01204211 Discrete Mathematics
Lecture 8a: Linear systems of equations

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Linear algebra

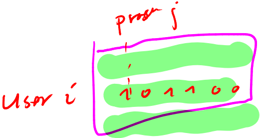
Linear algebra studies

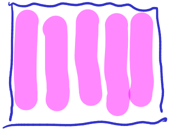
- matrices and operations with matrices
- systems of linear equations
- linear transformations
- linear spaces (and their structures) $\ast$

Why?

image processing
data science
ml
quantum complexity

- Lots of applications.
- Interesting perspectives.





A linear system of equations

x dogs

how many legs the dogs have?

$4x = y$ $\quad$ $4x - y = 0$

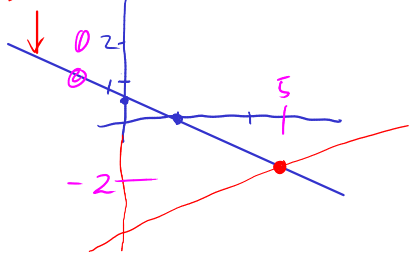
Let's start with a simple example with 2 variables:

$5x - 20 = 5$

$x = \frac{25}{5} = 5$

$$
\begin{aligned}
5x + 10y &= 5 \\
x - 3y &= 11
\end{aligned}
$$

constraints



How would you solve it?

Gaussian elimination

$\frac{\cancel{5}}{5}x + \frac{\cancel{10}^{2}}{5}y = \frac{\cancel{5}}{5}1$

$x + 2y = 1$

$-5y = 10$

$\boxed{\begin{aligned} y &= -2 \\ x &= 5 \end{aligned}}$

③ ← ① / 5

④ ← ② − ③

A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) =$$

A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 =$$

A linear system of equations

Let's start with a simple example with 2 variables:

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How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = \; - 50$$

Then you can conclude that $y = -2$. Substitute it to one of the equation, you can find out the value of x.

Gaussian elimination (1)

Let's consider a system with 3 variables:

$$
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \quad - ① \\
x_1 & + & & & 5x_3 & = & 12 \quad - ② \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \quad - ③
\end{array}
$$

x_1

$5 - 3 \times 2 = -1$

$12 - \cancel{2}\, 7/2 = -\cancel{4}\; 9.5$

$3 - 2 \cdot 3$

$3 - 3 \times (-3.5)$

$-4 - 3 \cdot 9.5$

$= 28.5$

$(② - ①/2) \qquad 0x_1 - 2x_2 - 3{,}5\, x_3 = 9.5 \quad - ④$

x_2

$(③ - 2 \cdot ①) \qquad 0x_1 - 6x_2 - 3x_3 = -4 \quad - ⑤$

$(⑤ - 3④) \qquad 0x_2 - 13{,}5\, x_3 = 32{,}5 \quad - ⑥$

x_3

Gaussian elimination (2)

$x_1 + 2x_2 + 1{,}5x_3 = 3{,}5$

Let's consider another system with 3 variables:

$$
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
3x_1 & + & 8x_2 & + & x_3 & = & 10
\end{array}
$$

$$
\boxed{\begin{array}{l}
x_1 - 2x_2 - x_3 = -2 \\
3x_1 + 2x_2 + 5x_3 = 14 \\
2x_2 - 2x_2 = 1
\end{array}}
$$

$$-2x_2 + 3{,}5x_3 = 8{,}5$$

$$2x_2 - 3{,}5x_3 = -0{,}5$$

$$
\boxed{\begin{array}{l}
8x_2 + 8x_3 = 20 \\
2x_2 + 2x_3 = 5
\end{array}}
$$

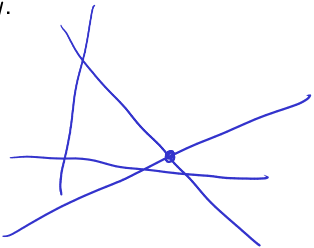
A closer look: 1st perspective

Consider

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

equations

Each equation (row) constraints certain values of x and y.



