

# 01204211 Discrete Mathematics

## Lecture 9a: Spans and Vector Spaces

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## Review: Linear combinations

### Definition

For any scalars

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **linear combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

## Review: Span

### Definition

A set of all linear combination of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **span** of that set of vectors.

It is denoted by  $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ .

## Exercise

The following vectors represent the amount of nutritions for 100ml of the healthy drink ingredients

$$\mathbf{v} = \begin{bmatrix} 100 \\ 50 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 300 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 50 \\ 0 \\ 50 \\ 10 \end{bmatrix}$$

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Write down the nutritions for a mixed drink that consists of 50ml of  $\boldsymbol{v}$ , 200ml of  $\boldsymbol{c}$  and 10ml of  $\boldsymbol{w}$ .

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Write down the nutritions for a mixed drink that consists of 50ml of  $\boldsymbol{v}$ , 200ml of  $\boldsymbol{c}$  and 10ml of  $\boldsymbol{w}$ .

Write that result as a matrix-vector product. (The matrix should be a  $4 \times 3$  matrix.)

## Example 1

Is  $\text{Span} \{[1, 2], [2, 5]\} = \mathbb{R}^2$ ?

## Example 2

Is  $\text{Span} \{[1, 0, 1], [1, 1, 0], [2, 3, 4]\} = \mathbb{R}^3$ ?



## Example 3

Is  $\text{Span} \{[1, 0, 1], [1, 1, 0], [4, 2, 2]\} = \mathbb{R}^3$ ?

# Elements in a vector

- ▶ We see examples of vectors over  $\mathbb{R}$ .
- ▶ However, elements in a vector can be from other sets with appropriate property. (I.e., they should behave a real numbers.)
- ▶ What do we want from an element in a vector?
  - ▶ We should be able to perform addition, subtraction, multiplication, and division.
  - ▶ Operations should be commutative and associative.
  - ▶ Additive and multiplicative identity should exist.
  - ▶ Addition and multiplication should have inverses.
- ▶ We refer to a set with these properties as a **field**.

# A field

## Definition

A set  $\mathbb{F}$  with two operations  $+$  and  $\times$  (or  $\cdot$ ) is a **field** iff these operations satisfy the following properties:

- ▶ (Associativity):  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- ▶ (Commutativity):  $a + b = b + a$  and  $a \cdot b = b \cdot a$
- ▶ (Identities): There exist two elements  $0 \in \mathbb{F}$  and  $1 \in \mathbb{F}$  such that  $a + 0 = a$  and  $a \cdot 1 = a$
- ▶ (Additive inverse): For every element  $a \in \mathbb{F}$ , there is an element  $-a \in \mathbb{F}$  such that  $a + (-a) = 0$
- ▶ (Multiplicative inverse): For every element  $a \in \mathbb{F} \setminus \{0\}$ , there is an element  $a^{-1}$  such that  $a \cdot a^{-1} = 1$
- ▶ (Distributive):  $a \cdot (b + c) = a \cdot b + a \cdot c$

## Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$ . I.e., it is a “bit” field.

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► We define  $b_1 + b_2$  to be XOR.

$$0 + 0 = 0$$

$$0 + 1 = 1 + 0 = 1$$

$$1 + 1 = 0$$

## Another useful field: $GF(2)$

$GF(2) = \{0, 1\}$ . I.e., it is a “bit” field.

What are  $+$  and  $\cdot$  in  $GF(2)$ ?

- We define  $b_1 + b_2$  to be XOR.

$$\begin{aligned}0 + 0 &= 0 \\0 + 1 &= 1 + 0 = 1 \\1 + 1 &= 0\end{aligned}$$

- We define  $b_1 \cdot b_2$  to be standard multiplication.

$$\begin{aligned}0 \cdot 0 &= 0 \cdot 1 = 1 \cdot 0 = 0 \\1 \cdot 1 &= 1\end{aligned}$$

You can check that  $GF(2)$  satisfies the axioms of fields.

## $2 \times 2$ Lights out

# Parity-check code



## Parity-check code

From message  $\mathbf{a} = [a_1, a_2, a_3, a_4]$ , we compute (in  $GF(2)$ ) the parity check bit

$$b = a_1 + a_2 + a_3 + a_4.$$

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Now our encoded message becomes

$$[a_1, a_2, a_3, a_4, a_5],$$

where  $a_5 = b = a_1 + a_2 + a_3 + a_4$ . It can detect a single-bit error.

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It is in fact a homogeneous linear equation (in  $GF(2)$ ):

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Now, what is the set of all possible codewords?

## Hamming code

You can detect and correct more errors with Hamming codes. In this version called a  $[7, 4]$  Hamming code, you encode 4-bit data  $[a_1, a_2, a_3, a_4]$  into a 7-bit codeword  $[p_1, p_2, a_1, p_3, a_2, a_3, a_4]$ . Using the formula:

$$p_1 = a_1 + a_2 + a_4$$

$$p_2 = a_1 + a_3 + a_4$$

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Let's see how this works.

