

01204211 Discrete Mathematics
Lecture 13a: Linear functions (II)

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Review: Linear functions

Linear functions

Consider vector spaces \mathcal{V} and \mathcal{W} over \mathbb{R} . A function $f : \mathcal{V} \rightarrow \mathcal{W}$ is **linear** if

1. for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ and
2. for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{V}$, $f(\alpha \mathbf{x}) = \alpha f(\mathbf{x})$.

Matrix-vector multiplication

Given an $m \times n$ matrix M over \mathbb{R} , consider a product

$$M\mathbf{x}.$$

Note that for the multiplication to work, \mathbf{x} must be in \mathbb{R}^n and the result vector is in \mathbb{R}^m . Therefore, we can define a function

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as

$$f(\mathbf{x}) = M\mathbf{x}.$$

Note that f is linear because:

$$f(\mathbf{x} + \mathbf{y}) = M(\mathbf{x} + \mathbf{y}) = M\mathbf{x} + M\mathbf{y} = f(\mathbf{x}) + f(\mathbf{y}),$$

and

$$f(\alpha\mathbf{x}) = M(\alpha\mathbf{x}) = \alpha M\mathbf{x} = \alpha f(\mathbf{x}).$$

The converse

Lemma 1

For any linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, there exists an $m \times n$ matrix M such that

$$f(\mathbf{x}) = M\mathbf{x}.$$

Example: a system of linear equations

Consider the following homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 3 & 4 & 7 & 5 & 3 \\ 6 & 7 & 13 & 8 & 6 \\ 2 & 4 & 6 & 14 & 6 \\ 4 & 6 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's try to solve it on Colab.

Example: a system of linear equations

Let's look at what we've got so far (after row permutation)

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

What is the rank of A ?

What does nullspace of A look like?

Example: nullspace of A

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's look at row 3:

$$2x_4 + x_5 = 0.$$

Let's look at row 2:

$$x_2 + x_3 + 2x_4 = 0.$$

Finally, let's look at row 1:

$$x_1 + 2x_2 + 3x_3 + 3x_4 + x_5 = 0.$$

How many “free” variable that you can set?

Ranks and nullities

Viewing matrix-vector multiplication as linear mapping

$$\begin{bmatrix} 1 & 2 & 3 & 3 & 1 \\ 3 & 4 & 7 & 5 & 3 \\ 6 & 7 & 13 & 8 & 6 \\ 2 & 4 & 6 & 14 & 6 \\ 4 & 6 & 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Structures of linear functions

Zero

Lemma 2

Consider any linear function $f : \mathcal{V} \rightarrow \mathcal{W}$. Let $0_{\mathcal{V}}$ denote the zero vector in \mathcal{V} and $0_{\mathcal{W}}$ denote the zero vector in \mathcal{W} . We have that linear function f always maps zero to zero, i.e., $f(0_{\mathcal{V}}) = 0_{\mathcal{W}}$.

Proof.

First note that $0_{\mathcal{V}} = 0_{\mathcal{V}} + 0_{\mathcal{V}}$. Since f is linear, we have that

$$f(0_{\mathcal{V}}) = f(0_{\mathcal{V}} + 0_{\mathcal{V}}) = f(0_{\mathcal{V}}) + f(0_{\mathcal{V}}).$$

Subtracting $f(0_{\mathcal{V}})$ from both sides, we conclude that

$$0_{\mathcal{W}} = f(0_{\mathcal{V}}).$$



One-to-one linear functions and Onto linear functions

One-to-one and onto functions

Consider a function $f : D \rightarrow R$ (i.e., the domain of f is D and its range is R).

- ▶ Function f is **one-to-one** (or **injective**) if for all $x, y \in D$, $f(x) = f(y)$ implies that $x = y$.
- ▶ Function f is **onto** (or **surjective**) if for all $x \in R$, there exists $y \in D$ such that $f(y) = x$.

For this course, we consider only linear functions; therefore, we consider $f : \mathcal{V} \rightarrow \mathcal{W}$, where \mathcal{V} and \mathcal{W} are vector spaces.

