

01204211 Discrete Mathematics

Lecture 10b: Dimensions

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Review: Vector spaces

Definition

A set \mathcal{V} of vectors over \mathbb{F} is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\mathbf{u} \in \mathcal{V}$,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- (V3) for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Review: Linearly independence

Definition

Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **linearly independent** if no vector \mathbf{u}_i can be written as a linear combination of other vectors.

(Another) Definition

Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are **linearly independent** if the only solution of equation

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Definition

A set of vectors $\{u_1, u_2, \dots, u_k\}$ is a **basis** for vector space \mathcal{V} if

- ▶ $\text{Span } \{u_1, u_2, \dots, u_k\} = \mathcal{V}$, and
- ▶ u_1, u_2, \dots, u_k are linearly independent.

Lemma 1 (Superfluous Vector Lemma)

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{v} \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$, then

$$\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

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Lemma 2

Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$. If $\mathbf{u}_n \in \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\}$, then

$$\text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}\} = \text{Span } \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$$

Lemma 3 (Unique representation)

Let u_1, u_2, \dots, u_k be a basis for vector space \mathcal{V} . For any $v \in \mathcal{V}$, there is a unique way to write v as a linear combination of u_1, \dots, u_k .

Examples in \mathbb{R}^2 and \mathbb{R}^3

Examples in $GF(2)$ - Vector spaces from graphs

Examples in $GF(2)$ - Cycles

Examples in $GF(2)$ - Basis

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- ▶ We have an observation that for a vector space \mathcal{V} , every basis has the same size.
- ▶ This is not a coincident.
- ▶ In this course, we will see two proofs.
- ▶ Remark: there are vector spaces whose basis has infinite size, but we are not dealing with those vector spaces in this course.

Theorem 4 (Main Theorem)

If u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_m are bases for vector space \mathcal{W} , then $n = m$.

Exchange Lemma

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Lemma 5 (Simplified Exchange Lemma)

Consider a set of vectors S and let z be a non-zero vector in $\text{Span } S$. There is a vector $w \in S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

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Lemma 6 (Exchange Lemma)

Consider a set of vectors S and its subset A . Let z be a non-zero vector in $\text{Span } S$ such that $A \cup \{z\}$ is linearly independent. There is a vector $w \in S - A$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

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If a set of vectors S spans a vector space \mathcal{W} and B is a linearly independent set of vectors in \mathcal{W} , then $|B| \leq |S|$.

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Proof.

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$. We show how to construct S_1, \dots, S_n such that for each i , $|S_i| = |S|$, $\text{Span } S_i = \text{Span } S$, and

$$\{\mathbf{u}_1, \dots, \mathbf{u}_i\} \subseteq S_i.$$

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$$\text{Span}(S_{i-1} \cup \{\mathbf{u}_i\} - \{\mathbf{w}\}) = \text{Span } S_{i-1}.$$

We then let $S_i = S_{i-1} \cup \{\mathbf{u}_i\} - \{\mathbf{w}\}$. (You can check that S_i has the properties as claimed above.)

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Morphing Lemma \Rightarrow Main Theorem

Theorem 8 (Main Theorem)

If u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_m are bases for vector space \mathcal{W} , then $n = m$.

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Therefore, $n = m$.



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Therefore we can let $w = u_i$ and the lemma follows. □

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Therefore we can let $\mathbf{w} = \mathbf{u}_i$ and the lemma follows. □

How can we prove the full lemma?

Dimensions

Definition

The **dimension** of a vector space \mathcal{V} is the size of its basis.

The dimension of \mathcal{V} is written as $\dim \mathcal{V}$.