

homogeneous system

$$Ax = \boxed{0}$$

solutions
↓
vector space

$$Ax = \boxed{b}$$

vector space

01204211 Discrete Mathematics

Lecture 9b: Affine Spaces

vector space + translation

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Review: Vector spaces

Definition

A set \mathcal{V} of vectors over \mathbb{F} is a **vector space** iff

- ▶ (V1) $\mathbf{0} \in \mathcal{V}$,
- ▶ (V2) for any $\mathbf{u} \in \mathcal{V}$,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- ▶ (V3) for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Examples of vector spaces:

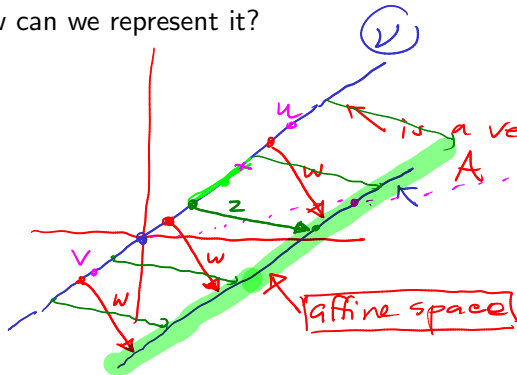
- ▶ A span of vectors is a vector space.
- ▶ A solution set to homogeneous linear equations is a vector space.

$$\boxed{A\mathbf{x} = \mathbf{0}}$$

$$\{\bar{\mathbf{x}} \mid A\bar{\mathbf{x}} = \mathbf{0}\}$$

Translation

If we have a line or a plane passing through a vector a , but not through the origin, how can we represent it?



$$A = \bar{w} + V$$

$$\text{is a vector space} = \{ \bar{x} + \bar{w} \mid \bar{x} \in V \}$$

affine space.

Translation



If we have a line or a plane passing through a vector \mathbf{a} , but not through the origin, how can we represent it?

- Translate the object so that it passes through the origin.

Translation

If we have a line or a plane passing through a vector \mathbf{a} , but not through the origin, how can we represent it?

- ▶ Translate the object so that it passes through the origin.
- ▶ We obtain a vector space \mathcal{V} .

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If we have a line or a plane passing through a vector \mathbf{a} , but not through the origin, how can we represent it?

- ▶ Translate the object so that it passes through the origin.
- ▶ We obtain a vector space \mathcal{V} .
- ▶ Then we translate it back so that it passes through \mathbf{a} .

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- ▶ We obtain a vector space \mathcal{V} .
- ▶ Then we translate it back so that it passes through \mathbf{a} .
- ▶ We get the set

$$\mathcal{A} = \{\mathbf{a} + \mathbf{u} : \mathbf{u} \in \mathcal{V}\}$$

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- ▶ *Question:* Is \mathcal{A} a vector space?

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- ▶ Translate the object so that it passes through the origin.
- ▶ We obtain a vector space \mathcal{V} .
- ▶ Then we translate it back so that it passes through \mathbf{a} .
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$$\mathcal{A} = \{\mathbf{a} + \mathbf{u} : \mathbf{u} \in \mathcal{V}\}$$

- ▶ *Question:* Is \mathcal{A} a vector space?
- ▶ We also write it as $\mathbf{a} + \mathcal{V}$.

Affine spaces

Definition

If a is a vector and \mathcal{V} is a vector space, then

$$\underline{a + \mathcal{V}}$$

is an affine space.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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Definition

The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **affine hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **convex combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Definition

The set of all convex combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **convex hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Writing an affine space using a span

Writing an affine space using a span

An affine space

An affine space passing through $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

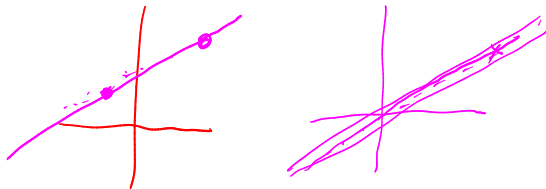
Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= b_2 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= b_m \end{aligned}$$

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned} \quad N(A)$$

What can you say about the solution sets of these two related linear systems?



Non-homogeneous linear system

Solutions to $A\bar{x} = \bar{b}$
 \bar{x}_1 \bar{x}_2

$$\bar{x}_1 - \bar{x}_2$$

Two linear systems:

✓ $A\bar{x}_1 = \bar{b}$

✓ $A\bar{x}_2 = \bar{b}$

$$A\bar{x} = \bar{b}$$

$$\begin{aligned} a_1 \cdot x &= b_1 \\ a_2 \cdot x &= b_2 \\ &\vdots \\ a_m \cdot x &= b_m \end{aligned}$$

non-homo.

$$\begin{aligned} a_1 \cdot x &= 0 \\ a_2 \cdot x &= 0 \\ &\vdots \\ a_m \cdot x &= 0 \end{aligned}$$

homo

$$A(\bar{x}_1 - \bar{x}_2)$$

$$\begin{aligned} &= A\bar{x}_1 - A\bar{x}_2 \\ &= \bar{b} - \bar{b} \\ &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

0 is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

This implies that $\mathbf{u}_1 - \mathbf{u}_2$ is a solution to the homogeneous linear system.

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{x : a_i x = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $u \in \mathcal{W}$ be one of the solutions, we have that

$$\{v - u : v \in \mathcal{W}\}$$

is a vector space, because

$$\{v - u : v \in \mathcal{W}\} = \{x : a_i x = 0, \text{ for } 1 \leq i \leq m\}$$

In other words,

$$\begin{aligned} \mathcal{W} &= u + \{v - u : v \in \mathcal{W}\} \\ &= \boxed{u} + \{x : a_i x = \boxed{0}, \text{ for } 1 \leq i \leq m\}, \end{aligned}$$

Solutions for non homogeneous

Sols of homogeneous

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\mathbf{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\}$$

is a vector space, because

$$\{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\}$$

In other words,

$$\begin{aligned} \mathcal{W} &= \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} \\ &= \mathbf{u} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\}, \end{aligned}$$

i.e., \mathcal{W} is an affine space.

Solutions to a non-homogeneous linear system

Lemma 1

If the solution set of a linear system is not empty, it is an affine space.