01204211 Discrete Mathematics
Lecture 8a: Linear systems of equations

Jittat Fakcharoenphol

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A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?

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Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) =$$

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How would you solve it?
Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

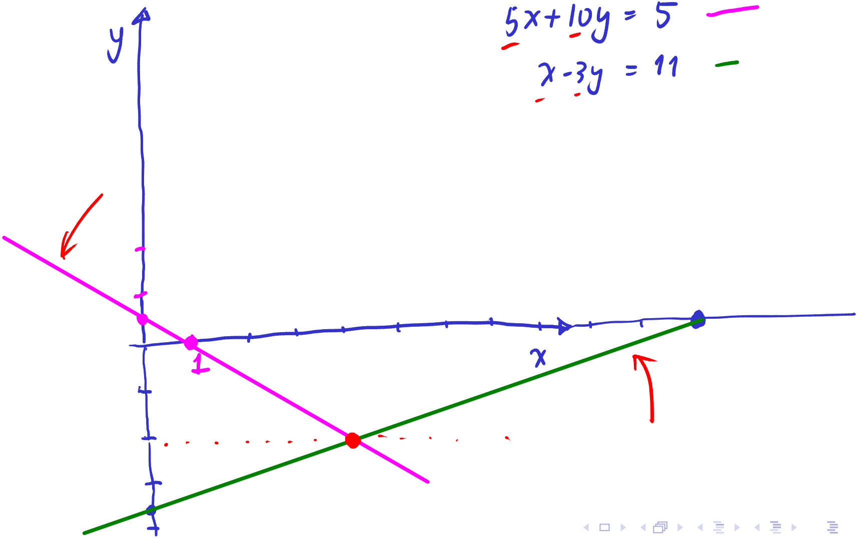
$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = \; - 50$$

Then you can conclude that $y = -2$. Substitute it to one of the equation, you can find out the value of x.

A closer look: 1st perspective

Each equation (row) constraints certain values of x and y.

$$5x + 10y = 5$$
$$x - 3y = 11$$



“Combining” two rows

Let’s focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{cccc} (& 5, & 10 &) & = \boldsymbol{u}_1 \\ (& 1, & -3 &) & = \boldsymbol{u}_2 \end{array}$$

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(\; 5, \quad 10 \;) & = \boldsymbol{u}_1 \\
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(\; 0, \quad 25 \;) & = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2
\end{array}$$

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Let's focus only on coefficients. This is how we obtain the third equation:

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The third equation is a "combination" of the other two rows. In fact, it is a **linear combination** of the first two.

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Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

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The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.

Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

Yes,

$$0.\cancel{2} \cdot \boldsymbol{u}_1 - \boldsymbol{u}_2 = (0, 1).$$

It turns out that you can obtain any (a, b) from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

A closer look: 2nd perspective

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$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this "vector" equation.

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Now, the goal is to find x and y satisfying this "vector" equation. But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1=\begin{bmatrix}5\\1\end{bmatrix},\quad \boldsymbol{v}_2=\begin{bmatrix}10\\-3\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}5\\11\end{bmatrix}$$

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We rewrite the system as

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Now, the goal is to find x and y satisfying this "vector" equation.
But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and with x and y, we now see that $\boldsymbol{b}$ is a **linear combination** of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
Finding x and y is essentially checking if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

A linear system with 3 variables

Gaussian Elimination

Let's consider a system with 3 variables:

$v_1 = (2, 4, 3)$

$$
\begin{array}{lccccccl}
v_1 & 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
v_2 & x_1 & + & & & 5x_3 & = & 12 \\
v_3 & 4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

