# 01204211 Discrete Mathematics
# Lecture 7c: Binomial Coefficients (3)

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# The binomial coefficients[1]

In this lecture, we discuss advanced counting with binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

# More on counting

We shall see more techniques for counting when we consider the following problems.

- How many anagrams does the word "KASETSARTUNIVERSITY" have? (They do not have to be real English words.)
- How can you give out $n$ different presents to $k$ students when student $i$ has to get $n_i$ pieces of presents?
- How many ways can you distribute $n$ baht coins to $k$ children?

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  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?
  - If we treat two $C$'s differently as $C_1$ and $C_2$, we can see that $CABC$ is counted twice as $C_1ABC_2$ and $C_2ABC_1$. This is true for any anagram of $ABCC$.

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  - Since each anagram is counted in $4!$ twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

# General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 $L$'s, 2 $O$'s, $H$, $E$, $W$, $R$, and $D$.)

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The number of permutation of alphabets in $HELLOWORLD$, treating each character differently is $10!$. However, each anagram is counted for $3!2!$ times because of the 3 copies of $L$ and the 2 copies of $O$. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

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- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.

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- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.
- Thus, the number of ways we can distribute presents is

$$\frac{9!}{3!3!3!}$$

# Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.

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- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

| Presents | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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| Children | A | B | A | B | C | B | C | A | C |

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- This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, we reach the same solution of

$$\frac{9!}{3!3!3!}.$$

# Distributing identical presents

Now suppose that I have $9$ identical presents. I want to give them to $3$ students: A, B, and C. I want to give each student $3$ presents. In how many ways can I do it?

- Note that when we state that the presents are identical, we mean that we do not distinguish them, i.e., the first present and the second present are indistinguishable.

# Distributing coins (1)

I have $9$ indentical coins. I want to give them to $3$ students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins.

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This is a fairly surprising use of binomial coefficients.

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Let's consider a general problem where we have $n$ identical coins to give out to $k$ students so that each student gets at least one coin. In how many ways can we do that?

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> There are $\binom{n-1}{k-1}$ ways to distribute $n$ identical coins to $k$ children so that each child get at least one coin.

# Distributing coins (4)

I have $9$ indentical coins. I want to give them to $3$ students: A, B, and C. In how many ways can I do it, given that some student may not get any coins?