

01204211 Discrete Mathematics

Lecture 8b: Vectors and Matrices

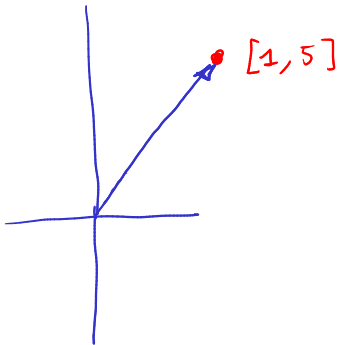
Jittat Fakcharoenphol

October 1, 2024

What is a vector?

You can think of a **vector** as an “ordered” list of elements (which are typically numbers). For example:

- ▶ $[1, 2, 5, 20]$
- ▶ $[0, 0, 1, 1, 0, 0, 0, 1]$



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You can also view a vector as a **function**, e.g., you can view $u = [1, 2, 5, 20]$ as a function u that maps

$$0 \mapsto 1, \quad 1 \mapsto 2, \quad 2 \mapsto 5, \quad 3 \mapsto 20.$$

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Each element in the vector is typically a real number (\mathbb{R}), but can be an element from other sets with appropriate property (more on this later).

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Each element in the vector is typically a real number (\mathbb{R}), but can be an element from other sets with appropriate property (more on this later).

Remark: Mathematically, a vector is an element of a vector space. We will understand this more later.

What can be represented as a vector?

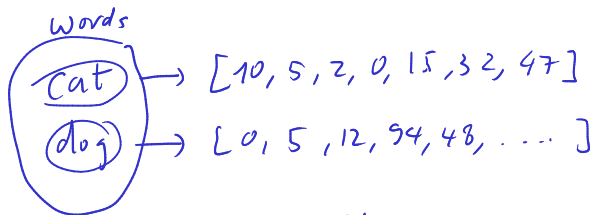


- Weight 76
- height 192
- age 18
- Shoe size 7.5
- # of girlfriends 0.5

$[76, 192, 18, 7.5, 0.5]$

1 data point.

Applications in machine learning

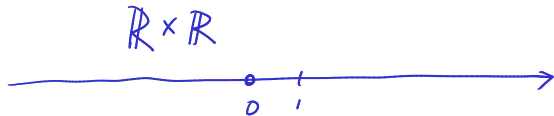
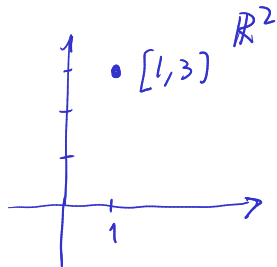


word embedding.

token embedding.

sentence embedding

Viewing vectors: vectors in \mathbb{R}^2



$[1, 3]$

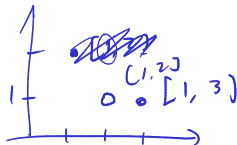
python str
ver

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

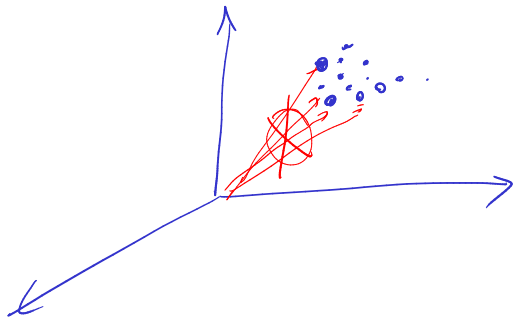
column vect

$[1 \ 3]$

row vector



Viewing vectors: vectors in \mathbb{R}^3 ~ space



n -vectors over \mathbb{R}

- ▶ We mostly deal with vectors with finite number of elements.
- ▶ This is a 4-vector: $[10, 20, 500, 4]$.

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n -vectors over \mathbb{R}

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- ▶ This is a **4-vector**: $[10, 20, 500, 4]$.
- ▶ We sometimes also write it as a column vector:

$$\begin{bmatrix} 10 \\ 20 \\ 500 \\ 4 \end{bmatrix}$$

- ▶ When every element of a vector is from some set, we say that it is a vector **over** that set. For example, $[10, 20, 500, 4]$ is a 4-vector over \mathbb{R} .

Vector operations

- ▶ As discussed in the previous slides, when working with a system of linear equations, we mostly deals with **linear combinations** of vectors.
- ▶ We will look at the operations we do to vectors to obtain their linear combinations.

