01204211 Discrete Mathematics
Lecture 5a: Mathematical Induction 4

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Review: Mathematical Induction

> Suppose that you want to prove that property $P(n)$ is true for every natural number n.
>
> Suppose that we can prove the following two facts:
> **Base case:** $P(1)$
> **Inductive step:** For any $k \geq 1$, $P(k) \Rightarrow P(k+1)$
>
> The **Principle of Mathematical Induction** states that $P(n)$ is true for every natural number n.

The assumption $P(k)$ in the inductive step is usually referred to as **the Induction Hypothesis**.

Review: Strengthening the Induction Hypothesis

To prove the following theorem with induction:

Theorem 1
For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.

You need to prove the following "stronger" version:
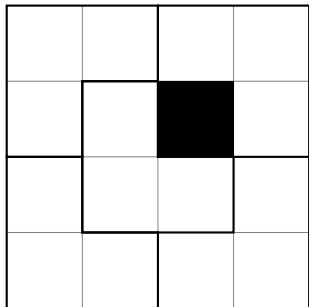
Theorem 2
For any integer $n \geq 1$, $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$.

A Lesson learned

- ▶ Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.
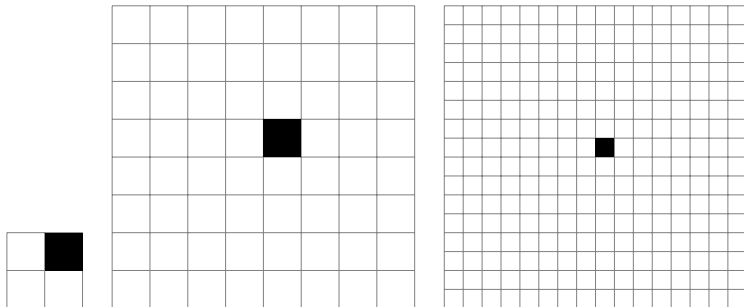
L-shaped tiles (1)[1]

A 4x4 area with a hole in the middle can be tiled with L-shaped tiles.

L-shaped tiles (2)

This is true for 2x2 area, 8x8 area, even 16x16 area.

This motivates us to try to prove that it is possible to use L-shaped tiles to tile a $2^n \times 2^n$ area.

Proving the fact?

Theorem 3
For integer $n \geq 1$, an area of size $2^n \times 2^n$ with one hole in the middle can be tiled with L-shaped tiles.

Proof: We prove by induction on n.

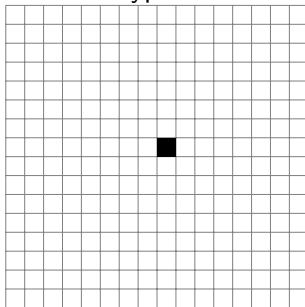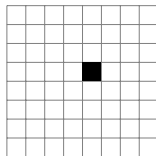**Base case:** For $n = 1$, $2^1 \times 2^1$ area with a hole in the middle can be tiled.

Inductive step: Assume that for $k \geq 1$, an $2^k \times 2^k$ area with a hole in the middle can be tiled. We shall prove the statement for $n = k + 1$, i.e., that an $2^{k+1} \times 2^{k+1}$ area with one hole in the middle can be tiled.
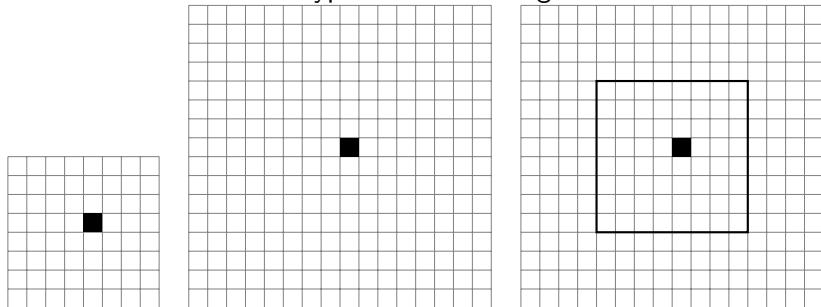
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Proof: (cont.)
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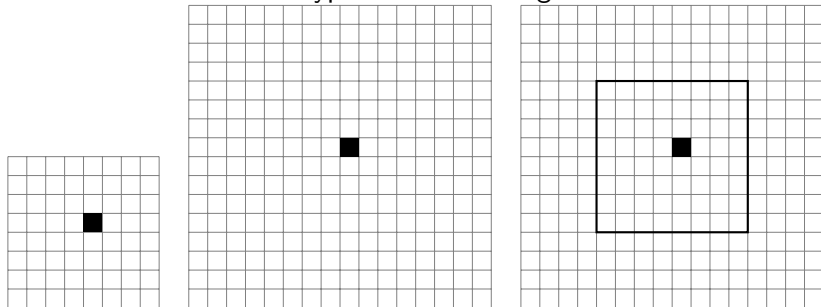

