

01204211 Discrete Mathematics  
Lecture 2a: Quantifiers

Jittat Fakcharoenphol

July 3, 2021

# Review (1)

- ▶ A *proposition* is a statement which is either **true** or **false**.
- ▶ We can use variables to stand for propositions, e.g.,  $P =$  "today is Tuesday".
- ▶ We can use connectives to combine variables to get propositional forms.
  - ▶ **Conjunction:**  $P \wedge Q$  (" $P$  and  $Q$ "),
  - ▶ **Disjunction:**  $P \vee Q$  (" $P$  or  $Q$ "), and
  - ▶ **Negation:**  $\neg P$  ("not  $P$ ")
  - ▶ **Implication:**  $P \Rightarrow Q$  (" $P$  implies  $Q$ ", "if  $P$ , then  $Q$ ", " $P$ , only if  $Q$ ")
  - ▶ **Equivalence:**  $P \Leftrightarrow Q$  (" $P$  if and only if  $Q$ ")

## Review (2): Testing primes

Consider the following code.

```
Algorithm CheckPrime2(n): // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
    while i <= s:
        if n is divisible by i:
            return False
        i = i + 1
    return True
```

How fast can it run? Note that  $s = \sqrt{n}$ ; therefore, it takes time approximately proportional to  $\sqrt{n}$  to run.

Ok, it should be faster. **But is it correct?**

# The goals

- ▶ Let's recall what we are trying to do.

**Original goal:** To show that Algorithm CheckPrime2 is correct.

**Current (sub) goal:** Consider a positive composite  $n$  and its positive divisor  $a$ , where  $a > \sqrt{n}$ . Let  $b = n/a$ . We want to show that  $2 \leq b \leq \sqrt{n}$ .

## The (sub) goal

- ▶ **Current (sub) goal:** Consider a positive composite  $n$  and its positive divisor  $a$ , where  $a > \sqrt{n}$ . Let  $b = n/a$ . We want to show that  $2 \leq b \leq \sqrt{n}$ .
- ▶ We can be more specific about what values of  $n$  and  $b$  that we want to consider.

### Revised statement

For all positive composite integer  $n$ , and for every divisor  $a$  of  $n$  such that  $\sqrt{n} < a < n$ ,

$$2 \leq b \leq \sqrt{n},$$

where  $b = n/a$ .

- ▶ Note that this revised statement is now “quantified,” that is, every variable in the statement has specific scope. Now the statement is either true or false.

# Predicates

- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ $x$  is even,” “ $p$  is prime,” or “ $s$  is a student.”
- ▶ As we previously did with propositions, we can use variables to represent these statements. E.g.,
  - ▶ let  $E(x) \equiv$  “ $x$  is even”,
  - ▶ let  $P(y) \equiv$  “ $y$  is prime, and
  - ▶ let  $S(w) \equiv$  “ $w$  is a student.

We call  $E(x)$ ,  $P(y)$ , and  $S(w)$  *predicates*. (You can think of predicates as statements that may be true or false depending on the values of its variables.)

## Quantifiers (1)

- ▶ As we note before, these predicates are not propositions. But if we know the values of their variables, then they become propositions. For example, if we let  $x = 5$ , then  $E(5)$  is a proposition which is false. Also,  $P(7)$  is true.
- ▶ Since the truth values of predicates depend on the assignments of their variables, we can put *quantifiers* to specify the scopes of these variables and how to interpret the truth values of the predicates over these values.

## Quantifiers (2): universal quantifiers

- ▶ Let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $E(2)$ ,  $E(4)$ ,  $E(6)$ , and  $E(8)$  are true, i.e.,  $E(x)$  is true for every  $x \in A$ .

In this case, we say that the following proposition is true:

$$(\forall x \in A)E(x).$$

- ▶ The quantifier  $\forall$  is called a universal quantifier. (We usually pronounce “for all  $x$ ”, or “for every  $x$ .”)



## Quantifiers (3): existential quantifiers

- ▶ Again, let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $P(2)$  is true. This means that  $P(y)$  is true for some  $y \in A$ .

In this case, we say that the following proposition is true:

$$(\exists y \in A)P(y).$$

- ▶ The quantifier  $\exists$  is called an existential quantifier. (We usually pronounce “for some  $x$ ”, or “there exists  $x$ .”)

When the universe  $A$  is clear, we can leave it out and just write  $\forall x E(x)$  or  $\exists y P(y)$ .

# The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input  $n$  is an integer. Thus, we are interested in every  $n \in \mathbb{Z}$ , where  $\mathbb{Z}$  denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where  $C(n) \equiv$  "CheckPrime2( $n$ ) returns True", and  
 $P(n) \equiv$  " $n$  is a prime."

## Quantified propositions with more than one variables

Let our universe be integers ( $\mathbb{Z}$ ). Which of the following statements is true?

- ▶  $\forall x \forall y (x = y)$
- ▶  $\forall x \exists y (x = y)$
- ▶  $\exists x \forall y (x = y)$
- ▶  $\exists x \exists y (x = y)$

When you have many quantifiers, we can interpret the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

$$\forall y \exists x P(x, y) \equiv \forall y (\exists x (P(x, y))).$$

Also note that usually,  $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$ .

## Quick check 4

We will consider the universe to be “everything”. Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.

## Quick check 5

- ▶ Let's consider the current subgoal. (Note that in this version, variable  $b$  is replaced with  $n/a$ .)

Another revised statement

For all positive composite integer  $n$ , and for every divisor  $a$  of  $n$  such that  $\sqrt{n} < a < n$ ,

$$2 \leq n/a \leq \sqrt{n}.$$

- ▶ Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

## Negations of quantified propositions (1)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $P(x) \equiv$  “ $x$  is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since  $P(4)$  is false,  $\forall x P(x)$  is false.

This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$

## Negations of quantified propositions (2)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $Q(x) \equiv$  “if  $x > 2$ , then  $x^2 \leq 2x$ .”

Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

When showing that an existential quantified proposition is false, we need to show that  $Q(x)$  is false for every possible values of  $x$ . In this case, since  $x^2 = x \cdot x > 2 \cdot x$  for every  $x > 2$ , we have that  $(\exists x)Q(x)$  is false.

This way of disproving a statement is equivalent to showing that

$$(\forall x)(\neg Q(x)).$$

## Negations of quantified propositions (3)

Thus, the following equivalences:

▶  $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$

▶  $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$



## Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.