NOTES OF OPERATION RESEARCH

运筹学笔记

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Contents

Chapter 1

Optimisation Theory

Assessment: 70% examination, 30% homework

Six assignments:

7% the first three assignments and 3% the last three assignments.

1.1 Linear Programming

LP in standard form:

$$\max f = \sum_{j=1}^{n} c_j x_j$$
$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ 1 \le i \le m$$
$$x_j \ge 0,\ 1 \le j \le n \tag{1.1}$$

In matrix form:

$$\max f = C^T x$$
$$Ax = b$$
$$x \ge 0 \tag{1.2}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given, $x \in \mathbb{R}^n$ is the **optimisation variable**. $m < n$, A has the full rank. That is $rank(A) = m$.

- $f = C^T x$ is the objective function.
- $R = \{x \in \mathbb{R}^n, Ax = b, x \ge 0\}$ is the feasible set. If $x \in R$, then x is feasible. Thus R is the intersection of a plane $Ax = b$ with the **positive quadrant** $x \ge 0$.

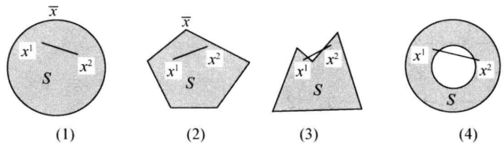


Figure 1.1: In this figure, the panel (1)-(2) are convex. Panel (3)-(4) are not convex.

1.1.1 Simplex Method

Aim: we want an effective method to solve the LP problem.

Def: The LP(1.1) is non-degenerate if b is not a linear combination of $< m$ columns of A. In other words, we can't have $Ax = b$ with x having $< m$ non-zeros entries.

All our problems will be non-degenerate.

Def Give the LP(1.1), a basic solution $x \in \mathbb{R}^n$ is as follows. Let $B \in \mathbb{R}^{m\times m}$ be a sub-matrix of A with full rank m. Suppose B occupies columns $j_1 < j_2 < \cdots < j_m$. Let $y = B^{-1}b$. Then

$$x_j = \begin{cases} 0 & \text{if } j \notin \{j_1, j_2, \cdots, j_m\} \\ y_l & \text{if } j = j_l \end{cases}$$

Note that $Ax = by = b$. If also $x \geq 0$, then $x \in \mathbb{R}$, and x is a basic feasible solution.

1.1.2 Definition

- $S \subset \mathbf{R}^n$ is convex if $\forall x, y \in S$ is an extreme point if $\exists x, y \in S$, $x \neq y$ and $\lambda \in (0,1)$, s.t. $z = (1-\lambda)x + \lambda y \in S$. Note that $S = \varnothing$ and $S = \{x\}$ are convex.
- If $S \subset \mathbb{R}^n$ is convex, then $z \in S$ is an extreme point if $\nexists x, y \in S$, $x \neq y$ and $\lambda \in (0,1)$, s.t. $z = (1-\lambda)x + \lambda y$.

1.1.3 Prop 1.1

Proposition 1.1. In the LP (1.1), R is convex.

Proof: Let $x, y \in R$ and $\lambda \in [0,1]$. Then

$$A\left((1-\lambda)x + \lambda y\right) = (1-\lambda)Ax + \lambda Ay = b$$

and

$$(1-\lambda)x + \lambda y \geq 0$$

since $x, y \geq 0$ and $1-\lambda, \lambda \geq 0$. Thus, $(1-\lambda)x + \lambda y \in R$.

By Prop 1.1, R is convex. Thus in 1.2, 1.3 below, we do have the notion of an extreme point of R.

1.1.4 Th 1.2

Theorem 1.1. Suppose the LP (1.1) has an optimal solution. Then R has an extreme point which is an optimal solution.

Proof. Suppose $x^* \in R$ is optimal, but not extreme. Then we have $x^* = (1-\lambda)y + \lambda z$, for some $y, z \in R$, $y \neq z$ and $\lambda \in (0,1)$. Now $c^T x^* = (1-\lambda)c^T y + \lambda c^T z \leq (1-\lambda)c^T x^* + \lambda c^T x^* = c^T x^*$. So $c^T x^* = c^T y = c^T z$. Thus, $c^T\left((1-\lambda')y + \lambda' z\right) = c^T x^*$, $\forall \lambda' \in R$. That is, all feasible points on the line through x^*, y, z are optimal.

Let $u = z - y$. Then $A(x^* + \mu u) = Ax^* + \mu Az - \mu Ay = b$. Since $u \neq 0$, assume $J = \{1 \leq j \leq n : u_j < 0\}$ satisfy $J \neq \varnothing$. (Otherwise consider $u = y - z$.) Then, we may choose $\mu' > 0$ maximum such that $x^* + \mu' u \geq 0$, that $(x^* + \mu' u)_j = 0$ for some $j \in J$. Now, for $1 \leq j \leq n$, $x_i^* = 0 \Longrightarrow y_i = z_i = 0$. (since $x_i^* = (1-\lambda)y_i + \lambda z_i$ and $y_i z_i \geq 0$) That means that $\Longrightarrow u_i = 0 \Longrightarrow (x^* + \mu' u)_i = 0$. Thus $i \notin J$ for such i. We see that $x^* + \mu' u \in R$ has more zero coordinates than x^*, and $x^* + \mu' u$ is also optimal.

Repeat this argument, we must stop since we can't have more that n zeros. Upon termination, we have an extreme point of R, which is optimal. □

1.1.5 Th 1.3

Theorem 1.2. Suppose the LP (1.1) is non-degenerate. Let $x \in R$, then x is extreme $\Longleftrightarrow$ x is a basic feasible solution.

Proof. $\Longrightarrow$ Suppose $x \in R$ is extreme, but not a basic feasible solution. Let $J = \{1 \leq j \leq n : x_j > 0\}$. Suppose first that $|J| \leq m$. Then either $|J| < m$ or $|J| = m$. And the columns of A indexed by J are linearly dependent. In either case, b is a linear combination of less than m columns of A, which contradicts non-degeneratecy.

Now, let $|J| > m$. Them the columns of A indexed by J are linearly dependent. If A' is the submatrix of A with these columns, then $\exists v' \in \mathbb{R}^{|J|} \setminus \{0\}$ such that $A'v' = 0$. Extend v' to $v \in \mathbb{R} \setminus \{0\}$ by adding zeros to the remaining coordinators (not indexed by J). Then $Av = 0$, and $x_i = 0 \Longrightarrow v_i = 0$. Now, $x \pm \varepsilon v \in R$ if $\varepsilon > 0$ is small: $A(x \pm \varepsilon v) = Ax = b$, and $x \pm \varepsilon v \geq 0$ for $\varepsilon > 0$ small (since $x_i = 0 \Longrightarrow v_i = 0$). Thus $x = \frac{1}{2}(x + \varepsilon v) + \frac{1}{2}(x - \varepsilon v)$, contradicting x extreme. □

$\Longleftarrow$ Suppose $x \in R$ is a basic feasible solution, but not extreme. Then $x = (1-\lambda)y + \lambda z$ for some $y, z \in R$, $y \neq z$ and $\lambda \in (0,1)$. Let $B \in \mathbb{R}^{m \times m}$ bet the submatrix of A corresponding to x, and $J \subset \{1, 2, \cdots, n\}$ be the index set of the columns of B. We have $|J| = m$ and $\operatorname{rank}(B) = m$. For $i \notin J$, we have $x_i = 0 \Rightarrow y_i = z_i = 0$(as in **Th 1.2**). Let $y', z' \in \mathbb{R}^m$ be obtained from y, z by deleting the zeros corresponding to $\{1, 2, \cdots, n\} \backslash J$. Then $b = Ay = By'$, $b = Az = Bz' \Rightarrow B(y' - z') = 0$. But $y' - z' \neq 0$, so $\mathrm{rank}(B) < m$, contradiction.

1.1.6 Cor 1.4

Now **1.1.4 and 1.1.5**$\Longrightarrow$

> **Corollary 1.1.1.** *Suppose the LP(1.1) has an optimal solution and is non-degenerate. Then $\exists$ basic feasible solution which is optimal.*

1.1.7 Simplex Method（单纯形法）

Now we present the **simplex method** for solving the LP(1.1). By **1.1.6**, to find the maximum of f, it suffices to check the value of f at basic feasible solutions. There are $\leq \binom{n}{m}$ such points, since there are $\leq \binom{n}{m}$ choices for the submatrix B of A. But the idea is, we move from one basic feasible solution to another, and strictly increase f each time. When this procedure stops, we either have an optimal solution, or conclude that there are none.

In (1.1), we treat f as another variable. We have the system:

$$\begin{cases} Ax = b \\ -c^T x + f = 0 \\ x \geq 0 \end{cases}$$

The system is the initial system of the LP 1.1. We describe the simplex method, using Example 1.

Example 1 Again consider Example 1. which has the standard from 1.2. The initial system is:

$$\begin{cases} \begin{array}{ccccccl} x_1 & +x_2 & +s_1 & & & & = 6 \\ 2x_1 & +4x_2 & & +s_2 & & & = 16 \\ & x_2 & & & +s_3 & & = 3 \\ -3x_1 & -4x_2 & & & & +f & = 0 \\ x_1, & x_2, & s_1, & s_2, & s_3 & & \geq 0 \end{array} \end{cases}$$

We form the simplex table as follows:

x_1	x_2	s_1	s_2	s_3	f	
1	1	1	0	0	0	6
2	4	0	1	0	0	16
0	1	0	0	1	0	3
-3	-4	0	0	0	1	0

Note that $(x_1, x_2, s_1, s_2, s_3) = (0, 0, 6, 16, 3)$ is a basic feasible solution, and $f = 0$. We require that the entries on the right are $\geq=$. We choose an entry called the **pivot** as follows.

① Choose the column containing the most negative entry in the bottom row, containing -4. We choose this since the objective function is $f = 3x_1 + 4x_2$, so increasing along x_2 is the most beneficial.

② Next, for each positive entry in the chosen column, (expect for the bottom one), divide the number on the right by the entry. Choose the row with the minimum answer, which is the third row, with $3/1 = 3$. We choose this row so that all three equality constraints of 1.2 can be satisfied when $x_2 \leq 3$.

The **pivot** is the entry in the chosen column and row.

③ We now reduce by making the pivot 1(here, it already is) and all other entries in the second column 0. We do not switch rows.

x_1	x_2	s_1	s_2	s_3	f	
1	0	1	0	-1	0	3
2(pivot)	0	0	1	-4	0	4
0	1	0	0	1	0	3
-3	0	0	0	-4	1	12

We have now moved to the basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (0, 3, 3, 4, 0)$ and $f = 12$. The value of f has increased.

If we still have negative entry in the entry in the bottom row, repeated the procedure. Continue until this ends.

x_1	x_2	s_1	s_2	s_3	f	
0	0	1	-1/2	1	0	1
1	0	0	1/2	-2(pivot)	0	2
0	1	0	0	1	0	3
0	0	0	3/2	-2	1	18

x_1	x_2	s_1	s_2	s_3	f	
0	0	1	-1/2	1	0	1
1	0	2	1/2	0	0	4
0	1	-1	0	0	0	2
0	0	0	2	1/2	1	20

We end at this table since there are no more negative entries in the bottom row, which gives:

$$f = 20 - 2s_2 - \frac{1}{2}s_3 \leq 20 \tag{1.3}$$

We can read off the final, optimal solution as $(x_1, x_2, s_1, s_2, s_3) = (4, 2, 0, 0, 1)$ and $f = 20$. This agrees with the answer in Example 1.

What happens if a LP has no optimal solution?

Example 1.1.

$$\begin{aligned} \min \quad & f = -5x_1 + x_2 \\ & x_1 - x_2 \leq 2 \\ & -3x_1 + x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution. We change the objective function to maximize $f' = 5x_1 - x_2$.

Initial system:

$$\begin{cases} x_1 - x_2 + s_1 & = 2 \\ -3x_1 + x_2 + s_2 & = 0 \\ -5x_1 + x_2 + f' & = 0 \\ x_1, x_2, s_1, s_2 & \geq 0 \end{cases}$$

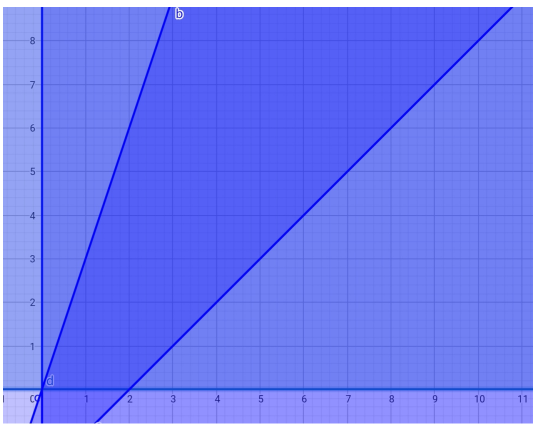
Simplex table:

x_1	x_2	s_1	s_2	f	
1(pivot)	-1	1	0	0	2
-3	1	0	1	0	0
-5	1	0	0	1	0

x_1	x_2	s_1	s_2	f	
1	-1	1	0	0	2
0	-2	3	1	0	6
0	-4	5	0	1	10

Since no entry in the second column is positive, the LP does not have a solution: there is no finite maximum for f', hence no finite minimum for f'. To see this, set $s_1 = 0$, since s_1 is a non-basic variable. Set $x_2 = M \geq 0$. Then $(x_1, x_2, s_1, s_2) = (2 + M, M, 0, 6 + 2M)$ is feasible, and $f' = 10 + 4M \to \infty$ as $M \to \infty$.

Picture of LP problem:



□

1.1.8 Procedure of simplex method

Procedure 1.1.1. Give a LP:

(1) Convert the LP to standard form, introducing slack variables and changing min to max, if necessary. Write down the initial system.

(2) Write down the simplex table. The entries on the right are ≥ 0, and some columns form on identity matrix, which gives a basic feasible solution.

(3) If there is a negative number in the bottom row, choose the column with the most negative number. If there is a tie, choose any such column. If there is no negative number, the optimal solution of the LP is found. Read off the values of the optimal solution and the optimal value.

(4) For every positive entry in the chosen column from Step 3(except the bottom one), divide the number on the right by the entry. Choose the row with the minimum answer. If there is a tie, choose any such row. If there is no positive entry in the column, then the LP has no optimal solution.

(5) The entry in the column and row from Step 3 and 4 is the pivot. Perform row operations to make the pivot 1, and other entries in the column 0. Do not exchange rows.

(6) Repeat Step 3 for the new simplex table. Continue until termination.

Remark 1.1. We note that the simplex method does indeed terminate. There are finitely many basic feasible solutions, $\leq \binom{n}{m}$. Enough to show that no basic feasible solution is visited more than once. For this, enough to show that f strictly increases in every move. At an iterate, the current value of f, say P, is in the bottom right of the simplex table. If the pivot is in row i and $P_i \geq 0$ is the entry in the right of row i, then the next value for f is $P' = P + kP_i$ for some $k > 0$. If $P_i = 0$, then we have a basic feasible solution with $< m$ non-zero entries, which contracts non-degeneracy. Hence, $P_i > 0$ and $P' = P + kP_i > P$.

1.1.9 Duality（对偶）in linear programming

For $x, y \in \mathbb{R}^n$, write $x \geq y$ to mean $x - y \geq 0$. We write $x \geq y$ to mean $y \geq x$.

Remark 1.2. Give a LP, we want to associate it with another LP, called the dual.

Consider the LP:

$$\begin{aligned} \max \quad & c^T \boldsymbol{x} \\ & A\boldsymbol{x} \leq b \\ & \boldsymbol{x} \geq 0 \end{aligned} \tag{1.4}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given, and $\boldsymbol{x} \in \mathbb{R}^n$ is variable. Let $p^* \in \mathbb{R}$ be the optimal value

Remark 1.3. $p^* = -\infty$ if (1.4) is not feasible, otherwise p^* is finite or $p^* = +\infty$.

We introduce a variable $\boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{y} \geq 0$. Then, $\forall \boldsymbol{x} \in \mathbb{R}^n$ feasible for (1.4).

$$\boldsymbol{y}^T A\boldsymbol{x} \leq \boldsymbol{y}^T b \quad \text{or} \quad \left(A^T \boldsymbol{y}\right)^T \boldsymbol{x} \leq b^T \boldsymbol{y} \tag{1.5}$$

If $\boldsymbol{y}$ also satisfies $A^T \boldsymbol{y} \geq c$, then

$$c^T \boldsymbol{x} \leq \left(A^T \boldsymbol{y}\right)^T \boldsymbol{x} \leq b^T \boldsymbol{y} \tag{1.6}$$

Thus, we want to find $\boldsymbol{y}$ such that (1.6) is best possible. We define the **dual** LP of the LP (1.4) to be:

$$\begin{aligned} \min \quad & b^T \boldsymbol{y} \\ & \begin{cases} A^T \boldsymbol{y} & \geq c \\ \quad \boldsymbol{y} & \geq 0 \end{cases} \end{aligned} \tag{1.7}$$

where $\boldsymbol{y} \in \mathbb{R}^n$ is a variable. Let d^* be the optimal value.

Remark 1.4. $d^* = \infty$ if (1.7) is not feasible, otherwise, d^* is finite or $d^* = -\infty$.

The LP (1.4) is the **primal** LP.

Proposition 1.2. The dual LP of the LP (1.7) is the LP (1.4).

Solution. By writing (1.7) as:

$$\begin{aligned} \max \quad & -b^T \boldsymbol{y} \\ & -A^T \boldsymbol{y} \leq c \\ & \boldsymbol{y} \geq 0 \end{aligned}$$

the dual is

$$\begin{aligned} \min \quad & -c^T\boldsymbol{x} \\ & -A\boldsymbol{x} \geq -b \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

which is (1.4). □

Theorem 1.3. Weak Duality: Given LP (1.4) and its dual (1.7), then we have:

(a) $p^* \leq d^*$;

(b) If $p^* = \infty$, then (1.7) is not feasible.

(c) If $d^* = \infty$, then (1.4) is not feasible.

(d) Suppose $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, $\bar{\boldsymbol{y}} \in \mathbb{R}^n$ are feasible for (1.4) and (1.7), and $c^T\bar{\boldsymbol{x}} = b^T\bar{\boldsymbol{y}}$. Then $c^T\bar{\boldsymbol{x}} = p^* = d^* = b^T\bar{\boldsymbol{y}}$.

Solution. (a) If either (1.4) or (1.7) is not feasible, then $p^* \leq d^*$ holds. Suppose (1.4) and (1.7) are both feasible. We have $c^T\boldsymbol{x} \leq b^T\boldsymbol{y}$, $\forall$ feasible $\boldsymbol{x} \in \mathbb{R}^n$ for (1.4), $\forall$ feasible $\boldsymbol{y} \in \mathbb{R}^n$ for (1.7). Thus $p^* \leq b^T\boldsymbol{y}$, $\forall$ feasible $\boldsymbol{y} \in \mathbb{R}^n$ for (1.7) and $p^* \leq d^*$.

(b) If (1.7) is feasible, then $p^* \leq b^T\boldsymbol{y}$, for all feasible $\boldsymbol{y} \in \mathbb{R}^m$ for (1.7), and thus $p^* < \infty$.

(c) Similar to (b).

(d) By (a), we have $c^T\bar{\boldsymbol{x}} \leq p^* \leq d^* \leq b^T\bar{\boldsymbol{y}} = c^T\bar{\boldsymbol{x}}$, so equality holds throughout. □

Theorem 1.4. Strong Duality: Given LP in (1.4) and its dual (1.7). If either LP is feasible and bounded, then so is the other, and $p^* = d^*$.

Remark 1.5. The LP (1.4) is BOUNDED if p^* is finite. Similarly for the LP (1.7) with d^*.

We will prove TH 1.7 later.

1.1.10 Dual simplex method

Example 1.2. Consider the LP:

$$\max \quad f = -4x_1 - 2x_2 - x_3$$
$$\begin{cases} -x_1 - x_2 + 2x_3 & \leq -3 \\ -4x_1 - 2x_2 + x_3 & \leq -4 \\ x_1 + x_2 - 2x_3 & \leq 2 \\ x_1, x_2, x_3 & \geq 0 \end{cases} \tag{1.8}$$

Introducing slack variables $s_1, s_2, s_3 \geq 0$, we have the form:

x_1	x_2	x_3	s_1	s_2	s_3	f	
-1	-1	2	1	0	0	0	-3
-4	-2	1	0	1	0	0	-4
1	1	-4	0	0	1	0	2
4	2	1	0	0	0	1	0

Table 1.1: table-1

But this says the basic solution is $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, -3, -4, 2)$, which is not feasible for (1.8). We can't use the simplex method.

We note that the dual of (1.8) is

$$\begin{aligned} \max \quad & y = -3y_1 - 4y_2 + 2y_3 \\ & \begin{cases} -y_1 - 4y_2 + y_3 & \leq -4 \\ -y_1 - 2y_2 + y_3 & \leq -2 \\ 2y_1 + y_2 - 4y_3 & \leq -1 \\ y_1, y_2, y_3 & \geq 0 \end{cases} \end{aligned} \tag{1.9}$$

Converting (1.9) to standard form, we can solve it by the simplex method. Then the Th 1.7$\Longrightarrow$ the solution to (1.9) gives the solution to (1.8).

However, the dual simplex method can solve both (1.8) and (1.9) simultaneously. We demonstrate the method with the example.

Procedure 1.1.2. STEP 1 Write out the dual simplex tables.

	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	-1	-1	2	1	0	0	-3
s_2	-4	-2	1	0	1	0	-4
s_3	1	1	-4	0	0	1	2
$-f$	-4	-2	-1	0	0	0	0

$y = 0$ is dual feasible simplex since all numbers are ≤ 0.

STEP 2 We aim for an operation about a pivot so that the primal LP "goes toward being feasible", while the dual LP remains feasible. Choose any row with a negative entry on the right, say row 1. This means that we will increase y, which decreases the value of the dual objective. Next, for every column with a negative entry in the chosen row (row 1), divide the entry in the bottom row by the entry (row 1). We obtain $\frac{-4}{-1} = 4$ and $\frac{-2}{-1} = 2$. Choose the column smallest answer, which is positive. This says that y_1 can increase by at most 2, so that the constraints of (1.9) are satisfied.

STEP 3 Row reduce as before. We obtain:

	x_1	x_2	x_3	s_1	s_2	s_3	
x_2 replace with variable in pivot column	1	1	-2	-1	0	0	3
s_2	-2	0	-3	-2	1	0	2
s_3	0	0	-2	1	0	1	-1
$-f$	-2	0	-5	-2	0	0	6

Note that this new table is still dual feasible, simplex choice of the pivot implies all entries in the bottom row remain ≤ 0. The primal has become "towards feasible", since now there is only one negative number on the right.

STEP 4 Repeat until procedure ends, when all numbers on the right are ≥ 0

	x_1	x_2	x_3	s_1	s_2	s_3	
x_2	1	1	0	-2	0	-1	4
$\rightarrow s_2$	-2	0	0	$-\frac{7}{2}$	1	$-\frac{3}{2}$	$\frac{7}{2}$
x_3	0	0	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
$-f$	-2	0	0	$-\frac{9}{2}$	0	$-\frac{5}{2}$	$\frac{17}{2}$

Final optimal solutions: $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 4, \frac{1}{2}, 0, \frac{7}{2}, 0)$, and $f = -\frac{17}{2}$, $(y_1, y_2, y_3) = (\frac{9}{2}, 0, \frac{5}{2})$.

Remark 1.6. In STEP 2, we could have chosen row 2.

If there is a tie for the ratios in STEP 2, we could choose any suitable column.

As before, if the dual simplex method does not terminate, we conclude that there is no optimal solution.

Example 1.3. Use the dual simplex method to solve the LP:

$$\min \quad f = 2x_1 + 3x_2 + x_3$$
$$\begin{cases} x_1 + x_2 - 2x_3 & \geq 3 \\ x_1 + 2x_2 - x_3 & \leq 2 \\ x_1, x_2, x_3 & \geq 0 \end{cases} \tag{1.10}$$

We change the objective to $\max f' = -2x_1 - 3x_2 - x_3$, and add slack variables $s_1, s_2 \geq 0$. We obtain:

	x_1	x_2	x_3	s_1	s_2	
s_1	-1(PIVOT)	-1	2	1	0	-3
s_2	-1	-2	1	0	1	-2
$-f'$	-2	-3	-1	0	0	0

Thus we have:

	x_1	x_2	x_3	s_1	s_2	
s_1	1	1	-2	-1	0	3
s_2(NO NEGATIVE NUMBER)	0	-1	-1	-1	1	-1
$-f'$	0	-1	-5	-2	0	6

This means the dual LP is unbounded. By Th 1.6, the primal LP(1.10) is not feasible.

Example 1.4.

$$\min \quad f = -3x_1 + x_2 - 2x_3$$
$$\begin{cases} x_1 + 3x_2 + x_3 & \leq 5 \\ -2x_1 + x_2 - x_3 & \leq -2 \\ 4x_1 - x_2 - 2x_3 & = 5 \\ x_1, \quad x_2, \quad x_3 & \geq 0 \end{cases} \tag{1.11}$$

By writing the objective as $\max f' = -3x_1 + x_2 - 2x_3$, and adding slack variables $s_1, s_2 \geq 0$, we find

x_1	x_2	x_3	s_1	s_2	f'	
1	3	1	1	0	0	5
-2	1	-1	0	1	0	-2
4	-1	2	0	0	0	5
-3	1	-2	0	0	-1	0

We can't use the simplex method or dual simplex method directly, since both of sets of coefficients $\{5, -2, 5\}$ and $\{-3, 1, -2\}$ have negative numbers.

