## 01204211 Discrete Mathematics Lecture 8b: Vectors and Matrices

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### What is a vector?

You can think of a **vector** as an "ordered" list of elements (which are typically numbers). For example:

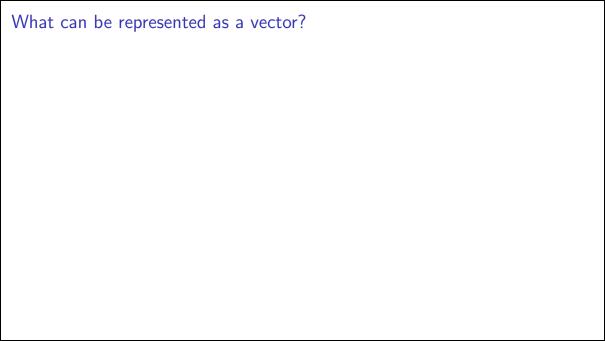
- ightharpoonup [1, 2, 5, 20]
- $\triangleright$  [0, 0, 1, 1, 0, 0, 0, 1]

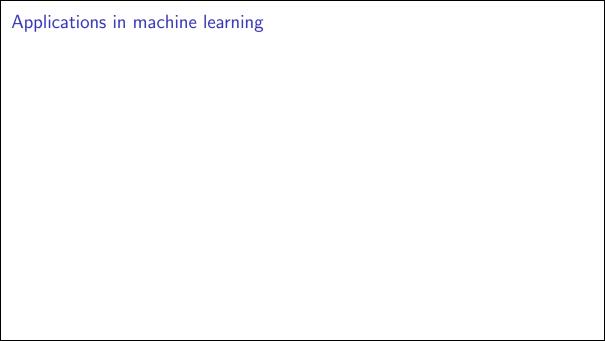
You can also view a vector as a **function**, e.g., you can view  ${\pmb u}=[1,2,5,20]$  as a function  ${\pmb u}$  that maps

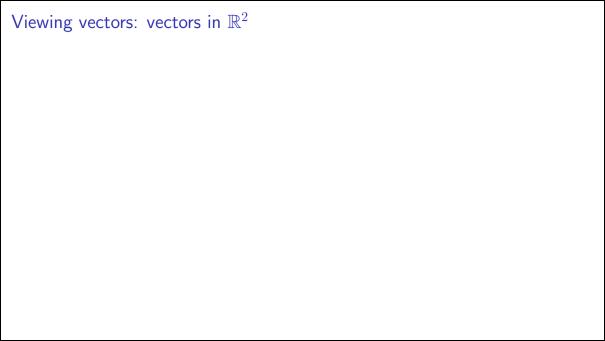
$$0 \mapsto 1$$
,  $1 \mapsto 2$ ,  $2 \mapsto 5$ ,  $3 \mapsto 20$ .

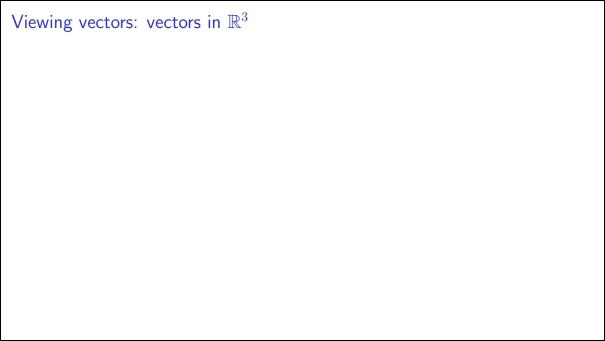
Each element in the vector is typically a real number  $(\mathbb{R})$ , but can be an element from other sets with appropriate property (more on this later).

Remark: Mathematically, a vector is an element of a vector space. We will understand this more later.









### n-vectors over $\mathbb R$

- We mostly deal with vectors with finite number of elements.
- ► This is a 4-vector: [10, 20, 500, 4].
- ▶ We sometimes also write it as a column vector:

$$\begin{bmatrix} 10\\20\\500\\4 \end{bmatrix}$$

When every element of a vector is from some set, we say that it is a vector **over** that set. For example, [10, 20, 500, 4] is a 4-vector over  $\mathbb{R}$ .

### Vector operations

- As discussed in the previous slides, when working with a system of linear equations, we mostly deals with **linear combinations** of vectors.
- ▶ We will look at the operations we do to vectors to obtain their linear combinations.
- ► The operations are:
  - Vector additions
  - Scalar multiplications
- ▶ These operations motivate the definition of vector spaces.

