

# 01204211 Discrete Mathematics

## Lecture 8b: Modular arithmetic

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## Quick check 1

If  $a|m$  and  $b|m$ , can we say that  $ab|m$ ? Prove this fact or provide a counter example.

## Quick check 2

If  $a|m$ ,  $b|m$ , and  $a \neq b$  are both prime, can we say that  $ab|m$ ? Prove this fact or provide a counter example.

# Prime factorization

One useful fact that we use over and over again is the following.

## Unique Factorization (or Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be written *uniquely* as a product of prime numbers (up to the order of factors).

Examples:

▶  $10 = 2 \cdot 5$

▶  $13 = 13$

▶  $112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^4 \cdot 7$

There are 3 clocks. At this moment, all three clocks ring at the same time. The first clock rings every 3 hours, the second clock rings every 4 hours, and the third clock rings every 10 hours. How long do you have to wait until you would hear all clocks ring at the same time again?

You have a large water container and two smaller buckets. The first bucket carries 3 litres of water and the second bucket carries 5 litres of water.  
Can you put exactly 1 litre of water in the water container?

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.  
What is the minimum volume of water you can exactly put in the water container?

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In general if you have two buckets of volumes  $x$  and  $y$ , the amount that you can exactly make must be in the form of

$$ax + by,$$

for some integers  $x$  and  $y$ . (Note that  $x$  and  $y$  may be negative.)



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for some integers  $x$  and  $y$ . (Note that  $x$  and  $y$  may be negative.)

Do you see why the sum must be divisible by any common divisor of  $x$  and  $y$ ?

## Useful fact

For any integer  $x$  and  $y$ , consider the term

$$a \cdot x + b \cdot y,$$

for some integer  $a$  and  $b$ .

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For any integer  $x$  and  $y$ , consider the term

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for some integer  $a$  and  $b$ .

When the term is non-zero, it must be divisible by  $\gcd(x, y)$ , so it has to be at least  $\gcd(x, y)$ .

It turns out that you can actually attain that value, i.e., there exist a pair of integer  $a$  and  $b$  such that

$$a \cdot x + b \cdot y = \gcd(x, y).$$

## Finding $a$ and $b$ : Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns  $a$  and  $b$  together with  $\gcd(x, y)$ .

```
Algorithm Euclid(x,y):  
  if x mod y == 0:  
  
    return y,          ,  
  else:  
    g, a', b' = Euclid(y, x mod y)  
  
    a =  
  
    b =  
  
    return g, a, b
```

## Notes:

We have  $a'$  and  $b'$  such that

$$a' \cdot y + b' \cdot (x \bmod y) = g.$$

# Secret sharing

# Secret sharing scheme based on straight lines

# Days

What day is it today?



# Days

What day is it today? Thursday.

# Days

What day is it today? Thursday.  
What day is 3 days after today?

# Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

# Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today?

# Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

# Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

What day is 10 days before today?

# Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

What day is 10 days before today? Monday.

# Clocks

Suppose that it is 1 o'clock.



# Clocks

Suppose that it is 1 o'clock.  
What time is the next 5 hours?

# Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

# Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours?

# Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

# Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

What time is the next 20 hours?

# Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

What time is the next 20 hours? 9 o'clock.

## Modular arithmetic

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Suppose that  $m = 7$ . We would like to say that

$$4 + 5 = 9 \bmod m = 2.$$

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$$2 - 6 = -4 \bmod 7 = 3 \bmod 7 = 3.$$

Note that when you view integers under the lense of modulus 7, these numbers

$$\dots, -19, -12, -5, 2, 9, 16, 23, \dots$$

are essentially **the same**.

## Properties (1)

$a \bmod m = b \bmod m$ , if and only if  $m \mid a - b$ .

# Properties (1)

$a \bmod m = b \bmod m$ , if and only if  $m|a - b$ .

Proof.

( $\Rightarrow$ ) Let  $r = a \bmod m$ . We can write

$$a = qm + r,$$

and

$$b = pm + r,$$

for some integers  $q$  and  $p$ . Thus, we have

$$a - b = qm + r - pm - r = (q - p)m.$$

Therefore  $m|a - b$ .

( $\Leftarrow$ ) Exercise.





## Properties (2)

- ▶  $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- ▶  $(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$
- ▶  $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$

# Congruences

## Definition (congruences)

For an integer  $m > 0$ , if integers  $a$  and  $b$  are such that

$$a \bmod m = b \bmod m,$$

we write

$$a \equiv b \pmod{m}.$$

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We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a - b)$$

# Congruences: properties (1)

- ▶ (reflexivity)

$$a \equiv a \pmod{m}.$$

- ▶ (symmetry)

$$a \equiv b \pmod{m} \text{ implies } b \equiv a \pmod{m}.$$

- ▶ (transitivity)

$$a \equiv b \pmod{m} \text{ and } b \equiv c \pmod{m} \text{ implies } a \equiv c \pmod{m}.$$

## Congruences: properties (2) – operations

If we have that

$$a \equiv b \pmod{m},$$

and

$$c \equiv d \pmod{m},$$

then

- ▶  $a + c \equiv b + d \pmod{m}$
- ▶  $a - c \equiv b - d \pmod{m}$
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We can pretty much think of this “congruence” as a normal equation.

*What is missing here?*

**Division!**



Also, we wish we can do “cancellation”, i.e., if

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then  $a \equiv b \pmod{m}$ . **BUT THIS IS NOT ALWAYS TRUE.**

Let's see the following example:

$$2 \cdot 1 \equiv 2 \cdot 3 \pmod{4},$$

but

$$1 \not\equiv 3 \pmod{4}.$$

## Multiplications as functions

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Which functions have inverses?

