

01204211 Discrete Mathematics
Lecture 9c: Linear Independence and Bases

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Previous Lemmas

Lemma 1

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are generators for \mathcal{V} , and for each i ,

$$\mathbf{v}_i \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\},$$

we have that $\mathcal{V} \subseteq \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$.

Lemma 2

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{v} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$, then

$$\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

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Lemma 3

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{u}_n \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\}$, then

$$\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

Proof of Lemma 2.

Since \mathbf{v} can be written as a linear combination of other vectors, there exist $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n.$$

Consider any vector $\mathbf{w} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\}$; thus, we can write

$$\mathbf{w} = \beta_0 \mathbf{v} + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_n \mathbf{u}_n.$$

Plugging in \mathbf{v} , we get that

$$\begin{aligned}\mathbf{w} &= \beta_0 (\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n) + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_k \mathbf{u}_k \\ &= (\beta_0 \alpha_1 + \beta_1) \mathbf{u}_1 + (\beta_0 \alpha_2 + \beta_2) \mathbf{u}_2 + \cdots + (\beta_0 \alpha_n + \beta_n) \mathbf{u}_n,\end{aligned}$$

implying that $\mathbf{w} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$.



Linearly independence

Definition

Vectors u_1, u_2, \dots, u_n are **linearly independent** if no vector u_i can be written as a linear combination of other vectors.

Linearly independence

Definition

Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **linearly independent** if no vector \mathbf{u}_i can be written as a linear combination of other vectors.

(Another) Definition

Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **linearly independent** if the only solution of equation

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Examples in \mathbb{R}^2

Examples in \mathbb{R}^3

Examples in $GF(2)$

Examples in linear systems

Subset of linearly independent vectors

Lemma 4

If $A = \{u_1, u_2, \dots, u_n\}$ be a set of linearly independent vectors, then any $B \subseteq A$ is also a set of linearly independent vectors.

Proof.

We prove by contradiction. Assume that B is **not** linearly independent. Without loss of generality, assume that $B = \{u_1, u_2, \dots, u_k\}$ where $k \leq n$.

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$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_k \mathbf{u}_k = \mathbf{0},$$

and some α_i 's is nonzero.

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$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_k \mathbf{u}_k = \mathbf{0},$$

and some α_i 's is nonzero. If we let $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$, we have that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0},$$

with some α_i 's being nonzero as well.

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and some α_i 's is nonzero. If we let $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$, we have that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0},$$

with some α_i 's being nonzero as well. This implies that vectors in A are not linearly independent; leading to a contradiction. □

Definition

A set of vectors $\{u_1, u_2, \dots, u_k\}$ is a **basis** for vector space \mathcal{V} if

- ▶ $\text{Span } \{u_1, u_2, \dots, u_k\} = \mathcal{V}$, and
- ▶ u_1, u_2, \dots, u_k are linearly independent.

Examples 1: \mathbb{R}^2 and \mathbb{R}^3

Examples 2

Lemma 5 (Unique representation)

Let u_1, u_2, \dots, u_k be a basis for vector space \mathcal{V} . For any $v \in \mathcal{V}$, there is a unique way to write v as a linear combination of u_1, \dots, u_k .

Proof of unique representation lemma.

We prove by contradiction.

Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector $v \in \mathcal{V}$ with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \dots, \alpha_k,$$

and

$$\beta_1, \beta_2, \dots, \beta_k,$$

that are not equal (i.e., there exists i where $\alpha_i \neq \beta_i$) such that $v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k$ and $v = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_k u_k$.

Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector $\mathbf{v} \in \mathcal{V}$ with more than one ways to be written as linear combinations of the basis. Thus, there exist

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that are not equal (i.e., there exists i where $\alpha_i \neq \beta_i$) such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k$ and $\mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k$. This implies that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k = \mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_k \mathbf{u}_k,$$

and

$$(\alpha_1 - \beta_1) \mathbf{u}_1 + (\alpha_2 - \beta_2) \mathbf{u}_2 + \dots + (\alpha_k - \beta_k) \mathbf{u}_k = 0.$$

Proof of unique representation lemma.

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and

$$(\alpha_1 - \beta_1) \mathbf{u}_1 + (\alpha_2 - \beta_2) \mathbf{u}_2 + \dots + (\alpha_k - \beta_k) \mathbf{u}_k = 0.$$

Since $\alpha_i \neq \beta_i$, we have that at least one of the coefficients is non-zero, implying that $\mathbf{u}_1, \dots, \mathbf{u}_k$ are not linearly independent. This contradicts the assumption that $\mathbf{u}_1, \dots, \mathbf{u}_k$ form a basis. □