01204211 Discrete Mathematics
Lecture 8b: Modular arithmetic

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Quick check 1

$2|2$ $2|2$ $2\cdot 2 \nmid 2$

ถ้า $a \neq b$

$6|12$ $4|12$ $24 \nmid 12$

If $a|m$ and $b|m$, can we say that $ab|m$? Prove this fact or provide a counter example.

X

Quick check 2

m มี a เป็นตัว
m มี b เป็นด้วย.

If $a|m$, $b|m$, and $a \neq b$ are both prime, can we say that $ab|m$? Prove this fact or provide a counter example. ✓

$$m = p_1^{k_1} p_2^{k_2} \cdot a^k \cdot b^{k'}.$$

Prime factorization

One useful fact that we use over and over again is the following.

Unique Factorization (or Fundamental Theorem of Arithmetic)

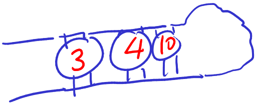
Every integer greater than 1 can be written *uniquely* as a product of prime numbers (up to the order of factors).

Examples:

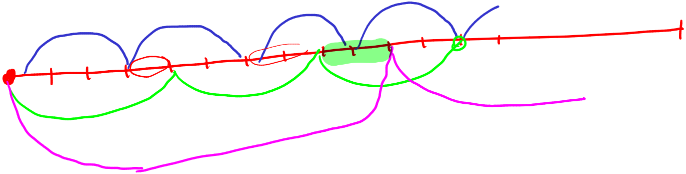
- $10 = 2 \cdot 5$
- $13 = 13$
- $112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^4 \cdot 7$

$7 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

* GF(p)



There are 3 clocks. At this moment, all three clocks ring at the same time. The first clock rings every 3 hours, the second clock rings every 4 hours, and the third clock rings every 10 hours. How long do you have to wait until you would hear all clocks ring a the same time again?



You have a large water container and two smaller buckets. The first bucket carries 3 litres of water and the second bucket carries 5 litres of water.

Can you put exactly 1 litre of water in the water container?

$$2 \cdot 3 - 5 = 1$$

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.

What is the minimum volume of water you can exactly put in the water container?

$$\underset{\in\mathbb{Z}}{a}\cdot 6 + \underset{\in\mathbb{Z}}{b}\cdot 15$$

$$\boxed{a\cdot 6 + b\cdot 15} > 0$$

$$\gcd(6,15) = 3$$

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.
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In general if you have two buckets of volumes x and y, the amount that you can exactly make must be in the form of

$$\boxed{ax + by,}$$

for some integers x and y. (Note that x and y may be negative.)

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In general if you have two buckets of volumes x and y, the amount that you can exactly make must be in the form of

$$ax + by,$$

for some integers x and y. (Note that x and y may be negative.) Do you see why the sum must be divisible by any common divisor of x and y?

Useful fact

For any integer x and y, consider the term

$$a \cdot x + b \cdot y,$$

for some integer a and b.

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For any integer x and y, consider the term

$$a \cdot x + b \cdot y,$$

for some integer a and b.
When the term is non-zero, it must be divisible by $gcd(x, y)$, so it has to be at least $gcd(x, y)$.

It turns out that you can actually attain that value, i.e., there exist a pair of integer a and b such that

$$\boxed{a \cdot x + b \cdot y = gcd(x, y).}$$

Finding a and b: Extended Euclid Algorithm

$3, 5 \Rightarrow 2 \cdot 3 + (-1) \cdot 5 = 1$

gcd, a, b

$a \cdot x + by = gcd(x, y)$

We will modify the Euclid algorithm so that it also returns a and b together with $gcd(x, y)$.

```
Algorithm Euclid(x,y):
  if x mod y == 0:
           gcd(x,y)
    return y,  0  ,  1
  else:
    g, a', b' = Euclid(y, x mod y)

    a = b'

    b = a' - b' ⌊x/y⌋

    return g, a, b
```

$0 \cdot x + 1 \cdot y = y$

$a' \cdot y + b' \cdot (x \bmod y) = g$

จะคืน g, a, b โดยที่ $g = gcd(x, y)$, $ax + by = gcd(x, y)$

Notes:

จาก a, b ที่ $a \cdot x + b \cdot y = g$

We have a' and b' such that

$$a' \cdot y + b' \cdot (x \bmod y) = g.$$

แทนค่า $x \bmod y = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y$

$$a' \cdot y + b'\left(x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y\right) = a' \cdot y + b' x - b' \left\lfloor \frac{x}{y} \right\rfloor \cdot y$$

$$= b' \cdot x + \left[a' - b' \left\lfloor \frac{x}{y} \right\rfloor\right] \cdot y = g$$

Days

What day is it today?

