01204211 Discrete Mathematics
Lecture 9a: Spans and Vector Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$\boldsymbol{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

Exercise

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Write down the nutritions for a mixed drink that consists of 50ml of $\boldsymbol{v}$, 200ml of $\boldsymbol{c}$ and 10ml of $\boldsymbol{w}$.

Exercise

$$0{,}5\begin{bmatrix}100\\50\\0\\0\end{bmatrix} + 2\cdot\begin{bmatrix}0\\0\\300\\0\end{bmatrix} + 0{,}1\begin{bmatrix}50\\0\\50\\10\end{bmatrix}$$

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

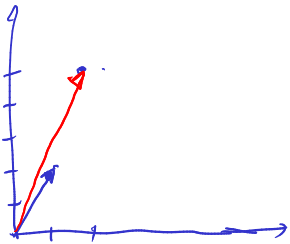
$$\boldsymbol{v} = \begin{bmatrix}100\\50\\0\\0\end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix}0\\0\\300\\0\end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix}50\\0\\50\\10\end{bmatrix}$$

1. Write down the nutritions for a mixed drink that consists of 50ml of $\boldsymbol{v}$, 200ml of $\boldsymbol{c}$ and 10ml of $\boldsymbol{w}$. (as a linear combination of v, c, & w)
2. Write that result as a matrix-vector product. (The matrix should be a 4×3 matrix.)

$$\begin{bmatrix}100 & 0 & 50\\50 & 0 & 0\\0 & 300 & 50\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix} = \begin{bmatrix}100 & 0 & 50\\50 & 0 & 0\\0 & 300 & 50\\0 & 0 & 10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix} = \begin{bmatrix}100\\50\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\300\\0\end{bmatrix}\begin{bmatrix}50\\0\\50\\10\end{bmatrix}\begin{bmatrix}0{,}5\\2\\0{,}1\end{bmatrix}$$

Example 1

Is $\text{Span}\ \{[1,2],[2,5]\} = \mathbb{R}^2$?



Example 2

Is $\text{Span}\ \{[1,0,1],[1,1,0],[2,3,4]\} = \mathbb{R}^3$?

Example 3

Is $\text{Span}\ \{[1,0,1],[1,1,0],[4,2,2]\} = \mathbb{R}^3$?

Elements in a vector

- We see examples of vectors over $\mathbb{R}$.
- However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- What do we want from an element in a vector?
  - We should be able to perform addition, subtraction, multiplication, and division.
  - Operations should be commutative and associative.
  - Additive and multiplicative identity should exist.
  - Addition and multiplication should have inverses.
- We refer to a set with these properties as a **field**.

$(ab)c = a(bc)$

A field

Definition

A set $\mathbb{F}$ with two operations $+$ and $\times$ (or $\cdot$) is a **field** iff these operations satisfy the following properties:

- (Associativity): $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (Commutativity): $a+b = b+a$ and $a \cdot b = b \cdot a$
- (Identities): There exist two elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that $a+0 = a$ and $a \cdot 1 = a$
- (Additive inverse): For every element $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$ such that $a+(-a) = 0$
- (Multiplicative inverse): For every element $a \in \mathbb{F} \setminus \{0\}$, there is an alement a^{-1} such that $a \cdot a^{-1} = 1$
- (Distributive): $a \cdot (b+c) = a \cdot b + a \cdot c$

Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$. I.e., it is a "bit" field.
What are $+$ and $\cdot$ in $GF(2)$?

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- We define $b_1 + b_2$ to be XOR.

$$0 + 0 = 0$$
$$0 + 1 = 1 + 0 = 1$$
$$1 + 1 = 0$$

Another useful field: $GF(2)$

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What are $+$ and $\cdot$ in $GF(2)$?

- We define $b_1 + b_2$ to be XOR.

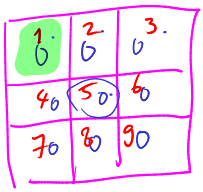
$$0 + 0 = 0$$
$$0 + 1 = 1 + 0 = 1$$
$$1 + 1 = 0$$

- We define $b_1 \cdot b_2$ to be standard multiplication.

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$

You can check that $GF(2)$ satisfies the axioms of fields.

