01204211 Discrete Mathematics
Lecture 8b: Finite automata[1]

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[1]Based on lecture notes of *Models of Computation* course by Jeff Erikson.

Example: syntax highlighting

```
cin >> x;
cout << x << "hello" << ...
for ( ( ) ) {
}
```

state (สถานะ)

HTML tokenizer

</ ____ >

<body>

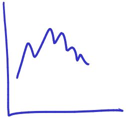
<a href=" ____ > ____ ">

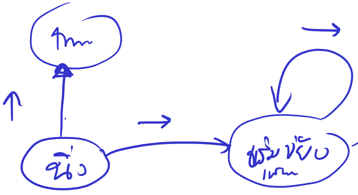
</body>

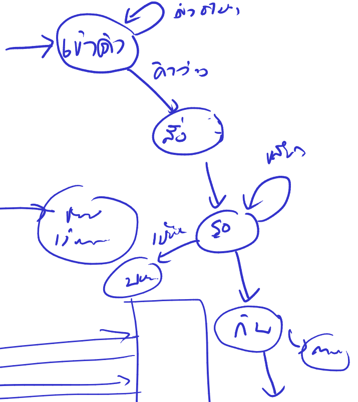
Game programming

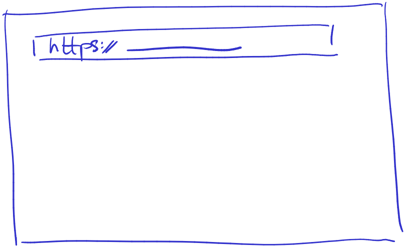
State Machine

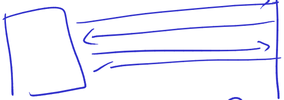
State-transition graphs

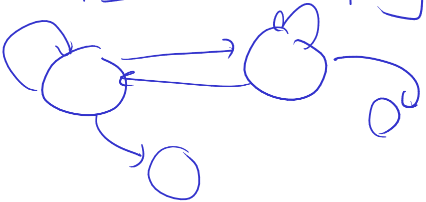






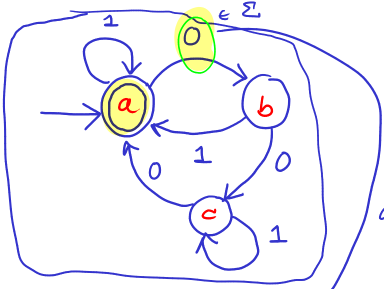






More examples over $\Sigma = \{0, 1\}$

All strings, except 010.



$\boxed{\delta(a,0) = b}$

01

00

001110

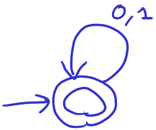
accepts:

ε, 1^*

$(000)^*$

Strings containing the subsequence 010.

Σ^*



$\{010\}$

Formal definitions

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A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

- the input alphabet Σ,
- a finite set of states Q,
- a transition function δ

state ปัจจุบัน   input symbol

domain: $Q \times \Sigma$

range: state ถัดไป Q

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Formal definitions

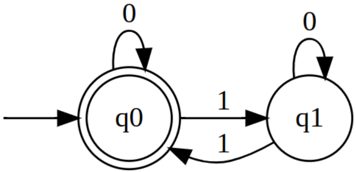
$$\underbrace{M = (\Sigma, Q, \delta, s, A)}_{\text{5-tuple}}$$

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

- the input alphabet Σ,
- a finite set of states Q,
- a transition function $\delta : Q \times \Sigma \longrightarrow Q$
- a start state $s \in Q$, and
- a subset $A \subseteq Q$ of accepting states.

