01204211 Discrete Mathematics
Lecture 7a: Binomial Coefficients (1)

Jittat Fakcharoenphol

August 7, 2021

The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- the Pascal's triangle,
- the binomial theorem

[1] This lecture mostly follows Chapter 3 of [LPV].

The equation

Last time we proved that, for $n, k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

While we can prove this equation algebraically using definitions of binomial coefficients, proving the fact by describing the process of choosing k-subsets reveals interesting insights. This equation also hints us how to compute the value of $\binom{n}{k}$ using values of $\binom{n-1}{\cdot}$'s. So, let's try to do it.

The table

We shall use the fact that $\binom{n}{0} = 1$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to fill in the following table.

n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

You can note that the table is left-right symmetric. This is true because of the fact that $\binom{n}{k} = \binom{n}{n-k}$.

The Triangle

If we move the numbers in the table slightly to the right, the table becomes the Pascal's triangle.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1
	$\vdots$		$\vdots$								$\vdots$	

The table and the binomial coefficients have many other interesting properties.

Polynomial expansions

Let's start by looking at polynomial of the form $(x+y)^n$. Let's start with small values of n:

- $(x+y)^1 = x + y$
- $(x+y)^2 = x^2 + 2 \cdot xy + y^2$
- $(x+y)^3 = x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + y^3$
- $(x+y)^4 = x^4 + 4 \cdot x^3y + 6 \cdot x^2y^2 + 4 \cdot xy^3 + y^4$.

Let's focus on the coefficient of each term. You may notice that terms x^n and y^n always have 1 as their coefficients. *Why is that?* Let's look further at the coefficients of terms $x^{n-1}y$. Do you see any pattern in their coefficients? *Can you explain why?*

Another way to look at it

Let's take a look at $(x+y)^4$ again. It is

$$(x+y)(x+y)(x+y)(x+y).$$

- How do we get x^4 in the expansion? For every factory, you have to pick x.
- How do we get x^3y in the expansion? Out of the 4 factors, you have to pick y in one of the factor (or you have to pick x in 3 of the factors). Thus there are $\binom{4}{3} = \binom{4}{1}$ ways to do so.

The binomial theorem

Theorem: If you expand $(x+y)^n$, the coefficient of the term $x^k y^{n-k}$ is $\binom{n}{k}$.

That is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$\binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y^1 + \binom{n}{n-2} x^{n-2} y^2 + \cdots + \binom{n}{1} x y^{n-1} + \binom{n}{0} y^n.$$

Additional applications of the binomial theorem

The binomial theorem can be used to prove various identities regarding the binomial coefficients. For example, if we let $x = 1$ and $y = 1$, we get that

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}.$$

Quick check. Can you prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

Note that this statements says that the number of odd subsets equals the number of even subsets.