01204211 Discrete Mathematics Lecture 10b: Dimensions

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a linear combination of u_1, \ldots, u_m .

Review: Span

Definition

A set of all linear combination of vectors $m{u}_1, m{u}_2, \dots, m{u}_m$ is called the span of that set of vectors.

It is denoted by $Span\{u_1, u_2, \dots, u_m\}$.

Review: Vector spaces

Definition

A set $\mathcal V$ of vectors over $\mathbb F$ is a **vector space** iff

- ightharpoonup (V1) $\mathbf{0} \in \mathcal{V}$,
- ightharpoonup (V2) for any $u\in\mathcal{V}$,

$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

ightharpoonup (V3) for any $u,v\in\mathcal{V}$,

$$u + v \in \mathcal{V}$$
.

Review: Linearly independence

Definition

Vectors u_1, u_2, \dots, u_n are **linearly independent** if no vector u_i can be written as a linear combination of other vectors.

(Another) Definition

Vectors u_1, u_2, \ldots, u_n are **linearly independent** if the only solution of equation

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_n \boldsymbol{u}_n = \boldsymbol{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Review: Bases

Definition

A set of vectors $\{m{u}_1, m{u}_2, \dots, m{u}_k\}$ is a basis for vector space $\mathcal V$ if

- $ightharpoonup \operatorname{Span} \left\{ oldsymbol{u}_1, oldsymbol{u}_2, \ldots, oldsymbol{u}_k \right\} = \mathcal{V}$, and
- $ightharpoonup u_1, u_2, \ldots, u_k$ are linearly independent.

Lemma 1 (Superfluous Vector Lemma)

Consider vectors u_1, u_2, \ldots, u_n . If $v \in \text{Span } \{u_1, u_2, \ldots, u_n\}$, then

Span
$$\{oldsymbol{u}_1,oldsymbol{u}_2,\ldots,oldsymbol{u}_n,oldsymbol{v}\}= ext{Span }\{oldsymbol{u}_1,oldsymbol{u}_2,\ldots,oldsymbol{u}_n\}$$

Span $\{u_1, u_2, \dots, u_{n-1}\}$ = Span $\{u_1, u_2, \dots, u_n\}$

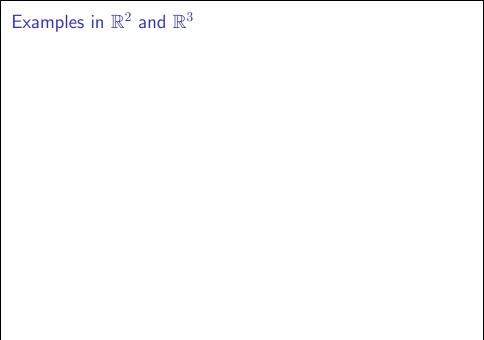
Lemma 2

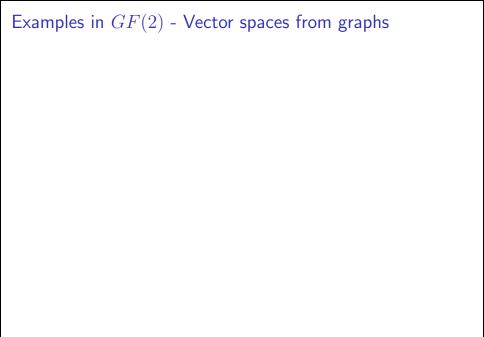
Consider vectors u_1, u_2, \ldots, u_n . If

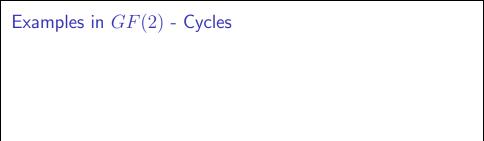
$$oldsymbol{u}_n \in \operatorname{Span} \ \{oldsymbol{u}_1, oldsymbol{u}_2, \dots, oldsymbol{u}_{n-1}\}$$
, then

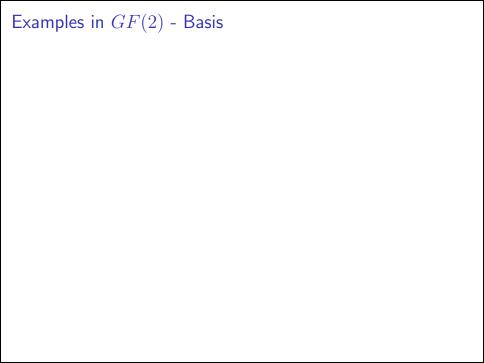
Lemma 3 (Unique representation)

Let u_1, u_2, \ldots, u_k be a basis for vector space \mathcal{V} . For any $v \in \mathcal{V}$, there is a unique way to write v as a linear combination of u_1, \ldots, u_k .









