01204211 Discrete Mathematics
Lecture 9b: Polynomials (1)[1]

Jittat Fakcharoenphol

October 17, 2023

[1]This section is from Berkeley CS70 lecture notes.

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

For any integer $a \neq 1$ and $n \geq 1$, $a - 1 | a^n - 1$.

Polynomials

A **single-variable polynomial** is a function $p(x)$ of the form

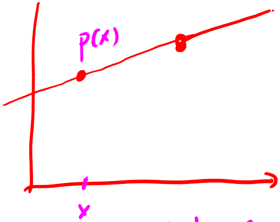
$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

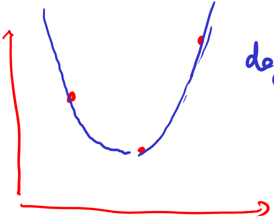
We call a_i's *coefficients*. Usually, variable x and coefficients a_i's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Examples

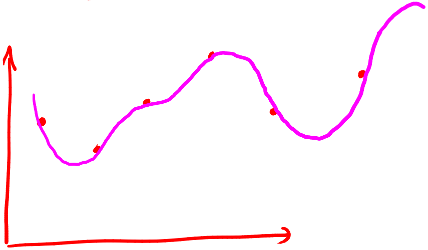
- $x^3 - 3x + 1$
- $x + 10$
- 10
- 0

Folklore





degree 2



Points $\xrightarrow{\text{"interpolation"}}$ polynomial

d+1 → degree d (uniquely)

$x \to p(x)$ evaluate p at x

$\cancel{\mathbb{R}}$

$\boxed{GF(p)}$

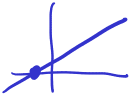
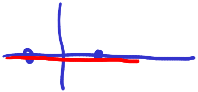
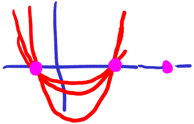
Applications

- Secret sharing (today)

Applications

- Secret sharing
- Error-correcting codes (พหุพัส)

Basic facts

Definition

a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

Properties

Property 1: A non-zero polynomial of degree d has at most d roots.
Property 2: Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.

Lemma 1

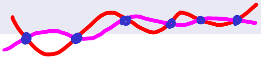
If two polynomials $f(x)$ and $g(x)$ of degree at most d that share $\boxed{d+1}$ points $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$, where all x_i's are distinct, i.e., $f(x_i) = g(x_i) = y_i$, then $f(x) = g(x)$.

Proof.

Suppose that $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $g(x) = b_d x^d + b_{d-1}x^{d-1} + \cdots + b_0$.
Let $h(x) = f(x) - g(x)$, i.e., let $h(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_0$, where $c_i = a_i - b_i$. Note that $h(x)$ is also a polynomial of degree (at most) d.
We claim that $h(x)$ has $d+1$ roots. Note that since $f(x_i) = g(x_i) = y_i$, we have that

$$h(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0,$$

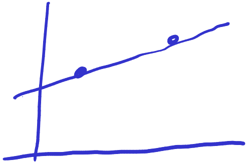
i.e., every x_i is a root of $h(x)$.
From **Property 1**, if $h(x)$ is non-zero it has at most d roots; therefore, $h(x)$ must be zero, i.e., $f(x) - g(x) = 0$ or $f(x) = g(x)$ as required. □



Polynomial interpolation - ideas

ให้ จุด $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$

หา polynomial degree $\leq d$ ที่ ผ่านจุดเหล่านี้ $\quad x_i \neq x_j$



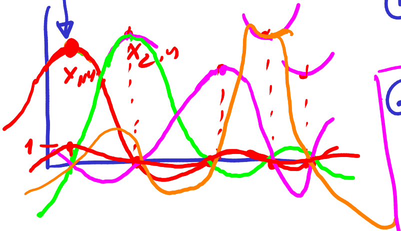
$P(x)$ degree $\leq d$ $\quad (x_1,y_1), \ldots (x_{d+1}, y_{d+1})$

ไม่มี

หา $F(x)$ degree $\leq d$

$F(x) = P(x)$

สนใจ (x_1, y_1) หา $\Delta_1(x)$ ที่



① $\Delta_1(x_1) = 1$

② $\Delta_1(x_2) = 0$

$\Delta_1(x_3) = 0$

$\vdots$

$\Delta_1(x_{d+1}) = 0$

$$\Delta_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)\cdots(x - x_{d+1})}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\cdots(x_1 - x_{d+1})}$$

$\Delta_1(x_1) = 1$, $\quad \Delta_1(x_i) = 0$ เมื่อ $x_i \neq x_1$

Lagrange polynomial

x_i หาย

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{d+1})}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d

$\Delta_i(x_i) = 1$

$\Delta_i(x_j) = 0$ ถ้า $x_j \neq x_i$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) =$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{d+1})}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) =$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{d+1})}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) = 1$.

$$P(x) = y_1 \Delta_1(x) + y_2 \cdot \Delta_2(x) + y_3 \Delta_3(x) + \cdots + y_{d+1} \Delta_{d+1}(x)$$

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_{d+1})}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) = 1$.

We can use $\Delta_i(x)$ to construct a degree-d polynomial

$$p(x) = y_1 \cdot \Delta_1(x) + y_2 \cdot \Delta_2(x) + \cdots y_{d+1} \cdot \Delta_{d+1}(x).$$

What can you say about $p(x_i)$?

มี 1 อันเท่านั้น
- มีอย่างน้อย 1 อัน
- ถ้ามี 2 อัน, จะเท่ากัน

Property 2 *

Given $d + 1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d + 1$.