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By dividing on both sides with 5:

$$2/5 = 5x/5 = x,$$

or equivalently, by multiplying with  $(1/5) = 5^{-1}$ :

$$2 \cdot 5^{-1} = 5x \cdot 5^{-1} = x \cdot 5 \cdot 5^{-1} = x \cdot 1 = x.$$

Here  $5^{-1}$  is a multiplicative inverse of 5.

## Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be  $m = 7$ . Note that

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod{7}.$$

Therefore,  $5^{-1} \equiv 3 \pmod{7}$ .

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Therefore,  $5^{-1} \equiv 3 \pmod{7}$ .

To find  $2/5$ , we can view our goal as to find the value of  $x$  such that

$$2 \equiv 5x \pmod{7}.$$

We can multiply both sides with  $5^{-1} \equiv 3$  to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

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We can multiply both sides with  $5^{-1} \equiv 3$  to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

Let's check:

$$5 \cdot 6 \equiv 30 \equiv 2 \pmod{7},$$

as required.

# Multiplicative inverse modulo $m$

## Definition

The multiplicative inverse modulo  $m$  of  $a$ , denoted by  $a^{-1}$ , is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

# Multiplicative inverse modulo 11

Let's try to figure out multiplicative inverse of every integer modulo 11.

$a$	$a^{-1} \pmod{11}$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

## Example: secret sharing

- ▶ Think of a secret number  $m \in \{0, 1, \dots, 10\}$ .
- ▶ Pick a random number  $a \in \{1, 2, \dots, 10\}$ .
- ▶ Your straight line function  $f(x) = (ax + m) \bmod 11$ .
- ▶ We will generate 3 points from  $f$  and give them to 3 of your friends, each with only 1 point. Pick 3 numbers  $x_1, x_2, x_3$  from  $\{1, 2, \dots, 10\}$ .
- ▶ Let's compute

$$(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)).$$

- ▶ Give them to 3 of your friends and challenge them to form a group of 2 people and figure out your number  $m$ .

## Theorem 1

*An integer  $a$  has a multiplicative inverse modulo  $m$  iff  $\gcd(a, m) = 1$ .*

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## Proof.

( $\Leftarrow$ ) Recall that there exist integers  $x$  and  $y$  such that

$$x \cdot a + y \cdot m = \gcd(a, m) = 1.$$

Thus,  $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$ , i.e.,  $x \cdot a \equiv 1 \pmod{m}$ . Therefore  $x$  is the inverse.

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Thus,  $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$ , i.e.,  $x \cdot a \equiv 1 \pmod{m}$ . Therefore  $x$  is the inverse.

( $\Rightarrow$ ) Let  $r = \gcd(a, m)$ . Suppose that  $b$  is the multiplicative inverse of  $a$  modulo  $m$ , i.e., we have that

$$b \cdot a \equiv 1 \pmod{m},$$

Thus,  $ba \bmod m = 1 \bmod m = 1$ , i.e., there exists an integer  $q$  such that

$$ba = qm + 1,$$

or  $ba - qm = 1$ . However,  $r$  since  $r|a$  and  $r|m$ ,  $r$  also divides  $ba - qm$  and 1. But it  $r \nmid 1$  because  $r > 1$  and we have the contradiction. □

## Examples: division in modular arithmetic

Since the requirement for an existence of  $a^{-1}$  modulo  $m$  is that  $\gcd(a, m) = 1$ , if we let  $m$  be a prime number, every  $a$  which is not a multiple of  $m$  has an inverse.

Can you solve this equation?

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Can you solve this equation?

$$4x + 9 \equiv 0 \pmod{11}.$$

We can even perform gaussian elimination (*which is very useful later*):

$$\begin{array}{rcl} 2x + y & \equiv & 3 \pmod{7} \\ x + 3y & \equiv & 5 \pmod{7} \end{array}$$

# Public-key cryptography

# RSA

## Quick recap: RSA

- ▶ Private key:  $(d, n)$ ,    Public key:  $(e, n)$
- ▶ Encryption  $E(m) = m^e \bmod n$ ,    Decryption:  $D(w) = w^d \bmod n$ .
- ▶ Goal: Select  $e, d, n$  such that  $D(E(m)) = m^{ed} \bmod n = m$ .

## Quick recap: RSA

- ▶ Private key:  $(d, n)$ , Public key:  $(e, n)$
- ▶ Encryption  $E(m) = m^e \bmod n$ , Decryption:  $D(w) = w^d \bmod n$ .
- ▶ Goal: Select  $e, d, n$  such that  $D(E(m)) = m^{ed} \bmod n = m$ .
- ▶ Pick two primes  $p$  and  $q$ . Let  $n = pq$ .
- ▶ Pick  $e$  (usually a small number)
- ▶ Pick  $d$  such that  $d = e^{-1} \pmod{(p-1)(q-1)}$ , i.e.,  $ed \equiv 1 \pmod{(p-1)(q-1)}$ , or
$$ed = k \cdot (p-1)(q-1) + 1,$$
for some integer  $k$ .
- ▶ What is  $m^{ed} \bmod n$ ?



## What's next?

- ▶ We will prove Fermat's Little Theorem and show how to efficiently test if a number is prime.
- ▶ We will also use Fermat's Little Theorem to prove the correctness of RSA.
- ▶ Modular arithmetic is also key to our usage of polynomials to perform secret sharing and error correcting codes, because now we can do Gaussian elimination using only integers.