

# 01204211 Discrete Mathematics

## Lecture 2: Quantifiers

Jittat Fakcharoenphol

June 27, 2021

# This lecture covers:

- ▶ More connectives: implications and equivalences
- ▶ Quantifiers

# Review (1)

- ▶ A *proposition* is a statement which is either **true** or **false**.
- ▶ We can use variables to stand for propositions, e.g.,  $P =$  “today is Tuesday”.
- ▶ We can use connectives to combine variables to get propositional forms.
  - ▶ **Conjunction:**  $P \wedge Q$  (“ $P$  and  $Q$ ”),
  - ▶ **Disjunction:**  $P \vee Q$  (“ $P$  or  $Q$ ”), and
  - ▶ **Negation:**  $\neg P$  (“not  $P$ ”)

## Review (2)

To represents values of propositional forms, we usually use truth tables.

And/Or/Not

$P$	$Q$	$P \wedge Q$	$P \vee Q$	$\neg P$
$T$	$T$	$T$	$T$	$F$
$T$	$F$	$F$	$T$	
$F$	$T$	$F$	$T$	$T$
$F$	$F$	$F$	$F$	

## Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1)  $P \wedge \neg P$  and (2)  $P \vee \neg P$ .

## Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1)  $P \wedge \neg P$  and (2)  $P \vee \neg P$ .

And/Or/Not

$P$	$\neg P$	$P \wedge \neg P$	$P \vee \neg P$
$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$

## Quick check 1

As we said before, the truth value of propositional forms may not depend on the values of its variables. As you can see in this exercise.

Use a truth table to find the values of (1)  $P \wedge \neg P$  and (2)  $P \vee \neg P$ .

And/Or/Not

$P$	$\neg P$	$P \wedge \neg P$	$P \vee \neg P$
$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$

Note that  $P \wedge \neg P$  is always false and  $P \vee \neg P$  is always true. A propositional form which is always true regardless of the truth values of its variables is called a *tautology*. On the other hand, a propositional form which is always false regardless of the truth values of its variables is called a *contradiction*.

# Implications<sup>1</sup>

Given  $P$  and  $Q$ , an implication

$$P \Rightarrow Q$$

stands for “if  $P$ , then  $Q$ ”. This is a very important propositional form.

It states that “when  $P$  is true,  $Q$  must be true”. Let’s try to fill in its truth table:

## Implications

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	



# Implications<sup>1</sup>

Given  $P$  and  $Q$ , an implication

$$P \Rightarrow Q$$

stands for “if  $P$ , then  $Q$ ”. This is a very important propositional form.

It states that “when  $P$  is true,  $Q$  must be true”. Let’s try to fill in its truth table:

## Implications

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	

# Implications<sup>1</sup>

Given  $P$  and  $Q$ , an implication

$$P \Rightarrow Q$$

stands for “if  $P$ , then  $Q$ ”. This is a very important propositional form.

It states that “when  $P$  is true,  $Q$  must be true”. Let’s try to fill in its truth table:

## Implications

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	

# Implications<sup>1</sup>

Given  $P$  and  $Q$ , an implication

$$P \Rightarrow Q$$

stands for “if  $P$ , then  $Q$ ”. This is a very important propositional form.

It states that “when  $P$  is true,  $Q$  must be true”. Let’s try to fill in its truth table:

Implications

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	

# Implications<sup>1</sup>

Given  $P$  and  $Q$ , an implication

$$P \Rightarrow Q$$

stands for “if  $P$ , then  $Q$ ”. This is a very important propositional form.

It states that “when  $P$  is true,  $Q$  must be true”. Let’s try to fill in its truth table:

Implications

$P$	$Q$	$P \Rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

---

<sup>1</sup>Materials in this lecture are mostly from Berkeley CS70's lecture notes.

# What?

- ▶ Yes, when  $P$  is false,  $P \Rightarrow Q$  is **always true** no matter what truth value of  $Q$  is.
- ▶ We say that in this case, the statement  $P \Rightarrow Q$  is *vacuously true*.

# What?

- ▶ Yes, when  $P$  is false,  $P \Rightarrow Q$  is **always true** no matter what truth value of  $Q$  is.
- ▶ We say that in this case, the statement  $P \Rightarrow Q$  is *vacuously true*.
- ▶ You might feel a bit uncomfortable about this, because in most natural languages, when we say that if  $P$ , then  $Q$  we sometimes mean something more than that in the logical expression " $P \Rightarrow Q$ ."

# One explanation

- ▶ But let's look closely at what it means when we say that:

if  $P$  is true,  $Q$  must be true.

- ▶ Note that this statement does not say anything about the case when  $P$  is false, i.e., it only considers the case when  $P$  is true.

