01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (II)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \left[\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

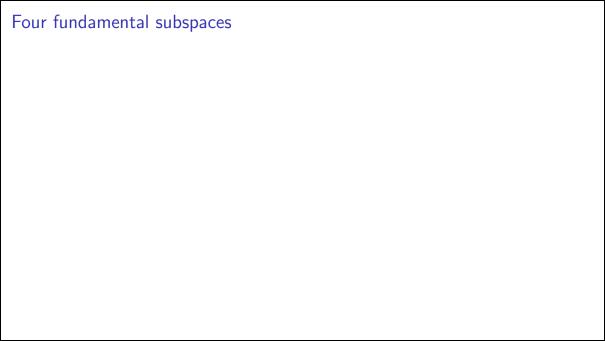# Four fundamental subspaces

Four fundamental subspaces

Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{C}(A) \subseteq \mathbb{R}^m$)
- The row space of A (denoted by $\mathcal{C}(A^T) \subseteq \mathbb{R}^n$)
- The nullspace of A

$$\mathcal{N}(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

- The left nullspace of A

$$\mathcal{N}(A^T) = \{\boldsymbol{y} \mid A^T\boldsymbol{y} = \boldsymbol{0}\} \subseteq \mathbb{R}^m$$

Four fundamental subspaces

Ranks

Definition

Consider an m-by-n matrix A.

- The **row rank** of A is the maximum number of linearly independent rows of A.
- The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{C}(A)$. The row rank of A is $\dim \mathcal{C}(A^T)$.

Theorem 1

For any matrix A, its row rank equals its column rank.

Proof.

Let r be the column rank. We will show that there are r n-vectors that span its row space. This implies that the row rank is at most r. We can use the same argument again on A^T to obtain that the column rank is at most the row rank; thus, they must be equal.

$\square$

Proof (cont.)

Rank and nullity

Given an m-by-n matrix A, the rank of A is $\dim \mathcal{C}(A)$. Let r be the rank of A. What is $\dim \mathcal{N}(A)$?

Dimensions

Four fundamental subspaces

Given an m-by-n matrix A of rank r, we have the following subspaces

- The column space of A (denoted by $\mathcal{C}(A) \subseteq \mathbb{R}^m$)
  dim $\mathcal{C}(A) = r$.
- The row space of A (denoted by $\mathcal{C}(A^T) \subseteq \mathbb{R}^n$)
  dim $\mathcal{C}(A^T) = r$.
- The nullspace of A (denoted by $\mathcal{N}(A) \subseteq \mathbb{R}^n$)
  dim $\mathcal{N}(A) = n - r$.
- The left nullspace of A (denoted by $\mathcal{N}(A^T) \subseteq \mathbb{R}^m$)
  dim $\mathcal{N}(A) = m - r$.

Application: Singular Value Decomposition (SVD)

Any n-by-d matrix A can be factored into the form of UDV^T, i.e.,

$$\left[\begin{array}{c} \\ A \\ \\ \end{array}\right] = \left[\begin{array}{c} \\ U \\ \\ \end{array}\right]\left[\begin{array}{c} \\ D \\ \\ \end{array}\right]\left[\begin{array}{c} \\ V^T \\ \\ \end{array}\right]$$

where

- U is an n-by-r matrix,
- D is a diagonal r-by-r matrix, and
- V is an d-by-r matrix (i.e., V^T is an r-by-d matrix)
- (Also, columns of U and D are "orthonormal.")

See demo.

$$\left[\begin{array}{c} A \end{array}\right] = \left[\begin{array}{cc} \boldsymbol{u}_i & U \end{array}\right] \left[\begin{array}{cc} d_{ii} & \\ & D \end{array}\right] \left[\begin{array}{c} \boldsymbol{v}_i^T \\ V^T \end{array}\right]$$

$$A = d_1\boldsymbol{u}_1\boldsymbol{v}_1^T + d_2\boldsymbol{u}_2\boldsymbol{v}_2^T + \cdots + d_r\boldsymbol{u}_r\boldsymbol{v}_r^T.$$