01204211 Discrete Mathematics
Lecture 12a: Undecidability (1)

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Non-context-free languages

While

$$\{\mathsf{0}^n\mathsf{1}^n \mid n \geq 0\}$$

is context free, the language

$$\{\mathsf{0}^n\mathsf{1}^n\mathsf{0}^n \mid n \geq 0\}$$

is not.

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is context free, the language

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is not.

Can we write a python program to check if a string w belongs to the language $\{\mathsf{0}^n\mathsf{1}^n\mathsf{0}^n \mid n \geq 0\}$?

Big question

Is there a python program that "solves" any possible problem?

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Is there a python program that "solves" any possible problem?
Can a computer solve any problem?
Is there an algorithm that solves every problem?
What is the limit of computation?

language.

$\forall$ language L,
มี algorithm ที่
solve L เสมอ
หรือไม่?

Answer by a counting argument

If there are “more” problems than any possible algorithms, then there should be some problem that algorithms cannot solve.
(Think of an algorithm as “a python program.)

Answer by a counting argument

If there are “more” problems than any possible algorithms, then there should be some problem that algorithms cannot solve.
(Think of an algorithm as “a python program.)
However, there are infinitely many python programs and there are infinitely many problems. It is not obvious how to much such an argument formally.

Bijections

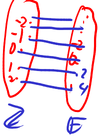
$\mathbb{Z}$ จน.เต็ม

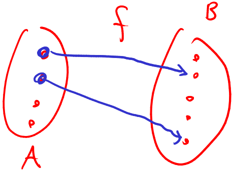
$\mathbb{N}$ จน.นับ.

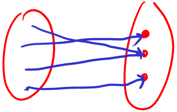
$\mathbb{E}$ จน.เต็มคู่

$f: \mathbb{Z} \to \mathbb{E}$

$f(x) = 2x$







Definition

- A function $f : A \longrightarrow B$ from domain A to range B is **one-to-one** if for any $x \neq y \in A$, $f(x) \neq f(y)$.
- A function $f : A \longrightarrow B$ from domain A to range B is **onto** if for any $x' \in B$, there exists $x \in A$ such that $f(x) = x'$.
- A function $f : A \longrightarrow B$ is a **bijection** (or bijective) if it is one-to-one and onto.

ถ้า A & B มีขนาดจำกัด

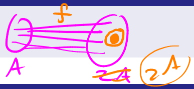
- one-to-one $\Rightarrow |A| \leq |B|$
- onto $\Rightarrow |A| \geq |B|$

$\Big\}$ bijective $|A| = |B|$

Bijection: examples

Lemma 1

For any set A, there is no bijective function $f : A \longrightarrow 2^A$.



Proof.

We prove by contradiction. Assume that there exists a bijective function f from A to 2^A. We construct a set $B \subseteq A$ such that there is no $x \in A$ such that $f(x) = B$.

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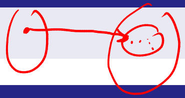

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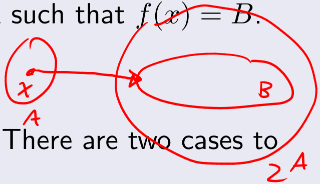
$$B = \{x \in A \mid \underline{x \notin f(x)}\}.$$



Now suppose that there exists $x \in A$ such that $\underline{f(x) = B}$. There are two cases to consider:

Case 1: $\underline{x \in B}$. . . . .

$f(x) = B \Rightarrow x \in f(x)$ เป็นไปไม่ได้

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Now suppose that there exists $x \in A$ such that $f(x) = B$. There are two cases to consider:
Case 1: $x \in B$.
Case 2: $\underline{x \notin B}$. $\quad x \notin \overset{B}{\overset{\shortparallel}{f(x)}} \Rightarrow \underline{x \in B}$

Lemma 1

For any set A, there is no bijective function $f : A \longrightarrow 2^A$.

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We define B as follows.

$$B = \{x \in A \mid x \notin f(x)\}.$$

Now suppose that there exists $x \in A$ such that $f(x) = B$. There are two cases to consider:
Case 1: $x \in B$.
Case 2: $x \notin B$.

