Base case: We can plug in $n = 1$ to check that $P(1)$ is true:
$1^2 = \frac{1}{6}(1+1)(2 \cdot 1 + 1)$.

Inductive step: We assume that $P(k)$ is true for $k \geq 1$ and show that $P(k+1)$ is true.

We first assume the Induction Hypothesis $P(k)$:
$\sum_{i=1}^{k} i^2 = \frac{k}{6}(k+1)(2k+1)$

(continue on the next page)

Example 1 (cont.)

Let's show $P(k+1)$. We write $\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2$.

Using the Induction Hypothesis, we know that this is equal to

$$\begin{aligned}
(k/6)(k+1)(2k+1) + (k+1)^2 &= \frac{(k+1)}{6}(k(2k+1) + 6(k+1)) \\
&\quad \text{(In this step, we factor out } (k+1)/6\text{)} \\
&= \frac{(k+1)}{6}(2k^2 + 7k + 6) \\
&= \frac{(k+1)}{6}((k+1)+1)(2(k+1)+1).
\end{aligned}$$

This implies $P(k+1)$ as required.
From the Principle of Mathematical Induction, this implies that $P(n)$ is true for every natural number n. ■

Not an example (1)

$\forall n \geq 1,$ $P(n) =$ "set ของวัวใดๆ ที่มีสมาชิก n ตัว วัวทุกตัวจะมีสีเดียวกัน"

Theorem 1

For any set of cows, all cows have the same color.

Proof.

We prove by induction on the size n of the set of cows.

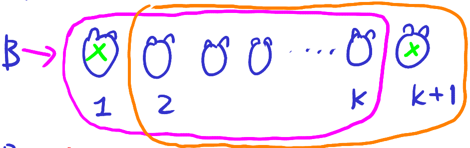
Base case: พิจารณา $n=1$ ✓

Inductive step: จะพิสูจน์ว่า $k \geq 1$, $P(k) \Rightarrow P(k+1)$ ✓

พิจารณาเซต A ของวัวที่มี $k+1$ ตัว

ให้ $B = A - \{$วัวตัวที่ $k+1\}$

ให้ $C = A - \{$วัวตัวที่ $1\}$



เนื่องจาก $|B| = k$, จาก $P(k)$ วัวทุกตัวในเซต B มีสีเดียวกัน

เนื่องจาก $|C| = k$, วัวทุกตัวใน C มีสีเดียวกัน

วัวตัวที่ 1 สีเดียวกับวัวตัวที่ 2 เพราะอยู่ใน B เหมือนกัน

วัวตัวที่ $k+1$ สีเดียวกับวัวตัวที่ 2 (อยู่ใน C) $\Rightarrow$ วัว 1 กับวัว $k+1$ สีเดียวกัน

Not an example (1)

Theorem 1

For any set of cows, all cows have the same color.

Proof.

We prove by induction on the size n of the set of cows.
Base case: For $n = 1$, clearly for any set of a single cow, every cow in the set has the same color.

Not an example (1)

Theorem 1

For any set of cows, all cows have the same color.

Proof.

We prove by induction on the size n of the set of cows.

Base case: For $n = 1$, clearly for any set of a single cow, every cow in the set has the same color.

Inductive step: Suppose that for every set of size k of cows, all cows in the set have the same color.

We will show that every set of size $k + 1$ of cows, all cows in this set have the same color.

Not an example (2)

Inductive step (cont.): Consider set A of $k+1$ cows.

Not an example (2)

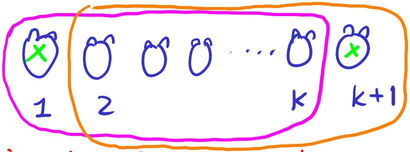
Inductive step (cont.): Consider set A of $k+1$ cows.

พิจารณาเซต A ของวัวที่มี k+1 ตัว ให้ $B = A - \{$วัวตัวที่ $k+1\}$

ให้ $C = A - \{$วัวตัวที่ $1\}$

เนื่องจาก $|B| = k$, จาก $P(k)$ วัวทุกตัวในเซต B มีสีเดียวกัน

$B \rightarrow$



เนื่องจาก $|C| = k$, วัวทุกตัวใน C มีสีเดียวกัน

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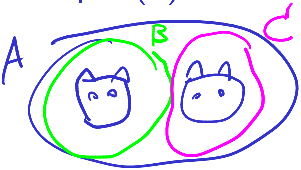
วัวตัวที่ k+1 สีเดียวกับวัวตัวที่ 2 (อยู่ใน C) $\Rightarrow$ วัว 1 กับวัว k+1 สีเดียวกัน

Because we have established that the base case and the inductive step is true, we can conclude that for any set of cows, all cows have the same color. ■

Not an example (3)



$P(1) \not\Rightarrow P(2) \Rightarrow P(3) \Rightarrow \dots$

$\underline{P(k) \Rightarrow P(k+1)}$

Clearly the following theorem cannot be true.

