

DFA: Formal definitions

NFA

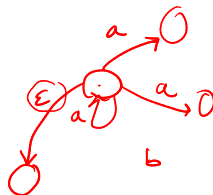
$P(Q)$
powerset

A **finite-state machine** or a **deterministic finite-state automaton** (DFA) has five components:

- ▶ the input alphabet Σ , ✓
- ▶ a finite set of states Q , ✓
- ▶ a transition function $\delta : Q \times \Sigma \rightarrow Q$ ✓
- ▶ a start state $s \in Q$, and ✓
- ▶ a subset $A \subseteq Q$ of accepting states. ✓

NFA

$$Q \times \Sigma_{\varepsilon} \rightarrow 2^Q$$



Acceptance

One step move: from state q with input symbol a , the machine changes its state to $\delta(q, a)$.

Extension: from state q with input string w , the machine changes its state to $\delta^*(q, w)$ defined as

$$\delta^*(q, w) = \begin{cases} q & \text{if } w = \varepsilon, \\ \delta^*(\delta(q, a), x) & \text{if } w = ax. \end{cases}$$

The signature of δ^* is $Q \times \Sigma^* \longrightarrow Q$.

accepting w

For a finite-state machine with starting state s and accepting states A , it accepts string w iff

$$\delta^*(s, w) \in A.$$

Language of a DFA

$L(M)$

For a DFA M , let $L(M)$ be the set of all strings that M accepts. More formally, for $M = (\Sigma, Q, \delta, s, A)$,

$$L(M) = \{w \in \Sigma^* \mid \delta^*(s, w) \in A\}.$$

We refer to $L(M)$ as the language of M .

Automatic languages²

Definition (for now)

A language L is “automatic” if there is a DFA M such that $L(M) = L$.

Lemma 1

If L_1 and L_2 are automatic languages over alphabet Σ , then

- ▶ $L_1 \cap L_2$,
- ✓▶ $L_1 \cup L_2$,
- ▶ $L_1 \setminus L_2$, and
- ▶ $\Sigma^* \setminus L_1$,

are also automatic.

✓▶ $L_1 \circ L_2$

✓▶ L_1^*

▶ \emptyset ✓

▶ language of
a string. ✓

The set of automatic languages is closed under these boolean operations.

²Taken directly from Erikson's lecture notes

Other ways to combine DFAs?

Given two languages L_1 and L_2 , we can combine them in various ways using Boolean operations (i.e., \cap , \cup , etc.).

What else can we do?

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$$\{x \cdot y \mid x \in L_1, y \in L_2\}$$

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Given two languages L_1 and L_2 , we can combine them in various ways using Boolean operations (i.e., \cap , \cup , etc.).

What else can we do?

- Concatenation: $L_1 \cdot L_2$, defined as

$$\{x \cdot y \mid x \in L_1, y \in L_2\}$$

- Kleene closure: L_1^* .

Interesting questions



$$L_1 \circ L_2$$



↗ L_1 નો DFA M_1 નો $L(M_1) = L_1$

L_2 નો DFA M_2 નો $L(M_2) = L_2$

(L_1 નો automatic)
(L_2 નો automatic)

We know that the set of automatic languages is closed under Boolean operations.

Questions

- ▶ Is it closed under concatenation?
- ▶ Is it closed under taking Kleene closure?

① ↗ DFA M' નો $L(M') = L_1 \circ L_2$
($L_1 \circ L_2$ નો automatic)

② ↗ DFA M' નો $L(M') = L_1^*$
(L_1^* નો automatic)

Interesting questions

We know that the set of automatic languages is closed under Boolean operations.

Questions

- ▶ Is it closed under concatenation?
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Spoiler:

Interesting questions

We know that the set of automatic languages is closed under Boolean operations.

Questions

- ▶ Is it closed under concatenation?
- ▶ Is it closed under taking Kleene closure?

Spoiler: Yes, it is (for both operations). We will see the proof, after we learn a required new concept.

Closure

Lemma 2

Given two automatic languages L_1 and L_2 , the following languages are automatic:

- ▶ $L_1 \cup L_2$,
- ▶ $L_1 \cdot L_2$, and
- ▶ L_1^* .

Closure

Lemma 2

Given two automatic languages L_1 and L_2 , the following languages are automatic:

- ▶ $L_1 \cup L_2$, ✓
- ▶ $L_1 \cdot L_2$, and ✓
- ▶ L_1^* . ✓

More over, \emptyset and a language containing a single string are also automatic.

Closure

More over, \emptyset and a language containing a single string are also automatic.

Lemma 2

These are automatic languages

- ▶ *The empty set,*
- ▶ *A language containing one string,*
- ▶ *$L_1 \cup L_2$ for automatic languages L_1 and L_2 ,*
- ▶ *$L_1 \cdot L_2$ for automatic languages L_1 and L_2 , and*
- ▶ *L^* for automatic languages L .*

Doese this look familiar?

