01204211 Discrete Mathematics
Lecture 7a: Languages and regular expressions[1]

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August 20, 2024

[1] Based on lecture notes of *Models of Computation* course by Jeff Erickson.

What is computation?

Models of computations

* ทำอะไรได้บ้าง

ทำอะไรไม่ได้

model 1 ⟷ equivalence ⟷ model 2

ง่าย/ ไม่เก่งมาก

เก่ง

Languages = specifications

$\boxed{\text{formal}}$ language

set ของ string

อยากให้ตอบว่า yes

accept :)

~~ภาษา (language)~~

ภาษาไทย
ภาษาอังกฤษ } natural language (NLP)

$\{x \mid x$ เป็นจำนวนเฉพาะ$\}$

$\{x \mid x$ เป็นจำนวนคู่$\}$

$\{x : x$ เป็น palindrome$\}$

$\{p : p$ เป็นโปรแกรมภาษา python ที่รับ input แล้วทำงานไม่สิ้นสุด$\}$

$\{\varepsilon\}$ $\{\} = \emptyset$

$\varepsilon \neq \emptyset$

Formal definition: strings

Intuitively, a string is a *finite* sequence of symbols. However, to be able to formally prove properties of strings we need a precise definition.

Let a finite set Σ be the **alphabet**. (E.g., for bit strings, $\Sigma = \{0, 1\}$; for digits, $\Sigma = \{0, 1, \ldots, 9\}$; for English string $\Sigma = \{a, b, \ldots, z\}$.)
The following is a recursive definition of strings.

Recursive definition of strings

A **string** w over alphabet Σ is either

- the empty string ε, or
- $a \cdot x$ where $a \in \Sigma$ and x is a string.

The set of all strings over alphabet Σ is denoted by Σ^*.

Review: more recursive definitions

Lengths

For a string w, let $|w|$ be the length of w defined as

$$|w| = \begin{cases} 0 & \text{when } w = \varepsilon \\ 1 + |x| & \text{when } w = a \cdot x \end{cases}$$

Concatenation

For strings w and z, the concatenation $w \bullet z$ is defiend recursively as

$$w \bullet z = \begin{cases} z & \text{when } w = \varepsilon \\ a \cdot (x \bullet z) & \text{when } w = a \cdot x \end{cases}$$

Review: proving facts about strings

Lemma 1 $\forall w, x$ def $|x|$

For strings w and x, $|w \bullet x| = |w| + |x|$.

Proof. จะพิสูจน์ by induction บนโครงสร้างของ $\boxed{w}$

- Induction hypothesis: Assume ว่า สำหรับทุก string y ที่ $|y| < |w|$, $|y \bullet x| = |y| + |x|$.

Case 1: $w = \varepsilon$: จากนิยามของ concatenation

$$w \bullet x = x \quad \text{เมื่อ } w = \varepsilon$$

ดังนั้น $|w \bullet x| = |x| = 0 + |x| = |w| + |x|$ ตามต้องการ ✓

เพราะจากนิยามของ $|\varepsilon| = 0$

Case 2: $w = a \cdot y$ สำหรับบาง string y

จากนิยามของ concat

$$w \bullet x = a \cdot (y \bullet x)$$

จาก induction hypothesis จะได้ว่า $|y \bullet x| = |y| + |x|$ ดังนั้น

$$\begin{aligned}
|w \bullet x| &= |a \cdot (y \bullet x)| \\
&= 1 + |y \bullet x| \quad \text{(จากนิยาม 1.1)} \\
&= (1 + |y|) + |x| \\
&= |w| + |x| \quad \text{(จากนิยาม 1.1)} \qquad \square
\end{aligned}$$

Formal languages

A **formal language** is a set of strings over some finite alphabet Σ.

Examples:

Careful...

empty language

These are different languages: $\emptyset, \{\varepsilon\}$

And ε is not a language.

language of empty string

How to describe languages? $\leftarrow$ "set" ของ string.

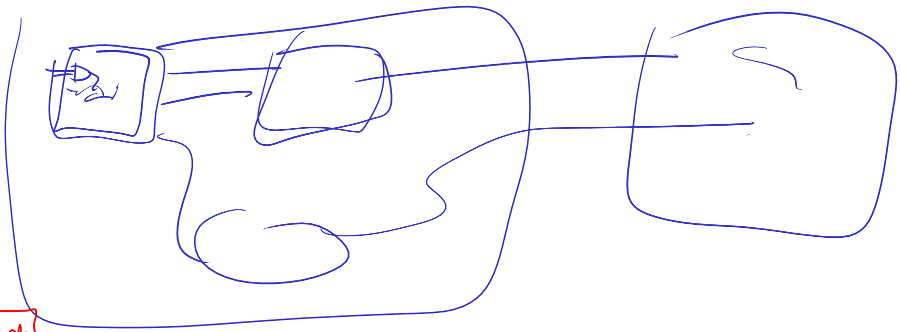
$\{1, \ldots\}$

$\{x : \_\_\_\_\}$

- ϕ
- set ที่มี string 1 string $\{\text{loveyou}\}$

⋮

Composition (concept) ← สร้างของที่ซับซ้อน จาก ของที่เบสิก



$\boxed{\text{set operat}}$

$A \cup B$, $A \cap B$, $A - B$

[illegible]

Combining languages

"hello" • "setha"

= "hello setha"

If A and B are languages over alphabet Σ.

- **Basic set operations:** $A \cup B$, $A \cap B$, $\bar{A} = \Sigma^* \setminus A$.
- **Concatenation:** $\underbrace{A}_{\text{set}} \cdot \underbrace{B}_{\text{set}}$.