One-to-one linear functions

Suppose that f is not one-to-one, i.e., there exists a pair $x, y \in \mathcal{V}$ such that $x \neq y$ and $f(x) = f(y)$. Since f is linear, we know that

$$f(x - y) = f(x) - f(y) = 0.$$

Since $x \neq y$, $x - y \neq 0$ and we have that there exists a non-zero element $z = x - y$ that f maps to 0. The contraposition of this fact is as follows.

If the only element in \mathcal{V} that f maps to $0_{\mathcal{W}}$ is $0_{\mathcal{V}}$, f is one-to-one (or injective).

Because the set of elements that f maps to zero is very important, we have a name for it.

Definition (Kernel)

The **kernel** of f , denoted by $\text{Ker } f$, is the set of all elements that f maps to zero, i.e.,

$$\text{Ker } f = \{v \in \mathcal{V} : f(v) = 0_{\mathcal{V}}\}.$$

We can now restate the condition for f to be one-to-one using this concept.

Lemma 3

A linear function f is one-to-one, if and only if $\text{Ker } f = \{0\}$.

The kernel is also a vector space

Lemma 4

Ker f is a vector space.

Proof.

First note that $f(0) = 0$; thus $0 \in \text{Ker } f$.

Suppose that $x \in \text{Ker } f$, i.e., $f(x) = 0$. Note that for any scalar α ,

$$f(\alpha x) = \alpha f(x) = \alpha 0 = 0.$$

Also suppose $y \in \text{Ker } f$. We have that

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0.$$



Onto linear functions

Definition (Image)

For any function g , its **image**, denoted by $\text{Im } g$, is the set of all elements that g maps to, i.e.,

$$\text{Im } g = \{y : \text{there exists } x \text{ such that } g(x) = y\}.$$

The image is also a vector space

Lemma 5

The image of linear function f , $\text{Im } f$, is a vector space.

Proof.

Since $f(0_V) = 0_W$, $0_W \in \text{Im } f$.

Consider $y \in \text{Im } f$. We have that there exists x such that $f(x) = y$. Consider any scalar α . We know that $\alpha y \in \text{Im } f$ because $f(\alpha x) = \alpha f(x) = \alpha y$.

Consider, also, $y' \in \text{Im } f$. Let x' be such that $f(x') = y'$. Since $y' \in \text{Im } f$, we know that x' exists. We have that

$$f(x + x') = f(x) + f(x') = y + y'.$$

This implies that $y + y' \in \text{Im } f$.



Kernels and images

Theorem 6 (Kernel-Image Theorem)

Consider a linear function $f : \mathcal{V} \rightarrow \mathcal{W}$. We have that

$$\dim \mathcal{V} = \dim \operatorname{Ker} f + \dim \operatorname{Im} f.$$

Completing the basis

Lemma 7

For a set of linearly independent vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$$

in \mathcal{V} with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ (where $k \leq n$), there exists a set of vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-k} \in B$ such that

$$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{n-k}\}$$

is also a basis for \mathcal{V} .

Proof.

Use the morphing lemma.



Theorem 8 (Kernel-Image Theorem)

For a linear function $f : \mathcal{V} \rightarrow \mathcal{W}$, $\dim \mathcal{V} = \dim \text{Ker } f + \dim \text{Im } f$.

Proof of Kernel-Image Theorem (1).

Let $n = \dim \mathcal{V}$ and $k = \dim \text{Ker } f$. Our goal is to show that $\dim \text{Im } f = n - k$.

Since $\text{Ker } f$ is a vector space, there is a basis

$B = \{v_1, v_2, \dots, v_k\}$. From the previous slide, we can find other $n - k$ vectors w_1, w_2, \dots, w_{n-k} to extend B to be a basis S for \mathcal{V} , i.e., we have that

$$S = \{v_1, v_2, \dots, v_k, w_1, w_2, \dots, w_{n-k}\}$$

is a basis for \mathcal{V} .



Proof of Kernel-Image Theorem (2).

