# 01204211 Discrete Mathematics Lecture 3a: Proof techniques 1

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# Proof techniques<sup>1</sup>

Using inference rules, we can prove facts in propositional logic. However, in many cases, we want to prove wider range of mathematical facts. Inference rules play crucial parts in providing high-level structures for our proofs.

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Using inference rules, we can prove facts in propositional logic. However, in many cases, we want to prove wider range of mathematical facts. Inference rules play crucial parts in providing high-level structures for our proofs.

In this lecture, we will focus on two general proof techniques that originate from two simple inference rules.

- Direct proofs
- Proofs by contraposition

## **Terminologies**

These are terminologies used when showing mathematical facts.

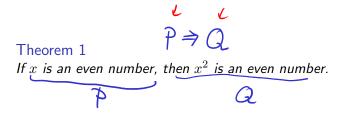
- ► A **theorem** is a statement that can be argued to be true.
- ▶ A **proof** is the sequence of statements forming that mathematical argument.
- ▶ An **axiom** is a statement that is assumed to be true. (Note that we do not prove an axiom; therefore, the validity of a theorem proved using an axiom relies of the validity of the axiom.)
- ➤ To prove a theorem, we may prove many simple lemmas to make our argument. A **lemma**, in this sense, is a smaller theorem (or a supportive one).
- ▶ A corollary is a theorem which is a "fairly" direct result of other theorems.
- ► A **conjecture** is a statement which we do not know if it is true or false.



## Direct proofs

When we want to prove a theorem of the form  $P\Rightarrow Q$ , we can assume that P is true, then use this to argue that Q has to be true as well.

Direct proofs	
Theorem: $P \Rightarrow Q$ .	
Proof. Assume $P$ (then show that $Q$ follows from $P$ )	



### Theorem 1

If x is an even number, then  $x^2$  is an even number.

Proof.

P

Assume that x is an even number.



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#### Proof.

Assume that x is an even number.

By definition, there exists an integer k such that x=2k. This implies that  $x^2=(2k)^2=4k^2$ . Since k is an integer,  $2k^2$  is also an integer. Hence we can write  $x^2=2\cdot(2k^2)$  where  $2k^2$  is an integer; this means that  $x^2$  is even.

## Theorem 2

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- ▶  $S(x) \wedge U \Rightarrow V(x)$ , where V = "there exists an integer k such that  $x^2 = 2 \cdot (2k^2)$  where  $2k^2$  is an integer."

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- ▶  $S(x) \wedge U \Rightarrow V(x)$ , where V = "there exists an integer k such that  $x^2 = 2 \cdot (2k^2)$  where  $2k^2$  is an integer."
- ▶ By definition,  $V(x) \Rightarrow Q(x)$ , where  $Q(x) = "x^2$  is even".



## Example 1: be careful

When we prove a statement with universal quantifiers like:

 $(\forall x)$  If x is an even number, then  $x^2$  is an even number

we have to be *extremely* careful not to assume anything about x except those state explicitly in the assumption.

## Practice: Back to our subgoal

Can you use direct proofs to show the following theorem?

### Theorem 3

For any positive number n and a such that  $a>\sqrt{n},$  then  $n/a\leq \sqrt{n}.$ 

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## Proof

Assume that  $a > \sqrt{n}$ .

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#### Theorem 3

For any positive number n and a such that  $a>\sqrt{n},$  then  $n/a\leq \sqrt{n}.$ 

### Proof.

Assume that  $a > \sqrt{n}$ . Since

$$n = n$$
,

by dividing the left side by a and the right side by  $\sqrt{n}$ , we get that

$$\frac{n}{a} < \frac{n}{\sqrt{n}},$$

because both a and  $\sqrt{n}$  are positive. Hence,  $n/a < \sqrt{n}$  as required.

# Practice: Divisibility by 3 (1)

Let's try to prove a well-known fact.

### Theorem 4

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- Theorem 4
An integer n is divisible by 3 if the sum of the digits of n is divisible by 3.