Days

What day is it today? Thursday.

Days

What day is it today? Thursday.
What day is 3 days after today?

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.

Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today?

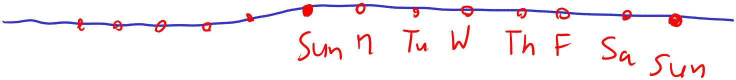
Days

What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.

Days

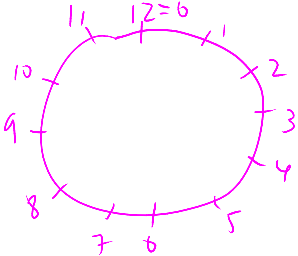
What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today?

Days



What day is it today? Thursday.
What day is 3 days after today? Sunday.
What day is 20 days after today? Wednesday.
What day is 10 days before today? Monday.

Clocks



Suppose that it is 1 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours?

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours? 6 o'clock.
What time is the next 10 hours? 11 o'clock.
What time is the next 20 hours? 9 o'clock.

Modular arithmetic

mod n

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Suppose that $m = 7$. We would like to say that

$$4 + 5 = 9 \bmod m = 2.$$

Or

$$3 \cdot 4 =$$

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Or

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$$2 - 6 = -4 \bmod 7 = 3 \bmod 7 = 3.$$

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Or

$$2 - 6 = \ -4 \bmod 7 = 3 \bmod 7 = 3.$$

-14 -7 0 7 14 21

Note that when you view integers under the lense of modulus 7, these numbers

$$\ldots, -19, -12, -5, 2, 9, 16, 23, \ldots$$

are essentially **the same**.

Properties (1)

$a \bmod m = b \bmod m$, if and only if $m | a - b$.

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$a \bmod m = b \bmod m$, if and only if $m|a-b$.

Proof.

($\Rightarrow$) Let $r = a \bmod m$. We can write

$$a = qm + r,$$

and

$$b = pm + r,$$

for some integers q and p. Thus, we have

$$a - b = qm + r - pm - r = \underline{(q-p)}m.$$

Therefore $m|a-b$.

($\Leftarrow$) Exercise. □

Properties (2)

- $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- $(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$
- $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$

Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

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We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m|(a-b)$$

Congruences: properties (1)

- (reflexivity)
  $a \equiv a \pmod{m}$.
- (symmetry)
  $a \equiv b \pmod{m}$ implies $b \equiv a \pmod{m}$.
- (transitivi~~e~~ty)
  $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ implies $a \equiv c \pmod{m}$.

Congruences: properties (2) – operations

$m \mid a - b$

If we have that

$$a \equiv b \pmod{m},$$

$m \mid c - d$

and

$$c \equiv d \pmod{m},$$

$m \mid a - b + c - a$

then

- $a + c \equiv b + d \pmod{m}$ $\leftarrow$ $m \mid (a+c) - (b+d)$
- $a - c \equiv b - d \pmod{m}$
- $ac \equiv bd \pmod{m}$

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$10x = 9$

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What is missing here?
Division!

Also, we wish we can do "cancellation", i.e., if

$$xa \equiv xb \pmod{m},$$

then $a \equiv b \pmod{m}$. **BUT THIS IS NOT ALWAYS TRUE.**

Also, we wish we can do "cancellation", i.e., if

$$xa \equiv xb \pmod{m},$$

then $a \equiv b \pmod{m}$. **BUT THIS IS NOT ALWAYS TRUE.** Let's see the following example:

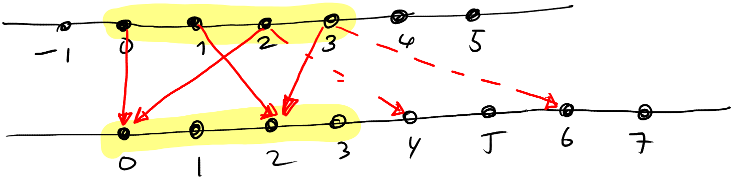
$$2 \cdot 1 \equiv 2 \cdot 3 \pmod{4},$$

but

$$1 \not\equiv 3 \pmod{4}.$$

Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \bmod 4$.



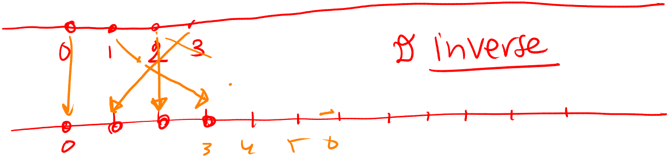
Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let
$f(x) = 2 \cdot x \bmod 4$.

