$\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$

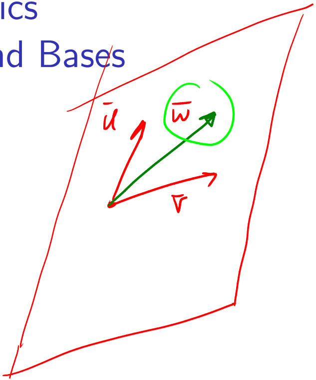
if $\bar{u}_n$ ------

$\boxed{\mathrm{Span}\{\bar{u}_1, \ldots, \bar{u}_n\}} = \mathrm{Span}\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$

i.e. $\bar{u}_n$ is not $\boxed{\text{needed}}$

$\{\bar{u}, \bar{v}, \cancel{\bar{w}}\}$

01204211 Discrete Mathematics
Lecture 9c: Linear Independence and Bases

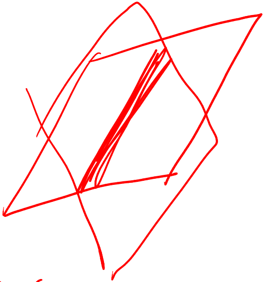


Jittat Fakcharoenphol

October 27, 2025

→ $x_1 + x_2 + x_3 = 0$ — (1)

→ $2x_1 + 3x_2 + 5x_3 = 0$ — (2)



× → $\boxed{x_1 + 2x_2 + 4x_3 = 0}$ — (3) = (2) − (1)

× → $\boxed{x_2 + 3x_3 = 0}$ — (4) = (2) − 2×(1)

Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Previous Lemmas

Lemma 1

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ *are generators for* $\mathcal{V}$, *and for each* i,

$$\boldsymbol{v}_i \in \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\},$$

we have that $\mathcal{V} \subseteq \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$.

Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

$$\operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

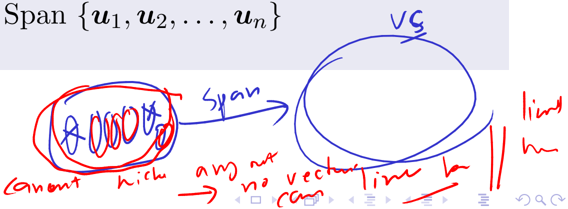
$$\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

Lemma 3

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{u}_n \in \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n-1}\}$, *then*

$$\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n-1}\} = \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

kicked out



Proof of Lemma 2.

Since $\boldsymbol{v}$ can be written as a linear combination of other vectors, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n.$$

Consider any vector $\boldsymbol{w} \in \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\}$; thus, we can write

$$\boldsymbol{w} = \beta_0 \boldsymbol{v} + \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \cdots + \beta_n \boldsymbol{u}_n.$$

Plugging in $\boldsymbol{v}$, we get that

$$\begin{aligned} \boldsymbol{w} &= \beta_0 \left(\alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_n \boldsymbol{u}_n\right) + \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \cdots + \beta_k \boldsymbol{u}_k \\ &= (\beta_0 \alpha_1 + \beta_1) \boldsymbol{u}_1 + (\beta_0 \alpha_2 + \beta_2) \boldsymbol{u}_2 + \cdots + (\beta_0 \alpha_n + \beta_n) \boldsymbol{u}_n, \end{aligned}$$

implying that $\boldsymbol{w} \in \mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$.

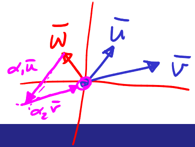
$\square$

Linearly independence

Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if no vector $\boldsymbol{u}_i$ can be written as a linear combination of other vectors.

Linearly independence



Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if no vector $\boldsymbol{u}_i$ can be written as a linear combination of other vectors.

(Another) Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if the only solution of equation

$$\alpha_1\boldsymbol{u}_1 + \alpha_2\boldsymbol{u}_2 + \cdots + \alpha_n\boldsymbol{u}_n = \boldsymbol{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Examples in $\mathbb{R}^2$

Examples in $\mathbb{R}^3$

Examples in $GF(2)$

Examples in linear systems

Subset of linearly independent vectors

Lemma 4

If $A = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ *be a set of linearly independent vectors, then any* $B \subseteq A$ *is also a set of linearly independent vectors.*

Proof.

We prove by contradiction. Assume that B is **not** linearly independent. Without loss of generality, assume that $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ where $k \leq n$.

Subset of linearly independent vectors

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Proof.

We prove by contradiction. Assume that B is **not** linearly independent. Without loss of generality, assume that $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ where $k \leq n$. This means that there exists $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k = \mathbf{0},$$

and some α_i's is nonzero.