We see that the solution $(x_1, x_2, x_3, s_1, s_2) = (0, 0, 0, 5, -2)$ is not feasible. We apply the TWO-PHASE METHOD where PHASE 1 tries to find a basic solution.

PHASE 1 We form the problem:

$$\max \quad h = -t_1 - t_2$$
$$\begin{cases} x_1 + 3x_2 + x_3 + s_1 & = 5 \\ 2x_1 - x_2 + x_3 - s_2 + t_1 & = 2 \\ 4x_1 - x_2 - 2x_3 + t_2 & = 5 \end{cases} \tag{1.12}$$

where $s_1 \geq 0$ is a slack variable, $s_2 \geq 0$ is a surplus variable and $t_1, t_2 \geq 0$ are artificial variables. If the optimal solution of (1.12) is $h = 0$, then $\exists x_1, x_2, x_3, s_1, s_2 \geq 0$, s.t. the constraints of (1.12) hold, with $t_1 = t_2 = 0$. This will be a basic feasible solution for (1.11). We form an extended version of simplex table.

x_1	x_2	x_3	s_1	s_2	t_1	t_2	f'	h	
1	3	1	1	0	0	0	0	0	5
2	-1	1	0	-1	1	0	0	0	2
4	-1	-2	0	0	0	1	0	0	5
-3	1	-2	0	0	0	0	-1	0	0
0	0	0	0	0	1	1	0	1	0

the bottom entries in the t_1, t_2 columns 0. Then we perform the simplex method procedure.

x_1	x_2	x_3	s_1	s_2	t_1	t_2	f'	h	
1	3	1	1	0	0	0	0	0	5
2(pivot)	-1	1	0	-1	1	0	0	0	2
4	-1	-2	0	0	0	1	0	0	5
-3	1	-2	0	0	0	0	-1	0	0
-6	2	1	0	1	0	0	0	1	7

Thus we have:

x_1	x_2	x_3	s_1	s_2	t_1	t_2	f'	h	
0	$\frac{7}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	4
1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	1
0	1	-4	0	2	-2	1	0	0	1
0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	1	0	3
0	-1	4	0	-2	3	0	0	1	-1

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x_1	x_2	x_3	s_1	s_2	t_1	t_2	f'	h	
0	$\frac{13}{4}$	$\frac{3}{2}$	1	0	0	$-\frac{1}{4}$	0	0	$\frac{15}{4}$
1	$-\frac{1}{4}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{4}$	0	0	$\frac{5}{4}$
0	$\frac{1}{2}$	-2	0	1	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
0	$\frac{1}{4}$	$-\frac{7}{2}$	0	0	0	$\frac{3}{4}$	1	0	$\frac{15}{4}$
0	0	0	0	0	1	1	0	1	0

This now says that we have the basic feasible solution $(x_1, x_2, x_3, s_1, s_2) = (\frac{5}{4}, 0, 0, \frac{15}{4}, \frac{1}{2})$, and $t_1 = t_2 = h = 0$. If we could not reach this point, then the optimal solution of (1.12) would be negative and (1.11) would not be feasible.

PHASE 2 We may not neglect t_1, t_2, h and use the simplex method.

x_1	x_2	x_3	s_1	s_2	f'	
0	$\frac{13}{4}$	$\frac{3}{2}$(Pivot)	1	0	0	$\frac{15}{4}$
1	$-\frac{1}{4}$	$-\frac{1}{2}$	0	0	0	$\frac{5}{4}$
0	$\frac{1}{2}$	-2	0	1	0	$\frac{1}{2}$
0	$\frac{1}{4}$	$-\frac{7}{2}$	0	0	1	$\frac{15}{4}$

Thus we have:

x_1	x_2	x_3	s_1	s_2	f'	
0	$\frac{13}{6}$	1	$\frac{2}{3}$	0	0	$\frac{5}{2}$
1	$\frac{5}{6}$	0	$\frac{1}{3}$	0	0	$\frac{5}{2}$
0	$\frac{29}{6}$	0	$\frac{4}{3}$	1	0	$\frac{11}{2}$
0	$\frac{47}{6}$	0	$\frac{7}{6}$	0	1	$\frac{25}{2}$

Final solution for (1.11): $\min f = -\frac{25}{2}$, when $(x_1, x_2, x_3) = (\frac{5}{2}, 0, \frac{5}{2})$. And $(s_1, s_2) = (0, \frac{11}{2})$.

Solution. For Th 1.7

By Prop 1.5 and Th 1.6(d), if suffices to assume LP(1.4) is feasible and bounded, and prove exists feasible $\bar{x} \in \mathbb{R}^n$ and $\bar{y} \in \mathbb{R}^m$ for (1.4) and (1.7), s.t. $c^T\bar{x} = b^T\bar{y}$. We apply either the simplex or two-Phase method to (1.4) by introducing $x \in \mathbb{R}^m$ which contains slack and surplus method. The simplex or two-phase method moves from one point of the set $S = \{\binom{x}{s} \in \mathbb{R}^{n+m} : Ax + s = b\}$ to another, which varying the objective function value c^Tx. When the algorithm ends, an optimal solution for (1.4) is found, and the objective function reads $p^* - \bar{z}^T\bar{x} - \bar{y}^T\bar{s}$ for some $\bar{x}, \bar{z} \in \mathbb{R}^n$, $\bar{s}, \bar{y} \in \mathbb{R}^m$ with $A\bar{x} + \bar{s} = b$ and $\bar{x}, \bar{z}, \bar{s}, \bar{y} \geq 0$. Thus $\forall \binom{x}{s} \in S$, we have $p^* - \bar{z}^Tx - \bar{y}^Ts = c^Tx$, so $\forall x \in \mathbb{R}^n$, we have:

$$p^* - \bar{z}^Tx - \bar{y}^T(b - Ax) = c^Tx.$$

Equating the constant terms gives $p^* = b^Ty$. Equating the x terms gives $A^T\bar{y} = c + \bar{z} \geq c$. So $\bar{y}$ is the feasible for (1.7). Finally, when the simplex or two-Phase method terminates, all variables with a non-zero coefficient in $z^T\bar{x} + \bar{y}^T\bar{s}$ is a non-basic variable, and hence equals zero in the solution $\binom{\bar{x}}{\bar{s}}$. Thus,

$$p^* = p^* - \bar{z}^T\bar{x} - \bar{y}^T\bar{s} = c^T\bar{s}$$

and $c^T\bar{x} = b^T\bar{y}$, as required. □

Remark 1.7. We developed the results and methods as follows. There were no circular chain of implications:

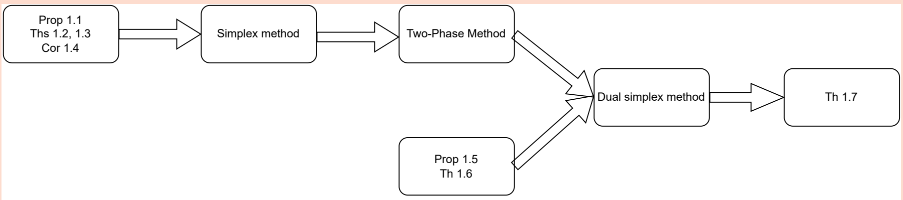


Figure 1.2: The chain of implications

Even though the two-phase method seems stronger than the dual simplex method, we actually derived the latter from the former!

1.2 Non-linear programming

Definition 1.1. convex: Let $X \in \mathbb{R}^n$ be convex. A function $f : X \to \mathbb{R}$ is convex if $\forall x, y \in X$ and $\lambda \in [0, 1]$

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

f is strictly convex if $\forall x, y \in X$ with $x \neq y$, and $\lambda \in (0, 1)$

$$f\left((1-\lambda)x + \lambda y\right) < (1-\lambda)f(x) + \lambda f(y)$$

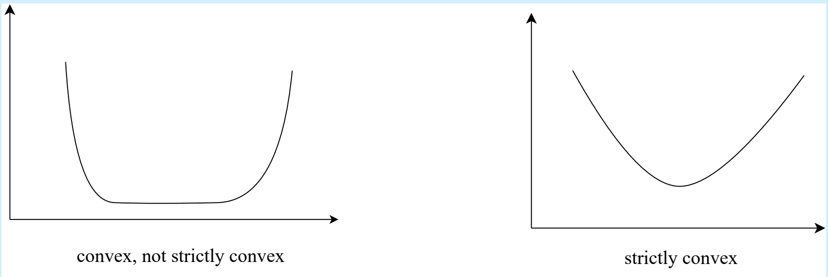


Figure 1.3: Convex function

f is concave/strictly concave if $-f$ is convex/strictly convex.

Definition 1.2. Differentiable: Let $X \in \mathbb{R}^n$ be an open set. A function $f : X \to \mathbb{R}$ is k times differentiable if all partial derivatives:

$$\frac{\partial^l f}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_n^{l_n}} \tag{1.13}$$

exist, where $0 \leq l_i, l \leq k$ and $l_1 + l_2 + \cdots + l_n = l$. Recall that $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$, $\forall 1 \leq i, j \leq n$, f is k times continuously differentiable if in addition, all derivatives (1.13) are continuous on X. If $k = 1$, we omit "k times". If $k = 2$, we write "twice" for "2 times".

The gradient of f is

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

The Hessian of f is

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}, \quad \text{which is symmetric.}$$

Definition 1.3 (symmetric definite matrix)**.** A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is:

Positive definite (正定) Written $A > 0$, if $x^T A x > 0$, $\forall x \in R^n \backslash \{0\}$.

Positive semi-definite Written $A \geq 0$, if $x^T A x \geq 0$, $\forall x \in \mathbb{R}^n$.

The following results each characterises convex functions.

Theorem 1.5. Let $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is convex. Then f is convex $\iff \forall x, y \in X$, $\forall \lambda \in [0, 1]$, the function:

$$h(\lambda) = f\left((1-\lambda)x + \lambda y\right)$$

is convex on $\lambda \in [0, 1]$. That is, f is convex when restricted to any line segment in X.

Theorem 1.6. Let $f : X \to \mathbb{R}$ be differentiable, where $X \subset \mathbb{R}$ is open. Then f is convex $\iff X$ is convex, and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

"f lies above the plane of contact at x"

Theorem 1.7. Let $f : X \to \mathbb{R}$ be twice differentiable, where $X \subset \mathbb{R}$ is open. Then f is convex $\iff X$ is convex, and

$$\nabla^2 f(x) \geq 0, \quad \forall x \in Xs$$

The standard form of a non-linear programming problem is

$$\begin{aligned} \min \quad & f(x) \\ & g_i(x) \leqslant 0, \quad \forall 1 \leqslant i \leqslant m \end{aligned} \tag{1.14}$$

where $x \in \mathbb{R}^n$ is the optimisation variable, and $f : X_0 \to \mathbb{R}$, $g_i : X_i \to \mathbb{R}$, where $X_0, X_1, \ldots, X_m \subset \mathbb{R}^n$ are the domains of $f, g_1, \cdots, g_n$. The function f is the objective function, and $g_i(x) \leq 0$ are the (inequality) constraints. The domain and feasible set of problem (1.14) are

$$D = \bigcap_{i=0}^{m} X_i$$
$$R = \{x \in D : g_i(x) \leq 0, \forall 1 \leq i \leq m\}$$

The optimal value of (1.14) is denoted by p^*. The optimal set of (1.14) is

$$R^* = \{x \in R : f(x) = p^*\}$$

Any elements of R^* is an OPTIMAL SOLUTION. Note that we may have $R^* = \varnothing$, even if p^* is finite.

Example 1.5. If $f(x) = e^{-x}$, then $p^* = 0$, but $R^* = \varnothing$.

Think of a series, its limitation isn't located on its set. Commonly appearing in Real Analysis and some courses. Secondly perhaps the function isn't continuous.

If $R^* \neq \varnothing$, then p^* is OTTAINED (by any element of R^*).

If f and all g_i in (1.14) are convex functions (so all sets X_i are convex), then (1.14) is a CONVEX PROGRAMMING (CP) PROBLEM.

Theorem 1.8. Th 1.11 Suppose (1.14) is a CP.

(a) The sets R and R^* are convex.

(b) If F is strictly convex, and $R^* \neq \varnothing$, then $\exists$ unique optimal solution.

Proof. (a) Let $x, y \in R$, and $\lambda \in [0,1]$. Then $x, y \in D$ and since $D = \bigcap_{i=1}^{m} X_i$ is convex[1], $z = (1-\lambda)x + \lambda y \in D$. Now $1 \leq i \leq m$, since g_i is convex. Then

$$g_i(z) = g_i\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)g_i(x) + \lambda g_i(y) \leq 0,$$

so $z \in R$ and R is convex.

Now let $x, y \in R^*$, and $\lambda \in [0,1]$. Then $x, y \in R$ and since R is convex, we have $z = (1-\lambda)x + \lambda y \in R$. Since f is convex,

$$p^* \leq f(z) = f\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)x + \lambda y = p^*$$

So $f(z) = p^*$ and $z \in R^*$. Thus R^* is also convex.

(b) Suppose that $\exists$ optimal solutions $x, y \in R^*$ with $x \neq y$. Let $\lambda \in (0,1)$. Then $(1-\lambda)x + \lambda y \in R^*$ by (a), and

$$f\left((1-\lambda)x + \lambda y\right) < (1-\lambda)f(x) + \lambda f(y) = p^*$$

a contradiction. ☐

Definition 1.4. In problem (1.14), $\bar{x} \in R$ is locally optimal if $\exists \delta > 0$, s.t.

$$f(x) \leq f(\bar{x}), \quad \forall x \in R \cap B^0(\bar{x}, \delta) \tag{1.15}$$

where $B^0(\bar{x}, \delta) = \{y \in \mathbb{R}^n : \|y - \bar{x}\| < \delta\}$ is the open ball with radius δ, centre $\bar{x}$.

[1] By HW1 in the homework

Remark 1.8. Recall that $\|z\| = \|z\|_2 = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}$ is the <u>Euclidean norm</u> of $z \in \mathbb{R}^n$

Theorem 1.9. Th 1.12 Suppose (1.14) is a CP. Then $\bar{x}$ is locally optimal $\iff$ $\bar{x}$ is an optimal solution.

Proof. Clearly it suffices to prove ($\Longrightarrow$). Let $\bar{x}$ be locally optimal. Then $\bar{x}$ satisfies (1.15) for some positive $\delta > 0$. Suppose $\bar{x} \notin R^*$, so $\exists y \in R$, $y \neq \bar{x}$, s.t. $f(y) < f(\bar{x})$. Then $\|y - \bar{x}\| \geq \delta$. Choose z where

$$z = (1-\lambda)x + \lambda y, \quad \lambda = \frac{\delta}{2\|y - \bar{x}\|}.$$

Then $\|z - \bar{x}\| = \frac{\delta}{2} < \delta$. Since R is convex by Th 1.11(a), $z \in R$. Since f is convex,

$$f(z) = f\left((1-\lambda)\bar{x} + \lambda y\right) \leq (1-\lambda)f(\bar{x}) + \lambda f(y) = f(\bar{x}) + \lambda\left(f(y) - f(\bar{x})\right) < f(\bar{x}).$$

which contradicts (1.15). Thus $\bar{\in} R^*$. ☐

Theorem 1.10. Th 1.13 Suppose that (1.14) is a CP, where f is differentiable. Then $\bar{x} \in R^* \iff$ $\bar{x} \in R$, and

$$\nabla f(\bar{x})^T (y - \bar{x}) \geq 0, \quad \forall y \in R. \tag{1.16}$$

Proof. $\Longleftarrow$

Suppose $\bar{x} \in R$ and satisfies (1.16). If $y \in R$, then

$$f(y) \underset{\text{Th 1.9}}{\geq} f(\bar{x}) + \nabla f(\bar{x})^T (y - \bar{x}) \underset{(1.16)}{\geq} f(\bar{x}),$$

so $\bar{x}$ is optimal.

$\Longrightarrow$

Suppose $\bar{x} \in R^*$. Then $\bar{x} \in R$. Suppose (1.16) is false. Then $\exists y \in R$ s.t. $\nabla f(\bar{x})^T (y - \bar{x}) < 0$. Consider $z(\lambda) = (1-\lambda)\bar{x} + \lambda y$, where $\lambda \in (-\varepsilon, 1-\varepsilon)$ for some small $\varepsilon > 0$. By Th 1.11(a), R is convex, so $z(\lambda) \in R$. Now

$$\begin{aligned} \frac{d}{d\lambda} f\left(z(\lambda)\right) &\underset{\text{Chain rule}}{=} Df(z(\lambda))Dz(\lambda) = \nabla f(z(\lambda))^T (y - \bar{x}) \\ \left.\frac{d}{d\lambda}\right|_{\lambda=0} &= \nabla f(\bar{x})^T (y - \bar{x}) < 0 \end{aligned}$$

so if $\lambda > 0$ is sufficiently small, then $f(z(\lambda)) < f(z(0)) = f(\bar{x})$. Thus $\bar{x} \notin R^*$, a contradiction. ☐

Theorem 1.11. Th 1.14 Suppose that (1.14) is a CP and is unconstrained, that is, $m = 0$. Let f be differentiable. Then $\bar{x} \in R^* \iff \nabla f(\bar{x}) = 0$.

Proof. ($\Longrightarrow$) Let $\bar{x} \in R^*$. Then Th 1.13 $\Longrightarrow \nabla f(\bar{x})^T(y - \bar{x}) \geq 0$, $\forall y \in R$. Since f is differentiable, $\exists$ open neighborhood U of $\bar{x}$, s.t. $\forall y \in U$, we have $y \in R$. Let $y = \bar{x} - t\nabla f(\bar{x})$ where $t > 0$ is small, and $y \in R$. Then

$$0 \leq \nabla f(\bar{x})^T(y - \bar{x}) = -t\|\nabla f(\bar{x})\|^2,$$

where implies $\nabla f(\bar{x}) = 0$.

($\Longleftarrow$) Suppose $\nabla f(\bar{x}) = 0$. Clearly $\bar{x} \in X_0$, so $\bar{x} \in R$. Also, $\nabla f(\bar{x})^T(y - \bar{x}) = 0$, $\forall y \in R$. Thus TH 1.13 $\Longrightarrow \bar{x}$ is optimal. $\square$

1.2.1 Duality and the Lagrangian

Given problem (1.14). suppose the domain $D \neq \varnothing$. We define the LAGRANGIAN $L : D \times \mathbb{R}^m \to \mathbb{R}$ by

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

Definition 1.5. The LAGRANGE DUAL FUNCTION is defined by

$$h(\lambda) = \inf_{x \in D} L(x, \lambda) = \inf_{x \in D} \left(f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right)$$

The (LAGRANGE) DUAL PROBLEM of (1.14) is

$$\begin{aligned} \max \quad & h(\lambda) \\ & \lambda \geq 0 \end{aligned} \tag{1.17}$$

where $\lambda \in \mathbb{R}^m$ is a variable. Let d^* denote the optimal value of problem (1.17). If $h(\bar{\lambda}) = d^*$ for some $\bar{\lambda} \in \mathbb{R}^m$, $\lambda \geq 0$, then d^* is ATTAINED (by $\bar{\lambda}$), and $\bar{\lambda}$ is an OPTIMAL SOLUTION of (1.17). We also call (1.14) the PRIMAL PROBLEM.

Theorem 1.12. Th 1.15 WEAK DUALITY

Suppose problem (1.14) has an optimal value p^*. Then $\forall \lambda \in \mathbb{R}^m$, $\lambda \geq 0$,

$$p^* \geq h(\lambda)$$

Thus, $p^* \geq d^*$. This property is WEAK DUALITY between problems (1.14) and (1.17).

Proof. The result holds if $h(\lambda) = -\infty$ or $p^* = \infty$, so assume otherwise. Then $R \neq \varnothing$. Let $y \in R$ in (1.14), so $g_i(y) \leq 0$, $\forall 1 \leq i \leq m$. Then for $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$,

$$h(\lambda) = \inf_{x \in D} L(x, \lambda) \leq L(y, \lambda) = f(y) + \sum_{i=1}^{m} \lambda_i g_i(y) \leq f(y).$$

Since $h(\lambda) \leq f(y)$. $\forall y \in R$, we have $h(\lambda) \leq p^*$. □

If $p^* = d^*$, then we have STRONG DUALITY between problems (1.14) and (1.17). We do not always have strong duality, even if (1.14) is a CP.

We can show that the Lagrange dual problem of the LP (1.4) is the dual LP (1.7)[2].

1.2.2 KT conditions

We will see that under some conditions, we have a necessary and sufficient condition for STRONG DUALITY between (1.14) and (1.17).

Theorem 1.13. TH 1.16 KUHN-TUCKER THEOREM

We have the primal problem (1.14) and its dual problem (1.17).

(a) Suppose that the optimal values p^* of (1.14) and d^* of (1.17) are both attained, and we have strong duality $p^* = d^*$. Let $\bar{x}$ and $\bar{\lambda}$ be optimal solutions. Then

$$\begin{aligned} g_i(\bar{x}) &\leq 0, \quad 1 \leq i \leq m \quad \text{Primal feasible} \quad (1.18\text{ a}) \\ \bar{\lambda}_i &\geq 0, \quad 1 \leq i \leq m \quad \text{Dual feasible} \quad (1.18\text{ b}) \\ \bar{\lambda} g_i(\bar{x}) &= 0 \quad 1 \leq i \leq m \quad \text{Complementary slacks} \quad (1.18\text{ c}) \end{aligned} \tag{1.18}$$

If in addition, f and g, $(1 \leq i \leq m)$ are differentiable (so the domains $X_i (0 \leq i \leq m)$ are open), then

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 \quad \text{Stationary condition} \quad (1.18\text{ d})$$

(b) Conversely, suppose that (1.14) is a CP, and f and $g_i (1 \leq i \leq m)$ are differentiable. If $\bar{x}$ and $\bar{\lambda}$ satisfy the conditions (1.18 a-d), then $\bar{x}$ and $\bar{\lambda}_i$ are optimal solutions yielding strong duality $p^* = d^*$.

The condition (1.18 a-d) are the KT conditions.

[2] HW 2 problem

Proof. (a) Clearly (1.18 a, b) hold. Now,

$$\begin{aligned} f(\bar{x}) = h(\bar{\lambda}) &= \inf_{x\in D} L(x,\bar{\lambda}) \leq L(x,\bar{\lambda}) \\ &= f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \leq f(\bar{x}) \end{aligned}$$

so equality holds throughout. We have $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) = 0$ and (1.18 a-b) $\Rightarrow \bar{\lambda}_i h_i(\bar{x}_i) \leq 0 \quad (1 \leq i \leq m)$, so (1.18 c) holds.

Also $\bar{x}$ minimises $L(x,\bar{\lambda})$. Suppose f and $g_i \quad (1 \leq i \leq m)$ are differentiable. Then $\nabla_x L(\bar{x},\bar{\lambda}) = 0$, which is (1.18 d).

(b) Since $L(x,\bar{\lambda})$ is a sum of convex functions in x, $L(x,\bar{\lambda})$ is convex in x[3]. Since (1.18 d) says $\nabla_x L(\bar{x},\bar{\lambda}) = 0$, Th 1.14$\Rightarrow$ $\bar{x}$ minimises $L(x,\bar{\lambda})$. Thus,

$$h(\bar{\lambda}) = L(\bar{x},\bar{\lambda}) = f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \underset{(1.18\text{ c})}{=} f(\bar{x}).$$

Thus $\bar{x}$ and $\bar{\lambda}$ are optimal solutions. and $p^* = d^*$. □

Remark 1.9. ∇_x means ∇ is taken w.r.t x.

Example 1.6. Solve the CP

$$\begin{aligned} \min \quad & f(x_1,x_2) = (x_1-6)^2 + (x_2-5)^2 \\ & \begin{cases} x_1^2 - 2x_1 + x_2 & \leq 8 \\ x_1, x_2 & \geq 0 \end{cases} \end{aligned} \tag{1.19}$$

By Th 1.16, the solution to (1.19) is achieved by any $\bar{x}$, $\bar{\lambda}$ satisfying the KT conditions (1.18). We have:

$$\begin{cases} g_1(x_1,x_2) = x_1^2 - 2x_2 + x_2 - 8 \leq 0 & (1.20\text{ a}) \\ g_2(x_1,x_2) = -x_1 \leq 0 & (1.20\text{ a'}) \\ g_3(x_1,x_2) = -x_2 \leq 0 & (1.20\text{ a''}) \end{cases} \tag{1.20}$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0 \qquad (1.20\text{ b})$$

$$\lambda_1(x_1^2 - 2x_1 + x_2 - 8) = \lambda_2 x_1 = \lambda_3 x_2 = 0 \qquad (1.20\text{ c})$$

$$\begin{pmatrix} 2(x_1-6) + \lambda_1(2x_2-2) - \lambda_2 \\ 2(x_2-5) + \lambda_1 - \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1.20\text{ d})$$

Solving this, we have $\bar{x}_1 = \frac{3+\sqrt{11}}{2}$, $\bar{x}_2 = \frac{12-\sqrt{11}}{2}$, $\lambda_1 = \sqrt{11} - 2$, $\lambda_2 = \lambda_3 = 0$. The minimum

[3] HW 2

value of f is:

$$(\bar{x}_1 - 6)^2 + (\bar{x}_2 - 5)^2 = \left(\frac{-9+\sqrt{11}}{2}\right)^2 + \left(\frac{2-\sqrt{11}}{2}\right)^2 = \frac{107 - 22\sqrt{11}}{4}$$

The detailed calculations is following:

Case 1. $\lambda_1 = 0$.