"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{lcl} (\ 5, & 10\ ) & = \boldsymbol{u}_1 \\ (\ 1, & -3\ ) & = \boldsymbol{u}_2 \end{array}$$

row operation

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$
\begin{array}{llll}
(& 5, & 10 &) = \boldsymbol{u}_1 \\
(& 1, & -3 &) = \boldsymbol{u}_2 \\
(& \underline{0,} & \underline{25} &) = \boldsymbol{u}_1 - \boxed{5 \cdot \boldsymbol{u}_2}
\end{array}
$$

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{llll} (& 5, & 10 &) = \boldsymbol{u}_1 \\ (& 1, & -3 &) = \boldsymbol{u}_2 \\ (& 0, & 25 &) = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2 \end{array}$$

The third equation is a "combination" of the other two rows. In fact, it is a **linear combination** of the first two.

"Combining" two rows

$(0,1) = \left(\frac{0}{25}, \frac{25}{25}\right) = \frac{u_1}{25} - \frac{5u_2}{25}$ $= \boxed{0{,}04\,u_1 - 0{,}2\,u_2}$

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{lcl} (\;\; 5, & 10 \;\;) & = \boldsymbol{u}_1 \\ (\;\; 1, & -3 \;\;) & = \boldsymbol{u}_2 \\ (\;\; 0, & 25 \;\;) & = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2 \end{array}$$

The third equation is a "combination" of the other two rows. In fact, it is a **linear combination** of the first two.

Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

“Combining” two rows

Let’s focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{lll} (\ 5, & 10 &) = \boldsymbol{u}_1 \\ (\ 1, & -3 &) = \boldsymbol{u}_2 \\ (\ 0, & 25 &) = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2 \end{array}$$

The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.

Can you obtain $(0,1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

Yes,

$$0.2 \cdot \boldsymbol{u}_1 - \boldsymbol{u}_2 = (0,1).$$

It turns out that you can obtain any (a,b) from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A closer look: 1st perspective (more example)

Consider

$$
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

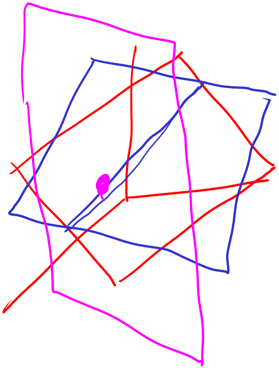
~~1 point~~

A 1 plane

B 1 plane

C 1 pl—

What are the row vectors?



A closer look: 2nd perspective

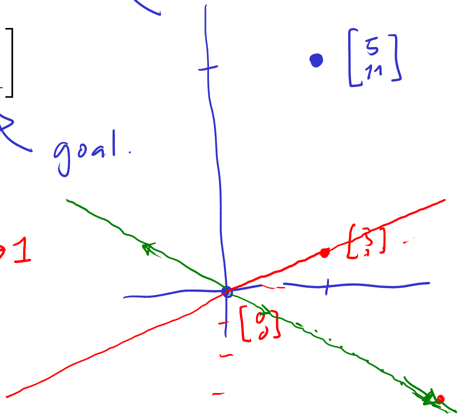
$$5x + 10y = 5$$
$$1 \cdot x - 3y = 11$$

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

goal.

$y \to 0 \to 1$



A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.
But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

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But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and with x and y, we now see that $\boldsymbol{b}$ is a **linear combination** of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
Finding x and y is essentially checking if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Example 2: a linear system with 2 variables

$$\begin{aligned} x &+ y &= 7 \\ x &- 2y &= 13 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ -2 \end{bmatrix} y = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$ ...hor