# One explanation

- ▶ But let's look closely at what it means when we say that:

if  $P$  is true,  $Q$  must be true.

- ▶ Note that this statement does not say anything about the case when  $P$  is false, i.e., it only considers the case when  $P$  is true.
- ▶ Therefore, having that  $P \Rightarrow Q$  is true is OK with the case that (1)  $Q$  is false when  $P$  is false, and (2)  $Q$  is true when  $P$  is false.



# One explanation

- ▶ But let's look closely at what it means when we say that:

if  $P$  is true,  $Q$  must be true.

- ▶ Note that this statement does not say anything about the case when  $P$  is false, i.e., it only considers the case when  $P$  is true.
- ▶ Therefore, having that  $P \Rightarrow Q$  is true is OK with the case that (1)  $Q$  is false when  $P$  is false, and (2)  $Q$  is true when  $P$  is false.
- ▶ This is an example when mathematical language is “stricter” than natural language.

## Noticing if-then

We can write “if  $P$ , then  $Q$ ” for  $P \Rightarrow Q$ , but there are other ways to say this. E.g., we can write (1)  $Q$  if  $P$ , (2)  $P$  only if  $Q$ , or (3) when  $P$ , then  $Q$ .

## Noticing if-then

We can write “if  $P$ , then  $Q$ ” for  $P \Rightarrow Q$ , but there are other ways to say this. E.g., we can write (1)  $Q$  if  $P$ , (2)  $P$  only if  $Q$ , or (3) when  $P$ , then  $Q$ .

### Quick check 2

For each of these statements, define propositional variables representing each proposition inside the statement and write the proposition form of the statement.

- ▶ If you do not have enough sleep, you will feel dizzy during class.
- ▶ If you eat a lot and you do not have enough exercise, you will get fat.
- ▶ You can get A from this course, only if you work fairly hard.

# Only-if

Let  $P$  be “you get A from this course.”

Let  $Q$  be “you work fairly hard.”

Let  $R$  be “You can get A from this course, only if you work fairly hard.”

Let's think about the truth values of  $R$ .

Only if you work fairly hard.

$P$	$Q$	$R$
$T$	$T$	
$T$	$F$	
$F$	$T$	
$F$	$F$	

# Only-if

Let  $P$  be “you get A from this course.”

Let  $Q$  be “you work fairly hard.”

Let  $R$  be “You can get A from this course, only if you work fairly hard.”

Let's think about the truth values of  $R$ .

Only if you work fairly hard.

$P$	$Q$	$R$
$T$	$T$	
$T$	$F$	
$F$	$T$	
$F$	$F$	

Thus,  $R$  should be logically equivalent to  $P \Rightarrow Q$ . (We write  $R \equiv P \Rightarrow Q$  in this case.)

## If and only if: ( $\Leftrightarrow$ )

Given  $P$  and  $Q$ , we denote by

$$P \Leftrightarrow Q$$

the statement “ $P$  if and only if  $Q$ .”

## If and only if: ( $\Leftrightarrow$ )

Given  $P$  and  $Q$ , we denote by

$$P \Leftrightarrow Q$$

the statement “ $P$  if and only if  $Q$ .” It is logically equivalent to

$$(P \Leftarrow Q) \wedge (P \Rightarrow Q),$$

i.e.,  $P \Leftrightarrow Q \equiv (P \Leftarrow Q) \wedge (P \Rightarrow Q)$ .

Let's fill in its truth table.

$P$	$Q$	$P \Rightarrow Q$	$P \Leftarrow Q$	$P \Leftrightarrow Q$
$T$	$T$			
$T$	$F$			
$F$	$T$			
$F$	$F$			

# An implication and its friends

When you have two propositions

- ▶  $P =$  “I own a cell phone”, and
- ▶  $Q =$  “I bring a cell phone to class”.

We have

- ▶ an implication  $P \Rightarrow Q \equiv$   
“If I own a cell phone, I’ll bring it to class”,
- ▶ its **converse**  $Q \Rightarrow P \equiv$   
“If I bring a cell phone to class, I own it”, and
- ▶ its **contrapositive**  $\neg Q \Rightarrow \neg P \equiv$   
“If I do not bring a cell phone to class, I do not own one”.



## Quick check 3

Let's consider the following truth table:

$P$	$Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
$T$	$T$			
$T$	$F$			
$F$	$T$			
$F$	$F$			

## Quick check 3

Let's consider the following truth table:

$P$	$Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
$T$	$T$			
$T$	$F$			
$F$	$T$			
$F$	$F$			

Do you notice any equivalence?

## Quick check 3

Let's consider the following truth table:

$P$	$Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
$T$	$T$			
$T$	$F$			
$F$	$T$			
$F$	$F$			

Do you notice any equivalence?

