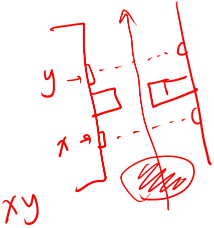
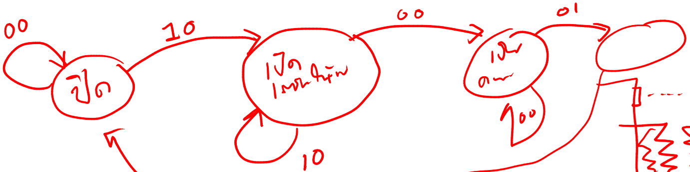
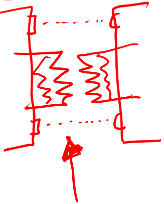
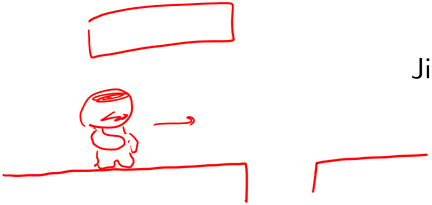
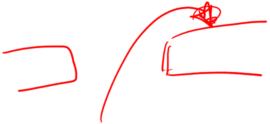






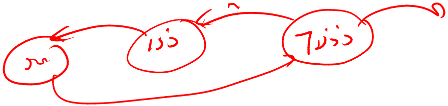
01204211 Discrete Mathematics
Lecture 9a: Finite automata[1]

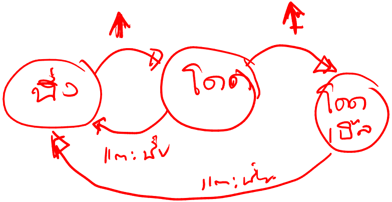




Jittat Fakcharoenphol

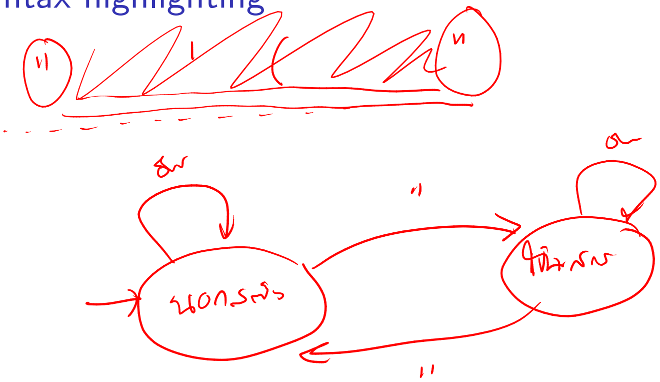
August 26, 2025





[1] Based on lecture notes of *Models of Computation* course by Jeff Erickson.

