01204211 Discrete Mathematics
Lecture 13b: Eigenvalues and Eigenvectors

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Review: Hamming codes (1)

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GF(2)

The code is defined by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_2 + x_3 + x_4 \\ x_1 + x_3 + x_4 \\ x_1 + x_2 + x_4 \end{bmatrix}$$

Consider the encoding function $e : GF(2)^4 \to GF(2)^7$. Let $e(\boldsymbol{x}) = G\boldsymbol{x}$. What is Ker e? $= \{ [0,0,0,0] \}$ dim Ker $e = 0$

nullspace via G

Review: Hamming codes (1)

$$\dim \mathcal{V} = \dim \operatorname{Im} e + \dim \operatorname{Ker} e$$
$$4 \qquad\qquad 4 \qquad\qquad 0$$
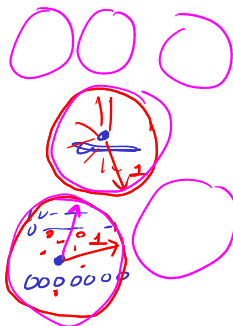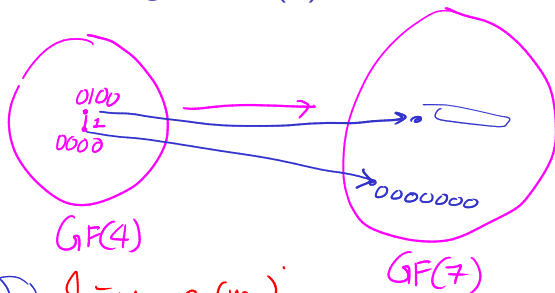
The code is defined by the generator matrix

$$
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\qquad
\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\qquad
G = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix} 1 \\ \\ 1 \\ 1 \end{bmatrix}
$$

Consider the encoding function $e : GF(2)^4 \to GF(2)^7$. Let $e(\boldsymbol{x}) = G\boldsymbol{x}$. What is Ker e?
What is $\dim \operatorname{Im} e$?

$\mathcal{V} \longrightarrow \mathcal{W}$

"4

e is injective

Review: Hamming codes (2)

GF(4)

GF(7)

$q_w - w_1 = e(m_1)$

$w_2 = e(m_2)$ קשה סיבה ווקטור

$\overline{w_1}' = e(m_1 - m_1) = e(0) = 0$

$w_2' = e(m_2 - m_1) = e(m_2) - e(m_1)$

$= w_2 - w_1$
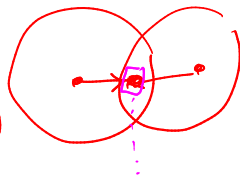
W_2 | | |

w_1 | |

$w_2 - w_1 = $ | 0 | 11111 |

Review: Hamming codes (2)

The code is defined by the generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

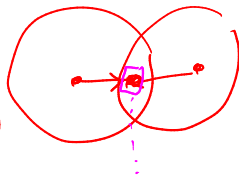What can you say about the minimum "distance"?

Examples: random walks

google
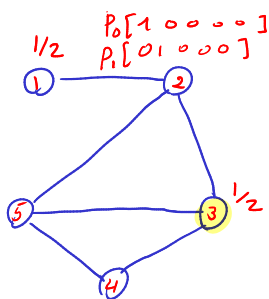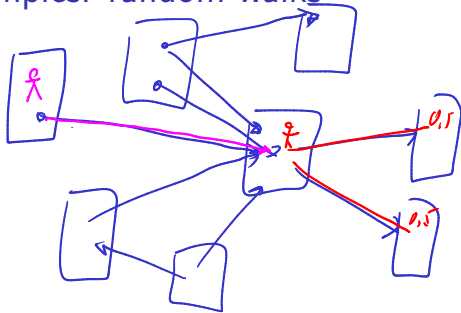
$P_0 [1 \ 0 \ 0 \ 0 \ 0]$

$P_1 [0 1 \ 0 \ 0 \ 0]$

1/2

0,5

0,5

1/2

P_i

$[1/2 \quad 0 \quad 1/2 \quad 0 \quad 0]$

$\downarrow$

$1/2 [0 \ 1 \ 0 \ 0 \ 0]$

$+ 1/2 [0 \ 1/3 \ 0 \ 1/3 \ 1/3]$

q, stationary distribution

$q = q^M$

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{array}$$

ﬠﬧ P_i ﬦﬧשׂ vector ﬦשׂשׂﬧ ﬠ﬩שׂ ﬤשׂﬦﬤ ﬠﬨﬤ
﬩ page ﬦשׂﬦ ﬦﬤﬦﬤﬧ i

$$P_{i+1} = P_i M$$

Examples: differential equations (1)

Let's start with a simple system with one variable.

$$\frac{du}{dt} = au,$$

with $u = u(0)$ when $t = 0$.