# Hamming code (encoding as matrix multiplication)



## Parity check

Suppose that we are given  $[d_1, d_2, d_3, d_4, d_5, d_6, d_7]$  Let

$$s_1 = d_1 + d_3 + d_5 + d_7$$

$$s_2 = d_2 + d_3 + d_6 + d_7$$

$$s_3 = d_4 + d_5 + d_6 + d_7$$

Given a codewords  $\mathbf{w} = [c_1, c_2, \dots, c_7]$ , if we compute  $s_1, s_2, s_3$ , we would get all zero's.

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What if there is an error? Let's try.

# Hamming code (parity check as matrix multiplication)

## Codewords from Hamming code

Turning the formula for  $p_1, p_2, p_3$  around, we have 3 homogeneous linear equations:

$$d_1 + d_3 + d_5 + d_7 = 0$$

$$d_2 + d_3 + d_6 + d_7 = 0$$

$$d_4 + d_5 + d_6 + d_7 = 0$$

and again the set of all possible codewords  $\mathcal{W}$  forms a vector space over  $GF(2)$ .

## Can you solve $2 \times 2$ Lights out?

Let  $\mathbf{u}_1 = [1, 1, 1, 0]$ ,  $\mathbf{u}_2 = [1, 1, 0, 1]$ ,  $\mathbf{u}_3 = [1, 0, 1, 1]$ , and  $\mathbf{u}_4 = [0, 1, 1, 1]$ .

Given  $\mathbf{b} = [b_1, b_2, b_3, b_4]$ , can you always find  $a_1, a_2, a_3, a_4 \in GF(2)$  such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

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$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

**Same question:** Is  $\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = GF(2)^4$ ?

## Can you solve $2 \times 2$ Lights out?

Let's try with an example. Let  $\mathbf{b} = [1, 0, 0, 0]$ . Can you find  $a_1, a_2, a_3, a_4 \in GF(2)$  such that

$$a_1 \cdot \mathbf{u}_1 + a_2 \cdot \mathbf{u}_2 + a_3 \cdot \mathbf{u}_3 + a_4 \cdot \mathbf{u}_4 = \mathbf{b}?$$

## Can you solve $2 \times 2$ Lights out?

Since

$$[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \in \text{Span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \},$$

and



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$$\text{Span} \{ [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \} = GF(2)^4,$$

what can we say about  $\text{Span} \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \}$ ?

# Generators

## Definition

Let  $\mathcal{V}$  be a set of vectors. Consider vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ .

If  $\text{Span} \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} = \mathcal{V}$ , we say that

- ▶  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  is a **generating set** for  $\mathcal{V}$
- ▶ vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  are **generators** for  $\mathcal{V}$

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## Examples

## Standard generators

Note that  $\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$  are generators for  $GF(2)^4$ .  
Why?

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They are called **standard generators** for  $GF(2)^4$ , written as  $e_1, e_2, e_3, e_4$ .

For  $\mathbb{R}^n$ , we also have  $[1, 0, 0, \dots, 0], [0, 1, 0, \dots, 0], [0, 0, 1, \dots, 0], \dots, [0, 0, 0, \dots, 1]$  as standard generators.

# Generators and spans

## Lemma 1

*Consider vectors  $u_1, u_2, \dots, u_n$ . If  $v_1, v_2, \dots, v_k$  are generators for  $\mathcal{V}$ , and for each  $i$ ,*

$$v_i \in \text{Span} \{u_1, u_2, \dots, u_n\},$$

*we have that  $\mathcal{V} \subseteq \text{Span} \{u_1, u_2, \dots, u_n\}$ .*



## Adding a vector into a span

### Lemma 2

*Consider vectors  $u_1, u_2, \dots, u_n$ . If  $v \in \text{Span} \{u_1, u_2, \dots, u_n\}$ , then*

$$\text{Span} \{u_1, u_2, \dots, u_n, v\} = \text{Span} \{u_1, u_2, \dots, u_n\}$$

## Geometry of spans: in $\mathbb{R}^2$

## Geometry of spans: in $\mathbb{R}^3$

## Two representations

There are two ways to represent a line, a plane, and a (hyper)plane, passing through the origin:

- ▶ as a span of vectors
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What are common properties of these geometric objects?

- ▶ they pass through the origin,
- ▶ if vector  $\mathbf{u}$  is in the objects,  $\alpha\mathbf{u}$  for any scalar  $\alpha$  is also in the objects, and
- ▶ if  $\mathbf{u}$  and  $\mathbf{v}$  are in the objects,  $\mathbf{u} + \mathbf{v}$  is also in the objects.

