Linear algebra

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field $\begin{cases} \mathbb{R} \\ GF(2) \text{ bits} \end{cases}$ elements → • vectors

$GF(p)$ — number theory

operations:
① scale (scalar mult)
② addition (element wise addition).
advanced: dot production

vectors → Vector space

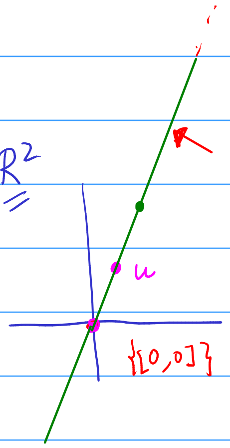
▶ V is a vector space if over $\mathbb{R}$

(V1) $0 \in V$

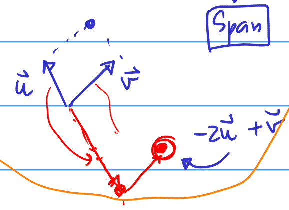
(V2) for any $\alpha \in \mathbb{R}$, any $u \in V$, $\alpha u \in V$.

(V3) for any $u, v \in V$, $u + v \in V$.

ex $\mathbb{R}^2$



linear combination → Span



linear independence

set of vectors

Examples of vector spaces

→ ① A span of a set of vectors ; $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k$, $\boxed{\mathrm{Span}\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}}$

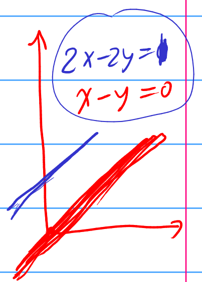
→ ② A set of solutions to a linear system

$Ax = 0$ $\quad \boxed{\{x : Ax = 0\}}$

homogeneous system of linear equation

Matrices

$Ax = b$



$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 5 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\ 2x_1 + 4x_2 + 1x_3 + 5x_4 &= 0 \\ 3x_1 + 2x_2 + 1x_3 + 1x_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

Linear system $\boxed{Ax = 0}$ ← homogeneous linear system of equation

$$\underbrace{\left.\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\right\} m}_{n} \left.\begin{bmatrix} \\ x \\ \\ \end{bmatrix}\right\} n = \left.\begin{bmatrix} \\ \\ \\ \end{bmatrix}\right\} m$$

Matrix A

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} r_1 \\ r_2 \\ A \ \vdots \\ r_m \end{bmatrix} \end{array}$$

$r_i \in \mathbb{R}^n$

$\boxed{\text{Span}\{r_1, r_2, \dots, r_m\}}$

$\|$

$\boxed{\text{Row space of } A}$

$C(A^T)$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} c_1 & c_2 & c_3 & A & \cdots & c_n \end{bmatrix} \end{array}$$

$c_i \in \mathbb{R}^m$

$\boxed{\text{Span}\{c_1, c_2, c_3, \dots, c_n\}}$

$\|$

$C(A)$ $\boxed{\text{Column Space of } A.}$

Row operations

$$\begin{bmatrix} r_1 \\ r_2 \\ A \ \vdots \\ r_m \end{bmatrix} \longrightarrow \longrightarrow \longrightarrow \begin{bmatrix} \quad \\ A' \\ \quad \end{bmatrix}$$

row space $= \boxed{\text{Span}\{r_1, r_2, \dots, r_m\}}$

r' is a result of row operation is a linear combination of $r_1, \dots, r_m$

$$\begin{array}{cc} & A \\ \text{basis of } C(A^T) & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} 2 & 5 & 6 & 7 \end{bmatrix}$$

Column space

$$\overset{A}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} =$$

Column space contains vectors of the form: $\boxed{\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_n c_n}$

$\|$

$$\alpha_1 \begin{bmatrix} c_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} c_2 \end{bmatrix} + \alpha_3 \begin{bmatrix} c_3 \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} c_n \end{bmatrix}$$

$$\boxed{\{Ax : x \in \mathbb{R}^n\}}$$

try

$Ax_1 \to b_1$

$Ax = b_1$

Column space ~~is~~ contains a set of vectors b s.t. $Ax = b$ has a solution.

