01204211 Discrete Mathematics
Lecture 13b: Eigenvalues and Eigenvectors

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Review: Hamming codes (1)

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GF(2)

The code is defined by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_3 + x_4 \\ x_1 + x_2 + x_4 \end{bmatrix}$$

Consider the encoding function $e : GF(2)^4 \to GF(2)^7$. Let $e(\boldsymbol{x}) = G\boldsymbol{x}$. What is Ker e? $= \{[0,0,0,0]\}$ dim Ker $e = 0$

"nullspace" ~~va~~ G

Review: Hamming codes (1)

$$\underbrace{\dim V}_{4} = \underbrace{\dim \mathrm{Im}\, e}_{4} + \underbrace{\dim \mathrm{Ker}\, e}_{0}$$

The code is defined by the generator matrix

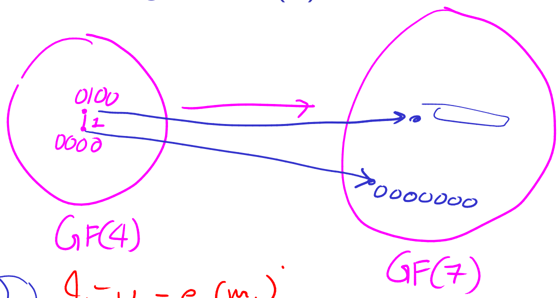
$$\begin{bmatrix}1\\1\\1\\0\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix} \qquad G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\ \\ \\ \\ \\1\\1\end{bmatrix}$$

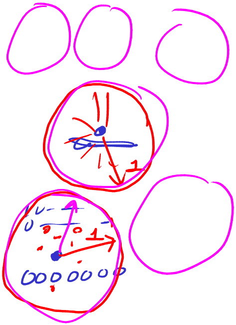
Consider the encoding function $e : GF(2)^4 \to GF(2)^7$. Let
$V \to W$
$e(\boldsymbol{x}) = G\boldsymbol{x}$. What is Ker e?
What is $\dim \mathrm{Im}\, e$?

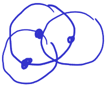
4

e is injective

Review: Hamming codes (2)

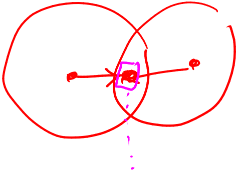






ให้ $w_1 = e(m_1)$

$w_2 = e(m_2)$ คู่ที่ระยะทางน้อยสุด



พิจารณา $w_1' = e(m_1 - m_1) = e(0) = 0$

$w_2' = e(m_2 - m_1) = e(m_2) - e(m_1)$

$= w_2 - w_1$

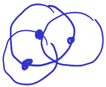
w_2	เหมือน	ต่าง
w_1		

$w_2 - w_1 =$ | 0 | 11111 |

Review: Hamming codes (2)

The code is defined by the generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

What can you say about the minimum "distance"?



ให้ $w_1 = e(m_1)$ $\quad - 7$

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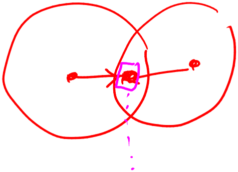

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$= w_2 - w_1$

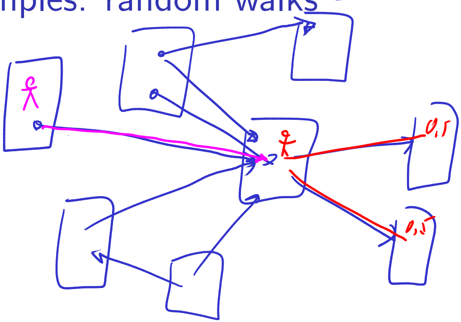
w_2 [ข้อมูล | [illegible]]

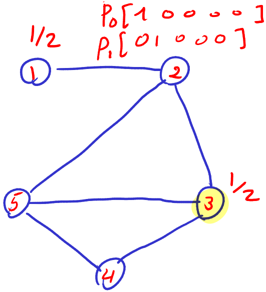
w_1 [|]

$w_2 - w_1 =$ [0 | 1 1 1 1 1]

