

01204211 Discrete Mathematics  
Lecture 4c: Mathematical Induction 3

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## Review: Mathematical Induction

Suppose that you want to prove that property  $P(n)$  is true for every natural number  $n$ .

Suppose that we can prove the following two facts:

**Base case:**  $P(1)$

**Inductive step:** For any  $k \geq 1$ ,  $P(k) \Rightarrow P(k + 1)$

The **Principle of Mathematical Induction** states that  $P(n)$  is true for every natural number  $n$ .

The assumption  $P(k)$  in the inductive step is usually referred to as **the Induction Hypothesis**.

# The Induction Hypothesis

## Theorem 1

For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$ .

## Proof.

The statement  $P(n)$  that we want to prove is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.$$

**Case case:** For  $n = 1$ , the statement is true because  $1 < 2$ .

**Inductive step:** For  $k \geq 1$ , let's assume  $P(k)$  and we prove that  $P(k+1)$  is true.

The induction hypothesis is:  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2$ .

We want to show  $P(k+1)$ , i.e.,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2.$$

Then...



# Strengthening the Induction Hypothesis (1)

- Is the assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 2.$$

“strong” enough to prove

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 \quad ?$$

Why?

- To prove  $P(k+1)$ , we need a “gap” between the LHS and 2, so that we can add  $1/(k+1)$  without blowing up the RHS.

## Strengthening the Induction Hypothesis (2)

- ▶ Let's see a few values of the sum:
  - ▶  $1/1 = 1.$
  - ▶  $1/1 + 1/4 = 1.25.$
  - ▶  $1/1 + 1/4 + 1/9 \approx 1.361.$
  - ▶  $1/1 + 1/4 + 1/9 + 1/16 \approx 1.4236.$
  - ▶  $1/1 + 1/4 + 1/9 + 1/16 + 1/25 \approx 1.4636.$

Yes, there is a gap. But how large?

- ▶ We need the gap to be large enough to insert  $1/(k+1)^2$ .
- ▶ After a “mysterious” moment, we observe that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

## Strengthening the Induction Hypothesis (3)

### Theorem 2

For any integer  $n \geq 1$ ,  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ .

### Proof.

(... the beginning is left out ...)

**Inductive step:** For  $k \geq 1$ , assume that  $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ .

Adding  $1/(k+1)^2$  on both sides, we get

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left( \frac{1}{k} - \frac{1}{(k+1)^2} \right).$$

Since  $1/k - 1/(k+1) = 1/(k(k+1))$ , we have that

$$1/(k+1) = 1/k - 1/(k(k+1)) < 1/k - 1/(k+1)^2.$$

Therefore, we conclude that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \left( \frac{1}{k} - \frac{1}{(k+1)^2} \right) \leq 2 - \frac{1}{k+1},$$

as required. □

## A Lesson learned

- ▶ Is a stronger statement easier to prove?
- ▶ In this case, the statement is indeed stronger, but the induction hypothesis gets stronger as well. Sometimes, this works out nicely.