$[1,0,0,1]$ basis of
~~$[2,0,0,2]$~~
$[0,1,5,1]$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 5 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \notin \text{Column spa}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\text{Useless}} \right\} = \boxed{\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$$\boxed{C(A)} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

① They span the vector space

② They are linearly independent

Basis of a vector space V

Thm: Every basis has the same size.
We refer to the size of the basis as the dimension of the vector space; denoted by $\dim V$

$\boxed{A}$ $\boxed{\text{Null spaces}}$

A column space row space

$$\boxed{\dim C(A) = \dim C(A^T)}$$

~~A~~ linear transformation function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$\xrightarrow{n} \boxed{f} \xrightarrow{m}$$

is a linear function if.

(1) $\forall \alpha, \forall x \in \mathbb{R}^n$, $f(\alpha x) = \alpha f(x)$

(2) $x, y \in \mathbb{R}^n$, $f(x+y) = f(x) + f(y)$

Thm: f is a linear function, there exists a matrix $M \in \mathbb{R}^{m \times n}$ s.t.

$$\boxed{f(x) = Mx}$$

[[a , b],
[c , d]]

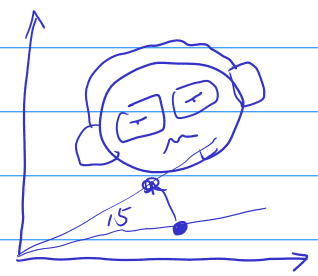
[x , y]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$$

Linear function

find matrix M such that

$$f(x) = Mx$$



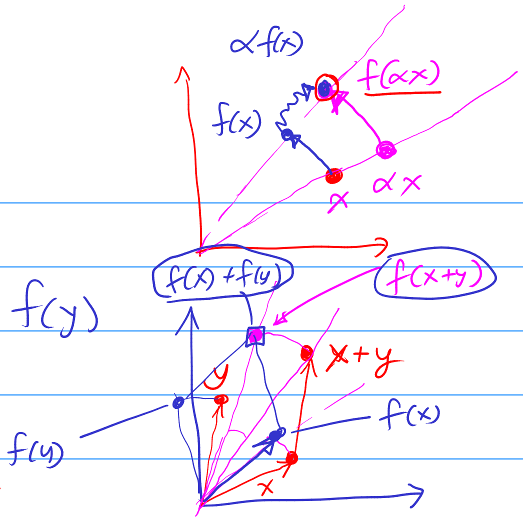
f

15° - rotation

$f(x) \Rightarrow$ rotated point

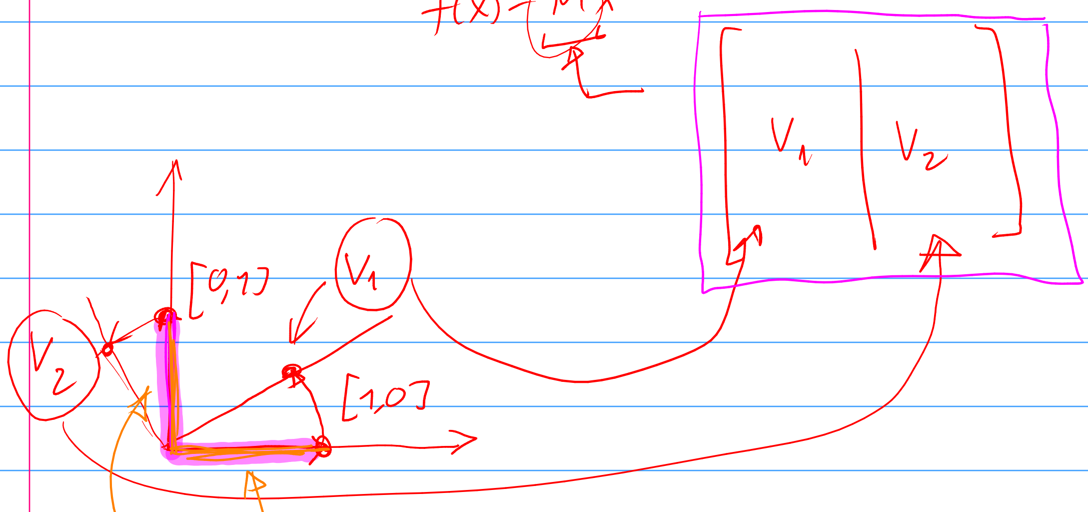
f is linear (1) $\boxed{f(\alpha x) = \alpha f(x)}$

(2) $f(x+y) = f(x) + f(y)$



there exists a matrix M such that

$f(x) = Mx$



spans $\mathbb{R}^2$, linearly indep $\Rightarrow$ basis of $\mathbb{R}^2$