

# 01204211 Discrete Mathematics

## Lecture 1: Introduction

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June 27, 2021

# What is this course about?

This is a math class.

But, what is mathematics?

Ah... that's a philosophical question.

IMHO, mathematics is a mean to communicate *precise* ideas.

## It's like learning a new language

- ▶ Do you remember the time when you start learning English?
- ▶ There are a few things you have to learn and get used to.
- ▶ They might not make so much sense in the beginning, but over time, you will get comfortable with how the language is used.
- ▶ As your knowledge of the language gets better, everything becomes more natural. Learning a new language sometimes expands your view of the world.
- ▶ I hope it is also true with this course.

# The goals of this course

There are two goals:

- ▶ To learn how to make mathematical arguments.
- ▶ To learn various fundamental mathematical concepts that are very useful in computer science.

## Why should we learn how to prove? (1)

Look at this program.

```
if a > b:  
    return a  
else:  
    return b
```

The author claims that this program takes two variables  $a$  and  $b$  and returns the larger one.

*Do you believe the author of the code? Why?*

## Finding the maximum value. (1)

Now look at this program.

```
if a > b:
    if a > c:
        return a
    else:
        return c
else:
    if c > b:
        return c
    else:
        return b
```

The author claims that this program takes three variables  $a$ ,  $b$  and  $c$  and returns the largest one.

*Do you believe the author of the code? Why?*

## Finding the maximum value. (2)

Finally, look at this program.

```
// Input: array A with n elements: A[0],...,A[n-1]
m = 0
for i = 0, 1, ..., n-1:
    if A[i] > m:
        m = A[i]
return m
```

The author claims that this program takes an array  $A$  with  $n$  elements and returns the maximum element.

*Do you believe the author of the code? Why?*

**Can we try to test the code with all possible inputs?**

## Finding the maximum value. (3)

Let's try again.

```
// Input: array A with n elements: A[0],...,A[n-1]
m = A[0]
for i = 1, 2, ..., n-1:
    if A[i] > m:
        m = A[i]
return m
```

*Do you believe the author of the code? Why?*

**Can we try to test the code with all possible inputs?**



## Another example: testing primes (1)

A *prime* is a natural number greater than 1 that has no positive divisors other than 1 and itself. E.g., 2,3,5,7,11 are primes.

```
Algorithm CheckPrime(n):      // Input: an integer n
    if n <= 1:
        return False
    i = 2
    while i <= n-1:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

The code above checks if  $n$  is a prime number. How fast can it run?

Note that if  $n$  is a prime number, the for-loop repeats for  $n - 2$  times. Thus, the running time is approximately proportional to  $n$ . Can we do better?

## Another example: testing primes (2)

Consider the following code.

```
Algorithm CheckPrime2(n): // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run? Note that  $s = \sqrt{n}$ ; therefore, it takes time approximately proportional to  $\sqrt{n}$  to run.

Ok, it should be faster. **But is it correct?**

## Informal arguments (1)

- ▶ Let's try to argue the Algorithm CheckPrime2 works correctly.
- ▶ Note that if  $n$  is a prime number, the algorithm answers correctly. (Why?)
- ▶ Therefore, let's consider the case when  $n$  is not prime (i.e.,  $n$  is a composite).
- ▶ If that's the case,  $n$  has some positive divisor which is not 1 or  $n$ . Let's call this number  $a$ .
- ▶ Now, if  $2 \leq a \leq \sqrt{n}$ , at some point during the execution of the algorithm,  $i = a$  and  $i$  should divide  $n$ ; thus the algorithm correctly returns False.
- ▶ Are we done?

## Informal arguments (2)

- ▶ Recall that we are left with the case that (1)  $n$  is not prime and (2) its positive divisor  $a$  is larger than  $\sqrt{n}$ .
- ▶ Let  $b = n/a$ . Since  $n$  and  $a$  are positive integers and  $a$  divides  $n$ ,  $b$  is also a positive integer.
- ▶ Note that if we can argue that  $2 \leq b \leq \sqrt{n}$ , we are done. (why?)
- ▶ How can we do that?

# The goals

- ▶ Let's take a break and look back at what we are trying to do.

**Original goal:** To show that Algorithm CheckPrime2 is correct.

**Current (sub) goal:** Consider a positive composite  $n$  and its positive divisor  $a$ , where  $a > \sqrt{n}$ . Let  $b = n/a$ . We want to show that  $2 \leq b \leq \sqrt{n}$ .

- ▶ Before we continue, I'd like to add a bit of formalism to our thinking process.

