# 01204211 Discrete Mathematics Lecture 6a: Counting 3

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### Quick recap

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  - ▶ We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.

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  - ▶ We count the number of ways one can choose a subset.
  - We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.
- For a set with n elements, the number of its permutations is n!.

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**Abbreviations:** We shall call a set with n elements as an n-set. We shall call a subset with k elements as a k-subset.



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▶ We will also discuss the inclusion-exclusion priciples.



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We can use the argument we used to derive the number of permutations here. We consider the process for selecting the winners.

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- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- ▶ Therefore, we conclude that the number of ways is  $10 \cdot 9 \cdot 8$ .

We can arrive at the same answer by a different way of counting.

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  - ► The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are 7! of them.
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- Let X be the set of ordered subsets with 3 elements of an 10-set. We then have  $|X| \times 7! = 10!$ , because they count the same objects. Solving this yields

$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

#### General answers: numbers of ordered subsets

Using the same arguments (either one), we have this theorem.

#### Theorem 1

The number of ordered subsets with k elements of an n-set is

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

- ▶ With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.<sup>2</sup>
- ► How can we start?

 $<sup>^2</sup>$ This section on estimation follows section 1.4 of [LPV]  $_{\bigcirc}$   $_{\bigcirc}$   $_{\bigcirc}$   $_{\bigcirc}$   $_{\bigcirc}$   $_{\bigcirc}$   $_{\bigcirc}$ 

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  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.) How about  $2^n$ ? Is it a lower bound? How about  $3^n$  or  $5^n$ ? Are they lower bounds of n!?

Recall that  $n! = 1 \cdot 2 \cdot 3 \cdots n$ . Since all its factor, except the first one is at least 2, we have that

$$2^{n-1} \le n!$$
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#### Bounds for n!

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n	$  2^{n-1}  $	n!	$n^{n-1}$
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	3,628,800	1,000,000,000

#### A better bound?

Let's consider n! again, but for simplicity, let's consider only the case when n is an even number:

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To get a better lower bound, we may move our cutting point from 2 to, say, n/2. Note that at least n/2 factors are at least n/2. Thus,

$$n! = 1 \cdot 2 \cdots n$$

$$\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2}$$

$$= (n/2)^{n/2} = \sqrt{(n/2)^n}.$$

## Better?

n	$2^{n-1}$	$\sqrt{(n/2)^n}$	n!	$n^{n-1}$
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	7,776
10	512	3,125	3,628,800	1,000,000,000
12	2,048	46,656	479,001,600	743,008,370,688

OK. A bit better.

An even better estimate for n! exists.

Theorem 2 (Stirling's formula)  $n! \sim \left(\frac{n}{2}\right)^n \sqrt{2\pi n}$ .

When we write  $a(n) \sim b(n)$ , we mean that  $\frac{a(n)}{b(n)} \to 1$  as  $n \to \infty$ .

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$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100\log(100/e) + \log(200\pi) \approx 157.9696.$$



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Note that the correct answer is 158 digits.



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▶ This upper bound of  $n^2$  is very good as the gaps between the upper bounds and the actual values will not be larger than 2, as  $\frac{n^2}{n(n+1)/2} < 2$ .

Theorem: The number of k-subsets of an n-set is

$$\frac{n\cdot (n-1)\cdot (n-2)\cdots (n-k+1)}{k!}=\frac{n!}{(n-k)!k!}.$$

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Consider the following process for choosing an ordered subsets with  $\boldsymbol{k}$  elements of an n-set.

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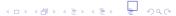
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$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of k-subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required.



The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

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- $\binom{n}{n} = 1$  (why?),
- $ightharpoonup \binom{n}{0} = 1 \text{ (why?), and,}$
- ightharpoonup when k > n,  $\binom{n}{k} = 0$ .

# Properties (1)

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}.$$

## Properties (2)

Theorem: When n, k > 0, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

# Properties (3)

Theorem: When n, k > 0, then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n.$$