Right,  $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$ .

# How about our subgoal?

- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ $x$  is even,” “ $p$  is prime,” or “ $s$  is a student.”

## How about our subgoal?

- ▶ In many cases, the statement we are interested in contains variables.
- ▶ For example, “ $x$  is even,” “ $p$  is prime,” or “ $s$  is a student.”
- ▶ As we previously did with propositions, we can use variables to represent these statements. E.g.,
  - ▶ let  $E(x) \equiv$  “ $x$  is even”,
  - ▶ let  $P(y) \equiv$  “ $y$  is prime, and
  - ▶ let  $S(w) \equiv$  “ $w$  is a student.

We call  $E(x)$ ,  $P(y)$ , and  $S(w)$  *predicates*. (You can think of predicates as statements that may be true or false depending on the values of its variables.)

# Quantifiers (1)

- ▶ As we note before, these predicates are not propositions. But if we know the values of their variables, then they become propositions. For example, if we let  $x = 5$ , then  $E(5)$  is a proposition which is false. Also,  $P(7)$  is true.
- ▶ Since the truth values of predicates depend on the assignments of their variables, we can put *quantifiers* to specify the scopes of these variables and how to interpret the truth values of the predicates over these values.

## Quantifiers (2): universal quantifiers

- ▶ Let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $E(2)$ ,  $E(4)$ ,  $E(6)$ , and  $E(8)$  are true, i.e.,  $E(x)$  is true for every  $x \in A$ .

## Quantifiers (2): universal quantifiers

- ▶ Let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $E(2)$ ,  $E(4)$ ,  $E(6)$ , and  $E(8)$  are true, i.e.,  $E(x)$  is true for every  $x \in A$ .

In this case, we say that the following proposition is true:

$$(\forall x \in A)E(x).$$



## Quantifiers (2): universal quantifiers

- ▶ Let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $E(2)$ ,  $E(4)$ ,  $E(6)$ , and  $E(8)$  are true, i.e.,  $E(x)$  is true for every  $x \in A$ .

In this case, we say that the following proposition is true:

$$(\forall x \in A)E(x).$$

- ▶ The quantifier  $\forall$  is called a universal quantifier. (We usually pronounce “for all  $x$ ”, or “for every  $x$ .”)

## Quantifiers (3): existential quantifiers

- ▶ Again, let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $P(2)$  is true. This means that  $P(y)$  is true for some  $y \in A$ .

## Quantifiers (3): existential quantifiers

- ▶ Again, let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $P(2)$  is true. This means that  $P(y)$  is true for some  $y \in A$ .

In this case, we say that the following proposition is true:

$$(\exists y \in A)P(y).$$

## Quantifiers (3): existential quantifiers

- ▶ Again, let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $P(2)$  is true. This means that  $P(y)$  is true for some  $y \in A$ .

In this case, we say that the following proposition is true:

$$(\exists y \in A)P(y).$$

- ▶ The quantifier  $\exists$  is called an existential quantifier. (We usually pronounce “for some  $x$ ”, or “there exists  $x$ .”)

## Quantifiers (3): existential quantifiers

- ▶ Again, let  $A = \{2, 4, 6, 8\}$ .
- ▶ Note that  $P(2)$  is true. This means that  $P(y)$  is true for some  $y \in A$ .

In this case, we say that the following proposition is true:

$$(\exists y \in A)P(y).$$

- ▶ The quantifier  $\exists$  is called an existential quantifier. (We usually pronounce “for some  $x$ ”, or “there exists  $x$ .”)

When the universe  $A$  is clear, we can leave it out and just write  $\forall x E(x)$  or  $\exists y P(y)$ .

# The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input  $n$  is an integer. Thus, we are interested in every  $n \in \mathbb{Z}$ , where  $\mathbb{Z}$  denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where  $C(n) \equiv$

# The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input  $n$  is an integer. Thus, we are interested in every  $n \in \mathbb{Z}$ , where  $\mathbb{Z}$  denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where  $C(n) \equiv$  "CheckPrime2( $n$ ) returns True", and  
 $P(n) \equiv$

# The main goal

- ▶ Let's try to be more specific about our main goal:

Algorithm CheckPrime2 is correct.

- ▶ Can we re-write this statement so that the input/output of the algorithm are explicit?
- ▶ Note that the set of its input  $n$  is an integer. Thus, we are interested in every  $n \in \mathbb{Z}$ , where  $\mathbb{Z}$  denote the set of all integers.
- ▶ Let's rewrite the goal as:

$$\forall n \in \mathbb{Z}, C(n) \Leftrightarrow P(n),$$

where  $C(n) \equiv$  "CheckPrime2( $n$ ) returns True", and  
 $P(n) \equiv$  " $n$  is a prime."