$A = \{$"hello", "goodbye", $\varepsilon\}$

$B = \{$"setha", "pom", "ink"$\}$

$A \bullet B = \{ x \cdot y \mid x \in A, y \in B \}$

$\phi \bullet B = \phi$

$\{\varepsilon\} \bullet B = B$

- **Kleene closure or Kleene star:** A^*.

string $w \in A^*$ ถ้า

(1) $w = \varepsilon$

(2) $w = x \bullet y$ ถ้าเป็น ขาว $x \in A$ และ $y \in A^*$

Also $A^+ = A \cdot A^*$

Examples

string $w \in \boxed{A^*}$ ถ้า

(1) $w = \varepsilon \leftarrow$

(2) $w = x \cdot y$ สำหรับบาง $x \in A$ และ $\underbrace{y \in A^*}$

$\Sigma = \{0, 1\}$

$\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$

$A = \{a, ab, c\}$

$A^* = \{\underline{\varepsilon}, a \cdot \varepsilon, ab \cdot \varepsilon, c \cdot \varepsilon, aa, aaa, aabacab, \ldots\}$

Regular languages

Definition: regular languages

A language L is **regular** if and only if it satisfies one of the following conditions:

- L is empty;
- L contains one string (can be the empty string ε);
- L is a union of two regular languages;
- L is the concatenation of two regular languages; or
- L is the Kleene closure of a regular language.

*

Examples

$\{\}$, $\{ink\}$, $\{pm\}$

$\underbrace{\{ink\}}\cup\left[\underbrace{\underbrace{\{taksin\}}\cup\underbrace{\{prayut\}}}_{reg}\right]$

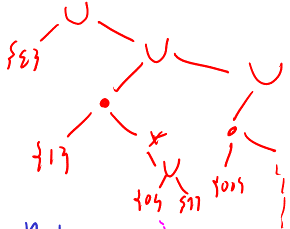
$\underbrace{\{ink\}}\bullet\underbrace{(\{pm\}\cup\underbrace{\{hello\}})}$

$\{0\}^* = \{\varepsilon, 0, 00, 000, \ldots\} = \{0^n \mid n \geq 0\}$

$\{0,1\}^* =$ bit string ทั้งหมด

bit string ทั้งหมด ที่ไม่มี 010 ← regular

$\{\varepsilon\}\cup(\{1\}\bullet\{0,1\}^*)\cup(\{00\}\bullet\{0,1\}^*)$
$\cup(\{011\}\bullet\{0,1\}^*)\cup\underline{\{01\}}$



$* \quad \underline{\overbrace{\{0,1\}^*}^{reg} - \overbrace{\{010\}}^{reg}} = \boxed{\{\varepsilon\}\cup(\{1\}\bullet\{0,1\}^*)\cup(\{00\}\bullet\{0,1\}^*)\cup(\{011\}\bullet\{0,1\}^*)\cup\underline{\{01\}}}$

Regular expressions

Let $\Sigma = \{0, 1\}$. Consider

$$((\{01\} \cup (\{1\} \cdot (\{0\} \cup \{10\}))) \cup (\{00\} \cdot (\{1\})^*)) \cdot ((\underbrace{\{0\}} \cdot \{0\}) \cdot \{1\})$$

$$\left(01 + 1(0 + 10) + 001^* \right) 001$$

Regular expressions

Regular language

$$((\{01\} \cup (\{1\} \bullet (\{0\} \cup \{10\}))) \cup (\{00\} \bullet (\{1\})^*)) \bullet ((\{0\} \bullet \{0\}) \bullet \{1\})$$

is represented as

$$(01 + 1(0 + 10) + 00(1)^*)001$$

Regular expressions

- omit braces around one-string sets
- use $+$ instead of $\cup$
- omit $\bullet$
- follow the precedence: Kleene star operator $*$, $\bullet$ (implicitly), and $+$.

Remark: $+$ and $\bullet$ are associative, i.e., $(A + B) + C = A + (B + C)$ and $(A \bullet B) \bullet C = A \bullet (B \bullet C)$.

Regular expressions: examples 1

Regular expressions: examples 2

All strings over $\{0, 1\}$ except 010.

Subexpressions

Regex is everywhere

Proofs about regular expressions - structural induction

Lemma 2

Every regular expression that does not use the symbol $\emptyset$ *represents a non-empty language.*

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Case 1: $R = \emptyset$.

Case 2: R is a single string.

Proof. (cont.2/4)
Case 3: $R = S + T$ for some regular expressions S and T.

Proof. (cont.3/4)
Case 4: $R = S \bullet T$ for some regular expressions S and T.

Proof. (cont.4/4)

Case 5: $R = S^*$ for some regular expression S.

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In every case, the language $L(R)$ is non-empty.

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Let R be a regular expression. We prove that if $L(R) \neq \emptyset$, then there exists a regular expression R' such that $L(R) = L(R')$ and R' does not contain $\emptyset$.
We prove by induction. What should the induction hypothesis be?

I.H.: For every subexpression S of R, if $L(S) \neq \emptyset$, there exists an $\emptyset$-free regular expression S' such that $L(S) = L(S')$.

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What are the cases that we have to consider?

- $R = \emptyset$
- R is a single string.
- $R = S + T$ for some regular expressions S and T.
- $R = S \bullet T$ for some regular expressions S and T.
- $R = S^*$ for some regular expression S.

(E-ex1-6) For string w, the reversal w^R is defined recursively as follows:

$$w^R = \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

For a language L, the reversal of L is defined as

$$L^R = \{w^R \mid w \in L\}.$$

You may assume the following facts.

- $L^* \bullet L^* = L^*$ for every language L.
- $(w^R)^R = w$ for every string w.
- $(x \bullet y)^R = y^R \bullet x^R$ for all strings x and y.

Prove that $(L^R)^* \subseteq (L^*)^R$.