→ ไม่มี x ที่ $f(x) = B$

In both case, we have a contradiction; therefore, our assumption is false. Thus, there is no bijection between A and 2^A. □

Example: finite set

Let $A = 1, 2, 3, 4, 5, 6, 7$. Consider function $f : A \longrightarrow 2^A$ defined as

$\{x \in A \mid x \notin f(x)\}$

$$
\begin{aligned}
f(1) &= \{\} && 1 \notin f(1) \ \checkmark \\
f(2) &= \{1, 2, 3\} && 2 \in f(2) \\
f(3) &= \{1, 2, 3, 4, 5, 6, 7\} && 3 \in f(3) \\
f(4) &= \{1, 3, 5, 7\} && 4 \notin f(4) \ \checkmark \\
f(5) &= \{2, 4, 6\} && 5 \notin f(5) \ \checkmark \\
f(6) &= \{7\} && 6 \notin f(6) \ \checkmark \\
f(7) &= \{1, 2, 3\} && 7 \notin f(7) \ \checkmark
\end{aligned}
$$

ไม่มี $y \in A$

ที่ $f(y) = B$

$B = \{1, 4, 5, 6, 7\}$

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\end{aligned}
$$

	1	2	3	4	5	6	7
1	Y						
2	✓	✓ N	✓				
3	✓	✓	✓ N	✓	✓	✓	✓
4	✓		✓	Y	✓		✓
5		✓		✓	Y	✓	
6						Y	✓
7	✓	✓	✓				Y

$B =$

$B = \{1 \quad 4 \quad 5 \quad 6 \quad 7\}$

Example: infinite set

Let $A = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Consider function $f : A \longrightarrow 2^A$ defined as

$$
\begin{array}{lcl}
f(1) & = & \{\} \\
f(2) & = & \{1, 2, 3\} \\
f(3) & = & \{1, 2, 3, 4, 5, 6, 7, \ldots\} \\
f(4) & = & \{1, 3, 5, 7, \ldots\} \\
f(5) & = & \{2, 4, 6, \ldots\} \\
f(6) & = & \{7\} \\
f(7) & = & \{1, 2, 3, 11, 12, 13, 21, 22, 23, \ldots\} \\
\vdots & & \vdots
\end{array}
$$

$B =$

	1	2	3	4	5	6	7	$\cdots$
1								
2								
3								
4								
5								
6								
7								
$\vdots$								

i

2^A

The previous lemma informally states that there are "more" subsets than the number of elements in the set. A

Let's think about:

- A set of all python programs, and
- A set of all languages.

$\subseteq$ set ของ strings ทั้งหมด A

language $L \subseteq A$

set of all languages $= 2^A$

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Since each python program “solves” at most one language, there are not “enough” python programs to solve all possible language.

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Since each python program “solves” at most one language, there are not “enough” python programs to solve all possible language.
But what exactly is a problem that cannot be “solved”?

Decision problems

- Given an integer x, is x odd?
- Given a string w, is w palindrome?
- Given a string w, is $w \in \{0^n 1^n \mid n \geq 0\}$?
- Given a map, a starting position s, a destination t, and an integer k, does there exist a path from s to t with distance at most k?
- Given a program P and input string w, when running P with w as an input, does P terminate?

Decision problems and languages

For this problem:

Given an integer x, is x ~~odd~~ even?

we can define a corresponding language

$$L_E = \{, \ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}.$$

To solve this problem, given x, we can ask if $x \in L_E$.

Languages and programs

We will talk about languages of particular programs. For example, let $\mathbb{P}$ be the set of all python programs. In this case, $\mathbb{P}$ is a language.

