

01204211 Discrete Mathematics
Lecture 8a: Linear systems of equations

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Linear algebra

Linear algebra studies

- ▶ matrices and operations with matrices
- ▶ systems of linear equations
- ▶ linear transformations
- ▶ linear spaces (and their structures)

Why?

- ▶ Lots of applications.
- ▶ Interesting perspectives.

A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?

Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

Then you can conclude that $y = -2$. Substitute it to one of the equations, you can find out the value of x .

Gaussian elimination (1)

Let's consider a system with 3 variables:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + & & 5x_3 = 12 \\ 4x_1 + 2x_2 + 3x_3 & = & 10 \end{array}$$

Gaussian elimination (2)

Let's consider another system with 3 variables:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + & & 5x_3 = 12 \\ 3x_1 + 8x_2 + x_3 & = & 10 \end{array}$$

A closer look: 1st perspective

Consider

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

Each equation (row) constraints certain values of x and y .

“Combining” two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{aligned} \begin{pmatrix} 5 & 10 \end{pmatrix} &= \mathbf{u}_1 \\ \begin{pmatrix} 1 & -3 \end{pmatrix} &= \mathbf{u}_2 \\ \begin{pmatrix} 0 & 25 \end{pmatrix} &= \mathbf{u}_1 - 5 \cdot \mathbf{u}_2 \end{aligned}$$

The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.

Can you obtain $(0, 1)$ from \mathbf{u}_1 and \mathbf{u}_2 ?

Yes,

$$0.2 \cdot \mathbf{u}_1 - \mathbf{u}_2 = (0, 1).$$

It turns out that you can obtain any (a, b) from \mathbf{u}_1 and \mathbf{u}_2 .

A closer look: 1st perspective (more example)

Consider

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + & & 5x_3 = 12 \\ 4x_1 + 2x_2 + 3x_3 & = & 10 \end{array}$$

What are the row vectors?

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Now, the goal is to find x and y satisfying this “vector” equation.

But if we change our focus to the vectors, we can see that we have 3 vectors:

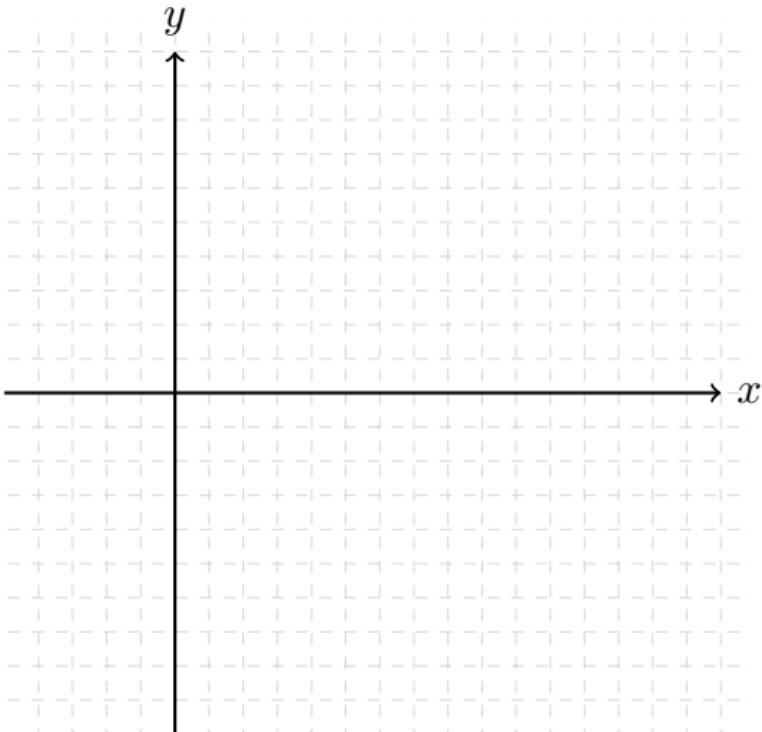
$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and with x and y , we now see that \mathbf{b} is a **linear combination** of \mathbf{v}_1 and \mathbf{v}_2 .

