

Fermat's Last Theorem

01204211 Discrete Mathematics Lecture 9a: Fermat's Little Theorem

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Quick recap

Number theory
↳ gcd
↳ mod

- ① modular arithmetic — mod
 - multiplicative inverse
 - { gaussian elimination / division }
- ② polynomial
- ③ Fermat LT, Euler → RSA.

For any integer x and y , there exist a pair of integers a and b such that

$$a \cdot x + b \cdot y = \boxed{gcd(x, y)} \rightarrow \text{Euclid alg}$$

Thm: n | \nexists $gcd(x, y) = 1 \Leftrightarrow$
 $x^{-1} \pmod{y}$:

x^{-1} : \pmod{y}

$$x \cdot x^{-1} \equiv 1 \pmod{y}$$

→ \Leftrightarrow multiplicative inverse of $x \pmod{y}$ for

Quick recap

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If $\boxed{\gcd(x, y) = 1}$ $\lceil a \cdot \underline{x} + b \cdot \underline{y} = \gcd(x, y). = 1 \rfloor$

$$a \cdot x \equiv a \cdot x + b \cdot y \equiv 1 \pmod{y}$$

$$\underbrace{a \text{ or } x^{-1} \pmod{y}}$$

Quick recap

For any integer x and y , there exist a pair of integers a and b such that

$$a \cdot x + b \cdot y = \gcd(x, y).$$

How to find a and b ? Use the extended GCD algorithm.

Finding a and b : Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $\gcd(x, y)$.

```
Algorithm Euclid(x,y):
    if x mod y == 0:
        return y, 0, 1
    else:
        g, a', b' = Euclid(y, x mod y)
        a = b'
        b = a' - b'*floor(x / y)
    return g, a, b
```

Recap: Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

$$\underline{a \bmod m} = \underline{b \bmod m},$$

we write

$$\underline{a \equiv b} \quad \boxed{\underline{(\bmod m)}}.$$

We also have that

$$a \equiv b \pmod{m} \iff m|(a - b)$$

Recap: Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo \underline{m} of \underline{a} , denoted by $\underline{a^{-1}}$, is an integer such that

$$\underline{a} \cdot \underline{a^{-1}} \equiv \underline{1} \pmod{\underline{m}}.$$

Theorem 1

An integer a has a multiplicative inverse modulo m iff $\underline{\gcd(a, m) = 1}$.

How to test if an integer n is prime

$$\frac{\log n}{\log 2} \approx \frac{\log n}{2}$$

- Try to find factors of \underline{n} . (Takes time $\underline{\sqrt{n}}$) \times

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How to test if an integer n is prime

- ▶ Try to find factors of n . (Takes time \sqrt{n})
- ▶ If there is a property that holds **iff** n is prime, we can check that property. If we can check that quickly, we can test if n is prime.
- ▶ If there is a property that holds **if** n is prime, how can we make use of that property?

* Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $\gcd(a, p) = 1$,

$$\underline{a^{p-1} \equiv 1 \pmod{p}}.$$

$$\downarrow \\ a^{p-2} \cdot a = a^{p-1} \equiv 1$$

Test(x):

if $a \in \{1, 2, \dots, x-1\}$

check $a^{x-1} \equiv 1 \pmod{x}$

→ yes: x prime? right

→ no; x prime?

Theorem 2 (Fermat's Little Theorem)

If p is prime and a is an integer such that $\gcd(a, p) = 1$,

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How can we use Fermat's Little Theorem to check if integer n is prime?

Fermat test

$$a^{n-1} \equiv 1 \pmod{n}$$

composite



```
Algorithm CheckPrime(n):  
    pick integer a from 2, ..., n-1
```

```
    if gcd(a,n) != 1:  
        return False
```

$O(\log^2 n)$

```
    if power(a, n-1, n) != 1: // FLT  
        return False  
    else:  
        return True
```

$O(1 \log^2 n)$

at times
false positive

$$\frac{1}{2} \frac{1}{10,000}$$

How good is the Fermat test?

When you call `CheckPrime(n)`:

- ▶ If n is prime,

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- ▶ If n is composite,

How good is the Fermat test?

When you call `CheckPrime(n)`:

- ▶ If n is prime, `CheckPrime` always return True.
- ▶ If n is composite, you want `CheckPrime` to return False, but **Fermat's Little Theorem does not guarantee that.**

Fermat test - when n is composite

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If n is composite, the algorithm returns False when

- ✓ ► $\gcd(a, n) \neq 1$, i.e., when you pick a with common factor with n .
- ✓ ► $a^{n-1} \bmod n \neq 1$, i.e., when you find \textcircled{a} that violates the property. We want to be in this case. How likely?