## Vector additions

Given two n-vectors

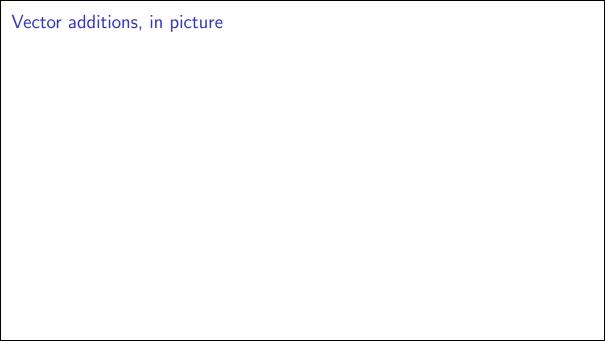
 $\boldsymbol{u} = [u_1, u_2, \dots, u_n]$ 

 $u + v = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n].$ 

and

 $\boldsymbol{v} = [v_1, v_2, \dots, v_n],$ 

we have that



### Zero vectors

A zero 
$$n$$
-vector  $\mathbf{0} = [0,0,\dots,0]$  is an additive identity, i.e., for any vector  $\boldsymbol{u}$ ,

$$0 + u = u + 0 = u$$
.

### Scalar multiplications

For a vector over  $\mathbb{R}$ , we refer to an element  $\alpha$  in  $\mathbb{R}$  as a scalar. For an n-vector

$$\boldsymbol{u} = [u_1, u_2, \dots, u_n],$$

we have that

$$\alpha \cdot \boldsymbol{u} = [\alpha \cdot u_1, \alpha \cdot u_2, \dots, \alpha \cdot u_n],$$



### Linear combinations

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is a linear combination of  $oldsymbol{u}_1,\dots,oldsymbol{u}_m.$ 

Examples:

## A linear system with 3 variables

Give the following linear system.

If we rewrite the system as

$$\begin{bmatrix} 2\\1\\4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4\\0\\2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3\\5\\3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7\\12\\10 \end{bmatrix}.$$

This becomes the problem of expressing a vector as linear combination of other vectors. I.e., given vectors

$$u_1 = [2, 1, 4], \quad u_2 = [4, 0, 2], \quad u_3 = [3, 5, 3]$$

we would like to find coefficients  $x_1, x_2, x_3$  such that

$$x_1 \cdot \boldsymbol{u}_1 + x_2 \cdot \boldsymbol{u}_2 + x_3 \cdot \boldsymbol{u}_3 = [7, 12, 10].$$

### Span

A set of all linear combination of vectors  $u_1, u_2, \dots, u_m$  is called the **span** of that set of vectors.

It is denote by  $\mathrm{Span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_m\}$ .

Examples:

### Convex combination

For any scalar

$$\alpha_1, \alpha_2, \ldots, \alpha_m,$$

such that  $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$  and  $\alpha_i \geq 0$  for all i, and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \dots + \alpha_m \boldsymbol{u}_m$$

is a convex combination of  $oldsymbol{u}_1,\dots,oldsymbol{u}_m.$ 

Examples:

### What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{5} & \frac{3}{6} \\ \frac{7}{10} & \frac{8}{11} & \frac{9}{12} \end{bmatrix}$$

## A matrix from a system of linear equations

Consider the following system of linear equations:

Again we can view it as a vector equation:

$$\begin{bmatrix} 1\\2\\3 \end{bmatrix} x_1 + \begin{bmatrix} 1\\1\\1 \end{bmatrix} x_2 + \begin{bmatrix} 1\\2\\2 \end{bmatrix} x_3 = \begin{bmatrix} 5\\10\\4 \end{bmatrix}$$

## A matrix from a system of linear equations

From the following system of linear equations

We can also view variables  $x_1,x_2,x_3$  as a vector, i.e., let  $m{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$  .

The coefficients form a nice rectangular "matrix" 
$$A$$
:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

and rewrite the system as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix}$$

### Size

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

The **size** of a matrix is determined by the number of rows and columns. A matrix with m rows and n columns is referred to as an m-by-n matrix or an  $m \times n$  matrix. We refers to m and n as its **dimensions**.

## Matrix-Vector Multiplication

How would we understand the multiplication

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

**By rows.** Consider the first row of A:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3.$$

Let's look at another two rows:

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3, \qquad \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3,$$

# Matrix-Vector Multiplication by Rows

We look at matrix-vector multiplication with "row perspective". This is a common way to view matrix-vector multiplication.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

Recall:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3.$$
$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3,$$

$$\begin{bmatrix} x_{1} \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_{1} \\ x_{2} \\ x_{3} \end{bmatrix} = 3 \cdot x_{1} + 1 \cdot x_{2} + 2 \cdot x_{3},$$

## Review: Dot product

### Definition

For n-vectors  ${\boldsymbol u}=[u_1,u_2,\dots,u_n]$  and  ${\boldsymbol v}=[v_1,v_2,\dots,v_n]$ , the **dot product** of  ${\boldsymbol u}$  and  ${\boldsymbol v}$ , denoted by  ${\boldsymbol u}\cdot{\boldsymbol v}$ , is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

## Matrix-Vector Multiplication by Rows

We look at matrix-vector multiplication with "row perspective", which can be written nicely with **dot product**.