By (1.20 d), we have $2(x_1 - 6) = \lambda_2$ and $2(x_2 - 5) = \lambda_3$. Then by (1.20 b, c), we have $x_1 = 6, \lambda_2 = 0$, and $x_2 = 5, \lambda_3 = 0$. But then $g_1(6,5) = 21$, and (1.20a) does not hold. Thus we cannot have the case $\lambda_1 = 0$.

Case 2. $\lambda_1 > 0$.

By (1.20 c), we have $x_1^2 - 2x_1 + x_2 - 8 = 0$. If $x_1 = 0$, then (1.20d) gives $2\lambda_1 + \lambda_2 = -12$, which contradicts (1.20 b). So $x_1 > 0$, and (1.20c) implies $\lambda_2 = 0$. Then if $x_2 = 0$, we have $0 = x_1^2 - 2x_1 - 8 = (x_1 - 4)(x_1 + 2)$, so $x_1 = 4$. By (1.20d), we have $-4 + 6\lambda_1 = 0$ and $-10 + \lambda_1 - \lambda_3 = 0$, which gives $\lambda_1 = \frac{2}{3}$, and $\lambda_3 = -10 + \frac{2}{3} < 0$, contradicting (1.20b). So $x_2 > 0$, and (1.20c) implies $\lambda_3 = 0$. Now (1.20d) gives $2(x_1 - 6) + \lambda_1(2x_1 - 2) = 0$ and $2(x_2 - 5) + \lambda_1 = 0$. Eliminating λ_1 and using $x_2 = 8 - x_1^2 + 2x_1$, we have $x_1(2x_1^2 - 6x_1 - 1) = 0$. Solving the quadratic gives $\bar{x}_1 = \frac{3+\sqrt{11}}{2}$, and then easy calculations give $\bar{x}_2 = \frac{12-\sqrt{11}}{2}$ and $\bar{\lambda}_1 = \sqrt{11} - 2$. We already know that $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$. This is the unique optimal solution, and the optimal value of f is

$$(\bar{x}_1 - 6)^2 + (\bar{x}_2 - 5)^2 = \left(\frac{-9+\sqrt{11}}{2}\right)^2 + \left(\frac{2-\sqrt{11}}{2}\right)^2 = \frac{107 - 22\sqrt{11}}{4}.$$

1.2.3 Algorithm Methods

In general, problem (1.14) can't be solved exactly. We consider algorithm methods we can give approximate solutions to (1.14).

We first consider the case when (1.14) is an unconstrained CP. That is:

$$\min \quad f(x) \tag{1.21}$$

where $f : X \to \mathbb{R}$ is a convex function, for some convex $X \subset \mathbb{R}^n$. Suppose f is twice continuously differentiable, so X is open, and (1.21) has an optimal solution $\bar{x}$. Suppose $x^{(0)} \in X$ and $x^{(0)} \notin R^*$. We iteratively find $x^{(0)}, x^{(1)}, x^{(2)}, \cdots \in X$, s.t. $f(x^{(k)} \to p^*$ as $k \to \infty$. Such $\{x^{(k)}\}_{k=0}^{\infty}$ is a MINIMISING SEQUENCE. We end the algorithm when some form of accuracy is achieved. The set

$$S = \{x \in X : f(x) \leq f(x^{(n)}\}$$

is convex[4]. Suppose S is compact. We want to find the $x^{(k)}$ s.t.

$$f(x^{(0)}) > f(x^{(1)}) > \cdots > p^*$$

Such an algorithm is a DESCENT METHOD. We have $x^{(k)} \in S$, $\forall k > 0$. Each iteration involves a direction $z^{(k)} \in \mathbb{R}^n$ and a step size $t^{(k)} \geq 0$. Thus

$$x^{(k+1)} = x^{(k)} + t^{(k)} z^{(k)},$$

or

$$x^+ = x + tz$$

if we want to focus on one iteration.

Th 1.8 $\Rightarrow$ f is convex on the line segment $L = \{x + \lambda z : \lambda \geq 0\}$, and $f\Big|_L$ attains its minimum in L. We usually choose

$$t \in \{\lambda \geq 0 : f(x + \lambda z) \text{ is minimised on } L \text{ at } \lambda\}. \tag{1.22}$$

We remark that there are other choices for t, since (1.22) is not always easy to compute.

We consider two possible choices for the direction z.

A. Steepest Descent Method

We choose $z = -\nabla f(x)$. Th 1.14 $\Rightarrow$ $\nabla f(x) \neq 0$, since $x \notin R^*$. For small $\lambda > 0$,

$$\begin{aligned} f(x^*) = f(x + tz) \leq f(x + \lambda z) &= f(x) + \lambda \nabla f(x)^T z + o(\lambda) \\ &= f(x) - \lambda \|\nabla f(x)\|^2 + o(\lambda) < f(x), \end{aligned}$$

where $o(\lambda)$ means that $\frac{0(\lambda)}{\lambda} \to 0$ as $\lambda \to 0$. So $z = -\nabla f(x)$ is a suitable choice.

B. Newton's Method

Assume $\nabla^2 f(x) > 0$, $\forall x \in S$. Note that $\nabla^2 f(x)$ is invertible. Taylor's theorem $\Rightarrow$

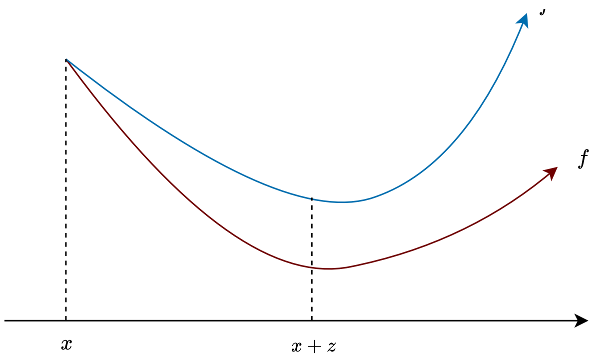
$$f(x + v) \approx f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v = \hat{f}(x + v)$$

$\hat{f}(x + v)$ is a convex quadratic function in v, minimised at $v = \nabla^2 f(x)^{-1} \nabla f(x)$[5]. We define the Newton Step:

$$z = -\nabla^2 f(x)^{-1} \nabla f(x).$$

[4] In HW1

[5] HW 2



For small $\lambda > 0$,

$$
\begin{aligned}
f(x^+) = f(x + tz) \leq f(x + \lambda z) &= f(x) + \lambda \nabla f(x)^T z + o(\lambda) \\
&= f(x) - \lambda \nabla f(x)^T \nabla^f(x)^{-1} \nabla f(x) + o(\lambda) < f(x),
\end{aligned}
$$

since $\nabla^2 f(x)^{-1} > 0$. So $z = -\nabla^2 f(x)^{-1} \nabla f(x)$ is a suitable choice.

$f(x) = o(g(x))$ means $\frac{f(x)}{g(x)} \to 0$ as $x \to c$ for some c. In most cases, $c = \infty$ or $c = 0$.

Note: "o depends on c"

Thus, we have the following algorithm for solving (1.21). Fix $\varepsilon > 0$. Suppose we found $x^{(0)}, x^{(1)}, \cdots, x^{(k)} \in S$. We find z using A or B above. A stopping criteria can be $\|z^{(k)}\| \leqslant \varepsilon$. If $\|z^{(k)}\| \geqslant \varepsilon$, thus we find $x^{(k+1)}$, and iterate.

Example 1.7. Consider

$$
\min \quad f(x_1, x_2) = x_1^4 + 8x_2^4
$$

where $(x_1, x_2) \in \mathbb{R}^2$. We have $p^* = 0$, where $(x_1, x_2) = (0, 0)$. Suppose we use the above algorithm with $\varepsilon = 0.1$ $x^{(0)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For steepest descent method, we have

$$
\begin{aligned}
\nabla f(x) = \begin{pmatrix} 4x_1^3 \\ 32x_2^3 \end{pmatrix}, z^{(0)} = -\nabla f\left(x^{(0)}\right) = -\begin{pmatrix} 32 \\ 32 \end{pmatrix} \\
f(2 - 32\lambda, 1 - 32\lambda) = (2 - 32\lambda)^4 + 8(1 - 32\lambda)^4 \\
\frac{df}{d\lambda} = -128(2 - 32\lambda)^3 - 1024(1 - 32\lambda)^3
\end{aligned}
$$

Solving $\frac{df}{d\lambda} = 0 \Rightarrow t^{(0)} = \frac{1}{24}$. Thus, $x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{24} \begin{pmatrix} 32 \\ 32 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$. We have

$z^{(1)} = -\nabla f\left(x^{(1)}\right) = \begin{pmatrix} -\frac{32}{27} \\ \frac{32}{27} \end{pmatrix}$ and $\left\|z^{(1)}\right\| = \frac{32}{27}\sqrt{2} > \varepsilon$. We iterate this until we find $z^{(k)}$ with $\left\|z^{(k)}\right\| < \varepsilon = 0.1$.

For Newton's method, we have:

$$\nabla^2 f(x) = \begin{pmatrix} 12x_1^2 & 0 \\ 0 & 96x_2^2 \end{pmatrix}, \quad \nabla^2 f(x)^{-1} = \begin{pmatrix} \frac{1}{12x_1^2} & 0 \\ 0 & \frac{1}{96x_2^2} \end{pmatrix}$$

$$-\nabla^2 f(x)^{-1}\nabla f(x) = -\begin{pmatrix} \frac{1}{12x_1^2} & 0 \\ 0 & \frac{1}{96x_2^2} \end{pmatrix}\begin{pmatrix} 4x_1^3 \\ 32x_2^3 \end{pmatrix} = -\begin{pmatrix} \frac{x_1}{3} \\ \frac{x_2}{3} \end{pmatrix}, z^{(0)} = -\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Thus $f(2-\frac{2}{3}\lambda, 1-\frac{1}{3}\lambda) = (2-\frac{2}{3}\lambda)^4 + 8(1-\frac{1}{3}\lambda)^4 = 24(1-\frac{1}{3}\lambda)^4$, which is minimised at $\lambda = 3$. So $t^{(0)} = 3 \implies x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and we terminate in one step.

Penalty function Method

Again, consider problem (1.14), with f and $g_i (1 \leq i \leq m)$ continuous. We introduce the penalty program, which is the unconstrained problem:

$$\min \quad q(c,x) = f(x) + cp(x) \tag{1.23}$$

over $x \in \mathbb{R}^n$, where $c > 0$, and $p : \mathbb{R}^n \to \mathbb{R}$ is the penalty function, s.t. p is continuous, $p(x) \geq 0$, $\forall x \in \mathbb{R}^n$ and $p(x) = 0 \iff x \in R$. The penalty term $cp(x)$ gives a cost if x violates the constraints of (1.14).

Remark 1.10. Recall: If $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$, then

$$\underset{x \in X}{\arg\min} f(x) = \{y \in X : f(y) \leqslant f(x), \forall x \in X\}.$$

Lemma 1.1. Let $0 < c_1 < c_2 < \cdots$ with $c_k \to \infty$. Let $q(c,x) = f(x) + cp(x)$ as in (1.23). Let $x_k \in \underset{x \in \mathbb{R}^n}{\arg\min} q(c_k, x)$. Then $\forall k \geqslant 1$,[a]

$$(a) q(c_k, x_k) \leqslant q(c_{k+1}, x_{k+1}).$$
$$(b) p(x_k) \geqslant p(x_{k+1})$$
$$(c) f(x_k) \leqslant f(x_{k+1})$$
$$(d) f(x^*) \geqslant q(c_k, x_k) \geqslant f(x_k), \quad \text{where } x^* \in R^* (\text{of problem (1.14)}).$$

[a] Proof HW2

Remark 1.11. Recall: For $S \subset \mathbb{R}^n$, a limit point of S is $x \in \mathbb{R}^n$ s.t.

$\forall$ open set U with $x \in U$, $\exists y \in S \backslash \{x\}$, s.t. $y \in U$.

Theorem 1.14. With the same notations as in (1.23) and Lemma 1.17, let $\bar{x}$ be a limit point of $\{x_k\}_{k=1}^{\infty}$. Then $\bar{x} \in R^*$ in problem (1.14).

Proof. Take a sub-sequence $x_{k_1}, x_{k_2}, \cdots$ of $\{x_k\}_{k=1}^{\infty}$ s.t. $x_{k_j} \to \bar{x}$[6]. Since f and p are continuous,

$$\begin{aligned}
&\bar{q} = \lim_{k_j \to \infty} q(c_{k_j}, x_{k_j}) = \lim_{k_j \to \infty} f(x_{k_j}) + \lim_{k_j \to \infty} c_{k_j} p(x_{k_j}) \underset{\text{Th 1.17(d)}}{\leqslant} f(x^*) \\
\Longrightarrow &\bar{q} - f(\bar{x}) = \lim_{k_j \to \infty} c_{k_j} p(x_{k_j}) \leqslant f(x^*) - f(\bar{x})
\end{aligned}$$

Thus $\lim\limits_{k_j \to \infty} c_{k_j} p(x_{k_j})$ is finite, and since $c_{k_j} \to \infty$, we have $p(x_{k_j}) \to 0$. Thus $p(\bar{x}) = 0$, and $\bar{x} \in R$. Moreover,

$$\begin{aligned}
&f(x^*) \geqslant f(\bar{x}) + \lim_{k_j \to \infty} c_{k_j} p(x_{k_j}) \geqslant f(\bar{x}) \\
\Longrightarrow &f(x^*) = f(\bar{x}), \quad \text{and } \bar{x} \in R^*
\end{aligned}$$

□

A commonly used penalty function is

$$p(x) = \sum_{i=1}^{m} \left[\max(0, g_i(x))\right]^t, \quad \text{for some } t \geqslant 1. \tag{1.24}$$

Especially, the cast $t = 2$.

Suppose (1.14) also has equality constraints:

$$\begin{aligned}
\min \quad & f(x) \\
& \begin{cases} g_i(x) \leqslant 0, & 1 \leqslant i \leqslant m \\ h_i(x) = 0 & 1 \leqslant i \leqslant r \end{cases}
\end{aligned} \tag{1.25}$$

where $h_i : Y_i \to \mathbb{R}$, with $Y_i \subset \mathbb{R}^n$ the domain of h_i. We may rewrite (1.25) in the form of (1.14).

$$\begin{aligned}
\min \quad & f(x) \\
& \begin{cases} g_i(x) \leqslant 0, & 1 \leqslant i \leqslant m \\ h_i(x) \leqslant 0 & 1 \leqslant i \leqslant r \\ -h_i(x) \leqslant 0 & 1 \leqslant i \leqslant r \end{cases}
\end{aligned}$$

[6] HW 2

and work from there. The penalty function (1.24) becomes

$$p(x) = \sum_{i=1}^{m} [\max(0, g_i(x))]^t + \sum_{i=1}^{r} |h_i(x)|^t \tag{1.26}$$

Example 1.8. Use the penalty function method to solve

$$\begin{aligned} \min \quad & -3x_1 + x_2 \\ & x_1^2 - 2x_2 \leqslant 4 \end{aligned}$$

From (1.23) and (1.26) with $t = 2$, we have

$$q(c_k, x) = \begin{cases} -3x_1 + x_2 & \text{if } x_1^2 - x_2 - 4 \leqslant 0 \\ -3x_1 + x_2 + c_k \left(x_1^2 - x_2 - 4\right)^2 & \text{if } x_1^2 - x_2 - 4 > 0 \end{cases}$$

We find $x_k \in \underset{x \in \mathbb{R}^n}{\arg\min} q(c_k, x)$.

- If $x_1^2 - 2x_2 - 4 \leqslant 0$, then $-3x_1 + x_2 \leqslant \frac{1}{2}x_1^2 - 3x_2 - 2 = \frac{1}{2}\left(x_1 - 3\right)^2 \geqslant -\frac{13}{2}$, with equality when $(x_1, x_2) = (3, \frac{5}{2})$. Minimum $q(c_k, x) = -3 \cdot 3 + \frac{5}{2} = -\frac{13}{2}$

- If $x_1^2 - 2x_2 - 4 > 0$. We solve $\nabla_x q(c_k, x) = 0$ (See Th 1.14). We have

$$\nabla_x q(c_k, x) = \begin{pmatrix} -3 + 4c_k x_1 (x_1^2 - 2x_2 - 4) \\ 1 - 4c_k (x_1^2 - 2x_1 - 4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  We can find $x_1 = 3$, $x_2 = \frac{5}{2} - \frac{1}{8c_k}$, and minimum $q(c_k, x) = -\frac{13}{2} - \frac{1}{16c_k}$. Thus $x_k = (3, \frac{5}{2} - \frac{1}{8c_k} \to (3, \frac{5}{2})$. Th 1.18

$$\Longrightarrow \bar{x} = (3, \frac{5}{2}) \in R^*, \text{and } p^* = -3 \cdot 3 + \frac{5}{2} = -\frac{13}{2}$$

Example 1.9. Use the penalty program method to solve

$$\begin{aligned} \min \quad & x_1^2 + 2x_2^2 + 4x_3^2 \\ & 1 - x - 1 - x_2 - x_3 = 0 \end{aligned}$$

From (1.23) and (1.26) with $t = 2$, we have

$$q(c_k, x) = x_1^2 + 2x_2^2 + 4x_3^2 + c_k(1 - x_1 - x_2 - x_3)^2$$

To find $x_k \in \underset{x\in\mathbb{R}^n}{\arg\min} q(c_k, x)$, we solve $\nabla_x q(c_k, x) = 0$. We have:

$$\nabla_x q(c_k, x) = \begin{pmatrix} 2x_1 - 2c_k(1 - x_1 - x_2 - x_3) \\ 4x_2 - 2c_k(1 - x_1 - x_2 - x_3) \\ 8x_3 - 2c_k(1 - x_1 - x_2 - x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then $x_1 = 2x_2 = 4x_3$, and $x_k = \left(\frac{4c_k}{4+7c_k}, \frac{2c_k}{4+7c_k}, \frac{c_k}{4+7c_k}\right)$. Since $x_k \to (\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$. $\bar{x} = (\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$. We have $p^* = \frac{4}{7}$.

Algorithm Method with linear constraints

Consider the CP

$$\begin{aligned} \min \quad & f(x) \\ Ax & \leqslant b \end{aligned} \tag{1.27}$$

where $f : X_0 \to \mathbb{R}$ with $X_0 \subset \mathbb{R}^n$ open, f convex and continuously differentiable, and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$. Let $R = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and assume $R \neq \varnothing$. Note that R is convex. Let $x^{(0)} \in R$ with $x^{(0)} \notin R^*$. Taylor's theorem $\Longrightarrow$

$$\begin{aligned} f(x) &\approx f(x^{(0)}) + \nabla f(x^{(0)})^T(x - x^{(0)}) \\ &= f(x^{(0)}) - \nabla f(x^{(0)})^T x^{(0)} + \nabla f(x^{(0)})^T x \end{aligned}$$

Thus we consider

$$\begin{aligned} \min \quad & f(x^{(0)})^T x \\ Ax & \leqslant b \end{aligned} \tag{1.28}$$

Let $y^{(0)}$ be an optimal solution of (1.28), so $\nabla f(x^{(0)})^T(y^{(0)} - x^{(0)}) \leqslant 0$. If $\nabla f(x^{(0)})^T(y^{(0)} - x^{(0)}) = 0$, then $\forall x \in R$,

$$\begin{aligned} f(x) &\underset{\text{Th 1.9}}{\geqslant} f(x^{(0)}) + \nabla f(x^{(0)})^T(x - x^{(0)}) \\ &\geqslant f(x^{(0)}) + \nabla f(x^{(0)})^T(y^{(0)} - x^{(0)}) \\ &= f(x^{(0)}), \end{aligned}$$

so $x^{(0)} \in R^*$, a contradiction. Thus $\nabla f(x^{(0)})^T(y^{(0)} - x^{(0)}) < 0$. Now R convex $\Longrightarrow (1-\lambda)x^{(0)} + \lambda y^{(0)} \in R$,

$\forall \lambda \in [0,1]$. Let

$$\lambda^{(0)} \in \underset{\lambda\in[0,1]}{\arg\min} f\left((1-\lambda)x^{(0)} + \lambda y^{(0)}\right),$$
$$x^{(1)} = (1-\lambda^{(0)})x^{(0)} + \lambda^{(0)}y^{(0)}$$

Then for $\lambda \in (0,1)$ small,

$$\begin{aligned}
f(x^{(1)}) &= f\left((1-\lambda^{(0)})x^{(0)} + \lambda^{(0)}y^{(0)}\right) \\
&\leqslant f\left((1-\lambda)x^{(0)} + \lambda y^{(0)}\right) \\
&= f\left(x^{(0)} + \lambda(y^{(0)} - x^{(0)})\right) \\
&= f(x^{(0)} + \lambda\nabla f(x^{(0)})^T(y^{(0)} - x^{(0)}) + o(\lambda) \\
&< f(x^{(0)})
\end{aligned}$$

Repeating, we obtain a minimising sequence $x^{(0)}, x^{(1)}, x^{(2)}, \cdots \in R$ for problem (1.27). A stopping criteria can be when we have k s.t. $\left|\nabla f(x^{(k)})^T(y^{(k)} - x^{(k)})\right| < \varepsilon$, for some given $\varepsilon > 0$.

Example 1.10. Consider

$$\min \quad f(x_1, x_2) = 4x_1^2 + (x_2-2)^2$$
$$\begin{cases} -2 \leqslant x_1 \leqslant 2 \\ -1 \leqslant x_2 \leqslant 1 \end{cases}$$

Let $x^{(0)} = (-2,-1)^T$. We have $\nabla f(x) = (8x_1, 2(x_2-2))^T$, and

$$\min \quad \nabla f(x^{(0)})^T x = -16x_1 - 6x_2$$
$$\begin{cases} -2 \leqslant x_1 \leqslant 2 \\ -1 \leqslant x_2 \leqslant 1 \end{cases}$$

has optimal solution $y^{(0)} = (2,1)^T$. Now

$$f((1-\lambda)x^{(0)} + \lambda y^{(0)}) = f(4\lambda - 2, 2\lambda - 1) = 68\lambda^2 - 76\lambda + 25$$

Solving $\frac{df}{d\lambda} = 0$ gives $\lambda^{(0)} = \frac{19}{34}$. Thus

$$x^{(1)} = \frac{15}{34}\begin{pmatrix}-2\\-1\end{pmatrix} + \frac{19}{34}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}\frac{4}{17}\\\frac{2}{17}\end{pmatrix}$$

We may repeat this procedure to obtain a minimising sequence $x^{(1)}, x^{(2)}, \cdots$.

1.3 Integer programming

Consider the LP

$$\max \quad c^T x \qquad \begin{cases} Ax \leqslant b \\ x \geqslant 0 \end{cases} \tag{1.5}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, with optimal value p^*. In many applications, we will be interested in the restriction $x \in \mathbb{Z}^n_{\geqslant 0}$. For example, suppose a company sells n different types of products, where x_i is the number of units that product of type i is sold. The company wants to maximise the profit, with some constraints are sold. Then we would be interested in $x_i \in \mathbb{Z}_{\geqslant 0}$.

Thus, we consider the problem.

$$\max \quad c^T x \qquad \begin{cases} Ax \leqslant b \\ x \in \mathbb{Z}^n_{\geqslant 0} \end{cases} \tag{1.29}$$

Let q^* be the optimal value. (1.29) is an integer programming problem (IP). The case

$$\max \quad c^T x \qquad \begin{cases} Ax \leqslant b \\ x_i \in \{0,1\}, \quad 1 \leqslant i \leqslant n \end{cases} \tag{1.30}$$

is a BINARY INTEGER PROGRAMMING PROBLEM (BIP). We say that (1.5) is that LP relaxation of (1.29). Note that $q^* \leqslant p^* \iff$ (1.5) has an optimal solution in $\mathbb{Z}^n_{\geqslant 0}$.

We shall consider two methods for solving (1.29) and (1.30).

A. Branch and bound method

Idea: We divide the IP (1.29) into several subproblems, and work towards finding the optimal value q^*.

Example 1.11. Solve

$$\max \quad f = -x_1 + 4x_2 \tag{1.31}$$

$$\begin{cases} -5x_1 + 10x_2 & \leqslant 11 \\ 5x_1 + 10x_2 & \leqslant 49 \\ x_1 & \leqslant 5 \\ x_1, x_2 & \in \mathbb{Z}_{\geqslant 0} \end{cases} \tag{1.31 a}$$

The optimal solution of (1.31 a) is $(2, 2)$, and the optimal value is $q^* = 6$. See the handout **??** for full details.

We may also consider the version of the IP (1.29) where some of the x_i are integers.

B. Mixed integer programming problem

A MIXED INTEGER PROGRAMMING PROBLEM (MIP) has the form

$$\max \quad c^T x$$
$$\begin{cases} Ax \leqslant b \\ x_i \in \mathbb{Z}_{\geqslant 0}, & i \in I \\ x_i \geqslant 0, & i \in \{1, 2, \cdots, n\} \backslash I \end{cases}$$

for some fixed $I \subset \{1, 2, \cdots, n\}$.

See Example in handout **??** where (1.31 a) is modified to $x_i \in \mathbb{Z}_{\geqslant 0}$, $x_2 \geqslant 0$.

C. BIP problem

For BIP, we may also use the same method. The following method is similar, but more simple.