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$$\Downarrow$$

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$$\{(P, x) \mid P \in \mathbb{P}, P(x) \text{ terminates}\}$$

$$\{(P, Q, x) \mid P, Q \in \mathbb{P}, P(x) \text{ and } Q(x) \text{ terminate with the same output.}\}$$

Programs and inputs

```
x = int(input())
if x % 2 == 0:
    print('yes')
else:
    print('no')
```

Programs and inputs

```
$ python le.py
10
yes
$ python le.py
7
no
```

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fjdsklfjsdf
Traceback (most recent call last):
  File "le.py", line 1, in <module>
    x = int(input())
ValueError: invalid literal for int()
with base 10: 'fjdsklfjsdf'
```

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```

```
$ python le.py < le.py
```

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Nice programs

We can systematically modify any python program P so that

- P contains a main function that works with the input as a string.
- P never crashes. (If the original P crashes, the modified P outputs `no`.)

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if x % 2 == 0:
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else:
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```

```
import sys
def main(w):
    try:
        x = int(w)
        if x % 2 == 0:
            print('yes')
        else:
            print('no')
    except:
        print('no')

if __name__ == '__main__':
    w = sys.stdin.read()
    main(w)
```

When running a program

When you run a program P with input x, there are three possible outcomes:

- P terminates and outputs **yes**,
- P terminates and outputs **no**, and
- P does not terminate. (It runs forever.)

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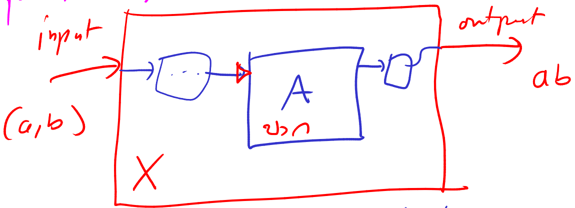
Remarks: if P crashes (even after modification), we treat it as if it terminates and outputs **no**.

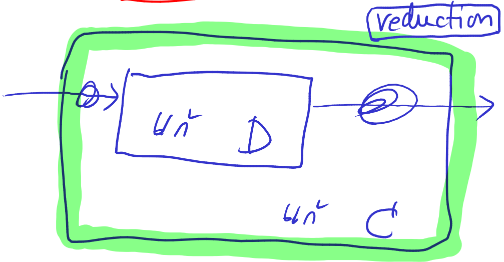
Proving impossibility

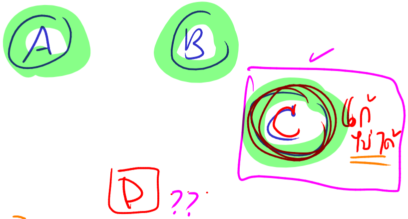
ปัญหา (language)

ที่ ไม่มี โปรแกรม python แก้ได้

possible → แสดงโปรแกรม







① ถ้าแก้ C ได้, จะแก้ D ได้

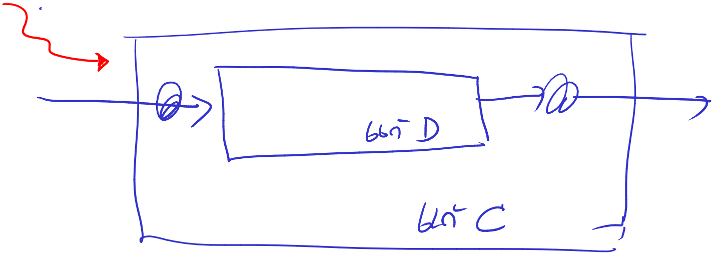
② ถ้าแก้ D ได้ จะแก้ C ได้

⇒

Reduction: rough idea

- ทราบว่า ปัญหา C แก้ไม่ได้
- อยาก พิสูจน์ว่า ปัญหา D แก้ไม่ได้

แสดงว่า: ถ้าแก้ D ได้, จะแก้ C ได้



ใช้ algo ที่แก้ D เป็น subroutine (function)

Language A

P.py

ท่ python p.py < p.py

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We use a function call notation $P(x)$ when refering to the execution of program P with input x.
We restate the definition of A as

$$\{P \in \mathbb{P} \mid P(P) \text{ terminates}\}.$$

Deciders

แก้ปัญหา

We say that a python program P **decides** the language L if for any input string x, P when running with x as an input,

- P always terminates,
- P outputs **yes**, if $x \in L$, and
- P outputs **no**, if $x \notin L$.

Deciders: more examples

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We restate the definition of A as

$$\{\underline{P} \in \mathbb{P} \mid \underline{P(P)} \text{ terminates}\}.$$

Not a decider for A

