01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (I)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

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Subspaces

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Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that $\mathcal{V}$ is a **subspace** of $\mathcal{W}$.

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Examples:

- $\mathrm{Span}\,\{[1,1]\}$ is a subspace of $\mathbb{R}^2$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1]\}$ is a subspace of $\mathbb{R}^3$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of $\mathbb{R}^3$.

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$$A\boldsymbol{x} = 0.$$

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The set of solutions $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product $A\boldsymbol{x}$. Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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- The column space of A (denoted by $\mathcal{R}(A)$)
- The row space of A (denoted by $\mathcal{R}(A^T)$)
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Linearly independent rows

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Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

Row rank $=$ Column rank

Theorem 1

For any matrix A, its row rank equals its column rank.

We will prove this theorem next time.