01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (preview)

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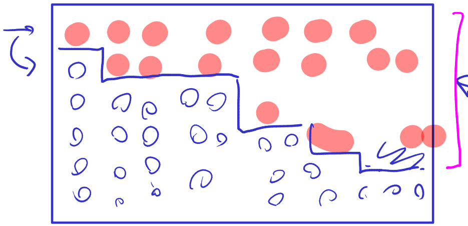
What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

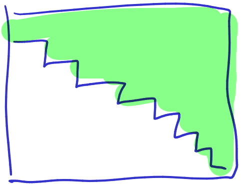
Matrices and elimination



r

row echelon form

upper triangular



row operation

$$R2 \leftarrow R2 + \alpha R1$$

Matrices and elimination

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

Matrices and elimination

$$\boxed{Ax = b}$$

if A is in row echelon form:

→ can find solution quickly.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{array}{c} u \\ v \\ w \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\boxed{3x_2 + x_3 = 1}$$

is u a linear comb of v & w ?

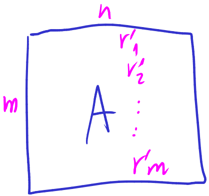
$$\boxed{u} = \alpha_1 v + \alpha_2 w$$

Are there α_1, α_2 such that $u = \alpha_1 v + \alpha_2 w$

$$\begin{aligned} x_1 + 2x_2 + \quad x_4 &= 3 \\ 3x_2 + x_3 &= 1 \\ 2x_4 &= 2 \end{aligned}$$

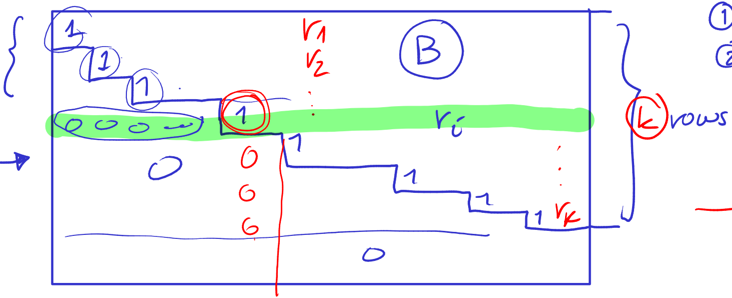
Back substitution.

Row echelon form



row space of A

row echelon form



Definition

A set of vectors $\{u_1, \dots, u_k\} = U$

(1) U spans vector space V

(2) $u_1, u_2, \dots$ are lin. indep.

$\Rightarrow u_1, u_2, \dots, u_k$ is a $\boxed{\text{basis}}$ of V

$r_1, r_2, \dots, r_k$ is a basis of $C(A^T)$

- $r_1, r_2, \dots, r_k$ are linearly independent.
- $\boxed{\text{Span}\{r_1, r_2, \dots, r_k\}} = \text{Span}\{r'_1, r'_2, \dots, r'_m\} = C(A^T)$

from $B \xrightarrow{\text{recreate.}} A$

$\boxed{C(A^T)} = \text{Span}\{r'_1, r'_2, \dots, r'_m\}$

Linearly independent rows

Vector spaces related to a matrix

Consider an m-by-n matrix A over $\mathbb{R}$.

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A

row space $C(A^T)$

$\dim C(A^T) = k$

col space

$\dim C(A) = k$

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- Row space: $\mathrm{Span}\,\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\} \subseteq \mathbb{R}^n$

Subspaces

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Examples:

- $\mathrm{Span}\,\{[1,1]\}$ is a subspace of $\mathbb{R}^2$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1]\}$ is a subspace of $\mathbb{R}^3$.
- $\mathrm{Span}\,\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of $\mathbb{R}^3$.

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$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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$$\mathcal{C}(A) = \{\alpha_1[1,0] + \alpha_2[2,1] + \alpha_3[4,3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

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We can think of A as a coefficient matrix of a system of homogenous linear equations:

$$A\boldsymbol{x} = 0.$$

In this case, we have

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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The set of solutions $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product $A\boldsymbol{x}$. Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{C}(A)$)
- The row space of A (denoted by $\mathcal{C}(A^T)$)
- The nullspace of A

$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$$

- The left nullspace of A

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f

$$\boxed{A}\begin{bmatrix} x \\ \vdots \end{bmatrix} = 0$$

- The left nullspace of A

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$$\boxed{Ax = 0}$$

$f(x) = Ax$

$$x \longrightarrow \boxed{f} \longrightarrow$$

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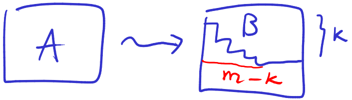

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$\dim C(A) = \dim C(A^T) = k$

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$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

$\dim N(A) = n-k$

- The left nullspace of A

$$\mathcal{N}(A^T) = \{\boldsymbol{y} \mid A^T\boldsymbol{y} = \boldsymbol{0}\} \subseteq \mathbb{R}^m$$

$\dim N(A^T) = m-k$