01204211 Discrete Mathematics
Lecture 9b: RSA Review and Euler's Theorem

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RSA

- Private key: (d, n),   Public key: (e, n)
- Encryption $E(m) = m^e \bmod n$,   Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

Recap: Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

$$a \bmod m = b \bmod m,$$

we write

$$a \equiv b \pmod{m}.$$

We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m|(a-b)$$

Recap: Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $gcd(a, m) = 1$.

Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Special case of Euler's theorem

Theorem 3 (Euler's theorem)

If p and q are different primes, for a such that $gcd(a, pq) = 1$, we have

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

Special case of Euler's theorem

Theorem 4 (Euler's theorem)

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Is this useful?

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Theorem 4 (Euler's theorem)

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Is this useful? Yes! In the RSA algorithm.

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- Pick two primes p and q. Let $n = pq$.
- Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or $ed = k \cdot (p-1)(q-1) + 1$, for some integer k.
- What is $m^{ed} \bmod n$?

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- What is $m^{ed} \bmod n$?

$$\begin{aligned} m^{ed} &\equiv m^{k(p-1)(q-1)+1} \pmod{n} \\ &\equiv (m^{(p-1)(q-1)})^k \cdot m \pmod{n} \\ &\equiv 1^k \cdot m \pmod{n} \\ &\equiv m \pmod{n} \end{aligned}$$

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What is the requirement for m? $gcd(m, n) = 1$, otherwise you can use the message to factor n.