01204211 Discrete Mathematics
Lecture 9a: Finite automata[1]

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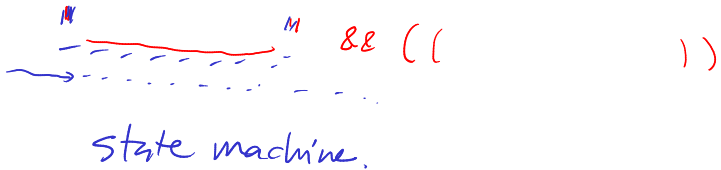
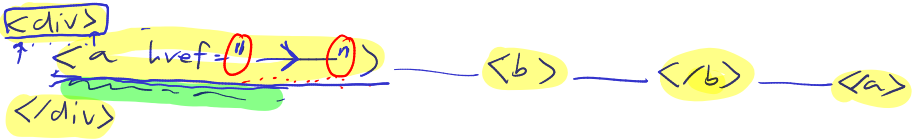
[1] Based on lecture notes of *Models of Computation* course by Jeff Erickson.

Example: syntax highlighting



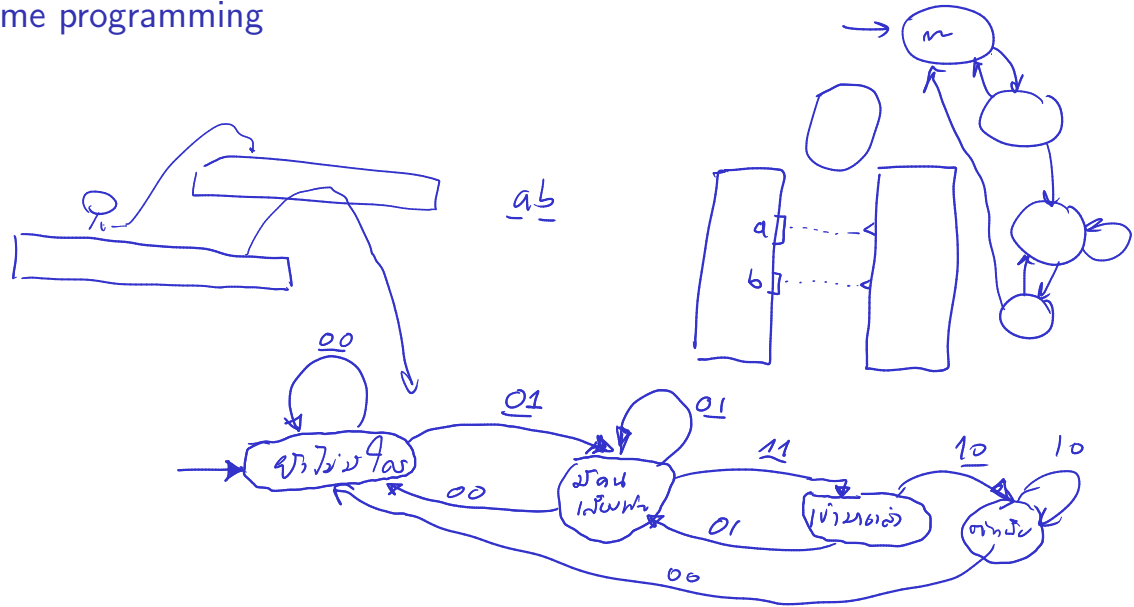
HTML tokenizer



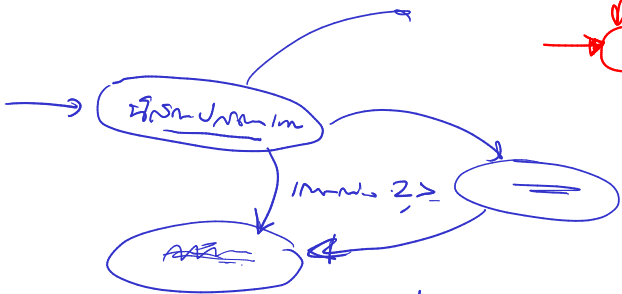

```
while ____ {
    if (xxx == ---):
        ...
        ...
        ...
        ...
        pass
}
```