Proof of Property 2.

→ Using Lagrange interpolation, we know that there exists a polynomial $p(x)$ of degree d such that $p(x_i) = y_i$ for all $1 \leq i \leq d + 1$.

→ For uniqueness, assume that there exists another polynomial $g(x)$ of degree d also satifying the condition. Since $p(x)$ and $g(x)$ agrees on more than d points, $p(x)$ and $g(x)$ must be equal from Lemma 1. □

Polynomials over a finite field $GF(p)$

$x \in GF(p)$

$a_0, \dots, a_d \in GF(p)$

$GF(p)$: $\{0, 1, \dots, p-1\}$

ทุก operation $+, -, \times$ mod p

$\div$ ต้องพึ่ง inverse.

ถ้า a มี multiplicative inverse mod m

$\Updownarrow$

$\gcd(a, m) = 1$

ถ้า p เป็น จน.เฉพาะ:

$x \in \{1, \dots, p-1\}$

มี $x^{-1} \pmod p$

mod 19

1	1	5	4	9	17	13	3	17	9
2	10	6	16	10	2	14	15	18	18
3	13	7	11	11	7	15	14		
4	5	8	12	12	8	16	6		

Examples - evaluation

Suppose that we work over $GF(m)$ where $m = 11$. Let $p(x) = 4 \cdot x^2 + 5 \cdot x + 3$. We have

x	$p(x)$	$p(x) \bmod m$
0	3	3
1	12	1
2	29	7
3	54	10
4	87	10
5	128	7
6	177	1
7	234	3
8	299	2
9	372	9
10	453	2
11	542	3

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2, 7)$, $(4, 10)$, and $(7, 3)$. Find $p(x)$.

$$\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} =$$

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2, 7), (4, 10)$, and $(7, 3)$. Find $p(x)$.
Let

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7),(4,10)$, and $(7,3)$. Find $p(x)$.

Let

- $\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2-11x+28}{(-2)\cdot(-5)} = \frac{x^2+6}{10} = 10x^2+5$

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7), (4,10)$, and $(7,3)$. Find $p(x)$.

Let

- $\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2-11x+28}{(-2)\cdot(-5)} = \frac{x^2+6}{10} = 10x^2 + 5$
- $\Delta_2(x) = \frac{(x-2)(x-7)}{(4-2)(4-7)} = \frac{x^2-9x+14}{2\cdot(-3)} = \frac{x^2+2x+3}{5} = 9x^2 + 7x + 5$

Examples - interpolation

Let $m = 11$. Suppose that $p(x)$ is a polynomial over $GF(m)$ of degree 2 passing through $(2,7),(4,10)$, and $(7,3)$. Find $p(x)$.
Let

- $\Delta_1(x) = \frac{(x-4)(x-7)}{(2-4)(2-7)} = \frac{x^2-11x+28}{(-2)\cdot(-5)} = \frac{x^2+6}{10} = 10x^2+5$
- $\Delta_2(x) = \frac{(x-2)(x-7)}{(4-2)(4-7)} = \frac{x^2-9x+14}{2\cdot(-3)} = \frac{x^2+2x+3}{5} = 9x^2+7x+5$
- $\Delta_2(x) = \frac{(x-2)(x-4)}{(7-2)(7-4)} = \frac{x^2-6x+8}{5\cdot 3} = \frac{x^2+5x+8}{4} = 3x^2+4x+2$

Thus,

$$\begin{aligned} p(x) &= 7\Delta_1(x) + 10\Delta_2(x) + 3\Delta_3(x) \\ &= (70x^2+35) + (90x^2+70x+50) + (9x^2+12x+6) \\ &= 4x^2+5x+3 \end{aligned}$$

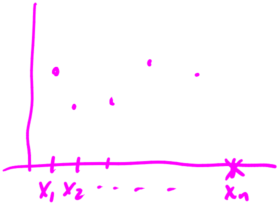
Secret sharing scheme - settings

Secret sharing scheme - settings

- polynomial P $\boxed{\text{degree } k-1}$

GF(m) ที่ • m เป็น จำนวนเฉพาะ
และ $m-1 \geq n$

• $m > s$



- There are n people, a secret s, and an integer k.
- We want to "distribute" the secret in such a way that any set of $k-1$ people cannot know anything about s, but any set of k people can reconstruct s.

$s = \boxed{12}$

$a_1 = 5$

$a_2 = 7$

$\boxed{7x^2 + 5x + 12}$

$P(0) = s$

$$P(x) = a_{k-1}x^{k-1} + \cdots + a_2x^2 + a_1x + s$$

สุ่ม $a_1, a_2, \ldots, a_{k-1}$

คนที่ i ได้ $(i, P(i))$

Secret sharing scheme

Secret sharing scheme

k all recover secret s

- Pick $\boxed{m}$ to be larger than n and s. (Much larger than s, i.e., $m >>> s$.)
- Pick a random polynomial of degree $\boxed{k-1}$ such that $\boxed{P(0) = s.}$
- Give $P(i)$ to person i, for $1 \leq i \leq n$.
- Correctness: for any set of k people,

Secret sharing scheme

- Pick m to be larger than n and s. (Much larger than s, i.e., $m >>> s$.)
- Pick a random polynomial of degree $k-1$ such that $P(0) = s$.
- Give $P(i)$ to person i, for $1 \leq i \leq n$.
- Correctness: for any set of k people,
- Correctness: for any set of $k-1$ people, how many possible candidate secrets compatible with the information these people have?