01204211 Discrete Mathematics Lecture 13b: Eigenvalues and Eigenvectors

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Review: Hamming codes (1)

The code is defined by the generator matrix

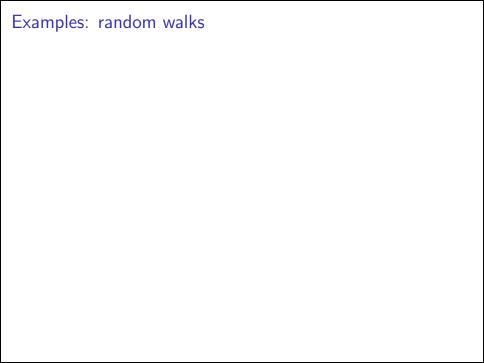
$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Consider the encoding function $e: GF(2)^4 \to GF(2)^7$. Let e(x) = Gx. What is Ker e? What is $\dim \operatorname{Im} e$?

Review: Hamming codes (2)

The code is defined by the generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

What can you say about the minimum "distance"?



Examples: differential equations (1)

Let's start with a simple system with one variable.

$$\frac{du}{dt} = au,$$

with u = u(0) when t = 0.

Examples: differential equations (2)

Now consider a system with two variables v and w:

$$\begin{array}{rcl} \frac{dv}{dt} & = & 4v - 5w \\ \frac{dw}{dt} & = & 2v - 3w \end{array}$$

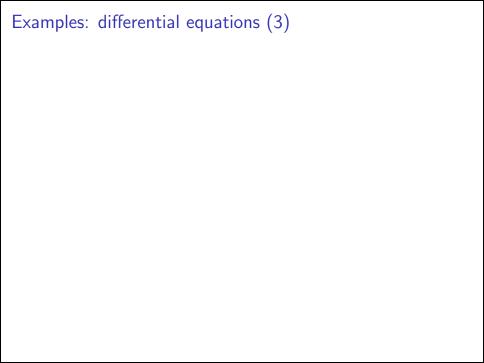
with v=5 and w=4 when t=0, or if we let $u(t)=\begin{bmatrix} v(t)\\w(t) \end{bmatrix}$ and

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix},$$

we have

$$\frac{du}{dt} = Au,$$

with
$$u(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$



Eigenvalues and eigenvectors

Definition

For an n-by-n matrix A, a vector ${\boldsymbol v}$ is an ${\bf eigenvector}$ of A if

$$A\mathbf{v} = \lambda \mathbf{v},$$

and $v \neq 0$. The scalar λ is called an **eigenvalue** associated with v.

Example

Consider matrix
$$A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$$
.

If we let $oldsymbol{v}_1 = egin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$A\mathbf{v}_1 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5+7 \\ -5+3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-2) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

If we let $oldsymbol{v}_2 = egin{bmatrix} 7 \\ 5 \end{bmatrix}$, we have

$$A\mathbf{v}_2 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 + 35 \\ 35 + 15 \end{bmatrix} = \begin{bmatrix} 70 \\ 50 \end{bmatrix} = 10 \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

See demo in colab.

Invariant subspace

Definition (invariant subspace)

For an n-by-n matrix A, subspace $\mathcal{V}\subseteq\mathbb{R}^n$ is called an **invariant subspace** under linear map $f(\boldsymbol{x})=A\boldsymbol{x}$ if for all $\boldsymbol{u}\in\mathcal{V}$, $f(\boldsymbol{u})=A\boldsymbol{u}\in\mathcal{V}$.

Eigenvector

If v is an eigenvector of matrix A, then

Span
$$\{v\}$$

is a 1-dimensional invariant subspace under linear map defined by ${\cal A}.$

Finding eigenvalues and eigenvectors

Given A, we want to find an eigenvalue λ and a vector $u \neq \mathbf{0}$ such that

$$A\mathbf{u} = \lambda \mathbf{u}.$$

After some writing, we want to solve this equation

$$(A - \lambda I)\boldsymbol{u} = 0,$$

where $\boldsymbol{u} \neq 0$.

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

$$Ax = 0.$$

Suppose that there exists $x \neq 0$ that satisfies the equation, what can you say about A?

Clearly, A cannot have an inverse because no matrix B can bring \boldsymbol{x} back from $A\boldsymbol{x}=0$. In this case, we say that A is **singular**. Equavilent conditions:

- ightharpoonup The rank of A is less than n.
- ▶ Rows of *A* are not linearly independent.
- ▶ The linear function f(x) = Ax is not injective.
- $\blacktriangleright \operatorname{Ker} f \neq \{\mathbf{0}\}.$
- $ightharpoonup \dim \operatorname{Ker} f \neq 0.$

Finding λ

From this equation

$$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}.$$

Since we want it to have nonzero solution x. Our goal is to find λ so that $A - \lambda I$ becomes singular.

Typically, the tool to use is the **determinant**. However, we do not cover this topic in this class. We will look at small examples and consider an iterative method instead.

Example: 2×2 matrix

Consider matrix $A=\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We want to find λ such that

$$\begin{bmatrix} 5 - \lambda & 7 \\ 5 & 3 - \lambda \end{bmatrix}$$

is singular. This amounts to solving

$$\frac{5-\lambda}{5} = \frac{7}{3-\lambda},$$

i.e.,

$$\lambda^2 - 8\lambda - 20 = 0.$$

The equation can be re-written as $(\lambda-10)(\lambda+2)=0$; thus, it has 2 roots: 10 and -2.

You can find associated eigenvectors by solving corresponding $(A-\lambda I)x=\mathbf{0}$ equations.

Matrix multiplication (again)

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We know that A has two eigenvectors

$$oldsymbol{v}_1 = egin{bmatrix} 7 \ 5 \end{bmatrix}, \qquad oldsymbol{v}_2 = egin{bmatrix} -1 \ 1 \end{bmatrix}.$$

with corresponding eigenvalues $\lambda_1=10$ and $\lambda_2=-2$. What can we say about

$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7-2 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \left(\begin{bmatrix} 7 \\ 5 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

Matrix multiplication (again and again)

Fact: An n-by-n matrix A has n linearly independent eigenvectors v_1, \ldots, v_n with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. (They might not be real vectors.)

Since v_1,\ldots,v_n form a basis, for any vector x there exist $\alpha_1,\alpha_2,\cdots,\alpha_n$ such that

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \dots + \alpha_n \boldsymbol{v}_n.$$

Let's multiply x with A:

$$Ax = A(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n)$$

= $A\alpha_1 \mathbf{v}_1 + A\alpha_2 \mathbf{v}_2 + \dots + A\alpha_n \mathbf{v}_n$
= $\lambda_1 \alpha_1 \mathbf{v}_1 + \lambda_2 \alpha_2 \mathbf{v}_2 + \dots + \lambda_n \alpha_n \mathbf{v}_n$.

We can keep multiplying with A many times:

$$A^k \mathbf{x} = \lambda_1^k \alpha_1 \mathbf{v}_1 + \lambda_2^k \alpha_2 \mathbf{v}_2 + \dots + \lambda_n^k \alpha_n \mathbf{v}_n.$$

The power method

If A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$|\lambda_1| > |\lambda_i|,$$

for $i \neq 1$. We call λ_1 the **dominant eigenvalue**. We also call the eigenvectors corresponding to λ_1 **dominant eigenvectors**.

The power method (or power iteration)

- ightharpoonup Start with a random vector x_0 .
- For $i=0,1,\ldots,k$, Let ${m x}_{i+1}=A{m x}_i$, with probably some scaling.