01204211 Discrete Mathematics
Lecture 5b: Counting 1

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- How to find these 2 representatives? One of your friends suggests that to be fair to everyone, you have to look at every possible pair and see how the 2 members of the pair play together as a team.
- It might take a very long time, you think. How many pairs are there?

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- The correct number of pairs is 780; too many possibilities to consider, you conclude.

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- Your friend argues that 15 is too small. Because the number of members is 40 and we will miss someone there.
- So you ask, how many pairs one have to randomly choose from 40 members so that it is very likely that every member is picked once?
- You try to calculate the number, but your friend starts writing a program to simulate.

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- Here's the table of the simulation. For each value of number of random pairs, 2,000 simulations has been conducted.

Number of pairs to random	% of choosing everyone once
20	0.00
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- You end up choosing randomly 100 pairs, as it has about 80% chance. You feel so tired, but you keep wondering if you can calculate the number without having to write a program.

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- To find the best team organization, you ask them to try all possible configurations of race choices against AI players. How many games do you have to watch?

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- Each member has 3 choices and this member's choice is independent of the other. Therefore, there are $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$ possible ways.
- You are still tired from watching 100 pairs of players. So you change your mind and ask them to try only configurations that contain all the three races. How many are there?

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  - Thus, the number of good configurations is $243 - 96 - 3 = 144$ ways.

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  - Thus, the number of good configurations is $243 - 96 - 3 = 144$ ways.
- That's not too many. So you let them play 144 games. It turns out that 144 is wrong.
- Can you spot the mistake?

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- In fact, it appears the first time we count single race configurations, appears when we count Protoss-Terrans, and finally appears when we count Protoss-Zerg.
- The number of bad configurations is $96 - 3$.
- The correct number of games would be $243 - (96 - 3) = 150$.

Sets: quick review (1)

- Sets are very important notions in mathematics. A **set** is a collection of elements.
- Common set of numbers: real numbers $\mathbb{R}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, positive integers $\mathbb{N}$.
- There are many ways to specify sets.
  - By listing all elements: $\{2, 3, 5, 7, 11\}$
  - By describing its elements: {all prime numbers}
  - By filtering elements from other sets: $\{p \in \mathbb{N} : p \text{ is a prime}\}$.

Sets: quick review (2)

- If a is an element of S, we write $a \in S$. The **cardinality** of a set is the number of its elements. We denote by $|A|$, the cardinality of A.
- Note that $|\{2, 3, 5, 7, 11\}| = 5$ and $|\mathbb{Z}| = \infty$. A set whose cardinality is zero is called an **empty set**, denoted by $\emptyset$.
- If every element of A is also an element of B, we say that A is a **subset** of B, denoted by $A \subseteq B$. For example,
  - $\{1, 3\} \subseteq \{1, 2, 3, 4\}$
  - $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$
  - $\{p \in \mathbb{N} : p \text{ is prime and } p > 2\} \subseteq \{x \in \mathbb{N} : x \text{ is odd}\}$

  Note that $A \subseteq A$ and $\emptyset \subseteq A$.
- If $A \subseteq B$ but $A \neq B$, we write $A \subset B$.

Set operations

Suppose that we are given two sets A and B.

- An **intersection**, denoted by $A \cap B$, is a set whose elements are elements of both A and B.
- A **union**, denoted by $A \cup B$, is a set whose elements are elements of A or B.
- A **difference** of A and B, denoted by $A \setminus B$ or $A - B$, is a set whose elements are elements of A but not elements of B.

Note that

- $A \cap B \subseteq A$
- $A \subseteq A \cup B$
- $A \setminus B \subseteq A$
- If $A \subseteq B$, then $A \setminus B = \emptyset$.

The number of subsets[1]

- Let's get back to counting. Our next question: *What is the number of all subsets of a set with n elements?*

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  - $\{1, 2, 3\}$ has 8 subsets.
- We can guess that the answer is 2^n. But how can we prove that?

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Choosing a subset (1)

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- There are many ways to do so, but let's follow an obvious one: consider each element and make a decision.

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  - 3rd step: let's consider 3. Whatever the decision that we make on 1 and 2, we again have 2 choices for 3.

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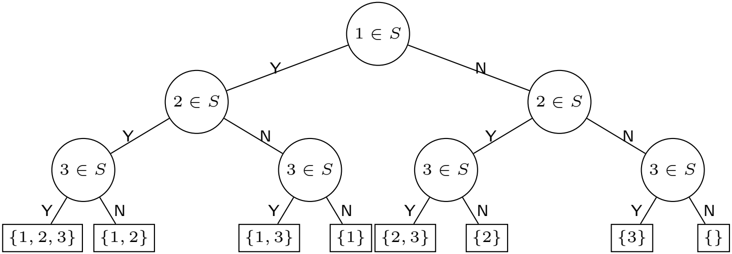
  - 3rd step: let's consider 3. Whatever the decision that we make on 1 and 2, we again have 2 choices for 3.
- This concludes that we have, in total, $2 \cdot 2 \cdot 2 = 8$ ways of choosing subsets of set $\{1, 2, 3\}$.

A decision tree

We describe the process as a decision tree for choosing subset S from $A = \{1, 2, 3\}$.



Choosing a subset (2)

- To be concrete, let's consider choosing a subset of set $\{1, 2, 3\}$.
- We make 3 decisions, for all elements in the set. The number of ways we can choose a subset is 8.
- While we know that it is the correct answer, let's look back on what we are trying to do.

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- To make sure that they are the same number, we need to make sure that:

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- We are trying to count the number of subsets. But we are actually counting the number of ways we can choose a subset.
- To make sure that they are the same number, we need to make sure that:
  - We count everything: for every possible subset, there is at least one way we can choose it.
  - We do not over count: any two different ways of choosing subsets produce two different subsets.

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Therefore, we have proved the following theorem.

> **Theorem:** The number of subsets of a set with n elements is 2^n.