01204211 Discrete Mathematics
Lecture 7c: Binomial Coefficients (3)

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August 7, 2021

The binomial coefficients[1]

In this lecture, we discuss advanced counting with binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

More on counting

We shall see more techniques for counting when we consider the following problems.

- How many anagrams does the word "KASETSARTUNIVERSITY" have? (They do not have to be real English words.)
- How can you give out n different presents to k students when student i has to get n_i pieces of presents?
- How many ways can you distribute n baht coins to k children?

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$4 \cdot 3 \cdot 2 \cdot 1$

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Harder anagrams

- How many anagrams does "ABC_1C_2" have? Is it $4!$?

Harder anagrams

- หยิบให้พอดี
- หยิบซ้ำ - จัดการกับตัวซ้ำ

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.

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  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?

$$\left.\begin{array}{l} C_1ABC_2 \\ C_2ABC_1 \end{array}\right\} \text{2 times}$$

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  - Since each anagram is counted in $4!$ twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

General anagrams

$3!2!$

$L_1 \; L_2 \; L_3$

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

The number of permutation of alphabets in $HELLOWORLD$, treating each character differently is $10!$. However, each anagram is counted for $3!2!$ times because of the 3 copies of L and the 2 copies of O. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

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- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents.

$$\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{9!}{\cancel{6!}\,3!} \cdot \frac{\cancel{6!}}{3!\,3!} \cdot \frac{\cancel{3!}}{\cancel{3!}\,1!}$$

$$= \frac{9!}{3!\,3!\,3!}$$

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- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.

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- Thus, the number of ways we can distribute presents is

$$\frac{9!}{3!3!3!}$$

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.

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- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

Presents	1	2	3	4	5	6	7	8	9
Children	A	B	A	B	C	B	C	A	C

anagram

to AAABBBCCC

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- This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, we reach the same solution of

$$\boxed{\frac{9!}{3!3!3!}.}$$

Distributing identical presents

Now suppose that I have 9 identical presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Note that when we state that the presents are identical, we mean that we do not distinguish them, i.e., the first present and the second present are indistinguishable.

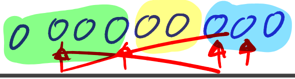
$$\left.\begin{array}{l} A - 3 \\ B - 3 \\ C - 3 \end{array}\right\} 1 \text{ way.}$$

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins.

Distributing coins (1)



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Thus, in how many ways can we do that?

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This is a fairly surprising use of binomial coefficients.

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There are $\binom{n-1}{k-1}$ ways to distribute n identical coins to k children so that each child get at least one coin.

Distributing coins (4)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it, given that some student may not get any coins?

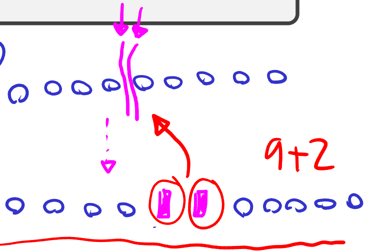
① - แจกเหรียญ โดยที่ทุกคนได้ อย่างน้อย 1 เหรียญ

- แจกเสร็จ – เก็บคืนคนละเหรียญ

ต้องใช้ 12 เหรียญ

$$\binom{11}{2}$$

②



9+2

$$\binom{9+2}{2} = \binom{11}{2}$$