01204211 Discrete Mathematics
Lecture 9b: Nonregular languages[1]

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August 28, 2025

[1]Based on lecture notes of *Models of Computation* course by Jeff Erickson.

DFA: Formal definitions

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

- the input alphabet Σ,
- a finite set of states Q,
- a transition function $\delta : Q \times \Sigma \longrightarrow Q$
- a start state $s \in Q$, and
- a subset $A \subseteq Q$ of accepting states.

Acceptance

One step move: from state q with input symbol a, the machine changes its state to $\delta(q,a)$.

Extension: from state q with input string w, the machine changes its state to $\delta^*(q,w)$ defined as

$$\delta^*(q,w) = \begin{cases} q & \text{if } w = \varepsilon, \\ \delta^*(\delta(q,a),x) & \text{if } w = ax. \end{cases}$$

The signature of δ^* is $Q \times \Sigma^* \longrightarrow Q$.

accepting w

For a finite-state machine with starting state s and accepting states A, it accepts string w iff

$$\delta^*(s,w) \in A.$$

Language of a DFA

$L(M)$

For a DFA M, let $L(M)$ be the set of all strings that M accepts. More formally, for $M = (\Sigma, Q, \delta, s, A)$,

$$L(M) = \{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

We refer to $L(M)$ as the language of M.

Automatic languages[2]

Definition (for now)

A language L is **"automatic"** if there is a DFA M such that $L(M) = L$.

Lemma 1

If L_1 and L_2 are automatic languages over alphabet Σ, then

- $L_1 \cap L_2$,
- $L_1 \cup L_2$,
- $L_1 \setminus L_2$, *and*
- $\Sigma^* \setminus L_1$,

are also automatic.

The set of automatic languages is closed under these boolean operations.

[2] Taken directly from Erikson's lecture notes

Other ways to combine DFAs?

Given two languages L_1 and L_2, we can combine them in various ways using Boolean operations (i.e., $\cap$, $\cup$, etc.).
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- Kleene closure: L_1^*.

Interesting questions

We know that the set of automatic languages is closed under Boolean operations.

Questions

- Is it closed under concatenation?
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Questions

- Is it closed under concatenation?
- Is it closed under taking Kleene closure?

Spoiler: Yes, it is (for both operations). We will see the proof, after we learn a required new concept.

Closure

Lemma 2

Given two automatic languages L_1 and L_2, the following languages are automatic:

- $L_1 \cup L_2$,
- $L_1 \cdot L_2$, *and*
- L_1^*.

Closure

Lemma 2

Given two automatic languages L_1 and L_2, the following languages are automatic:

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More over, $\emptyset$ and a language containing a single string are also automatic.

Closure

More over, $\emptyset$ and a language containing a single string are also automatic.

Lemma 2

These are automatic languages

- *The empty set,*
- *A language containing one string,*
- $L_1 \cup L_2$ *for automatic languages* L_1 *and* L_2*,*
- $L_1 \bullet L_2$ *for automatic languages* L_1 *and* L_2*, and*
- L^* *for automatic languages* L*.*

Doese this look familiar?

Regular languages

Definition: regular languages

A language L is **regular** if and only if it satisfies one of the following conditions:

- L is empty;
- L contains one string (can be the empty string ε);
- L is a union of two regular languages;
- L is the concatenation of two regular languages; or
- L is the Kleene closure of a regular language.

$\Longrightarrow$

Every regular language is automatic

Big question:

$\Rightarrow$

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Big question:

$\Leftarrow$

Is every automatic language regular?

Spoiler:

$\Rightarrow$

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Spoiler: Yes, it is. We will see some idea on how to prove this.

$\Rightarrow$

Every regular language is automatic

Big question:

$\Leftarrow$

Is every automatic language regular?

Spoiler: Yes, it is. We will see some idea on how to prove this.

Theorem 3

A language L is regular if and only if there exists a DFA M such that $L(M) = L$.

