01204211 Discrete Mathematics
Lecture 8a: Linear systems of equations

Jittat Fakcharoenphol

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A linear system of equations

Let's start with a simple example with 2 variables:

$$\begin{aligned} 5x + 10y &= 5 \\ x - 3y &= 11 \end{aligned}$$

How would you solve it?

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Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) =$$

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$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 =$$

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Using basic techniques you learned from high school, you may multiply the second equation with 5 and subtract it to the first equation; yielding:

$$5x + 10y - (5x - 5 \cdot 3y) = 25y = 5 - 5 \cdot 11 = -50$$

Then you can conclude that $y = -2$. Substitute it to one of the equation, you can find out the value of x.

A closer look: 1st perspective

Each equation (row) constraints certain values of x and y.

"Combining" two rows

Let's focus only on coefficients. This is how we obtain the third equation:

$$\begin{array}{rl} \begin{pmatrix} 5, & 10 \end{pmatrix} & = \boldsymbol{u}_1 \\ \begin{pmatrix} 1, & -3 \end{pmatrix} & = \boldsymbol{u}_2 \end{array}$$

"Combining" two rows

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$$
\begin{array}{lrrll}
(& 5, & 10 &) & = \boldsymbol{u}_1 \\
(& 1, & -3 &) & = \boldsymbol{u}_2 \\
(& 0, & 25 &) & = \boldsymbol{u}_1 - 5 \cdot \boldsymbol{u}_2
\end{array}
$$

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The third equation is a “combination” of the other two rows. In fact, it is a **linear combination** of the first two.

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Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

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The third equation is a "combination" of the other two rows. In fact, it is a **linear combination** of the first two.
Can you obtain $(0, 1)$ from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?
Yes,

$$0.2 \cdot \boldsymbol{u}_1 - \boldsymbol{u}_2 = (0, 1).$$

It turns out that you can obtain any (a, b) from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

A closer look: 2nd perspective

We rewrite the system as

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot x + \begin{bmatrix} 10 \\ -3 \end{bmatrix} \cdot y = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

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Now, the goal is to find x and y satisfying this “vector” equation.

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Now, the goal is to find x and y satisfying this "vector" equation. But if we change our focus to the vectors, we can see that we have 3 vectors:

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

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$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 10 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

and with x and y, we now see that $\boldsymbol{b}$ is a **linear combination** of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
Finding x and y is essentially checking if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

A linear system with 3 variables

Let's consider a system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array}$$

Row perspective

$$
\begin{array}{ccccccc}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
4x_1 & + & 2x_2 & + & 3x_3 & = & 10
\end{array}
$$

Each equation becomes a **plane** in 3 dimensional space.

Row perspective: the goal of Gaussian Elimination

From vectors:

$$(2,4,3), \quad (1,0,5), \quad (4,2,3)$$

We want to linearly combine them to obtain

$$(1,0,0), \quad (0,1,0), \quad (0,0,1)$$

Row perspective: the goal of Gaussian Elimination

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$$(2,4,3), \quad (1,0,5), \quad (4,2,3)$$

We want to linearly combine them to obtain

$$(1,0,0), \quad (0,1,0), \quad (0,0,1)$$

In other words, what are the possible linear combinations of

$$(2,4,3), \quad (1,0,5), \quad (4,2,3)$$

Column perspective

From

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \end{array},$$

we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 + = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

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Our goal is to find a way to linear combine 3 vectors to obtain

$$\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}.$$

In other words, the vector $\boldsymbol{b}$, for a successful Gaussian Elimination, should be in the set of all possible linear combinations of the 3 column vectors.

More example

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 3x_1 & + & 8x_2 & + & x_3 & = & 10 \end{array}$$

More example 2

Let's consider another system with 3 variables:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\ 5x_1 & + & 2x_2 & + & 8x_3 & = & 22 \end{array}$$

More failed example 3

Let's consider the last system with 3 variables:

$$\begin{array}{rcrcrcl}
2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\
x_1 & + & & & 5x_3 & = & 12 \\
2x_1 & + & & & 10x_3 & = & 24
\end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

More failed example 3 (cont.)

The last equation (constraint) can be derived from the other two.

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions.

More failed example 3 (cont.)

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.

More failed example 3 (cont.)

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$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions?

More failed example 3 (cont.)

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This system has many solutions. Suppose that $\boldsymbol{u} = [u_1, u_2, u_3]$ and $\boldsymbol{v} = [v_1, v_2, v_3]$ are both solutions but $\boldsymbol{u} \neq \boldsymbol{v}$.
What does it mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions? It means that, for $\boldsymbol{u}$, you can plug in $x_1 = u_1, x_2 = u_2, x_3 = u_3$ and that satisfies the system of equations.

More failed example 3 (cont. 1)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 \\ x_1 & + & & & 5x_3 & = & 12 \end{array}$$

I.e.,

$$\begin{array}{ccccccc} 2u_1 & + & 4u_2 & + & 3u_3 & = & 7 \\ u_1 & + & & & 5u_3 & = & 12 \end{array} \qquad \begin{array}{ccccccc} 2v_1 & + & 4v_2 & + & 3v_3 & = & 7 \\ v_1 & + & & & 5v_3 & = & 12 \end{array}$$

Consider $\boldsymbol{u} - \boldsymbol{v}$.

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =$$

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \end{array}$$

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Consider $\boldsymbol{u} - \boldsymbol{v}$. We see that

$$\begin{array}{l} (2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) = \\ \quad 2(u_1 - v_1) \quad + \quad 4(u_2 - v_2) \quad + \quad 3(u_3 - v_3) \quad = \quad (7 - 7) = 0 \\ (u_1 + 5u_3) - (v_1 + 5v_3) = \end{array}$$

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$$\begin{array}{ccccccc} \multicolumn{7}{l}{(2u_1 + 4u_2 + 3u_3) - (2v_1 + 4v_2 + 3v_3) =} \\ 2(u_1 - v_1) & + & 4(u_2 - v_2) & + & 3(u_3 - v_3) & = & (7 - 7) = 0 \\ \multicolumn{7}{l}{(u_1 + 5u_3) - (v_1 + 5v_3) =} \\ (u_1 - v_1) & + & & & 5(u_1 - v_3) & = & (12 - 12) = 0 \end{array}$$

More failed example 3 (cont. 2)

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are different solutions to the system:

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It turns out that $\boldsymbol{u} - \boldsymbol{v}$ is a solution to the following system:

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It is the same system with all right-hand-side constants equal to zero. This type of linear systems is called a **homogeneous system of linear equations**.

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It would play a central role when dealing with linear systems with many solutions.

Key take away

- There are 2 ways to look at how we solve linear systems: row perspective and column perspective.
- **Linear combination** is the main operation.