

01204211 Discrete Mathematics

Lecture 8b: Modular arithmetic

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Quick check 1

If $a|m$ and $b|m$, can we say that $ab|m$? Prove this fact or provide a counter example.

Quick check 2

If $a|m$, $b|m$, and $a \neq b$ are both prime, can we say that $ab|m$? Prove this fact or provide a counter example.

Prime factorization

One useful fact that we use over and over again is the following.

Unique Factorization (or Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be written *uniquely* as a product of prime numbers (up to the order of factors).

Examples:

▶ $10 = 2 \cdot 5$

▶ $13 = 13$

▶ $112 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 7 = 2^4 \cdot 7$

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When inputs contain a few numbers

GCD and Power

Days

What day is it today?

Days

What day is it today? Thursday.

Days

What day is it today? Thursday.
What day is 3 days after today?

Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today?

Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

What day is 10 days before today?

Days

What day is it today? Thursday.

What day is 3 days after today? Sunday.

What day is 20 days after today? Wednesday.

What day is 10 days before today? Monday.

Clocks

Suppose that it is 1 o'clock.

Clocks

Suppose that it is 1 o'clock.
What time is the next 5 hours?

Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours?

Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

What time is the next 20 hours?

Clocks

Suppose that it is 1 o'clock.

What time is the next 5 hours? 6 o'clock.

What time is the next 10 hours? 11 o'clock.

What time is the next 20 hours? 9 o'clock.

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$$4 + 5 = 9 \bmod m = 2.$$

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$$2 - 6 = -4 \bmod 7 = 3 \bmod 7 = 3.$$

Note that when you view integers under the lense of modulus 7, these numbers

$$\dots, -19, -12, -5, 2, 9, 16, 23, \dots$$

are essentially **the same**.

Properties (1)

$a \bmod m = b \bmod m$, if and only if $m \mid a - b$.

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$a \bmod m = b \bmod m$, if and only if $m|a - b$.

Proof.

(\Rightarrow) Let $r = a \bmod m$. We can write

$$a = qm + r,$$

and

$$b = pm + r,$$

for some integers q and p . Thus, we have

$$a - b = qm + r - pm - r = (q - p)m.$$

Therefore $m|a - b$.

(\Leftarrow) Exercise.



Properties (2)

- ▶ $(a + b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$
- ▶ $(a - b) \bmod m = ((a \bmod m) - (b \bmod m)) \bmod m$
- ▶ $(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$

Congruences

Definition (congruences)

For an integer $m > 0$, if integers a and b are such that

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we write

$$a \equiv b \pmod{m}.$$

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We also have that

$$a \equiv b \pmod{m} \quad \Leftrightarrow \quad m \mid (a - b)$$

Congruences: properties (1)

- ▶ (reflexivity)

$$a \equiv a \pmod{m}.$$

- ▶ (symmetry)

$$a \equiv b \pmod{m} \text{ implies } b \equiv a \pmod{m}.$$

- ▶ (transitivity)

$$a \equiv b \pmod{m} \text{ and } b \equiv c \pmod{m} \text{ implies } a \equiv c \pmod{m}.$$

Congruences: properties (2) – operations

If we have that

$$a \equiv b \pmod{m},$$

and

$$c \equiv d \pmod{m},$$

then

- ▶ $a + c \equiv b + d \pmod{m}$
- ▶ $a - c \equiv b - d \pmod{m}$
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What is missing here?

Division!

Also, we wish we can do “cancellation”, i.e., if

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Let's see the following example:

$$2 \cdot 1 \equiv 2 \cdot 3 \pmod{4},$$

but

$$1 \not\equiv 3 \pmod{4}.$$

Multiplications as functions

Let's view multiplication by 2 as a function, i.e., let $f(x) = 2 \cdot x \bmod 4$.

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Which functions have inverses?

Multiplicative inverses (standard arithmetic)

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By dividing on both sides with 5:

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or equivalently, by multiplying with $(1/5) = 5^{-1}$:

$$2 \cdot 5^{-1} = 5x \cdot 5^{-1} = x \cdot 5 \cdot 5^{-1} = x \cdot 1 = x.$$

Here 5^{-1} is a multiplicative inverse of 5.

Multiplicative inverses (modular arithmetic)

You can do the same thing in modular arithmetic. Let the modulus be $m = 7$. Note that

$$5 \cdot 3 \equiv 15 \equiv 1 \pmod{7}.$$

Therefore, $5^{-1} \equiv 3 \pmod{7}$.

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To find $2/5$, we can view our goal as to find the value of x such that

$$2 \equiv 5x \pmod{7}.$$

We can multiply both sides with $5^{-1} \equiv 3$ to get

$$2 \cdot 5^{-1} \equiv 2 \cdot 3 \equiv 6 \equiv 5^{-1} \cdot 5x \equiv x \pmod{7}.$$

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Let's check:

$$5 \cdot 6 \equiv 30 \equiv 2 \pmod{7},$$

as required.