Example 1.12. Solve

$$\max \quad f = -8x_1 - 2x_2 - 4x_3 - 7x_4 - 5x_5 + 10$$
$$\begin{cases} -3x_1 - 3x_2 + x_3 3 + 2x_4 + 3x_5 \leqslant -2 \\ -5x_1 - 3x_2 - 2x_2 - x_4 + x_5 \leqslant -4 \\ x_i \in \{0, 1\}, \quad 1 \leqslant i \leqslant 5 \end{cases}$$

Note that to maximise f, we would "prefer $x_i = 0$ for as many i as possible".

1. Consider $x_1 = 0$ and $x_1 = 1$.

2. If $x_1 = 1$, then setting $x_2 = x_3 = x_4 = x_5 = 0$, the constraints are satisfied. We have an incumbent optimal value of 2.

3. Let $x_1 = 0$. Consider $x_2 = 0$ and $x_2 = 1$.

4. If $x_2 = 0$, then first constraint can't be satisfied.

5. Let $x_2 = 1$. Consider $x_3 = 0$ and $x_3 = 1$.

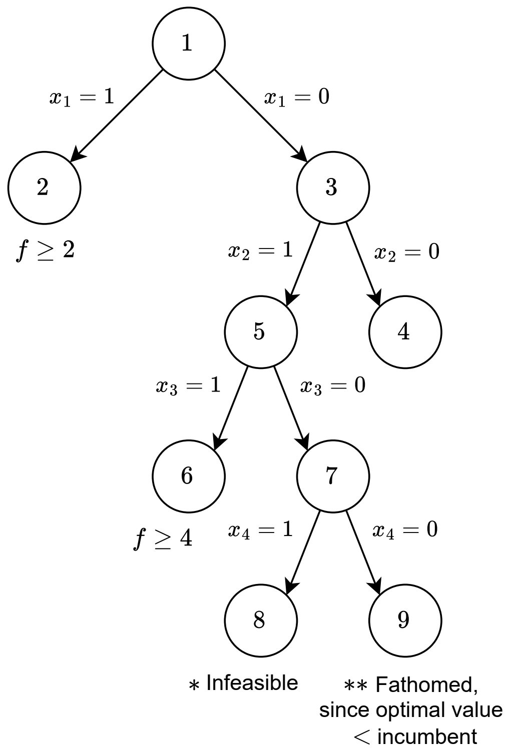
6. If $x_3 = 1$, then setting $x_4 = x_5 = 0$, the constraints are satisfied. We have a new incumbent optimal value of 4.

7. Let $x_3 = 0$. Consider $x_4 = 0$ and $x_4 = 1$.

8. If $x_4 = 1$, then the objective value is $\leqslant 1$, $<$ current incumbent optimal value of 4.

9. Let $x_4 = 0$. We currently have $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$. Second constraint is not satisfied.

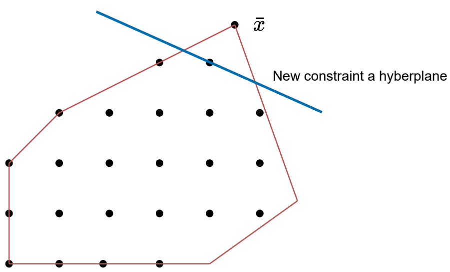
We conclude that the optimal solution is $(0, 1, 1, 0, 0)$, and the optimal value is 4. We have the following branching procedure:



Note: If the coefficient of some x_i in the objective function is positive, we may substitute $x_i' = 1 - x_i$.

1.3.1 B. Cutting plane method

Idea: Suppose that in the IP (1.29), the LP relaxation (1.4) has an optimal solution $\bar{x} \notin \mathbb{Z}^n_{\geq 0}$. We add a new constraint into (1.4) which defines a hyberplane. The hyberplane "cuts off" $\bar{x}$ from the feasible set of (1.4) but none of the feasible integer points of (1.29).



We solve the new LP, and repeat the procedure until we find an optimal solution in $\mathbb{Z}^n_{\geqslant 0}$ for some LP.

Here, we assume the entries of A, b, c in (1.29) are in $\mathbb{Z}$, or equivalently, in $\mathbb{Q}$ (since we may clear fractions by multiplying by a suitable integer).

Lemma 1.2. Lemma 1.19 Let $x_i \in \mathbb{Z}_{\geq 0}$, $a_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$, and $b \in \mathbb{R}$. If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, then

$$\begin{aligned} & \lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \leqslant \lfloor b \rfloor, \\ \text{and} \quad & (a_1 - \lfloor a_1 \rfloor) x_1 + \cdots + (a_n - \lfloor a_n \rfloor) x_n \geqslant b - \lfloor b \rfloor. \end{aligned}$$

Proof. We have

$$\begin{aligned} b &= (a_1 - \lfloor a_1 \rfloor) x_1 + \cdots + (a_n - \lfloor a_n \rfloor) x_n + \lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \\ &\geqslant \lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n. \end{aligned}$$

Since $\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \in \mathbb{Z}$, we have $\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \leqslant \lfloor b \rfloor$.

Multiplying by -1, and adding $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ gives $(a_1 - \lfloor a_1 \rfloor x_1 + \cdot + (a_n - \lfloor a_n \rfloor x_n \geqslant b_n - \lfloor b \rfloor$ ☐

Example 1.13. Use the cutting plane method to solve

$$\max \quad f = 2x_1 + 3x_2 \tag{1.32}$$
$$\begin{cases} -2x_1 + 2x_2 & \leqslant 3 \\ x_1 + 2x_2 & \leqslant 6 \\ 4x_1 + 5x_2 & \leqslant 20 \\ x_1, x_2 & \in \mathbb{Z}_{\geqslant 0} \end{cases}$$

Introduce slack variables $s_1, s_2, s_3 \geqslant 0$ to the LP relaxation of (1.32). By simplex method, we have

x_1	x_2	s_1	s_2	s_3	f	
-2	2	1	0	0	0	3
1	2	0	1	0	0	6
4	5	0	0	1	0	20
-2	-3	0	0	0	1	0

After some work, we have

x_1	x_2	s_1	s_2	s_3	f	
0	1	0	$\frac{4}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
1	0	0	$-\frac{5}{3}$	$\frac{3}{2}$	0	$\frac{10}{3}$
0	0	1	-6	2	0	7
0	0	0	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{32}{3}$

The final table says that we have not found an integer solution to the LP relaxation. The entries on the right of rows 1, 2, 4 are non-integers. Note that $f \in \mathbb{Z}_{\geqslant 0}$. We have

$$\begin{aligned} x_2 + \tfrac{4}{3}s_2 - \tfrac{1}{3}s_3 &= \tfrac{4}{3} & & \tfrac{1}{3}s_2 + \tfrac{2}{3}s_3 \geqslant \tfrac{1}{3} \quad (1) \\ x_1 - \tfrac{5}{3}s_2 + \tfrac{2}{3}s_2 &= \tfrac{10}{3} & \underset{\text{Lemma 1.19}}{\Longrightarrow} \quad & \tfrac{1}{3}s_2 + \tfrac{2}{3}s_3 \geqslant \tfrac{1}{3} \quad (2) \\ \tfrac{2}{3}s_2 + \tfrac{1}{3}s_3 + f &= \tfrac{32}{3} & & \tfrac{2}{3}s_2 + \tfrac{1}{3}s_3 \geqslant \tfrac{2}{3} \quad (3) \end{aligned}$$

Each of (1), (2), (3) is a Gomory fractional cut. We see that (1) and(2) are the same. Note that (3)$\Longrightarrow$ (1), (2), since (3) $\Longrightarrow \frac{1}{3}s_2 + \frac{1}{6}s_3 \geqslant \frac{1}{3}$, which $\Longrightarrow$ (1), (2). Now

$$(3) \Longleftrightarrow \frac{2}{3}(6 - x_1 - 2x_2) + \frac{1}{3}(20 - 4x_1 - 5x_2) \geqslant \frac{2}{3} \Longleftrightarrow 2x_1 + 3x_2 \leqslant 10$$

Now we solve a new LP by adding the constraint (3), which we write as $-\frac{2}{3}s_2 - \frac{1}{3}s_3 + s_4 = -\frac{2}{3}$ for a slack variable $s_3 \geqslant 0$. We use the dual simplex method:

	x_1	x_2	s_1	s_2	s_3	s_4	
x_2	0	1	0	$\frac{4}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
x_1	1	0	0	$-\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{10}{3}$
s_1	0	0	1	-6	2	0	-7
s_4	0	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$(pivot)	1	$-\frac{2}{3}$
$-f$	0	0	0	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$-\frac{32}{3}$

Then we have:

	x_1	x_2	s_1	s_2	s_3	s_4	
x_2	0	1	0	2	0	-1	2
x_1	1	0	0	-3	0	2	2
s_1	0	0	1	-10	0	6	3
s_3	0	0	0	2	1	-3	2
$-f$	0	0	0	0	0	-1	-10

We find the integer optimal solution $(x_1, x_2) = (2, 2)$ for (1.32). The optimal values is $f = 10$.

Remark 1.12. (1), (2) $\implies 3x_1 + 4x_2 \leqslant 15$, which gives a weaker cutting line that $2x_1 + 3x_2 \leqslant 10$ from (3). In general, we may obtain several cutting planes in dimension n which are incomparable.

If we did not find an integer optimal solution, we may repeat the whole procedure.

1.4 Dynamic Programming

A dynamic programming (DP) problem can be described as a problem that can be solved by solving a sequence of subproblems. DPs occur in many areas of OR, not just in optimisation theory. We consider some important example.

1.4.1 A. Knapsack Problem

We would like to prepare a knapsack of items, such as water bottles, apples, etc, for a camping trip. There are n types of items, and each item of type i has weight a_i and a value of importance, or profit $c_i (1 \leqslant i \leqslant n)$. The knapsack can hold a total weight of b.

Thus we want to solve the DP

$$\begin{aligned} \max \quad & c^T x \\ & \begin{cases} a^T x & \leqslant b \\ x & \in \mathbb{Z}^n_{\geqslant 0} \end{cases} \end{aligned} \tag{1.33}$$

where $c, a \in \mathbb{R}^n_{\geqslant 0}$, $b \in \mathbb{R}$. We have x_i is the number of items of type i.

We may solve (1.33) with a recurrence relation, as follows. For $a \leqslant d \leqslant b$, $1 \leqslant k \leqslant n$. Let $S_k(d) =$ maximum profits that can be achieved by item types $k, k+1, \cdots, n$, if the knapsack has remaining weight d. We want $S_1(b)$.

Suppose we have assigned that values for $x_1, \cdots, x_{k-1}$. We want to assign a value for x_k, and the knapsack has remaining weight d. Then we require $0 \leqslant x_k \leqslant \lfloor \frac{d}{a_k} \rfloor$. Adding the type k items increases the profit by $c_k x_k$ and decreases the remaining weight $a_k x_k$. The best "future contribution" to the total profit in the knapsack is $S_{k+1}(d - a_k x_k)$. Thus

$$S_k(d) = \max\{c_k x_k + S_{k+1}(d - a_k x_k) : \ 0 \leqslant x_k \leqslant \lfloor \frac{d}{a_k} \rfloor\}, \ 1 \leqslant k \leqslant n-1 \tag{1.34}$$

Example 1.14. Solve the IP

$$\max \quad 4x_1 + 5x_2 + 6x_3$$
$$\begin{cases} 3x_1 + 4x_2 + 5x_3 & \leqslant 10 \\ x_1,\ x_2,\ x_3 & \in \mathbb{Z}_{\geqslant 0} \end{cases}$$

Solution. Stage 3 Find $S_3(d)$ for $0 \leqslant d \leqslant 10$.

$$S_3(d) = \begin{cases} 0, & 0 \leqslant d < 5, & (x_3 = 0) \\ 6, & 5 \leqslant d < 10, & (x_3 = 1) \\ 12, & d = 10, & (x_3 = 2) \end{cases}$$

Stage 2 Find $S_2(d)$ for $0 \leqslant d \leqslant 10$. (1.34) $\Rightarrow$

$$S_2(d) = \max\{5x_2 + S_3(d - 4x_2) : 0 \leqslant x_2 \leqslant \lfloor \frac{d}{4} \rfloor\}.$$

$0 \leqslant d < 4$: $x_2 = 0$, $S_2(d) = S_3(d) = 0$.

$4 \leqslant d < 8$: $x_2 \in \{0,1\}$, $S_2(d) = \max(S_3(d), 5 + S_3(d-4)) = \max(S_3(d), 5) = \begin{cases} 5, & 4 \leqslant d < 5 \\ 6, & 5 \leqslant d < 8 \end{cases}$

$8 \leqslant d < 10$: $x_2 \in \{0,1,2\}$,

$$S_2(d) = \max(S_3(d), 5 + S_3(d-4), 10 + S_3(d-8)) = \begin{cases} \max(6,5,10) = 10, & 8 \leqslant d < 9, \\ \max(6,11,10) = 11, & 9 \leqslant d < 10, \\ \max(12,11,10) = 12, & d = 10. \end{cases}$$

Stage 1 Find $S_1(10)$. (1.34)$\Rightarrow$

$$\begin{aligned} S_1(10) &= \max\{4x_1, S_2(10 - 3x_1) :\ 0 \leqslant x_1 \leqslant \lfloor \frac{10}{3} \rfloor\} \\ &= \max\{S_2(10), 4 + S_2(7), 8 + S_2(4), 12 + S_2(1)\} \\ &= \max\{12, 4+6, \boxed{8+5}, 12+0\} \\ &= 13. \end{aligned}$$

Optimal value $= 13$. Optimal solution: $x_1 = 2$.

$$S_2(4) = \max\{0, \boxed{5}\} = 5 \Rightarrow\ x_2 = 1$$
$$x_3 = 0 \text{ from } 4x_1 + 5x_2 + 6x_3 = 13$$

□

We may more conveniently use a table method, which is a common approach in DP problems. The values of $S_1(d)$ $(0 \leqslant d \leqslant 10)$ are also found.

k \ d	0	1	2	3	4	5	6	7	8	9	10
3	$\underline{0}^{+5}$	0	0	0	0	6	6	6	6	6	12
2	0	0	0	0	$\underline{5}^{+8}$	6	6	6	10	$\underline{11}$	12
1	0	0	0	4	5	6	8	9	10	12	$\underline{13}$

Table 1.2: Example

Again, maximum= $S_1(10) = 13$. To calculate the values of the x_1 achieving this maximum, we backtrack. See the entries in red. Note that if there is q tie during the back tracking, we may consider all possibilities to obtain all optimal solutions.

Table Method 1. Fill in the first row $(k = n)$: $S_n(d) = c_n \lfloor \frac{d}{a_n} \rfloor$.

2. Fill in each successive row $(n > k \geqslant 1)$ by using (1.34). Note that each row depends only on the entries in the row above.

We may add more constraints to problem (1.33). For two constraints, we have the ID:

$$\begin{aligned} \max \quad & c^T x \\ & \begin{cases} a^T x \leqslant b \\ p^T x \leqslant q \\ x \in \mathbb{Z}^n_{\geqslant 0} \end{cases} \end{aligned} \tag{1.35}$$

where $c, a, q \in \mathbb{R}^n_{\geqslant 0}$, $b, q \in \mathbb{R}_{\geqslant 0}$.

We may interpret the second constraint as: Each item of type i $(1 \leqslant i \leqslant n)$ has volume p_i, and the knapsack can also hold a maximum volume of q.

For $0 \leqslant d \leqslant b$, $0 \leqslant r \leqslant q$, $1 \leqslant k \leqslant n$, let $S_k(d, r)$ be the maximum profit from item types $k, k+1, \cdots, n$ with remaining weight d and volume r. We want $S_1(b, q)$. We may similarly obtain the following recurrence relation for (1.35)

$$S_k(d, r) = \max\{c_k x_k + S_{k-1}(d - a_k x_k, r - p_k x_k) : \; 0 \leqslant x_k \leqslant \min\left(\lfloor \frac{d}{a_k} \rfloor, \lfloor \frac{r}{p_k} \rfloor\right), \; 1 \leqslant k \leqslant n-1 \tag{1.36}$$

Example 1.15. Solve the IP

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ & \begin{cases} 3x_1 + 4x_2 \leqslant 12 \\ x_1 + 5x_2 \leqslant 10 \\ x_1, x_2 \in \mathbb{Z}_{\geqslant 0} \end{cases} \end{aligned} \tag{1.37}$$

Solution. Stage 2 For $0 \leqslant d \leqslant 12$, $0 \leqslant r \leqslant 10$,

$$S_2(d,r) = \begin{cases} 0, & \text{if } (d,r) \in ([0,12]\times[0,10]) \setminus ([4,12]\times[5,10]) \\ 3, & \text{if } (d,r) \in ([4,12]\times[5,10]) \setminus ([8,12]\times\{10\}) \\ 6, & \text{if } (d,r) \in [9,12]\times\{10\} \end{cases}$$

Stage 1 (1.36) $\Rightarrow$

$$\begin{aligned} S_1(12,10) &= \max\{2x_1 + S_2(12-3x_1, 10-x_1) : \ 0 \leqslant x_1 \leqslant \min\left(\lfloor\frac{12}{3}\rfloor, \lfloor\frac{10}{1}\rfloor\right) = 4\} \\ &= \max\{S_2(12,10), 2+S_2(9,9), 4+S_2(6,8), 6+S_2(3,7), 8+S_2(0,6)\} \\ &= \max(6, 2+3, 4+3, 6+0, 8+0) \\ &= 8 \end{aligned}$$

Optimal value $= 8$

Optimal solution: $x_1 = 4, x_2 = 0$ for $2_x1 + 3x_2 = 8$.

We may also use a table method by considering $k = 2$ and computing d versus r, then repeat for $k = 1$. □

1.4.2 B. 0-1 knapsack Problem

We have n items available, where item i has weight a_i and profit c_i $(1 \leqslant i \leqslant n)$. We want to place some of the items into a knapsack which holds a maximum weight b, and maximise the profit. We have the version of the knapsack problem where repetitions of the items are not allowed, or the 0-1 knapsack problem (a BIP).

$$\begin{aligned} \max \quad & c^T x \\ & \begin{cases} a^T x \leqslant b \\ x_i \in \{0,1\} \quad 1 \leqslant i \leqslant n \end{cases} \end{aligned} \tag{1.38}$$

where $c, a \in \mathbb{R}^n_{\geqslant 0}$, $b \in \mathbb{R}_{\geqslant 0}$. The recurrence relation for (1.38) is slightly different. For $0 \leqslant d \leqslant b$, $1 \leqslant k \leqslant n$, let $S_k(d)$ be the maximum profit from items $k, k+1, \cdots, n$ if the knapsack has remaining weight d.

Similar to (1.34), we have, for $1 \leqslant k \leqslant n-1$,

$$S_k(d) = \begin{cases} S_{k+1}(d), & \text{if } 0 \leqslant d < a_k \\ \max(S_{k+1}(d), c_k + S_{k+1}(d-a_k)), & \text{if } a_k \leqslant d \leqslant b \end{cases} \tag{1.39}$$

Example 1.16. Solve the BIP

$$\max \quad 8x_1 + 11x_2 + 6x_3 + 4x_4$$
$$\begin{cases} 5x_1 + 7x_2 + 4x_3 + 3x_4 & \leqslant 14 \\ x_i \in \{0,1\}, & 1 \leqslant i \leqslant 4 \end{cases}$$

STAGE 4

$$S_4(d) = \begin{cases} 0, & \text{if } 0 \leqslant d < 3 \quad (x_4 = 0) \\ \boxed{4}, & \text{if } 3 \leqslant d \leqslant 14 \quad (x_4 = 1) \end{cases}$$

STAGE 3

$$(1.39) \implies$$
$$S_3(d) = S_4(d) = \begin{cases} 0, & \text{if } 0 \leqslant d < 3 \quad (x_4 = 0) \\ 4, & \text{if } 3 \leqslant d \leqslant 4 \quad (x_4 = 1) \end{cases}$$
$$S_3(d) = \max(S_4(d), \boxed{6 + S_4(d-4)})$$
$$= \begin{cases} \max(4, 6+0) = 6, & \text{if } 4 \leqslant d < 7 \\ \max(4, \boxed{6+4}) = 10, & \text{if } 7 \leqslant d \leqslant 14 \end{cases}$$

STAGE 2

$$S_2(d) = S_3(d) = \begin{cases} 0, & \text{if } 0 \leqslant d < 3 \\ 4, & \text{if } 3 \leqslant d < 4 \\ 6, & \text{if } 4 \leqslant d < 7 \end{cases}$$
$$S_2(d) = \max(S_3(d), \boxed{11 + S_3(d-7)})$$
$$= \begin{cases} \max(10, 11+0) = 11, & \text{if } 7 \leqslant d < 10 \\ \max(10, 11+4) = 14, & \text{if } 10 \leqslant d < 11 \\ \max(10, 11+6) = 17, & \text{if } 11 \leqslant d \leqslant 14 \\ \max(10, \boxed{11+10}) = 21, & \text{if } d = 14 \end{cases}$$

STAGE 1

$$(1.39) \implies S_1(14) = \max(\boxed{S_2(14)}, 8 + S_2(14-5)) = \max(\boxed{21}, 8+11) = 21$$

Optimal value $= 21$. Optimal solution: $(x_1, x_2, x_3, x_4) = (0, 1, 1, 1)$.

TABLE METHOD

k \ d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Optimal solution
4	$\boxed{0}$	0	0	$\boxed{4}$	4	4	4	4	4	4	4	4	4	4	4	$x_4 = 1$
3	0	0	0	4	6	6	6	$\boxed{10}$	10	10	10	10	10	10	10	$x_3 = 1$
2	0	0	0	4	6	6	6	11	11	11	15	17	17	17	$\boxed{21}$	$x_2 = 1$
1	0	0	0	4	6	8	8	11	12	14	15	17	19	19	$\boxed{21}$	$x_1 = 0$

Table 1.3: Table Method

As before, we may generalise (1.38) by having two or more constraints.

1.4.3 C. Resource allocation problem

Suppose we have a total budget (or resource) $b \in \mathbb{Z}_{\geqslant 0}$, to be spent on n types of activities. If $x_i \in \mathbb{Z}_{\geqslant 0}$ units of activity type i are performed, there is a profit of $f_i(x_i)$, where $f_i : \{0, 1, \cdots, b\} \to \mathbb{R}_{\geqslant 0}$ is a non-decreasing function with $f_i(0) = 0$, $\forall\ 1 \leqslant i \leqslant n$.

We want to solve the IP:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} f_i(x_i) \\ & \begin{cases} \sum_{i=1}^{n} x_i & = b \\ \qquad x & \in \mathbb{Z}^n_{\geqslant 0} \end{cases} \end{aligned} \tag{1.40}$$

For $0 \leqslant d \leqslant b$, $1 \leqslant k \leqslant n$, let $S_k(d)$ be the maximum profit from activities of type $k, k+1, \cdots, n$, with a budget of d remaining. We want $S_1(b)$. Similarly, we have the following recurrence relation for (1.40):

$$S_k(d) = \max\{f_k(x_k) + S_{k+1}(d - x_k) :\ 0 \leqslant x_k \leqslant d\},\ 1 \leqslant k \leqslant n - 1 \tag{1.41}$$

Note that $S_n(d) = f_n(d)$ for $d = 1, 2, \cdots, b$, and $S_k(0) = 0$ for $k = 1, 2, \cdots, n$.

Example 1.17. Suppose 4 doctors are assigned to 3 hospitals. For $i = 1, 2, 3$ and $y = 1, 2, 3, 4$, let $f_i(y)$ be the number of patients that y doctors can serve per hour in hospital i given by the table

y = Number pf doctors	$f_1(y)$	$f_2(y)$	$f_3(y)$
0	0	$\boxed{0}$	0
1	4	2	$\boxed{5}$
2	6	4	7
3	$\boxed{9}$	7	8
4	10	11	9

Table 1.4: number of patients that y doctors can serve per hour

What is the maximum umber of patients that can be served per hour, across 3 hospitals?

We have problem (1.40) where $b = 4$, and x_1 = number of doctors assigned to hospital i.

Stage 3

$$S_3(d) = f_3(d),\ 0 \leqslant d \leqslant 4$$

Stage 2

$$S_2(0) = 0$$
$$(1.41) \Longrightarrow S_2(d) = \max\{f_2(x_2) + S_3(d - x_2) :\ 0 \leqslant x_2 \leqslant d\}$$
$$S_2(1) = \max\{\boxed{0 + S_3(1)}, 2 + f_3(0)\} = \max\{\boxed{5}, 2\} = 5$$
$$S_2(2) = \max\{0 + S_3(2), 2 + S_3(1), 4 + S_3(0)\} = \max\{7, 7, 4\} = 7$$
$$S_2(3) = \max\{0 + S_3(3), 2 + S_3(2), 4 + S_3(1), 7 + S_3(0)\} = \{8, 9, 9, 7\} = 9$$
$$S_2(4) = \max\{0 + S_3(4), 2 + S_3(3), 4 + S_3(2), 7 + S_3(1), 11 + S_3(0)\} = \{9, 10, 11, 12, 11\} = 12$$

Stage 1

$$(1.41) = S_1(4) = \max\{f_1(x_1) + S_2(4 - x_1) :\ 0 \leqslant x_1 \leqslant 4\}$$
$$= \max\{0 + S_2(4), 4 + S_2(3), 6 + S_2(2), \boxed{9 + S_2(1)}, 10 + S_2(0)\} = \max\{12, 13, 13, \boxed{14}, 10\} = 14$$

Optimal value = 14. Optimal solution : $(x_1, x_2, x_3) = (3, 0, 1)$. See the circled entries in the above table.