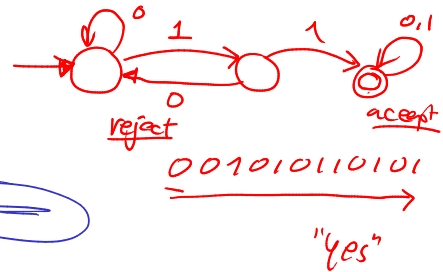
state machine

Game programming



State-transition graphs

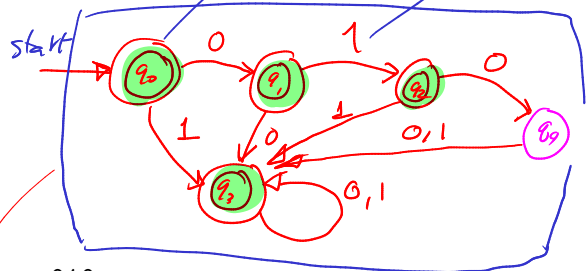




More examples over $\Sigma = \{0, 1\}$

All strings, except 010.

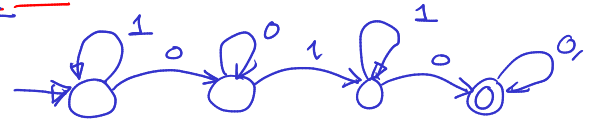
✓ 0100 →



δ	0	1
q_0	q_1	q_3
q_1	q_3	q_2
q_2	q_4	q_3
q_3	q_3	q_3
q_4	q_3	q_3

→ Strings containing the subsequence 010.

00111101



$Q = \{q_0, q_1, q_2, q_3, q_4\}$

$Q \times \Sigma = \{(q_0, 0), (q_0, 1), (q_1, 0), \ldots\}$

Formal definitions

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

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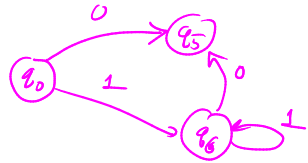

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Formal definitions

$\delta : A \to B$

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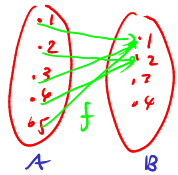
- a transition function δ

Formal definitions



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Formal definitions

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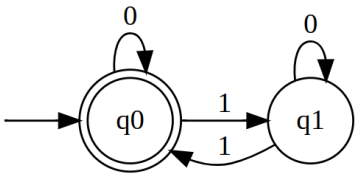
- the input alphabet Σ,
- a finite set of states Q,
- a transition function $\delta : Q \times \Sigma \longrightarrow Q$
- a start state $s \in Q$, and
- a subset $A \subseteq Q$ of accepting states.

Example 1



- $\Sigma = \{0, 1\}$
- $Q = \{q_0, q_1\}$
- δ:

	0	1
q_0	q_0	q_1
q_1	q_1	q_0

- start state $s = q_0$
- accepting states $A = \{q_0\}$

$(\{0,1\}, \{q_0, q_1\}, \delta, s, A)$

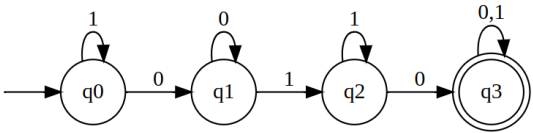
Example 2

$\Sigma = \{0, 1\}$

$Q = \{$ ——

start state q_0

accepting state $\{q_3\}$



Moves

One step move: from state q with input symbol a, the machine changes its state to

$$\delta(q, a)$$

Moves

- **One step move**: from state q with input symbol a, the machine changes its state to $\delta(q, a)$.
- **Extension:** from state q with input string w, the machine changes its state to $\delta^*(q, w)$ defined as

Case: $w = \varepsilon$, $\delta^*(q, w) = q$

Case: $w = a \cdot x$ where $a \in \Sigma$, x is a string

$$\delta^*(q, a \cdot x) = \delta^*(\delta(q, a), x)$$

Moves

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Extension: from state q with input string w, the machine changes its state to $\delta^*(q, w)$ defined as

$$\delta^*(q, w) = \begin{cases} q & \text{if } w = \varepsilon, \\ \delta^*(\delta(q, a), x) & \text{if } w = ax. \end{cases}$$

The signature of δ^* is $Q \times \Sigma^* \longrightarrow Q$.