$\mathcal{U} = \boxed{e^{\alpha t}}$

$a = 0.1 \ , \ $

$u_0 = 100$

$u_1 = u_0 + 0.1\,u_0 = 110$

$u_2 = 110 + 11 = 121$

$u_3 = 121 + 12.1 = 1\cdots$

$u_t = \underbrace{(1.1)}_{\text{exponential}}{}^{t} \, 100$

Examples: differential equations (2)

Now consider a system with two variables v and w:

$$\begin{array}{rcl} \frac{dv}{dt} & = & 4v - 5w \\ \frac{dw}{dt} & = & 2v - 3w \end{array}$$

with $v = 5$ and $w = 4$ when $t = 0$,

Examples: differential equations (2)

Now consider a system with two variables v and w:

$$\frac{dv}{dt} = 4v - 5w$$
$$\frac{dw}{dt} = 2v - 3w$$

with $v = 5$ and $w = 4$ when $t = 0$, or if we let $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ and

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix},$$

we have

$$\frac{du}{dt} = Au,$$

with $u(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Handwritten annotations:

$$\{u'_i\} \quad v = xe^{\lambda t}$$
$$w = ye^{\lambda t}$$

$$\lambda xe^{\lambda t} = 4xe^{\lambda t} - 5ye^{\lambda t}$$
$$\lambda ye^{\lambda t} = 2xe^{\lambda t} - 3ye^{\lambda t}$$

$$\lambda x = 4x - 5y$$
$$\lambda y = 2x - y$$

$$\lambda [u] = [A][u]$$

Examples: differential equations (3)

$$(\lambda)u = A u$$

$$q = q^M$$

$$q = M q$$

Eigenvalues and eigenvectors

Definition

For an n-by-n matrix A, a vector $\boldsymbol{v}$ is an **eigenvector** of A if

$$A\boldsymbol{v} = \lambda\boldsymbol{v},$$

and $\boldsymbol{v} \neq \boldsymbol{0}$. The scalar λ is called an **eigenvalue** associated with $\boldsymbol{v}$.

Example

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$.

If we let $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$A\boldsymbol{v}_1 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

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If we let $\boldsymbol{v}_2 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, we have

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$$A\boldsymbol{v}_2 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 + 35 \\ 35 + 15 \end{bmatrix} =$$

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$$A\boldsymbol{v}_2 = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 + 35 \\ 35 + 15 \end{bmatrix} = \begin{bmatrix} 70 \\ 50 \end{bmatrix} = 10 \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

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See demo in colab.

Invariant subspace

Definition (invariant subspace)

For an n-by-n matrix A, subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called an **invariant subspace** under linear map $f(\boldsymbol{x}) = A\boldsymbol{x}$ if for all $\boldsymbol{u} \in \mathcal{V}$, $f(\boldsymbol{u}) = A\boldsymbol{u} \in \mathcal{V}$.

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Eigenvector

If $\boldsymbol{v}$ is an eigenvector of matrix A, then

$$\text{Span} \left\{ \boldsymbol{v} \right\}$$

is a 1-dimensional invariant subspace under linear map defined by A.

Finding eigenvalues and eigenvectors

Given A, we want to find an eigenvalue λ and a vector $\boldsymbol{u} \neq \boldsymbol{0}$ such that

$$A\boldsymbol{u} = \lambda\boldsymbol{u}.$$

$$A u - \lambda u = \cancel{0} \quad A u - \lambda I u$$
$$= (A - \lambda I) u = 0$$

$\boxed{Ax = b}$

$x_1 \neq x_2$

$A x_1 = b, \quad A x_2 = b$

$A(x_1 - x_2) = 0$

homogeneous

Finding eigenvalues and eigenvectors

$n \times n$

Given A, we want to find an eigenvalue λ and a vector $\boldsymbol{u} \neq \boldsymbol{0}$ such that

$$A\boldsymbol{u} = \lambda\boldsymbol{u}.$$

After some writing, we want to solve this equation

$$(A - \lambda I)\boldsymbol{u} = 0,$$

where $\boldsymbol{u} \neq 0$.

