01204211 Discrete Mathematics

Lecture 8a: Linear systems of equations

Jittat Fakcharoenphol

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solving equations & matrices ← linear algebra

↓

Vector spaces
- definition
- span
- linear combination
- independence
- dimension
- bases.

n

m

[redacted]

$m \times n$

$m + n$

1,000,000

60,000

$$
\begin{array}{c} \\ 750 \end{array}
\begin{array}{c} 1,200 \\ \begin{bmatrix} \quad \end{bmatrix} \end{array}
=
\begin{array}{c} \\ 750 \end{array}
\begin{array}{c} 30 \\ \begin{bmatrix} \quad \end{bmatrix} \end{array}
\begin{array}{c} \\ 30 \end{array}
\begin{array}{c} 1,200 \\ \begin{bmatrix} \quad \end{bmatrix} \end{array}
$$

LORA

Linear algebra

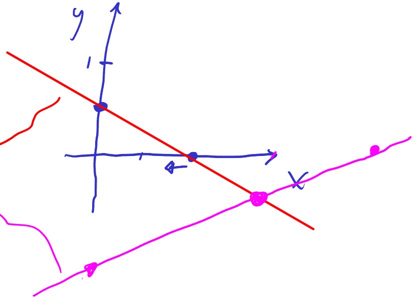
Linear algebra studies

- matrices and operations with matrices
- systems of linear equations
- linear transformations
- linear spaces (and their structures)

Why?

- Lots of applications.
- Interesting perspectives.

A linear system of equations

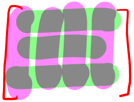


Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?

A linear system of equations



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How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) =$$

A linear system of equations

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Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 =$$

A linear system of equations

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$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

A linear system of equations

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How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

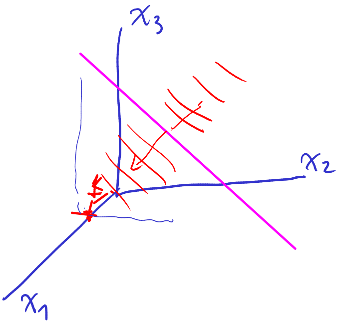
Then you can conclude that $y = -2$. Substitute it to one of the equation, you can find out the value of x.

Gaussian elimination (1)

Let's consider a system with 3 variables:

$\cancel{-2}\ \cancel{2\cdot 1} \times \boxed{4x_2 + 3x_3 = 7} - 2 = 5$

$$
\begin{array}{lrcrcrcl}
R1 & 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
R2 & x_1 & + & & & 5x_3 & = & 12 \\
R3 & 4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$



$R1 \leftarrow R1 \times 0{,}5$ $\qquad x_1 + 2x_2 + 1{,}5x_3 = 3{,}5$

$R2 \leftarrow R2 - R1$ $\qquad -2x_2 + 3{,}5x_3 = 8{,}5$

$R3 \leftarrow R3 - 4R1$ $\qquad -6x_2 - 3x_3 = -18$

$R2 \leftarrow R2(-0{,}5)$ $\qquad x_2 - 1{,}75x_3 = -4{,}25$

$R3 \leftarrow R3 + 6R2$

$\vdots$

Gaussian elimination (1)

Let's consider a system with 3 variables:

$$
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

$$
\underbrace{\begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 4 & 2 & 3 \end{bmatrix}}_{\text{Matrix}}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{Vector}}
=
\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}
$$

Gaussian elimination (1)

Let's consider a system with 3 variables:

$$\begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

-3

$$\begin{array}{r} 4 \\ 3 \\ 1{,}75 \times \\ 6 \\ \hline 10{,}50 \end{array}$$

$R_1/2$ $\begin{bmatrix} 1 & 2 & 1{,}5 \\ 1 & 0 & 5 \\ 4 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 1{,}5 \\ 0 & -2 & 3{,}5 \\ 4 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 4R_1} \begin{bmatrix} 1 & 2 & 1{,}5 \\ 0 & -2 & 3{,}5 \\ 0 & -6 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{-2}} \begin{bmatrix} 1 & 2 & 1{,}5 \\ 0 & 1 & -1{,}75 \\ 0 & -6 & -3 \end{bmatrix}$