Vector operations

- ▶ As discussed in the previous slides, when working with a system of linear equations, we mostly deal with **linear combinations** of vectors.
- ▶ We will look at the operations we do to vectors to obtain their linear combinations.
- ▶ The operations are:
 - ▶ Vector additions
 - ▶ Scalar multiplications
- ▶ These operations motivate the definition of vector spaces.

Vector additions

Given two n -vectors

size of the vector = n

$$\underline{\mathbf{u}} = [u_1, u_2, \dots, u_n]$$

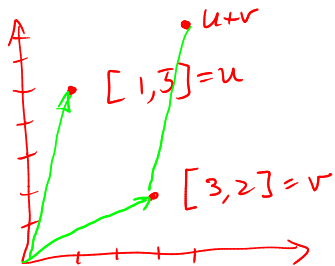
and

$$\underline{\mathbf{v}} = [v_1, v_2, \dots, v_n],$$

we have that

$$\underline{\mathbf{u} + \mathbf{v}} = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n].$$

Vector additions, in picture



$$u+v = [1+3, 5+2] = [4, 7]$$

Zero vectors

A zero n -vector $\mathbf{0} = [0, 0, \dots, 0]$ is an additive identity, i.e., for any vector \mathbf{u} ,

$$\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}.$$

Scalar multiplications

$$x_1 + x_3 = 20$$

$$x_1 + x_3 = 50$$

0.5u



+

0.5v



$[0, 1, 1, 0]$

$[0, 1, 1, 1]$



For a vector over \mathbb{R} , we refer to an element α in \mathbb{R} as a scalar. For an n -vector

$$\mathbf{u} = [u_1, u_2, \dots, u_n],$$

we have that

$$\alpha \cdot \mathbf{u} = [\alpha \cdot u_1, \alpha \cdot u_2, \dots, \alpha \cdot u_n],$$

Tana's secret mocktail.

- 10 drops of dragon tears
- 20 litres of Coca cola
- 50 ml of 100% alcohol.
- 70 spoons of red haló's blueberry

$[10, 20, 50, 70]$

$[1, 0, 0, 0.5]$

$$x_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 50 \\ 70 \end{bmatrix}$$

Scalar multiplications, in pictures

Linear combinations

For any scalar

$$\alpha_1, \alpha_2, \dots, \alpha_m$$

and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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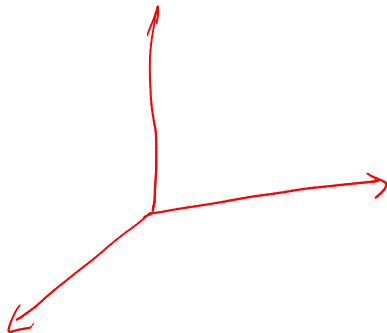
is a **linear combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

Examples:

A linear system with 3 variables

Give the following linear system.

$$\begin{array}{rcccccccl} 2x_1 & + & 4x_2 & + & 3x_3 & = & 7 & \leftarrow \\ x_1 & + & & & 5x_3 & = & 12 & \leftarrow \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 10 & \leftarrow \end{array}$$



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If we rewrite the system as

$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \cdot x_3 = \begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$$

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This becomes the problem of expressing a vector as linear combination of other vectors. I.e., given vectors

$$\mathbf{u}_1 = [2, 1, 4], \quad \mathbf{u}_2 = [4, 0, 2], \quad \mathbf{u}_3 = [3, 5, 3]$$

we would like to find coefficients x_1, x_2, x_3 such that

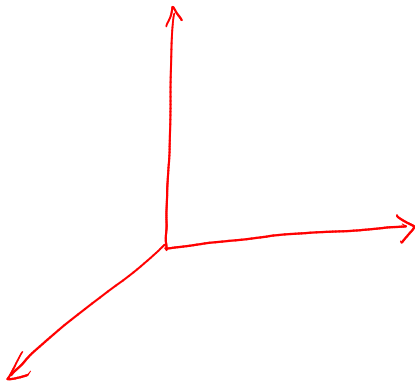
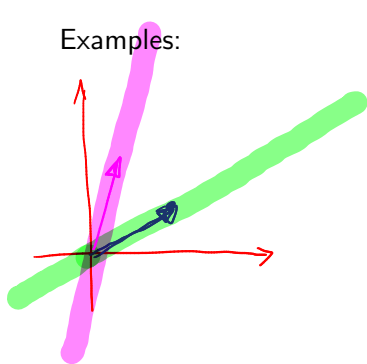
$$x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2 + x_3 \cdot \mathbf{u}_3 = [7, 12, 10].$$

Span

A set of all linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$ is called the **span** of that set of vectors.

