01204211 Discrete Mathematics
Lecture 1: Introduction

Jittat Fakcharoenphol

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Ah... that's a philosopical question.
IMHO, mathematics is a mean to communicate *precise* ideas.

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- I hope it is also true with this course.

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- To learn how to make mathematical arguments.
- To learn various fundamental mathematical concepts that are very useful in computer science.

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    return a
else:
    return b
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The author claims that this program takes two variables a and b and returns the larger one.

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Finding the maximum value. (1)

Now look at this program.

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        return c
else:
    if c > b:
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The author claims that this program takes three variables a, b and c and returns the largest one.

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Finding the maximum value. (2)

Finally, look at this program.

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// Input: array A with n elements: A[0],...,A[n-1]
m = 0
for i = 0, 1, ..., n-1:
  if A[i] > m:
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Can we try to test the code with all possible inputs?

input → Code → output

expected output ✓ ✗

Finding the maximum value. (3)

Let's try again.

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Another example: testing primes (1)

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Algorithm CheckPrime(n):       // Input: an integer n
    if n <= 1:
        return False
    i = 2
    while i <= n-1:
        if n is divisible by i:
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The code above checks if n is a prime number. How fast can it run?

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- The code above checks if n is a prime number. How fast can it run?
- Note that if n is a prime number, the for-loop repeats for $n-2$ times. Thus, the running time is approximately proportional to n.
- Can we do better?

Another example: testing primes (2)

Consider the following code.

```
Algorithm CheckPrime2(n):  // Input: an integer n
    if n <= 1:
        return False
    let s = square root of n
    i = 2
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```

$\sqrt{n}$

$\sqrt{n}$ sov.

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10 3
100 10
1000 [illegible]
1000000 1000

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```

1, 2, 2, 3, ..., n-1

2, 3, ..., √n

ถ้า n เป็น prime | ถ้า n ไม่เป็น prime

false

✓

How fast can it run? Note that $s = \sqrt{n}$; therefore, it takes time approximately proportional to $\sqrt{n}$ to run.

Ok, it should be faster. **But is it correct?** ????

Informal arguments (1)

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- Now, if $\boxed{2 \leq a \leq \sqrt{n},}$ at some point during the execution of the algorithm, $i = a$ and i should divides n; thus the algorithm correctly returns `False`. ✓

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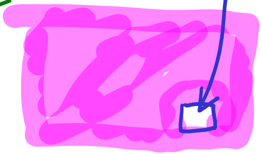
- Now, if $2 \leq a \leq \sqrt{n}$, at some point during the execution of the algorithm, $i = a$ and i should divides n; thus the algorithm correctly returns `False`. ✓
- Are we done?

hole



Informal arguments (2)

ex: $10 = 5 \times \boxed{2}$

$2, 3, \cancel{4}$

ABSTRACT

$\updownarrow$

CONCRETE

- Recall that we are left with the case that (1) n is not prime and (2) its positive divisor a is larger than $\sqrt{n}$.

Informal arguments (2)

- Recall that we are left with the case that (1) n is not prime and (2) its positive divisor a is larger than $\sqrt{n}$.
- Let $b = n/a$. Since n and a are positive integers and a divides n, b is also a positive integer.

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- Note that if we can argue that $\boxed{2 \le b \le \sqrt{n},}$ we are done. (why?)

ถ้า n เป็น composite

$\Rightarrow$ จะมี a ที่หาร n

$\rightarrow$ ถ้า $a \le \sqrt{n}$ ✓

$\rightarrow$ มี $b = n/a$ ที่ $b \le \sqrt{n}$

และ: b สามารถ หาร ลงตัว

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- Let $b = n/a$. Since n and a are positive integers and a divides n, b is also a positive integer.
- Note that if we can argue that $2 \leq b \leq \sqrt{n}$, we are done. (why?)
- How can we do that?

The goals

- Let's take a break and look back at what we are trying to do.

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[illegible]
$2, 3, \ldots \sqrt{n}$

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- Before we continue, I'd like to add a bit of formalism to our thinking process.

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> **"Algorithm `CheckPrime2` is correct."**

- Note that this statement can either be "true" or "false." If we can demonstrate, using logical/mathematical arguments that this statement is true, we can say that we **prove** the statement.

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- If we only look at this statement, it is unclear if the statement is true or false because there are variables b and n in the statement. It can be true in some case and it can be false in some case depending on the values of n and b.
- Are we doom? Not really. The statement above is not precisely the statement we want to prove.

The (sub) goal (second try)

- **Current (sub) goal:** Consider a positive composite n and its positive divisor a, where $a > \sqrt{n}$. Let $b = n/a$. We want to show that $2 \leq b \leq \sqrt{n}$.
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Revised statement

For all positive composite integer n, and for every divisor a of n such that $\sqrt{n} < a < n$,

bound

"$\forall n$"

$$2 \le b \le \sqrt{n},$$

free

where $\boxed{b = n/a.}$

- Note that this revised statement is now "quantified," that is, every variable in the statement has specific scope. Now the statement is either true or false.

Propositions[1] ประพจน์

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- It is our basic unit of mathematical "facts".
- Examples:
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  - $10^2 = 90$.
  - $\sqrt{2}$ is irrational.

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- Examples:
  - Algorithm `CheckPrime2` is correct.
  - $10^2 = 90$.
  - $\sqrt{2}$ is irrational.
- Examples of statements which are not propositions (why?):
  - $x > 10$. ← x คืออะไร ?
  - $1 + 2 + \cdots + 10$. ← ไม่เป็น true/false
  - This algorithm is fast. ← ขึ้นกับค่าที่เร็ว
  - Run, run quickly.

↑ คำสั่ง

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Combining propositions

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- For example, we can join P and Q with "and" (denoted by "$\wedge$") and get
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which stands for "it rains and the road is wet".
- An expression $P \wedge Q$ is an example of *propositional forms.* The logical value of a propositional form "usually" depends on the truth value of its variables.

Connectives: "and", "or", "not"

Given propositions P and Q, we can use connectives to form more complex propositions:

- **Conjunction:** $P \wedge Q$ ("P and Q"), (True when both P and Q are true)
- **Disjunction:** $P \vee Q$ ("P or Q"), (True when at least one of P and Q is true)
- **Negation:** $\boxed{\neg P}$ ("not P") (True only when P is false)

$\sim P$

$\boxed{\neg P}$

เข้มงวด

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If P stands for "today is Tuesday" and Q stands for "dogs are animals", then

- $P \wedge Q$ stands for "today is Tuesday and dogs are animals",
- $P \vee Q$ stands for "today is Tuesday or dogs are animals", and
- $\neg P$ stands for "today is not Tuesday".

Truth tables

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	
F	T	F	T	T
F	F	F	F	

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Quick check 1

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- All prime numbers are larger than 0 and all natural numbers is at least one.
- You are smart or you won't be taking this class.

P

Q

$P \wedge Q$

S

C "you take this class"

$S \vee (\neg C)$

Next lecture...

if
only-if

- We will discuss other ways to join two propositions, i.e., implications ($\Rightarrow$) and equivalences ($\Leftrightarrow$). iff
- We will look at two forms of quantifiers: universal quantifiers $\forall$ and existential quantifiers. $\exists$