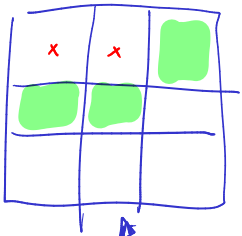
3 × 3 Lights out



$b_1 =$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$b_2 =$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$b_5 =$ $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

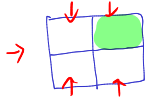


$b_1 + b_2 =$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $+$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $=$ $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$b_1, b_2 \ldots b_9$

2×2 Lights out

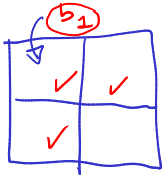
4-vector



$[0,1,0,0] + b_1$

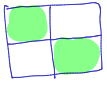
$=$

b_1

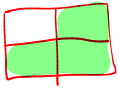


b_1

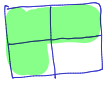
$= [1, 1, 1, 0]$



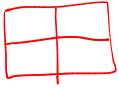
$[1, 0, 0, 1]$



b_1	b_2
b_3	b_4



$[1, 1, 1, 0]$

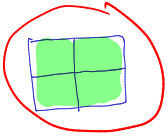


$b_1 = [1, 1, 1, 0]$

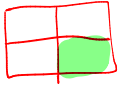
$b_2 = [1, 1, 0, 1]$

$b_3 = [1, 0, 1, 1]$

$b_4 = [0, 1, 1, 1]$



$[1, 1, 1, 1]$



2×2 Lights out

$b_1 = [1, 1, 1, 0]$

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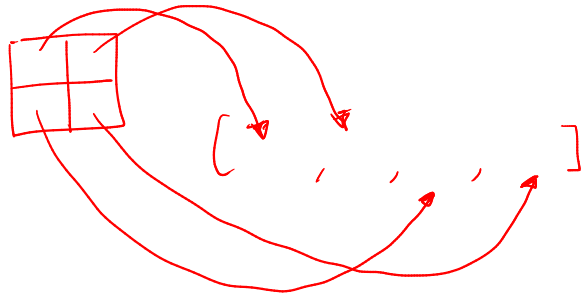
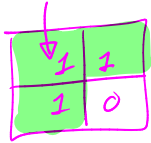
$b_3 = [1, 0, 1, 1]$

$b_4 = [0, 1, 1, 1]$





$[1, 1, 1, 0]$

Press b_1 for x_1 time

b_2 for x_2 time

b_3 for x_3 time

b_4 for x_4 time.

$[0,0,0,0]$

$[1,1,1,1] = x_1 b_1 + x_2 b_2 + x_3 b_3 + x_4 b_4$

2×2 Lights out

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + x_4 &= 1 \\ x_1 + x_3 + x_4 &= 1 \\ x_2 + x_3 + x_4 &= 1 \end{aligned}$$

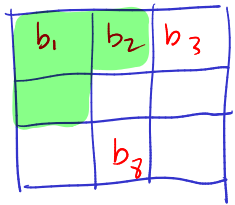
$b_1 = [1, 1, 1, 0]$
$b_2 = [1, 1, 0, 1]$
$b_3 = [1, 0, 1, 1]$
$b_4 = [0, 1, 1, 1]$

2×2 Lights out

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

3×3 Lights out

9-vector



$$b_1 = \begin{bmatrix} 1,1,0 \\ 1,0,0 \\ 0,0,0 \end{bmatrix}$$

$$b_3 = \begin{bmatrix} 0,0,0 \\ 0,1,0 \\ 1,1,1 \end{bmatrix}$$

Parity-check code

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From message $\boldsymbol{a} = [a_1, a_2, a_3, a_4]$, we compute (in $GF(2)$) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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Now our encoded message becomes

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where $a_5 = b = a_1 + a_2 + a_3 + a_4$. It can detects a single-bit error.

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What can we say about the condition on a_5?
It is in fact a homogeneous linear equation (in $GF(2)$):

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0$$

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It is in fact a homogeneous linear equation (in $GF(2)$):

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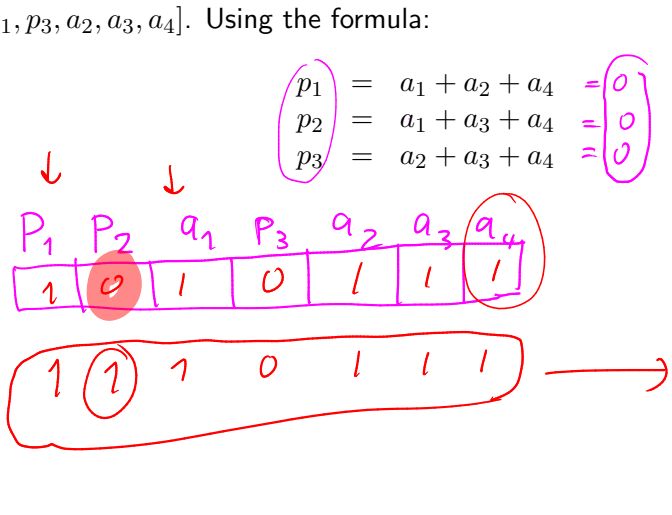
Now, what is the set of all possible codewords?

