01204211 Discrete Mathematics
Lecture 11b: Four fundamental subspaces (preview)

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November 4, 2024

What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\left[\begin{array}{c|c|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \end{array}\right]$$

Matrices and elimination

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$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & 1 \\ 2 & 10 & 2 & 4 \\ 2 & 7 & 1 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

Linearly independent rows

Vector spaces related to a matrix

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Subspaces

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Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces such that $\mathcal{V} \subseteq \mathcal{W}$. We say that $\mathcal{V}$ is a **subspace** of $\mathcal{W}$.

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Examples:

- $\text{Span}\,\{[1,1]\}$ is a subspace of $\mathbb{R}^2$.
- $\text{Span}\,\{[1,0,0],[0,1,1]\}$ is a subspace of $\mathbb{R}^3$.
- $\text{Span}\,\{[1,0,0],[0,1,1],[1,1,2]\}$ is a subspace of $\mathbb{R}^3$.

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The set of solutions $\{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\}$ form a vector space.

Example 1 (cont.)

Given a matrix A, we can look at the matrix-vector product $A\boldsymbol{x}$.
Consider

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Four fundamental subspaces

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Given an m-by-n matrix A, we have the following subspaces

- The column space of A (denoted by $\mathcal{C}(A)$)
- The row space of A (denoted by $\mathcal{C}(A^T)$)
- The nullspace of A

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