Acceptance

For a finite-state machine with starting state s and accepting states A, it accepts string w iff

$$\delta^*(s, w) \in A$$

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Multiple of 5

$\Sigma = \{0, 1\}$

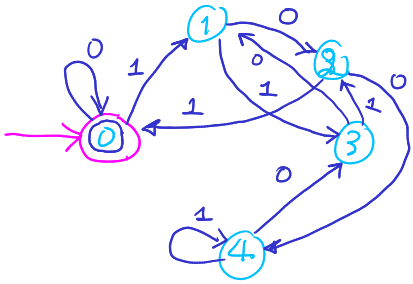
5, 10, 15, 25

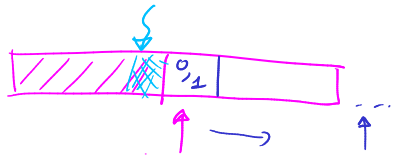
101, 1010, 1111

11001

x ↓1 2x+1

x ↓0 2x





$\delta(q, a)$

Multiple of 5

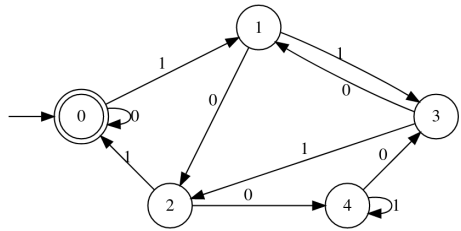
[0,1,1,0,1,0]

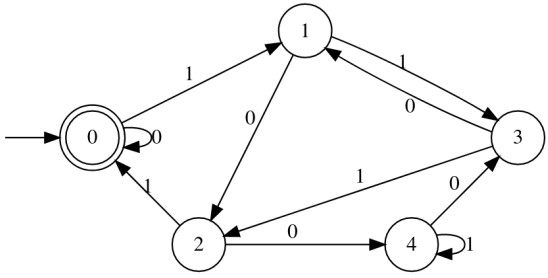
```
def multiple_of_5(w):
    r = 0
    for i in w:
        r = (2*r + w) % 5
    return r == 0
```

// $r = \delta(r, w)$

Multiple of 5

```
def multiple_of_5(w):
    r = 0
    for i in w:
        r = (2*r + w) % 5
    return r == 0
```




0
1
2
3
4
1
0
1
0
1
0
1
0
0
1

Digital design: Implementation

Digital design: Moore and Mealy machines

In the digital design class, you will encounter finite-state machines as well. The version we consider in this class is refered to as a **Moore machine**.
In practices, there is another variant of FSM called **Mealy machines**, whose outputs depend on input symbols as well.

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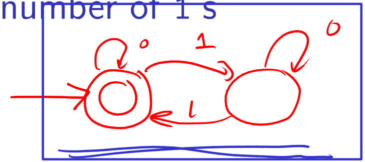
Formally, they differ in output function.

- Moore machine: $G : Q \longrightarrow [0, 1]$
- Mealy machine: $G : Q \times \Sigma \longrightarrow [0, 1]$

Example: even number of 1's



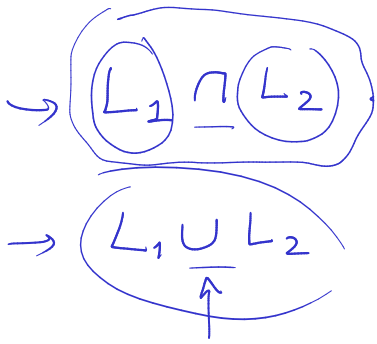
M_1

$(0^*10^*10^*)^*$

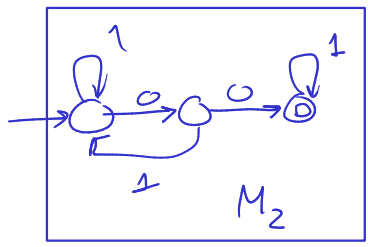
$L_1 = \{ w \mid w$ มี จน. 1 เป็นคู่ $\}$