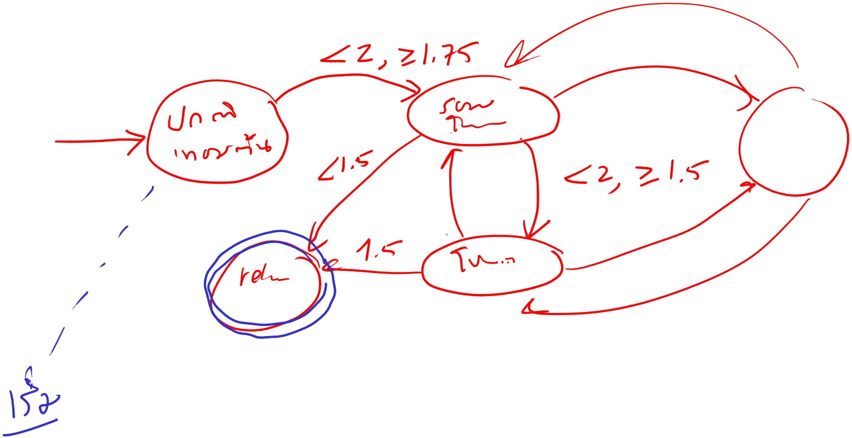
Example: syntax highlighting



HTML tokenizer

Game programming

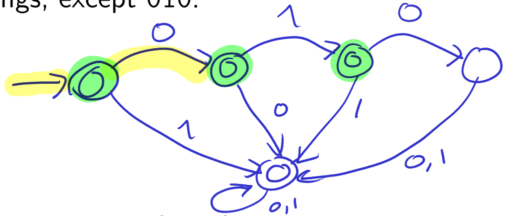
State-transition graphs

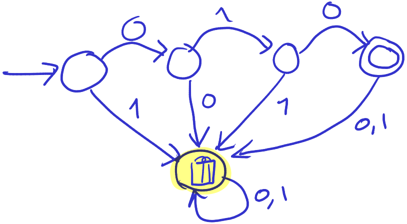


More examples over $\Sigma = \{0, 1\}$

String = 010

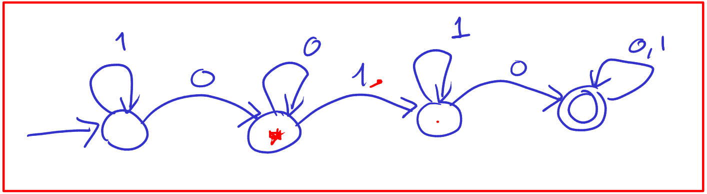
All strings, except 010.

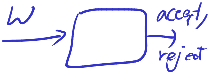




Strings containing the subsequence 010.

001011





Formal definitions

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

Formal definitions

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Formal definitions

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

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- a finite set of states Q,

Formal definitions

def transition (current_state,
input_symbol):

=

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

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- a finite set of states Q,
- a transition function δ : $\underbrace{Q \times \Sigma}_{\text{domain}} \rightarrow \underbrace{Q}_{\text{range}}$

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Formal definitions

$\forall$ A, B $\quad A \times B = \{(a,b) \mid a \in A, b \in B\}$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

- the input alphabet Σ,
- a finite set of states Q,
- a transition function $\delta : \underline{Q \times \Sigma} \longrightarrow Q$
- a start state $s \in Q$, and
- a subset $A \subseteq Q$ of accepting states.

Example 1

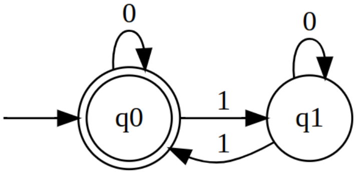
q_0

$w = $ "0100010111"

$\Sigma = \{0, 1\}$

$Q = \{q_0, q_1\}$

$\delta : Q \times \Sigma \to Q$



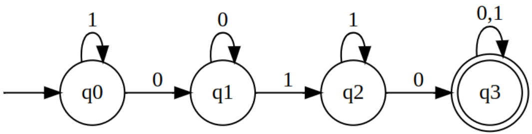
δ	0	1
q_0	q_0	q_1
q_1	q_1	q_0

DFA $M = (\Sigma, Q, \delta, q_0, \{q_0\})$

start state $s = q_0$

accepting states $A = \{q_0\}$

Example 2



$\Sigma = \{0,1\}$

$Q = \{q_0, q_1, q_2, q_3\}$

δ

	0	1
q_0	q_1	q_0
q_1	q_1	q_2
q_2		
q_3		

$s = q_0$

$A = \{q_3\}$

Moves

$(\Sigma, Q, \delta, s, A)$

One step move: from state q with input symbol a, the machine changes its state to

Moves

$q \xrightarrow{a} q'$ $\quad q' = \delta(q, a)$

One step move: from state q with input symbol a, the machine changes its state to $\boxed{\delta(q, a)}$.