Nonregular languages

Can you design a DFA that accepts strings from language

$$\{0^n 1^n \mid n \geq 0\}$$

Key idea

If you have finite states, you can't possibly distinguish between strings in the language and strings not in the language.

Basic question

How can you show that you need at least two states?

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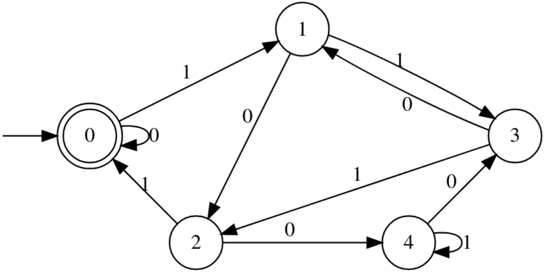
Let's see how a DFA works.

Another example



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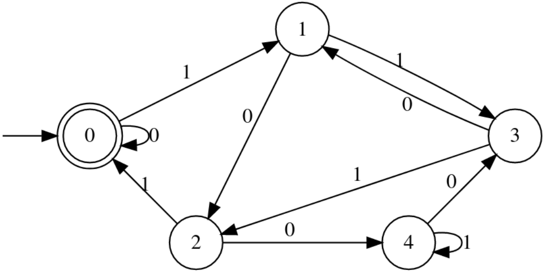

If string x and y reach the same state in a DFA, for any string z, both xz and yz must reach the same state.

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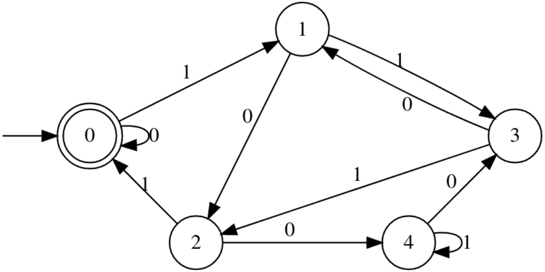

If string x and y reach the same state in a DFA, for any string z, both xz and yz must reach the same state.
In other words, a DFA accepts xz iff it accepts yz.

Distinguishing suffixes

Consider language $L = \{0^n 1^n \mid n \geq 0\}$.
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What can you say about a DFA M such that $L(M) = L$?

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What can you say about a DFA M such that $L(M) = L$?

Definition

For strings x and y, string z is a **distinguishing suffix** with respect to L if exactly one of xz and yz is in L.

Fooling sets

A **fooling set** for a language L is set F of strings such that every pair of strings in F has a distinguishing suffix.

Example: The set $\{0, 00, 000\}$ is a fooling set for $L = \{0^n1^n \mid n \geq 0\}$.

A large fooling set

Lemma 4

The set $\{0^n \mid n \geq 0\}$ *is a fooling set for* $L = \{0^n 1^n \mid n \geq 0\}$.

Proof.

$\square$

Observation

If language L has an infinite fooling set, L is not regular

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Lemma 5

Language $L = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof.

We previously establish that the set $F = \{0^n \mid n \geq 0\}$ is a fooling set for L.
Since F has infinite size, from the observation, we know that L is not regular. $\square$

Lemma 6

For $\Sigma = \{0, 1\}$, *the language* $L = \{ww^R \mid w \in \Sigma^*\}$ *is not regular.*

Proof.

$\square$

$L = \{0^{2^n} \mid n \geq 0\}$: Proof 1

Lemma 7

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.

$\square$

$L = \{0^{2^n} \mid n \geq 0\}$: Proof 2

Lemma 8

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.

$\square$

$L = \{0^{2^n} \mid n \geq 0\}$: Proof 3

Lemma 9

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.

$\square$

$L = \{0^p \mid p$ is prime$\}$: Proof 1

Lemma 10

For $\Sigma = \{\mathtt{0}\}$, the language $L = \{0^p \mid p$ is prime$\}$ is not regular.

Proof.

$\square$

$L = \{0^p \mid p$ is prime$\}$: Proof 2

Lemma 11

For $\Sigma = \{0\}$, the language $L = \{0^p \mid p$ is prime$\}$ is not regular.

Proof.

$\square$