Subset of linearly independent vectors

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If $A = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ *be a set of linearly independent vectors, then any* $B \subseteq A$ *is also a set of linearly independent vectors.*

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$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k = \mathbf{0},$$

and some α_i's is nonzero. If we let $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$, we have that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n = \mathbf{0},$$

with some α_i's being nonzero as well.

Subset of linearly independent vectors

Lemma 4

If $A = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a set of linearly independent vectors, then any $B \subseteq A$ is also a set of linearly independent vectors.

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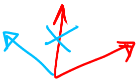
and some α_i's is nonzero. If we let $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$, we have that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n = \mathbf{0},$$

with some α_i's being nonzero as well. This implies that vectors in A are not linearly indepedent; leading to a contradiction. $\square$

Bases



Set of vectors V

- $0 \in V$
- if $\bar{u} \in V$, $\alpha \bar{u} \in V$
- if $\bar{u}, \bar{v} \in V$, $\bar{u} + \bar{v} \in V$

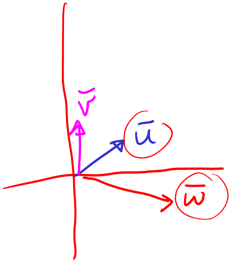
Definition

A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a **basis** for vector space $\mathcal{V}$ if

- $\mathrm{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \mathcal{V}$, and
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

Examples 1: $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\text{Span}\{\bar{u}, \bar{v}\} = \mathbb{R}^2$$



$$\text{Span}\{\bar{v}, \bar{u}, \bar{w}\} = \mathbb{R}^2$$

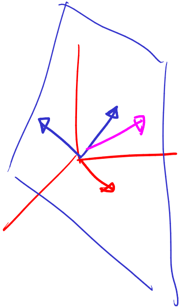
$$\{\bar{u}, \bar{w}\}$$

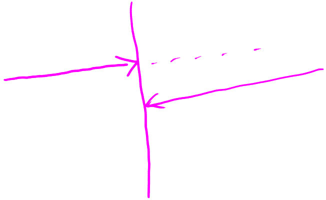
Examples 1: $\mathbb{R}^2$ and $\mathbb{R}^3$

Less vectors ⟷ ~~incr~~ more vec

−span +span

+ — −indep





Examples 2

Lemma 5 (Unique representation)

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be a basis for vector space $\mathcal{V}$. For any $\boldsymbol{v} \in \mathcal{V}$, there is a unique way to write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$.

Proof of unique representation lemma.

We prove by contradiction.

Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector $\boldsymbol{v} \in \mathcal{V}$ with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

and

$$\beta_1, \beta_2, \ldots, \beta_k,$$

that are not equal (i.e., there exists i where $\alpha_i \neq \beta_i$) such that $\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k$ and $\boldsymbol{v} = \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \cdots + \beta_k \boldsymbol{u}_k$.

Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector $\boldsymbol{v} \in \mathcal{V}$ with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

and

$$\beta_1, \beta_2, \ldots, \beta_k,$$

that are not equal (i.e., there exists i where $\alpha_i \neq \beta_i$) such that $\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k$ and $\boldsymbol{v} = \beta_1 \boldsymbol{u}_1 + \beta_2 \boldsymbol{u}_2 + \cdots + \beta_k \boldsymbol{u}_k$. This implies that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k = \boldsymbol{v} = \beta_1 \boldsymbol{u}_1 + \beta \boldsymbol{u}_2 + \cdots + \beta_k \boldsymbol{u}_k,$$

and

$$(\alpha_1 - \beta_1)\boldsymbol{u}_1 + (\alpha_2 - \beta_2)\boldsymbol{u}_2 + \cdots + (\alpha_k - \beta_k)\boldsymbol{u}_k = 0.$$

Proof of unique representation lemma.

We prove by contradiction. Assume that there exists a vector $\boldsymbol{v} \in \mathcal{V}$ with more than one ways to be written as linear combinations of the basis. Thus, there exist

$$\alpha_1, \alpha_2, \ldots, \alpha_k,$$

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$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_k \boldsymbol{u}_k = \boldsymbol{v} = \beta_1 \boldsymbol{u}_1 + \beta \boldsymbol{u}_2 + \cdots + \beta_k \boldsymbol{u}_k,$$

and

$$(\alpha_1 - \beta_1)\boldsymbol{u}_1 + (\alpha_2 - \beta_2)\boldsymbol{u}_2 + \cdots + (\alpha_k - \beta_k)\boldsymbol{u}_k = 0.$$

Since $\alpha_i \neq \beta_i$, we have that at least one of the coefficients is non-zero, implying that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ are not linearly independent. This contradicts the assumption that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ form a basis. $\square$