01204211 Discrete Mathematics
Lecture 9b: Affine Spaces

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Review: Linear combinations

Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Review: Span

Definition

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

Review: Vector spaces

Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,
$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$
  for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,
$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

Examples of vector spaces:

- A span of vectors is a vector space.
- A solution set to homogeneous linear equations is a vector space.

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$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$

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- We obtain a vector space $\mathcal{V}$.
- Then we translate it back so that it passes through $\boldsymbol{a}$.
- We get the set
$$\mathcal{A} = \{\boldsymbol{a} + \boldsymbol{u} : \boldsymbol{u} \in \mathcal{V}\}$$
- *Question:* Is $\mathcal{A}$ a vector space?
- We also write it as $\boldsymbol{a} + \mathcal{V}$.

Affine spaces

Definition

If $\boldsymbol{a}$ is a vector and $\mathcal{V}$ is a vector space, then

$$\boldsymbol{a} + \mathcal{V}$$

is an **affine space**.

An affine space and convex combination: 2 dimensions

An affine space and convex combination: 3 dimensions

Affine combination

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is an **affine combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

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Definition

The set of all affine combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **affine hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Convex combination: review

Definition

For any scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ such that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$$

and vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, we say that a linear combination

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **convex combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

Definition

The set of all convex combinations of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **convex hull** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$.

Writing an affine space using a span

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An affine space

An affine space passing through $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ is

$$\boldsymbol{u}_1 + \operatorname{Span}\{\boldsymbol{u}_2 - \boldsymbol{u}_1, \boldsymbol{u}_3 - \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n - \boldsymbol{u}_1\}.$$

Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= b_1 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= b_2 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= b_m \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{a_1} \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{a_2} \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{a_m} \cdot \boldsymbol{x} &= 0 \end{aligned}$$

What can you say about the solution sets of these two related linear systems?

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What can you say about the solution sets of these two related linear systems?
$\mathbf{0}$ is always a solution to the linear system on the right.
Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i\boldsymbol{u}_1 - \boldsymbol{a}_i\boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

Solutions of the two systems

Recall that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are both solutions to the non-homogeneous linear system, we have that for any i

$$\boldsymbol{a}_i \boldsymbol{u}_1 - \boldsymbol{a}_i \boldsymbol{u}_2 = b_i - b_i = 0 = \boldsymbol{a}_i(\boldsymbol{u}_1 - \boldsymbol{u}_2).$$

This implies that $\boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution to the homogeneous linear system.

Suppose that $\mathcal{W}$ is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let $\boldsymbol{u} \in \mathcal{W}$ be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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In other words,

$$\begin{aligned} \mathcal{W} &= \boldsymbol{u} + \{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\} \\ &= \boldsymbol{u} + \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = 0, \text{ for } 1 \le i \le m\}, \end{aligned}$$

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i.e., $\mathcal{W}$ is an affine space.

Solutions to a non-homogeneous linear system

Lemma 1

If the solution set of a linear system is not empty, it is an affine space.