```
Input: python program P (as a string)

1. Load module P as Pmod
2. Call Pmod.main(P)
3. print('yes')             # we reach this line,
                            #  only if M.main(P) terminates
```

P(P)

Lemma 2

There is no python program that decides A.

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We will see the proof at the end of class.

Undecidability

If we believe that anything that a computer can do can be written as a python program,

Undecidability

If we believe that anything that a computer can do can be written as a python program, and there is no python program that decides A, when we say that

A is undecidable.

Language A will be very important later on, we give it a proper name as $\boxed{\textsc{HaltA}}$.

The proof as a table

List all python programs in $\mathbb{P}$ as $P_1, P_2, P_3, \ldots$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	$\ldots$
P_1								
P_2								
P_3								
P_4								
P_5								
P_6								
$\vdots$								
(B)								

What does B do on each input program P_i?

Another language HALT

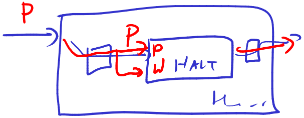
$$\rightarrow \text{HALTA} = \{ P \mid P \in \mathbb{P},\ P(P) \text{ terminates} \}$$

Let

$$\text{HALT} = \{(P, w) \mid P \text{ is a python program such that } P(w) \text{ terminates}\}$$

We shall prove that HALT is also undecidable (if we believe that python programs represent all possible computation). แก้ไม่ได้

ถ้าแก้ HALT ได้, จะแก้ HALTA ได้



สร้างโปรแกรมที่แก้ HALTA

โดยใช้โปรแกรมที่แก้ HALT

Lemma 3

HALT *is undecidable.*

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides HALT.

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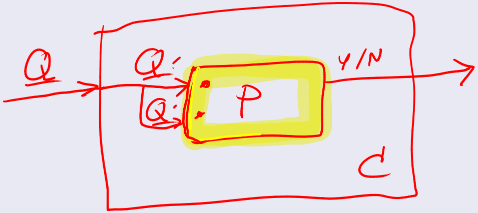
Su

We construct a program C as follows → decidas HALTA

```
Program C
Input Q
1.    Load P as module Pmod
2.    if Pmod.main(Q,Q) == 'yes':
3.        print('yes')
4.    else
5.        print('no')
```



□

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides HALT.
We construct a program C as follows

קוד input

```
Program C
Input Q
1.    if P(Q,Q) == 'yes':
2.        print('yes')
3.    else
4.        print('no')
```

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides $\textsc{Halt}$.
We construct a program C as follows

```
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Input Q
1.   if P(Q,Q) == 'yes':
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```

Given program P, we can construct a program C that decides $\textsc{HaltA}$.

Proof.

We prove the lemma by contradiction. Assume that there is a python program P that decides $\textsc{Halt}$.
We construct a program C as follows

```
Program C
Input Q
1.   if P(Q,Q) == 'yes':
2.       print('yes')
3.   else
4.       print('no')
```

Given program P, we can construct a program C that decides $\textsc{HaltA}$. However, we know that $\textsc{HaltA}$ is undecidable; thus, we reach a contradiction.
We conclude that there does not exist a python program P that decides $\textsc{Halt}$. $\square$

Reduction

Reduction

- We show that if HALT is decidable, then HALTA is also decidable.

มี $\boxed{P}$ ที่ decides HALT
(แล้ว)

มี $\boxed{C}$ ที่ decide HALTA
(แล้ว)

Reduction

- We show that if HALT is decidable, then HALTA is also decidable.
- However, HALTA IS UNDECIDABLE.

Reduction

- We show that if HALT is decidable, then HALTA is also decidable.
- However, HALTA IS UNDECIDABLE.
- We conclude that HALT is also undecidable.

Reduction in picture

HALT $= \{(P,w) \mid P \in \mathbb{P},\ P(w)$ terminates $\}$

ans yes

HALT(A) $= \{ P \mid P \in \mathbb{P}$

$P(P)$

termt$\}$

Let $\textsc{Accept} = \{(P, w) \mid P \in \mathbb{P}$ and $P(w)$ terminates with acceptance$\}$.