Example 1

$\{ w \in \{0,1\}^* \mid w$ มี 1 เป็นจน.คู่ $\}$



$\Sigma = \{0, 1\}$

$Q = \{q_0, q_1\}$

δ:

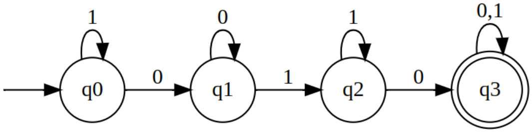
	0	1
q_0	q_0	q_1
q_1	q_1	q_0

$s = q_0$

$A = \{q_0\}$

Example 2

010 เป็น substring



$\Sigma = \{0, 1\}$

$Q = \{q_0, q_1, q_2, q_3\}$

δ

	0	1
q_0	q_1	q_0
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_3	q_3

$s = q_0$

$\boxed{A = \{q_3\}}$

ทำไม ถ้า 0110

ไม่ 101

$q_1 = \delta(s, 0)$

$s = q_0 \to \delta(s, 0)$

$\downarrow$

$\delta(q_1, 1) \to \delta(q_2, 1) \to \dots q_3 \in \boxed{A}$

$\delta(\delta(s, 0), 1) \to \delta(\delta(\delta(s, 0), 1), 1)$

Moves

One step move: from state q with input symbol a, the machine changes its state to

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Extension: from state q with input string q, the machine changes its state to $\delta^*(q, w)$ defined as

$$\delta^*(q, w) = \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), x) & \text{if } w = ax, \text{ [illegible] } a \in \Sigma,\ x \in \Sigma^* \end{cases}$$

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The signature of δ^* is $Q \times \Sigma^* \longrightarrow Q$.

Acceptance

For a finite-state machine with starting state s and accepting states A, it accepts string w iff

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input

$$\delta^*(s, w) \in A.$$

เริม

Multiple of 5

Multiple of 5

[0, 1, 1, 0, 1, 0, 1, 1]

01101011

```
def multiple_of_5(w):
    r = 0
    for i in w:
        r = (2*r + w) % 5
    return r == 0
```

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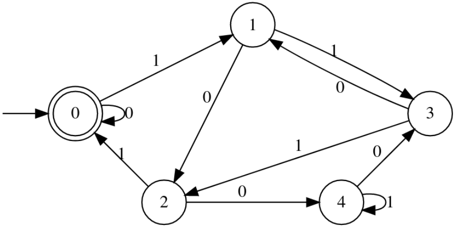
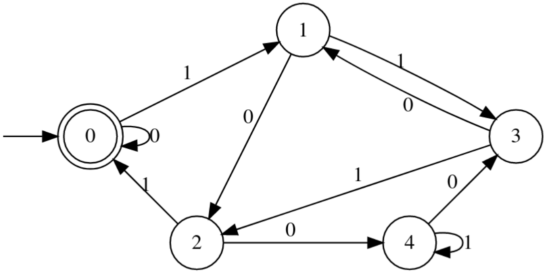
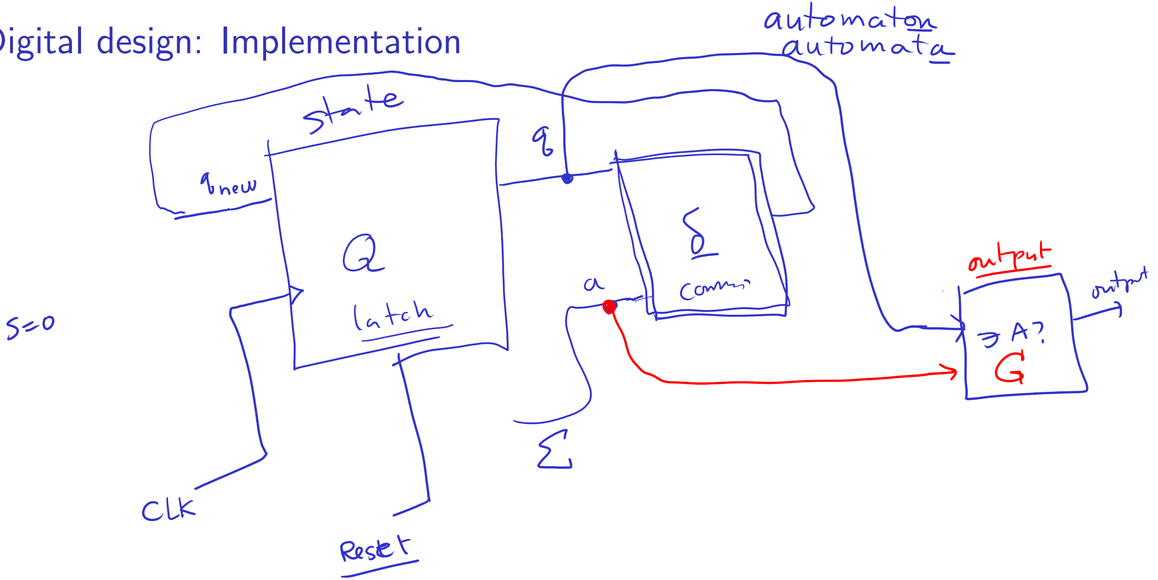



