01204211 Discrete Mathematics
Lecture 11b: Context-free languages and grammars (2)[1]

Jittat Fakcharoenphol

September 11, 2025

[1] Based on lecture notes of *Models of Computation* course by Jeff Erickson.

Review: Definition

A **context-free grammer** consists of the following components:

- a finite set Σ, a set of *symbols* (or *terminals*),
- a finite set Γ disjoint from Σ, a set of *non-terminals* (you can think of them as variables),
- a finite set R of *production rules* of the form $A \to w$ where $A \in \Gamma$ and $w \in (\Sigma \cup \Gamma)^*$ is a string of symbols and variable, and
- a *starting* non-terminal (usually the non-terminal of the first production rule).

Review: Applying the rules

If you have strings $x, y, z \in (\Sigma \cup \Gamma)^*$ and the production rule

$$A \to y,$$

You can apply the rule to the string xAz. This yields the string

$$xyz.$$

We use the notation

$$xAz \rightsquigarrow xyz$$

to describe this application.

Review: Derivation

We say that z derives from x if we can obtain z from x by production rule applications, denoted by $x \rightsquigarrow^* z$.
Formally, for any string $x, z \in (\Sigma \cup \Gamma)^*$, we say that $x \rightsquigarrow^* z$ if either

- $x = z$, or
- $x \rightsquigarrow y$ and $y \rightsquigarrow^* z$ for some string $y \in (\Sigma \cup \Gamma)^*$.

Review: $L(w)$

The *language* $L(w)$ of string $w \in (\Sigma \cup \Gamma)^*$ is the set of all strings in Σ^* that derive from w, i.e.,

$$L(w) = \{x \in \Sigma^* \mid w \rightsquigarrow^* x\}.$$

The language **generated by** a context-free grammar G, denoted by $L(G)$ is the language of its starting non-terminal.
A language L is **context-free** if there exists some context-free grammar G such that $L(G) = L$.

Review: Parse tree

- 00011

$$
\begin{aligned}
S &\rightarrow A \mid B \\
A &\rightarrow 0A \mid 0C \\
B &\rightarrow B1 \mid C1 \\
C &\rightarrow \varepsilon \mid 0C1
\end{aligned}
$$

Ambiguity

- $1 + 1 + 1 + 1 + 1$

$$S \quad \rightarrow \quad 1 \mid S + S \mid S * S$$

- A string w is **ambiguous** with respect to a grammar G if more than one parse tree for w exists.
- A grammar G is **ambiguous** if some string is ambiguous with respect to G.

More example

11011

$S \to \varepsilon \mid 0S0 \mid 1S1 \mid 0 \mid 1$

Palindrome in $\{0, 1\}^*$

More example

$S \to 11$

$L = \{ w \mid w = w^R,\ w \in \{0,1\}^* \}$

Palindrome in $\{0, 1\}^*$

Grammar $\boxed{G}$

$$S \rightarrow 0S0 \mid 1S1 \mid 1 \mid 0 \mid \varepsilon$$

อยากจะพิสูจน์ว่า $L(G) = L$

- $L(G) \subseteq L$
- $L \subseteq L(G)$

Consider the following grammar

$$S \longrightarrow \mathtt{0}S\mathtt{1} \mid \varepsilon$$

To show that

$$L(S) = \{\mathtt{0}^n\mathtt{1}^n \mid n \geq 0\},$$

we have to prove

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

To show that

$$L(S) = \{0^n1^n \mid n \geq 0\},$$

we have to prove

- $L(S) \supseteq \{0^n1^n \mid n \geq 0\}$, and

  ทุก string ที่หน้าตาตามรูปนี้ สร้างได้
- $L(S) \subseteq \{0^n1^n \mid n \geq 0\}$.