It is denote by $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$.

Examples:



Convex combination

For any scalar

$$\alpha_1, \alpha_2, \dots, \alpha_m,$$

such that $\alpha_1 + \alpha_2 + \dots + \alpha_m = 1$ and $\alpha_i \geq 0$ for all i , and vectors

$$\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m,$$

we say that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_m \mathbf{u}_m$$

is a **convex combination** of $\mathbf{u}_1, \dots, \mathbf{u}_m$.

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Examples:

What is a matrix?

Matrices arise in many places. We will see that there are essentially two ways to look at matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

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A matrix from a system of linear equations

Consider the following system of linear equations:

$$\begin{array}{rrcrcl} x_1 & + & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & x_2 & + & 2x_3 & = & 10 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 4 \end{array}$$

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Again we can view it as a vector equation:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix}$$

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The coefficients form a nice rectangular “matrix” A .

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix},$$

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix},$$

and rewrite the system as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 4 \end{bmatrix}$$

Size

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Size

$$\begin{matrix} & n \\ m & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix} \end{matrix}$$

The **size** of a matrix is determined by the number of rows and columns. A matrix with m rows and n columns is referred to as an m -by- n matrix or an $m \times n$ matrix. We refers to m and n as its **dimensions**.

Matrix-Vector Multiplication

How would we understand the multiplication

$$\overset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

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By rows. Consider the first row of A :

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By rows. Consider the first row of A :

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3.$$

Let's look at another two rows:

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Let's look at another two rows:

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3, \quad \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix-Vector Multiplication

How would we understand the multiplication

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \text{red oval} \\ \text{blue oval} \\ \text{purple oval} \end{bmatrix}$$

By rows. Consider the first row of A :

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Let's look at another two rows:

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3, \quad \begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3,$$

Matrix-Vector Multiplication by Rows

We look at matrix-vector multiplication with “row perspective”. This is a common way to view matrix-vector multiplication.

$$\begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Recall:

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$$\begin{bmatrix} 1 & 1 & 1 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ \end{bmatrix}$$

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Recall:

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$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

Recall:

$$\underline{\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3,$$

Review: Dot product

Definition

For n -vectors $\mathbf{u} = [u_1, u_2, \dots, u_n]$ and $\mathbf{v} = [v_1, v_2, \dots, v_n]$, the **dot product** of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} \cdot \mathbf{v}$, is

$$u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

Matrix-Vector Multiplication by Rows

We look at matrix-vector multiplication with “row perspective”, which can be written nicely with **dot product**.

I.e., from:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

we have

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \mathbf{r}_3 \cdot \mathbf{x} \end{bmatrix},$$

where

$$\mathbf{r}_1 = [1 \quad 1 \quad 1], \quad \mathbf{r}_2 = [2 \quad 1 \quad 2], \quad \mathbf{r}_3 = [3 \quad 1 \quad 2].$$

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we have

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where

$$\mathbf{r}_1 = [1 \quad 1 \quad 1], \quad \mathbf{r}_2 = [2 \quad 1 \quad 2], \quad \mathbf{r}_3 = [3 \quad 1 \quad 2].$$

Dot-product perspective

The matrix-vector product is a vector of **dot products** between each rows and the vector.

Matrix-Vector Multiplication **by Columns**

However, another nice way to look at matrix-vector multiplication is **by columns**.

Notice that:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \\ 2 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \\ 3 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 \end{bmatrix}$$

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can be written as

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Matrix-Vector Multiplication **by Columns**

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Linear combination perspective

The matrix-vector product is a **linear combination** of column vectors.

Two perspectives: Matrix-Vector multiplication

Dot products between rows and the vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

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Dimensions

If the matrix has n columns, the vector should be an n -vector.

Document search

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- ▶ You need some way to measure document **similarity**.
- ▶ Suppose that you have N documents in the library: d_1, d_2, \dots, d_N . Given a query document q , you want to find document d_i that maximize

$$\text{sim}(d_i, q),$$

where $\text{sim}(d, d')$ is the similarity score between documents d and d' .

