01204211 Discrete Mathematics
Lecture 9a: Fermat's Little Theorem

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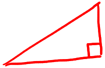
$a^2+b^2=c^2$



3, 4, 5

$a^n+b^n=c^n$

ไม่มีคำตอบที่เป็น จน.เต็ม เมื่อ $n \geq 3$

Fermat's Last Theorem

Quick recap

For any integer x and y, there exist a pair of integers a and b such that

$$a \cdot x + b \cdot y = gcd(x, y).$$

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How to find a and b? Use the extended GCD algorithm.

Finding a and b: Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $gcd(x, y)$.

```
Algorithm Euclid(x,y):
  if x mod y == 0:

    return y, 0, 1
  else:
    g, a', b' = Euclid(y, x mod y)

    a = b'

    b = a' - b'*floor(x / y)

    return g, a, b
```

Recap: Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

$$a \bmod m = b \bmod m,$$

we write

$$a \equiv b \pmod{m}.$$

We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m | (a - b)$$

Recap: Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $gcd(a, m) = 1$.

How to test if an integer n is prime

- Try to find factors of n. (Takes time $\sqrt{n}$)

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- If there is a property that holds **iff** n is prime, we can check that property. If we can check that quickly, we can test if n is prime.

How to test if an integer n is prime

exponential fn $\log n \sim 1024$, 2048

- Try to find factors of n. (Takes time $\sqrt{n}$)

$2^{\log n / 2}$

- If there is a property that holds iff n is prime, we can check that property. If we can check that quickly, we can test if n is prime.
- If there is a property that holds if n is prime, how can we make use of that property?

n ✓ – ยังไม่แน่ใจ

n ✗ – n ไม่เป็น prime

Fermat's little Thm

Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $\gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

$p \mid (a^{p-1} - 1)$

Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $gcd(a, p) = 1$,

$$a^{p-1} \equiv 1 \pmod{p}.$$

How can we use Fermat's Little Theorem to check if integer n is prime?