$\rightarrow$ rank $\text{vo } B = (r)$

$Bu = 0$

$\rightarrow$ dim group nullspace $\text{vo } B = p$

$u \in \mathbb{R}^n$

$n = r + p$

si $r \neq n$
$p > 0$

Review: ranks and invertible matrices

Consider an n-by-n matrix A and the following linear system of equations

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Suppose that there exists $\boldsymbol{x} \neq \boldsymbol{0}$ that satisfies the equation, what can you say about A?

$$A^{-1} A x = 0 = x$$

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Clearly, A cannot have an inverse because no matrix B can bring $\boldsymbol{x}$ back from $A\boldsymbol{x} = 0$.

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Equavilent conditions:

- ▶ The rank of A is less than n.
- ▶ Rows of A are not linearly independent.
- ▶ The linear function $f(\boldsymbol{x}) = A\boldsymbol{x}$ is not injective.
- ▶ Ker $f \neq \{\boldsymbol{0}\}$.
- ▶ $\dim$ Ker $f \neq 0$.

From this equation

$$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}.$$

Since we want it to have nonzero solution $\boldsymbol{x}$. Our goal is to find λ so that $A - \lambda I$ becomes singular.

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Typically, the tool to use is the **determinant**. However, we do not cover this topic in this class. We will look at small examples and consider an iterative method instead.

Example: 2×2 matrix

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We want to find λ such that

$$[5 - \lambda, 7] = \alpha [5, 3 - \lambda]$$

$$\alpha \begin{bmatrix} 5 - \lambda & 7 \\ 5 & 3 - \lambda \end{bmatrix}$$

$$\frac{5 - \lambda}{5} = \frac{7}{3 - \lambda}$$

is singular.

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i.e.,

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The equation can be re-written as $(\lambda - 10)(\lambda + 2) = 0$; thus, it has 2 roots: 10 and -2.
You can find associated eigenvectors by solving corresponding $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ equations.

Matrix multiplication (again)

Consider matrix $A = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix}$. We know that A has two eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

with corresponding eigenvalues $\lambda_1 = 10$ and $\lambda_2 = -2$.

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$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7 - 2 \\ 5 + 2 \end{bmatrix} =$$

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$$\begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 7-2 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 5 & 3 \end{bmatrix} \left(\begin{bmatrix} 7 \\ 5 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$= 10 \begin{bmatrix} 7 \\ 5 \end{bmatrix} + 2 \cdot (-2) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Matrix multiplication (again and again)

Fact: An n-by-n matrix A has n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. (They might not be real vectors.)

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Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis, for any vector $\boldsymbol{x}$ there exist $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$\boldsymbol{x} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n.$$

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Let's multiply $\boldsymbol{x}$ with A:

$$
\begin{aligned}
A\boldsymbol{x} &= A(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_n \boldsymbol{v}_n) \\
&= A\alpha_1 \boldsymbol{v}_1 + A\alpha_2 \boldsymbol{v}_2 + \cdots + A\alpha_n \boldsymbol{v}_n \\
&= \lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 \; \text{- - - -}
\end{aligned}
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\end{aligned}
$$

$= \lambda_1 \alpha_1 v_1 = \lambda_2^2$

We can keep multiplying with A many times:

$$A^k \boldsymbol{x} =$$

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\end{aligned}
$$

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$$A^k\boldsymbol{x} = \lambda_1^k\alpha_1\boldsymbol{v}_1 + \lambda_2^k\alpha_2\boldsymbol{v}_2 + \cdots + \lambda_n^k\alpha_n\boldsymbol{v}_n.$$

The power method

$$\chi_j = (a_1, a_2 \ldots, a_n)$$

$$\chi_1' = \frac{1}{\sqrt{a_1^2 + a_2^2 \cdots a_n}} (a_1, \cdots, a_n)$$

If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$|\lambda_1| > |\lambda_i|,$$

for $i \neq 1$. We call λ_1 the **dominant eigenvalue**. We also call the eigenvectors corresponding to λ_1 **dominant eigenvectors**.

The power method (or power iteration)

- ▶ Start with a random vector $\boldsymbol{x_0}$.
- ▶ For $i = 0, 1, \ldots, k$,
     Let $\boldsymbol{x_{i+1}} = A\boldsymbol{x_i}$, with probably some scaling.