01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (II)

Jittat Fakcharoenphol

October 28, 2024

What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$)
- The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
- The nullspace of A

$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

- The left nullspace of A

$$\mathcal{N}(A^T) = \{\boldsymbol{y} \mid A^T\boldsymbol{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

Four fundamental subspaces

Ranks

Definition

Consider an m-by-n matrix A.

- The **row rank** of A is the maximum number of linearly independent rows of A.
- The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

Theorem 1

For any matrix A, its row rank equals its column rank.

Proof.

Let r be the column rank. We will show that there are r n-vectors that span its row space. This implies that the row rank is at most r. We can use the same argument again on A^T to obtain that the column rank is at most the row rank; thus, they must be equal. $\square$

Proof (cont.)

$\square$

Rank and nullity

Given an m-by-n matrix A, the rank of A is $\dim\ \mathcal{R}(A)$. Let r be the rank of A. What is $\dim\ \mathcal{N}(A)$?

Dimensions

Four fundamental subspaces

Given an m-by-n matrix A of rank r, we have the following subspaces

- The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$)
  $\dim\ \mathcal{R}(A) = r.$
- The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
  $\dim\ \mathcal{R}(A^T) = r.$
- The nullspace of A (denoted by $\mathcal{N}(A) \subseteq \mathbb{R}^n$)
  $\dim\ \mathcal{N}(A) = n - r.$
- The left nullspace of A (denoted by $\mathcal{N}(A^T) \subseteq \mathbb{R}^m$)
  $\dim\ \mathcal{N}(A) = m - r.$

Application: Singular Value Decomposition (SVD)

Any n-by-d matrix A can be factored into the form of UDV^T, i.e.,

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & D & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & V^T & \\ & & \end{bmatrix}$$

where

- U is an n-by-r matrix,
- D is a diagonal r-by-r matrix, and
- V is an d-by-r matrix (i.e., V^T is an r-by-d matrix)
- (Also, columns of U and D are "orthonormal.")

See demo.

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & & \\ & \boldsymbol{u}_i & U & \\ & & & \end{bmatrix} \begin{bmatrix} d_{ii} & \\ & D \\ & \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_i^T \\ V^T \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{u}_i & U & \quad \\ \\ \end{bmatrix} \begin{bmatrix} d_{ii} & \\ & D \\ & \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_i^T \\ V^T \\ \\ \end{bmatrix}$$

$$A = d_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + d_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + d_r \boldsymbol{u}_r \boldsymbol{v}_r^T .$$