01204211 Discrete Mathematics
Lecture 9b: Polynomials (1)[1]

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[1] This section is from Berkeley CS70 lecture notes.

Quick exercise

For any integer $a \neq 1$, $a - 1 | a^2 - 1$.

For any integer $a \neq 1$ and $n \geq 1$, $a - 1 | a^n - 1$.

Polynomials

A **single-variable polynomial** is a function $p(x)$ of the form

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0.$$

We call a_i's *coefficients*. Usually, variable x and coefficients a_i's are real numbers. The **degree** of a polynomial is the largest exponent of the terms with non-zero coefficients.

Examples

- $x^3 - 3x + 1$
- $x + 10$
- 10
- 0

Folklore

Applications

- Secret sharing
- Error-correcting codes

Basic facts

Definition

a is a **root** of polynomial $f(x)$ if $f(a) = 0$.

Properties

Property 1: A non-zero polynomial of degree d has at most d roots.
Property 2: Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.

Lemma 1

If two polynomials $f(x)$ and $g(x)$ of degree at most d that share $d+1$ points $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$, where all x_i's are distinct, i.e., $f(x_i) = g(x_i) = y_i$, then $f(x) = g(x)$.

Proof.

Suppose that $f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$ and $g(x) = b_d x^d + b_{d-1} x^{d-1} + \cdots + b_0$.
Let $h(x) = f(x) - g(x)$, i.e., let $h(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_0$, where $c_i = a_i - b_i$. Note that $h(x)$ is also a polynomial of degree (at most) d.
We claim that $h(x)$ has $d+1$ roots. Note that since $f(x_i) = g(x_i) = y_i$, we have that

$$h(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0,$$

i.e., every x_i is a root of $h(x)$.

From **Property 1**, if $h(x)$ is non-zero it has at most d roots; therefore, $h(x)$ must be zero, i.e., $f(x) - g(x) = 0$ or $f(x) = g(x)$ as required. $\square$

Polynomial interpolation - ideas

Lagrange polynomial

For $d+1$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$ where all x_i's are distinct, let

$$\Delta_i(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{d+1})}.$$

Note that $\Delta_i(x)$ is a polynomial of degree d. Also we have that

- For $j \neq i$, $\Delta_i(x_j) = 0$, and
- $\Delta_i(x_i) = 1$.

We can use $\Delta_i(x)$ to construct a degree-d polynomial

$$p(x) = y_1 \cdot \Delta_1(x) + y_2 \cdot \Delta_2(x) + \cdots y_{d+1} \cdot \Delta_{d+1}(x).$$

What can you say about $p(x_i)$?

Property 2

Given $d+1$ pairs $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$ with distinct x_i's, there is a *unique* polynomial $p(x)$ of degree at most d such that $p(x_i) = y_i$ for $1 \leq i \leq d+1$.

Proof of Property 2.

Using Lagrange interpolation, we know that there exists a polynomial $p(x)$ of degree d such that $p(x_i) = y_i$ for all $1 \leq i \leq d+1$.
For uniqueness, assume that there exists another polynomial $g(x)$ of degree d also satifying the condition. Since $p(x)$ and $g(x)$ agrees on more than d points, $p(x)$ and $g(x)$ must be equal from Lemma 1. $\square$