01204211 Discrete Mathematics
Lecture 8b: Modular arithmetic

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September 25, 2025

Quick check 1

$6 \mid 12$ $4 \mid 12$ $24 \nmid 12$

If $a|m$ and $b|m$, can we say that $ab|m$? Prove this fact or provide a counter example.

Quick check 2

If $a|m$, $b|m$, and $a \neq b$ are both prime, can we say that $ab|m$? Prove this fact or provide a counter example.

Prime factorization

One useful fact that we use over and over again is the following.

Unique Factorization (or Fundamental Theorem of Arithmetic)

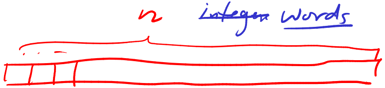
Every integer greater than 1 can be written *uniquely* as a product of prime numbers (up to the order of factors).

Examples:

- $10 = 2 \cdot 5$
- $13 = 13$
- $112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^4 \cdot 7$

Problem size

- Sort n numbers



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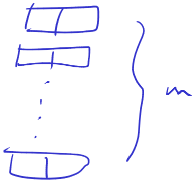
- Check if vertex u is reachable from vertex v in a graph with n vertices and m edges



$2m + n$

Problem size

- Sort n numbers
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- Multiply two $n \times n$ matrices

$\boxed{n \times n} \times 2$ $\quad 2n^2$

n^2

Problem size

- Sort n numbers
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- Multiply two $n \times n$ matrices
- Add two integers

$\leq n$ $\leq n$

a $\square$ b $\square$ - - - $\boxed{2}$ words

$\log n$ bits ค่า จน. $\leq n$

Problem size

- Sort n numbers
- Check if vertex u is reachable from vertex v in a graph with n vertices and m edges
- Multiply two $n \times n$ matrices
- Add two integers
- Check if an integer n is prime

n

$\underline{\text{ขนาด(พุ)}}$ $\underline{\text{bits}}$

1 $\log n$

$O(\sqrt{n})$

$n^{1/2} = (2^{\log n})^{1/2} = 2^{\log n/2}$

Problem size

- Sort n numbers
- Check if vertex u is reachable from vertex v in a graph with n vertices and m edges
- Multiply two $n \times n$ matrices
- Add two integers
- Check if an integer n is prime
- Find $gcd(a, b)$ for inputs a and b

$a > b$

$\underbrace{\qquad}_{\log a}$ $\underbrace{\qquad}_{\log b}$

2

$\log a + \log b$

recursive $2 \cdot \log a$

– [illegible] $\log^2 a$

$\boxed{\log^3 a}$ ✓

Polynomial times

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When inputs contain a few numbers

size = bits

GCD and Power

$x^{y} \bmod m = x^{y/2} \cdot x^{y/2} \bmod m$

$\log m$

recurse TV $\log y$.

$O(\log y \cdot \log^2 m)$

Days

What day is it today?

Days

What day is it today? Thursday.

Days

What day is it today? Thursday.
What day is 3 days after today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today? Monday.

Clocks

Suppose that it is 1 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.

Clocks

Suppose that it is 1 o'clock.
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What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours? 9 o'clock.

Modular arithmetic

As in the days of weeks and clocks examples (and also as the modulo in RSA algorithm in our experiment), when working under modular arithmetic, we start with a **modulus** m.

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$$4 + 5 = 9 \bmod m = 2.$$

Or

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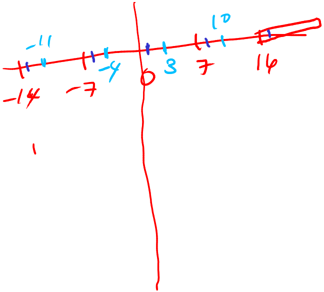

Or

$$3 \cdot 4 = 12 \bmod m = 5.$$

Or

$$2 - 6 = -4 \bmod 7 = 3 \bmod 7 = 3.$$

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$$4 + 5 = 9 \bmod m = 2.$$

Or

$$3 \cdot 4 = 12 \bmod m = 5.$$

Or

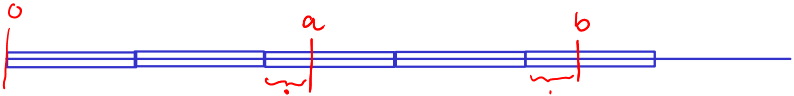
$$2 - 6 = \; -4 \bmod 7 = 3 \bmod 7 = 3.$$

Note that when you view integers under the lense of modulus 7, these numbers

$$\ldots, -19, -12, -5, 2, 9, 16, 23, \ldots$$

are essentially **the same**.