Regular languages

Definition: regular languages

A language L is **regular** if and only if it satisfies one of the following conditions:

- ▶ L is empty;
- ▶ L contains one string (can be the empty string ε);
- ▶ L is a union of two regular languages;
- ▶ L is the concatenation of two regular languages; or
- ▶ L is the Kleene closure of a regular language.



Every regular language is automatic

Big question:

⇒

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⇐

Is every automatic language regular?

Spoiler:

⇒ $\left[\begin{array}{l} \text{if regular lang is automatic reg ex } R \\ \Rightarrow \exists \text{ DFA } M \text{ s.t. } L(M) = L(R) \end{array} \right] \leftarrow$

⇒

Every regular language is automatic

Big question:

←

Is every automatic language regular?

Spoiler: Yes, it is. We will see some idea on how to prove this.

$\left[\begin{array}{l} \text{if DFA } M \text{ then } \exists \text{ regular expression } R \\ \text{s.t. } L(R) = L(M) \end{array} \right] \leftarrow$

$\{0^n 1^n \mid n \geq 0\}$ ← specification
 math

⇒

Every regular language is automatic

Big question:

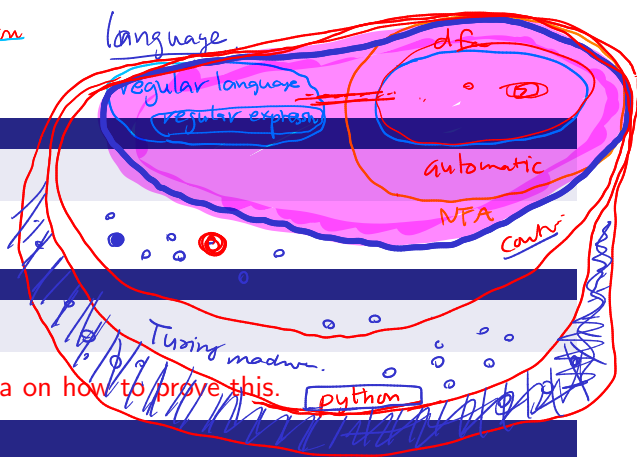
←

Is every automatic language regular?

Spoiler: Yes, it is. We will see some idea on how to prove this.

Theorem 3

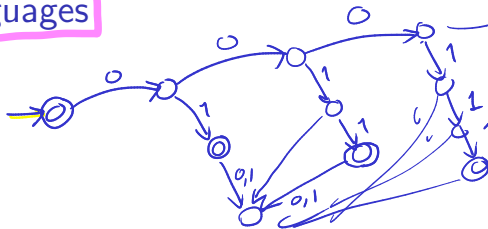
A language L is regular if and only if there exists a DFA M such that $L(M) = L$.



$\{p \mid p \text{ เป็นโปรแกรมที่ } \epsilon \text{ ที่สามารถ}$
 $\text{ทำงานได้และไม่มีปัญหาการคำนวณ}\}$

การคำนวณ.

Nonregular languages



Can you design a DFA that accepts strings from language

string L non regular

$$\{0^n 1^n \mid n \geq 0\}$$

▷ This language is not regular

▷ This DFA can not accept L

▷ This NFA can not accept L

Key idea

If you have finite states, you can't possibly distinguish between strings in the language and strings not in the language.

Basic question

How can you show that you need at least two states?

