

01204211 Discrete Mathematics Lecture 9b: Affine Spaces

"vector space + translation

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denoted by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Review: Vector spaces

Definition

A set \mathcal{V} of vectors over \mathbb{F} is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\mathbf{u} \in \mathcal{V}$,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- (V3) for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

Examples of vector spaces:

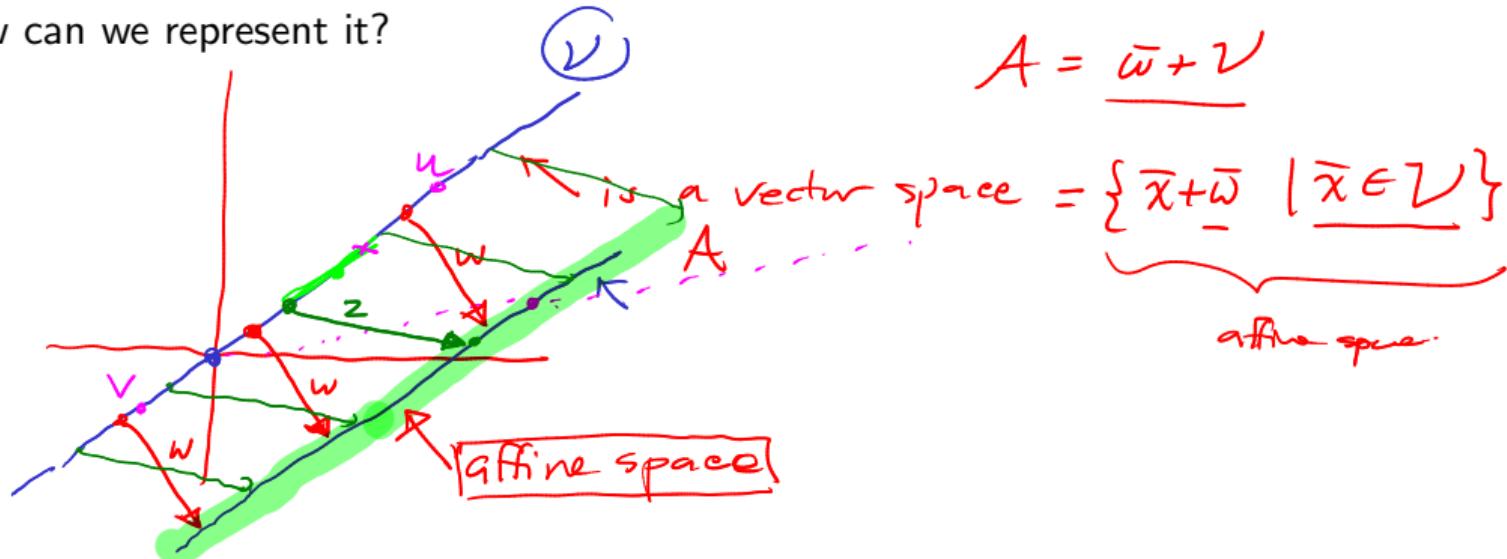
- A span of vectors is a vector space.
- A solution set to homogeneous linear equations is a vector space.

$$Ax = 0$$

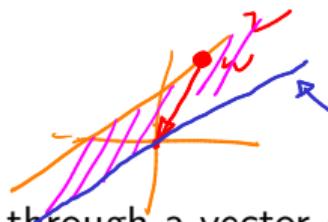
$$\{ \bar{x} \mid A\bar{x} = 0 \}$$

Translation

If we have a line or a plane passing through a vector a , but not through the origin, how can we represent it?



Translation



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- ▶ Translate the object so that it passes through the origin.

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- ▶ We obtain a vector space \mathcal{V} .
- ▶ Then we translate it back so that it passes through a .