# The main goal

- ▶ **Original goal:** To show that Algorithm CheckPrime2 is correct.
- ▶ Let's focus on the statement we want to argue for:

**“Algorithm CheckPrime2 is correct.”**

- ▶ Note that this statement can either be “true” or “false.” If we can demonstrate, using logical/mathematical arguments that this statement is true, we can say that we **prove** the statement.

## The (sub) goal

- ▶ **Current (sub) goal:** Consider a positive composite  $n$  and its positive divisor  $a$ , where  $a > \sqrt{n}$ . Let  $b = n/a$ . We want to show that  $2 \leq b \leq \sqrt{n}$ .
- ▶ Let's focus only on the statement we want to argue for:

$$2 \leq b \leq \sqrt{n}.$$

- ▶ If we only look at this statement, it is unclear if the statement is true or false because there are variables  $b$  and  $n$  in the statement. It can be true in some case and it can be false in some case depending on the values of  $n$  and  $b$ .
- ▶ Are we doom? Not really. The statement above is not precisely the statement we want to prove.

## The (sub) goal (second try)

- ▶ **Current (sub) goal:** Consider a positive composite  $n$  and its positive divisor  $a$ , where  $a > \sqrt{n}$ . Let  $b = n/a$ . We want to show that  $2 \leq b \leq \sqrt{n}$ .
- ▶ We can be more specific about what values of  $n$  and  $b$  that we want to consider.

### Revised statement

For all positive composite integer  $n$ , and for every divisor  $a$  of  $n$  such that  $\sqrt{n} < a < n$ ,

$$2 \leq b \leq \sqrt{n},$$

where  $b = n/a$ .

- ▶ Note that this revised statement is now “quantified,” that is, every variable in the statement has specific scope. Now the statement is either true or false.



# Propositions<sup>1</sup>

- ▶ A *proposition* is a statement which is either **true** or **false**.
- ▶ It is our basic unit of mathematical “facts”.
- ▶ Examples:
  - ▶ Algorithm CheckPrime2 is correct.
  - ▶  $10^2 = 90$ .
  - ▶  $\sqrt{2}$  is irrational.
- ▶ Examples of statements which are not propositions (why?):
  - ▶  $x > 10$ .
  - ▶  $1 + 2 + \cdots + 10$ .
  - ▶ This algorithm is fast.
  - ▶ Run, run quickly.

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<sup>1</sup>This section follows the expositions in Berkeley's CS70 lecture notes.

## Combining propositions

- ▶ We usually use a variable to refer to a proposition. For example, we may use  $P$  to stand for “it rains” or  $Q$  to stand for “the road is wet.”
- ▶ The truth value of a variable is the truth value of the proposition it stands for.
- ▶ Many propositions can be combined to get a complex statement using logical operators.
- ▶ For example, we can join  $P$  and  $Q$  with “and” (denoted by “ $\wedge$ ”) and get

$$P \wedge Q,$$

which stands for “it rains and the road is wet”.

- ▶ An expression  $P \wedge Q$  is an example of *propositional forms*. The logical value of a propositional form “usually” depends on the truth value of its variables.

## Connectives: “and”, “or”, “not”

Given propositions  $P$  and  $Q$ , we can use connectives to form more complex propositions:

- ▶ **Conjunction:**  $P \wedge Q$  (“ $P$  and  $Q$ ”),  
(True when both  $P$  and  $Q$  are true)
- ▶ **Disjunction:**  $P \vee Q$  (“ $P$  or  $Q$ ”),  
(True when at least one of  $P$  and  $Q$  is true)
- ▶ **Negation:**  $\neg P$  (“not  $P$ ”),  
(True only when  $P$  is false)

If  $P$  stands for “today is Tuesday” and  $Q$  stands for “dogs are animals”, then

- ▶  $P \wedge Q$  stands for “today is Tuesday and dogs are animals”,
- ▶  $P \vee Q$  stands for “today is Tuesday or dogs are animals”, and
- ▶  $\neg P$  stands for “today is not Tuesday”.

# Truth tables

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

| $P$ | $Q$ | $P \wedge Q$ | $P \vee Q$ | $\neg P$ |
|-----|-----|--------------|------------|----------|
| $T$ | $T$ | $T$          | $T$        | $F$      |
| $T$ | $F$ | $F$          | $T$        |          |
| $F$ | $T$ | $F$          | $T$        | $T$      |
| $F$ | $F$ | $F$          | $F$        |          |

## Quick check 1

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- ▶ All prime numbers are larger than 0 and all natural numbers is at least one.
- ▶ You are smart or you won't be taking this class.

## Next lecture...

- ▶ We will discuss other ways to join two propositions, i.e., implications ( $\Rightarrow$ ) and equivalences ( $\Leftrightarrow$ ).
- ▶ We will look at two forms of quantifiers: universal quantifiers and existential quantifiers.