Proof of Fermat's Little Thm: Idea

$$a^{p-1} \equiv 1 \pmod{p}$$

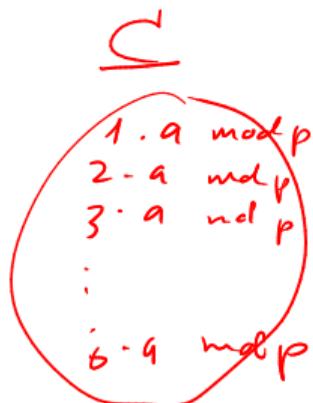
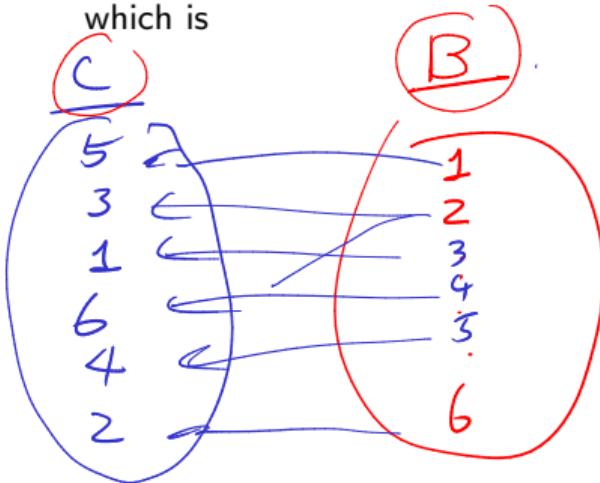
Let $p = 7$ and $a = 5$. Consider set

$$B = \{1, 2, 3, \dots, p-1\} = \{1, 2, 3, 4, 5, 6\}$$

Also consider set

$$C = \{1 \cdot 5 \pmod{7}, 2 \cdot 5 \pmod{7}, 3 \cdot 5 \pmod{7}, \dots, 6 \cdot 5 \pmod{7}\},$$

which is



$$a \not\equiv a^{-1} \pmod{p}$$

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$$C = \{5, 3, 1, 6, 4, 2\}$$

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Is this coincidental?

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which is

$$\underline{C} = \{5, 3, 1, 6, 4, 2\} = \underline{(B)}$$

Is this coincidental? No. (We will prove that. But can you quickly tell why.)

Since $B = C$, the following terms are equal:

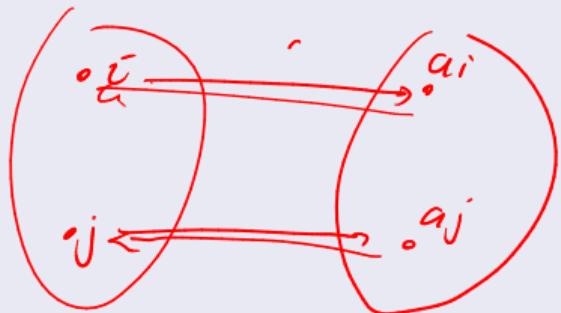
$$\left(\prod_{i \in B} i \right) \bmod 7 = \underline{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \bmod 7,$$

and

$$\begin{aligned} \left(\prod_{i \in C} i \right) \bmod 7 &= 5 \cdot 3 \cdot 1 \cdot 6 \cdot 4 \cdot 2 \bmod 7 \\ &= \underline{(1a)} \cdot \underline{(2a)} \cdot \underline{(3a)} \cdot \underline{(4a)} \cdot \underline{(5a)} \cdot \underline{(6a)} \bmod 7 \\ &= \underline{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6)} \cdot a^6 \bmod 7. \end{aligned}$$

Proof of Fermat's Little Thm.

Recall that $\gcd(a, p) = 1$, i.e., there exists a multiplicative inverse a^{-1} of a modulo p . This implies that for $i \neq j \pmod{p}$, $ai \not\equiv aj \pmod{p}$. Also note that $a \cdot 0 \equiv 0 \pmod{p}$.



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Let $B = \{1, 2, \dots, p - 1\}$. Let

$$C = \{a \cdot i \bmod p \mid i \in B\}.$$

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$$C = \{a \cdot i \bmod p \mid i \in B\}.$$

Since for different $i, j \in B$, we have different $ai \bmod p, aj \bmod p$, we know that $|C| = p - 1$. Also, $C \subseteq B$ because $0 \leq ai \bmod p \leq p - 1$ and $0 \notin C$. Thus, we can conclude that $C = B$.