Finding x and y is essentially checking if \mathbf{b} is a linear combination of \mathbf{v}_1 and \mathbf{v}_2 .

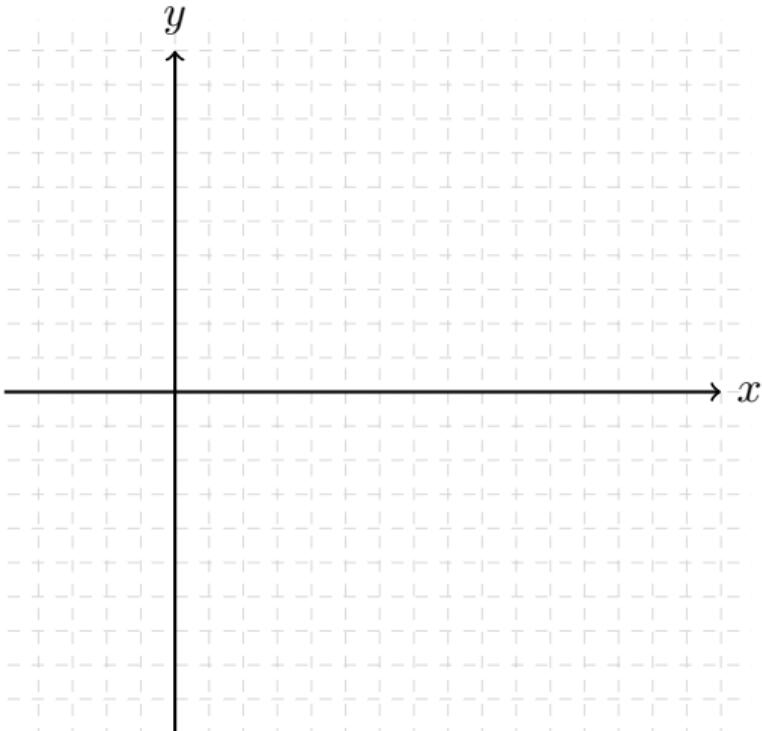
Example 2: a linear system with 2 variables

$$\begin{array}{rcrcl} x & + & y & = & 7 \\ x & - & 2y & = & 13 \end{array}$$



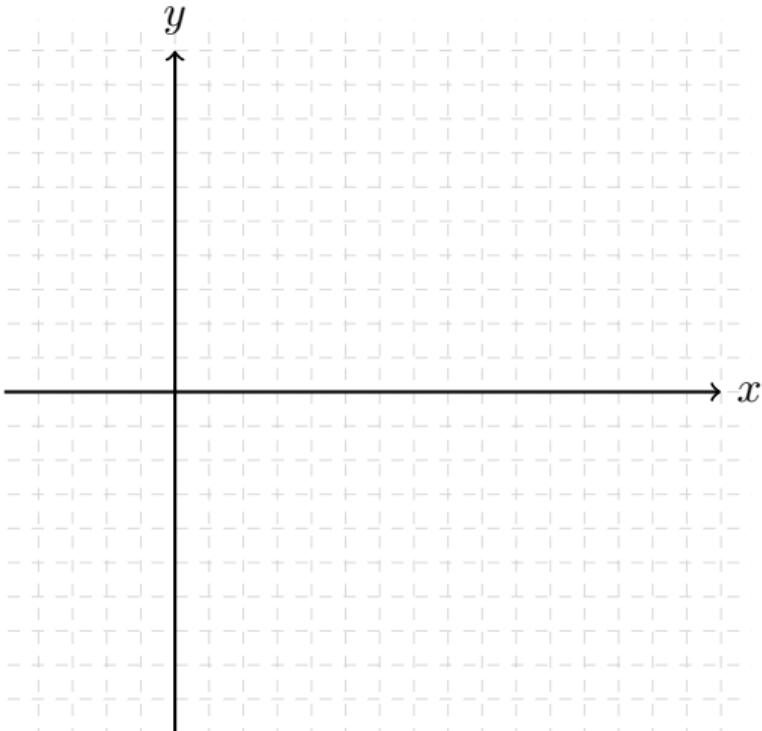
Example 3: a linear system with 2 variables