# Quantified propositions with more than one variables

Let our universe be integers ( $\mathbb{Z}$ ). Which of the following statements is true?

- ▶  $\forall x \forall y (x = y)$
- ▶  $\forall x \exists y (x = y)$
- ▶  $\exists x \forall y (x = y)$
- ▶  $\exists x \exists y (x = y)$

# Quantified propositions with more than one variables

Let our universe be integers ( $\mathbb{Z}$ ). Which of the following statements is true?

- ▶  $\forall x \forall y (x = y)$
- ▶  $\forall x \exists y (x = y)$
- ▶  $\exists x \forall y (x = y)$
- ▶  $\exists x \exists y (x = y)$

When you have many quantifiers, we can interpret the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

$$\forall y \exists x P(x, y) \equiv \forall y (\exists x (P(x, y))).$$

# Quantified propositions with more than one variables

Let our universe be integers ( $\mathbb{Z}$ ). Which of the following statements is true?

- ▶  $\forall x \forall y (x = y)$
- ▶  $\forall x \exists y (x = y)$
- ▶  $\exists x \forall y (x = y)$
- ▶  $\exists x \exists y (x = y)$

When you have many quantifiers, we can interpret the statement by nesting the quantifiers. E.g,

$$\exists x \forall y P(x, y) \equiv \exists x (\forall y (P(x, y))).$$

$$\forall y \exists x P(x, y) \equiv \forall y (\exists x (P(x, y))).$$

Also note that usually,  $\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$ .

## Quick check 4

We will consider the universe to be “everything”. Consider the following statements. Define appropriate predicates and rewrite them using the defined predicates and quantifiers. (Note: the predicates may have more than one variables.)

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.

## Quick check 5

- ▶ Let's consider the current subgoal. (Note that in this version, variable  $b$  is replaced with  $n/a$ .)

Another revised statement

For all positive composite integer  $n$ , and for every divisor  $a$  of  $n$  such that  $\sqrt{n} < a < n$ ,

$$2 \leq n/a \leq \sqrt{n}.$$

- ▶ Define all required predicates and describe a quantified proposition equivalent to the revised statement above.

# Negations of quantified propositions (1)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $P(x) \equiv$  “ $x$  is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

# Negations of quantified propositions (1)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $P(x) \equiv$  “ $x$  is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since  $P(4)$  is false,  $\forall x P(x)$  is false.

# Negations of quantified propositions (1)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $P(x) \equiv$  “ $x$  is a prime number.”

Consider this proposition

$$(\forall x \in \mathbb{Z}^+)P(x).$$

How can we show that this is false?

When showing that a universally quantified proposition is false, we need to show “one” counter example. In this case, since  $P(4)$  is false,  $\forall x P(x)$  is false.

This way of disproving a statement is equivalent to showing that

$$(\exists x)(\neg P(x)).$$



## Negations of quantified propositions (2)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $Q(x) \equiv$  “if  $x > 2$ , then  $x^2 \leq 2x$ .”

Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

## Negations of quantified propositions (2)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $Q(x) \equiv$  “if  $x > 2$ , then  $x^2 \leq 2x$ .”

Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

When showing that an existential quantified proposition is false, we need to show that  $Q(x)$  is false for every possible values of  $x$ . In this case, since  $x^2 = x \cdot x > 2 \cdot x$  for every  $x > 2$ , we have that  $(\exists x)Q(x)$  is false.

## Negations of quantified propositions (2)

Let consider a set of positive integers  $\mathbb{Z}^+$  as our universe. Let predicate  $Q(x) \equiv$  “if  $x > 2$ , then  $x^2 \leq 2x$ .”

Consider this proposition

$$(\exists x \in \mathbb{Z}^+)Q(x).$$

How can we show that this is false?

When showing that an existential quantified proposition is false, we need to show that  $Q(x)$  is false for every possible values of  $x$ . In this case, since  $x^2 = x \cdot x > 2 \cdot x$  for every  $x > 2$ , we have that  $(\exists x)Q(x)$  is false.

This way of disproving a statement is equivalent to showing that

$$(\forall x)(\neg Q(x)).$$

## Negations of quantified propositions (3)

Thus, the following equivalences:

►  $\neg(\forall x P(x)) \equiv \exists x(\neg P(x))$

►  $\neg(\exists x P(x)) \equiv \forall x(\neg P(x))$

## Quick check 6

Consider the following statements with the quantified propositions that you have written previously. Write down their negations in quantified propositional forms, and then translate them back to English sentences.

- ▶ Every human must die.
- ▶ Some animal eats other animals.
- ▶ If a student works hard, that student will be successful.
- ▶ Everyone has someone that care about him or her.