DFA M is
on x has y

Two state machine

$\forall z, xz, yz$

z : Two state machine

Goal: string x
no string y

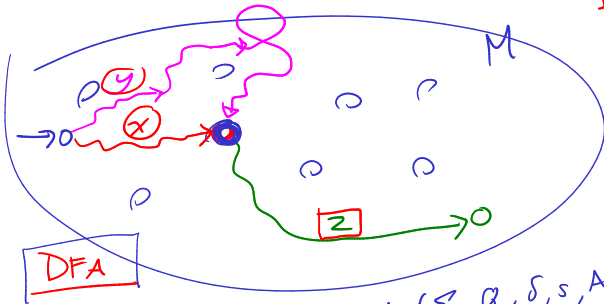
only 2 state
 $xz \in L$
 $yz \notin L$

DFA $M = (\Sigma, Q, \delta, s, A)$

$$\delta^*(s, x) = \delta^*(s, y)$$

$$\delta^*(s, xz) = \delta^*(s, yz)$$

\forall string z ,



DFA

Proof by contradiction

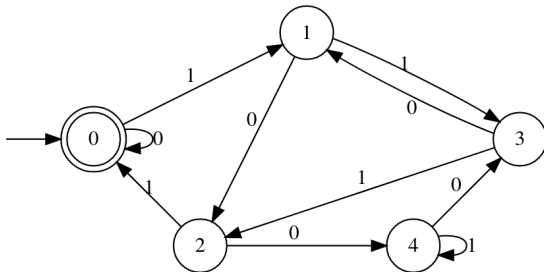
Goal: P

Assume $\neg P$

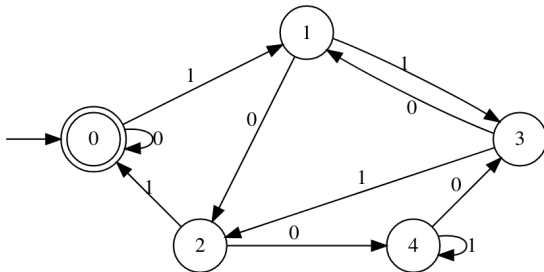
Basic question

How can you show that you need at least two states?
Let's see how a DFA works.

Another example

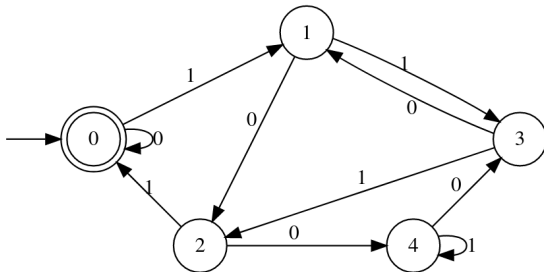


Another example



If string x and y reach the same state in a DFA, for any string z , both xz and yz must reach the same state.

Another example



If string x and y reach the same state in a DFA, for any string z , both xz and yz must reach the same state.

In other words, a DFA accepts xz iff it accepts yz .

Distinguishing suffixes

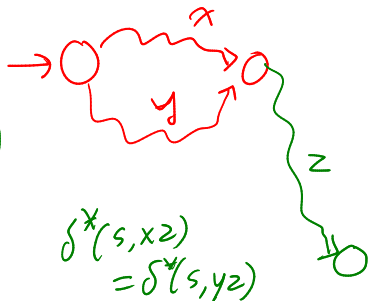
Consider language $L = \{0^n 1^n \mid n \geq 0\}$.

Consider $x = 0$ and $y = 00$. Consider suffix $z = 11$.

$$\underline{x}z = 011 \notin L$$

$$\underline{y}z = 0011 \in L$$

$$\begin{aligned} \nabla \text{ in } \delta^*(s, x) \\ = \delta^*(s, y) \end{aligned}$$



$$\text{in } xz \in L \Leftrightarrow yz \in L$$

Distinguishing suffixes

Consider language $L = \{0^n 1^n \mid n \geq 0\}$.

Consider $x = 0$ and $y = 00$. Consider suffix $z = 11$.

We have that

$$xz = 0\underline{1}1 \notin L,$$

but

$$yz = 00\underline{1}1 \in L.$$

Distinguishing suffixes

Consider language $L = \{0^n 1^n \mid n \geq 0\}$.

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What can you say about a DFA M such that $L(M) = L$?

Distinguishing suffixes

Consider language $L = \{0^n 1^n \mid n \geq 0\}$.

Consider $x = 0$ and $y = 00$. Consider suffix $z = 11$.

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but

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What can you say about a DFA M such that $L(M) = L$?

Definition

For strings x and y , string z is a **distinguishing suffix** with respect to L if exactly one of xz and yz is in L .

Fooling sets

A fooling set for a language L is set F of strings such that every pair of strings in F has a distinguishing suffix.

Example: The set $\{0, 00, 000\}$ is a fooling set for $L = \{0^n 1^n \mid n \geq 0\}$.

A large fooling set

Lemma 4

The set $\{0^n \mid n \geq 0\}$ is a fooling set for $L = \{0^n 1^n \mid n \geq 0\}$.

Proof.

• ឱ្យយើង យក $x, y \in F$ រួច $x \neq y$. តើយើង យក $x = 0^i$ ឬ $y = 0^j$ ដែល $i \neq j$

• យើង យក $z = 1^i$

យើង យក $xz = 0^i 1^i \in L$

ប៉ុន្តែ $yz = 0^j 1^i \notin L$

តើយើង យក distinguishing suffix z យើង យក x ឬ y

យើង F គឺជា fooling set ពិតប្រាកដ

▷ (Goal: យើង យក distinguishing suffix z យើង យក x ឬ y)
▷ បើ z យើង $xz \in L$ ឬ $yz \in L$
ឬ $xz \notin L$ ឬ $yz \in L$

✓ 000 (111) ✓
00000 111 x





Observation

If language L has an infinite fooling set, L is not regular

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Lemma 5

Language $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof.