Table Method

k \ d	0	1	2	3	4	Optimal solution
3	0	5	7	8	9	$x_3 = 1$
2	0	5	7	9	12	$x_2 = 0$
1	0	5	9	11	14	$x_1 = 3$

Table 1.5: Table Method

Chapter 2

Game Theory

2.1 Introduction

Suppose two players Alice and Bob play a game. The set of choices, called STRATEGIES, available to Alice and Bob, are $S_1 = \{\alpha_1, \cdots, \alpha_m\}$ and $S_2 = \{\beta_1, \cdots, \beta_n\}$. If Alice and Bob choose α_i and β_j, then they win a_{ij} and b_{ij} respectively. The matrices $A, B \in \mathbb{R}^{m \times n}$ are the PAYOFF MATRICES for Alice and Bob.

What should each player do to try and maximise their winnings?

Example 2.1 (Rock-Scissors-Paper)**.** We have $S_1 = S_2 = \{\text{Rock}, \text{Scissors}, \text{Paper}\}$. Rock beats scissors, scissors beats paper, paper beats rock, and the same choices is a tie. If there is a win, the loser pays the winner ¥1. We may superimpose the payoff matrices as

Alice \ Bob	$\beta_1 = \text{Rock}$	$\beta_2 = \text{Scissors}$	$\beta_3 = \text{Paper}$
$\alpha_1 = \text{Rock}$	(0, 0)	(1, -1)	(-1, 1)
$\alpha_2 = \text{Scissors}$	(-1, 1)	(0, 0)	(1, -1)
$\alpha_2 = \text{Paper}$	(1, -1)	(-1, 1)	(0, 0)

Table 2.1: Rock-Scissor-Paper

(2.1) is also called the PAYOFF MATRICES.

Example 2.2 (Prisoner's Dilemma)**.** Alice and Bob have been arrested and accused of jointly committing a robbery. The police have offered each of them to either remain silent, or testify the other suspect.

- If they both remain silent, they are both jailed for 1 year.
- If they both testify, they are both jailed for 3 years.

- If one suspect testifies, and the other remains silent, then one who testifies is set free, while the other is jailed for 6 years.

Thus, $S_1 = S_2 = \{\text{Silent}, \text{Testify}\}$. They are then taken into separate rooms and questioned, so they are unaware of the other's action.

The payoff matrix is

Alice \ Bob	$\beta_1 =$ Silent	$\beta_2 =$ Testify
$\alpha_1 =$ Silent	(-1, -1)	(-6, 0)
$\alpha_2 =$ Testify	(0, -6)	(-3, 3)

Table 2.2: The payoff matrix

In games such as Examples 1 and 2, the following may be assumed:

- The games are full information: Both Alice and Bob know that the other player knows both strategy sets S_1, S_2, and the payoff matrix.
- The games are "non-cooperative": Alice and Bob are not in an alliance, and each plays to promote only their own winnings.
- Alice and Bob both play "rationally". That is, if Alice chooses α_i, then Bob should try to choose β_j to promote his winnings, and vice versa.

We may also consider dropping any of these assumptions. Such games may be extended to more than two players. These games have applications in economics, political science (eg: voting), computer science, etc.

2.2 Two-player zero-sum and constant-sum games

If in the game between Alice and Bob, we have $A + B = 0$, then we have a zero-sum game. Thus, Alice wins $a_{ij} \implies$ Bob loses a_{ij}, or Bob pays Alice a_{ij}. Example 1 is a zero-sum game.

It suffices to study only the matrix A. A "rational" situation may occur as follows. If Alice chooses α_i, she should expect Bob to choose β_j to maximise his winnings (thus minimise Alice's winnings). So Alice can win $\geqslant \max_i \min_j a_{ij}$. Likewise, if Bob choose β_j, he should expect Alice to choose α_i to maximise her winnings and Bob's losses. So Bob can lose $\leqslant \min_j \max_i a_{ij}$.

If we have

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v, \text{ says}$$

then any entry of A is the same row as $\max_i \min_j a_{ij}$, and same column as $\min_j \max_i a_{ij}$, is a saddle point, or an equilibrium point. In this case, we say the game has value v.

Indeed, we have the following result.

Theorem 2.1. Suppose we have a zero-sum game between Alice and Bob, where $A \in \mathbb{R}^{m\times n}$ is Alice's payoff matrix.

(a) We have $\max_i \min_j a_{ij} \leqslant \min_j \max_i a_{ij}$.

(b) An entry $a_{ij} \in A$ is a saddle point if and only if a_{ij} is the smallest number is row i, and the largest number in column j.

(c) Any two saddle points have the same value.

Proof. HW3 □

Thus, if the game has a saddle point with value v, then we have rational situation that Alice/Bob are guaranteed to win/lose v, if they choose α_i and β_j corresponding to a saddle point. If Alice deviates to some α_k, then Th 2.1 (b) $\Rightarrow$ her winnings can become $< v$. A similar situation applies to Bob if her deviates to some β_l.

Example 2.3. Suppose we have the payoff matrix A:

Bob / Alice	β_1	β_2	β_3	β_4	Row min
α_1	3	4	10	5	3
α_2	8	5	-2	1	-2
α_3	6	$\boxed{5}$ Saddle Point	7	9	$\boxed{5} \leftarrow \max_i \min_j a_{ij}$
Column max$\rightarrow$	8	$\boxed{5} \leftarrow \min_j \max_i a_{ij}$	10	9	

Table 2.3: Payoff Matrix

We see that there is a saddle point with value 5 as indicated. Note that there are other entries also with value 5. If the game is played rationally, the Alice win 5 units by choosing α_3, and Bob loses 5 units by choosing β_2.

Similarly, suppose we have the payoff matrices

$$A = \begin{pmatrix} 8 & \boxed{4} & 4 & \boxed{4} \\ 9 & 3 & 2 & \boxed{4} \\ 5 & \boxed{4} & 7 & \boxed{4} \\ 0 & \boxed{4} & 8 & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & \boxed{1} & -1 \\ -1 & 0 & \boxed{1} \\ \boxed{1} & -1 & 0 \end{pmatrix}$$

The first matrix has four saddle points with value 4. The second, which is from Example 1, has no saddle point.

Now suppose that $A + B = C$, where every entry of C is the same, say equal to c. Then we have a constant-sum game. A zero-sum game is thus a constant-sum game with $c = 0$. Constant-sum games generalise zero-sum games, since Bob aims to maximise $b_{ij} = c - a_{ij} \iff$ Bob aims to minimise a_{ij}. Then notions of rationality, and saddle points of A, remain valid. We also have Th 2.1 with "constant-sum" in place of "zero-sum".

Back to a zero-sum game, what happens if the matrix A has no saddle point? If Alice and Bob still play rationally, then they would use mixed strategy, where they make their choices according to some probability distributions. In the case where they choose α_i and β_j with probability 1, we have pure strategy. Thus in particular, if A has a saddle point, then we have pure strategy since Alice and Bob would always choose α_i and β_j corresponding to a saddle point.

We have

Alice \ Bob	β_1	$\cdots$	β_n
α_1	a_{11}	$\cdots$	a_{1n}
$\vdots$	$\vdots$		$\vdots$
α_m	a_{m1}	$\cdots$	a_{mn}

For $1 \leqslant 1 \leqslant m$, suppose Alice choose α_i with probability $x_i \in [0, 1]$, so that $x_1 + \cdots + x_m = 1$. For example, $m = 2$, $x_1 = x_2 = \frac{1}{2}$ means Alice throws a fair coin to make her decision. Then Alice's expected winnings, if Bob chooses β_j $(1 \leqslant j \leqslant n)$, is $a_{1j}x_1 + \cdots + a_{mj}x_m$. So Alice is expected to win at least

$$u = \min\{a_{1j}x_1 + \cdots + a_{mj}x_m : \ 1 \leqslant j \leqslant n\}$$

Thus, Alice wants to choose $x_1, x_2, \cdots, x_m$ s.t. u is maximised. We have Alice's LP

$$\max \quad u$$
$$\begin{cases} a_{11}x_1 + \cdots + a_{m1}x_m & \geqslant u \\ \qquad\vdots & \\ a_{1n}x_1 + \cdots + a_{mn}x_m & \geqslant u \\ x_1 + \cdots + x_m & = 1 \\ x_i \geqslant 0 \ (1 \leqslant i \leqslant m), & u \in \mathbb{R} \end{cases} \tag{2.3}$$

Writing $u = u_1 - u_2$ where $u_1, u_2 \geqslant 0$, we convert to the form of (1.1):

$$\max \quad c^T\tilde{x}$$
$$\begin{cases} A\tilde{x} & \leqslant b \\ \tilde{x} & \geqslant 0 \end{cases} \tag{2.4}$$

where

$$A = \begin{pmatrix} -A^T & \mathbb{1}_n & -\mathbb{1}_n \\ \mathbb{1}_m^T & 0 & 0 \\ -\mathbb{1}_m^T & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+2)\times(m+2)}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^{n+2}, \quad c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^{m+2}$$

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^{m+2}, \quad \text{and} \quad \mathbb{1}_k = \underbrace{(1 \ \cdots \ 1)}_{k}^T$$

Similarly, for $1 \leqslant i \leqslant n$, suppose Bob choose β_j with probability $y_j \in [0,1]$, so that $y_1 + \cdots + y_n = 1$. Then Bob's expected loss, if Alice chooses α_i $(1 \leqslant i \leqslant m)$, is $a_{i1}y_1 + \cdots + a_{in}y_n$. So Bob is expected to lose at most

$$w = \max\{a_{i1}y_1 + \cdots + a_{in}y_n : \ 1 \leqslant i \leqslant m\}$$

Thus, Bob wants to choose $y_1, \cdots, y_n$ s.t. w is minimised. We have Bob's LP:

$$\min \quad w$$
$$\begin{cases} a_{11}y_1 + \cdots + a_{1n}y_n & \leqslant w \\ \qquad \cdots & \\ a_{m1}y_1 + \cdots + a_{mn}y_n & \leqslant w \\ y_1 + \cdots y_n & = 1 \\ y_j \geqslant 0 \ (1 \leqslant j \leqslant n), & w \in \mathbb{R} \end{cases} \tag{2.5}$$

Writing $w = w_1 - w_2$, where $w_1, w_2 \geqslant 0$, we convert to the form of (1.7):

$$\min \quad b^T\tilde{y}$$
$$\begin{cases} \tilde{A}^T\tilde{y} & \succeq c \\ \tilde{y} & \succeq 0 \end{cases} \tag{2.6}$$

where $\tilde{y} = (y_1 \ \cdots \ y_n \ w_1 \ w_2)^T \in \mathbb{R}^{n+2}$.

Now clearly (2.4) and (2.6) are dual LPs. It is easy to show that both (2.4) and (2.6) are feasible and bounded. By Th 1.7, we have strong duality. Let both optimal values of (2.4) and (2.6) be v. We say that v is the <u>value</u> of the game.

We have the following important result in game theory.

Theorem 2.2. (Von Neumann Minimax Theorem) Suppose that Alice and Bob play a zero-sum game rationally, using mixed strategy. Let $A \in \mathbb{R}^{m\times n}$ be Alice's payoff matrix. Then $\exists$ optimal probability distributions $\bar{x} = (x_1, \cdots, x_m)$ and $\bar{y} = (y_1, \cdots, y_n)$ for Alice and Bob, s.t. Alice is expected to win $\geqslant v$, and Bob is expected to lose $\leqslant v$, where v is the value of the game.

Moreover,

$$\max_x \min_y x^T Ay = \min_y \max_x x^T Ay = v \tag{2.7}$$

where the max and min are taken over $x \in [0,1]^m$, $y \in [0,1]^n$ with $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j = 1$. Also, $\bar{x}$ and $e_j\mathbb{R}^n$ attain $\max_x \min_y x^A y = v$ for some j, and $e_i \in \mathbb{R}^m$ and $\bar{y}$ attain $\min_y \max_x x^T Ay = v$ from some i. (**Recall:** $e_i \in \mathbb{R}^k$ is the standard basis vector, $e_i = \overbrace{(0 \ \cdots \ 1 \ 0 \ \cdots \ 0)^T}^{k}$).

Proof. From the above discussion, it suffices to prove (2.7). Note that the set of $y \in [0,1]^n$ with $\sum_{j=1}^n y_j = 1$ defines a simplex in $\mathbb{R}^n$, whose extreme points are $e_1, \cdots, e_n$. For any fixed x, since $x^T Ay$ is a linear function in y, we have $\min_y x^T Ay = \min_j x^T Ae_j$ by Th 1.2. Thus,

$$\max_x \min_y x^T Ay = \max_x \min_j x^T Ae_j \underset{(2.3),\ \text{Th } 1.7}{=} v.$$

Similarly,

$$\min_y \max_x x^T Ay = \min_y \max_x x^T Ay \underset{(2.5),\ \text{Th } 1.7}{=} v,$$

and (2.7) holds. $\square$

Remark 2.1.
- If A has a saddle point a_{ij}, then the mixed strategy reduces to pure strategy. We have the LP (2.3), (2.5) and value v, where $\binom{\bar{x}}{\bar{u}} = \binom{e_i}{v} \in \mathbb{R}^{m+1}$, $\binom{\bar{y}}{\bar{w}} = \binom{e_j}{v} \in \mathbb{R}^{n+1}$ are optimal solutions. The two definitions of "value" coincide.
- Since Alice and Bob both know the matrix A, they also know the optimal probabilities distributions $\bar{x} \in \mathbb{R}^m$, $\bar{y} \in \mathbb{R}^n$, since they can be computed from A.
- If instead, we have a constant-sum game, say $A + B = C$, then we may consider the zero-sum game with $A' + B' = 0$, where $A' = A - \frac{1}{2}C$, $B' = B - \frac{1}{2}C$. We may then apply all arguments above on A'.

Example 2.4. Again, consider (2.1)

Alice \ Bob	Rock	Scissors	Paper
Rock	$(0,0)$	$(1,-1)$	$(-1,1)$
Scissors	$(-1,1)$	$(0,0)$	$(1,-1)$
Paper	$(1,-1)$	$(-1,1)$	$(0,0)$

A convenient method to solve for $\bar{x}, \bar{y}, v$ is as follows:

We add $|-1| = 1$ to every entry of A, B to obtain $\hat{A}, \hat{B}$ (-1 being the most negative entry in A). We have a constant-sum game with $\hat{A}, \hat{B}$, where $\hat{A}$ has non-negative entries. We have

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Using a similar argument for (2.3) on $\hat{A}$, we have the LP for Alice:

$$\begin{aligned} &\max \quad u' \\ &\begin{cases} x_1 & & +2x_3 & \geqslant u' \\ 2x_1 & +x_2 & & \geqslant u' \\ & 2x_2 & +x_3 & \geqslant u' \\ x_1 & +x_2 & +x_3 & = 1 \\ x_1, & x_2, & x_3, & u' \geqslant 0 \end{cases} \end{aligned}$$

We note that $u' \geqslant 0$ since x_1, x_2, x_3, and the entries of A are $\geqslant 0$. Eliminating x_3 gives:

$$\begin{aligned} &\max \quad u' \\ &\begin{cases} x_1 & +2x_2 & +u' & \leqslant 2 \\ -2x_1 & +x_2 & +u' & \leqslant 1 \\ x_2 & -x_2 & +u' & \leqslant 1 \\ x_1, & x_2, & u' & \geqslant 0 \end{cases} \end{aligned} \tag{2.8}$$

Similarly, using $\hat{A}$, similar to (2.5), we have the LP for Bob:

$$\min \quad w'$$
$$\begin{cases} -y_1 & -2y_2 & +w' & \geqslant 0 \\ 2y_1 & +y_2 & +w' & \geqslant 2 \\ -y_1 & +y_2 & +w' & \geqslant 1 \\ y_1, & y_2, & w' & \geqslant 0 \end{cases} \tag{2.9}$$

We can then use some method to solve (2.8), (2.9), such as the simplex or two-phase methods. We find the optimal solutions $(\bar{x}, \bar{u}') = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1\right)$ and $(\bar{y}, \bar{w}') = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1\right)$. The value of the original zero-sum game is $v = \bar{u}' - 1 = \bar{w}' - 1 = 0$. All of this make sense, since the moves Rock, Scissors and Paper are "symmetric", so they each should be made with probability $\frac{1}{3}$. If the game is played many times, say 100, the overall expected winnings for both players should be zero (the value).

Definition 2.1 (DOMINATE). The strategy α_i DOMINATES the strategy α_k if $a_{ij} \geqslant a_{kj}$, $\forall 1 \leqslant j \leqslant n$. Similarly, β_j DOMINATES β_l if $b_{ij} \geqslant b_{il}$, $\forall 1 \leqslant i \leqslant m$.

We note that in a zero-sum game, if α_i DOMINATES α_k, then in (2.5), the constraint $a_{i1}y_1 + \cdots + a_{in}y_n \leqslant w$ is stronger than the constraint $a_{k1}y_1 + \cdots + a_{kn}y_n \leqslant w$, and we may ignore the latter. Likewise, if β_j DOMINATES β_l, then in (2.3), $a_{1k}x_1 + \cdots + a_{mj}x_m \geqslant u$ is stronger than $a_{1l}x_1 + \cdots + a_{ml}x_m \geqslant u$, and we may ignore the latter.

In other words, to simplify a zero-sum game with matrix A, we may delete any dominated rows, then any DOMINATING columns, and repeat to obtain a simpler matrix.

Example 2.5. A zero-sum game has

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -1 & 3 & 2 \\ 4 & -2 & -2 \\ 2 & 1 & 0 \end{pmatrix} \tag{2.10}$$

We see that rows 2 and 3 dominate rows 4 and 1. Delete rows 1 and 4.

$$\begin{pmatrix} -1 & 3 & 2 \\ 4 & -2 & -2 \end{pmatrix}$$

Now, column 2 dominates column 3. Delete column 2.

$$\begin{pmatrix} -1 & 2 \\ 4 & -2 \end{pmatrix}$$

Now proceed as before. Add $|-2| = 2$ to all entries. Note also the "surviving" variables are x_2, x_3, y_1, y_3:

$$\begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix}$$

The LPs in the forms (2.3), (2.5) are

$$\max \quad u'$$
$$\begin{cases} x_2 + 6x_3 & \geqslant u' \\ 4x_2 & \geqslant u' \\ x_2 + x_3 & = 1 \\ x_2, x_3, u' & \geqslant 0 \end{cases}$$

$$\min \quad w'$$
$$\begin{cases} y_1 + 4y_3 & \leqslant w' \\ 6y_1 & \leqslant w' \\ y_1 + y_3 & = 1 \\ y_1, y_3, w' & \geqslant 0 \end{cases}$$

Using $x_3 = 1 - x_2$, $y_3 = 1 - y_2$, we have

$$\max \quad u'$$
$$\begin{cases} 5x_2 + u' & \leqslant 6 \\ -4x_2 + u' & \leqslant 0 \\ x_2, u' \geqslant 0, & x_2 \leqslant 1 \end{cases}$$

$$\min \quad w'$$
$$\begin{cases} 3y_1 + w' & \geqslant 4 \\ -6y_1 + w' & \geqslant 0 \\ y_1, w' \geqslant 0, & y_1 \leqslant 1 \end{cases}$$

We can easily obtain the optimal solutions $(\bar{x}_2, \bar{u}') = (\frac{2}{3}, \frac{8}{3})$, $(\bar{y}_1, \bar{w}' = (\frac{4}{9}, \frac{8}{3})$. For the game (2.10), the optimal probability distributions are $\bar{x} = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}^T$, $\bar{y} = \begin{pmatrix} \frac{4}{9} & 0 & \frac{5}{9} \end{pmatrix}^T$, and the value is $v = \frac{8}{3} - 2 = \frac{2}{3}$.

2.3 Two-player non-constant-sum games

A NON-CONSTANT-SUM GAME, often also called a NON-ZERO-SUM GAME, is a game which is not constant-sum. Thus Alice and Bob may possibly both win or both lose from a particular pair of strategies.

We continue to assume that Alice and Bob play rationally, and they are non-cooperative.

Definition 2.2 (Nash Equilibrium)**.** In a two-player game (A, B), a (PURE STRATEGY) NASH EQUILIBRIUM (NE) is an entry (a_{ij}, b_{ij}) s.t. $a_{ij} \geqslant a_{kj}, \forall 1 \leqslant k \leqslant m$, and $b_{ij} \geqslant b_{il}, \forall 1 \leqslant l \leqslant n$. Thus if Alice changes strategy from a NE and Bob remains, then Alice cannot win more, and similarly for Bob. A NE is precisely a saddle point in a constant-sum game.

In the game (2.2), $(-3, -3)$ is the only NE.

If a game does not have a NE, we may again consider mixed strategy.

Definition 2.3. Let

$$\Delta_k = \{z \in [0,1]^k : \sum_{i=1}^{k} z_i = 1\}$$

be the PROBABILITY SIMPLEX, the space of all probability distributions for k outcomes.

Definition 2.4. Consider the game (A, B) $(A, B \in \mathbb{R}^{m \times n})$, played with mixed strategy. Let $x \in \Delta_m, y \in \Delta_n$.

- For fixed x, y, the EXPECT UTILITY of Alice and Bob are

$$\mathbb{E}_A(x, y) = \sum_{i,j} a_{ij} x_i y_j = x^T A y, \text{ and } \mathbb{E}_B(x, y) = \sum_{i,j} b_{ij} x_i y_j = x^T B y.$$

- $\bar{x} \in \Delta_m$ is Alice's BEST RESPONSE to $y \in \Delta_n$ if

$$\mathbb{E}_A(x, y) \geqslant \mathbb{E}_B(x, y), \ \forall x \in \Delta_m.$$

Similarly for $\bar{y} \in \Delta_n$ for Bob: $\mathbb{E}_B(x, \bar{y}) \geqslant \mathbb{E}_B(x, y), \forall y \in \Delta_n$.

- $(\bar{x}, \bar{y})$ is a MIXED STRATEGY NASH EQUILIBRIUM (MSNE) if $\bar{x}$ is Alice's best response to $\bar{y}$, and $\bar{y}$ is Bob's best response to $\bar{x}$. The NASH EQUILIBRIUM PAYOFFS for Alice and Bob are $\mathbb{E}_A(\bar{x}, \bar{y})$ and $\mathbb{E}_B(\bar{x}, \bar{y})$.

Example 2.6. Consider

Alice \ Bob	β_1	β_2
α_1	$(2,-3)$	$(1,2)$
α_2	$(1,1)$	$(4,-1)$

Suppose Alice and Bob choose α_1, β_1 with probabilities $p, q \in [0,1]$. Then

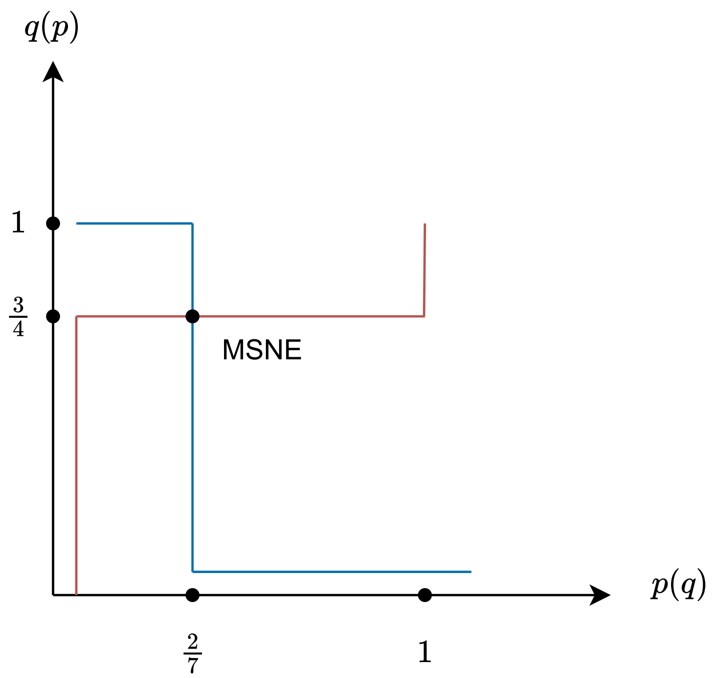
$$\begin{aligned}\mathbb{E}_A(p,q) &= 2pq + p(1-q) + (1-p)q + 4(1-p)(1-q)\\ &= (4q-3)p + (-3q+4)\end{aligned}$$

$$\implies \text{Best response } p = \begin{cases} 0, & \text{if } q < \frac{3}{4} \\ 1, & \text{if } q > \frac{3}{4} \\ \in [0,1] & \text{if } q = \frac{3}{4} \end{cases}$$

$$\begin{aligned}\mathbb{E}_B(p,q) &= -3pq + 2p(1-q) + (1-p)q - (1-p)(1-q)\\ &= (-7p+2)q + (3p-1)\end{aligned}$$

$$\implies \text{Best response } q = \begin{cases} 0, & \text{if } p > \frac{2}{7} \\ 1, & \text{if } p < \frac{2}{7} \\ \in [0,1], & \text{if } p = \frac{2}{7} \end{cases}$$

We have



and one MSNE $\bar{x} = (\frac{2}{7}, \frac{5}{7})$, $\bar{y} = (\frac{3}{4}, \frac{1}{4})$. The NE payoffs for Alice and Bob are $\mathbb{E}_A(\frac{2}{7}, \frac{3}{4}) = \frac{7}{4}$, $\mathbb{E}_B(\frac{2}{7}, \frac{3}{4}) = -\frac{1}{7}$.

