01204211 Discrete Mathematics
Lecture 11a: Guassian Elimination and LU Decomposition

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Review: A Linear System

Consider the following system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 10 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 4 \end{array}$$

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Again we can view it as a vector equation:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix}$$

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Solving this system is equivalent to testing if

$$[5, 10, 4] \in \mathrm{Span}\ \{[1, 2, 3], [1, 1, 1], [1, 2, 2]\}.$$

Review: Testing spans

Problem

Given a set of n-vectors $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ over $\mathbb{F}$ and an n-vector $\boldsymbol{v}$, can we check if $\boldsymbol{v} \in \text{Span } S$?

Example: Consider 3-vectors over $\mathbb{R}$. Let

$$\boldsymbol{u}_1 = [1, 2, 3], \boldsymbol{u}_2 = [1, 1, 1], \boldsymbol{u}_3 = [1, 2, 2].$$

We would like to check if

$$\boldsymbol{v} = [10, 13, 29] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}.$$

Let us define variables $\alpha_1, \alpha_2, \alpha_3$ such that $\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$; i.e., we want $\alpha_1, \alpha_2, \alpha_3$ to be such that

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 29 \end{bmatrix}$$

Matrix form

We can write these constraints down as the following linear equations.

$$\begin{array}{ccccccc} 1\alpha_1 & + & 1\alpha_2 & + & 1\alpha_3 & = & 10 \\ 2\alpha_1 & + & 1\alpha_2 & + & 2\alpha_3 & = & 13 \\ 3\alpha_1 & + & 1\alpha_2 & + & 2\alpha_3 & = & 29 \end{array}$$

We also can, equivalently, write them in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 29 \end{bmatrix}$$

Review: Properties of matrix multiplications

$1+2+5+7$

Let A, B, C be matrices.

- (Associative) $(AB)C = A(BC)$
- (Distributive) $A(B+C) = AB + AC$
- In general not commutative: Usually $AB \neq BA$

Identity Matrix

Definition

An n-by-n matrix I is an **identity matrix** if

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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$$I = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

such that $a_{ii} = 1$ for $1 \leq i \leq n$ and $a_{ij} = 0$ for all $i \neq j$.

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For any n-by-m matrix A,

$$IA = A,$$

and for any m-by-n matrix B,

$$BI = B.$$

Inverses

$\frac{1}{2} = 2^{-1}$

Definition

For an n-by-n matrix A, the **inverse of A**, denoted by A^{-1}, is an n-by-n matrix such that

$$AA^{-1} = A^{-1}A = I.$$

Remark 1: You can think of an inverse as a matrix that let you "get back" anything after multiplying with A, i.e.,

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Remark 3: A might not have an inverse. If A has an inverse, we say that A is **invertible.**

Lemma 1

If A and B are invertible, we have that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

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- $(AB)^T = B^T A^T$.
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Definition

Matrix A is **symmetric** if

$$A = A^T.$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 2 & 7 \\ 0 & 7 & 1 \end{bmatrix}$$

Solving a system of linear equations

For an $n \times n$ matrix A and a system of linear equations

$$\boxed{A\boldsymbol{x} = \boldsymbol{b},}$$

with n variables $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$, if A^{-1} is an inverse of A, we can use it to solve for $\boldsymbol{x}$.

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Note that we can multiply on the left of both sides with A^{-1} to obtain

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Thus

$$\boldsymbol{x} = A^{-1}\boldsymbol{b},$$

is the solution of the system.

$$\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ 3 & 1 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}10\\ 13\\ 29\end{bmatrix}$$

Let's perform Gaussian elimination.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 29 \end{bmatrix}$$

Let's perform Gaussian elimination.

$R_2 \leftarrow R_2 + (-2)R_1 \rightarrow$

$R_3 \leftarrow R_3 + (-3)R_1 \rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ -1 \end{bmatrix}$$

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$$R_3 \leftarrow R_3 - 2R_2 \rightarrow \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}10\\ 7\\ 13\end{bmatrix}$$

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We can perform backward substitution to find that $\alpha_1 = 16, \alpha_2 = 7$, and $\alpha_3 = -13$.

Gaussian elimination and matrix operations

We will look closer to see how we could "describe" the steps from Gaussian elimination using matrix multiplications. This would be very useful later. We start with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}.$$

The first row operation we did is: $R_2 \leftarrow R_2 - 2R_1$
Can we explain this step with a matrix multiplication?

Gaussian elimination and matrix operations

We will look closer to see how we could "describe" the steps from Gaussian elimination using matrix multiplications. This would be very useful later. We start with

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}.$$

The first row operation we did is: $R_2 \leftarrow R_2 - 2R_1$
Can we explain this step with a matrix multiplication? I.e., can we find M such that

$$MA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}. = R_2 + (-2)R_1$$

We currently have

$$MA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$$

The next row operation we did is: $R_3 \leftarrow R_3 - 3R_1$
Can we explain this step with a matrix multiplication?

We currently have

$M' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $\quad MA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$

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Elementary matrices

Operations	Result	Elementary matrix	~~Remarks~~ inverse.
$R_2 \leftarrow R_2 - 2R_1$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$	$E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$E_{12}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$R_3 \leftarrow R_3 - 3R_1$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$	$E_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$	
$R_3 \leftarrow R_3 - 2R_2$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$	

Let's denote the final upper triangular matrix by B. Therefore, we have

$$E_{23}E_{13}E_{12}A = B,$$

or equivalently,

goal

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{23}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E_{13}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{12}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{B}.$$

Recall that we have

$$E_{\underset{23}{\cancel{12}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_{\underset{12}{\cancel{23}}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $E_{12}^{-1}, E_{13}^{-1}, E_{23}^{-1}$ be inverses of E_{12}, E_{13}, E_{23}, respectively.
It is not hard to see that

$$E_{12}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Therefore, we can write

$$E_{12}^{-1} E_{13}^{-1} E_{23}^{-1} E_{23} E_{13} E_{12} A = A = E_{12}^{-1} E_{13}^{-1} E_{23}^{-1} B,$$

After working out the multiplication

$$E_{12}^{-1}E_{13}^{-1}E_{23}^{-1} = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}$$

we see that

$$\overset{A}{\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ 3 & 1 & 2\end{bmatrix}} = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

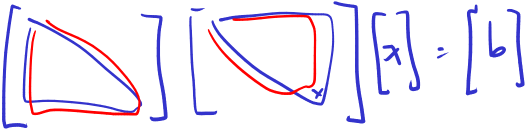
The matrix A is “factored” into two matrices. We denote the first matrix L (for lower triangular) and the second one U (for upper triangular).

This is called an **LU decomposition** of A.

Why is an LU decomposition useful? (1)

$$Ax = b$$

$A = LU$ $\qquad \boxed{L(Ux) = b}$



$$\boxed{\begin{aligned} Ux &= b \\ Lx &= b \end{aligned}}$$

solve in

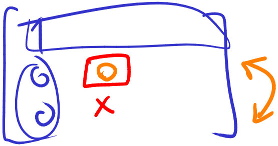
1st $\quad Ly = b$ in y

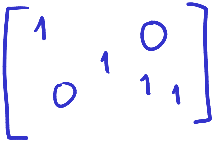
2nd $\quad Ux = y$ in x

Why is an LU decomposition useful? (2)

$$A = LU \quad , \quad A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$$

LU decomposition - pivots





permutation matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ \hline r_4 \\ \hline r_2 \\ \hline r_3 \end{bmatrix}$$