พิสูจน์เมื่อ $k=1$

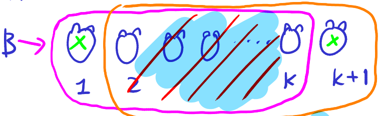
> **Theorem 2**
>
> *For any set of cows, all cows have the same color.*

What is wrong with its proof based on mathematical induction?

พิจารณาเซต A ของวัวที่มี ~~$k+1$~~ ตัว ($k=1$, 2)

ให้ $B = A - \{$วัวตัวที่ $k+1\}$

ให้ $C = A - \{$วัวตัวที่ $1\}$



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Unused facts

- Let's informally think about how proving $P(1)$ and $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$ implies that $P(n)$ is true for all natural number n.

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \cdots\cdots P(k)$$

I.S. I.S.

$$P(k) \Rightarrow P(k+1)$$

Unused facts

- Let's informally think about how proving $P(1)$ and $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$ implies that $P(n)$ is true for all natural number n.

- One may notice that when we prove a statement $P(n)$ for all natural number n by induction, during the inductive step where we want to show $\boxed{P(k+1)}$ from $\boxed{P(k)}$, we usually have that $P(1), P(2), \ldots, P(k)$ is true at hands as well.

Unused facts

- Let's informally think about how proving $P(1)$ and $P(k) \Rightarrow P(k+1)$ for all $k \geq 1$ implies that $P(n)$ is true for all natural number n.

- One may notice that when we prove a statement $P(n)$ for all natural number n by induction, during the inductive step where we want to show $P(k+1)$ from $P(k)$, we usually have that $P(1), P(2), \ldots, P(k)$ is true at hands as well.
- Then why don't we use them as well?

Strong Mathematical Induction

Strong Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts:

Strong Mathematical Induction

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Suppose that we can prove the following two facts: **Base case:** $P(1)$

Strong Mathematical Induction

$R \Rightarrow S$

Strong Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts: **Base case:** $P(1)$
Inductive step: For any $k \geq 1$,

$$\underbrace{P(1) \wedge P(2) \wedge \cdots \wedge P(k)} \Rightarrow P(k+1).$$

Strong Mathematical Induction

Strong Induction

Suppose that you want to prove that property $P(n)$ is true for every natural number n.

Suppose that we can prove the following two facts: **Base case:** $P(1)$
Inductive step: For any $k \geq 1$,

$$P(1) \wedge P(2) \wedge \cdots \wedge P(k) \Rightarrow P(k+1).$$

Then $P(n)$ is true for every natural number n.

Example 2

$P(n) \equiv$ "สามารถใช้เหรียญ 2 บาท & 3 บาท รวมเป็น n บาท"

Theorem: For any integer $n \geq 4$, one can use only 2-baht coins and 3-baht coins to obtain exactly n baht.

Proof: We prove by strong induction on n.

Base case: ~~P(3)~~ $P(4)$ ✓ ② ②

$P(5)$? ✓ ② ③

Induction Step: Assume ~~$P(k)$~~ $P(4), P(5), \ldots, P(k-1), P(k)$, จะแสดง $P(k+1)$, $k \geq \cancel{4}\ 5$

~~P(3)~~
$P(4)$
$P(5)$
$P(k-1)$
$P(k)$
$P(k+1)$

$k+1$ บาท

รวม $(k-1)$ บาท ได้จาก เหรียญ ② ③ + ②

เนื่องจากสามารถรวมเหรียญ 2 บาท 3 บาท ให้เป็น $k-1$ บาท ได้ เพิ่มเหรียญ 2 บาท เข้าไป จะรวมได้ $k+1$ บาท ตามต้องการ

จาก P.M.I, จะได้ว่า ประโยคที่ต้องการพิสูจน์ จริง.

Example 2

Theorem: For any integer $n \geq 4$, one can use only 2-baht coins and 3-baht coins to obtain exactly n baht.

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Inductive step: Assume that for $k \geq 5$, we can obtain exactly ℓ baht, for $4 \leq \ell \leq k$, using only 2-baht and 3-baht coins. We will show how to obtain a set of $k + 1$ baht.

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Since $k \geq 5$, we have that $k - 1 \geq 4$. Therefore from the Induction Hypothesis, we can use only 2-baht coins and 3-baht coins to form a set of coins of total value $k - 1$ baht.

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From the Principle of Strong Mathematical Induction, we conclude that the theorem is true. ■

Is strong induction more powerful?

- Can we prove the previous theorem without using the strong induction?

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- Can we prove the previous theorem without using the strong induction? Yes, you can (homework).
- In fact, if you can prove that $P(n)$ is true for all natural number n with strong induction, you can always prove it with mathematical induction.
- Hint: Let $\underbrace{Q(n)} = \underbrace{P(1) \land P(2) \land \cdots \land P(n)}$.