Example: strings containing 00 as a substring

$L_2 = \{w \mid w$ has 00 as a substring$\}$



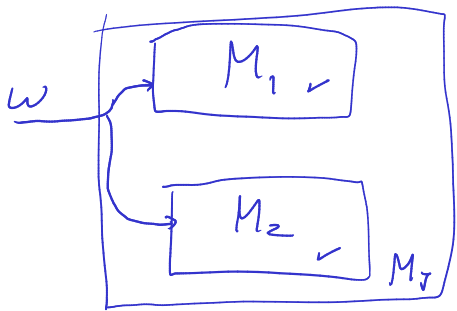
$(0+1)^* 00 (0+1)^*$

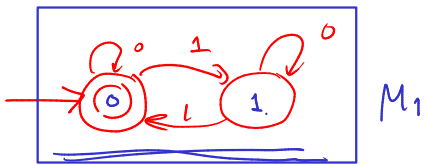


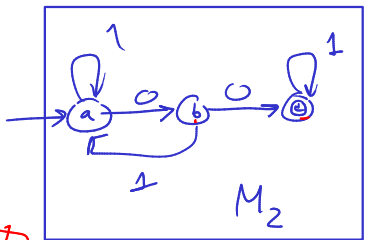
Combining DFAs

011001

What if we want to build a DFA that accepts strings with an even number of 1's and containing 00 as a substring?







(q_0, r_0)

0 ↓

(q_0, r_1)

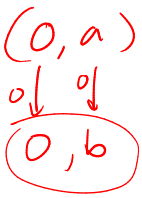
1 ↓

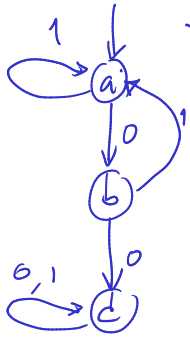
(q_1, r_0) —1→ (q_0, r_0) —0→ (

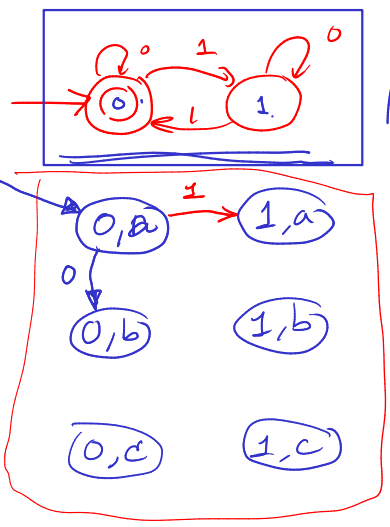
(q, r)

→Product construction

$Q \times \Sigma$







Product construction (formally)

Given a DFA $M_1 = (\Sigma, Q_1, \delta_1, s_1, A_1)$ that accepts strings from language L_1 and $M_2 = (\Sigma, Q_2, \delta_2, s_2, A_2)$ that accepts strings from language L_2, we can construct a DFA $M = (\Sigma, Q, \delta, s, A)$ that accepts strings from $L_1 \cap L_2$ as follows:

- Let $Q = Q_1 \times Q_2$,

$\delta : Q \times \Sigma \to Q$

$\delta : (Q_1 \times Q_2) \times \Sigma \to Q_1 \times Q_2$

(q_i, r_j) $\quad a$ $\quad (\delta_1(q_i, a), \delta_2(q_j, a))$

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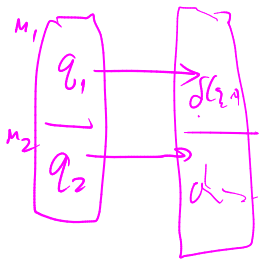
$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)),$$



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- Let $s = (s_1, s_2)$, and

Product construction (formally)

δ – one step
δ^*

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$$\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)),$$

- Let $s = (s_1, s_2)$, and
- Let $A = A_1 \times A_2$.