Hamming code

You can detect and correct more errors with Hamming codes. In this version called a $[7, 4]$ Hamming code, you encode 4-bit data $[a_1, a_2, a_3, a_4]$ into a 7-bit codeword $[p_1, p_2, a_1, p_3, a_2, a_3, a_4]$. Using the formula:

$$\begin{aligned} p_1 &= a_1 + a_2 + a_4 \\ p_2 &= a_1 + a_3 + a_4 \\ p_3 &= a_2 + a_3 + a_4 \end{aligned}$$



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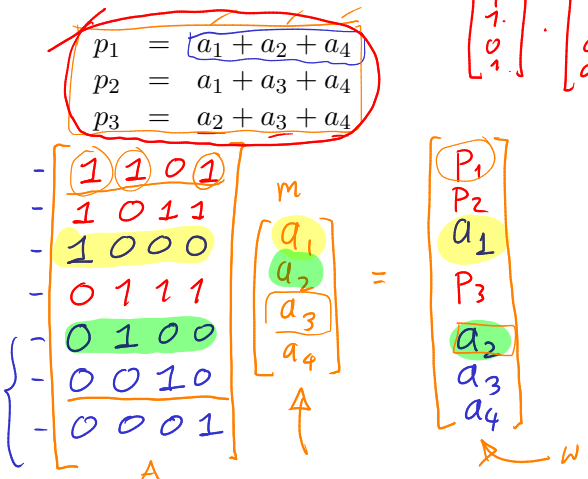
$$
\begin{aligned}
p_1 &= a_1 + a_2 + a_4 \\
p_2 &= a_1 + a_3 + a_4 \\
p_3 &= a_2 + a_3 + a_4
\end{aligned}
$$

Let's see how this works.



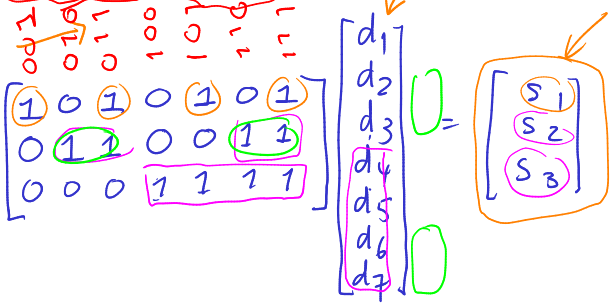
Hamming code (encoding as matrix multiplication)

Parity check

Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$
\begin{aligned}
s_1 &= d_1 + d_3 + d_5 + d_7 \\
s_2 &= d_2 + d_3 + d_6 + d_7 \\
s_3 &= d_4 + d_5 + d_6 + d_7
\end{aligned}
$$

Given a codewords $w = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero s.



S_3	S_2	S_1
1	1	1

Parity check

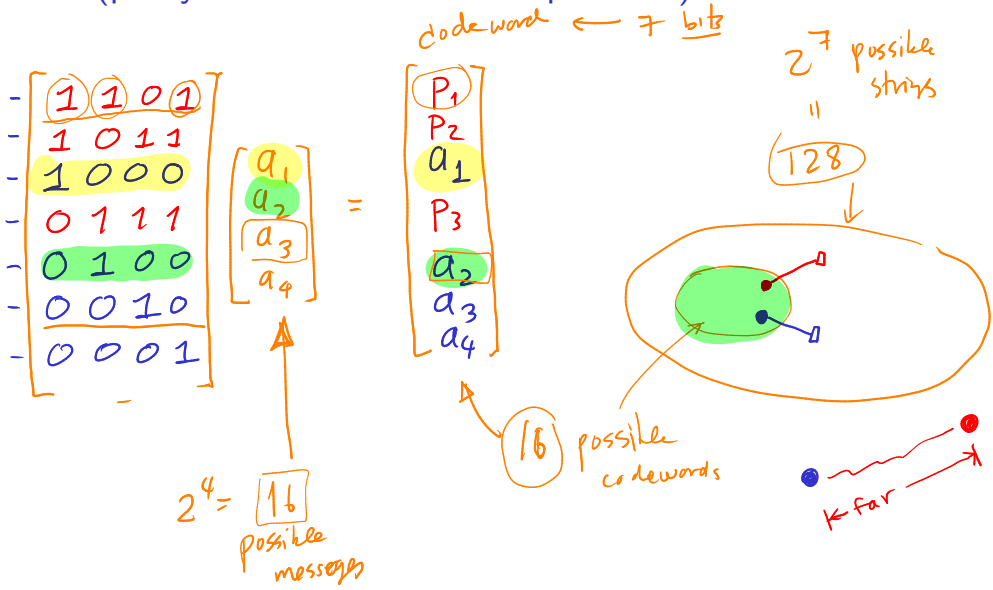
Suppose that we are given $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$ Let