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- ▶ Translate the object so that it passes through the origin.
- ▶ We obtain a vector space \mathcal{V} .
- ▶ Then we translate it back so that it passes through a .
- ▶ We get the set

$$\mathcal{A} = \{a + u : u \in \mathcal{V}\}$$

Translation

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- ▶ Translate the object so that it passes through the origin.
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- ▶ *Question:* Is \mathcal{A} a vector space?

Translation

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$$\mathcal{A} = \{\mathbf{a} + \mathbf{u} : \mathbf{u} \in \mathcal{V}\}$$

- ▶ *Question:* Is \mathcal{A} a vector space?
- ▶ We also write it as $\mathbf{a} + \underline{\mathcal{V}}$.

Affine spaces

Definition

If a is a vector and \mathcal{V} is a vector space, then

$$\underline{a + \mathcal{V}}$$

is an affine space.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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Definition

The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **affine hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$, we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **convex combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Definition

The set of all convex combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **convex hull** of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$.

Writing an affine space using a span

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An affine space

An affine space passing through $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

Non-homogeneous linear system

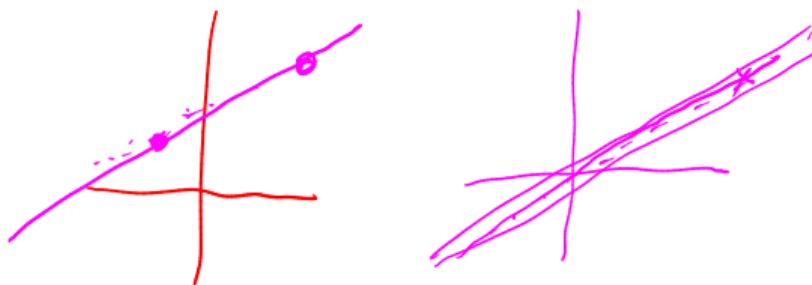
Two linear systems:

$$\begin{array}{lcl} \mathbf{a}_1 \cdot \mathbf{x} & = & b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} & = & b_2 \\ \vdots & & \\ \mathbf{a}_m \cdot \mathbf{x} & = & b_m \end{array}$$

\mathbf{A}

$$\begin{array}{lcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ \mathbf{a}_2 \cdot \mathbf{x} & = & 0 \\ \vdots & & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array} \quad N(\mathbf{A})$$

What can you say about the solution sets of these two related linear systems?



Non-homogeneous linear system

Solutions to $A\bar{x} = \bar{b}$
 $\underline{\bar{x}_1}$
 $\underline{\bar{x}_2}$

$$\boxed{\bar{x}_1 - \bar{x}_2}$$

Two linear systems:

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non homo - .

$$\checkmark A\bar{x}_1 = \bar{b}$$

$$A\bar{x}_2 = \bar{b}$$

$$\begin{array}{lcl} a_1 \cdot x & = & 0 \\ a_2 \cdot x & = & 0 \\ \vdots & & \\ a_m \cdot x & = & 0 \end{array}$$

homo

$$A(\bar{x}_1 - \bar{x}_2)$$

$$\begin{aligned} &= A\bar{x}_1 - A\bar{x}_2 \\ &= \bar{b} - \bar{b} \\ &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

0 is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$

Solutions of the two systems

Recall that if \mathbf{u}_1 and \mathbf{u}_2 are both solutions to the non-homogeneous linear system, we have that for any i

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This implies that $\mathbf{u}_1 - \mathbf{u}_2$ is a solution to the homogeneous linear system.

Suppose that \mathcal{W} is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\mathbf{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\}$$

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In other words,

$$\begin{aligned} \mathcal{W} &= \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} \\ &= \boxed{\mathbf{u}} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = \boxed{0}, \text{ for } 1 \leq i \leq m\}, \end{aligned}$$

*Solutions for
non homo*

Sols for hom

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i.e., \mathcal{W} is an affine space.

Solutions to a non-homogeneous linear system

Lemma 1

If the solution set of a linear system is not empty, it is an affine space.