Examples: random walks

google





$$M = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 3 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 4 & 0 & 0 & 1/2 & 0 & 1/2 \\ 5 & 0 & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

p_i

$$[\,1/2 \quad 0 \quad 1/2 \quad 0 \quad 0\,]$$

$$\downarrow$$

$$\tfrac{1}{2}[\,0 \quad 1 \quad 0 \quad 0 \quad 0\,]$$

$$+\tfrac{1}{2}[\,0 \quad 1/3 \quad 0 \quad 1/3 \quad 1/3\,]$$

ให้ p_i เป็น vector แทนความน่าจะเป็นที่อยู่ที่ page ต่างๆ ในเวลา i

$$\boxed{p_{i+1} = p_i M}$$

$\Leftarrow$

q stationary distribution

$$\boxed{q = qM}$$

Examples: differential equations (1)

$a=0.1$, $u_0 = 100$
$u_1 = u_0 + 0.1 u_0 = 110$
$u_2 = 110 + 11 = 121$
$u_3 = 121 + 12.1 = 1\ldots$

$u_t = \underbrace{(1.1)^t}_{\text{exponential.}} 100$

Let's start with a simple system with one variable.

$$\frac{du}{dt} = au,$$

$u = \boxed{e^{\alpha t}}$

with $u = u(0)$ when $t = 0$.

Examples: differential equations (2)

Now consider a system with two variables v and w:

$$\begin{aligned}\frac{dv}{dt} &= 4v - 5w \\ \frac{dw}{dt} &= 2v - 3w\end{aligned}$$

with $v = 5$ and $w = 4$ when $t = 0$,

Examples: differential equations (2)

$v = xe^{\lambda t}$
$w = ye^{\lambda t}$

Now consider a system with two variables v and w:

$$\begin{aligned}\frac{dv}{dt} &= 4v - 5w\\ \frac{dw}{dt} &= 2v - 3w\end{aligned}$$

with $v = 5$ and $w = 4$ when $t = 0$, or if we let $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ and

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix},$$

$\lambda x e^{\lambda t} = 4 e^{\lambda t} x - 5 e^{\lambda t} y$
$\lambda y e^{\lambda t} = 2 e^{\lambda t} x - 3 e^{\lambda t} y$

we have

$$\frac{du}{dt} = Au,$$

$\lambda x = 4x - 5y$
$\lambda y = 2x - y$

with $u(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\lambda [u] = [A][u]$

Examples: differential equations (3)

$$\boxed{\lambda u = A u}$$

$$q = q^M$$

$$\boxed{q = M q}$$

Eigenvalues and eigenvectors

Definition

For an n-by-n matrix A, a vector $\boldsymbol{v}$ is an **eigenvector** of A if

$$A\boldsymbol{v} = \lambda\boldsymbol{v},$$

and $\boldsymbol{v} \neq \mathbf{0}$. The scalar λ is called an **eigenvalue** associated with $\boldsymbol{v}$.

Example

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$.

If we let $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$A\boldsymbol{v}_1 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

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See demo in colab.

Invariant subspace

Definition (invariant subspace)

For an n-by-n matrix A, subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called an **invariant subspace** under linear map $f(\boldsymbol{x}) = A\boldsymbol{x}$ if for all $\boldsymbol{u} \in \mathcal{V}$, $f(\boldsymbol{u}) = A\boldsymbol{u} \in \mathcal{V}$.

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Eigenvector

If $\boldsymbol{v}$ is an eigenvector of matrix A, then

$$\mathrm{Span}\ \{\boldsymbol{v}\}$$

is a 1-dimensional invariant subspace under linear map defined by A.

Finding eigenvalues and eigenvectors

Given A, we want to find an eigenvalue λ and a vector $\boldsymbol{u} \neq \mathbf{0}$ such that

$$\underbrace{A\boldsymbol{u}} = \lambda \boldsymbol{u}.$$

$$Au - \lambda u = \cancel{0} \quad Au - \lambda I u$$

$$= (A - \lambda I)u = 0$$

Finding eigenvalues and eigenvectors

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After some writing, we want to solve this equation

$$(A - \lambda I)\boldsymbol{u} = 0,$$

where $\boldsymbol{u} \neq 0$.