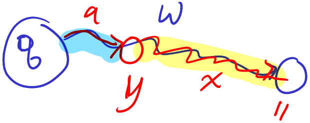
Extension: from state q with input string $\boxed{w}$, the machine changes its state to $\boxed{\delta^*(q, w)}$ defined as

Case 1: $w = \varepsilon$

$$\delta^*(q, w) = q$$

Case 2: $w = a \cdot x$ where x is the string

$$\delta^*(q, w) = \delta^*(\underbrace{\delta(q, a)}, x)$$



$\| \; \delta^*(y, x)$

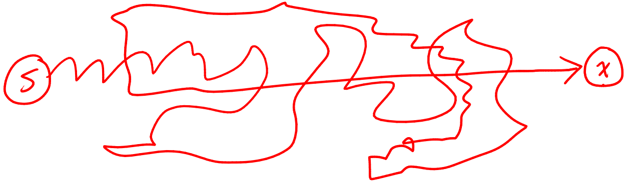
Moves

One step move: from state q with input symbol a, the machine changes its state to $\delta(q, a)$.
Extension: from state q with input string w, the machine changes its state to $\delta^*(q, w)$ defined as

$$\delta^*(q, w) = \begin{cases} q & \text{if } w = \varepsilon, \\ \delta^*(\delta(q, a), x) & \text{if } w = ax. \end{cases}$$

The signature of δ^* is $Q \times \Sigma^* \longrightarrow Q$.

Acceptance



For a finite-state machine with starting state s and accepting states A, it accepts string w iff

Acceptance

For a finite-state machine with starting state s and accepting states A, it accepts string w iff

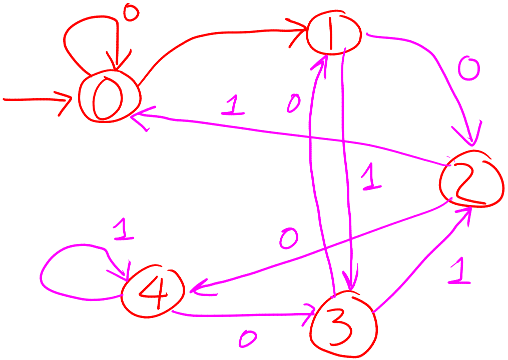
$$\delta^*(s, w) \in A.$$

Multiple of 5

$\Sigma = \{0, 1\}$

10

	1010	
	000	✓
5	101	✓
15	1111	✓
	1110	X
20	10100	✓
3	11	X



Multiple of 5

```
def multiple_of_5(w):
    r = 0
    for i in w:
        r = (2*r + w) % 5
    return r == 0
```