I.e., from:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

we have

$$\left[ egin{array}{c} rac{oldsymbol{r}_1}{oldsymbol{r}_2} \ \hline oldsymbol{r}_3 \end{array} 
ight] oldsymbol{x} = \left[ egin{array}{c} rac{oldsymbol{r}_1 \cdot oldsymbol{x}}{oldsymbol{r}_2 \cdot oldsymbol{x}} \ \hline oldsymbol{r}_3 \cdot oldsymbol{x} \end{array} 
ight],$$

where

$$r_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}.$$

### Dot-product perspective

The matrix-vector product is a vector of **dot products** between each rows and the vector.

## Matrix-Vector Multiplication by Columns

However, another nice way to look at matrix-vector multiplication is **by columns**. Notice that:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

can be written as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix}$$

### Linear combination perspective

The matrix-vector product is a linear combination of column vectors.

### Two perspectives: Matrix-Vector multiplication

#### Dot products between rows and the vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 \\ a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 \\ a_{41} \cdot x_1 + a_{42} \cdot x_2 + a_{43} \cdot x_3 \end{bmatrix}$$

#### Linear combination of column vectors

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \cdot x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix} \cdot x_2 + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \end{bmatrix} \cdot x_3$$

#### **Dimensions**

If the matrix has n columns, the vector should be an n-vector.

### Document search

- ➤ You have 1,000,000 documents in a library. Given another document, you would like to find similar documents from the library. How can you do that?
- You need some way to measure document similarity.
- ▶ Suppose that you nave N documents in the library:  $d_1, d_2, \ldots, d_N$ . Given a query document q, you want to find document  $d_i$  that maximize

$$sim(d_i, q),$$

where sim(d, d') is the similarity score between documents d and d'.

### Document vector models

What is a document? It's just a list of words. If you throw all the ordering away, a document is simply a set of words.

Let's start with an example. Suppose that we only care about 5 words: dog, cat, food, restaurant, and coffee.

Consider the following 4 (very short) documents:

- $lacksquare d_1$ : People love pets. Most famous pets are cats and dogs.
  - $d_1 = \{ \mathtt{dog}, \mathtt{cat} \}$
- $ightharpoonup d_2$ : Bar Mai has many restaurants with cheap foods.

$$d_2 = \{ \texttt{restaurant}, \texttt{food} \}$$

lacksquare  $d_3$ : Cat cafe used to be popular in Thailand. People buy coffee and play with cats there.

$$d_3 = \{ coffee, cat \}$$

lacksquare d4: Dogs are human's best friends. They were around in civilization for a long long time.

$$d_4 = \{ \mathsf{dog} \}$$

How can we translate these sets into vectors?

### Document vector models

We assign a fixed co-ordinate for each word, and if a set contain a particular word, we put 1 in that co-ordinate.

Here are our 5 words: dog, cat, food, restaurant, and coffee. Each document becomes:

- $d_1$ : People love pets. Most famous pets are cats and dogs.  $d_1 = \{\deg, \mathsf{cat}\}$ ,  $oldsymbol{d}_1 = [1,1,0,0,0]$
- $m{d}_2$ : Bar Mai has many restaurants with cheap foods.  $d_2=\{ ext{restaurant}, ext{food}\}, m{d}_2=[0,0,1,1,0]$
- $m d_3$ : Cat cafe used to be popular in Thailand. People buy coffee and play with cats there.  $d_3=\{ exttt{coffee}, exttt{cat}\}, \ m d_3=[0,1,0,0,1]$
- $d_4$ : Dogs are human's best friends. They were around in civilization for a long long time.  $d_4=\{\deg\},\ m{d}_4=[1,0,0,0,0]$

### Document vector models

Words: dog, cat, food, restaurant, and coffee.

Suppose that we have query document:

```
q\colon I love cats and coffee. What restaurant should I visit?
```

as a set:  $q = \{ \texttt{cat}, \texttt{coffee}, \texttt{restaurant} \}$  as a vector:  ${\boldsymbol q} = [0, 1, 0, 1, 1]$ 

#### Our documents are:

- $d_1$ : People love pets. Most famous pets are cats and dogs.  $d_1 = \{ \mathsf{dog}, \mathsf{cat} \}$   $d_1 = [1, 1, 0, 0, 0]$
- $ightharpoonup d_2$ : Bar Mai has many restaurants with cheap foods.
- $oldsymbol{d}_2 = \{ exttt{restaurant}, exttt{food} \} \qquad oldsymbol{d}_2 = [0,0,1,1,0]$
- $lack d_3$ : Cat cafe used to be popular in Thailand. People buy coffee and play with cats there.
- $d_3 = \{ \mathtt{coffee}, \mathtt{cat} \} \qquad {m d}_3 = [0, 1, 0, 0, 1]$
- $d_4$ : Dogs are human's best friends. They were around in civilization for a long long time.  $d_4=\{\log\}$   $d_4=[1,0,0,0,0]$

How can we define "similarity" measure?