$2a \equiv 2b \pmod 4$

$\underline{3a} \equiv \underline{3b} \pmod 4$

Let's also see $g(x) = 3 \cdot x \bmod 4$.



inverse

Multiplications as functions

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$\times$ $f(x) = 2 \cdot x \bmod 4$.

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Which functions have inverses?

Multiplicative inverses (standard arithmetic)

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By dividing on both sides with 5:

5 พา x ไปเที่ยว.

$$2/5 = 5x/5 = x,$$

or equivalently, by multiplying with $(1/5) = 5^{-1}$:

พากลับบ้าน

$$2 \cdot 5^{-1} = 5x \cdot 5^{-1} = x \cdot \underbrace{5 \cdot 5^{-1}}_{} = x \cdot 1 = x.$$

Here 5^{-1} is a multiplicative inverse of 5.

$= 1/5 = 0.2$

Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be $m = 7$. Note that

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod{7}.$$

Therefore, $5^{-1} \equiv 3 \pmod{7}$.

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To find $2/5$, we can view our goal as to find the value of x such that

$$2 \equiv 5x \pmod{7}.$$

We can multiply both sides with $5^{-1} \equiv 3$ to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be $m = 7$. Note that

$2/3 \pmod 7$

$a(5 \cdot 3) = a$

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod 7.$$

$2(3^{-1})$

Therefore, $5^{-1} \equiv 3 \pmod 7$.

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We can multiply both sides with $5^{-1} \equiv 3$ to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod 7.$$

Let's check:

$$5 \cdot 6 \equiv 30 \equiv 2 \pmod 7,$$

as requied.

Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a, denoted by a^{-1}, is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

$$x \cdot a - y \cdot m =$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $gcd(a, m) = 1$.

Proof.

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Proof.

($\Leftarrow$) Recall that there exist integers x and y such that

$$x \cdot a + y \cdot m = gcd(a, m) = 1.$$

0

Thus, $(x \cdot a + \cancel{y \cdot m}) \bmod m = x \cdot a \bmod m = 1 \bmod m$, i.e., $x \cdot a \equiv 1 \pmod{m}$. Therefore x is the inverse.

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Thus, $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$, i.e., $x \cdot a \equiv 1 \pmod{m}$. Therefore x is the inverse.

($\Rightarrow$) Let $r = gcd(a, m)$. Suppose that b is the multiplicative inverse of a modulo m, i.e., we have that

$$b \cdot a \equiv 1 \pmod{m},$$

Thus, $ba \bmod m = 1 \bmod m = 1$, i.e., there exists an integer q such that

$$ba = qm + 1,$$

or $ba - qm = 1$. However, r since $r|a$ and $r|m$, r also divides $bd - qm$ and 1. But it $r \not| 1$ because $r > 1$ and we have the contradiction. $\square$

Examples: division in modular arithmetic

Since the requirement for an existance of a^{-1} modulo m is that $gcd(a, m) = 1$, if we let m be a prime number, every a which is not a multiple of m has an inverse.
Can you solve this equation?

$$4x + 9 \equiv 0 \pmod{11}.$$

$4x \equiv -9 \equiv 2 \pmod{11}$

$4^{-1} \equiv 3 \pmod{11}$

$3 \cdot \cancel{4} x \equiv \cancel{4^{-1} \cdot 4}\, x \equiv 4^{-1} \cdot 2 \equiv 3 \cdot 2 \equiv 6 \pmod{11}$

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Can you solve this equation?

$$4x + 9 \equiv 0 \pmod{11}.$$

We can even perform gaussian elimination (*which is very useful later*):

$$\begin{aligned} 2x + y &\equiv 3 \pmod{7} \\ x + 3y &\equiv 5 \pmod{7} \end{aligned}$$

Quick recap: RSA

- Private key: (e, n), Public key: (d, n)
- Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

Quick recap: RSA

- Private key: (e, n), Public key: (d, n)
- Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

- Pick two primes p and q. Let $n = pq$.
- 65537 Pick e (usually a small number)
- Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or

$$ed = k \cdot (p-1)(q-1) + 1,$$

for some integer k.

- What is $m^{ed} \bmod n$?

$$m^{ed} = \cancel{\left[m^{k(p-1)(q-1)}\right]}^{\,1} \cdot m \pmod n$$

What's next?

- We will prove Fermat's Little Theorem and show how to efficiently test if a number is prime.
- We will also use Fermat's Little Theorem to prove the correctness of RSA.
- Modular arithmetic is also key to our usage of polynomials to perform secret sharing and error correcting codes, because now we can do Gaussian elimination using only integers.