01204211 Discrete Mathematics
Lecture 9b: Affine Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Review: Vector spaces

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

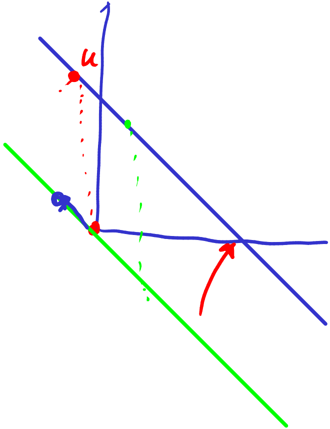
- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,
$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$
for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,
$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

Examples of vector spaces:

- A span of vectors is a vector space.
- A solution set to homogeneous linear equations is a vector space.

Translation

If we have a line or a plane passing through a vector $\boldsymbol{a}$, but not through the origin, how can we represent it?



Translation

If we have a line or a plane passing through a vector $\boldsymbol{a}$, but not through the origin, how can we represent it?

- Translate the object so that it passes through the origin.

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- Then we translate it back so that it passes through $\boldsymbol{a}$.

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- Translate the object so that it passes through the origin.
- We obtain a vector space $\mathcal{V}$.
- Then we translate it back so that it passes through $\boldsymbol{a}$.
- We get the set

$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

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- *Question:* Is $\mathcal{A}$ a vector space?

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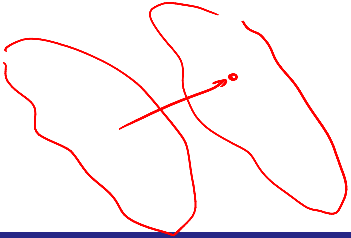
If we have a line or a plane passing through a vector $\boldsymbol{a}$, but not through the origin, how can we represent it?

- Translate the object so that it passes through the origin.
- We obtain a vector space $\mathcal{V}$.
- Then we translate it back so that it passes through $\boldsymbol{a}$.
- We get the set
$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$
- *Question:* Is $\mathcal{A}$ a vector space?
- We also write it as $\underbrace{\boldsymbol{a} + \mathcal{V}}$.

