01204211 Discrete Mathematics Lecture 11b: Four fundamental subspaces (II)

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What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{2}{5} & \frac{3}{6} \\ \frac{7}{7} & \frac{8}{9} & \frac{9}{10} \end{bmatrix}$$

Four fundamental subspaces

Four fundamental subspaces

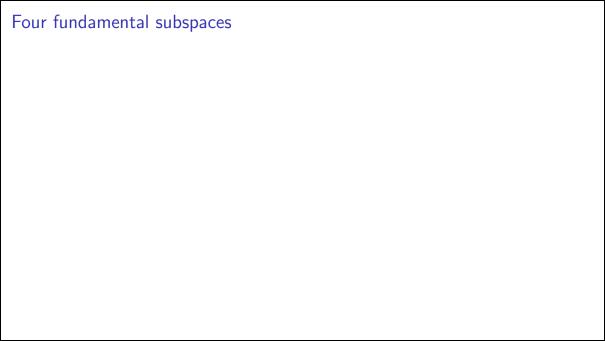
Given an m-by-n matrix A, we have the following subspaces

- ightharpoonup The column space of A (denoted by $\mathcal{R}(A)\subseteq\mathbb{R}^m$)
- ▶ The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$)
- ightharpoonup The nullspace of A

$$\mathcal{N}(A) = \{ \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{0} \} \subseteq \mathbb{R}^n$$

ightharpoonup The left nullspace of A

$$\mathcal{N}(A^T) = \{ \boldsymbol{y} \mid A^T \boldsymbol{y} = \boldsymbol{0} \} \subseteq \mathbb{R}^m$$



Ranks

Definition

Consider an m-by-n matrix A.

- ightharpoonup The row rank of A is the maximum number of linearly independent rows of A.
- ightharpoonup The **column rank** of A is the maximum number of linearly independent columns of A.

Remark: The column rank of A is $\dim \mathcal{R}(A)$. The row rank of A is $\dim \mathcal{R}(A^T)$.

Theorem 1

For any matrix A, its row rank equals its column rank.

Proof.

Let r be the column rank. We will show that there are r n-vectors that span its row space. This implies that the row rank is at most r. We can use the same argument again on A^T to obtain that the column rank is at most the row rank; thus, they must be equal.

Proof (cont.)

Rank and nullity

Given an m-by-n matrix A, the rank of A is $\dim \mathcal{R}(A)$. Let r be the rank of A. What is $\dim \mathcal{N}(A)$?

Dimensions

Four fundamental subspaces

Given an m-by-n matrix A of rank r, we have the following subspaces

- ▶ The column space of A (denoted by $\mathcal{R}(A) \subseteq \mathbb{R}^m$) dim $\mathcal{R}(A) = r$.
- The row space of A (denoted by $\mathcal{R}(A^T) \subseteq \mathbb{R}^n$) dim $\mathcal{R}(A^T) = r$.
- ▶ The nullspace of A (denoted by $\mathcal{N}(A) \subseteq \mathbb{R}^n$) dim $\mathcal{N}(A) = n r$.
- ▶ The left nullspace of A (denoted by $\mathcal{N}(A^T) \subseteq \mathbb{R}^m$) dim $\mathcal{N}(A) = m r$.

Application: Singular Value Decomposition (SVD)

Any n-by-d matrix A can be factored into the form of UDV^T , i.e.,

$$\left[\begin{array}{c}A\end{array}\right]=\left[\begin{array}{c}U\end{array}\right]\left[\begin{array}{c}D\end{array}\right]\left[\begin{array}{c}V^T\end{array}\right]$$

where

- ightharpoonup U is an n-by-r matrix,
- ightharpoonup D is a diagonal r-by-r matrix, and
- ightharpoonup V is an d-by-r matrix (i.e., V^T is an r-by-d matrix)
- ightharpoonup (Also, columns of U and D are "orthonormal.")

See demo.

$$\left[egin{array}{c} A \end{array}
ight] = \left[egin{array}{ccc} oldsymbol{u}_i & U \end{array}
ight] \left[egin{array}{ccc} d_{ii} & & \ & D \end{array}
ight] \left[egin{array}{c} oldsymbol{v}_i^T \ V^T \end{array}
ight]$$

 $A = d_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + d_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \dots + d_r \boldsymbol{u}_r \boldsymbol{v}_r^T.$