$R_3 \leftarrow R_3 + 6R_2$ $\rightarrow \begin{bmatrix} 1 & 2 & 1{,}5 \\ 0 & 1 & -1{,}75 \\ 0 & 0 & -10{,}5 \end{bmatrix}$ rows are linear combinations

1 solution

Gaussian elimination (2)

Let's consider another system with 3 variables:

$$
\left\{
\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
3x_1 & + & 8x_2 & + & x_3 & = & 10
\end{array}
\right.
$$

$2R1 - R2$

$4x_1 + 8x_2 + 6x_3$

NO SOLUTIONS

$\boxed{3x_1 + 8x_2 + x_3} = 7 \times 2 - 12 = 2$

Gaussian elimination (2)

Let's consider another system with 3 variables:

$$\begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 5 \\ 3 & 8 & 1 \end{bmatrix}$$

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 3x_1 & + & 8x_2 & + & x_3 & = & \cancel{10} \; 2 \end{array}$$

$R_1 \leftarrow R_1/2$

$$\to \begin{bmatrix} 1 & 2 & 1,5 \\ 1 & 0 & 5 \\ 3 & 8 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 1,5 \\ 0 & -2 & 3,5 \\ 3 & 8 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 1,5 \\ 0 & -2 & 3.5 \\ 0 & 2 & -3,5 \end{bmatrix} \leftarrow R_2 \cdot (-1)$$

$R_3 \leftarrow R_3 + R_2$

$$\to \begin{bmatrix} 1 & 2 & 1,5 \\ 0 & -2 & 3,5 \\ 0 & 0 & 0 \end{bmatrix}$$

3

2 eq

zero

~~$0 \neq 0$~~ no soln

no soln

$\boxed{0 = 0}$ many solutions

Gaussian elimination ~~(2)~~ 3

Let's consider another system with 3 variables:

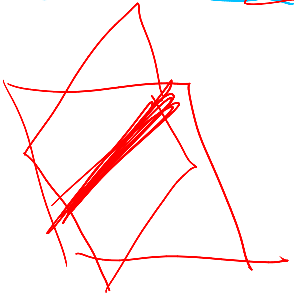
$$
\begin{cases}
2x_1 + 4x_2 + 3x_3 = 7 \\
x_1 + 5x_3 = 12 \\
3x_1 + 8x_2 + x_3 = \cancel{10}\ 2
\end{cases}
$$

$2R1 - R2$

1 unique solution

no solution

(infinitely) many solutions.



A closer look: 1st perspective

Consider

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

Each equation (row) constraints certain values of x and y.

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{rcl} (\quad 5, & 10 &) = \boldsymbol{u}_1 \\ (\quad 1, & -3 &) = \boldsymbol{u}_2 \end{array}$$

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{aligned}
(\quad 5, \quad 10 \quad) &= \boldsymbol{u}_1 \\
(\quad 1, \quad -3 \quad) &= \boldsymbol{u}_2 \\
(\quad 0, \quad 25 \quad) &= \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2
\end{aligned}$$

“Combining” two rows

Let’s focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{cccc} (& 5, & 10 &) & = \boldsymbol{u}_1 \\ (& 1, & -3 &) & = \boldsymbol{u}_2 \\ (& 0, & 25 &) & = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2 \end{array}$$

The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.

$(0, 1)$

“Combining” two rows

Let’s focus only on coefficients. This is how we obtain the third equation:

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The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.
Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

“Combining” two rows

Let’s focus only on coefficients. This is how we obtain the third equation:

$$
\begin{aligned}
(\quad 5, \quad 10 \quad) &= \boldsymbol{u}_1 \\
(\quad 1, \quad -3 \quad) &= \boldsymbol{u}_2 \\
\underline{(\quad 0, \quad \overset{2}{\cancel{1}}5 \quad)} &= \boxed{\boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2}
\end{aligned}
$$

The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.
Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?
Yes,

$$\boxed{0.2 \cdot \boldsymbol{u}_1 - \boldsymbol{u}_2 = (0, 1).}$$

It turns out that you can obtain any (a, b) from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A closer look: 1st perspective (more example)

Consider

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

What are the row vectors?