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

$$A\boldsymbol{x} = \mathbf{0}.$$

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

$$A\boldsymbol{x} = \mathbf{0}.$$

Suppose that there exists $\boldsymbol{x} \neq \mathbf{0}$ that satisfies the equation, what can you say about A?

$$\cancel{A^{-1}} A x = 0 = x$$

$$\cancel{A^{-1}} A x = A^{-1} 0 = 0$$

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

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Clearly, A cannot have an inverse because no matrix B can bring $\boldsymbol{x}$ back from $A\boldsymbol{x} = 0$.

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Clearly, A cannot have an inverse because no matrix B can bring $\boldsymbol{x}$ back from $A\boldsymbol{x} = 0$. In this case, we say that A is **singular**.
Equavilent conditions:

- The rank of A is less than n.
- Rows of A are not linearly independent.
- The linear function $f(\boldsymbol{x}) = A\boldsymbol{x}$ is not injective.
- Ker $f \neq \{\mathbf{0}\}$.
- $\dim \text{Ker } f \neq 0$.

Finding λ

From this equation

$$(A - \lambda I)\boldsymbol{x} = \mathbf{0}.$$

Since we want it to have nonzero solution $\boldsymbol{x}$. Our goal is to find λ so that $A - \lambda I$ becomes singular.

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Typically, the tool to use is the **determinant**. However, we do not cover this topic in this class. We will look at small examples and consider an iterative method instead.

Example: 2×2 matrix

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We want to find λ such that

$[5-\lambda, 7] = \alpha[5, 3-\lambda]$

$$\alpha \quad \begin{bmatrix} 5-\lambda & 7 \\ 5 & 3-\lambda \end{bmatrix}$$

$$\frac{5-\lambda}{5} = \frac{7}{3-\lambda}$$

is singular.

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You can find associated eigenvectors by solving corresponding $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ equations.

Matrix multiplication (again)

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We know that A has two eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

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$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 - 2 \\ 5 + 2 \end{bmatrix} =$$

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$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7-2 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \left(\begin{bmatrix} 7 \\ 5 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

Matrix multiplication (again and again)

Fact: An n-by-n matrix A has n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. (They might not be real vectors.)

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Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis, for any vector $\boldsymbol{x}$ there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

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Let's multiply $\boldsymbol{x}$ with A:

$$\begin{aligned} A\boldsymbol{x} &= A(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n) \\ &= \underbrace{A\alpha_1 \boldsymbol{v}_1} + A\alpha_2 \boldsymbol{v}_2 + \cdots + A\alpha_n \boldsymbol{v}_n \\ &= \lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 - - - - \end{aligned}$$

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We can keep multiplying with A many times:

$$A^k \boldsymbol{x} =$$

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Let's multiply $\boldsymbol{x}$ with A:

$$\begin{aligned} A\boldsymbol{x} &= A(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n) \\ &= A\alpha_1 \boldsymbol{v}_1 + A\alpha_2 \boldsymbol{v}_2 + \cdots + A\alpha_n \boldsymbol{v}_n \\ &= \lambda_1 \alpha_1 \boldsymbol{v}_1 + \lambda_2 \alpha_2 \boldsymbol{v}_2 + \cdots + \lambda_n \alpha_n \boldsymbol{v}_n. \end{aligned}$$

We can keep multiplying with A many times:

$$A^k \boldsymbol{x} = \lambda_1^k \alpha_1 \boldsymbol{v}_1 + \lambda_2^k \alpha_2 \boldsymbol{v}_2 + \cdots + \lambda_n^k \alpha_n \boldsymbol{v}_n.$$

The power method

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$|\lambda_1| > |\lambda_i|,$$

for $i \neq 1$. We call λ_1 the **dominant eigenvalue**. We also call the eigenvectors corresponding to λ_1 **dominant eigenvectors**.

The power method (or power iteration)

- Start with a random vector $\boldsymbol{x}_{\mathbf{0}}$.
- For $i = 0, 1, \ldots, k$,
  Let $\boldsymbol{x}_{i+1} = A\boldsymbol{x}_i$, with probably some scaling.