01204211 Discrete Mathematics
Lecture 7b: Binomial Coefficients (2)

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The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- identities on binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

Identities in the Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Odd and even subsets

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Let's try to prove this identity with the Pascal's triangle

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

A more formal proof

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

ดูจากสไลด์ที่แล้ว.

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$1^2 = 1$$

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$\begin{aligned} 1^2 &= 1 \\ 1^2 + 1^2 &= 2 \end{aligned}$$

The next experiment

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Let's try to compute the sum of squares of numbers in each row.

$$\begin{aligned} 1^2 &= 1 \\ 1^2 + 1^2 &= 2 \\ 1^2 + 2^2 + 1^2 &= 6 \end{aligned}$$

The next experiment

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 & \binom{1}{1} \\
1^2 + 1^2 &= 2 & \binom{2}{1} \\
1^2 + 2^2 + 1^2 &= 6 & \binom{4}{2} \\
1^2 + 3^2 + 3^2 + 1^2 &= 20 & \binom{6}{3}
\end{aligned}
$$

The next experiment

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

70

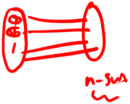
Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2 \\
1^2 + 2^2 + 1^2 &= 6 \\
1^2 + 3^2 + 3^2 + 1^2 &= 20 \\
1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70 \qquad \binom{8}{4}
\end{aligned}
$$

$\binom{4}{0}^2 + \binom{4}{1}^2 \cdots + \binom{4}{4}^2$

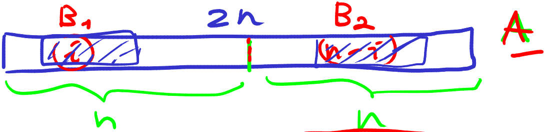
$\binom{n}{k} = \binom{n}{n-k}$

Theorem:



$$\underbrace{\binom{n}{0}^2} + \underbrace{\binom{n}{1}^2} + \underbrace{\binom{n}{2}^2 + \cdots + \binom{n}{n}^2} = \underbrace{\binom{2n}{n}}.$$

Proof: RHS คือ จน. n-subsets ของ 2n-set



พิจารณา $\underbrace{\binom{n}{i}^2} = \binom{n}{i}\binom{n}{i} = \boxed{\underbrace{\binom{n}{i}}\binom{n}{n-i}}$

จน n-subset ของ A ที่มีสมาชิกใน B_1 = i ตัว

เลือก i ตัว จาก B_1

เลือก $n-i$ ตัว จาก B_2

$\Rightarrow$ LHS นับ จน. n-subsets ของ A เช่นเดียวกัน ดังนั้น $= \binom{2n}{n}$

Another identity

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Another identity

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

This suggests

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Theorem:

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+k}{k}=\binom{n+k+1}{k}.$$

Proof: by induction บน k พิจารณา n ใดๆ

base case: $k=0$ $\quad \binom{n}{0}=1$ ✓ $\quad \binom{n+0+1}{0}=1$ ✓

inductive step: Assume ว่าจริง ที่ $k=m$ นั่นคือ

$P(m)$ $\quad \binom{n}{0}+\binom{n+1}{1}+\cdots+\binom{n+m}{m}=\binom{n+m+1}{m}$

จะแสดง $P(m+1)$

พิจารณา $\quad \underbrace{\binom{n}{0}+\binom{n+1}{1}+\cdots+\binom{n+m}{m}}+\binom{n+m+1}{m+1}$

(จาก I.H) $\quad =\binom{n+m+1}{m}+\binom{n+m+1}{m+1}=\binom{n+m+2}{m+1}$

ตามต้องการ

จาก Principle of Math Induction ประโยคจริงกับทุกๆ $k\geq 0$. ∎