0
1
2
3
4
0
1
0
1
1
0
1
0
0
1

Digital design: Implementation



Digital design: Moore and Mealy machines

In the digital design class, you will encounter finite-state machines as well. The version we consider in this class is refered to as a **Moore machine**.
In practices, there is another variant of FSM called **Mealy machines**, whose outputs depend on input symbols as well.

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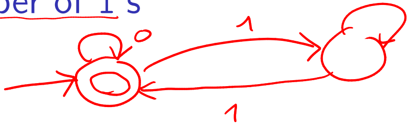
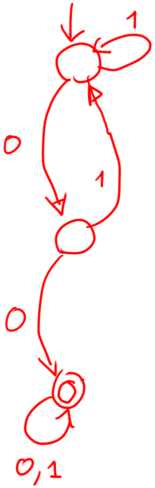
In practices, there is another variant of FSM called **Mealy machines**, whose outputs depend on input symbols as well.
Formally, they differ in output function.

- Moore machine: $G : Q \longrightarrow [0, 1]$
- Mealy machine: $G : Q \times \Sigma \longrightarrow [0, 1]$

Example: even number of 1's



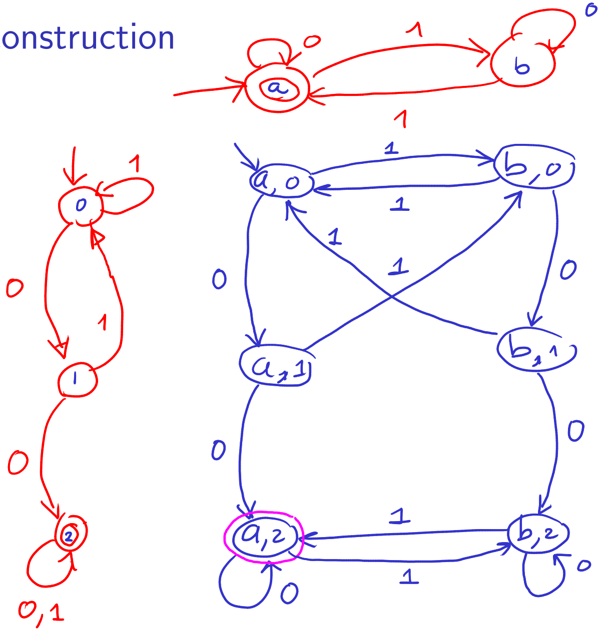
Example: strings containing 00 as a substring



Combining DFAs

What if we want to build a DFA that accepts strings with an even number of 1's and containing 00 as a substring?

Product construction



$\boxed{\text{ต่างที่ set } A}$

Product construction (formally)

Given a DFA $M_1 = (\Sigma, Q_1, \delta_1, s_1, A_1)$ that accepts strings from language L_1 and $M_2 = (\Sigma, Q_2, \delta_2, s_2, A_2)$ that accepts strings from language L_2, we can construct a DFA $M = (\Sigma, Q, \delta, s, A)$ that accepts strings from $L_1 \cap L_2$ as follows:

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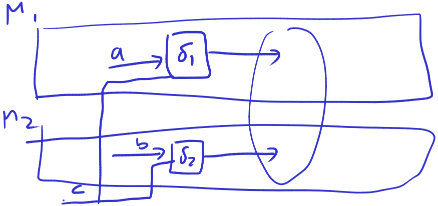
- Let $Q = Q_1 \times Q_2$,
- Let $\delta : \underbrace{(Q_1 \times Q_2)} \times \underbrace{\Sigma} \longrightarrow \underbrace{(Q_1 \times Q_2)}$ be such that

$$\delta((a, b), c) = (\delta_1(a, c), \delta_2(b, c))$$



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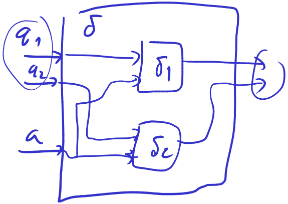
- Let $Q = Q_1 \times Q_2$,
- Let $\delta : (Q_1 \times Q_2) \times \Sigma \longrightarrow (Q_1 \times Q_2)$ be such that

$$\boxed{\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)),}$$



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- Let $s = (s_1, s_2)$, and
- Let $A = A_1 \times A_2$.

$L_1 \cup L_2$

$A = \{ (q_1, q_2) \mid q_1 \in A_1 \text{ or } q_2 \in A_2 \}$

$L_1 \setminus L_2$

$A = \{ (q_1, q_2) \mid q_1 \in A_1, q_2 \notin A_2 \}$

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Lemma 1

M_1 M_2

$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$ *for any string* w.

Proof.

We prove by induction. I.H.: Assume that $\delta^*((q_1, q_2), x) = (\delta_1^*(q_1, x), \delta_2^*(q_2, x))$, for any string x such that $|x| < |w|$.

Case 1: $w = \varepsilon$

$\delta^*((q_1, q_2), w) = (q_1, q_2)_\varepsilon$ จาก def ของ *

$= (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$ จาก def ของ δ_1^*

Case 2: $w = ax$

$\delta^*((q_1, q_2), w) = \delta^*((q_1, q_2), ax)$ ตาม IH

$= \delta^*(\delta((q_1, q_2), a), x)$ ตามนิยาม *

$= \delta^*((\underbrace{\delta_1(q_1, a)}_{q_1}, \underbrace{\delta_2(q_2, a)}_{q_2}), x)$ จากนิยาม δ

$= (\delta_1^*(\delta_1(q_1, a), x), \delta_2^*(\delta_2(q_2, a), x)) = (\delta_1^*(q_1, w), \delta_1^*(q_2, w))$ จาก I.H. ตาม δ^* $\square$

Correctness

From the previous lemma, we have that

$$\delta^*(s, w) \quad = \quad \delta^*((s_1, s_2), w)$$

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$$\begin{aligned}
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&= (\delta_1^*(s_1, w), \delta_2^*(s_2, w))
\end{aligned}$$

Correctness

From the previous lemma, we have that

$$\begin{aligned}\delta^*(s,w) &= \delta^*((s_1,s_2),w)\\ &= (\delta_1^*(s_1,w),\delta_2^*(s_2,w))\end{aligned}$$

Thus, for an input w, M would reach the state $(\delta_1^*(s_1,w),\delta_2^*(s_2,w))$; it accepts w when

$$(\overset{\in A_1}{\delta_1^*(s_1,w)},\overset{\in A_2}{\delta_2^*(s_2,w)}) \in A_1 \times A_2.$$

This implies that M accepts w when $\delta_1^*(s_1,w) \in A_1$ and $\delta_2^*(s_2,w) \in A_2$, i.e., M accepts w iff M_1 and M_2 accept w.
Finally, we conclude that M accepts strings from language $L_1 \cap L_2$.

Language of a DFA

$L(M)$

For a DFA M, let $L(M)$ be the set of all strings that M accepts. More formally, for $M = (\Sigma, Q, \delta, s, A)$,

$$L(M) = \{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

We refer to $L(M)$ as the language of M.

Closure

$L_1 = L(M_1)$
$L_2 = L(M_2)$

Lemma 2

If L_1 and L_2 are languages of some DFAs M_1 and M_2, we have that

- *there is a DFA M that accepts $L_1 \cap L_2$,*

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- *there is a DFA M that accepts $\Sigma^* \setminus L_1$,*

ลข

Automatic languages[2]

Definition (for now)

A language L is **"automatic"** if there is a DFA M such that $L(M) = L$.

[2] Taken directly from Erikson's lecture notes

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are also automatic.

The set of automatic languages is closed under these boolean operations.

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