Definition 2.5. The strategies $\alpha_{i_1}, \cdots, \alpha_{i_t}$ STRICTLY DOMINATE the strategy α_k $(k \neq i_1, i_2, \cdots, i_t)$ if $\exists \gamma \in \Delta_t$, s.t.

$$\gamma_1 a_{i_1,j} + \cdots + \gamma_t a_{i_t,j} > a_{kj}, \ \forall 1 \leqslant j \leqslant n$$

We similar define $\beta_{j_1}, \cdots, \beta_{j_t}$ STRICTLY DOMINATE β_l.

Lemma 2.3. Suppose that in the game (A, B), the strategies $\alpha_{i_1}, \cdots, \alpha_{i_t}$ strictly dominate the strategy α_k. Then every MSNE $(\bar{x}, \bar{y})$ must have $\bar{x}_k = 0$. Similarly, if $\beta_{j_1}, \cdots, \beta_{j_t}$ strictly dominates β_l, then every MSNE $(\bar{x}, \bar{y})$ must have $\bar{y}_l = 0$.

Proof. By permuting the α_i, we may assume $\alpha_1, \cdots, \alpha_t$ $(1 \leqslant t < m)$ strictly dominate α_m. Then $\exists \gamma \in \Delta_t$, s.t. $\gamma_1 a_{1j} + \cdots + \gamma_t a_{tj} > a_{mj}$, $\forall 1 \leqslant j \leqslant n$. Suppose $\exists$ a MSNE $(\bar{x}, \bar{y})$ s.t. $\exists \bar{x}_m > 0$. Consider $x' \in \Delta_m$ where

$$x'_i = \begin{cases} x_1 + \varepsilon \gamma_i \bar{x}_m, & \text{if } 1 \leqslant i \leqslant t, \\ \bar{x}_i & \text{if } t < i < m, \\ (1-\varepsilon)\bar{x}_m & \text{if } i = m \end{cases}$$

for some small $\varepsilon > 0$. Note that $x' \in \Delta_m$ since $\bar{x}_m > 0 \implies \bar{x}_i < 1, \forall 1 \leqslant i \leqslant m$. Then

$$\begin{aligned}
\mathbb{E}_A(x', \bar{y}) &= \sum_{\substack{1\leqslant i\leqslant t,\\ 1\leqslant j\leqslant n}} a_{ij}(\bar{x}_i + \varepsilon\gamma_i\bar{x}_m)\bar{y}_i + \sum_{\substack{t<i<m,\\ 1\leqslant j\leqslant n}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\leqslant j\leqslant n} a_{mj}(1-\varepsilon)\bar{x}_m\bar{y}_j \\
&= \sum_{\substack{1\leqslant i<m,\\ 1\leqslant j\leqslant n}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\leqslant j\leqslant n} a_{mj}(1-\varepsilon)\bar{x}_m\bar{y}_j + \sum_{\substack{1\leqslant i\leqslant t,\\ 1\leqslant j\leqslant n}} a_{ij}\varepsilon\gamma_i\bar{x}_m\bar{y}_j \\
&\underset{(*)}{>} \sum_{\substack{1\leqslant i<m,\\ 1\leqslant j\leqslant n}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\leqslant j\leqslant n} a_{mj}(1-\varepsilon)\bar{x}_m\bar{y}_j + \sum_{1\leqslant j\leqslant n} a_{mj}\varepsilon\bar{x}_m\bar{y}_j = \mathbb{E}_A(\bar{x}, \bar{y})
\end{aligned}$$

which contradicts that $\bar{x}$ is Alice's best response to $\bar{y}$.

Note that $(*)$ holds since $\bar{x}_m\bar{y}_j > 0$ for some j. A similar proof holds if there is a strict domination among the β_j. $\square$

Example 2.6. Find the MSNEs of

$$\begin{pmatrix}
(-1,-1) & (1,-2) & (5,1) \\
(1,2) & (2,1) & (2,4) \\
(0,-2) & (1,-3) & (3,2) \\
(3,-2) & (4,0) & (2,-2)
\end{pmatrix}$$

We say that $(\frac{1}{3} \times \text{row } 1) + (\frac{2}{3} \times \text{row } 4)$ and $(\frac{1}{2} \times \text{row } 1) + (\frac{1}{2} \times \text{row } 4)$ strictly dominate rows 2 and 3. Delete row 2 and 3:

$$\begin{pmatrix}
(-1,-1) & (1,-2) & (5,1) \\
(3,-2) & (4,0) & (2,-2)
\end{pmatrix}$$

Next, $(\frac{1}{2} \times \text{column } 2) + (\frac{1}{2} \times \text{column } 3)$ strictly dominates column 1. Delete column 1.

	y_2	y_3
x_1	$(1,-2)$	$(5,1)$
x_4	$(4,0)$	$(2,-2)$

The surviving variables are x_1, x_4, y_2, y_3. Now proceed as in Example (2.6). Suppose Alice and Bob choose α_1, β_2 with probabilities $p, q \in [0,1]$. Then

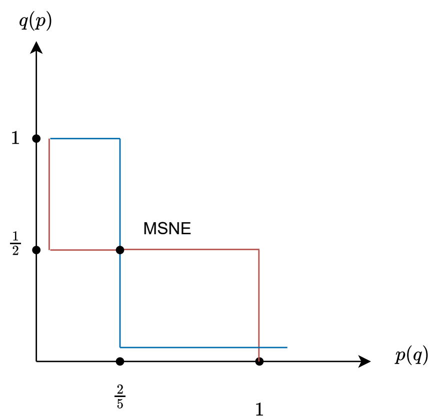
$$\begin{aligned}
\mathbb{E}_A(p,q) &= pq + 5p(1-q) + 4(1-p)q + 2(1-p)(1-q) \\
&= (-6q+3)p + (2q+2)
\end{aligned}$$

$$\implies \text{Best response } p = \begin{cases} 0, & \text{if } q > \frac{1}{2}, \\ 1, & \text{if } q < \frac{1}{2}, \\ \in [0,1], & \text{if } q = \frac{1}{2}. \end{cases}$$

$$\begin{aligned} \mathbb{E}_B(p,q) &= -2pq + p(1-q) - 2(1-p)(1-q) \\ &= (-5p+2)q + (3p-2) \end{aligned}$$

$$\implies \text{Best response } q = \begin{cases} 0, & \text{if } q > \frac{2}{5}, \\ 1, & \text{if } q < \frac{2}{5}, \\ \in [0,1], & \text{if } q = \frac{2}{5}. \end{cases}$$

We have



There are three MSNEs, two of which are also NEs.

MSNE	NE	NE payoffs	
		Alice	Bob
$\bar{x} = (0\ 0\ 0\ 1)^T$, $\bar{y} = (0\ 1\ 0)^T$	$(4,0)$	$\mathbb{E}_A(0,1) = 4$	$\mathbb{E}_B(0,1) = 0$
$\bar{x} = (1\ 0\ 0\ 0)^T$, $\bar{y} = (0\ 0\ 1)^T$	$(5,1)$	$\mathbb{E}_A(1,0) = 5$	$\mathbb{E}_B(0,1) = 1$
$\bar{x} = (\frac{2}{5}\ 0\ 0\ \frac{3}{5}$, $\bar{y} = (0\ \frac{1}{2}\ \frac{1}{2})^T$	\	$\mathbb{E}_A(\frac{2}{5},\frac{1}{2}) = 3$	$\mathbb{E}_B(\frac{2}{5},\frac{1}{2}) = -\frac{4}{5}$

2.3.1 Cooperative Games

For non-constant-sum games, it is interesting if the players can "cooperate".

Example 2.7. Recall that in Prisoner's Dilemma, we have

Alice \ Bob	β_1 = Silent	β_2 = Testify
α_1 = Silent	(-1, -1)	(-6, 0)
α_2 = Testify	(0, -6)	(-3, 3)

Table 2.4: (2.2)

Suppose that, before being taken into separate rooms and questioned, Alice and Bob can communicate privately with each other. From the table (2.2), it seems a good compromise for them to both remain silence, so they each spend a short time of 1 year in jail.

BUT, can they keep their promise (cooperate) when they are questioned separately? If one of them decides to defect and testify (non-cooperate), while the other keeps the promise, the one who defects "gains" and does not go to jail, while the other is "suckered", and gets 6 years in jail. However, if they defect, then they are both worse off than if they had cooperated, since they each have to go to jail for 3 years ($3 > 1$).

We see that Prisoner's Dilemma is interesting since it asks the question "Can Alice and Bob really trust each other?".

In general, the payoff matrix may be represented as follows:

Alice \ Bob	C	NC
C	(r, r)	(s, t)
NC	(t, s)	(p, p)

C = Cooperate

NC = Non-Cooperate

r = Reward for cooperating

t = Temptation for defecting

s = Punishment for getting suckered

p = Punishment for non-cooperation

We see that t and s should be the largest and smallest payoffs. Also, for cooperation to be possible, we need $r > p$. Thus, $t > r > p > s$. We see that (p, p) is the only NE. Moreover, by strict domination, we see that $\bar{x} = \bar{y} = (0\ 1)^T$ is the only MSNE, giving the NE payoffs p, p. Thus, it is wisest if they both defect.

Example 2.7 (The game of Chicken)**.** After a tough F1 race, Livid Lewis accuses Mad Max of cheating towards to win. Lewis promises Max that in the next race, he will cause a collision so that Max cannot win the race, he will cause a collision so that Max cannot win the race, to which Max replies that he will also promise to cause a collision.

In the next race, the pair are side by side in their cars.

- In one of them swerves away[a] to avoid a collision ("chickens[b] out"), and the other remains committed to the promise, the one who swerves is overtaken by the other. The score "mind games points" of -3 and 5.
- If they both swerve, they both spin out of the race, and they each score 0 mind games points.
- But is they both kept the promise, they cause a collision and each suffers an injury. They both score -10 mind games points.

The payoff matrix is

Lewis \ Max	Don't swerve	Swerve
Don't swerve	$(-10,-10)$	$(5,-3)$
Swerve	$(-3,5)$	$(0,0)$

We may generally represent the matrix as:

Lewis \ Max	C	NC
C	(r,r)	(s,t)
NC	(t,s)	(p,p)

Clearly, s and r should be the largest and smallest payoffs. We should have $p>t$, since if one chickens out, it is better if the other chickens out than not. Thus $s>p>t>r$. We see that (t,s) and (s,t) are NEs. If say $s=2$, $p=1$, $t=-1$, $r=-M$ ($M>0$ is large), then we can find that $\bar{x}=\bar{y}=\left(\frac{1}{M},1-\frac{1}{M}\right)$ is a MSNE. Thus it is wisest if they both swerve.

[a] 突然转向

[b] a man who is not brave

2.4 Games with t players

Suppose $t \geqslant 2$ players play a game. Their strategy sets are $S_1,\cdots,S_t$. Let $S=S_1\times\cdots\times S_t$. The PAYOFF FUNCTION for Player i ($1\leqslant i\leqslant t$) is $u_i:\ S\to\mathbb{R}$, where Player i wins $u_i(s)$ when the strategies

$s = (s_1, \cdots, s_t) \in S$ are chosen by the t players. Thus for a 2-player game (A, B), u_1, u_2 correspond to A, B.

We assume the players are non-cooperative, and they play rationally. Also, the sets S_i are finite.

Definition 2.6. • For $s \in S$ $(1 \leqslant i \leqslant t)$, write

$$s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_t) \in S_1 \times \cdots S_{i-1} \times S_i \times \cdots S_t$$

For $s_i' \in S_i$, write

$$(s_i', s_{-i}) = (s_1, \cdots, s_{i-1}, s_i', s_{i+1}, \cdots, s_t) \in S$$

• A (pure strategy) Nash equilibrium (NE) is $\bar{s} \in S$ (or $(u_1(\bar{s}), \cdots, u_t(\bar{s}))$) s.t.

$$u_i(\bar{s}) = u_i(\bar{s}_i, \bar{s}_{-i}) \geqslant u_i(s_i, \bar{s}_{-i}), \ \forall 1 \leqslant i \leqslant t.$$

Thus if Player i changes strategy from a NE and all other players remain, then Player i cannot win more.

Definition 2.7. • For probability distributions $\pi_i : S_i \to [0, 1]$ $(1 \leqslant i \leqslant t)$, the EXPECTED UTILITY of Player i is

$$\mathbb{E}_i(\pi_1, \cdots, \pi_t) = \sum_{s \in S} u_i(s) \prod_{j=1}^{t} \pi_j(s_j).$$

• $\bar{pi}_i$ is Player i's best response to $\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_t$ if

$$\mathbb{E}_i(\pi_1, \cdots, \pi_{i-1}, \bar{\pi}_i, \pi_{i+1}, \cdots, \pi_t) \geqslant \mathbb{E}_i(\pi_1, \cdots, \pi_{i-1}, \pi_i, \pi_{i+1}, \cdots, \pi_t), \ \forall \pi_i$$

• $(\bar{\pi}_1, \cdots, \bar{\pi}_t)$ is a MIXED STRATEGY NASH EQUILIBRIUM (MSNE) if $\bar{\pi}_i$ is Player i's best response to $\bar{\pi}_1, \cdots, \bar{\pi}_{i-1}, \bar{\pi}_{i+1}, \cdots, \bar{\pi}_t$, $\forall 1 \leqslant i \leqslant t$. The NASH EQUILIBRIUM PAYOFF for Player i is $\mathbb{E}_i(\bar{\pi}_1, \cdots, \bar{\pi}_t)$.

Example 2.8. Suppose we have a 3-player game, where $S_1 = \{\alpha_1, \alpha_2\}$, $S_2 = \{\beta_1, \beta_2\}$, $S_3 = \{\gamma_1, \gamma_2\}$, and

P_1 \ P_2	β_1	β_2
α_1	$(3,3,3)$	$\boxed{(3,\ 4,\ 3)}$
α_2	$\boxed{(4,\ 3,\ 3)}$	$(1,1,0)$

Table 2.5: If P_3 choose γ_1

P_1 \ P_2	β_1	β_2
α_1	$\boxed{(3,\ 3,\ 4)}$	$(0,1,1)$
α_2	$(1,0,1)$	$\boxed{(1,\ 1,\ 1)}$

Table 2.6: If P_3 choose γ_3

There are four NEs as indicated. Now consider mixed strategies. Let $x = (p, 1-p),\ y = (q, 1-q),\ z = (r, 1-r)$, be the probability distributions.

$$\begin{aligned}
\mathbb{E}_1(p,q,r) &= [3qr + 3(1-q)r + 3q(1-r)p]\, p \\
&\quad + [4qr + (1-q)r + q(1-r) + (1-q)(1-r)]\,(1-p) \\
&= (3qr + 3r - 6qr - 1)p + (3qr + 1)
\end{aligned}$$

$$\Longrightarrow \text{Best response } p = \begin{cases} 0, & \text{if } 3qr + 3r - 6qr - 1 < 0 (*) \\ 1, & \text{if } 3qr + 3r - 6qr - 1 > 0 \\ \in [0,1], & \text{if } 3qr + 3r - 6qr - 1 = 0 \end{cases}$$

Similarly (by symmetry of p, q, r)

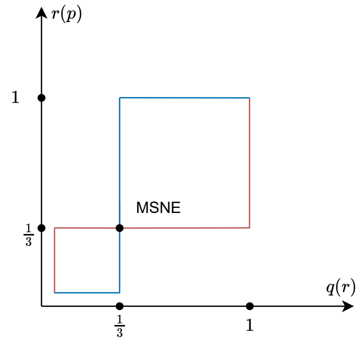
$$\Longrightarrow \ \text{BR } q = \begin{cases} 0, & \text{if } 3p + 3r - 6pr - 1 < 0 \\ 1, & \text{if } 3p + 3r - 6pr - 1 > 0 \\ \in [0,1] & \text{if } 3p + 3r - 6pr - 1 = 0 \end{cases}$$

$$\Longrightarrow \ \text{BR } r = \begin{cases} 0, & \text{if } 3p + 3q - 6pq - 1 < 0 \\ 1, & \text{if } 3p + 3q - 6pq - 1 > 0 \\ \in [0,1] & \text{if } 3p + 3q - 6pq - 1 = 0 \end{cases}$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be a MSNE. If $\bar{p} = 0$,

$$\text{BR } q = \begin{cases} 0, & \text{if } r < \frac{1}{3} \\ 1, & \text{if } r > \frac{1}{3} \\ \in [0,1], & \text{if } r = \frac{1}{3} \end{cases},$$

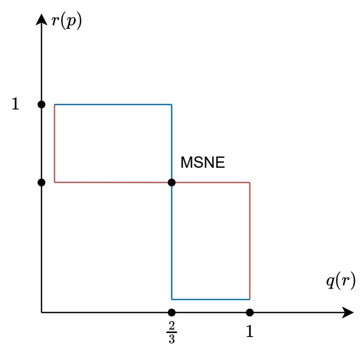
$$\text{BR } e = \begin{cases} 0, & \text{if } q < \frac{1}{3} \\ 1, & \text{if } q > \frac{1}{3} \\ \in [0,1], & \text{if } q = \frac{1}{3} \end{cases}.$$



$\implies (p, q, r) = (0, 0, 0),\ (0, \frac{1}{3}, \frac{1}{3}),\ (0, 1, 1)$. We discord $(0, \frac{1}{3}, \frac{1}{3})$ since it does not satisfy (*). If $\bar{p} = 1$,

$$\text{BR } q = \begin{cases} 0, & \text{if } r > \frac{2}{3} \\ 1, & \text{if } r < \frac{2}{3} \\ \in [0,1], & \text{if } r = \frac{2}{3} \end{cases},$$

$$\text{BR } r = \begin{cases} 0, & \text{if } q > \frac{2}{3} \\ 1, & \text{if } q < \frac{2}{3} \\ \in [0,1], & \text{if } q = \frac{2}{3} \end{cases}$$



$\implies (\bar{p}, \bar{q}, \bar{r}) = (1, 0, 1),\ (1, \frac{2}{3}, \frac{2}{3}),\ (1, 1, 0)$.

Now by symmetry of p, q, r, suppose $\bar{p}, \bar{q}, \bar{r} \in (0,1)$. Then

$$\begin{aligned} 3q + 3r - 6qr - 1 = 0 \quad (1) \\ 3p + 3r - 6pr - 1 = 0 \quad (2) \\ 3p + 3q - 6pq - 1 = 0 \quad (3) \end{aligned}$$

$(1),(2) \implies 3q - 6qr = 3p - 6pr \implies (p-q)(1-2r) = 0$.

- If $p = q$, $(3) \implies 6p - 6p^2 - 1 = 0 \implies \bar{p} = \bar{q} = \frac{3 \pm \sqrt{3}}{6} \in (0,1)$.

  $(1) \implies \bar{p} = \bar{q} = \bar{r} = \frac{3 \pm \sqrt{3}}{6} \in (0,1)$.
- If $r = \frac{1}{2}$, $(1) \implies \frac{1}{2} = 0$, which is absurd.

Conclusion NEs: $(\bar{p}, \bar{q}, \bar{r}) = (0,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,0)$.

MSNEs: $(\bar{p}, \bar{q}, \bar{r}) = (1, \frac{2}{3}, \frac{2}{3}),\ (\frac{2}{3}, 1, \frac{2}{3}),\ (\frac{2}{3}, \frac{2}{3}, 1),\ \left(\frac{3\pm\sqrt{3}}{6}, \frac{3\pm\sqrt{3}}{6}, \frac{3\pm\sqrt{3}}{6}\right)$.

We may obtain all possible $(\bar{x}, \bar{y}, \bar{z})$ and NE payoffs from these.

We have the celebrated result.

Theorem 2.4 (Nash's theorem)**.** If a game with $t \geqslant 2$ players has finite strategy sets $S_1, \cdots, S_t$ then $\exists$ MSNE.

Note that 2.4 says that a MSNE exists, but does not tell us how to find one!

A proof of Th 2.4 requires:

Theorem 2.5 (Brouwer's fixed-point theorem)**.** Let $K \subset \mathbb{R}^n$ be a compact, convex set, and $f : K \to K$ be continuous. Then $\exists x \in K$, s.t. $f(x) = x$. Such an x is a FIXED POINT.

Proof. For Th 2.4 (Using Th 2.5)

First, consider the case of two players Alice and Bob, with matrices $A, B \in \mathbb{R}^{m \times n}$. Let $K = \Delta_m \times \Delta_n$. For $(x,y) \in K$, we define $f(x,y) = (x', y')$ as follows. Let

$$c_i = c_i(x,y) = \max\left(A_i y - x^T A y, 0\right),\ 1 \leqslant i \leqslant m$$

where A_i is the i^{th} row of A. Note that c_i is Alice's possible gain by switching from mixed strategy x to pure strategy α_i, if such a gain is positive. Let $x' \in \Delta_m$ where

$$x_i' = \frac{x_i + c_i}{1 + \sum\limits_{k=1}^{m} c_k}$$

Similarly, let

$$d_j = d_j(x,y) = \max\left(x^T B^j - x^T B y, 0\right),\ 1 \leqslant j \leqslant n$$

where B^j is the j^{th} column of B, and let $y' \in \Delta_n$, where

$$y'_j = \frac{y_j + d_j}{1 + \sum\limits_{k=1}^{n} d_k}.$$

Now, if $c_i = 0$ (or $x^T Ay \geqslant A_i y$), $\forall i$, then $x' = x$ is a best response to y. Otherwise, $c = \sum\limits_{i=1}^{m} c_i > 0$. Then

$$\begin{aligned}
& \sum_{i=1}^{m} c_i Ay > cx^T Ay = \sum_{i=1}^{m} c_i x^T Ay \\
\implies & \sum_{i=1}^{m} (x_i + c_i) y > (1 + c) x^T Ay \\
\implies & \sum_{i=1}^{m} x'_i A_i y > x^T Ay
\end{aligned}$$

so x' is a better response to y than x is. Similarly, either $y' = y$ is a best response to x, or y' is a better response to x than y is.

Clearly, K is compact and convex, and f is continuous since c_i and d_j are. Thus Th 2.5 $\implies$ f has a fixed point, which is a MSNE.

For $t \geqslant 3$ players, we define for Player j with pure strategy l, the quantity $c_l^{(i)}$, which is the gain that Player j gets by switching from current strategy $x^{(j)}$ to pure strategy l, is positive, while fixing the current strategies of all other players. The argument is similar. $\square$

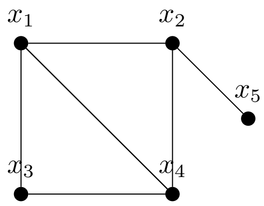
Chapter 3

Graph Theory and Networks

3.1 Terminology

A GRAPH is a pair $G = (V, E)$ where V is a set, and E is a set of ordered pairs from V.

For example



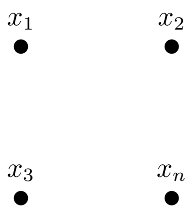
$$V = \{x_1, x_2, x_3, x_4, x_5\}$$
$$E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}, \{x_1, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5\}\}$$

V is the VERTEX SET and E is the EDGE SET. We may also write $V(G)$ and $E(G)$ for V and E. $|V|$ and $|E|$ are the ORDER and SIZE of G.

Note: No loops.

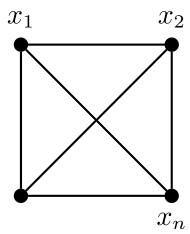
No multi edges.

Important examples: The EMPTY GRAPH E_n of order n:



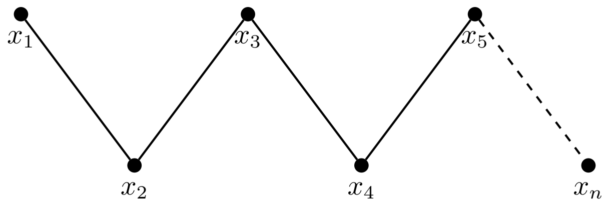
$$V = \{x_1, \cdots, x_n\}$$

The COMPLETE GRAPH K_n of order n:



$$V = \{x_1, \cdots, x_5\}, \quad E = \{x_i x_j \mid 1 \leqslant i < j \leqslant n\}$$
$$|V| = n, \ |E| = \binom{n}{2}$$

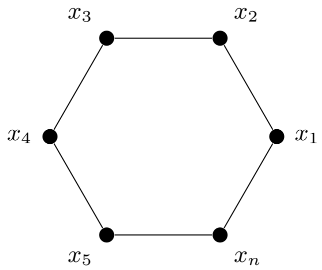
The PATH P_n of order n, LENGTH $n - 1$:



$$V = \{x_1, \cdots, x_5\}, \quad E = \{x_i x_{i+1} \mid 1 \leqslant i \leqslant n\}$$
$$|V| = n, \; |E| = n - 1$$

We may write $P_n = x_1 x_2 \cdots x_n$. We say that x_1, x_n are the END-VERTICES of P_n, and P_n is an $x_1 - x_n$ PATH.

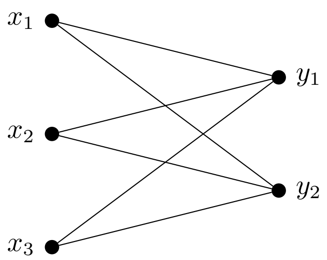
The CYCLE C_n of order and length $n \geqslant 3$:



$$V = \{x_1, \cdots, x_n\}, \quad E = \{x_i x_{i+1} \mid 1 \leqslant i \leqslant n\} \cup \{x_n x_1\}$$
$$|V| = n, \; |E| = n$$

We may write $C_n = x_1 x_2 \cdots x_n x_1$.