$$
\begin{aligned}
0 = \quad s_1 &= d_1 + d_3 + d_5 + d_7 \\
0 = \quad s_2 &= d_2 + d_3 + d_6 + d_7 \\
0 = \quad s_3 &= d_4 + d_5 + d_6 + d_7
\end{aligned}
$$

Given a codewords $\boldsymbol{w} = [c_1, c_2, \ldots, c_7]$, if we compute s_1, s_2, s_3, we would get all zero's.

What if there is an error? Let's try.

Hamming code (parity check as matrix multiplication)



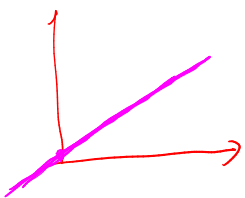
Codewords from Hamming code

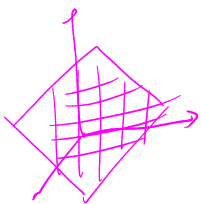
$[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$

Turning the formula for p_1, p_2, p_3 around, we have 3 homogeneous linear equations:

$$\begin{aligned} d_1 + d_3 + d_5 + d_7 &= 0 \\ d_2 + d_3 + d_6 + d_7 &= 0 \\ d_4 + d_5 + d_6 + d_7 &= 0 \end{aligned}$$

and ~~again the set of all possible codewords $\mathcal{W}$ forms a vector space over $GF(2)$.~~





Can you solve 2×2 Lights out?

(Skipped)

Let $\boldsymbol{u}_1 = [1, 1, 1, 0]$, $\boldsymbol{u}_2 = [1, 1, 0, 1]$, $\boldsymbol{u}_3 = [1, 0, 1, 1]$, and $\boldsymbol{u}_4 = [0, 1, 1, 1]$.

Given $\boldsymbol{b} = [b_1, b_2, b_3, b_4]$, can you always find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Can you solve 2×2 Lights out?

Let $\boldsymbol{u}_1 = [1, 1, 1, 0]$, $\boldsymbol{u}_2 = [1, 1, 0, 1]$, $\boldsymbol{u}_3 = [1, 0, 1, 1]$, and $\boldsymbol{u}_4 = [0, 1, 1, 1]$.

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$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Same question: Is $\text{Span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = GF(2)^4$?

Can you solve 2×2 Lights out?

Let's try with an example. Let $\boldsymbol{b} = [1, 0, 0, 0]$. Can you find $a_1, a_2, a_3, a_4 \in GF(2)$ such that

$$a_1 \cdot \boldsymbol{u}_1 + a_2 \cdot \boldsymbol{u}_2 + a_3 \cdot \boldsymbol{u}_3 + a_4 \cdot \boldsymbol{u}_4 = \boldsymbol{b}?$$

Can you solve 2×2 Lights out?

Since

$$[1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$$

and

Can you solve 2×2 Lights out?

Since

$$[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$$

and

$$\text{Span } \{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\} = GF(2)^4,$$

Can you solve 2×2 Lights out?

Since

$$[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1] \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\},$$

and

$$\text{Span } \{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\} = GF(2)^4,$$

what can we say about $\text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$?

Generators

Definition

Let $\mathcal{V}$ be a set of vectors. Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$.
If $\operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\} = \mathcal{V}$, we say that

- $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a **generating set** for $\mathcal{V}$
- vectors $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_n$ are **generators** for $\mathcal{V}$

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Examples

Standard generators

Note that $\{[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ are generators for $GF(2)^4$. Why?

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They are called **standard generators** for $GF(2)^4$, written as $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$.

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For $\mathbb{R}^n$, we also have $[1,0,0,\ldots,0],[0,1,0,\ldots,0],[0,0,1,\ldots,0],\ldots,[0,0,0,\ldots,1]$ as standard generators.

Generators and spans

Lemma 1

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ *are generators for* $\mathcal{V}$, *and for each* i,

$$\boldsymbol{v}_i \in \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\},$$

we have that $\mathcal{V} \subseteq \operatorname{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$.