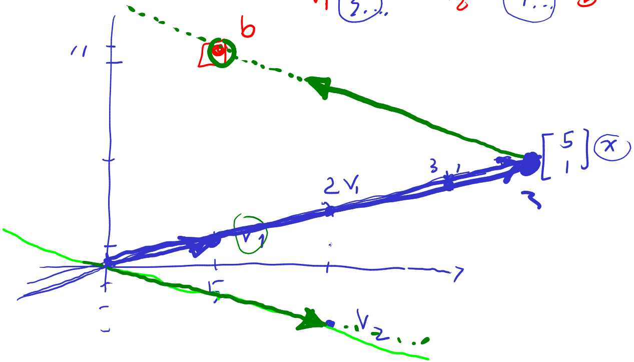
A closer look: 2nd perspective

$$\bar{v}_1 \cdot x + \bar{v}_2 \cdot y$$

linear combinat of v_1 & v_2

We rewrite the system as

$$\underset{\bar{v}_1}{\begin{bmatrix} 5 \\ 1 \end{bmatrix}} \cdot \underset{3..}{x} + \underset{\bar{v}_2}{\begin{bmatrix} 10 \\ -3 \end{bmatrix}} \cdot \underset{-1..}{y} = \underset{\bar{b}}{\begin{bmatrix} 5 \\ 11 \end{bmatrix}}$$



A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix}5\\1\end{bmatrix} \cdot x + \begin{bmatrix}10\\-3\end{bmatrix} \cdot y = \begin{bmatrix}5\\11\end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.
But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix}5\\1\end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix}10\\-3\end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix}5\\11\end{bmatrix}$$

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.
But if we change our focus to the vectors, we can see that we have 3 vectors:

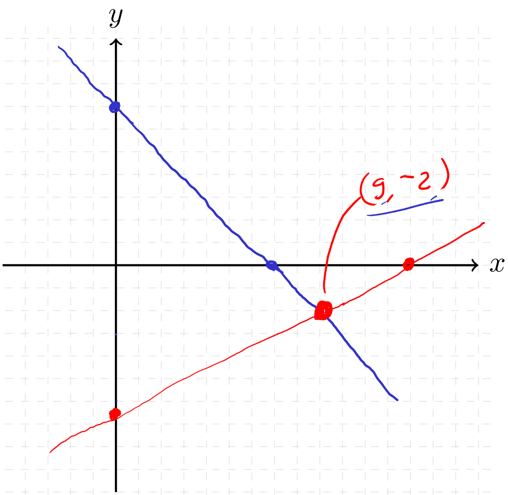
$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

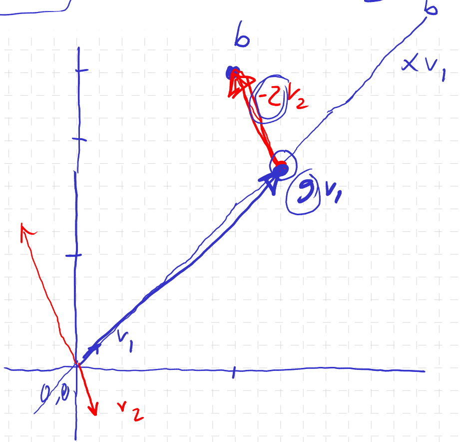
and with x and y, we now see that $\boldsymbol{b}$ is a **linear combination** of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
Finding x and y is essentially checking if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Example 2: a linear system with 2 variables