$v_2 - 0{,}5v_1$

$\parallel$

$(0, -2, 3{,}5)$

$\cancel{0x_1} - 2x_2 + 3{,}5x_3 = 12 - 3{,}5 = 8.5$

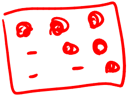
$v_3 - 2v_1$

$(0, -6, -3)$

$0x_1 - 6x_2 - 3x_3 = 10 - 14 = -4$

$0x_2 - 13{,}5x_3 = -4 - 3 \cdot 8{,}5$

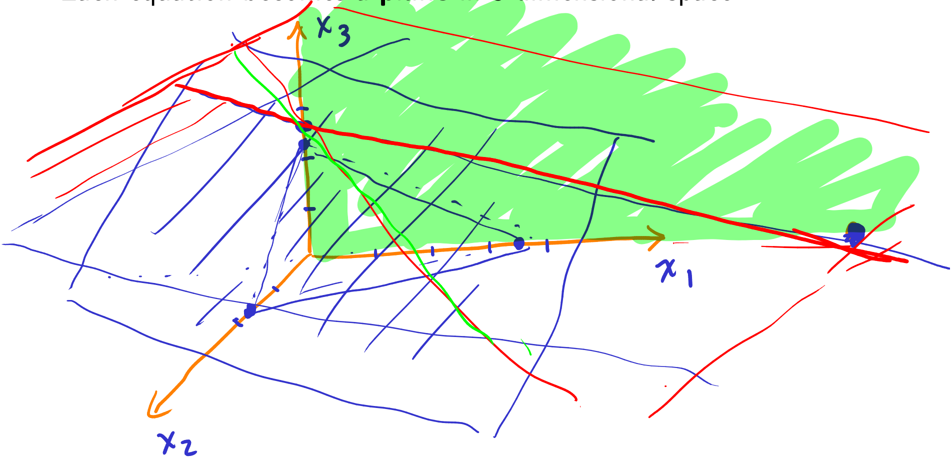
$(0, 0, -13.5)$



Row perspective

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Each equation becomes a **plane** in 3 dimensional space.



Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

Row perspective: the goal of Gaussian Elimination

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$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

In other words, what are the possible linear combinations of

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

Column perspective

From
$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$
we rewrite the system as
$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$
Our goal is to find a way to linear combine 3 vectors to obtain
$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

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Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

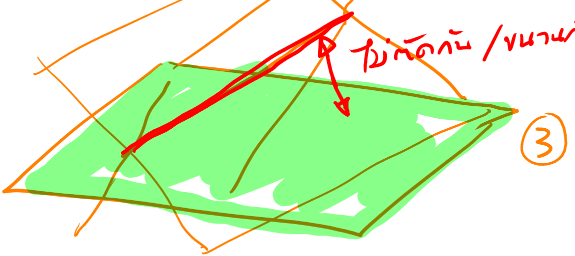
In other words, the vector $\boldsymbol{b}$, for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \quad -① \\ x_1 & + & & & 5x_3 & = & 12 \quad -② \\ 3x_1 & + & 8x_2 & + & x_3 & = & 10 \quad -③ \end{array}$$

$2 \times ① - ②$ $\quad 3x_1 + 8x_2 + x_3 = 14 - 12 = 2$



More example 2

Let's consider another system with 3 variables:

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\
\cancel{5x_1} & \cancel{+} & \cancel{2x_2} & \cancel{+} & \cancel{8x_3} & \cancel{=} & \cancel{22}
\end{array}
$$

inf

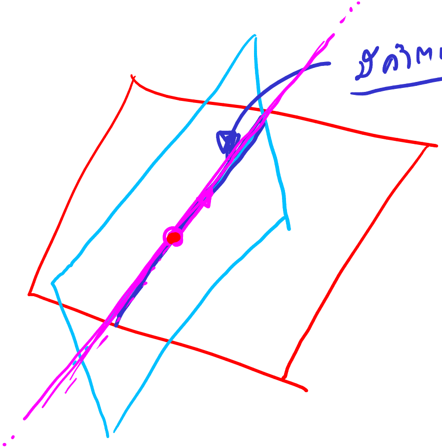
More failed example 3

Let's consider the last system with 3 variables:

$$
\begin{array}{rcrcrcr}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
\cancel{2x_1} & \cancel{+} & & & \cancel{10x_3} & \cancel{=} & \cancel{24}
\end{array}
$$

ซ้ำ

ชุดคำตอบไม่จำกัด



More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

More failed example 3 (cont.)

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.

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What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions?

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions? It means that, for $\boldsymbol{u}$, you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

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I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$.

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =$$

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\overset{7}{(2u_1 + 4u_2 + 3u_3)} - \overset{7}{(2v_1 + 4v_2 + 3v_3)} =$$
$$2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7-7) = 0$$

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$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \end{array}$$

More failed example 3 (cont. 1)

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Consider $\underline{\boldsymbol{u}-\boldsymbol{v}}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7-7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \\ \quad (u_1 - v_1) \quad + \quad 5(u_1 - v_3) \quad = \quad (12 - 12) = 0 \end{array}$$

$w = [u_1 - v_1,\ u_2 - v_2,\ u_3 - v_3]$

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ x_1 & + & & & 5x_3 & = & 0 \end{array}$$

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It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.

More failed example 3 (cont. 2)

$x + y \cdot \alpha$

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

x

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y

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It would play a central role when dealing with linear systems with many solutions.

Key take away

- There are 2 ways to look at how we solve linear systems: row perspective and column perspective.
- **Linear combination** is the main operation.