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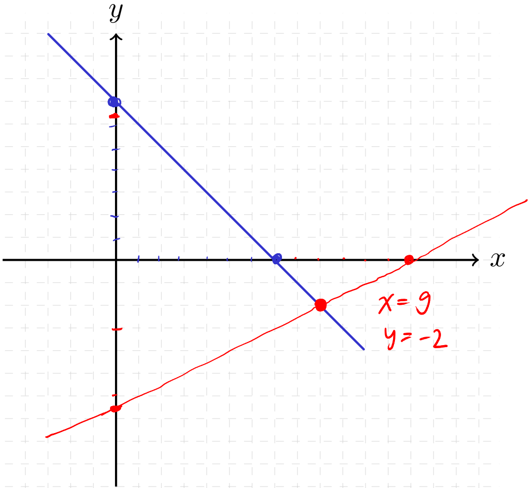
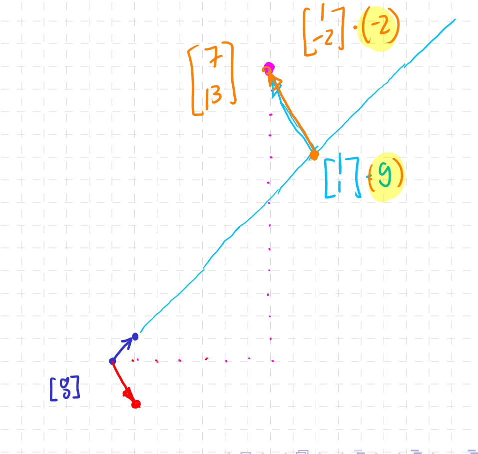
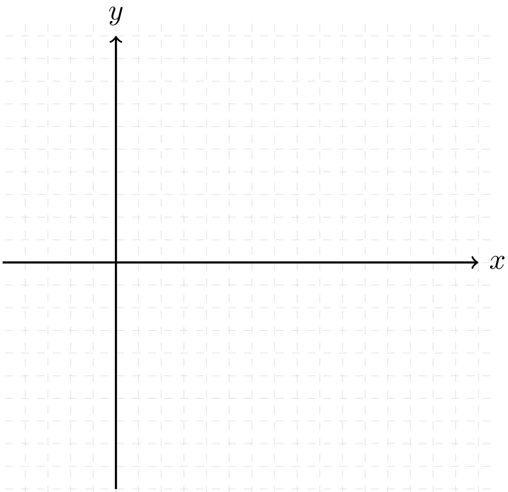




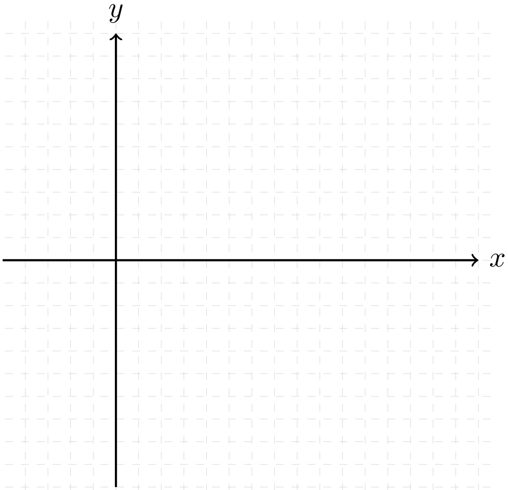
Example 3: a linear system with 2 variables

$$\begin{array}{rcrcr} 2x & + & y & = & 5 \\ 4x & + & 2y & = & 10 \end{array}$$



Example 4: a linear system with 2 variables

$$\begin{array}{rcrcl} x & + & 3y & = & 6 \\ 0.5 \cdot x & + & 1.5 \cdot y & = & 9 \end{array}$$



A linear system with 3 variables

Let's consider a system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Row perspective

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Each equation becomes a **plane** in 3 dimensional space.

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2,4,3), \quad (1,0,5), \quad (4,2,3)$$

We want to linearly combine them to obtain

$$(1,0,0), \quad (0,1,0), \quad (0,0,1)$$

In other words, what are the possible linear combinations of

$$(2,4,3), \quad (1,0,5), \quad (4,2,3)$$

Column perspective

linear combinations of $u_1, u_2,$ and u_3

From

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$

we rewrite the system as

$$\overset{u_1}{\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}} \cdot x_1 + \overset{u_2}{\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}} \cdot x_2 + \overset{u_3}{\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Column perspective

From

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$

we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

In other words, the vector $\boldsymbol{b}$, for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 10 \end{array}$$

More example 2

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\ 5x_1 & + & 2x_2 & + & 8x_3 & = & 22 \end{array}$$

More failed example 3

Let's consider the last system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 2x_1 & + & & & 10x_3 & = & 24 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions.

More failed example 3 (cont.)

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$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions?

More failed example 3 (cont.)

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$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions? It means that, for $\boldsymbol{u}$, you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \end{array}$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{rcrcrcl} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{rcrcrcl} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \end{array}$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{rcrcrcl} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{rcrcrcl} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \\ \quad (u_1 - v_1) \quad + \quad 5(u_1 - v_3) \quad = \quad (12 - 12) = 0 \end{array}$$

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ x_1 & + & & & 5x_3 & = & 0 \end{array}$$

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ x_1 & + & & & 5x_3 & = & 0 \end{array}$$

It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ x_1 & + & & & 5x_3 & = & 0 \end{array}$$

It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.
It would play a central role when dealing with linear systems with many solutions.

Key take away

- There are 2 ways to look at how we solve linear systems: row perspective and column perspective.
- **Linear combination** is the main operation.