We previously establish that the set $F = \{0^n \mid n \geq 0\}$ is a fooling set for L .
Since F has infinite size, from the observation, we know that L is not regular. □

Lemma 6

For $\Sigma = \{0, 1\}$, the language $L = \{\underline{w}w^R \mid w \in \Sigma^*\}$ is not regular.

$w: w^R$
 $001100 \quad 6L$
 $00001100 \quad 6L$
 (Note: The second string is circled in red with a red 'X' over it, indicating it is not in the language because the first and last characters don't match.)

Proof.

Let $F = \{0^n \mid n \geq 0\}$ is a fooling set

for L . Pick $x, y \in F$ s.t. $x = 0^i, y = 0^j$ for $i \neq j$

Let $z = 110^i$

$$\bullet xz = 0^i 110^i \in L$$

$$\bullet yz = 0^j 110^i \notin L$$

} cannot distinguish
suffix of x from y

$$x = 001100 \quad \checkmark$$

$$y = 00001100 \quad \times$$

$$z = 1100$$

$$x = 0^i$$

$$y = 0^j \text{ s.t. } i \neq j$$

$$z = 110^i$$

Plan:

→ Pick a
 fooling set F
 → Pick $x, y \in F$
 s.t. $x \neq y$

Since F is a fooling set, L is not regular.

Since F is a fooling set, L is not regular.

→ L is regular. ~~contradiction~~



Lemma 6

For $\Sigma = \{0, 1\}$, the language $L = \{ww^R \mid w \in \Sigma^*\}$ is not regular.

Proof.

$$\text{b} \quad \underline{F = \{0^n 1 \mid n \geq 0\}}$$

$$x = 0^{\bar{i}} 1$$

$$y = 0^{\bar{j}} 1 \quad \bar{j} \neq \bar{i}$$

$$z = 10^{\bar{i}}$$

$$xz = \underbrace{0^{\bar{i}} 1} \underbrace{10^{\bar{i}}} \in L$$

$$yz = 0^{\bar{j}} 110^{\bar{i}} \notin L$$



Lemma 6

For $\Sigma = \{0, 1\}$, the language $L = \{ww^R \mid w \in \Sigma^*\}$ is not regular.

Proof.

$$\text{Let } \textcircled{F} = \{0^n 11 \mid n \geq 0\}$$

$$\underline{x} = 0^{\bar{i}} 11$$

$$\underline{y} = 0^{\hat{j}} 11 \quad \bar{i} \neq \bar{v}$$

$$\textcircled{z} = 0^{\bar{v}}$$

$$xz = \underline{0^{\bar{i}} 11} \underline{0^{\bar{v}}} \in L$$

$$yz = \underline{0^{\hat{j}} 11} \underline{0^{\bar{v}}} \notin L.$$



$L = \{0^{2^n} \mid n \geq 0\}$: Proof 1

~~0000~~

$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} = 0^8$

Lemma 7

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.

พิจารณา $F = \{0^{2^n} \mid n \geq 0\}$ เลือก $x, y \in F$ ที่ $x = 0^{2^i}$ และ $y = 0^{2^j}$, $i < j$

ให้ $z = 0^{2^i}$ สังเกตว่า

$$xz = 0^{2^i} \cdot 0^{2^i} = 0^{2 \cdot 2^i} = 0^{2^{i+1}} \in L$$

$yz = 0^{2^j} \cdot 0^{2^i} = 0^{2^j + 2^i}$ สังเกตว่า จำนวนเลข 0 ใน yz มากกว่า 2
เนื่องจาก 2^j ที่ $2^{j+1} = 2^j + 2^j > 2^j + 2^i$ เมื่อ $j > i$
ดังนั้น $2^j + 2^i$ ไม่ใช่กำลังของ 2 และ $yz \notin L$.

.....



$L = \{0^{2^n} \mid n \geq 0\}$: Proof 2

Lemma 8

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.



$L = \{0^{2^n} \mid n \geq 0\}$: Proof 3

Lemma 9

For $\Sigma = \{0\}$, the language $L = \{0^{2^n} \mid n \geq 0\}$ is not regular.

Proof.




$L = \{0^p \mid p \text{ is prime}\}$: Proof 1

Lemma 10

For $\Sigma = \{0\}$, the language $L = \{0^p \mid p \text{ is prime}\}$ is not regular.

Proof.

$$F = \{0^n \mid n \geq 0\} \quad \dots \quad x = 0^i \quad y = 0^j \quad i < j$$


$$\begin{aligned} \boxed{i+k} & \text{ is prime} \\ j+k & \text{ is prime} \\ &= \boxed{i+k} + \underline{(j-i)} \end{aligned}$$



$L = \{0^p \mid p \text{ is prime}\}$: Proof 2

Lemma 11

For $\Sigma = \{0\}$, the language $L = \{0^p \mid p \text{ is prime}\}$ is not regular.

Proof.