Multiplicative inverse modulo m

Definition

The multiplicative inverse modulo m of a , denoted by a^{-1} , is an integer such that

$$a \cdot a^{-1} \equiv 1 \pmod{m}.$$

Multiplicative inverse modulo 11

Let's try to figure out multiplicative inverse of every integer modulo 11.

a	$a^{-1} \pmod{11}$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Examples: division in modular arithmetic

Suppose that we know that every non-zero integer a has an inverse modulo m .
Can you solve this equation?

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$$\begin{array}{rcl} 2x + y & \equiv & 3 \pmod{7} \\ x + 3y & \equiv & 5 \pmod{7} \end{array}$$

There are 3 clocks. At this moment, all three clocks ring at the same time. The first clock rings every 3 hours, the second clock rings every 4 hours, and the third clock rings every 10 hours. How long do you have to wait until you would hear all clocks ring a the same time again?

You have a large water container and two smaller buckets. The first bucket carries 3 litres of water and the second bucket carries 5 litres of water.
Can you put exactly 1 litre of water in the water container?

You have a large water container and two smaller buckets. The first bucket carries 6 litres of water and the second bucket carries 15 litres of water.
What is the minimum volume of water you can exactly put in the water container?

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In general if you have two buckets of volumes x and y , the amount that you can exactly make must be in the form of

$$ax + by,$$

for some integers x and y . (Note that x and y may be negative.)

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Do you see why the sum must be divisible by any common divisor of x and y ?

Useful fact

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When the term is non-zero, it must be divisible by $\gcd(x, y)$, so it has to be at least $\gcd(x, y)$.

It turns out that you can actually attain that value, i.e., there exist a pair of integer a and b such that

$$a \cdot x + b \cdot y = \gcd(x, y).$$

Finding a and b : Extended Euclid Algorithm

We will modify the Euclid algorithm so that it also returns a and b together with $\gcd(x, y)$.

```
Algorithm Euclid(x,y):  
  if x mod y == 0:  
  
    return y,          ,  
  else:  
    g, a', b' = Euclid(y, x mod y)  
  
    a =  
  
    b =  
  
    return g, a, b
```

Notes:

We have a' and b' such that

$$a' \cdot y + b' \cdot (x \bmod y) = g.$$

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(\Leftarrow) Recall that there exist integers x and y such that

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Thus, $(x \cdot a + y \cdot m) \bmod m = x \cdot a \bmod m = 1 \bmod m$, i.e., $x \cdot a \equiv 1 \pmod{m}$. Therefore x is the inverse.

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(\Rightarrow) Let $r = \gcd(a, m)$. Suppose that b is the multiplicative inverse of a modulo m , i.e., we have that

$$b \cdot a \equiv 1 \pmod{m},$$

Thus, $ba \bmod m = 1 \bmod m = 1$, i.e., there exists an integer q such that

$$ba = qm + 1,$$

or $ba - qm = 1$. However, r since $r|a$ and $r|m$, r also divides $ba - qm$ and 1. But it $r \nmid 1$ because $r > 1$ and we have the contradiction. □

Examples: division in modular arithmetic

Since the requirement for an existence of a^{-1} modulo m is that $\gcd(a, m) = 1$, if we let m be a prime number, every a which is not a multiple of m has an inverse.

Can you solve this equation?

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Public-key cryptography

RSA

RSA: steps

- ▶ Private key: (d, n) , Public key: (e, n)
- ▶ Encryption $E(m) = m^e \bmod n$, Decryption: $D(w) = w^d \bmod n$.
- ▶ Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.

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- ▶ Goal: Select e, d, n such that $D(E(m)) = m^{ed} \bmod n = m$.
- ▶ Pick two primes p and q . Let $n = pq$.
- ▶ Pick e (usually a small number)
- ▶ Pick d such that $d = e^{-1} \pmod{(p-1)(q-1)}$, i.e., $ed \equiv 1 \pmod{(p-1)(q-1)}$, or
$$ed = k \cdot (p-1)(q-1) + 1,$$
for some integer k .
- ▶ What is $m^{ed} \bmod n$?

Secret sharing

Secret sharing scheme based on straight lines

Example: secret sharing

- ▶ Think of a secret number $m \in \{0, 1, \dots, 10\}$.
- ▶ Pick a random number $a \in \{1, 2, \dots, 10\}$.
- ▶ Your straight line function $f(x) = (ax + m) \bmod 11$.
- ▶ We will generate 3 points from f and give them to 3 of your friends, each with only 1 point. Pick 3 numbers x_1, x_2, x_3 from $\{1, 2, \dots, 10\}$.
- ▶ Let's compute

$$(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)).$$

- ▶ Give them to 3 of your friends and challenge them to form a group of 2 people and figure out your number m .

What's next?

- ▶ We will prove Fermat's Little Theorem and show how to efficiently test if a number is prime.
- ▶ We will also use Fermat's Little Theorem to prove the correctness of RSA.
- ▶ Modular arithmetic is also key to our usage of polynomials to perform secret sharing and error correcting codes, because now we can do Gaussian elimination using only integers.