# 01204211 Discrete Mathematics
# Lecture 10b: Dimensions

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# Review: Linear combinations

## Definition

For any scalars

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

and vectors

$$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m,$$

we say that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_m \boldsymbol{u}_m$$

is a **linear combination** of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$.

# Review: Span

**Definition**

A set of all linear combination of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ is called the **span** of that set of vectors.
It is denoted by $\mathrm{Span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m\}$.

# Review: Vector spaces

## Definition

A set $\mathcal{V}$ of vectors over $\mathbb{F}$ is a **vector space** iff

- (V1) $\mathbf{0} \in \mathcal{V}$,
- (V2) for any $\boldsymbol{u} \in \mathcal{V}$,
$$\alpha \cdot \boldsymbol{u} \in \mathcal{V}$$
for any $\alpha \in \mathbb{F}$, and
- (V3) for any $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$,
$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V}.$$

# Review: Linearly independence

## Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if no vector $\boldsymbol{u}_i$ can be written as a linear combination of other vectors.

## (Another) Definition

Vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are **linearly independent** if the only solution of equation

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n = \boldsymbol{0}$$

is

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

# Review: Bases

**Definition**

A set of vectors $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a **basis** for vector space $\mathcal{V}$ if

- $\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \mathcal{V}$, and
- $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

## Lemma 1 (Superfluous Vector Lemma)

*Consider vectors* $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{v} \in \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, *then*

$$\mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \boldsymbol{v}\} = \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$$

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**Lemma 2**

*Consider vectors* $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. *If* $\boldsymbol{u}_n \in \mathrm{Span}\ \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n-1}\}$, *then*

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## Lemma 3 (Unique representation)

*Let* $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ *be a basis for vector space* $\mathcal{V}$. *For any* $\boldsymbol{v} \in \mathcal{V}$, *there is a unique way to write* $\boldsymbol{v}$ *as a linear combination of* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$.

# Examples in $\mathbb{R}^2$ and $\mathbb{R}^3$

# Examples in $GF(2)$ - Vector spaces from graphs

# Examples in $GF(2)$ - Cycles

# Examples in $GF(2)$ - Basis

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- This is not a coincident.
- In this course, we will see two proofs.
- Remark: there are vector spaces whose basis has infinite size, but we are not dealing with those vector spaces in this course.

**Theorem 4 (Main Theorem)**

*If $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are bases for vector space $\mathcal{W}$, then $n = m$.*

# Exchange Lemma

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## Lemma 5 (Simplified Exchange Lemma)

*Consider a set of vectors* $S$ *and let* $\boldsymbol{z}$ *be a non-zero vector in* $\text{Span } S$. *There is a vector* $\boldsymbol{w} \in S$ *such that* $\text{Span } (S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\}) = \text{Span } S$.

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**Lemma 6 (Exchange Lemma)**

*Consider a set of vectors $S$ and its subset $A$. Let $\boldsymbol{z}$ be a non-zero vector in* $\text{Span } S$ *such that $A \cup \{\boldsymbol{z}\}$ is linearly independent. There is a vector $\boldsymbol{w} \in S - A$ such that* $\text{Span } (S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\}) = \text{Span } S$.

**Lemma 7 (Morphing Lemma)**

*If a set of vectors $S$ spans a vector space $\mathcal{W}$ and $B$ is a linearly independent set of vectors in $\mathcal{W}$, then $|B| \leq |S|$.*

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## Proof.

Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. We show how to construct $S_1, \ldots, S_n$ such that for each $i$, $|S_i| = |S|$, $\mathrm{Span}\ S_i = \mathrm{Span}\ S$, and

$$\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_i\} \subseteq S_i.$$

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$$\text{Span } (S_{i-1} \cup \{\boldsymbol{u_i}\} - \{\boldsymbol{w}\}) = \text{Span } S_{i-1}.$$

We then let $S_i = S_{i-1} \cup \{\boldsymbol{u_i}\} - \{\boldsymbol{w}\}$. (You can check that $S_i$ has the properties as claimed above.)

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Since $|S_n| = |S|$ and $B \subseteq S_n$, we have that $|B| \leq |S|$. □

# Morphing Lemma $\Rightarrow$ Main Theorem

**Theorem 8 (Main Theorem)**

*If* $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ *and* $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ *are bases for vector space* $\mathcal{W}$*, then* $n = m$*.*

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Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis, it spans $\mathcal{W}$. Also, vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are linearly independent because they also form a basis. Thus, from the Morphing Lemma, $m \leq n$.

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Therefore, $n = m$. $\square$

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How can we prove the full lemma?

# Dimensions

**Definition**

The **dimension** of a vector space $\mathcal{V}$ is the size of its basis. The dimension of $\mathcal{V}$ is written as $\dim \mathcal{V}$.