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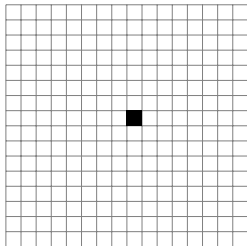
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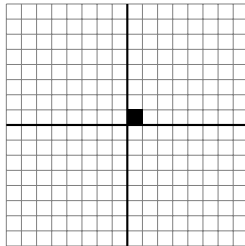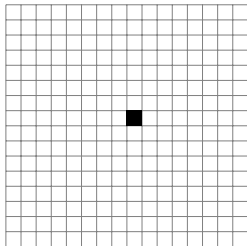Let's see the Induction Hypothesis and the goal:

With the current form of the Induction Hypothesis, this is probably the way to use it. But it seems hard to go further with this approach....
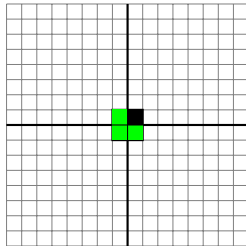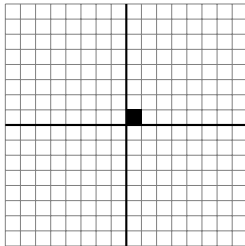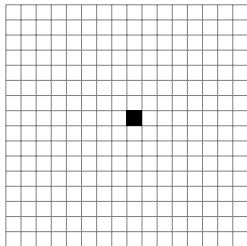
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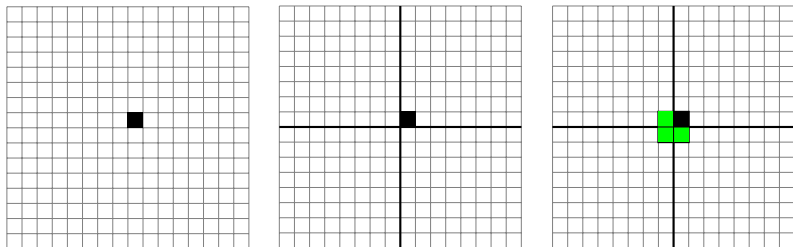
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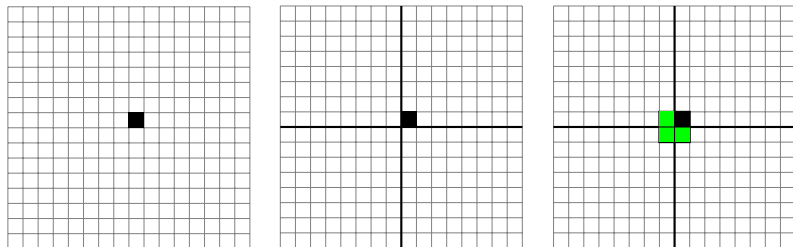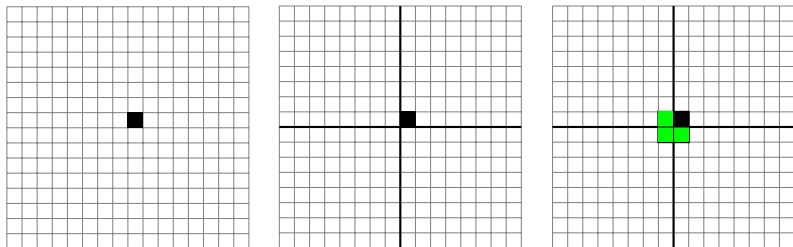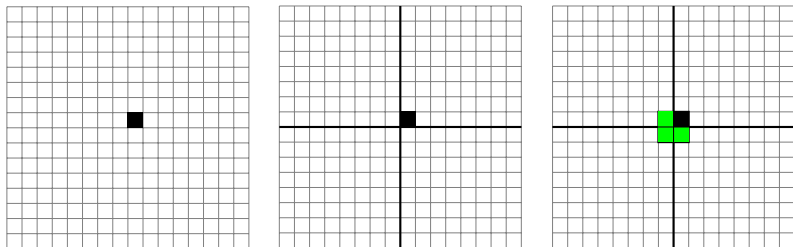
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Current Inductive Hypothesis: Assume that for $k \geq 1$, an $2^k \times 2^k$ area with "a hole in the middle" can be tiled.

A Stronger Inductive Hypothesis: Assume that for $k \geq 1$, an $2^k \times 2^k$ area with one hole can be tiled.

A stronger statement

Theorem: For integer $n \geq 1$, an area of size $2^n \times 2^n$ with one hole can be tiled with L-shaped tiles.

Proof: We prove by induction on n.

Base case: For $n = 1$, $2^1 \times 2^1$ area with one hole can be tiled;

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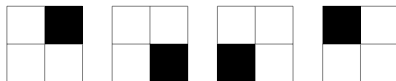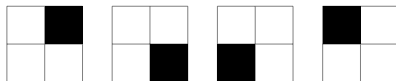
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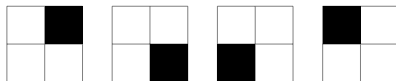

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(Try to finish it in homework.)

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Is this proof correct?

The Well-Ordering Property

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▶ Previously, we use the well-ordering property of natural numbers to prove the Principle of Mathematical Induction, but it turns out that we can use the induction to prove the well-ordering property as well. Therefore, we can take one as an axiom, and use it to prove the other.