

Vectors

↓  
spans

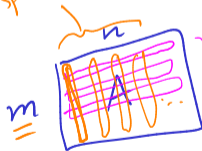
solutions  
homogen  
lin eq

↓  
vector spaces

01204211 Discrete Mathematics

## Lecture 11b: Four fundamental subspaces (preview)

Column  $\subset \mathbb{R}^m$   
spaces



row  
space  
 $\subset \mathbb{R}^n$

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# What is a matrix?

basis

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$\subseteq \mathbb{R}^4$

$\text{Span}\{[1, 4, 7, 10], [2, 5, 8, 11], [3, 6, 9, 12]\} = \text{column space}$

$$\left[ \begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array} \right]$$

$\text{span of rows of a mat}$

$\text{row space}$

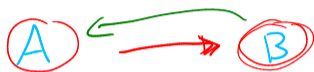
$\text{Span}\{[1, 2, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]\} \subseteq \mathbb{R}^3$

# Matrices and elimination

# Matrices and elimination

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

# Matrices and elimination

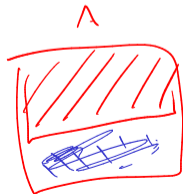


$r = k$

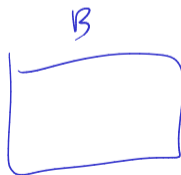
row sp

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$k$   
linear  
in  
rows



$k \geq r$



$k$   
linear  
in  
cols

# Matrices and elimination

row space of A

Span  $\{ [1, 2, 0, 1], [1, 5, 1, 1], [2, 10, 2, 4], [2, 7, 1, 10] \}$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B

$\dim$   
row space of B = 3

Span  $\{ [1, 2, 0, 1], [0, 3, 1, 0], [0, 0, 0, 2] \}$

$\{ [1, 2, - -], [ - - - ] \}$

is a basis of  
row of B

# Matrices and elimination

Row space of  $A$   $\mathcal{C}(A^T)$

$$= \text{Span} \{ \bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4 \}$$

dimension of  $\mathcal{C}(A^T)$

$$\boxed{\dim \mathcal{C}(A^T)} \\ = 3$$

row rank of  $A$   $= 3$

= max number  
of linearly  
independent  
rows of  $A$ .

$$\mathcal{C}(A^T) = \mathcal{C}(B^T)$$

$A$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

$B$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (1) \mathcal{C}(A^T) &\supseteq \mathcal{C}(B^T) \\ (2) \mathcal{C}(A^T) &\subseteq \mathcal{C}(B^T) \end{aligned}$$

$V$  is a vector space

- ①  $0 \in V$
- ②  $\forall \bar{u} \in V, \alpha \bar{u} \in V$
- ③  $\forall \bar{u}, \bar{v} \in V, \bar{u} + \bar{v} \in V$ .

Row space of  $B$

$$\mathcal{C}(B^T) = \text{Span} \{ \bar{r}_1, \bar{r}_2, \bar{r}_3 \}$$

$\bullet \{ \bar{r}_1, \bar{r}_2, \bar{r}_3 \}$  is a basis of  $\mathcal{C}(B^T)$

$$\bullet \dim \mathcal{C}(B^T) = 3$$

$$\bullet \text{row rank of } B = 3$$

# Matrices and elimination

column space of  $A$

$$\mathcal{C}(A)$$

$$= \text{Span} \{ \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4 \}$$

$A$

$$\dim \mathcal{C}(A) = 3$$

//

$$\dim \mathcal{C}(A^T) = 3$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4$

## Matrices and elimination

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B

Column space

Span  $\{ [1, 0, 0, 0],$   
 $[2, 2, 0, 0]$   
 $[0, 1, 0, 0],$   
 $[1, 0, 2, 0] \}$

## Matrices and elimination

$$N(A) = \{\bar{x} \mid A\bar{x} = 0\} = \{\bar{x} \mid B\bar{x} = 0\} = \left\{ \alpha \begin{bmatrix} 2/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$$

$$\dim N(A) = 1$$

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{array} \right] \end{array} \Rightarrow \begin{array}{c} B \\ \left[ \begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_3 \\ -\frac{1}{3}x_3 \\ x_3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ 3x_2 + x_3 &= 0 \\ 2x_4 &= 0 \end{aligned}$$

$x_4 = 0$   
 $x_2 = -\frac{x_3}{3}$

$$\begin{aligned} x_1 - \frac{2x_3}{3} &= 0 \\ x_1 &= \frac{2}{3}x_3 \end{aligned}$$

## Matrices and elimination

column rank  
||  
row rank

column rank = 3  
of B

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix on the right is annotated with blue arrows pointing to the first, second, and fourth columns, and a red 'X' above the third column. The third column is highlighted in yellow.

# Row echelon form

# Linearly independent rows

## Vector spaces related to a matrix

Consider an  $m$ -by- $n$  matrix  $A$  over  $\mathbb{R}$ .

