

01204211 Discrete Mathematics

Lecture 10a: Polynomials (2)¹

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¹This section is from Berkeley CS70 lecture notes. ◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ≡ ↺ 🔍 ↻

Fun fact: Check digit for Thai National ID

Review: Polynomials

A **single-variable polynomial** is a function $p(x)$ of the form

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

We call a_i 's *coefficients*. Usually, variable x and coefficients a_i 's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Review: Basic facts

Definition

a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

Properties

Property 1: A non-zero polynomial of degree d has at most d roots.

Property 2: Given $d + 1$ pairs $(x_1, y_1), \dots, (x_{d+1}, y_{d+1})$ with distinct x_i 's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

Polynomial division

Polynomial division

If you have a polynomial $p(x)$ of degree d , you can divide it with a polynomial $q(x)$ of degree $\leq d$. You have that there exists a pair of polynomial $q'(x)$ and $r(x)$ such that

$$p(x) = q'(x)q(x) + r(x),$$

and $r(x)$ is of degree **less** than $q(x)$'s degree.

Lemma 1

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Dividing $p(x)$ with $(x - a)$, we get that

$$p(x) = q'(x)(x - a) + r(x),$$

where $r(x)$ is of degree at most $1 - 1 = 0$, i.e., $r(x)$ must be a constant; thus, we assume that $r(x) = c$. Let's evaluate $p(a)$; note that $p(a) = c$, since

$$p(a) = q'(a)(a - a) + c = 0 + c = c.$$

However we know that a is a root of $p(x)$, i.e., $p(a) = 0$. Therefore $c = 0$, or $r(x) = 0$. Thus, the lemma follows. □

Lemma 2

If $p(x)$ is a polynomial of degree d with d distinct roots a_1, a_2, \dots, a_d , $p(x)$ can be written as $c(x - a_1)(x - a_2) \cdots (x - a_d)$.

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Property 1

Polynomials over a finite field $GF(p)$

Examples - evaluation

Suppose that we work over $GF(m)$ where $m = 11$. Let $p(x) = 4 \cdot x^2 + 5 \cdot x + 3$. We have

x	$p(x)$	$p(x) \bmod m$
0	3	3
1	12	1
2	29	7
3	54	10
4	87	10
5	128	7
6	177	1
7	234	3
8	299	2
9	372	9
10	453	2
11	542	3

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2, 7)$, $(4, 10)$, and $(7, 3)$. Find $p(x)$.

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$$\blacktriangleright \Delta_3(x) = \frac{(x-2)(x-4)}{(7-2)(7-4)} = \frac{x^2-6x+8}{5 \cdot 3} = \frac{x^2+5x+8}{4} = 3x^2 + 4x + 2$$

Thus,

$$\begin{aligned} p(x) &= 7\Delta_1(x) + 10\Delta_2(x) + 3\Delta_3(x) \\ &= (70x^2 + 35) + (90x^2 + 70x + 50) + (9x^2 + 12x + 6) \\ &= 4x^2 + 5x + 3 \end{aligned}$$

Practice

Let's work modulo 11.

Use the last 3 digits of your student ID. Suppose that they are a_2, a_1, a_0 . Find a polynomial $p(x)$ of degree 2 such that $p(i) = a_i$ for $i = 0, 1, 2$.

How many?

Two ways of specifying a polynomial $p(x)$ of degree d :

- Specify its coefficients a_0, a_1, \dots, a_d , i.e., the polynomial is

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- Specify $d + 1$ points, i.e., $(x_1, y_1), (x_2, y_2), \dots, (x_{d+1}, y_{d+1})$, where all x_i are distinct. There is a *unique* polynomial $p(x)$ of degree at most d that passes through these points (from Property 2).

For polynomials of degree at most d over $GF(m)$, if you specify q points, there are:

q	numbers of polynomials
$d + 1$	1
d	m
$d - 1$	m^2
$d - 2$	m^3
\vdots	\vdots
1	m^d
0	m^{d+1}

Secret sharing scheme - settings

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- ▶ There are n people, a secret s , and an integer k .
- ▶ We want to “distribute” the secret in such a way that any set of $k - 1$ people cannot know anything about s , but any set of k people can reconstruct s .

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- ▶ Pick m to be larger than n and s . (Much larger than s , i.e., $m \gg s$.)
- ▶ Pick a random polynomial of degree $k - 1$ such that $P(0) = s$.
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- ▶ Correctness: for any set of $k - 1$ people, how many possible candidate secrets compatible with the information these people have?

A more complex secret sharing scheme

Suppose that a company has 3 VPs and 5 senior members. You want to distribute a secret such that (1) any 2 VPs can obtain the secret or (2) a single VP with 3 senior members can also obtain the secret. How can you do that?

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Lossy internet:

Erasure codes

Suppose that we want to send a message m_1, m_2, \dots, m_n where $m_i \leq p - 1$ for some prime p .

However, we know that our communication channel is lossy, i.e., some messages can be *dropped*. How can we send this message?