Number of vectors in bases

- ightharpoonup We have an observation that for a vector space $\mathcal V$, every basis has the same size.
- This is not a coincident.
- In this course, we will see two proofs.
- ► Remark: there are vector spaces whose basis has infinite size, but we are not dealing with those vector spaces in this course.

Theorem 4 (Main Theorem)

If $m{u}_1, m{u}_2, \dots, m{u}_n$ and $m{v}_1, m{v}_2, \dots, m{v}_m$ are bases for vector space \mathcal{W} , then n=m.

Exchange Lemma

We will prove the main theorem using the "exchange" lemma.

Lemma 5 (Simplified Exchange Lemma)

Consider a set of vectors S and let z be a non-zero vector in $\operatorname{Span} S$. There is a vector $w \in S$ such that $\operatorname{Span} (S \cup \{z\} - \{w\}) = \operatorname{Span} S$.

Lemma 6 (Exchange Lemma)

Consider a set of vectors S and its subset A. Let z be a non-zero vector in $\operatorname{Span} S$ such that $A \cup \{z\}$ is linearly independent. There is a vector $w \in S - A$ such that $\operatorname{Span} (S \cup \{z\} - \{w\}) = \operatorname{Span} S$.

Lemma 7 (Morphing Lemma)

If a set of vectors S spans a vector space \mathcal{W} and B is a linearly independent set of vectors in \mathcal{W} , then $|B| \leq |S|$.

Proof.

Let $B = \{u_1, u_2, \dots, u_n\}$. We show how to construct S_1, \dots, S_n such that for each $i, |S_i| = |S|$, $\operatorname{Span} S_i = \operatorname{Span} S_i$, and

$$\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_i\}\subseteq S_i.$$

Let $S_0=S$. We construct S_i from S_{i-1} . Note that since B is linearly independent, $\{u_1,\ldots,u_{i-1}\}\subseteq S_{i-1}$ is also linearly independent. We can use the Exchange Lemma to state that there exist $\boldsymbol{w}\in S_{i-1}-\{u_1,\ldots,u_{i-1}\}$ such that

Span
$$(S_{i-1} \cup \{u_i\} - \{w\}) = \text{Span } S_{i-1}.$$

We then let $S_i = S_{i-1} \cup \{u_i\} - \{w\}$. (You can check that S_i has the properties as claimed above.)

Since
$$|S_n| = |S|$$
 and $B \subseteq S_n$, we have that $|B| \le |S|$.

Morphing Lemma ⇒ Main Theorem

Theorem 8 (Main Theorem)

If u_1, u_2, \ldots, u_n and v_1, v_2, \ldots, v_m are bases for vector space W, then n = m.

Proof.

Since $\{u_1, u_2, \ldots, u_n\}$ is a basis, it spans \mathcal{W} . Also, vectors v_1, v_2, \ldots, v_m are linearly independent because they also form a basis. Thus, from the Morphing Lemma, $m \leq n$. We can reverse the roles of u_i 's and v_i 's to obtain that $n \leq m$. Therefore, n = m.

Proof of the Simplified Exchange Lemma.

Let $S = \{u_1, u_2, \dots, u_n\}$. Since $z \in \operatorname{Span} S$, we note that $\operatorname{Span} S = \operatorname{Span} (S \cup \{z\})$. We can also write

$$\boldsymbol{z} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \ldots + \alpha_n \boldsymbol{u}_n.$$

Because z is non-zero, there exists some non-zero α_i . We can rewrite the above equation as

$$\alpha_i \boldsymbol{u}_i = \boldsymbol{z} - \alpha_1 \boldsymbol{u}_1 - \ldots - \alpha_{i-1} \boldsymbol{u}_{i-1} - \alpha_{i+1} \boldsymbol{u}_{i+1} - \cdots - \alpha_n \boldsymbol{u}_n,$$

or

$$\boldsymbol{u}_i = \left(\frac{1}{\alpha_i}\boldsymbol{z} - \frac{\alpha_1}{\alpha_i}\boldsymbol{u}_1 - \ldots - \frac{\alpha_{i-1}}{\alpha_i}\boldsymbol{u}_{i-1} - \frac{\alpha_{i+1}}{\alpha_i}\boldsymbol{u}_{i+1} - \cdots - \frac{\alpha_n}{\alpha_i}\boldsymbol{u}_n\right),$$

i.e., $u_i \in \mathrm{Span}\ (S \cup \{z\})$. In this case, we can remove u_i , i.e.,

$$\operatorname{Span} (S \cup \{z\} - \{u_i\}) = \operatorname{Span} (S \cup \{z\}) = \operatorname{Span} S.$$

Therefore we can let $oldsymbol{w} = oldsymbol{u}_i$ and the lemma follows.

How can we prove the full lemma?

Dimensions

Definition

The dimension of a vector space $\mathcal V$ is the size of its basis. The dimension of $\mathcal V$ is written as $\dim \mathcal V$.