Multiple of 5

```
def multiple_of_5(w):
    r = 0
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```
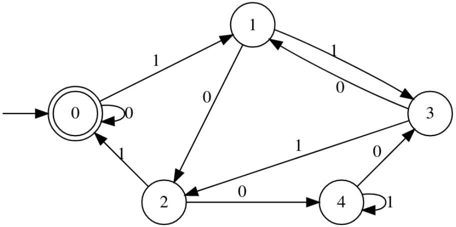
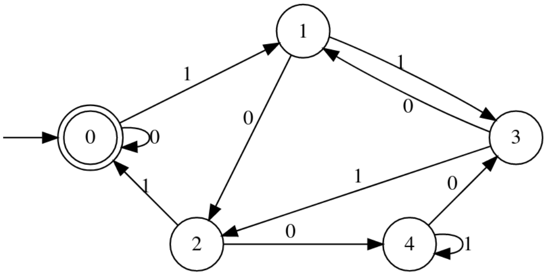
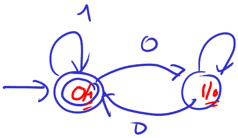
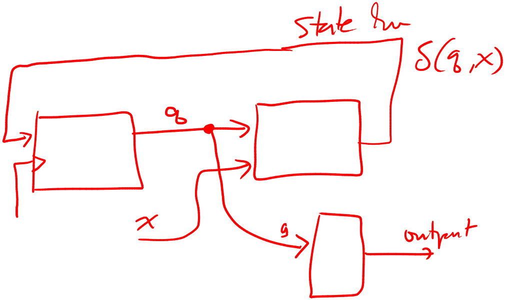




0
1
2
3
4
1
0
1
0
1
0
1
0
0
1

Digital design: Implementation





Digital design: Moore and Mealy machines

In the digital design class, you will encounter finite-state machines as well. The version we consider in this class is refered to as a **Moore machine**.

In practices, there is another variant of FSM called **Mealy machines**, whose outputs depend on input symbols as well.

Digital design: Moore and Mealy machines

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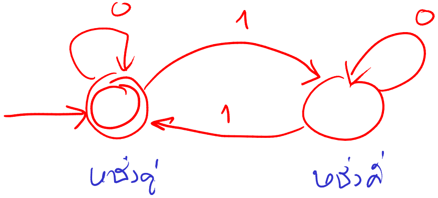
In practices, there is another variant of FSM called **Mealy machines**, whose outputs depend on input symbols as well.
Formally, they differ in output function.

- Moore machine: $G : Q \longrightarrow [0, 1]$
- Mealy machine: $G : Q \times \Sigma \longrightarrow [0, 1]$

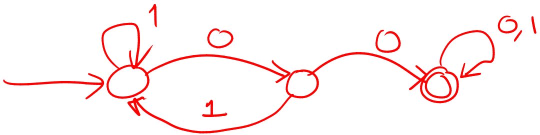
Example: even number of 1’s $\longrightarrow$ language L_1



111111

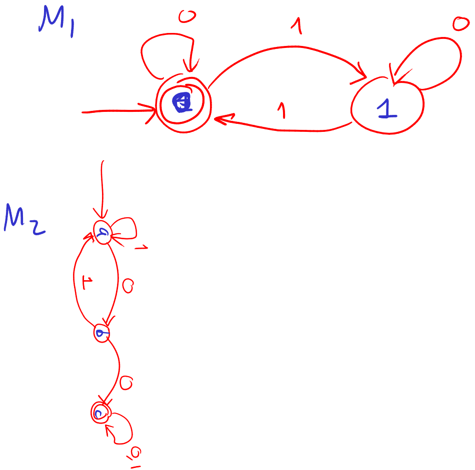
10011

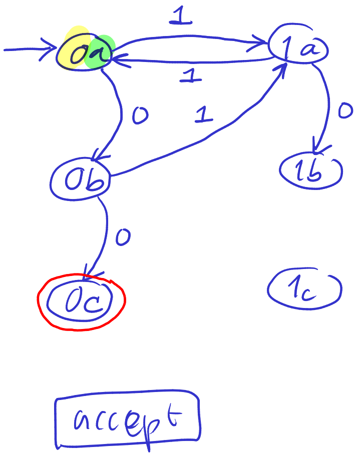
Example: strings containing 00 as a substring — language L_2



Example: strings containing 00 as a substring

$Q_1 \times Q_2$





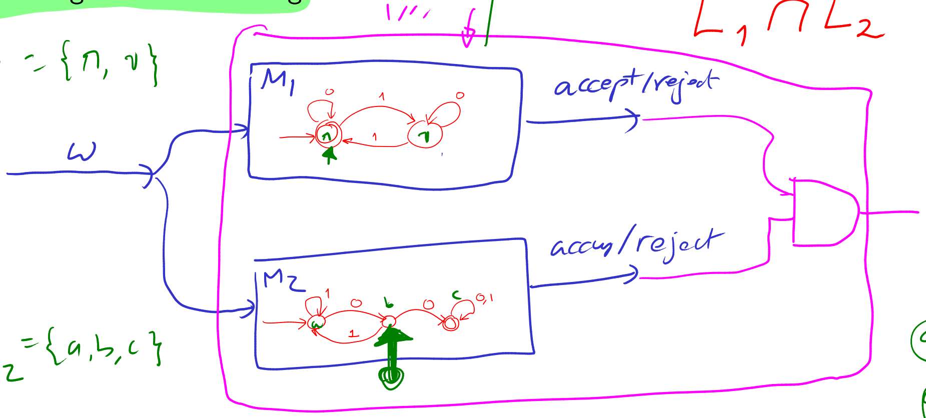
Combining DFAs

$Q_1 \times Q_2$

What if we want to build a DFA that accepts strings with an even number of 1's and containing 00 as a substring?

$L_1 \cap L_2$

$Q_1 = \{n, v\}$



$Q_2 = \{a, b, c\}$

$q_1 \in Q_1$

$q_2 \in Q_2$

Product construction

ให้ DFA $M_1 = (\Sigma, Q_1, \delta_1, s_1, A_1)$

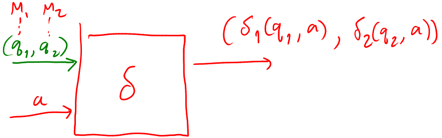
DFA $M_2 = (\Sigma, Q_2, \delta_2, s_2, A_2)$

สร้าง DFA $M = (\Sigma, Q_1 \times Q_2, \delta, s, A)$ โดยที่

1) $\boxed{\delta : (Q_1 \times Q_2) \times \Sigma \to (Q_1 \times Q_2)}$

$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$$

สำหรับทุก $q_1 \in Q_1$, $q_2 \in Q_2$, $a \in \Sigma$



2) $s = (s_1, s_2)$

3) $A = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\} = A_1 \times A_2$

Product construction (formally)

Given a DFA $M_1 = (\Sigma, Q_1, \delta_1, s_1, A_1)$ that accepts strings from language L_1 and $M_2 = (\Sigma, Q_2, \delta_2, s_2, A_2)$ that accepts strings from language L_2, we can construct a DFA $M = (\Sigma, Q, \delta, s, A)$ that accepts strings from $L_1 \cap L_2$ as follows:

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- Let $s = (s_1, s_2)$, and
- Let $A = A_1 \times A_2$.

- $L_1 \cup L_2$

$A = \{(q_1, q_2) \mid q_1 \in A_1 \text{ or } q_2 \in A_2\}$

- $L_1 - L_2$

$A = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \notin A_2\}$

Recall the definition of $\delta^*(q, w)$, i.e.,

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Lemma 1

$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$ *for any string* w.

Proof.

We prove by induction. I.H.: Assume that $\delta^*((q_1, q_2), x) = (\delta_1^*(q_1, x), \delta_2^*(q_2, x))$, for any string x such that $|x| < |w|$.

Case 1: $w = \varepsilon$: $\delta^*((q_1, q_2), \varepsilon) = (q_1, q_2) = (\delta_1^*(q_1, \varepsilon), \delta_2^*(q_2, \varepsilon))$ ✓

Case 2: $w = a \cdot x$,

$\delta^*((q_1, q_2), ax) = \delta^*(\delta((q_1, q_2)a), x)$ [จากนิยาม δ^*]

$= \delta^*((\delta_1(q_1, a), \delta_2(q_2, a)), x)$ [นิยาม δ]

$= (\delta_1^*(\delta_1(q_1, a), x), \delta_2^*(\delta_2(q_2, a), x)) =$ [I.H]

□

Correctness

From the previous lemma, we have that

$$\delta^*(s, w) \quad = \quad \delta^*((s_1, s_2), w)$$

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Correctness

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Thus, for an input w, M would reach the state $(\delta_1^*(s_1,w),\delta_2^*(s_2,w))$; it accepts w when

$$(\delta_1^*(s_1,w),\delta_2^*(s_2,w)) \in A_1 \times A_2.$$

This implies that M accepts w when $\delta_1^*(s_1,w) \in A_1$ and $\delta_2^*(s_2,w) \in A_2$, i.e., M accepts w iff M_1 and M_2 accept w.
Finally, we conclude that M accepts strings from language $L_1 \cap L_2$.

Language of a DFA

$L(M)$

For a DFA M, let $L(M)$ be the set of all strings that M accepts. More formally, for $M = (\Sigma, Q, \delta, s, A)$,

$$L(M) = \{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

We refer to $L(M)$ as the language of M.

Closure

Lemma 2

If L_1 and L_2 are languages of some DFAs M_1 and M_2, we have that

- *there is a DFA M that accepts $L_1 \cap L_2$,*

Closure

(q_1', q_2')

$M_1 \quad M_2$

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- *there is a DFA M that accepts $\Sigma^* \setminus L_1$,*

Automatic languages[2]

Definition (for now)

A language L is **"automatic"** if there is a DFA M such that $L(M) = L$.

[2] Taken directly from Erickson's lecture notes

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are also automatic.

The set of automatic languages is closed under these boolean operations.

[2] Taken directly from Erickson's lecture notes