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Since for different $i, j \in B$, we have different $ai \bmod p, aj \bmod p$, we know that $|C| = p - 1$. Also, $C \subseteq B$ because $0 \leq ai \bmod p \leq p - 1$ and $0 \notin C$. Thus, we can conclude that $C = B$. Since $B = C$, we have that $\prod_{i \in B} i \equiv \prod_{i \in C} i \pmod{p}$, i.e.

$$\cancel{1 \cdot 2 \cdots (p-1)} \equiv (a1) \cdot (a2) \cdot (a3) \cdots (a(p-1)) \pmod{p}$$
$$\equiv \cancel{(1 \cdot 2 \cdots (p-1))} \cdot a^{p-1} \pmod{p}.$$

Since each of $1, 2, \dots, p - 1$ has an inverse modulo p , we can multiply both sides with $1^{-1}, 2^{-1}, \dots, (p-1)^{-1}$ to obtain

$$1 \equiv a^{p-1} \pmod{p},$$

as required.

Exercise

Prove that for any integer a and prime p ,

$$a^p \equiv a \pmod{p}.$$

How good is the Fermat test when n is composite?

To answer correctly, we want a to be such that $\gcd(a, n) \neq 1$ or

$$\underline{a^{n-1} \not\equiv 1 \pmod{n}}.$$

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We refer to $a \in \{1, 2, \dots, p-1\}$ such that $\gcd(a, n) = 1$ and $a^{n-1} \not\equiv 1 \pmod{n}$ as a **witness**. The other element b such that $b^{n-1} \equiv 1 \pmod{n}$ is called a **non-witness**.
How likely that we randomly choose an element and get a witness?

Number of witnesses

Suppose that there exists a witness a ; we know that $a^{n-1} \not\equiv 1 \pmod{n}$. How can we find other witnesses?

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Suppose that there exists a witness a ; we know that $a^{n-1} \not\equiv 1 \pmod{n}$. How can we find other witnesses?

Consider a non-witness b such that $b^{n-1} \equiv 1 \pmod{n}$.

Carmichael Number

A Carmicheal number is a composite number n where

$$b^{n-1} \equiv 1 \pmod{n},$$

for every b which are relatively prime to n .

Carmicheal numbers are rare. The smallest is $561 = 3 \cdot 11 \cdot 17$. The next ones are 1105, 1729, and 2465. There are 20,138,200 Carmicheal numbers between 1 and 10^{21} .

So, if we ignore Carmicheal numbers, the Fermat test is very good. There are other probabilistic tests (e.g, Miller-Rabin test) that uses other properties that works for all numbers and there are deterministic algorithms for testing primes.

Lemma 3

If n is not a Carmicheal number, the Fermat test returns that n is a composite with probability at least $1/2$.

Note that if you repeat the test for k times, the probability that it gives the wrong answer is at most $1/2^k$.

Running time

```
Algorithm CheckPrime(n):
    pick integer a from 2,...,n-1

    if gcd(a,n) != 1:
        return False

    if power(a,n-1,n) != 1:
        return False
    else:
        return True
```

Special case of Euler's theorem

$$a^{p-1} \equiv 1 \pmod{p}$$

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}$$

Theorem 4 (Euler's theorem)

If p and q are different primes, for a such that $\gcd(a, pq) = 1$, we have

$$a^{(p-1)(q-1)} \equiv 1 \pmod{pq}.$$

↳ ryptography RSA.

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Is this useful? Yes! In the RSA algorithm.

RSA

- ▶ Private key: (d, n) , Public key: (e, n)
- ▶ Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- ▶ Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

RSA

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- ▶ Pick two primes p and q . Let $n = pq$.
- ▶ Pick e (usually a small number)
- ▶ Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or $ed = k \cdot (p-1)(q-1) + 1$, for some integer k .
- ▶ What is $m^{ed} \text{ mod } n$?

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$$\begin{aligned}m^{ed} &\equiv m^{k(p-1)(q-1)+1} \pmod{n} \\&\equiv (m^{(p-1)(q-1)})^k \cdot m \pmod{n} \\&\equiv 1^k \cdot m \pmod{n} \\&\equiv m \pmod{n}\end{aligned}$$

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What is the requirement for m ? $\gcd(m, n) = 1$, otherwise you can use the message to factor n ↗ ↘ ↙ ↘