$$\begin{aligned} x + y &= 7 \\ x - 2y &= 13 \end{aligned}$$

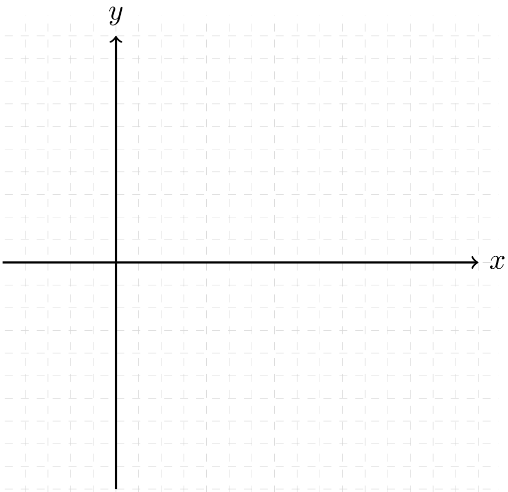
$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{v_1} \underbrace{x}_{=9} + \underbrace{\begin{bmatrix}1\\-2\end{bmatrix}}_{v_2} \underbrace{y}_{=-2} = \underbrace{\begin{bmatrix}7\\13\end{bmatrix}}_{b}$$





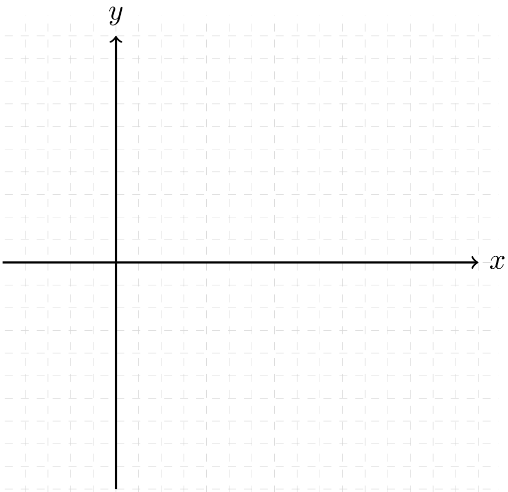
Example 3: a linear system with 2 variables

$$\begin{array}{rcrcr} 2x & + & y & = & 5 \\ 4x & + & 2y & = & 10 \end{array}$$



Example 4: a linear system with 2 variables

$$\begin{array}{rcrcl} x & + & 3y & = & 6 \\ 0.5 \cdot x & + & 1.5 \cdot y & = & 9 \end{array}$$



A linear system with 3 variables

Let's consider a system with 3 variables:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Row perspective

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Each equation becomes a **plane** in 3 dimensional space.

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

In other words, what are the possible linear combinations of

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

Column perspective

From

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$

we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Column perspective

From

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$

we rewrite the system as

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Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

In other words, the vector $\boldsymbol{b}$, for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 10 \end{array}$$

More example 2

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\ 5x_1 & + & 2x_2 & + & 8x_3 & = & 22 \end{array}$$

More failed example 3

Let's consider the last system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 2x_1 & + & & & 10x_3 & = & 24 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions.

More failed example 3 (cont.)

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$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions?

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions? It means that, for $\boldsymbol{u}$, you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{rcrcrcl} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{rcrcrcl} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \end{array}$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \end{array}$$

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{lllllll} \multicolumn{7}{l}{(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =} \\ \quad 2(u_1 - v_1) & + & 4(u_2 - v_2) & + & 3(u_3 - v_3) & = & (7 - 7) = 0 \\ \multicolumn{7}{l}{(u_1 + 5u_3) - (v_1 + 5v_3) =} \\ \quad (u_1 - v_1) & + & & & 5(u_1 - v_3) & = & (12 - 12) = 0 \end{array}$$

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ x_1 & + & & & 5x_3 & = & 0 \end{array}$$

More failed example 3 (cont. 2)

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It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

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It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.
It would play a central role when dealing with linear systems with many solutions.

Key take away

- There are 2 ways to look at how we solve linear systems: row perspective and column perspective.
- **Linear combination** is the main operation.