Lemma 4

$\textsc{Accept}$ is undecidable.

Proof.

We prove the lemma by contradiction. Assume that there is a python program Q that decides $\textsc{Accept}$.

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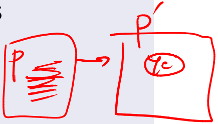


```
Program C    → จะถามว่า P(w) terminate?
Input P, w
1.    Replace every "print('no')" statement in P with "print('yes')"
1.    if Q(P,w) == 'yes':
2.       print('yes')
3.    else
4.       print('no')
```

Proof (cont.)

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Program C
Input P,w
1.   Replace every "print('no')" statement in P with "print('yes')"
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We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.

→ โปรแกรม P หลังแก้ไข จะตอบ yes $\Rightarrow$ $Q(P,w) = \text{yes}$

$\Rightarrow$ C จะตอบ yes

Proof (cont.)

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We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.

ถ้า P แบบเก่า ไม่ จะไม่ halt ด้วย, $Q(P,w) == $ 'no'

$\Rightarrow$ C ตอบ no

Proof (cont.)

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Program C
Input P,w
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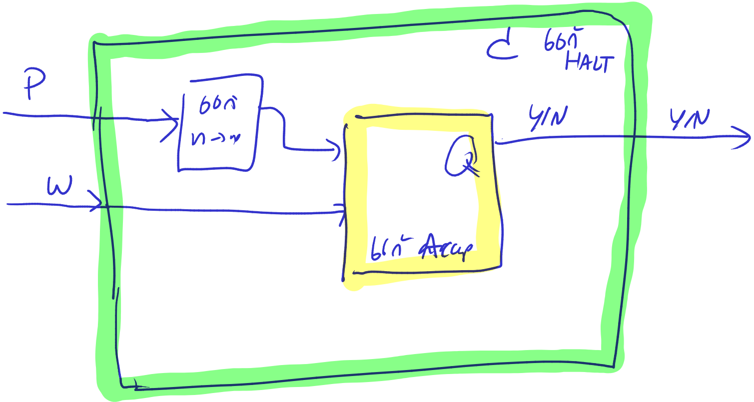
We have to make sure that our reduction is correct by considering two cases.
Case 1: when $P(w)$ halts.
Case 2: when $P(w)$ does not halt.
Since in both cases, C answers correctly, we know that given program Q deciding ACCEPT, we can construct a program C that decides HALT. However, we know that HALT is undecidable; thus, we reach a contradiction. We conclude that ACCEPT is also undecidable. □

Reduction from HALT to ACCEPT in picture



How about $\textsc{Reject}$?

Let

$$\boxed{\textsc{Reject}} = \{(P, w) \mid P \in \mathbb{P} \text{ and } P \text{ rejects } w\}.$$

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There is no python program that decides A.

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We prove by contradiction. Assume that there is a python program P that decides A.

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We prove by contradiction. Assume that there is a python program P that decides A. We describe a python program B that reads a string Q as an input as follows:

```
Program B
Input Q
1.    Load P as module Pmod
2.    if Pmod.main(Q) == 'yes':     # when Pmod outputs yes
3.        while True: pass          #   loop forever
4.    else:                         # when Pmod outputs no
5.        quit()                    #   halt
```

Given program Q as an input, B loops forever when

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Given program Q as an input, B loops forever when

It terminates when

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Proof.

We know that

- $B(Q)$ loops when $Q(Q)$ terminates, and
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Does running B using B as an input terminate?

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Since either $B(B)$ loops or terminates, and we cannot be in any of the cases, we obtain a contradiction.
Therefore, we conclude that program P does not exist. $\square$

Python as computation

Do you believe in this assumption:

Anything that a computer can do can be written as a python program.

Turing machines

Anything that a computer can do can be carried out using Turing machines.

Turing machines

Anything that a computer can do can be carried out using Turing machines.

Any possible computation can be performed by Turing machines.