

01204211 Discrete Mathematics

Lecture 5c: Counting 2

Jittat Fakcharoenphol

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Listing all subsets¹

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- ▶ Let's try to enumerate them.

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In this case, we order by their cardinalities, then use dictionary ordering for subsets with the same numbers of elements.

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A different representation (1)

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- ▶ Also, this representation can be considered backwards, i.e., if we start with an integer 6, we can write down its binary representation: 110_2 and turns it into a subset $\{a, b\}$.

A correspondence

Let's see a full list of correspondence between $\{0, 1, 2, \dots, 7\}$ and subsets of $\{a, b, c\}$.

- ▶ $0 \leftrightarrow 000_2 \leftrightarrow \{\}$
- ▶ $1 \leftrightarrow 001_2 \leftrightarrow \{c\}$
- ▶ $2 \leftrightarrow 010_2 \leftrightarrow \{b\}$
- ▶ $3 \leftrightarrow 011_2 \leftrightarrow \{c, b\}$
- ▶ $4 \leftrightarrow 100_2 \leftrightarrow \{a\}$
- ▶ $5 \leftrightarrow 101_2 \leftrightarrow \{a, c\}$
- ▶ $6 \leftrightarrow 110_2 \leftrightarrow \{a, b\}$
- ▶ $7 \leftrightarrow 111_2 \leftrightarrow \{a, b, c\}$

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Do you notice anything interesting?

A general case

Similarly, we can describe a representation for each subset of a set A with n elements. As we consider each element a of A , we put 1 if $a \in A$ and put 0 if $a \notin A$.

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There are 2^n bit strings; hence, the number of subsets is also 2^n .

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subset \rightarrow bit strings

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How many bit strings of length n are there?

There are 2^n bit strings; hence, the number of subsets is also 2^n . This is another proof of the following theorem:

Theorem: The number of subsets of a set with n elements is 2^n .

Two proofs

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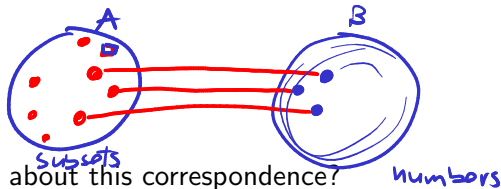
Really, it does not make a statement stronger, truer, “more” correct. But each proof usually reveals additional facts related to the statement.

- ▶ The first proof considers a procedure for constructing subsets.
- ▶ The second proof introduces a nice technique for counting. I.e., instead of counting subsets directly, we show that we have a “special” correspondence between subsets and binary numbers, and then just count the numbers.

A bijection

What is so special about this correspondence?

A bijection



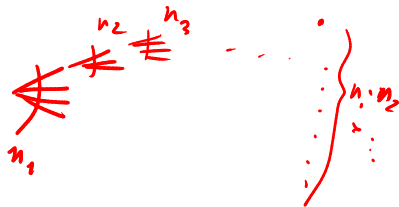
What is so special about this correspondence?

- ▶ For each number, there is exactly **one** subset that corresponds to it. $|A| \geq |B|$
- ▶ For each subset, there is exactly **one** number that it corresponds to. $|B| \geq |A|$

With these two properties, we can conclude that both sets have the same cardinality.

This type of correspondence is called a one-to-one correspondence or bijection.

Sequences of choices



Previously, when we want to count the number of bit strings of length n , we use this argument:

Suppose that to select an object, you have to make k decisions. The first decision has n_1 choices, the second decision has n_2 choices, and so on. More precisely, for $1 \leq i \leq k$, the i -th decision has n_i choices. Then the number of ways you can select an object is $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$.

Example 1

A car license number consists of two English letters and one number from 1 to 9999. How many possible license numbers are there?

$$26 \times 26 \times 9999 =$$

Example 2

10 students stand in a line. You want to give them ice cream. There are 4 flavours, but you don't want to give the same flavour to any consecutive students. In how many ways can you give out the ice cream to these students?



$$4 \times \underbrace{3 \times 3 \times \dots \times 3}_9 = 4 \cdot 3^9$$

Permutations — 50000 variations 24 no. of terms

$\{a, b, c, d\}$

a b c d

b c a d

d a c b

x a b c (111)

x a a b c d (111)

Counting permutations: an example

We want to count the number of permutations. Let's try with a small example: permutations of set $\{a, b, c\}$.

$a b c$

$a c b$

$b a c$

$b c a$

$c a b$

$c b a$

Counting permutations

- นับ # วิธีกลับ permutation.
- สำหรับ A เป็น set ที่มี n elements

→ ① เลือก 1 หนึ่ง 1 — n วิธี

→ ② เลือก 2 หนึ่ง 2 — $n-1$ วิธี

⋮
⋮
⋮
เลือก i หนึ่ง i — $n-i-1$ วิธี

วิธีกลับทั้งหมด

$$n \cdot (n-1)(n-2) \cdots (2)(1)$$
$$= n!$$

Number of permutations

We have proved this theorem.

Theorem: The number of permutations of a set with n elements is $n!$.