01204211 Discrete Mathematics
Lecture 9a: Spans and Vector Spaces

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August 30, 2022

Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Example 1

Is $\mathrm{Span}\ \{[1,2],[2,5]\} = \mathbb{R}^2$?

Example 2

Is Span $\{[1, 0, 1], [1, 1, 0], [2, 3, 4]\} = \mathbb{R}^3$?

Example 3

Is $\mathrm{Span}\ \{[1,0,1],[1,1,0],[4,2,2]\} = \mathbb{R}^3$?

Elements in a vector

- ▶ We see examples of vectors over $\mathbb{R}$.
- ▶ However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
  - ▶ We should be able to perform addition, subtraction, multiplication, and division.
  - ▶ Operations should be commutative and associative.
  - ▶ Additive and multiplicative identity should exist.
  - ▶ Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a **field**.

A field

Definition

A set $\mathbb{F}$ with two operations $+$ and $\times$ (or $\cdot$) is a **field** iff these operations satisfy the following properties:

- (Associativity): $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (Commutativity): $a + b = b + a$ and $a \cdot b = b \cdot a$
- (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$
- (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$
- (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an alement a^{-1} such that $a \cdot a^{-1} = 1$
- (Distributive): $a \cdot (b + c) = a \cdot b + a \cdot c$

Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$. I.e., it is a "bit" field.
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Another useful field: $GF(2)$

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What are $+$ and $\cdot$ in $GF(2)$?

▶ We define $b_1 + b_2$ to be XOR.

$$0 + 0 = 0$$
$$0 + 1 = 1 + 0 = 1$$
$$1 + 1 = 0$$

▶ We define $b_1 \cdot b_2$ to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$

You can check that $GF(2)$ satisfies the axioms of fields.

2×2 Lights out

Can you solve 2×2 Lights out?

Let $\boldsymbol{u}_1 = [1, 1, 1, 0]$, $\boldsymbol{u}_2 = [1, 1, 0, 1]$, $\boldsymbol{u}_3 = [1, 0, 1, 1]$, and $\boldsymbol{u}_4 = [0, 1, 1, 1]$.

Given $\boldsymbol{b} = [b_1, b_2, b_3, b_4]$, can you always find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

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$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Same question: Is $\mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\boldsymbol{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Can you solve 2×2 Lights out?

Since

$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \mathrm{Span} \ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$

and

Can you solve 2×2 Lights out?

Since

$$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$$

and

$$\text{Span } \{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\} = GF(2)^4,$$

Can you solve 2×2 Lights out?

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$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$

and

$\text{Span } \{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\} = GF(2)^4,$

what can we say about $\text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$?

Generators

Definition

Let $\mathcal{V}$ be a set of vectors. Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$.
If $\mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$, we say that

- $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a **generating set** for $\mathcal{V}$
- vectors $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_n$ are **generators** for $\mathcal{V}$

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Examples

Standard generators

Note that $\{[1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1]\}$ are generators for $GF(2)^4$. Why?

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They are called **standard generators** for $GF(2)^4$, written as e_1, e_2, e_3, e_4.

For $\mathbb{R}^n$, we also have $[1,0,0,\ldots,0], [0,1,0,\ldots,0], [0,0,1,\ldots,0], \ldots, [0,0,0,\ldots,1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are generators for $\mathcal{V}$, and for each i,

$$\boldsymbol{v}_i \in \operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\},$$

we have that $\mathcal{V} \subseteq \operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$.

Adding a vector into a span

Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. If $\boldsymbol{v} \in \operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, then

$$\operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \operatorname{Span} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

Geometry of spans: in $\mathbb{R}^2$

Geometry of spans: in $\mathbb{R}^3$

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.

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What are common properties of these geometric objects?

Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
- ▶ as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- ▶ they pass through the origin,
- ▶ if vector u is in the objects, αu for any scalar α is also in the objects, and
- ▶ if u and v are in the objects, $u + v$ is also in the objects.

Vector spaces

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,

$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

  for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,

$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

Span of vectors is a vector space

Consider n-vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$,

$$\text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$$

is a vector space.

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Let's check if properties V1, V2, and V3 are satisfied.

Solutions to homogeneous linear equations is a vector space

Consider a set $\mathcal{S}$ of all n-vectors in the form $[x_1, x_2, \ldots, x_n]$ where

$$
\begin{aligned}
a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= 0 \\
a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n &= 0 \\
\vdots &= \vdots \\
a_{m1} x \cdot_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n &= 0
\end{aligned}
$$

Let's check if properties V1, V2, and V3 are satisfied.

Dot product

Definition

For n-vectors $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$ and $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$, the **dot product** of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

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Using dot products, the previous set $\mathcal{S}$ can be written as

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} = 0, \boldsymbol{a}_2 \cdot \boldsymbol{x} = 0, \ldots, \boldsymbol{a}_m \cdot \boldsymbol{x} = 0\}$$

and we know that $\mathcal{S}$ is a vector space.