Consider any $\mathbf{u} \in \mathcal{V}$. We can write \mathbf{u} as

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \cdots + \beta_{n-k} \mathbf{w}_{n-k},$$

because S is a basis. Consider $f(\mathbf{u})$. We have that

$$\begin{aligned} f(\mathbf{u}) &= f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{w}_1 + \cdots + \beta_{n-k} \mathbf{w}_{n-k}) \\ &= f(\alpha_1 \mathbf{v}_1) + \cdots + f(\alpha_k \mathbf{v}_k) + f(\beta_1 \mathbf{w}_1) + \cdots + f(\beta_{n-k} \mathbf{w}_{n-k}) \\ &= f(\beta_1 \mathbf{w}_1) + f(\beta_2 \mathbf{w}_2) + \cdots + f(\beta_{n-k} \mathbf{w}_{n-k}) \\ &= \beta_1 f(\mathbf{w}_1) + \beta_2 f(\mathbf{w}_2) + \cdots + \beta_{n-k} f(\mathbf{w}_{n-k}) \end{aligned}$$

(Note that the second step follows because $\mathbf{v}_i \in \text{Ker } f$. Other steps use the fact that f is linear.)

This calculation shows that an image of f can be written as a linear combination of $f(\mathbf{w}_1), \dots, f(\mathbf{w}_{n-k})$. That is

$$\text{Im } f = \text{Span } \{f(\mathbf{w}_1), \dots, f(\mathbf{w}_{n-k})\}.$$



Proof of Kernel-Image Theorem (3).

Let $S' = \{f(\mathbf{w}_1), \dots, f(\mathbf{w}_{n-k})\}$. If we can show that S' is a basis for $\text{Im } f$, we are done because that would imply that $\dim \text{Im } f = n - k$ as required.

We already know that S' spans $\text{Im } f$. To show that S' is a basis we still need to show that S' is linearly independent.

Suppose that there exist $\beta_1, \dots, \beta_{n-k}$ such that

$$\beta_1 f(\mathbf{w}_1) + \beta_2 f(\mathbf{w}_2) + \dots + \beta_{n-k} f(\mathbf{w}_{n-k}) = 0_{\mathcal{W}}.$$

Since f is linear we know that

$$\begin{aligned} 0_{\mathcal{W}} &= \beta_1 f(\mathbf{w}_1) + \beta_2 f(\mathbf{w}_2) + \dots + \beta_{n-k} f(\mathbf{w}_{n-k}) \\ &= f(\beta_1 \mathbf{w}_1) + f(\beta_2 \mathbf{w}_2) + \dots + f(\beta_{n-k} \mathbf{w}_{n-k}) \\ &= f(\beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \dots + \beta_{n-k} \mathbf{w}_{n-k}), \end{aligned}$$

i.e., $\beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \dots + \beta_{n-k} \mathbf{w}_{n-k}$ is in $\text{Ker } f$.



Proof of Kernel-Image Theorem (4).

Suppose that some $\beta_i \neq 0$.

Since

$$\beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \cdots + \beta_{n-k} \mathbf{w}_{n-k} \in \text{Ker } f,$$

we know that it is a linear combination of vectors from B , as B is a basis for vector space $\text{Ker } f$.

From here, we can reach a contradiction using the fact that vectors in S are linearly independent.

Therefore, we conclude that all $\beta_1, \dots, \beta_{n-k}$ must be 0. Hence, $S' = \{f(\mathbf{w}_1), \dots, f(\mathbf{w}_{n-k})\}$ is linearly independent as needed. □

Direct sum (optional)

Consider two subspaces \mathcal{V} and \mathcal{W} of a vector space \mathcal{Z} . If $\mathcal{V} \cap \mathcal{W} = \{0\}$, we can define their *direct sum* to be another vector space $\mathcal{V} \oplus \mathcal{W}$ as

$$\mathcal{V} \oplus \mathcal{W} = \{\boldsymbol{v} + \boldsymbol{u} : \boldsymbol{v} \in \mathcal{V}, \boldsymbol{u} \in \mathcal{W}\}.$$

Note, again, that $\mathcal{V} \oplus \mathcal{W}$ is defined only when $\mathcal{V} \cap \mathcal{W} = \{0\}$.