$$\begin{aligned} 2x + y &= 5 \\ 4x + 2y &= 10 \end{aligned}$$



Example 4: a linear system with 2 variables

$$\begin{aligned}x &+ 3y = 6 \\0.5 \cdot x &+ 1.5 \cdot y = 9\end{aligned}$$



A linear system with 3 variables

Let's consider a system with 3 variables:

$$\begin{array}{rcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Row perspective

$$\begin{array}{rclcl} 2x_1 & + & 4x_2 & + & 3x_3 = 7 \\ x_1 & + & & 5x_3 & = 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 = 10 \end{array}$$

Each equation becomes a **plane** in 3 dimensional space.

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

We want to linearly combine them to obtain

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1)$$

In other words, what are the possible linear combinations of

$$(2, 4, 3), \quad (1, 0, 5), \quad (4, 2, 3)$$

Column perspective

From

$$\begin{array}{ccc|c} 2x_1 & + & 4x_2 & + & 3x_3 = 7 \\ x_1 & + & & 5x_3 = 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 = 10 \end{array},$$

we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

In other words, the vector b , for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$\begin{array}{rcl} 2x_1 & + & 4x_2 & + & 3x_3 = 7 \\ x_1 & + & & & 5x_3 = 12 \\ 3x_1 & + & 8x_2 & + & x_3 = 10 \end{array}$$

More example 2

Let's consider another system with 3 variables:

$$2x_1 + 4x_2 + 3x_3 = 7$$

$$x_1 + \quad \quad \quad 5x_3 = 12$$

$$4x_1 + 2x_2 + 3x_3 = 10$$

$$5x_1 + 2x_2 + 8x_3 = 22$$

More failed example 3

Let's consider the last system with 3 variables:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + & & 5x_3 = 12 \\ 2x_1 + & & 10x_3 = 24 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ & & x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\mathbf{u} = [u_1, u_2, u_3]$ and $\mathbf{v} = [v_1, v_2, v_3]$ are both solutions but $\mathbf{u} \neq \mathbf{v}$.

What does it mean that \mathbf{u} and \mathbf{v} are solutions? It means that, for \mathbf{u} , you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that \mathbf{u} and \mathbf{v} are different solutions to the system:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + 5x_3 & = & 12 \end{array}$$

i.e.,

$$\begin{array}{rcl} 2u_1 + 4u_2 + 3u_3 & = & 7 \\ u_1 + 5u_3 & = & 12 \end{array} \quad \begin{array}{rcl} 2v_1 + 4v_2 + 3v_3 & = & 7 \\ v_1 + 5v_3 & = & 12 \end{array}$$

Consider $\mathbf{u} - \mathbf{v}$. We see that

$$\begin{aligned} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) &= \\ 2(u_1 - v_1) + 4(u_2 - v_2) + 3(u_3 - v_3) &= (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) &= \\ (u_1 - v_1) + 5(u_3 - v_3) &= (12 - 12) = 0 \end{aligned}$$

More failed example 3 (cont. 2)

Suppose that \mathbf{u} and \mathbf{v} are different solutions to the system:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 7 \\ x_1 + \quad \quad \quad 5x_3 & = & 12 \end{array}$$

It turns out that $\mathbf{u} - \mathbf{v}$ is a solution to the following system:

$$\begin{array}{rcl} 2x_1 + 4x_2 + 3x_3 & = & 0 \\ x_1 + \quad \quad \quad 5x_3 & = & 0 \end{array}$$

It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.

It would play a central role when dealing with linear systems with many solutions.

Key take away

- ▶ There are 2 ways to look at how we solve linear systems: row perspective and column perspective.
- ▶ **Linear combination** is the main operation.