# Vector spaces

## Definition

A set  $\mathcal{V}$  of vectors over  $\mathbb{F}$  is a **vector space** iff

- ▶ (V1)  $\mathbf{0} \in \mathcal{V}$ ,
- ▶ (V2) for any  $\mathbf{u} \in \mathcal{V}$ ,

$$\alpha \cdot \mathbf{u} \in \mathcal{V}$$

for any  $\alpha \in \mathbb{F}$ , and

- ▶ (V3) for any  $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ ,

$$\mathbf{u} + \mathbf{v} \in \mathcal{V}.$$

# Span of vectors is a vector space

Consider  $n$ -vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ ,

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Let's check if properties V1, V2, and V3 are satisfied.

# Solutions to homogeneous linear equations is a vector space

Consider a set  $\mathcal{S}$  of all  $n$ -vectors in the form  $[x_1, x_2, \dots, x_n]$  where

$$\begin{aligned}a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n &= 0 \\a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n &= 0 \\&\vdots = \vdots \\a_{m1}x \cdot 1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n &= 0\end{aligned}$$

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# Dot product

## Definition

For  $n$ -vectors  $\mathbf{u} = [u_1, u_2, \dots, u_n]$  and  $\mathbf{v} = [v_1, v_2, \dots, v_n]$ , the **dot product** of  $\mathbf{u}$  and  $\mathbf{v}$ , denoted by  $\mathbf{u} \cdot \mathbf{v}$ , is

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Using dot products, the previous set  $\mathcal{S}$  can be written as

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_1 \cdot \mathbf{x} = 0, \mathbf{a}_2 \cdot \mathbf{x} = 0, \dots, \mathbf{a}_m \cdot \mathbf{x} = 0\}$$

and we know that  $\mathcal{S}$  is a vector space.

An object not passing through the origin: 2 dimensions

An object not passing through the origin: 3 dimensions

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- ▶ Then we translate it back so that it passes through  $\mathbf{a}$ .

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- ▶ We get the set

$$\mathcal{A} = \{\mathbf{a} + \mathbf{u} : \mathbf{u} \in \mathcal{V}\}$$

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- ▶ *Question:* Is  $\mathcal{A}$  a vector space?
- ▶ We also write it as  $\mathbf{a} + \mathcal{V}$ .

# Affine spaces

## Definition

If  $\mathbf{a}$  is a vector and  $\mathcal{V}$  is a vector space, then

$$\mathbf{a} + \mathcal{V}$$

is an **affine space**.

## An affine space and convex combination: 2 dimensions

# An affine space and convex combination: 3 dimensions



# Affine combination

## Definition

For any scalars  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ , we say that a linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is an **affine combination** of  $\mathbf{u}_1, \dots, \mathbf{u}_m$ .

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## Definition

The set of all affine combinations of vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  is called the **affine hull** of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ .

# Convex combination: review

## Definition

For any scalars  $\alpha_1, \alpha_2, \dots, \alpha_m \geq 0$  such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$$

and vectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ , we say that a linear combination

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# Writing an affine space using a span

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## An affine space

An affine space passing through  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$  is

$$\mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1, \dots, \mathbf{u}_n - \mathbf{u}_1 \}.$$

# Non-homogeneous linear system

Two linear systems:

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= b_2 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= b_m \end{aligned}$$

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

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What can you say about the solution sets of these two related linear systems?

$\mathbf{0}$  is always a solution to the linear system on the right.

Note: A linear equation whose right-hand-side is zero is called a **homogeneous linear equation**. A system of linear homogeneous equations is called a **homogeneous linear system**.

## Solutions of the two systems

Recall that if  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are both solutions to the non-homogeneous linear system, we have that for any  $i$

$$\mathbf{a}_i \mathbf{u}_1 - \mathbf{a}_i \mathbf{u}_2 = b_i - b_i = 0 = \mathbf{a}_i (\mathbf{u}_1 - \mathbf{u}_2).$$



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This implies that  $\mathbf{u}_1 - \mathbf{u}_2$  is a solution to the homogeneous linear system.

Suppose that  $\mathcal{W}$  is the set of all solution to the non-homogeneous linear system, i.e.,

$$\mathcal{W} = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} = b_i, \text{ for } 1 \leq i \leq m\},$$

and let  $\boldsymbol{u} \in \mathcal{W}$  be one of the solutions, we have that

$$\{\boldsymbol{v} - \boldsymbol{u} : \boldsymbol{v} \in \mathcal{W}\}$$

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In other words,

$$\begin{aligned} \mathcal{W} &= \mathbf{u} + \{\mathbf{v} - \mathbf{u} : \mathbf{v} \in \mathcal{W}\} \\ &= \mathbf{u} + \{\mathbf{x} : \mathbf{a}_i \mathbf{x} = 0, \text{ for } 1 \leq i \leq m\}, \end{aligned}$$

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i.e.,  $\mathcal{W}$  is an affine space.

# Solutions to a non-homogeneous linear system

## Lemma 3

*If the solution set of a linear system is not empty, it is an affine space.*