The COMPLETE BIPARTITE GRAPH $K_{m,n}$:



$$V = X \cup Y, \text{ where } X = \{x_1, \cdots, x_m\}, \; Y = \{y_1, \cdots, y_n\}$$
$$E = \{x_i y_j : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$$
$$|V| = m + n, \; |E| = mn$$

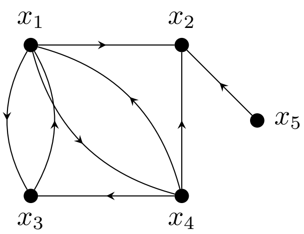
In general, a graph $G = (V, E)$ is BIPARTITE if we have $V = X \cup Y$ s.t. $E \subset \{xy \mid x \in X, y \in Y\}$.

$G' = (V', E')$ is a SUBGRAPH of $G = (V, E)$, written $G' \subset G$, if $V' \subset V$ and $E' \subset E$. If $V' = V$ and $E' \subset E$, then G' is a SPANNING SUBGRAPH of G.

Let $G = (V, E)$. If $xy \in E$, then x and y are NEIGHBOURS or ADJACENT. The NEIGHBOURHOOD of $x \in V$ is $\Gamma(x)$ (or $N(x)$) $= \{y \in V \mid xy \in E\}$. The DEGREE of x is $d(x) = |\Gamma(x)|$.

Definition 3.1 (Directed Graph)**.** A DIRECTED GRAPH or DIGRAPH is a pair $D = (V, A)$, where V is a set and A is a set of ordered pairs from V.

For example:



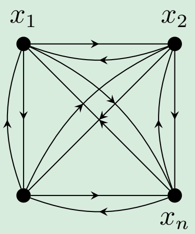
$$V = \{x_1, \cdots, x_5\},$$
$$E = \{x_1x_2, x_1x_3, x_1x_4, x_3x_1, x_4x_1, x_4x_2, x_4x_3, x_5x_2\}$$

V is the VERTEX SET, and A is the ARC SET. We may also write $V(D)$ and $A(D)$ for V and A. $|V|$ and $|A|$ are the ORDER and SIZE of D.

Note: No loops.

No multiple arcs. But $x1 - x2, x2 - x1$ is acceptable.

Example 3.1. The COMPLETE DIAGRAPH K_n of order n:

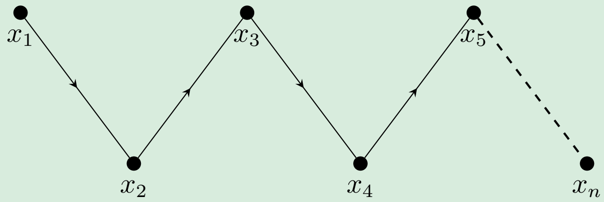


$V = \{x_1, \cdots, x_n\}$,

$E = \{x_i x_j \mid 1 \leqslant i \neq j \leqslant n\}$

$|V| = n$, $|A| = n(n_1)$

Example 3.2. The DIRECTED PATH, or PATH $\hat{P}_n$ of order n, length $n - 1$:



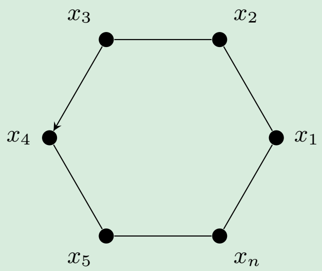
$V = \{x_1, \cdots, x_5\}$,

$E = \{x_i x_{i+1} \mid 1 \leqslant i \leqslant n\}$,

$|V| = n$, $|E| = n - 1$

We may write $\hat{P}_n = x_1 x_2 \cdots x_n$. We say that x_1, x_n are the SOURCE and SINK of $\hat{P}_n$, and $\hat{P}_n$ is an $x_1 - x_n$ Path.

Example 3.3. The DIRECTED CYCLE, or DICYCLE $\hat{C}_n$ of order and length $n \geqslant 3$ (Sometimes $n = 2$ are allowed).



$V = \{x_1, \cdots, x_n\}$, $E = \{x_i x_{i+1} \mid 1 \leqslant i \leqslant n\} \cup \{x_n x_1\}$

$|V| = n$, $|E| = n$

We may write $\hat{C}_n = x_1 x_2 \cdots x_n x_1$.

$D' = (V', A')$ is a SUBDIGRAPH of $D = (V, A)$, written $D' \subset D$, if $V' \subset V$ and $A' \subset A$. If $V' = V$ and $A' \subset A$, then D' is a SPANNING SUBDIGRAPH of D.

Let $D = (V, A)$. If $xy \in A$, then Y is an OUT-NEIGHBOUR of x and x is an IN-NEIGHBOUR of y. The OUT-NEIGHBOURHOOD/IN-NEIGHBOURHOOD of $x \in V$ are $\Gamma^+(x) = \{y \in V : xy \in A\}$ and $\Gamma^-(x) = \{y \in V : yx \in A\}$. The OUT-DEGREE/IN-DEGREE of $x \in V$ are $d^+(x) = |\Gamma^+(x)|$ and $d^-(x) = |\Gamma^-(x)|$.

For a digraph $D = (V, A)$, the UNDERLYING GRAPH is $G = (V, E)$ where $E = \{xy \mid xy \in A \textit{ or } yx \in A\}$.

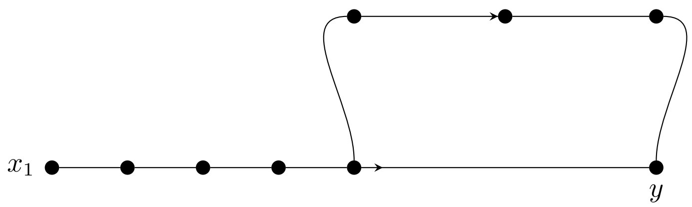
A graph $G = (V, E)$ is CONNECTED if $\forall x, y \in V$, $\exists x - y$ path $P \subset G$. A digraph $D = (V, A)$ is CONNECTED if the underlying graph of D is connected, and STRONG-CONNECTED if $\forall x, y \in V$, $\exists x - y$ path $\hat{P} \subset D$.

A NETWORK GRAPH is a pair (G, w), where G is a graph, and $w \mid E(G) \to \mathbb{R}$. A NETWORK DIGRAPH is a pair (D, w), where D is a digraph, and $w \mid A(D) \to \mathbb{R}$. In each case, w is a WEIGHT FUNCTION.

From now on, we assume V is finite, $\forall$ graphs and digraphs. In many situations, we consider connected graphs and digraphs.

3.2 Shortest path problem

Definition 3.2. Let (D, w) be a network digraph and $x, y \in V(D)$. A WALK from x to y, or an $x - y$ WALK, is a sequence vertices $x = v_0, v_1, \cdots, v_k = y$ s.t. $v_{i-1}v_i \in A(D)$, $\forall 1 \leqslant i \leqslant k$. Note that $v_0, v_1, \cdots, v_k$ and $v_0 v_1 \cdots v_{k-1} v_k$ are not necessarily distinct.



We say that y is REACHABLE from x if $\exists$ an $x - y$ walk (equivalently, $\exists x - y$ path) in D. In this case, a SHORTEST PATH from x to y is an $x - y$ walk. $x = v_0, v_1, \cdots, v_k = y$ with minimum weight, that is, $\sum_{i=1}^{k} w(v_{i-1}, v_i)$ is minimum. Note that D is strong connected $\iff$ y is reachable from x, $\forall\ x, yV(D)$.

We consider the following problem.

3.2.1 Problem 3.1 Shortest path problem

Let (D, w) be a network digraph. Let $x \in V(D)$ s.t. $\forall\ y \in V(D) \backslash \{x\}$, y is reachable from x. Find a shortest path from x to every $y \in V(D) \backslash \{x\}$. We call x the SOURCE VERTEX.

We consider two algorithms for solving Problem 3.1.

A. Dijkstra's Algorithm

We assume $w : A(D) \to \mathbb{R}_{\geqslant 0}$ in Problem 3.1. As Dijkstra's Algorithm is performed, every vertex $y \in V(D)$ is given a temporary label $T(y)$ which gets updated, then eventually a permanent label $P(y)$, which will be the weight of a shortest path from x to y. Then algorithm is as follows.

1. Set $P(x) = 0$, and

$$T(y) = \begin{cases} w(xy), & \text{if } y \in \Gamma^+(x) \\ \infty, & \text{if } y \in V(D) \backslash \{\Gamma^+(x) \cup \{x\}\} \end{cases}$$

Then, choose $y \in \Gamma^+(x)$ with $T(y)$ minimum, and set $P(y) = T(y)$.

2 Now, suppose $z \in V(D)$ is the most recent vertex to receive a permanent label, that is, to have $P(z)$ defined. $\forall v \in \Gamma^+(z)$ s.t. $T(v)$ but not $P(v)$ is defined, update

$$T(v) = \min\left(T(v), P(z) + w(zv)\right)$$

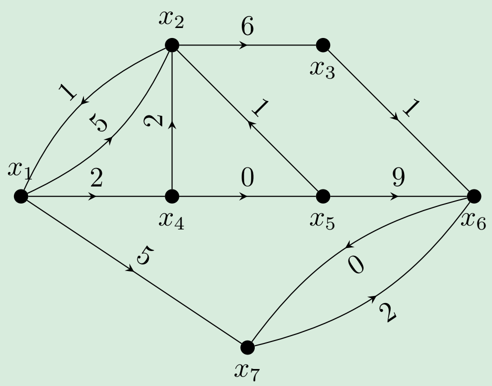
Then $\forall u \in V(D)$ s.t. $T(u)$ but not $P(u)$ is defined, choose y where $T(y)$ is minimum, and set $P(y) = T(y)$.

3 Return to Step 2. Repeat procedure until termination.

Remark 3.1.

- In Step 1 and 2 if there is a tie for the minimum, we may not choose any such suitable y.
- The assumption $w : A(D) \to \mathbb{R}_{\geqslant 0}$ implies that any shortest path from x is in fact a path.

Example 3.1. Find the shortest paths from x_1 to all other vertices, where (D, w) is:



We initially set

x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\underline{0^*}$	5	∞	$2^{\boxed{*}}$	∞	∞	5

where _ means the out-neighbour of this vertex (z in Step 2) are being considered, and * is a permanent label, with $\boxed{*}$ the newest one. The next iteration is

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0^*	4	∞	$\underline{2^*}$	$2^{\boxed{*}}$	∞	5

New temporary labels:

$$x_2 : \min(5, 2+2) = 4$$

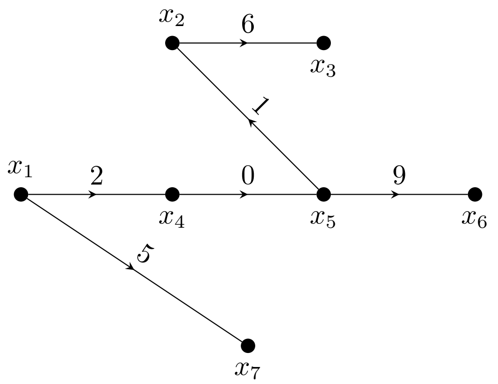
$$x_5 : \min(\infty, 2+0) = 2$$

$2 = \min(4, 2, 5)$, this becomes permanent.

Repeating, we obtain:

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0^*	$3\boxed{*}$	∞	2^*	$\underline{2^*}$	5	5
0^*	$\underline{3^*}$	9	2^*	2^*	$5\boxed{*}$	5
0^*	3^*	9	2^*	2^*	$\underline{5^*}$	$5\boxed{*}$
0^*	3^*	$9\boxed{*}$	2^*	2^*	5^*	$\underline{5^*}$

We obtain a <u>shortest paths tree from x_1 as</u>:



To obtain a shortest path from x_1 to y, look at when y is given $\boxed{*}$, and backtrack.

Theorem 3.1. Let (D, w) be a network digraph, where $w : A(D) \to \mathbb{R}_{\geqslant 0}$. Then Dijkstra's Algorithm solves Problem 3.1.

Proof. Let $R_t \subset V(D)$ be the set of permanently labelled vertices after t iterations of Dijkstra's Algorithm. Note that $R_t = R_{t-1} \cup \{y\}$, where y becomes permanently labelled at the t^{th} iteration $(1 < t \leqslant |V(D)|)$. We use induction on t to show that $P(z) = W(z)$, $\forall z \in R_t \backslash \{x\}$, where $W(z)$ is the weight of a shortest path from x to z.

Base cases: $t = 1 : R_1 = \{x\}, not \exists z \in R_1 \backslash \{x\}$.

$t = 2$: Let $R_2 = \{x, z\}$. Then clearly, $w(xz) \leqslant$ weight of any $x - z$ walk since $w : A(D) \to \mathbb{R}_{\geqslant 0}$.

Induction Step Let $t \geqslant 3$, and suppose $P(z) = W(z)$, $\forall z \in R_{t-1} \backslash \{x\}$. Let $R_t = R \cup \{y\}$. We show that $P(y) = W(y)$. Clearly, $P(y) \geqslant W(y)$. Suppose that $P(y) > W(y)$. Let $v_0 v_1 \cdots v_k$ be a shortest path from x to y. $(v_0 = x, v_k = y)$ with $\sum_{i=1}^{k} w(v_{i-1}v_i) = W(y)$. Since $y \in R_{t-1}$, $\exists 0 \leqslant l \leqslant k$ s.t. $v_i \in R_{t-1}$, $\forall 0 \leqslant i \leqslant l$, and $v_{l+1} \notin R_{t+1}$, then

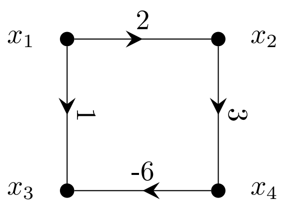
$$P(y) > \sum_{i=1}^{k} w(v_{i-1}v_i) \geqslant \sum_{i=1}^{k} w(v_{i-1}v_i) + w(v_l v_{l+1}) \geqslant P(v_l) + w(v_l v_{l+1}) \geqslant T(v_{l+1}) \geqslant P(y)$$

where $T_(v_{l+1})$ is the temporary label of v_{l+1} at the t^{th} iteration. To see $(*)$, note that at some iteration before t, v_l was given the permanent label $P(v_l)$, and v_{l+1} was given a temporary label $\leqslant P(v_l)+w(v_lv_{l+1})$. $+$ holds since at t^{th} iteration, y is given a permanent label over v_{l+1} or $y = v_{l+1}$.

We have a contradiction. The induction step holds. The proof follows by letting $R_t = V(D)$. □

Bellman-Fold Algorithm

Now, let $w : A(D) \to \mathbb{R}$ in Problem 3.1, so arcs of D may have negative weights. Then Dijkstra's Algorithm fails. A major problem is if D has a negative dicycle $\vec{C} = v_1v_2\cdots v_kv_1$ $(k \geqslant 2)$, where $\sum_{i=1}^{k} w(v_iv_{i+1}) < 0$ $(v_{k+1} = v_1)$. Then no shortest path exists from x to any v_i, since we may reach v_i from x, then go around $\vec{C}$ as much as we like. Dijkstra's Algorithm does not detect this problem, since at each iteration, only vertices without a permanent label are explored. Even if D has no dicycle, there is still a problem like:



If x_1 is the source vertex, then Dijkstra's Algorithm gives $P(x_3) = 1$. But the shortest path from x_1 to x_3 is $x_1x_2x_4x_3$, with weight -1. Then proof Th 3.2 breaks down at the base case $t = 2$, and also at the induction step.

The Bellman-Ford Algorithm can overcome these problems, and detect the presence of a negative dicycle, but takes more work than Dijkstra's Algorithm. As we run the algorithm, each vertex $y \in V(D)$ has its label $L(y)$ updated, which is the currently known weight of an $x - y$ walk. The algorithm is as follows:

1 Fix an ordering of the arcs of D, say $a_1, \cdots, a_m$, where $m = |A(D)|$.

2 Set $L(x) = 0$ and $L(y) = \infty$, $\forall y \in V(D)\backslash\{x\}$.

3 In one round, we run through $a_1, \cdots, a_m$. When we consider $a_i = uv$ say, update $L(v)$ by

$$L(v) = \min(L(v), L(u) + w(uv))$$

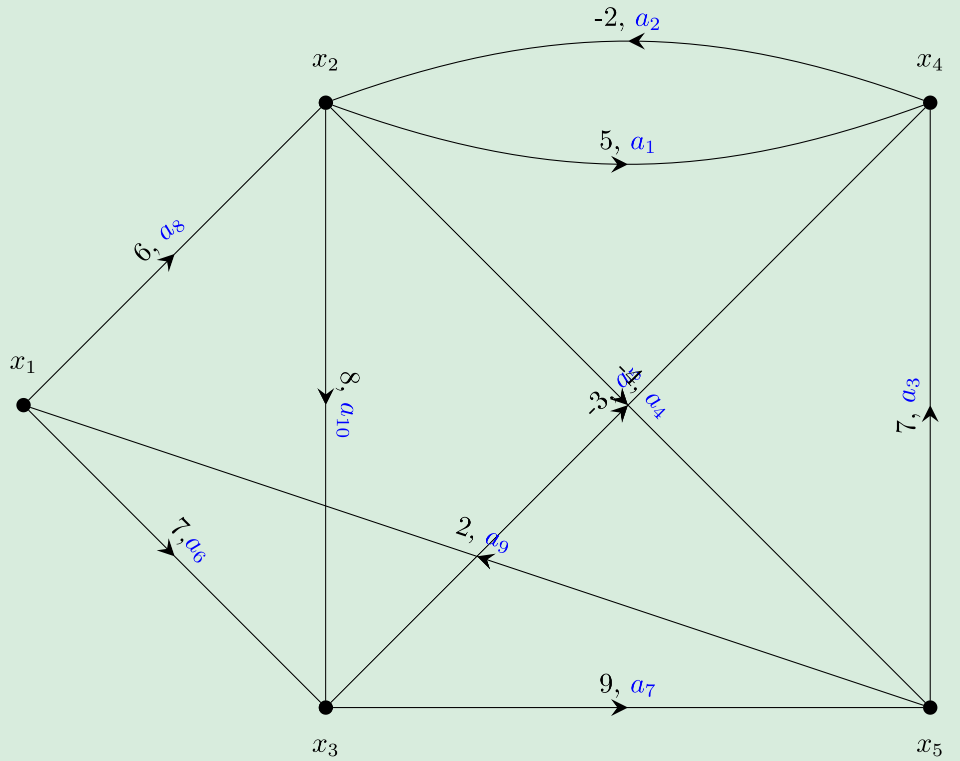
4 Repeat Step 3. If after $\leqslant |V(D)| - 1$ rounds, $L(y)$ does not decrease $\forall y \in V(D)$, then the resulting $L(y)$ are the weights of these shortest paths from x to every $y \in V(D)\backslash\{x\}$. Otherwise, $\exists$ negative dicycle in D.

Note that:

- After r rounds step 3, all walks from x with $\leqslant r$ arcs are found.

- If at round $|V(D)|$, some $L(y)$ is updated, then an x-y walk with $\geqslant |V(D)|$ arcs and weight $L(y)$ has been found. Such a walk contains a dicycle, and then can only exist if D has a negative dicycle.

Example 3.2. Find the shortest paths from x_1 to all others vertices when (D, w) is the following:



Fix the arc ordering $a_1, \cdots, a_{10}$ as shown. In round 1, we have

	x_1	x_2	x_3	x_4	x_5	
Initially	0	∞	∞	∞	∞	
After $a_1, \cdots, a_6$	$\boxed{0}$	∞	$\boxed{7}$	∞	∞	a_6
After a_7	0	∞	7	∞	16	
After a_8	0	6	7	∞	16	
After a_9, a_{10}	0	6	7	∞	16	

Table 3.1: Round 1

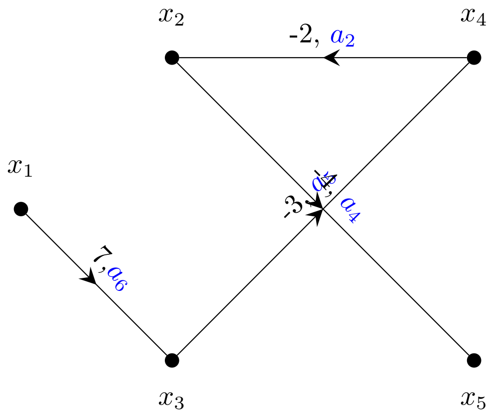
	x_1	x_2	x_3	x_4	x_5	
Initially	0	6	7	∞	16	
After a_1	0	6	7	11	16	
After a_2, a_3, a_4	0	6	7	11	2	
After a_5	0	6	$\boxed{7}$	$\boxed{4}$	2	a_5
After $a_6, \cdots, a_{10}$	0	6	7	4	2	

Table 3.2: Round 2

	x_1	x_2	x_3	x_4	x_5	
Initially	0	6	7	4	2	
After a_1, a_2	0	$\boxed{2}$	7	$\boxed{4}$	2	a_2
After a_3, a_4	0	$\boxed{2}$	7	4	$\boxed{-2}$	a_4
After $a_5, \cdots, a_{10}$	0	2	7	4	-2	

Table 3.3: Round 3

We can then show that round 4 does not update the bottom row of 3.3. The weights of the shortest paths from x_1, are given by the bottom row of 3.3. We can easily backtrack to obtain the shortest paths tree from x_1.



For example, for x_5, see the circle entries. The shortest path has arcs a_6, a_5, a_2, a_4.

Theorem 3.2. Let (D, w) be a network digraph, where $w : A((D) \to R$. Suppose we apply the Bellman-Ford Algorithm to solve Problem 3.1.

(a) Suppose for some $1 \leqslant r \leqslant |V(D)| - 1$, when going from round r to round $r+1$, the algorithm does not update $L(y)$, $\forall y \in V(D)$. Then $L(y)$ after round r is the weight of the shortest path from x to y, $\forall y \in V(D)\backslash\{x\}$, and $L(x) = 0$ at all times.

(b) If when going from round $|V(D)|-1$ to $|V(D)|$, the algorithm updates $L(y)$ for some $y \in V(D)$, then $\exists$ negative dicycle $\vec{C} \subset D$ $(|V(\vec{C}) \geqslant 2)$. Moreover, for $z \in V(D)\backslash\{x\}$, $\exists$ shortest path (with finite weight) from x to $z \iff z$ is not reachable from any vertex of a negative dicycle of D.

Proof. **(a)** Note that no $L(y)$ is updated when we go from round s to round $s+1$, $\forall s \geqslant r$. Suppose $\exists\ z \in V(D)\backslash\{x\}$, s.t. at round s, $\exists s \geqslant r$, we have $L(z) > W(z)$, where

$$W(z) = \begin{cases} \text{Minimum weight of an } x - z \text{ walk}, & \text{if finite}, \quad (1) \\ -\infty, & \text{otherwise} \quad (2) \end{cases}$$

If (1) holds, then $\exists$ x-z walk with weight $W(z)$, which is in fact an x-z path $\vec{P}$ (see HW5), so

$|A(\vec{P})| \leqslant |V(D)| - 1$. But then $\vec{P}$ would be realised by the algorithm at round $|V(D)| - 1$, when we have $L(z) \leqslant W(z) < L(z)$, a contradiction.

If (2) holds, then $\exists$ sequence of x-z walks, whose weights strictly decrease and $\to -\infty$. By running the algorithm for as many rounds as we like, all these weights will be realised, and so $L(z) \to -\infty$, a contradiction.

Finally, if $L(x) < 0$ after some number of rounds, then $\exists$ negative dicycle containing x. Running the algorithm as long as we like, we have $L(x) \to -\infty$, a contradiction.

(b) If $y = x$, then proof of last part of (a) $\implies \exists$ negative dicycle. Now let $y \neq x$. If $\exists v \in V(D)\backslash\{x\}$ s.t. $\neg\exists$ shortest path from x to v. ($\forall M > 0$, $\exists$ $x - v$ walk with weight $\leqslant -M$), then $\exists$ negative dicycle (HW5). Otherwise, $\forall y \in V(D)\backslash\{x\}$, $\exists$ shortest path from x to y, and this is an $x - y$ path(HW5), with $\leqslant |V(D)| - 1$ arcs. Hence, the algorithm cannot update after round $|V(D)| - 1$, a contradiction.

Final part: ($\Longrightarrow$) If z is reachable from some vertex u of a negative dicycle $\vec{C}$, then we may reach u from x, go around $\vec{C}$ as much as we like, then reach z from u. Then $\neg\exists$ shortest path from x to z.

($\Longleftarrow$) If $\neg\exists$ shortest path from x to z, then $\exists$ negative dicycle $\vec{C}$, and $\exists v \in V(\vec{C})$ s.t. z is reachable from v (HW5). $\square$

3.3 Maximum Flow Problem

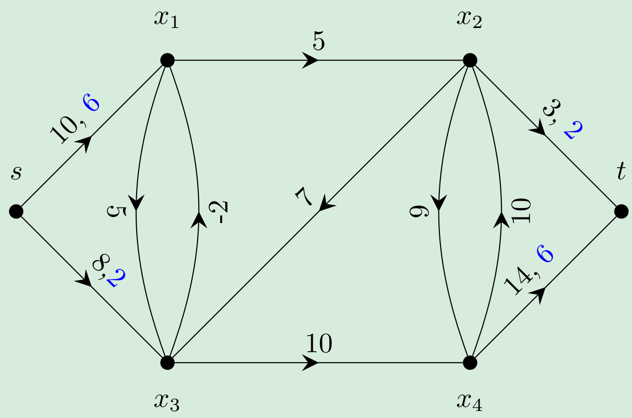
Definition 3.3. A FLOW NETWORK is a 4-tuple (D, c, s, t), where (D, c) is a network digraph, $c: A(D) \to \mathbb{R}_{\geqslant 0}$ is the CAPACITY FUNCTION, and $s, t \in V(D)$ are the SOURCE and SINK, s.t. t is reachable from s.