Let's start by proving this lemma.

### Lemma 5

For any integer 
$$k \geq 0$$
,  $10^k - 1$  is divisible by  $3$ .

Prove 
$$10^{k}-1 = 9 \cdot \cdot \cdot \cdot 9 = 3 \times 33 \cdot \cdot \cdot 3$$
 $k-1 = 7$ 

Practice: Divisibility by 3 (2) 
$$q_i(10^{k-i}) = q_i(10^{k-i}-1)$$

### Proof.

Assume that the sum of the digits of n is divisible by 3. We will show that n is divisible by 3.

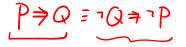
Let k be the number of digits of n. Let  $a_1, a_2, \ldots, a_k$  be the digits of n where  $a_1$  is the most significant digit and  $a_k$  is the least significant one. Therefore, we can write

$$n = a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \dots + a_{k-1} \cdot 10^1 + a_k \cdot 10^0.$$

Consider the i-th term:  $a_i \cdot 10^{k-i}$ . From Lemma 5, we know that  $10^{k-i}-1$  is divisible by 3. Thus  $a_i \cdot (10^{k-i}-1)$  is also divisible by 3. Therefore, the remainder of  $a_i \cdot 10^{k-i}$  divided by 3 is equal to the remainder of  $a_i$  divided by 3.

Summing all terms, the remainder of the division of n by 3 is  $a_1+a_2+\cdots+a_k$ . Since 3 divides this number, the remainder of n/3 is 0; thus, 3 divides n.

## Proof by contraposition



When we want to prove a theorem of the form  $P \Rightarrow Q$ , we can assume that Q is false, then use this to argue that P has to be false as well.

Proof by contraposition	
Theorem:	
$P \Rightarrow Q$ .	
Proof	
Assume $\bigcirc Q$	
(then show that $(\neg P)$ follows from $\neg Q$ )	

### Theorem 6

If  $x^2$  is an even number, then x is an even number,

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Asser x2 15x 1'

Surv y k" x x2=2.k
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### Proof.

Assume that  $x^2$  is an even number...

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Before we try to prove by contraposition, let's try to use direct proof to show this theorem.

### Proof.

Assume that  $x^2$  is an even number...

... doesn't seem to go very well.

#### Theorem 7

If  $x^2$  is an even number, then x is an even number,

# Theorem 7



If  $x^2$  is an even number, then x is an even number,

## Proof.



We will prove by contraposition. Assume that x is not an even number. 17 DU. NU K 12'

$$\chi = 2k+1$$



warm x2 1:15-57

$$\chi^2 = (2k+1)^2 = 2k^2 + 4k + 1$$
  
= 2(2k2+2k)+1

$$= 2(2k^2+2k)+1$$

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# Proving iff statements

How can we prove a statement of the form  $P \Leftrightarrow Q$ ? For example:

### Theorem 8

x is an even number iff  $x^2$  is an even number.

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#### Theorem 8

x is an even number iff  $x^2$  is an even number.

### Proof.

We will prove that the statement is true in both directions.

- $(\Rightarrow)$  This direction is true from Theorem 1.
- $(\Leftarrow)$  This direction is true from Theorem 5.

Theorem 9 For any numbers x and y, x = y.

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### Proof.

Assume that

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Multiplying both terms by 0, we get that

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and this implies

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which is clearly true.



### Theorem 9

For any numbers x and y, x = y.

### Proof.

Assume that

$$x = y$$
.

Multiplying both terms by 0, we get that

$$0 \cdot x = 0 \cdot y$$

and this implies

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What is wrong with this (non) proof?

In a way, proving theorems is like solving puzzles. There is no general rules on how to prove theorems.

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But you can get better by (1) trying to read and understand good proofs and by (2) practicing.

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- Understand why the proof needs each step

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### There are many levels of understandings:

- Understand each step of the proof and how each step follows from previous ones
- Understand why the proof needs each step
- ► Can apply techniques or proof strategies learned from this proof for proving other statements