Parity-check code

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From message $\boldsymbol{a} = [a_1, a_2, a_3, a_4]$, we compute (in $GF(2)$) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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Now our encoded message becomes

$$[a_1, a_2, a_3, a_4, a_5],$$

where $a_5 = b = a_1 + a_2 + a_3 + a_4$. It can detects a single-bit error.

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What can we say about the condition on a_5?
It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

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It is in fact a homogeneous linear equation (in $GF(2)$):

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Now, what is the set of all possible codewords?

Hamming code

You can detect and correct more errors with Hamming codes. In this version called a $[7, 4]$ Hamming code, you encode 4-bit data $[a_1, a_2, a_3, a_4]$ into a 7-bit codeword $[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$. Using the formula:

$$\begin{aligned} a_5 &= a_1 + a_2 + a_4 \\ a_6 &= a_1 + a_3 + a_4 \\ a_7 &= a_2 + a_3 + a_4 \end{aligned}$$

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\begin{aligned}
a_5 &= a_1 + a_2 + a_4 \\
a_6 &= a_1 + a_3 + a_4 \\
a_7 &= a_2 + a_3 + a_4
\end{aligned}
$$

Let's see how this works.

Parity check

Let

$$\begin{aligned}
s_1 &= a_1 + a_2 + a_4 + a_5 \\
s_2 &= a_1 + a_3 + a_4 + a_6 \\
s_3 &= a_2 + a_3 + a_4 + a_7
\end{aligned}$$

Given a codewords $w = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero's.

Parity check

Let

$$s_1 = a_1 + a_2 + a_4 + a_5$$
$$s_2 = a_1 + a_3 + a_4 + a_6$$
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Given a codewords $w = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero's.

What if there is an error? Let's try.

Codewords from Hamming code

Turning the formula for a_5, a_6, a_7 around, we have 3 homogeneous linear equations:

$$
\begin{aligned}
a_1 + a_2 + a_4 + a_5 &= 0 \\
a_1 + a_3 + a_4 + a_6 &= 0 \\
a_2 + a_3 + a_4 + a_7 &= 0
\end{aligned}
$$

and again the set of all possible codewords $\mathcal{W}$ forms a vector space over $GF(2)$.

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

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If we have a line or a plane passing through a vector a, but not through the origin, how can we represent it?

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▶ Then we translate it back so that it passes through a.

Translation

If we have a line or a plane passing through a vector $\boldsymbol{a}$, but not through the origin, how can we represent it?

▶ Translate the object so that it passes through the origin.

▶ We obtain a vector space $\mathcal{V}$.

▶ Then we translate it back so that it passes through $\boldsymbol{a}$.

▶ We get the set

$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

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- ▶ Translate the object so that it passes through the origin.
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- ▶ Then we translate it back so that it passes through a.
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- ▶ *Question:* Is $\mathcal{A}$ a vector space?

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▶ We obtain a vector space $\mathcal{V}$.

▶ Then we translate it back so that it passes through a.

▶ We get the set

$$\mathcal{A} = \{a + u : u \in \mathcal{V}\}$$

▶ *Question:* Is $\mathcal{A}$ a vector space?

▶ We also write it as $a + \mathcal{V}$.

Affine spaces

Definition

If a is a vector and $\mathcal{V}$ is a vector space, then

$$a + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

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Definition

The set of all affine combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **affine hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Definition

The set of all convex combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **convex hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Writing an affine space using a span

Writing an affine space using a span

An affine space

An affine space passing through $u_1, u_2, \ldots, u_n$ is

$$u_1 + \operatorname{Span} \{u_2 - u_1, u_3 - u_1, \ldots, u_n - u_1\}.$$

Non-homogeneous linear system

Two linear systems:

$$
\begin{array}{rcl}
\boldsymbol{a_1} \cdot \boldsymbol{x} & = & b_1 \\
\boldsymbol{a_2} \cdot \boldsymbol{x} & = & b_2 \\
& \vdots & \\
\boldsymbol{a_m} \cdot \boldsymbol{x} & = & b_m
\end{array}
\qquad\qquad
\begin{array}{rcl}
\boldsymbol{a_1} \cdot \boldsymbol{x} & = & 0 \\
\boldsymbol{a_2} \cdot \boldsymbol{x} & = & 0 \\
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$$

What can you say about the solution sets of these two related
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\boldsymbol{a_m} \cdot \boldsymbol{x} & = & 0
\end{array}
$$

What can you say about the solution sets of these two related linear systems?

0 is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

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$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

This implies that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution to the homogeneous linear system.

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i\boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

is a vector space, because

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \le i \le m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

is a vector space, because

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = 0, \text{ for } 1 \le i \le m\}$$

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In other words,

$$
\begin{aligned}
\mathcal{W} &= \boldsymbol{u} + \{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\} \\
&= \boldsymbol{u} + \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = 0, \text{ for } 1 \leq i \leq m\},
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i.e., $\mathcal{W}$ is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.