Fermat test

- ความถูกต้อง
- ประสิทธิภาพ

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
        return False
    else:
        return True
```

??

ทายผิด

คำนวณ $\boxed{a^{n-1} \bmod n}$

ถ้า n เป็นจน.เฉพาะ: FLT รับประกันว่า

$a^{n-1} \bmod n == 1$

รันซ้ำ .....

How good is the Fermat test?

When you call `CheckPrime(n)`:

- If n is prime,

How good is the Fermat test?

When you call `CheckPrime(n)`:

- If n is prime, `CheckPrime` always return True.
- If n is composite,

How good is the Fermat test?

When you call `CheckPrime(n)`:

- If n is prime, `CheckPrime` always return True.
- If n is composite, you want `CheckPrime` to return False, but **Fermat's Little Theorem does not guarantee that.**

Fermat test - when n is composite

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If n is composite, the algorithm returns False when

- $gcd(a, n) \neq 1$, i.e., when you pick a with common factor with n.
- $a^{n-1} \bmod n \neq 1$, i.e., when you find a that violates the property. We want to be in this case. How likely?

Proof of Fermat's Little Thm: Idea

$a^{p-1} \equiv 1 \pmod{p}$

Let $p = 7$ and $a = 5$. Consider set

$5, 3, 1, 6, 4, 2$

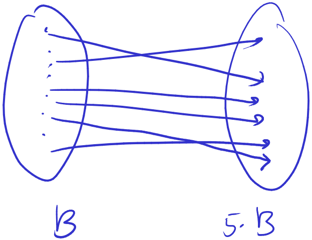
$$B = \{1, 2, 3, \ldots, p-1\} = \{1, 2, 3, 4, 5, 6\}$$

$\gcd(5, 7) = 1$

Also consider set

$$C = \{1 \cdot 5 \bmod 7,\ 2 \cdot 5 \bmod 7,\ 3 \cdot 5 \bmod 7, \ldots, 6 \cdot 5 \bmod 7\}.$$

which is



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which is

$$C = \{5, 3, 1, 6, 4, 2\}$$

1

$\shortparallel$

$\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}$

$5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2$

$\shortparallel$

$= (\cancel{1} \cdot a)(\cancel{2} \cdot a) \cdot (\cancel{3} \cdot a)(\cancel{4} \cdot a)(\cancel{5} \cdot a)(\cancel{6} \cdot a) = a^{p-1}$

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Is this coincidental?

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which is

$$C = \{5, 3, 1, 6, 4, 2\} = B.$$

Is this coincidental? No. (We will prove that. But can you quickly tell why.)
Since $B = C$, the following terms are equal:

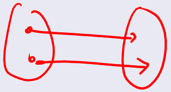
$$\left(\prod_{i \in B} i\right) \bmod 7 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \bmod 7,$$

and

$$\begin{aligned} \left(\prod_{i \in C} i\right) \bmod 7 &= 5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \bmod 7 \\ &= (1a) \cdot (2a) \cdot (3a) \cdot (4a) \cdot (5a) \cdot (6a) \bmod 7 \\ &= (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6) \cdot a^6 \bmod 7. \end{aligned}$$

Proof of Fermat's Little Thm.

Recall that $gcd(a, p) = 1$, i.e., there exists a multiplicative inverse a^{-1} of a modulo p. This implies that for $i \not\equiv j \pmod{p}$, $ai \not\equiv aj \pmod{p}$. Also note that $a \cdot 0 \equiv 0 \pmod{p}$.



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Let $B = \{1, 2, \ldots, p-1\}$. Let

$$C = \{a \cdot i \bmod p | i \in B\}.$$

Claim $B = C$

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Let $B = \{1, 2, \ldots, p-1\}$. Let

$$C = \{a \cdot i \bmod p | i \in B\}.$$

$i \neq j$

Since for different $i, j \in B$, we have different $ai \bmod p, aj \bmod p$, we know that $|C| = p - 1$. Also, $C \subseteq B$ because $0 \leq ai \bmod p \leq p - 1$ and $0 \notin C$. Thus, we can conclude that $C = B$.

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$$\begin{aligned} 1 \cdot 2 \cdots (p-1) &\equiv (a1) \cdot (a2) \cdot (a3) \cdots (a(p-1)) \pmod p \\ &\equiv (1 \cdot 2 \cdots (p-1)) \cdot a^{p-1} \pmod p. \end{aligned}$$

Since each of $1, 2, \ldots, p-1$ has an inverse modulo p, we can multiply both sides with $1^{-1}, 2^{-1}, \ldots, (p-1)^{-1}$ to obtain

$$1 \equiv a^{p-1} \pmod p,$$

as required.

Exercise

HW

Prove that for any integer a and prime p,

$$a^p \equiv a \pmod{p}.$$

FLT: $\gcd(a,p)=1$

$\Rightarrow a^{p-1} \equiv 1 \pmod{p}$

How good is the Fermat test when n is composite?

To answer correctly, we want a to be such that $gcd(a, n) \neq 1$ or

$$a^{n-1} \not\equiv 1 \pmod{n}.$$

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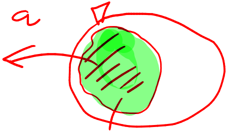

$\leq (\frac{1}{2})^{100}$

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We only consider the case where $gcd(a, n) = 1$ because when $gcd(a, n) \neq 1$, the test works perfectly.

We refer to $a \in \{1, 2 \ldots, p-1\}$ such that $gcd(a, n) = 1$ and $a^{n-1} \not\equiv 1 \pmod{n}$ as a **witness**. The other element b such that $b^{n-1} \equiv 1 \pmod{n}$ is called a **non-witness**.
How likely that we randomly choose an element and get a witness?

หลักฐานว่า n ไม่ใช่ จน.เฉพาะ

Number of witnesses

Suppose that there exists a witness a; we know that $a^{n-1} \not\equiv 1 \pmod{n}$. How can we find other witnesses?

Number of witnesses

(n เป็นจน.ประกอบ)

Suppose that there exists a witness a; we know that $a^{n-1} \not\equiv 1 \pmod{n}$. How can we find other witnesses?

Consider a non-witness b such that $b^{n-1} \equiv 1 \pmod{n}$.

พิจารณา $c = ab$

witness, non-witness

$c^{n-1} \equiv (ab)^{n-1} \pmod{n}$

$\equiv a^{n-1} \cancel{b^{n-1}}$

$\equiv a^{n-1} \pmod{n}$

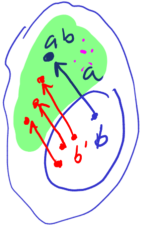
$\not\equiv 1 \pmod{n}$

จะได้ว่า c เป็น witness

พิจารณา non-witness b, b' ที่ $b \neq b'$

ทราบว่า $ab \not\equiv ab' \pmod{n}$

เพราะ $\boxed{\gcd(a,n) = 1}$ ทำให้ a มี inverse.



$\Rightarrow$ จำนวน witness $\geq$ จำนวน non-witness | ถ้ามี a, จะได้ witness

$\geq 1/2$

Carmichael Number

עדין witness

A **Carmicheal number** is a composite number n where

$$b^{n-1} \equiv \underline{\underline{1}} \pmod{n},$$

for every b which are relatively prime to n.

Carmicheal numbers are rare. The smallest is $561 = 3 \cdot 11 \cdot 17$. The next ones are $1105, 1729$, and 2465. There are $20,138,200$ Carmicheal numbers between 1 and 10^{21}.
So, if we ignore Carmicheal numbers, the Fermat test is very good. There are other probabilistic tests (e.g, Miller-Rabin test) that uses other properties that works for all numbers and there are deterministic algorithms for testing primes.

Lemma 3

If n is not a Carmicheal number, the Fermat test returns that n is a composite with probability at least $1/2$.

Note that if you repeat the test for k times, the probability that it gives the wrong answer is at most $1/2^k$.

Running time

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
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    else:
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```

Special case of Euler's theorem

Theorem 4 (Euler's theorem)

If p and q are different primes, for a such that $gcd(a, pq) = 1$, we have

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

Special case of Euler's theorem

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Is this useful?

Special case of Euler's theorem

Theorem 4 (Euler's theorem)

If p and q are different primes, for a such that $gcd(a, pq) = 1$, we have

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Is this useful? Yes! In the RSA algorithm.

RSA

- Private key: (d, n), Public key: (e, n)
- Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

RSA

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- Pick two primes p and q. Let $n = pq$.
- Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or $ed = k \cdot (p-1)(q-1) + 1$, for some integer k.
- What is $m^{ed} \bmod n$?

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$$
\begin{aligned}
m^{ed} &\equiv m^{k(p-1)(q-1)+1} \pmod{n} \\
&\equiv (m^{(p-1)(q-1)})^k \cdot m \pmod{n} \\
&\equiv 1^k \cdot m \pmod{n} \\
&\equiv m \pmod{n}
\end{aligned}
$$

* ความถูกต้อง ①

Euler's thm

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What is the requirement for m?

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What is the requirement for m? $gcd(m, n) = 1$, otherwise you can use the message to factor n.