  ทุกๆ string ที่ generate ได้ หน้าตาตามรูปแบบนี้

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ for every non-negative integer n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ *for every non-negative integer* n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$. $0^0 1^0 = \varepsilon$ ต้องแสดงว่า $S \rightsquigarrow \varepsilon$ ได้โดยใช้ production rule $S \to \varepsilon$

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ *for every non-negative integer* n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$.
- Case 2: $n > 0$. จาก induction hypothesis ทราบ $S \rightsquigarrow^* 0^k 1^k$ ที่ $k = n-1$ ได้

เนื่องจาก $S \rightsquigarrow 0S1$ และ $S \rightsquigarrow^* 0^{n-1} 1^{n-1}$

จะได้ว่า $S \rightsquigarrow^* 0 0^{n-1} 1^{n-1} 1 = 0^n 1^n$ ตามต้องการ

เนื่องจากประโยคจริงในทุกกรณี, จะได้ว่า lemma จริง.

Consider the grammar $S \longrightarrow 0S1 \mid \varepsilon$.

Lemma 1

$S \rightsquigarrow^* 0^n 1^n$ *for every non-negative integer* n.

Proof.

Consider any non-negative integer n.
Induction Hypothesis: Assume that for every non-negative integer $k < n$, $S \rightsquigarrow^* 0^k 1^k$.
There are two cases to consider.

- Case 1: $n = 0$.
- Case 2: $n > 0$. From I.H., we know that

$$S \rightsquigarrow^* 0^{n-1} 1^{n-1}.$$

We can apply rule $S \longrightarrow 0S1$ to obtain $0^n 1^n$, i.e.,

$$S \longrightarrow 0S1 \rightsquigarrow^* 00^{n-1} 1^{n-1} 1 = 0^n 1^n.$$

In both cases, we conclude that $S \rightsquigarrow^* 0^n 1^n$, as required.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

แสดงให้เห็นว่า $\{0^n1^n \mid n \geq 0\} \subseteq L(S)$ เพื่อแสดงว่า $L(S) \subseteq \{0^n1^n \mid n \geq 0\}$

ซึ่งจะตรวจสอบ แสดงว่า ถ้า $S \Rightarrow^* y$, y อยู่ในรูป 0^n1^n

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.
Case 2: $w = 0x1$ for some $x \in L(S)$.

Consider the following grammar

$$S \longrightarrow 0S1 \mid \varepsilon$$

Lemma 2

$L(S) = \{0^n 1^n \mid n \geq 0\}$

Proof.

Consider any string $w \in L(S)$. We show that $w = 0^n 1^n$ for some non-negative integer n.
I.H.: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, $x = 0^k 1^k$ for some non-negative integer k.
There are 2 cases:
Case 1: $w = \varepsilon$.
Case 2: $w = 0x1$ for some $x \in L(S)$. Since $|x| = |w| - 2 < |w|$, we can apply I.H., and get that $x = 0^k 1^k$;thus $w = 00^k 1^k 1$, i.e., $w = 0^n 1^n$ where $n = k + 1$, as required. $\square$

Careful

- When using inductive proof, you have to ensure that each part of the string w is shorter than w.
- Consider this grammar
$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$
- When w is created by rule $S \to SS$, we know that $w = xy$ for $x, y \in L(S)$.
- Do we know that $|x| < |w|$ and $|y| < |w|$?

Careful

- When using inductive proof, you have to ensure that each part of the string w is shorter than w.
- Consider this grammar
$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$
- When w is created by rule $S \to SS$, we know that $w = xy$ for $x, y \in L(S)$.
- Do we know that $|x| < |w|$ and $|y| < |w|$?
- We can consider a **minimum-length derivation** in the proof to avoid this problem.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0,w) = \#(1,w)$, where $\#(a,w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0,x) = \#(1,x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because in that case we can shorten the derivation of w.

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because in that case we can shorten the derivation of w.

  From I.H.,

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because in that case we can shorten the derivation of w.

  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\#(0, w) \quad = \quad \#(0, x) + \#(0, y)$$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because in that case we can shorten the derivation of w.