Document vector models

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Let's start with an example. Suppose that we only care about 5 words: dog, cat, food, restaurant, and coffee.

Consider the following 4 (very short) documents:

- ▶ d_1 : People love pets. Most famous pets are cats and dogs.
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How can we translate these sets into vectors?

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Words: dog, cat, food, restaurant, and coffee.

Suppose that we have query document:

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How can we define “similarity” measure?

Dot products as a similarity measure

From the previous example, we see that the dot products between \mathbf{d}_i 's and \mathbf{q} count the number of common words.

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- ▶ We can increase our “dictionary”'s size to include more words.

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This simple idea can be extended in many ways.

- ▶ We can increase our “dictionary”’s size to include more words.
- ▶ We can group similar words into the same “co-ordinates”.

Dot products as a similarity measure

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This simple idea can be extended in many ways.

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- ▶ We can group similar words into the same “co-ordinates”.
- ▶ In fact, the dot product measures the “angle” between vectors. For vectors over \mathbb{R} , we have that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta,$$

where θ is the angle between vectors \mathbf{u} and \mathbf{v} .

Computing all similarity scores

If we have documents $\mathbf{d}_1, \mathbf{d}_1, \dots, \mathbf{d}_N$, as vectors, and a query \mathbf{q} , how can we compute all similarity scores?

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By performing matrix-vector multiplication:

$$\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_N \end{bmatrix} \begin{bmatrix} \mathbf{q} \end{bmatrix} = \begin{bmatrix} \text{sim}(\mathbf{d}_1, \mathbf{q}) \\ \text{sim}(\mathbf{d}_2, \mathbf{q}) \\ \vdots \\ \text{sim}(\mathbf{d}_N, \mathbf{q}) \end{bmatrix}$$

Vector-matrix multiplication

Let's consider another direction.

What is

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} ?$$

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As dot products

Matrix-matrix multiplication

Consider

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Matrix-matrix multiplication (based on matrix-vector multiplication)

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Matrix transpose

If A is an $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix},$$

the **transpose** of A , denoted by A^T is an $n \times m$ matrix

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Remark: We usually view a vector as a column vector. Therefore, a dot product between m -vectors can be viewed also as a matrix multiplication:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Matrix multiplication and transpose

What is $(AB)^T$?

Key-Value database

Suppose you have a database of key-value pairs:

$$\{(somchai, 10), (somying, 14), (sommuk, 23), (somjai, 50), (somsom, -40)\}$$

Given a query q , you want to find a value v such that (q, v) is in the database. E.g.,

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Let's see how we could do that (very **inefficiently**) with matrix multiplication.

Vector encodings of keys and queries

- ▶ You want to have **distinct** keys: k_1, k_2, \dots, k_n
- ▶ You want a query q to **match** with an appropriate key. (Maybe the key which is exactly the same.)

Example

- ▶ Key encoding:

$$somchai = [0, 1, 0, 0, 0, 0], \quad somying = [0, 0, 0, 0, 1, 0], \quad somnuk = [1, 0, 0, 0, 0, 0],$$

$$somjai = [0, 0, 1, 0, 0, 0], \quad somsom = [0, 0, 0, 0, 0, 1]$$

- ▶ A value table (or vector): $v = \begin{bmatrix} 10 \\ 14 \\ 23 \\ 50 \\ -40 \end{bmatrix}$

- ▶ A query q is a 5-vector. A query matches key k_i if

$$k_i^T q = 1$$

Example (cont)

- ▶ Key matrix $K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
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► Let's try querying with various q

► The final formula is

$$(Kq)^T v = (q^T K^T) v$$

Key-Value database (with vector values)

Suppose you have a database of key-value pairs, where a value is a 2-vector:

$$\{(somchai, [10, 20]), (somying, [14, -2]), (sornnuk, [23, 3]), (somjai, [50, -10])\}$$

Given a query q , can you find a 2-vector v such that (q, v) is in the database?

Understanding self-attention formula

Self-attention mechanisms are key steps in transformers, work horses for all chatbots you have been using recently. The formula looks like (from wikipedia)

$$Attention(Q, K, V) = softmax\left(\frac{QK^T}{\sqrt{d_k}}\right) V$$