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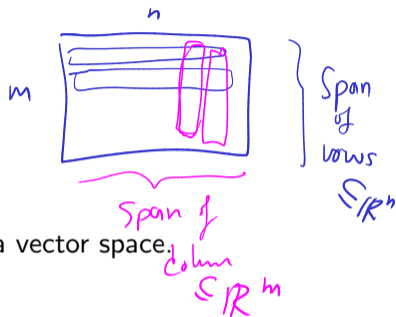
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- ▶ Column space:  $\text{Span} \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subseteq \mathbb{R}^m$
- ▶ Row space:  $\text{Span} \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\} \subseteq \mathbb{R}^n$



$$\text{Span}\{\mathbf{r}_1, \cancel{\mathbf{r}_2}, \dots, \cancel{\mathbf{r}_m}\} = \text{row space}$$

# Subspaces

## Definition

Let  $\mathcal{V}$  and  $\mathcal{W}$  be vector spaces such that  $\mathcal{V} \subseteq \mathcal{W}$ . We say that  $\mathcal{V}$  is a **subspace** of  $\mathcal{W}$ .

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## Examples:

- ▶  $\text{Span} \{[1, 1]\}$  is a subspace of  $\mathbb{R}^2$ .
- ▶  $\text{Span} \{[1, 0, 0], [0, 1, 1]\}$  is a subspace of  $\mathbb{R}^3$ .
- ▶  $\text{Span} \{[1, 0, 0], [0, 1, 1], [1, 1, 2]\}$  is a subspace of  $\mathbb{R}^3$ .

## Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

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$$\mathcal{C}(A) = \{\alpha_1[1, 0] + \alpha_2[2, 1] + \alpha_3[4, 3] \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}$$

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► Row space:

$$\mathcal{C}(A^T) = \{\alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

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Vector spaces



α Basis

⊗ dimension

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► Row space:

$$\mathcal{C}(A^T) = \{\alpha_1[1, 2, 4] + \alpha_2[0, 1, 3] \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

## Example 1 (cont.)

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left\{ \begin{array}{l} 1(x_1) + 2(x_2) + 4(x_3) = 0 \\ 1x_2 + 3x_3 = 0 \end{array} \right. \leftarrow$$

$1x_2 + 3x_3 = 0 \quad \leftarrow$

Let

$N(A)$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

homogeneous system

$$\bar{x} = [x_1, x_2, x_3]$$

Is there any other way to obtain vector spaces from  $A$ ?

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \{ \bar{x} \mid A\bar{x} = 0 \}$$

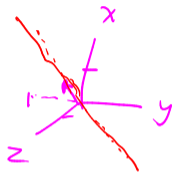
$$1(x_1) + 2(x_2) + 4(x_3) = 0 \quad \text{--- (1)}$$

$$1x_2 + 3x_3 = 0 \quad \text{--- (2)}$$

from (2)  $x_2 = -3x_3$

plug into (1)  $x_1 + 6x_3 + 4x_3 = 0$

$$x_1 = -2x_3$$



$$\underline{N(A)} = \{ \bar{x} \mid A\bar{x} = 0 \} = \left\{ \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\} = \left\{ x_3 \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \mid x_3 \in \mathbb{R} \right\}$$

## Example 1 (cont.)

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We can think of  $A$  as a coefficient matrix of a system of homogenous linear equations:

$$A\mathbf{x} = \mathbf{0}.$$

In this case, we have

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The set of solutions  $\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$  form a vector space.

Null space of  $A$

## Example 1 (cont.)

Given a matrix  $A$ , we can look at the matrix-vector product  $A\mathbf{x}$ .

Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

# Four fundamental subspaces

## Four fundamental subspaces

Given an  $m$ -by- $n$  matrix  $A$ , we have the following subspaces

- ▶ The column space of  $A$  (denoted by  $\mathcal{C}(A)$  )
- ▶ The row space of  $A$  (denoted by  $\mathcal{C}(A^T)$  )
- ▶ The nullspace of  $A$

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$$

- ▶ The left nullspace of  $A$

$$\mathcal{N}(A^T) = \{\mathbf{y} \mid A^T\mathbf{y} = \mathbf{0}\}$$

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# Four fundamental subspaces



$$\text{rank}(A) = r$$

## Four fundamental subspaces

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$$\dim \mathcal{C}(A) = r$$

► The row space of  $A$  (denoted by  $\mathcal{C}(A^T) \subseteq \mathbb{R}^n$ )

$$\dim \mathcal{C}(A^T) = r$$

→ ► The nullspace of  $A$

$$\mathcal{N}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

$$\dim \mathcal{N}(A) = n - r$$

→ ► The left nullspace of  $A$

$$\mathcal{N}(A^T) = \{\mathbf{y} \mid A^T \mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

$$\dim \mathcal{N}(A^T) = m - r$$

dim  $n-r$

$$N(A) = \{x \mid Ax=0\} \subseteq \mathbb{R}^n$$

Homogeneous  
system  
lik  
9.

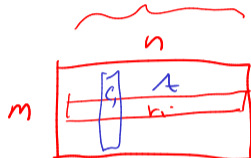
dim  $\boxed{r}$

column space  
 $\text{Span}\{c_1, c_2, \dots, c_n\} \subseteq \mathbb{R}^m$

span

dim  $\boxed{r}$

row space  
 $\text{Span}\{r_1, \dots, r_m\} \subseteq \mathbb{R}^n$



The diagram shows a matrix  $A$  with dimensions  $m$  (rows) and  $n$  (columns). A specific column is highlighted and labeled  $c_i$ . A bracket above the matrix indicates the column dimension  $n$ . A bracket to the left indicates the row dimension  $m$ . A dashed arrow points from the column space box to the row space box, indicating the relationship between the two spaces.

$$N(A^T) = \{y \mid A^T y = 0\} \subseteq \mathbb{R}^m$$

dim  $m-r$