  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\begin{aligned} \#(0, w) &= \#(0, x) + \#(0, y) \\ &= \#(1, x) + \#(1, y) \end{aligned}$$

Consider grammar $S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0$. For every string $w \in L(S)$, we have $\#(0, w) = \#(1, w)$, where $\#(a, w)$ is the number of occurrences of a in w.

Proof.

Consider $w \in L(S)$. Fix a minimum-length derivation of w.
Induction Hypothesis: Assume that for any string $x \in L(S)$ such that $|x| < |w|$, we have $\#(0, x) = \#(1, x)$.
There are four cases to consider, depending on the first production in this derivation.

- Case 1: The first production is $S \longrightarrow \varepsilon$.
- Case 2: The first production is $S \longrightarrow 0S1$. Case 3: The first production is $S \longrightarrow 1S0$.
- Case 4: The first production is $S \longrightarrow SS$. In this case $w = xy$ for some $x, y \in L(S)$. Since we assume the minimum-length derivation, x and y cannot be ε because in that case we can shorten the derivation of w.

  From I.H., we know that $\#(0, x) = \$(1, x)$ an $\#(0, y) = \#(1, y)$; thus,

$$\begin{aligned} \#(0, w) &= \#(0, x) + \#(0, y) \\ &= \#(1, x) + \#(1, y) = \#(1, w) \end{aligned}$$

In all cases, we conclude that $\#(0, w) = \#(1, w)$.

Examples: Not palindromes

Strings in $(0+1)^*$ that are not palindromes.

$$\begin{array}{rcl} S & \longrightarrow & 0S0 \mid 1S1 \mid 0Z1 \mid 1Z0 \\ Z & \longrightarrow & \varepsilon \mid 0Z \mid 1Z \end{array}$$

Why does this work?

Strings with the same number of 0s and 1s

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

We already show that every string in $L(S)$ contains the same number of 0s and 1s.
Why does it contain all possible required strings?

Strings in which the number of 0s is greater than or equal to the number of 1s

We can start with the previous grammar

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

And try to add more rules.

Strings in which the number of 0s is greater than or equal to the number of 1s

We can start with the previous grammar

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0.$$

And try to add more rules.

$$S \longrightarrow \varepsilon \mid SS \mid 0S1 \mid 1S0 \mid 0S \mid S0.$$

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid 0E1 \mid 1E0.$$

There are two cases.

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid \mathtt{0}E\mathtt{1} \mid \mathtt{1}E\mathtt{0}.$$

There are two cases.

$$S \longrightarrow O \mid I$$

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid 0E1 \mid 1E0.$$

There are two cases.

$$S \longrightarrow O \mid I$$

$$O \longrightarrow E0O \mid E0E$$

How about I?

Strings with different numbers of 0s and 1s

We can start with the previous grammar E of strings with equal number of 0 and 1.

$$E \longrightarrow \varepsilon \mid EE \mid 0E1 \mid 1E0.$$

There are two cases.

$$S \longrightarrow O \mid I$$

$$O \longrightarrow E0O \mid E0E$$

How about I?

$$I \longrightarrow E1I \mid E1E$$

Balanced parentheses

$$S \longrightarrow (S) \mid SS \mid \varepsilon$$

Balanced parentheses

$$S \longrightarrow (S) \mid SS \mid \varepsilon$$

$$S \longrightarrow (S)S \mid \varepsilon$$

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?
From inspection, we may guess that $L(S) = (\mathtt{01})^*$. But how can we prove that?

Mutual induction

Consider grammar

$$S \longrightarrow \mathtt{0}A\mathtt{1} \mid \varepsilon \qquad\qquad A \longrightarrow \mathtt{1}S\mathtt{0} \mid \varepsilon$$

What is $L(S)$?
From inspection, we may guess that $L(S) = (\mathtt{01})^*$. But how can we prove that?
To prove $L(S) = (\mathtt{01})^*$, we must also prove $L(A) = (\mathtt{10})^*$ *at the same time*.