Affine spaces



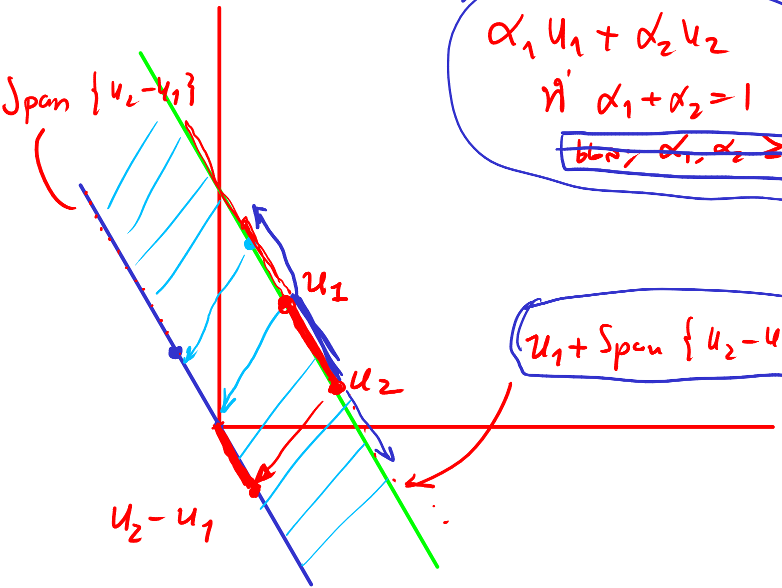
Definition

If $\boldsymbol{a}$ is a vector and $\mathcal{V}$ is a vector space, then

$$\boldsymbol{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions



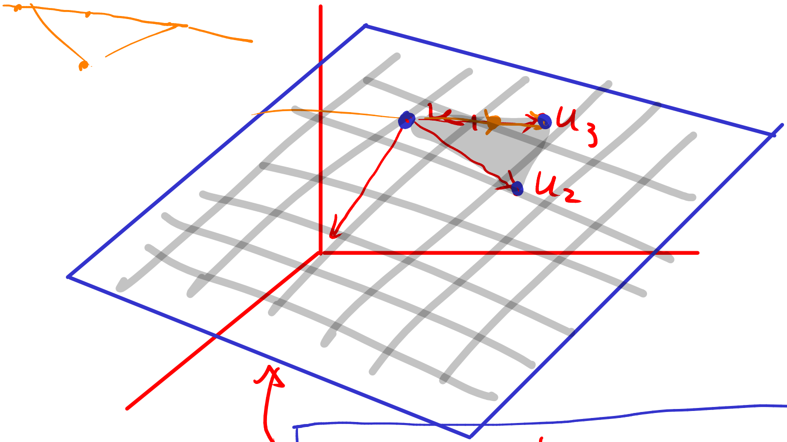
$$\alpha_1 u_1 + \alpha_2 u_2$$

if $\alpha_1 + \alpha_2 = 1$

$\cancel{\text{but } \alpha_1, \alpha_2 \geq 0}$

$$u_1 + \operatorname{Span}\{u_2 - u_1\}$$

An affine space and convex combination: 3 dimensions



$$u_1 + \operatorname{Span}\{u_3 - u_1, u_2 - u_1\}$$

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

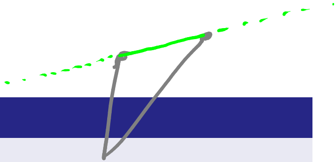
$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Affine combination



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is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Definition

The set of all affine combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **affine hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Definition

The set of all convex combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **convex hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Writing an affine space using a span

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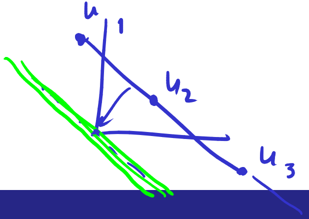

An affine space A

An affine space passing through $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is

$$\boldsymbol{u}_1 + \operatorname{Span}\{\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{u}_3 - \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n - \boldsymbol{u}_1\}.$$

พิจารณา $v \in u_1 + \operatorname{Span}\{u_2 - u_1, u_3 - u_1, \ldots, u_n - u_1\}$

มี $\alpha_2, \alpha_3, \ldots, \alpha_n$ ที่

$$\Rightarrow v = u_1 + \alpha_2(u_2 - u_1) + \alpha_3(u_3 - u_1) + \cdots + \alpha_n(u_n - u_1)$$

$$= \underbrace{[1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n]}_{= \alpha_1} u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n$$

$$\Rightarrow \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

Non-homogeneous linear system

$x = [x_1, x_2, x_3 \ldots x_n]$

$a_1 = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]$

Two linear systems: w & v

1stw sol. m

$$\begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m \end{aligned} \qquad \begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

$a_{11} x_1 + a_{12} x + - \cdots + a_{1n} x_n = b_1$

$a_{11} w_1 + a_{12} \cdot w_2 + \cdots + a_{1n} w_n = b_1$

$a_{11} v_1 + \cdots \qquad a_{12} v_n = b_1$

$w = [w_1, w_2 \ldots, w_n]$ $\quad v = [v_1, v_2 \ldots, v_n]$

Non-homogeneous linear system

u_1 & u_2

$a_1 \cdot u_1 = b_1$
$a_2 \cdot u_2 = b_1$ $\Big\} \Rightarrow$ $a_1(u_1 - u_2) = 0$

Two linear systems:

$$\begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

$\mathbf{0}$ is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i \boldsymbol{u}_1 - \boldsymbol{a}_i \boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

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Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

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This implies that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution to the homogeneous linear system.

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i\boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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$v-u$ $\quad a_i(v-u) = 0$

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In other words,

$$\begin{aligned} \mathcal{W} &= \boldsymbol{u} + \{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\} \\ &= \boldsymbol{u} + \underline{\{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = 0, \text{ for } 1 \le i \le m\}}, \end{aligned}$$

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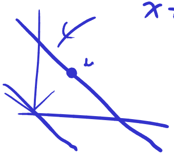
i.e., $\mathcal{W}$ is an affine space.

Solutions to a non-homogeneous linear system

$x + 5y = 10$



$x + 5y = 0$

Lemma 1

If the solution set of a linear system is not empty, it is an affine space.