# 01204211 Discrete Mathematics
# Lecture 11a: Context-free languages and grammars (1)[1]

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August 29, 2024

[1]Based on lecture notes of *Models of Computation* course by Jeff Erickson.

# How to program a computer?

# High-level programming languages

- 1954 - FORTRAN

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- 1960 - ALGO60

# Backus-Naur form

# Python grammar

# Building up languages

$$\boxed{\{ 0^n 1^n \mid n \ge 0 \}}$$

Regular languages: $\left\{ \begin{array}{l} \phi \\ \text{string เดียว} \end{array} \right.$

- ▶ Contenation
- ▶ Union
- ▶ Kleene star

# Building up languages

Regular languages:

- Contenation
- Union
- Kleene star

Context-free grammar:

- Contenation
- Union
- $\boxed{\text{Recursion}}$

# Example

กฎการแทนที่ substitution..rule.

$$\boxed{S} \to 0S1$$
$$S \to \varepsilon$$

variable

$\cancel{S}$

$0\cancel{S}1$

$00S11$

$000S111$

$000\varepsilon111$

$=$

$000111$

# Example

$$S \rightarrow 0S1$$
$$S \rightarrow \varepsilon$$

You can use "|" to write production rules more succinctly.

$$S \rightarrow 0S1 \mid \varepsilon$$

# Definition

A **context-free grammer** consists of the following components:

- a finite set $\Sigma$, a set of *symbols* (or *terminals*),
- a finite set $\Gamma$ disjoint from $\Sigma$, a set of *non-terminals* (you can think of them as variables),
- a finite set $R$ of *production rules* of the form $A \to w$ where $A \in \Gamma$ and $w \in (\Sigma \cup \Gamma)^*$ is a string of symbols and variable, and string
- a *starting* non-terminal (usually the non-terminal of the first production rule).

# Another example

$$
\begin{array}{rcl}
S & \rightarrow & A \mid B \\
A & \rightarrow & 0A \mid 0C \\
B & \rightarrow & B1 \mid C1 \\
\boxed{C} & \boxed{\rightarrow} & \boxed{\varepsilon \mid 0C1}
\end{array}
$$

Here $\Sigma = \{0, 1\}$ and $\Gamma = \{S, A, B, C\}$.

$S$

$A$

$0A$

$00A$

$000A$

$0000A$

$00000C$

$000000C1$

$0000000C11$

$\boxed{00000011}$

# Applying the rules

If you have strings $x, y, z \in (\Sigma \cup \Gamma)^*$ and the production rule

$$A \to y,$$

You can apply the rule to the string $xAz$. This yields the string

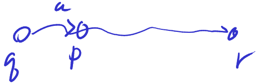
$$xyz.$$

We use the notation

$$xAz \rightsquigarrow xyz$$

to describe this application.

# Derivation




We say that $z$ derives from $x$ if we can obtain $z$ from $x$ by production rule applications, denoted by $x \rightsquigarrow^* z$.
Formally, for any string $x, z \in (\Sigma \cup \Gamma)^*$, we say that $x \rightsquigarrow^* z$ if either

- $x = z$, or
- $x \rightsquigarrow y$ and $y \rightsquigarrow^* z$ for some string $y \in (\Sigma \cup \Gamma)^*$.

1 production

# $L(w)$

The *language* $L(w)$ of string $w \in \underline{(\Sigma \cup \Gamma)^*}$ is the set of all strings in $\Sigma^*$ that derive from $w$, i.e.,

$$L(w) = \{\underline{x \in \Sigma^*} \mid w \rightsquigarrow^* x\}.$$

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The language **generated by** a context-free grammar $G$, denoted by $L(G)$ is the language of its starting non-terminal.
A language $L$ is **context-free** if there exists some context-free grammar $G$ such that $L(G) = L$.

## Grammar $G_1$

$$
\begin{aligned}
S &\rightarrow NP\,VP \\
NP &\rightarrow CN | CN\ PP \\
VP &\rightarrow CV | CV\ PP \\
PP &\rightarrow PREP\ CN \\
CN &\rightarrow ART\ N \\
CV &\rightarrow V | V\ NP \\
ART &\rightarrow \texttt{a} | \texttt{the} \\
N &\rightarrow \texttt{boy} | \texttt{girl} | \texttt{flower} \\
V &\rightarrow \texttt{touches} | \texttt{likes} | \texttt{sees} \\
PREP &\rightarrow \texttt{with}
\end{aligned}
$$

# Small English grammar

$$
\begin{aligned}
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- Examples of strings in $L(G_2)$ are:
  - `a boy sees`

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- Examples of strings in $L(G_2)$ are:
  - `a boy sees`
  - `the boy sees a flower`
  - `a girl with a flower likes the boy`

# Parse tree

$$\begin{aligned} S &\rightarrow A \mid B \\ A &\rightarrow \mathtt{0}A \mid \mathtt{0}C \\ B &\rightarrow B\mathtt{1} \mid C\mathtt{1} \\ C &\rightarrow \varepsilon \mid \mathtt{0}C\mathtt{1} \end{aligned}$$

# Parse tree

- 00011

$$
\begin{aligned}
S &\rightarrow A \mid B \\
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# Parse tree

`a girl with a flower likes the boy`

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# Ambiguity

- $1 + 1 * 1$

$$S \rightarrow 1 \mid S + S \mid S * S$$

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$$S \quad \rightarrow \quad 1 \mid S + S \mid S * S$$

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- $1 + 1 * 1$
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$$S \quad \rightarrow \quad 1 \mid S + S \mid S * S$$

- A string $w$ is **ambiguous** with respect to a grammar $G$ if more than one parse tree for $w$ exists.
- A grammar $G$ is **ambiguous** if some string is ambiguous with respect to $G$.