01204211 Discrete Mathematics Lecture 11b: Four fundamental subspaces (I)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{5} & \frac{3}{6} \\ \frac{7}{6} & \frac{8}{6} & \frac{9}{10} \end{bmatrix}$$

Vector spaces related to a matrix

Consider an m-by-n matrix A over \mathbb{R} .

We can view A as

- ightharpoonup n columns of m-vectors: c_1, c_2, \ldots, c_n
- ightharpoonup m rows of *n*-vectors: r_1, r_2, \ldots, r_m

When we have a set of vectors, recall that its span forms a vector space.

We have

- ightharpoonup Column space: Span $\{c_1, c_2, \ldots, c_n\} \subseteq \mathbb{R}^m$
- ightharpoonup Row space: Span $\{m{r}_1, m{r}_2, \dots, m{r}_m\} \subseteq \mathbb{R}^n$

Subspaces

Definition

Let \mathcal{V} and \mathcal{W} be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that \mathcal{V} is a subspace of \mathcal{W} .

Examples:

- ▶ Span $\{[1,1]\}$ is a subspace of \mathbb{R}^2 .
- ▶ Span $\{[1,0,0],[0,1,1]\}$ is a subspace of \mathbb{R}^3 .
- ▶ Span $\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of \mathbb{R}^3 .

Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Column space:

$$C(A) = \{\alpha_1[1,0] + \alpha_2[2,1] + \alpha_3[4,3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} = \mathbb{R}^2.$$

Note that: $\dim \mathcal{C}(A) = 2$

► Row space:

$$C(A^T) = \{ \alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R} \} \subseteq \mathbb{R}^3.$$

Note that: $\dim \mathcal{C}(A^T) = 2$

Example 1 (cont.)

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

Is there any other way to obtain vector spaces from A?

We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$A\mathbf{x} = 0.$$

In this case, we have

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The set of solutions $\{x \mid Ax = 0\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product $A\boldsymbol{x}$. Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Four fundamental subspaces

Four fundamental subspaces

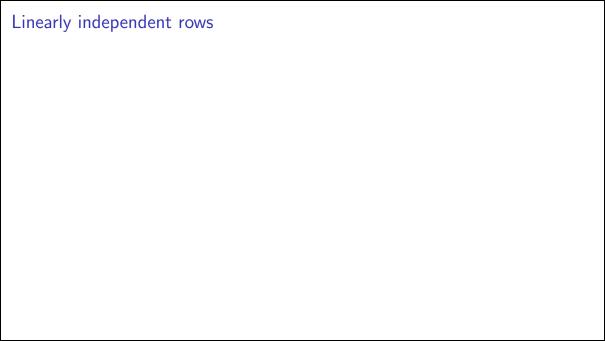
Given an m-by-n matrix A, we have the following subspaces

- ightharpoonup The column space of A (denoted by $\mathcal{C}(A)\subseteq\mathbb{R}^m$)
- ightharpoonup The row space of A (denoted by $\mathcal{C}(A^T)\subseteq\mathbb{R}^n$)
- ightharpoonup The nullspace of A

$$\mathcal{N}(A) = \{ \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{0} \} \subseteq \mathbb{R}^n$$

ightharpoonup The left nullspace of A

$$\mathcal{N}(A^T) = \{ \boldsymbol{y} \mid A^T \boldsymbol{y} = \boldsymbol{0} \} \subseteq \mathbb{R}^m$$



Ranks

Definition

Consider an m-by-n matrix A.

- ightharpoonup The row rank of A is the maximum number of linearly independent rows of A.
- ightharpoonup The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{C}(A)$. The row rank of A is $\dim \mathcal{C}(A^T)$.

Row rank = Column rank

Theorem 1

For any matrix A, its row rank equals its column rank.

We will prove this theorem next time.