01204211 Discrete Mathematics Lecture 12b: Linear functions (I)

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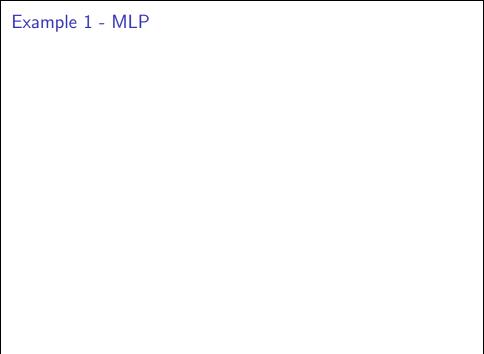
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Linear functions

Linear functions

Consider vector spaces $\mathcal V$ and $\mathcal W$ over $\mathbb R$. A function $f:\mathcal V\to\mathcal W$ is linear if

- 1. for all $x, y \in \mathcal{V}$, f(x + y) = f(x) + f(y) and
- 2. for all $\alpha \in \mathbb{R}$ and $x \in \mathcal{V}$, $f(\alpha x) = \alpha f(x)$.





Example 2 - Page rank (1)



Example 2 - Page rank (2)

Matrix-vector multiplication

Given an $m \times n$ matrix M over \mathbb{R} , consider a product

Mx.

Note that for the multiplication to work, x must be in \mathbb{R}^n and the result vector is in \mathbb{R}^m . Therefore, we can define a function $f: \mathbb{R}^n \to \mathbb{R}^m$ as

$$f(\boldsymbol{x}) = M\boldsymbol{x}.$$

Note that f is linear because:

$$f(\boldsymbol{x} + \boldsymbol{y}) = M(\boldsymbol{x} + \boldsymbol{y}) = M\boldsymbol{x} + M\boldsymbol{y} = f(\boldsymbol{x}) + f(\boldsymbol{y}),$$

and

$$f(\alpha x) = M(\alpha x) = \alpha Mx = \alpha f(x).$$

The converse

Lemma 1

For any linear function $f:\mathbb{R}^n \to \mathbb{R}^m$, there exists an $m \times n$ matrix M such that

$$f(\boldsymbol{x}) = M\boldsymbol{x}.$$

Proof.

Consider any $x \in \mathbb{R}^n$. Let $\boldsymbol{x} = [x_1, x_2, \dots, x_n]$. Note that

$$\mathbf{x} = [x_1, 0, \dots, 0] + [0, x_2, 0, \dots, 0] + \dots + [0, \dots, 0, x_n].$$

Let $e_1,e_2,\ldots,e_n\in\mathbb{R}^n$ be standard generators, i.e., e_i be a vector with 1 at the i-th row and 0 at every other positions. (For example $e_1=[1,0,\ldots,0]$ and $e_3=[0,0,1,0,\ldots,0]$.)

We thus have

$$\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \dots + x_n \boldsymbol{e}_n.$$

Since f is linear, this implies that

$$f(x) = x_1 f(e_1) + x_2 f(e_2) + \dots + x_n f(e_n).$$

Proof (cont.)

Define M as follows

$$M = \left[\begin{array}{c|c} f(\boldsymbol{e}_1) & f(\boldsymbol{e}_2) & \cdots & f(\boldsymbol{e}_n) \end{array} \right].$$

Hence.

$$Mx = \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) = f(x),$$

as required.

