01204211 Discrete Mathematics
Lecture 5c: Counting 2

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Listing all subsets[1]

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- Let's try to enumerate them.

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    In this case, we order by their cardinalities, then use dictionary ordering for subsets with the same numbers of elements.

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- Also, this representation can be considered backwards, i.e., if we start with an integer 6, we can write down its binary representation: 110_2 and turns it into a subset $\{a, b\}$.

A correspondence

Let's see a full list of correspondence between $\{0, 1, 2, \ldots, 7\}$ and subsets of $\{a, b, c\}$.

- $0 \leftrightarrow 000_2 \leftrightarrow \{\}$
- $1 \leftrightarrow 001_2 \leftrightarrow \{c\}$
- $2 \leftrightarrow 010_2 \leftrightarrow \{b\}$
- $3 \leftrightarrow 011_2 \leftrightarrow \{c, b\}$
- $4 \leftrightarrow 100_2 \leftrightarrow \{a\}$
- $5 \leftrightarrow 101_2 \leftrightarrow \{a, c\}$
- $6 \leftrightarrow 110_2 \leftrightarrow \{a, b\}$
- $7 \leftrightarrow 111_2 \leftrightarrow \{a, b, c\}$

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Do you notice anything interesting?

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This is another proof of the following theorem:

Theorem: The number of subsets of a set with n elements is 2^n.

Two proofs

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Really, it does not make a statement stronger, truer, “more” correct. But each proof usually reveals additional facts related to the statement.

- The first proof considers a procedure for constructing subsets.
- The second proof introduces a nice technique for counting. I.e., instead of counting subsets directly, we show that we have a “special” correspondence between subsets and binary numbers, and then just count the numbers.

A bijection

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What is so special about this correspondence?

- For each number, there is exactly **one** subset that corresponds to it.
- For each subset, there is exactly **one** number that it corresponds to.

With these two properties, we can conclude that both sets have the same cardinality.

> This type of correspondence is called a **one-to-one correspondence** or **bijection**.

Sequences of choices

Previously, when we want to count the number of bit strings of length n, we use this argument:

> Suppose that to select an object, you have to make k decisions. The first decision has n_1 choices, the second decision has n_2 choices, and so on. More precisely, for $1 \leq i \leq k$, the i-th decision has n_i choices. Then the number of ways you can select an object is $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$.

Example 1

A car license number consists of two English letters and one number from 1 to 9999. How many possible license numbers are there?

Example 2

10 students stand in a line. You want to give them ice cream. There are 4 flavours, but you don't want to give the same flavour to any consecutive students. In how many ways can you give out the ice cream to these students?

Permutations

Counting permutations: an example

We want to count the number of permutations. Let's try with a small example: permutations of set $\{a, b, c\}$.

Counting permutations

Number of permutations

We have proved this theorem.

Theorem: The number of permutations of a set with n elements is $n!$.