## Dot products as a similarity measure

From the previous example, we see that the dot products between  $d_i$ 's and q count the number of common words.

This simple idea can be extended in many ways.

- ▶ We can increase our "dictionary" 's size to include more words.
- ▶ We can group similar words into the same "co-ordinates".
- ▶ In fact, the dot product measures the "angle" between vectors. For vectors over  $\mathbb{R}$ , we have that

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta,$$

where  $\theta$  is the angle between vectors  $\boldsymbol{u}$  and  $\boldsymbol{v}$ .

## Computing all similarity scores

If we have documents  $d_1, d_1, \ldots, d_N$ , as vectors, and a query q, how can we compute all similarity scores?

By performing matrix-vector multiplication:

$$egin{bmatrix} egin{array}{c} egin{array}{c} d_1 \ \hline d_2 \ \hline \vdots \ \hline d_N \ \end{bmatrix} egin{bmatrix} egin{array}{c} sim(oldsymbol{d}_1,oldsymbol{q}) \ sim(oldsymbol{d}_2,oldsymbol{q}) \ dots \ sim(oldsymbol{d}_N,oldsymbol{q}) \ \end{bmatrix}$$

### Vector-matrix multiplication

Let's consider another direction. What is

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}?$$

As a linear combination

As dot products

## Matrix-matrix multiplication

Consider

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

## Matrix-matrix multiplication (based on matrix-vector multiplication)

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

## Matrix-matrix multiplication (based on vector-matrix multiplication)

$$\begin{bmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix}.$$

### Matrix transpose

If A is an  $m \times n$  matrix

$$\left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}\right],$$

the **transpose** of A, denoted by  $A^T$  is an  $n \times m$  matrix

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Remark: We usually view a vector as a column vector. Therefore, a dot product between m-vectors can be viewed also as a matrix multiplication:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$$

# Matrix multiplication and transpose

What is  $(AB)^T$ ?

### Key-Value database

Suppose you have a database of key-value pairs:

```
\{(somchai,10),(somying,14),(somnuk,23),(somjai,50),(somsom,-40)\}
```

Given a query q, you want to find a value v such that (q, v) is in the database. E.g., Let's see how we could do that (very **inefficiently**) with matrix multiplication.

# Vector encodings of keys and queries

ightharpoonup You want to have **distinct** keys:  $oldsymbol{k}_1, oldsymbol{k}_2, \dots, oldsymbol{k}_n$ 

▶ You want a query **q** to **match** with an appropriate key. (Maybe the key which is exactly the same.)

### Example

► Key encoding:

$$somchai = [0, 1, 0, 0, 0, 0], \ somying = [0, 0, 0, 0, 1, 0], \ somnuk = [1, 0, 0, 0, 0, 0],$$

$$som jai = [0, 0, 1, 0, 0, 0], \qquad som som = [0, 0, 0, 0, 0, 1]$$

A value table (or vector): 
$$v = \begin{bmatrix} 10\\14\\23\\50\\-40 \end{bmatrix}$$

lacktriangle A query  $m{q}$  is a 5-vector. A query matches key  $m{k}_i$  if

$$\boldsymbol{k}_i^T \boldsymbol{q} = 1$$

# Example (cont)

A value table (or vector): 
$$v = \begin{bmatrix} 10\\14\\23\\50\\-40 \end{bmatrix}$$

- Let's try querying with various q
- ► The final formula is

$$(K\boldsymbol{q})^T \boldsymbol{v} = (\boldsymbol{q}^T K^T) \boldsymbol{v}$$

# Key-Value database (with vector values)

Suppose you have a database of key-value pairs, where a value is a 2-vector:

```
\{(somchai, [10, 20]), (somying, [14, -2]), (somnuk, [23, 3]), (somjai, [50, -10])\}
```

Given a query q, can you find a 2-vector v such that (q, v) is in the database?

### Understanding self-attention formula

Self-attention mechanisms are key steps in transformers, work horses for all chatbots you have been using recently. The formula looks like (from wikipedia)

$$Attention(Q, K, V) = softmax\left(\frac{QK^T}{\sqrt{d_k}}\right)V$$