Properties (1)



$a \bmod m = b \bmod m$, if and only if $m | a - b$.

Properties (1)

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Proof.

($\Rightarrow$) Let $r = a \bmod m$. We can write

$$a = qm + r,$$

and

$$b = pm + r,$$

for some integers q and p. Thus, we have

$$a - b = qm + r - pm - r = \boxed{(q - p)m.}$$

Therefore $m | a - b$.

($\Leftarrow$) Exercise. $\square$

Properties (2)

- $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- $(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$
- $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$

Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

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we write

$$a \equiv b \pmod{m}.$$

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We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a - b)$$

Congruences: properties (1)

- (reflexivity)
  $a \equiv a \pmod{m}$.
- (symmetry)
  $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$.
- (transitiviey)
  $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ implies $a \equiv c \pmod{m}$.

Congruences: properties (2) – operations

If we have that

$$a \equiv b \pmod{m},$$

and

$$c \equiv d \pmod{m},$$

then

- $a + c \equiv b + d \pmod{m}$
- $a - c \equiv b - d \pmod{m}$
- $ac \equiv bd \pmod{m}$

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We can pretty much think of this "congruence" as a normal equation.

What is missing here?
Division!

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{12}{5}$$

Also, we wish we can do "cancellation", i.e., if

$$xa \equiv xb \pmod{m},$$

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$$xa \equiv xb \pmod{m},$$

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Let's see the following example:

$$2 \cdot 1 \equiv 2 \cdot 3 \pmod{4},$$

but

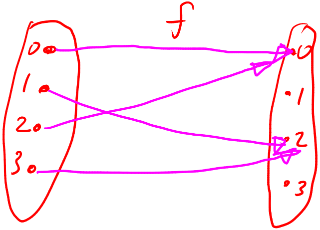
$$1 \not\equiv 3 \pmod{4}.$$

Multiplications as functions

$\underline{2 \cdot x}$

$x \longrightarrow \boxed{f} \longrightarrow 2x$

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \boxed{\bmod 4}$.

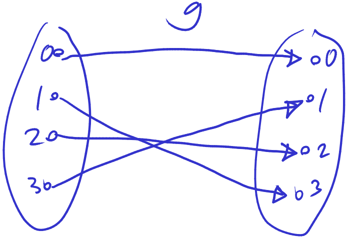


$2x \equiv 2 \pmod 4$

Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \bmod 4$.

Let's also see $g(x) = 3 \cdot x \bmod 4$.



$3x \equiv 2 \pmod 4$

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Which functions have inverses?

Multiplicative inverses (standard arithmetic)

In standard arithmetic, what is $2/5$?

Multiplicative inverses (standard arithmetic)

$5x = 2$

$x = 2/5$

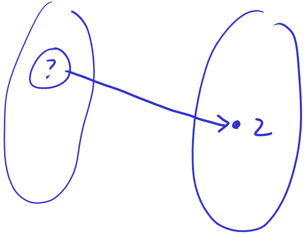
In standard arithmetic, what is $2/5$?

We are looking to a number x such that $2 = 5x$. How can we do that?

$\times 5$

$$x = 1 \cdot x = \left(\frac{1}{5}\right)5x = \left(\frac{1}{5}\right)2$$

$$x = (5^{-1})5x = (5^{-1})2$$



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By dividing on both sides with 5:

$$2/5 = 5x/5 = x,$$

or equivalently, by multiplying with $(1/5) = 5^{-1}$:

$$2 \cdot 5^{-1} = 5x \cdot 5^{-1} = x \cdot 5 \cdot 5^{-1} = x \cdot 1 = x.$$

Here 5^{-1} is a multiplicative inverse of 5.

Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be $m = 7$. Note that

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod{7}.$$

Therefore, $5^{-1} \equiv 3 \pmod{7}$.

$x = (5^{-1})5x = (5^{-1})2 = 6 \pmod{7}$

$x = \frac{2}{5}$ (m

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To find $2/5$, we can view our goal as to find the value of x such that

$$2 \equiv 5x \pmod 7.$$

We can multiply both sides with $5^{-1} \equiv 3$ to get

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$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod 7.$$

Let's check:

$$5 \cdot 6 \equiv 30 \equiv 2 \pmod 7,$$

as requied.

Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Multiplicative inverse modulo 11

Let's try to figure out multiplicative inverse of every integer modulo 1~~1~~7.

a	$a^{-1} \pmod{1\cancel{1}7}$
1	1
2	9
3	6
4	13
5	
6	3
7	
8	
9	2
10	
11	
12	
13	
14	
15	

$2 \times 9 \equiv 18 \equiv 1 \pmod{17}$

Examples: division in modular arithmetic

Suppose that we know that every non-zero integer a has an inverse modulo m.
Can you solve this equation?

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We can even perform gaussian elimination (*which is very useful later*):

$$\begin{aligned} 2x + y &\equiv 3 \pmod 7 \\ x + 3y &\equiv 5 \pmod 7 \end{aligned}$$

There are 3 clocks. At this moment, all three clocks ring at the same time. The first clock rings every 3 hours, the second clock rings every 4 hours, and the third clock rings every 10 hours. How long do you have to wait until you would hear all clocks ring a the same time again?

You have a large water container and two smaller buckets. The first bucket carries 3 litres of water and the second bucket carries 5 litres of water.
Can you put exactly 1 litre of water in the water container?

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.
What is the minimum volume of water you can exactly put in the water container?

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In general if you have two buckets of volumes x and y, the amount that you can exactly make must be in the form of

$$ax + by,$$

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for some integers x and y. (Note that x and y may be negative.)
Do you see why the sum must be divisible by any common divisor of x and y?

Useful fact

For any integer x and y, consider the term

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For any integer x and y, consider the term

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for some integer a and b.
When the term is non-zero, it must be divisible by $gcd(x, y)$, so it has to be at least $gcd(x, y)$.

It turns out that you can actually attain that value, i.e., there exist a pair of integer a and b such that

$$a \cdot x + b \cdot y = gcd(x, y).$$

Finding a and b: Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $gcd(x, y)$.

```
Algorithm Euclid(x,y):
  if x mod y == 0:

    return y,          ,
  else:
    g, a', b' = Euclid(y, x mod y)

    a =

    b =

    return g, a, b
```

Notes:

We have a' and b' such that

$$a' \cdot y + b' \cdot (x \bmod y) = g.$$

Theorem 1

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($\Leftarrow$) Recall that there exist integers x and y such that

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Thus, $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$, i.e., $x \cdot a \equiv 1 \pmod{m}$. Therefore x is the inverse.

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($\Rightarrow$) Let $r = gcd(a, m)$. Suppose that b is the multiplicative inverse of a modulo m, i.e., we have that

$$b \cdot a \equiv 1 \pmod{m},$$

Thus, $ba \bmod m = 1 \bmod m = 1$, i.e., there exists an integer q such that

$$ba = qm + 1,$$

or $ba - qm = 1$. However, r since $r|a$ and $r|m$, r also divides $bd - qm$ and 1. But it $r \not| 1$ because $r > 1$ and we have the contradiction. $\square$

Examples: division in modular arithmetic

Since the requirement for an existance of a^{-1} modulo m is that $gcd(a, m) = 1$, if we let m be a prime number, every a which is not a multiple of m has an inverse.
Can you solve this equation?

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Public-key cryptography

RSA

RSA: steps

- Private key: (d, n),   Public key: (e, n)
- Encryption $E(m) = m^e \mod n$, Decryption: $D(w) = w^d \mod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \mod n = m$.

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- Pick two primes p and q. Let $n = pq$.
- Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or

  $$ed = k \cdot (p-1)(q-1) + 1,$$

  for some integer k.
- What is $m^{ed} \bmod n$?

Secret sharing

Secret sharing scheme based on straight lines

Example: secret sharing

- Think of a secret number $m \in \{0, 1, \ldots, 10\}$.
- Pick a random number $a \in \{1, 2, \ldots, 10\}$.
- Your straight line function $f(x) = (ax + m) \bmod 11$.
- We will generate 3 points from f and give them to 3 of your friends, each with only 1 point. Pick 3 numbers x_1, x_2, x_3 from $\{1, 2, \ldots, 10\}$.
- Let's compute
$$(x_1, f(x_1)), \quad (x_2, f(x_2)), \quad (x_3, f(x_3)).$$
- Give them to 3 of your friends and challenge them to form a group of 2 people and figure out your number m.

What's next?

- We will prove Fermat's Little Theorem and show how to efficiently test if a number is prime.
- We will also use Fermat's Little Theorem to prove the correctness of RSA.
- Modular arithmetic is also key to our usage of polynomials to perform secret sharing and error correcting codes, because now we can do Gaussian elimination using only integers.