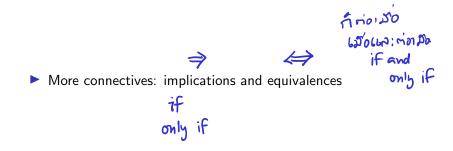
01204211 Discrete Mathematics Lecture 1b: Implications and equivalences

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This lecture covers:



Review (1)

- A proposition is a statement which is either true or false.
- We can use variables to stand for propositions, e.g., P = "today is Tuesday".
- We can use connectives to combine variables to get propositional forms.
 - **Conjunction:** $P \wedge Q$ ("P and Q"),
 - **Disjunction:** $P \lor Q$ ("P or Q"), and
 - **Negation:** $\neg P$ ("not P")



Review (2)

To represents values of propositional forms, we usually use truth tables.

And,	/Or/	Not						
P	Q	$P \wedge Q$	$P \lor Q$	$\neg P$	7			
T	T	T	T	F	1			
$\mid T$	$\mid F \mid$	\overline{F}	T					
$\mid F$	$\mid T \mid$	F	T	T				
$oxedsymbol{F}$	$\mid F \mid$	F	F					

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1) $P \land \neg P$ and (2) $P \lor \neg P$.

P	PATP	PV7P
T	F	T
F	T F	T

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And/Or/N	ot			
$ \begin{array}{c c} P & \neg P \\ T & F \\ F & T \end{array} $	$ \begin{array}{c c} P \land \neg P \\ \hline F \\ F \end{array} $	$\begin{array}{c c} P \vee \neg P \\ \hline T \\ T \end{array}$		

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$\neg P$
7
7

Note that $P \wedge \neg P$ is always false and $P \vee \neg P$ is always true. A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

Given P and Q, an implication P implies Q $P \Rightarrow Q \qquad \qquad \text{ on } P \text{ this } Q$

stands for "if P, then Q". This is a very important propositional form.

It states that "when ${\cal P}$ is true, ${\cal Q}$ must be true". Let's try to fill in its truth table:



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mplica	tions		
$\begin{array}{c c} P & Q \\ \hline T & T \end{array}$	$P \Rightarrow Q$		
$T \mid F$	ll .		

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Implications		
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		

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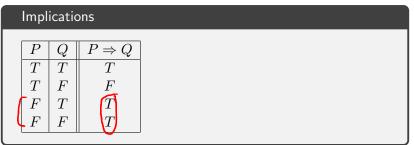
Implications		
$egin{array}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{c} P \Rightarrow Q \\ \hline T \\ F \\ T \end{array}$	

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It states that "when ${\cal P}$ is true, ${\cal Q}$ must be true". Let's try to fill in its truth table:



¹Materials in this lecture are mostly from Berkeley CS70's lecture notes.



What?

- Yes, when P is false, $P \Rightarrow Q$ is always true no matter what truth value of Q is.
- We say that in this case, the statement $P\Rightarrow Q$ is *vacuously true*.

What?

- ▶ Yes, when P is false, $P \Rightarrow Q$ is **always true** no matter what truth value of Q is.
- ▶ We say that in this case, the statement $P \Rightarrow Q$ is *vacuously true*.
- You might feel a bit uncomfortable about this, because in most natural languages, when we say that if P, then Q we sometimes mean something more than that in the logical expression " $P\Rightarrow Q$."

One explanation

▶ But let's look closely at what it means when we say that:

if P is true, Q must be true.

▶ Note that this statement does not say anything about the case when *P* is false, i.e., it only considers the case when *P* is true.

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- ▶ Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.

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- Note that this statement does not say anything about the case when *P* is false, i.e., it only considers the case when *P* is true.
- ▶ Therefore, having that $P \Rightarrow Q$ is true is OK with the case that (1) Q is false when P is false, and (2) Q is true when P is false.
- ► This is an example when mathematical language is "stricter" than natural language.

Noticing if-then

We can write "if P, then Q" for $P \Rightarrow Q$, but there are other ways to say this. E.g., we can write (1) Q if P, (2) P only if Q, or (3) when P, then Q.

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Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- If you do not have enough sleep, you will feel dizzy during class.
 A
- If you eat a lot and you do not have enough exercise, you will get fat △ A ↑ ¬ B → C
- You can get A from this course, only if you work fairly hard.



Only-if

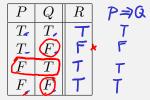
Let P be "you get A from this course."

Let ${\cal Q}$ be "you work fairly hard."

Let R be "You can get A from this course, only if you work fairly hard."

Let's think about the truth values of (R)

Only if you work fairly hard.



Only-if

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Let P be "you get A from this course."

Let Q be "you work fairly hard."

Let R be "You can get A from this course, only if you work fairly hard."

Let's think about the truth values of R.

 $A \Rightarrow B$

4, only if B

Only if you work fairly hard.

P	Q	R
T	T	
$\mid T \mid$	F	
F	$\mid T \mid$	
F	F	

Thus, R should be logically equivalent to $P\Rightarrow Q$. (We write $R\equiv P\Rightarrow Q$ in this case.)

If and only if: (\Leftrightarrow)

Given \underline{P} and \underline{Q} , we denote by



the statement "P if and only if Q."

If and only if: (\Leftrightarrow)

Given P and Q, we denote by

P = Q ... Q ⇒ P

$$P \Leftrightarrow Q$$

the statement "P if and only if Q." It is logically equivalent to

$$(P \Leftarrow Q) \land (P \Rightarrow Q),$$

i.e., $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \land (P \Rightarrow Q)$.

Let's fill in its truth table.

P	Q	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
T	$\left \frac{T}{F} \right $		1	+ "
$\begin{array}{ c c }\hline & I \\ F \end{array}$	$\left \begin{array}{c} T \\ T \end{array} \right $	I	F	F
F	F	T		Т

An implication and its friends

When you have two propositions

- ightharpoonup P = "I own a cell phone", and
- ightharpoonup Q = "I bring a cell phone to class".

We have

- ▶ an implication $P \Rightarrow Q \equiv$ "If I own a cell phone, I'll bring it to class",
- its converse $Q \Rightarrow P \equiv$ "If I bring a cell phone to class, I own it", and
- its contrapositive $\neg Q \Rightarrow \neg P \equiv$ "If I do not bring a cell phone to class, I do not own one".

Let's consider the following truth table:

	CONVENSE	Contrapositio.
$P \mid Q \mid P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T T	T	T
	Ť	(🗲 \
$\mid F \mid T \mid \mid \neg \mid$	۲	
	7	
	'	

Let's consider the following truth table:

D		D . O	() , D	O > D
P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
$\mid T \mid$	$T \mid$			
$\mid T \mid$	F			
$\mid F \mid$	T			
$\mid F \mid$	F			

Do you notice any equivalence?

Let's consider the following truth table:

\overline{P}	· T	\overline{Q}	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	1		- / 40	90 / -	9 / -
		F			
$\mid F \mid$	1	T			
$oxedsymbol{oxed{L}}$		F			

Do you notice any equivalence?

$$\mathsf{Right},\, P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P.$$