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state q_0 M_1 — state M_2

Lemma 1

$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$ *for any string* w.

Proof.

We prove by induction. I.H.: Assume that $\delta^*((q_1, q_2), x) = (\delta_1^*(q_1, x), \delta_2^*(q_2, x))$, for any string x such that $|x| < |w|$.

Case: $w = \varepsilon$, $\delta^*((q_1, q_2), \varepsilon) = (q_1, q_2) = (\delta_1^*(q_1, \varepsilon), \delta_2^*(q_2, \varepsilon))$ 성립함

Case: $w = ax$ for $x \neq \varepsilon$

$$\delta^*((q_1, q_2), ax) = \delta^*(\delta((q_1, q_2), a), x) = \delta^*((\delta_1(q_1, a), \delta_2(q_2, a)), x)$$

$$\text{귀납적 가정} = (\delta_1^*(\delta_1(q_1, a), x), \delta_2^*(\delta_2(q_2, a), x)) = (\delta_1^*(q_1, ax), \delta_2^*(q_2, ax))$$

$$= (\delta_1^*(q_1, w), \delta_2^*(q_2, w))$$

i/Correctness ← M proves M accepts string w iff M_1 & M_2 accept w

$L_1 \cap L_2$.

From the previous lemma, we have that

$$\delta^*(s, w) = \delta^*((s_1, s_2), w)$$

$$= (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$$

Correctness

From the previous lemma, we have that

$$\begin{aligned} \delta^*(s, w) &= \delta^*((s_1, s_2), w) \\ &= (\delta_1^*(s_1, w), \delta_2^*(s_2, w)) \end{aligned}$$

Correctness

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$$\begin{aligned}\delta^*(s, w) &= \delta^*((s_1, s_2), w) \\ &= (\delta_1^*(s_1, w), \delta_2^*(s_2, w))\end{aligned}$$

Thus, for an input w, M would reach the state $(\delta_1^*(s_1, w), \delta_2^*(s_2, w))$; it accepts w when

$$(\delta_1^*(s_1, w), \delta_2^*(s_2, w)) \in A_1 \times A_2.$$

This implies that M accepts w when $\delta_1^*(s_1, w) \in A_1$ and $\delta_2^*(s_2, w) \in A_2$, i.e., M accepts w iff M_1 and M_2 accept w.
Finally, we conclude that M accepts strings from language $L_1 \cap L_2$.

Language of a DFA

$L(M)$

For a DFA M, let $L(M)$ be the set of all strings that M accepts. More formally, for $M = (\Sigma, Q, \delta, s, A)$,

$$L(M) = \{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

We refer to $L(M)$ as the language of M.

Closure

Lemma 2

If L_1 and L_2 are languages of some DFAs M_1 and M_2, we have that

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- *there is a DFA M that accepts $L_1 \cup L_2$,*

$A = \{(q_1, q_2) \mid q_1 \in A_1 \text{ and } q_2 \in A_2\}$

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- *there is a DFA M that accepts $\Sigma^* \setminus L_1$,*

Automatic languages[2]

Definition (for now)

A language L is **"automatic"** if there is a DFA M such that $L(M) = L$.

[2] Taken directly from Erickson's lecture notes

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- $L_1 \setminus L_2$, *and*

[2] Taken directly from Erickson's lecture notes

Automatic languages[2]

Definition (for now)

A language L is **"automatic"** if there is a DFA M such that $L(M) = L$.

Lemma 3

If L_1 and L_2 are automatic languages over alphabet Σ, then

- $L_1 \cap L_2$,
- $L_1 \cup L_2$,
- $L_1 \setminus L_2$, *and*
- $\Sigma^* \setminus L_1$,

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The set of automatic languages is closed under these boolean operations.

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