Adding a vector into a span

Lemma 2

Consider vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

$$\text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

Geometry of spans: in $\mathbb{R}^2$

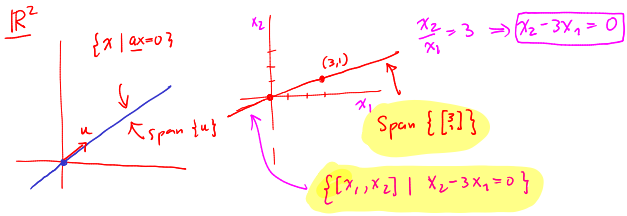
Geometry of spans: in $\mathbb{R}^3$

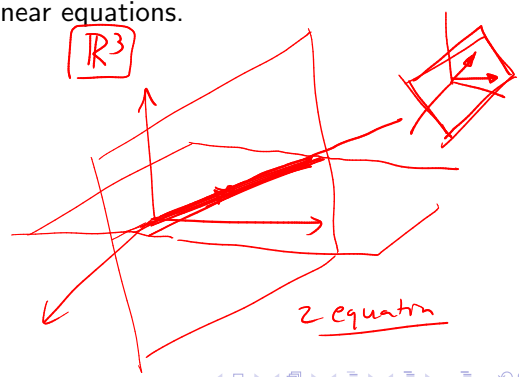
Two representations

Vector space

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.





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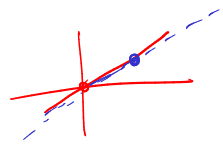


There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- as a span of vectors
- as solutions of a system of homogeneous linear equations.

What are common properties of these geometric objects?

- they pass through the origin,
- if vector $\boldsymbol{u}$ is in the objects, $\alpha\boldsymbol{u}$ for any scalar α is also in the objects, and
- if $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the objects, $\boldsymbol{u}+\boldsymbol{v}$ is also in the objects.

closed under scalar multiplication

closed under addition

Vector space

Vector spaces

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,

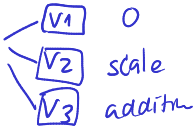
$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$

for any $\alpha \in \mathbb{F}$, and

- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,

$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

Span of vectors is a vector space



Consider n-vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$,

$$\text{Span } \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$$

is a vector space.

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is a vector space.

Let's check if properties V1, V2, and V3 are satisfied.

(V1) $0 \in \text{Span}\{u_1, u_2, \ldots, u_m\}$ ✓

(V2) If $u \in \text{Span}\{u_1, \ldots, u_m\}$, any α

Proof: there exist $\alpha_1, \alpha_2 \ldots, \alpha_m$ s.t.

$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$

$\beta u = \beta(\quad) = \beta\alpha_1 u_1 + \beta\alpha_2 u_2 + \cdots \in \text{Span}\{\ldots\}$

(V3) if $x \in$ Span ... $y \in$ ...

Solutions to homogeneous linear equations is a vector space

Consider a set $\mathcal{S}$ of all n-vectors in the form $[x_1, x_2, \ldots, x_n]$ where

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= 0 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n &= 0 \\ \vdots &= \vdots \\ a_{m1} x \cdot_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n &= 0 \end{cases}$$

m equation

homogeneous

Let's check if properties V1, V2, and V3 are satisfied.

V1

Dot product

Definition

For n-vectors $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]$ and $\boldsymbol{v} = [v_1, v_2, \ldots, v_n]$, the **dot product** of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \cdots + u_n \cdot v_n$$

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Using dot products, the previous set $\mathcal{S}$ can be written as

$$\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} = 0, \boldsymbol{a}_2 \cdot \boldsymbol{x} = 0, \ldots, \boldsymbol{a}_m \cdot \boldsymbol{x} = 0\}$$

and we know that $\mathcal{S}$ is a vector space.

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

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$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

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$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

- *Question:* Is $\mathcal{A}$ a vector space?
- We also write it as $\boldsymbol{a} + \mathcal{V}$.

Affine spaces

Definition

If $\boldsymbol{a}$ is a vector and $\mathcal{V}$ is a vector space, then

$$\boldsymbol{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

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Definition

The set of all affine combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **affine hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

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Writing an affine space using a span

Writing an affine space using a span

An affine space

An affine space passing through $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is

$$\boldsymbol{u}_1 + \text{Span } \{\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{u}_3 - \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n - \boldsymbol{u}_1\}.$$

Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m \end{aligned} \qquad \begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

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What can you say about the solution sets of these two related linear systems?
$\mathbf{0}$ is always a solution to the linear system on the right.
Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

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This implies that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution to the homogeneous linear system.

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

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In other words,

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i.e., $\mathcal{W}$ is an affine space.

Solutions to a non-homogeneous linear system

Lemma 3

If the solution set of a linear system is not empty, it is an affine space.