Definition 3.4. A FLOW of (D, c, s, t) is $f : A(D) \to \mathbb{R}_{\geqslant 0}$ s.t. $0 \leqslant f(a) \leqslant c(a)$, $\forall a \in A(D)$, and

$$\sum_{z \in \Gamma_-(x)} f(zx) = \sum_{y \in \Gamma^+(x)} f(xy), \quad \forall x \in V(D)\backslash\{s, t\} \tag{3.2}$$

That is, under f, the amount flowing into x and out of x is same. Note that we always have the flow $f \equiv 0$.

Example 3.3. A flow network, and a possible flow (reduced capacities in blue):



Flow $= 8$.

Lemma 3.4. Let f be a flow of (D, s, c, t). Then

$$\sum_{y \in \Gamma^+(s)} f(sy) - \sum_{z \in \Gamma^-(s)} f(zs) = \sum_{z \in \Gamma^-(t)} f(zt) - \sum_{y \in \Gamma^+(t)} f(ty) \tag{3.3}$$

That is, net amount flowing out of s =net amount flowing into t. The VALUE $v(f)$ of f is the common value of 3.3. If particular, if $\Gamma^+(s) = \Gamma^+(t) = \varnothing$, then

$$\sum_{y \in \Gamma^+(s)} f(sy) = \sum_{z \in \Gamma^-(t)} f(zt)$$

Proof. Note that

$$\sum_{v \in V(D)} \underbrace{\left(\sum_{y \in \Gamma^+(x)} f(xy) - \sum_{z \in \Gamma^-(x)} f(zx) \right)}_{(*)} = 0$$

since at x, then terms $f(xy)$ and $-f(zx)$ are cancelled by the terms $-f(xy)$ and $f(zx)$ from y and z. Thus 3.2 $\Longrightarrow$ The terms $(*) = 0$, except when $x = s, t$ and $\Longrightarrow$ 3.3 $\square$

By, lemma 3.4, we may ask:

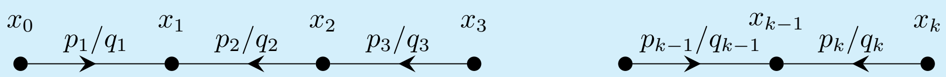
3.3.1 Problem 3.5

Given a flow network (D, c, s, t), find a maximum flow, that is, a flow f s.t. $v(f)$ is maximum.

Folk-Fulkerson Algorithm

Definition 3.5. Given (D, c, s, t), let $a_1, \cdots, a_k \in A(D)$ be s.t. when their directions are ignored, we have a graph path $x_0 x_1 \cdots x_k$, where $a_i = x_{i-1}x_i$ or $x_i x_{i-1}$, $\forall 1 \leqslant i \leqslant k$. Assign a symbol p_i/q_i to a_i $(1 \leqslant i \leqslant k)$, where $p_i + q_i = c(a_i)$, and $p_i, q_i \geqslant 0$. The a_i form an x_0-x_k augmenting path if

$$\begin{cases} p_i > 0, & \text{if } a_i = x_{i-1}x_i : \quad a_i \text{ is a } \underline{\text{forward arc}} \\ q_i > 0, & \text{if } a_i = x_i x_{i-1} : \quad a_i \text{is a } \underline{\text{backward arc}} \end{cases}$$



The Ford-Fulkerson Algorithm for solving Problem 3.5 is as follows.

- Label the arcs of D as $a_1, \cdots, a_m$ $(m = |A(D)|)$. Assign $c(a_i)/0$ to a_i, $\forall 1 \leqslant i \leqslant m$.
- Suppose a_i is labelled with p_i/q_i $(1 \leqslant i \leqslant m)$. Choose an s-t augmenting with arcs $b_1, \cdots, b_k$, labelled with p'_i/q'_i $(1 \leqslant i \leqslant k)$. Let

$$r = \min\left(\min\{p'_i : \ b_i \text{ is forward}\}, \ \min\{q'_i : \ b_i \text{ is backward}\}\right) > 0$$

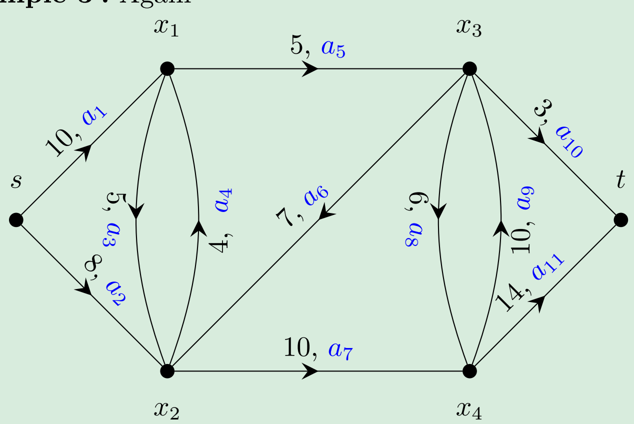
  Update p'_i/q'_i to

$$p'_i/q'_i = \begin{cases} p'_i - r/q'_i + r, & \text{if } b_i \text{ is forward,} \\ p'_i + r/q'_i - r, & \text{if } b_i \text{ is backward.} \end{cases}$$

  This directs a flow of $r > 0$ from s to t along the s-t augmenting path.

- Repeat Step 2 until $\neg\exists$ s-t augmenting path. The maximum flow is the sum of all the r's, achieved by the final q_i's

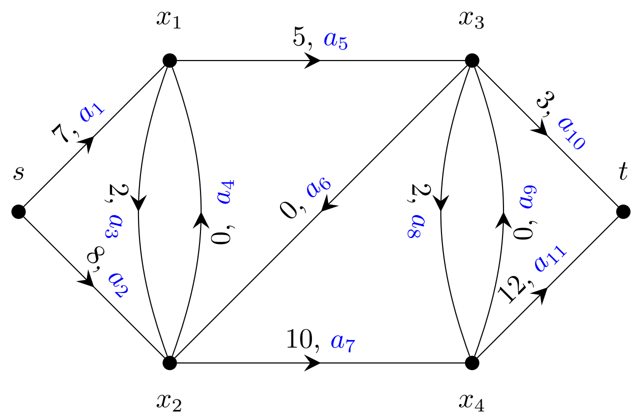
Example 3'. Again



We proceed as follows.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	s-t augmenting path	flow
$\underline{10}/0$	$8/0$	$5/0$	$4/0$	$\underline{5}/0$	$7/0$	$10/0$	$6/0$	$10/0$	$\underline{3}/0$	$14/0$	$a_1a_5a_{10}$	3
$7/3$	$\underline{8}/0$	$5/0$	$4/0$	$2/3$	$7/0$	$\underline{10}/0$	$6/0$	$10/0$	$0/3$	$\underline{14}/0$	$a_2a_7a_{11}$	8
$\underline{7}/3$	$0/8$	$5/0$	$4/0$	$\underline{2}/3$	$\underline{7}/0$	$\underline{2}/8$	$6/0$	$2/8$	$0/3$	$\underline{6}/8$	$a_1a_5a_6a_7a_{11}$	2
$\underline{5}/5$	$0/8$	$\underline{5}/0$	$4/0$	$0/5$	$5/\underline{2}$	$0/10$	$\underline{6}/0$	$2/8$	$0/3$	$\underline{4}/10$	$a_1a_3\overleftarrow{a_6}a_8a_{11}$	2
$3/7$	$0/8$	$3/2$	$4/0$	$0/5$	$7/0$	$0/10$	$4/2$	$10/0$	$0/3$	$2/12$		

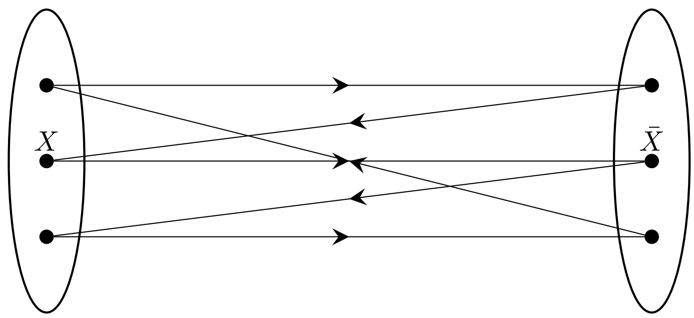
The bottom row has no s-t augmenting path. The maximum flow is $3 + 8 + 2 + 2 = 15$, achieved as follows:



Definition 3.6. Let (D, c, s, t) be a flow network. For disjoint $X, Y \subset V(D)$, let

$$A(X, Y) = \{xy \in A(D) : x \in X, y \in Y\},$$
$$c(X, Y) = \sum_{xy \in A(X,Y)} c(xy)$$

A CUT (SEPARATING s AND t is a set $A(X, \bar{X})$, where $s \in X$, $t \in \bar{X}$ and $\bar{X} = V(D) \backslash X$. Thus by deleting a cut $A(X, \bar{X})$ from D, the remaining subdigraph $D' = (V(D), A(D) \backslash A(X, \bar{X}))$ is s.t. t is not reachable from s. A MINIMUM CUT of (D, c, s, t) is a cut $A(X, \bar{X})$ s.t. $c(X, \bar{X})$ is minimum.

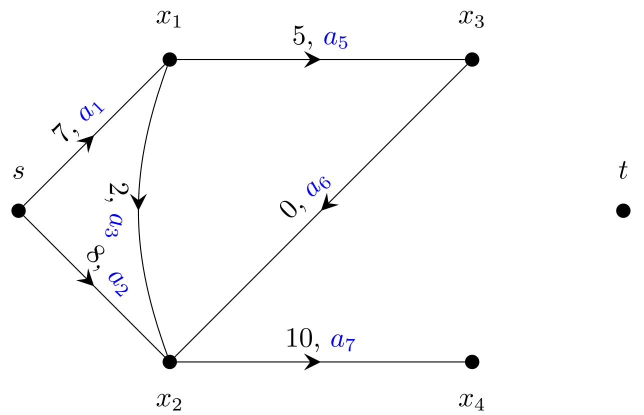


The Fold-Fulkersoon Algorithm also finds a minimum cut of (D, c, s, t). When the algorithm terminates, let

$$X = \{x \in V(D) : \exists\ s - x \text{ augmenting path w.r.t. final flow}\} \cup \{s\}$$

Note that $t \in \bar{X}$, otherwise $\exists$ s-t augmenting path when the algorithm terminated, a contradiction. So $A(X, \bar{X})$ is a cut, and in fact, a minimum cut.

Example 3′. After the algorithm terminates in Example 3', we see that $\exists$ s-t augmenting path $\Longleftrightarrow x \in \{x_1, x_2\}$.



We have $X = \{s, x_1, x_2\}$, $\bar{X} = \{t, x_3, x_4\}$, and $A(X, \bar{X}) = \{a_5, a_7\}$ with $c(X, \bar{X}) = 5 + 10 = 15$.

We have the following result

Theorem 3.6 (Max-flow min-cut Theorem)**.** For a flow network (D, c, s, t), the maximum flow is equal to $c(X, \bar{X})$, where $A(X, \bar{X})$ is a minimum cut.

We first prove:

Theorem 3.7 (Flow value lemma)**.** Let f be a flow and $A(X, \bar{X})$ be a cut of (D, c, s, t). Then

$$v(f) = \sum_{xy \in A(X, \bar{X})} f(xy) - \sum_{yx \in A(\overline{X}, X)} f(yx)$$

Proof. We have

$$\begin{aligned} v(f) &= \sum_{y \in \Gamma^+(s)} f(sy) - \sum_{z \in \Gamma^-_(s)} f(zs) \\ &= \sum_{x \in X} \underbrace{\left(\sum_{y \in \Gamma^+(s)} f(xy) - \sum_{z \in \Gamma^-_(s)} f(zx) \right)}_{(*)}, \text{ the terms } (*) = 0, \text{ except for } x = s \\ &= \sum_{xy \in A(X, \bar{X})} f(xy) - \sum_{yx \in A(X, \bar{X})} f(yx), \text{ arcs within } X \text{ cancel.} \end{aligned}$$

□

Proof. of Th 3.6. For any flow g and any cut $A(Y, \bar{Y})$, we have $v(g) \leqslant c(Y, \bar{Y})$(HW5). So $v = \sup\{v(g) : g \text{ is a flow of } (D, c, s, t)\} < \infty$. We may take a sequence of flow $g_j : A(D) \to \mathbb{R}_{\geqslant 0}$ s.t. $v(g_j) \to v$ and $g_j(xy) \to f(xy)$, $\forall xy \in A(D)$ (f is the point-wise limit of the g_j). Then f is a minimum flow. Now, we construct a cut $A(X, \bar{X})$ s.t. $v(f) = c(X, \bar{X})$. Define

$$X = \{x \in V(D) : \exists\ s - x \text{ augmenting path w.r.t. } f\} \cup \{s\}$$

We claim that $A(X, \bar{X})$ is a cut. Otherwise, suppose $t \in X$. Then $\exists s = x_0, x_1, \cdots, x_k = t \in V(D)$ s.t.

$$r_i = c(x_{i-1}x_i) - f(x_{i-1}x_i) \text{ or } r_i = f(x_i x_{i-1})$$

satisfies $r_i > 0$, $\forall 1 \leqslant i \leqslant k$. Let $r = \min\limits_i r_i > 0$. We define the flow f^* by

$$\begin{aligned}
&f^*(x_{i-1}x_i) = f(x_{i-1}x_i) + r \leqslant c(x_{i-1}x_i), && \text{if } r_i = c(x_{i-1}x_i) - f(x_{i-1}x_i) \\
&f^*(x_{i-1}x_i) = f(x_i x_{i-1}) - r \geqslant 0, && \text{if} r_i = f(x_i x_{i-1}), \\
&f^*(a) = f(a), && \forall \text{ other arcs } a.
\end{aligned}$$

Then f^* is also a flow, and $v(f^*) = v(f) + r$ since either sx_1 increased by r, or $x_1 s$ decreased by r. This contradicts that f is a maximum flow.

Now,

$$v(f) \underset{=}{\text{Th } 3.7} \sum_{xy \in A(X, \bar{X})} \underbrace{f(xy)}_{=c(xy)} - \sum_{yx \in A(X, \bar{X}} \underbrace{f(yx)}_{=0} = c(X, \bar{X}),$$

as required. □

> **Theorem 3.8.** The Ford-Fulkerson Algorithm, <u>If</u> it terminates, solves Problem 3.5 for $(D,\ c,\ s,\ t)$ and also finds a minimum cut. Precisely, let f be the final flow, and
>
> $$X = \{x \in V(D) :\ \exists\ s - x \text{ augmenting path w.r.t } f\} \cup \{s\}$$
>
> Then f is a maximum flow, and $A(X,\ \overline{X})$ is a minimum cut.

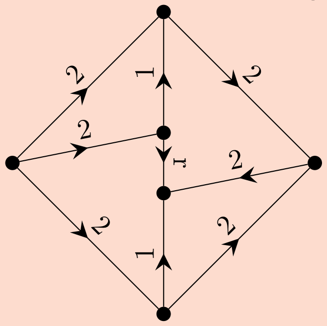
Proof. From the discussion before Example 3", $A(X,\ \overline{X})$ is a cut. Note that if $xy \in A(X,\ \overline{X})$ then $f(xy) = c(xy)$, and if $yx \in A(X,\ \overline{X})$ then $f(yx) = 0$. Otherwise, in either case, $\exists\ s - y$ augmenting path:

Take an $s-x$ augmenting path, which must be within X, then go along xy or yx to y. Thus

$$v(f) \underset{L\ 3.7}{=} \sum_{xy\in A(X,\overline{X})} \underbrace{f(xy)}_{=c(xy)} - \sum_{yx\in A(\overline{X},X)} \underbrace{f(yx)}_{=0} = c(X,\overline{X})$$

If $A(Y,\ \overline{Y})$ is a minimum cut, then $c(X,\ \overline{X}) = v(f) \leqslant c(Y,\ \overline{Y})$ (HW5). So $A(X,\ \overline{X})$ is a minimum cut, and Th 3.6 $\Rightarrow$ f is a maximum flow. $\square$

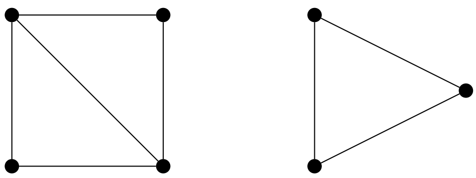
Remark 3.2. The Fold-Fulkersoon Algorithm terminates if $c:\ A(D) \to \mathbb{Z}_{\geqslant 0}$, since each iteration increase the currently known flow by $\geqslant 1$. The algorithm also terminates if $c:\ A(D) \to \mathbb{Q}_{\geqslant 0}$, since we may consider $c' : A(D) \to \mathbb{Z}_{\geqslant 0}$ by multiplying the denominators of $c(xy)$, $\forall\ xy \in A(D)$ by a common multiple. But the algorithm may possible not terminate if $c : A(D) \to \mathbb{R}_{\geqslant 0}$. For example:



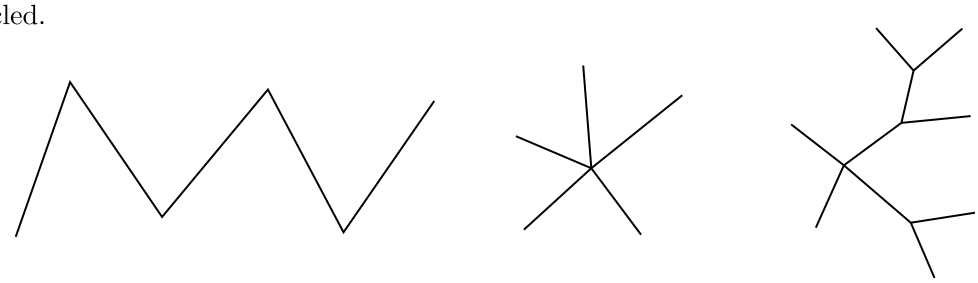
$r = \frac{\sqrt{5}-1}{2}$, which satisfies $r^2 = 1-r$.

3.4 Minimum Spanning Tree Problem

Definition 3.7. A <u>(connected) component</u> of a graph G is a "maximal connected subgraph" $H \subset G$ in the following sense: H is connected, and $\neg\exists\ xy \in E(G)\backslash E(H)$ s.t. $H' = (V(H)\cup\{x,\ y\},\ E(H)\cup\{xy\})$ is connected.



A graph T is a tree if $T$is connected; and does not contain q cycle as q subgraph. A graph F is a forest if all components of F are trees. A leaf of a graph is a vertex with degree 1. Trees, with leaves circled.



3.4.1 Problem 3.9 (Minimum Spanning Tree Problem)

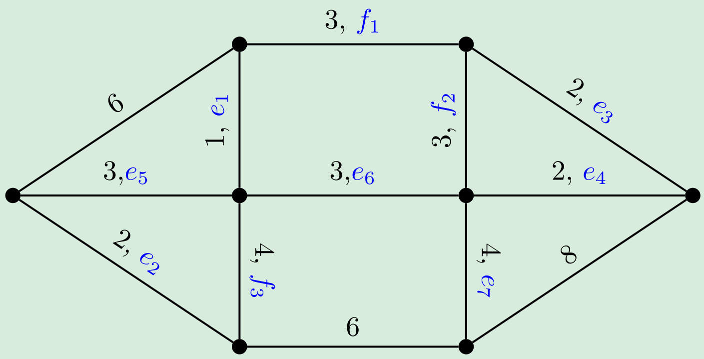
Let (G, W) be a network graph, where G is connected and $w: E(G) \to \mathbb{R}$. Find a minimum spanning tree (MST), that is, a spanning tree (ST) T of G s.t. $\sum_{e \in E(T)} w(e)$ is minimum.

Kruskal's Algorithm can solve Problem 3.9, and is as follows.

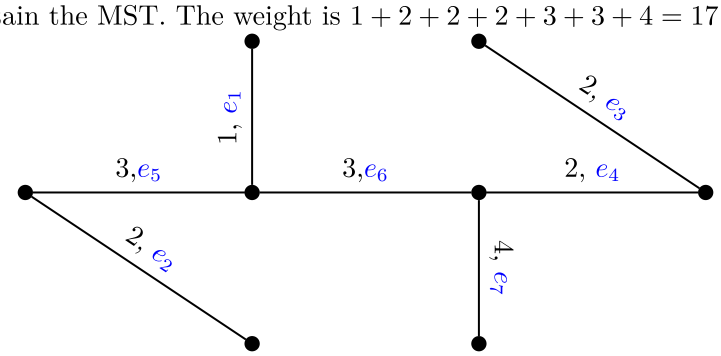
Kruskal's Algorithm

- Choose an edge $e_1 \in E(G)$ with $w(e_i)$ minimum.
- Suppose edges $e_1, \cdots, e_k \in E(G)$ have been chosen, for some $k \geqslant 1$. Choose $e_{k+1} \in E(G) \backslash \{e_1, \cdots, e_k\}$ s.t. e_{k+1} does not from a cycle with some of $e_1, \cdots, e_k$ and $w(e_{k+1})$ is minimum.
- Continue until $|V(G)| - 1$ edges are found. The result is a MST.

Example 4′. Let (G, W) be



We "greedily" choose $e_1, \cdots, e_6$, since they have the smallest weights, and do not from a cycle. Then we cannot choose f_1 or f_2 since adding either edge forms a cycle. We can then choose e_7 (but not f_3) to obtain the MST. The weight is $1 + 2 + 2 + 2 + 3 + 3 + 4 = 17$.



Theorem 3.9. Kruskal's Algorithm solves Problem 3.9.

Proof. Let $|V(G)| = n$. We first show that when the algorithm terminates, we have a *ST* T of G.

Initially, we have a forest with n single vertices as components. Suppose that after t iterations $(0 \leqslant t \leqslant n-2)$, the current graph F is a forest with $n - t$ components. If we add a new edge xy to F to form F^+, then we cannot have x, y belonging to the same component of F, otherwise we create a cycle. Also, since G is connected, we may add xy where x, y belong to two different components F_1, F_2 of F. The new component $F' = (V(F_1) \cup V(F_2),\ E(F_1) \cup E(F_2) \cup \{xy\})$ of F^+ is also a tree. Clearly F' is connected, and if $\exists$ cycle $C \subset F'$, then $\exists\ e, e' \in E(C)$, each one connecting F_1 and F_2, a contradiction since xy is the only such edge in F'. Thus, F^+ is a forest with $n-t-1$ components. Iterating $n-1$ times gives a ST T of G.

Now, we prove that T is a MST. Let $e_1, \cdots, e_{n-1}$ chosen in that order, so $w(e_1) \leqslant \cdots \leqslant w(e_{n-1})$. If T is not a MST, choose a MST $S \subset G$ s.t. $|E(S) \cap E(T)|$ is maximum. Let $1 \leqslant k \leqslant n-1$ be s.t. $e_1, \cdots, e_{k-1} \in E(S) \cap E(T)$, and $e_k \in E(T) \backslash E(S)$. Then $S' = (V(S), E(S) \cup \{e_k\})$ has a cycle C'. Now, $\exists\ f \in E(C') \backslash E(T)$. Then $S'' = (V(S',\ E(S') \backslash \{f\})$ is also a ST, since S'' is connected and has $n-1$ edges (HW6). To see that S'' is connected, let $u,\ v \in V(S''$. Then $\exists\ u-v$ path $P \subset S'$. Either $P \subset S''$, or $f \in E(P)$. If the latter, let $f = zz'$. Then $\exists\ u-z$ and $z'-v$ paths $\subset P$, and a $z-z'$ path in C', not using f. These paths $\implies \exists\ u-v$ path in S''.

Now $\sum\limits_{e \in E(S'')} w(e) \geqslant \sum\limits_{e \in E(S)} w(e) \implies w(e_k) \geqslant w(f)$. Since $e_1, \cdots, e_{k-1}, e_k \in E(T)$ and $e_1, \cdots, e_{k-1}, f \in E(S)$, neither of these sets of edges creates a cycle, and since the algorithm chose e_k instead of f, we have $w(e_k) \geqslant w(f)$. So $w(e_k) = w(f)$, and S'' is also a MST. But then $|E(S'') \cap E(T)| = |E(S) \cap E(T)| + 1$, which contradicts the choice of S. $\square$

Remark 3.3.

- Kruskal's Algorithm also work if some edges of G has negative weights.
- If these is a tie for the choice of edges with minimum weight at an iteration, we may choose any suitable edge.
- If G has q components, then Kruskal's Algorithm finds a minimum spanning forest in $n-q$ iterations.

Proposition 3.11. Let T be a tree with $n \geqslant 2$ vertices.

(a) T has a leaf.

(b) Deleting a leaf x and the edge e incident to x from T gives another tree T'.

Proof. True for $n = 2$. Assume $n \geqslant 3$.

(a) Let $P = x_0 x_1 \cdots x_k \subset T$ be a longest path, for some $k \geqslant 2$. Then $\Gamma(x_k) \subset \{x_0, \cdots, x_{k-1}\}$, otherwise we have a path longer than P. Also, $x_k x_i \notin E(T)$, $\forall 0 \leqslant i \leqslant k-2$, otherwise T has a cycle. So $\Gamma(x_k) = x_{k-1}$, and x_k is a leaf.

(b) Clearly $\neq \exists$ cycle in $T \implies \neq \exists$ cycle in T'. Let $u, v \in V(T') = V(T) \backslash \{x\}$. Then $\exists u-v$ path $Q \subset T$, and Q cannot use e and x since $d(x) = 1$ in T. So $G \subset T'$ and T' is connected.

$\square$

Prop 3.11 is very useful for induction proofs involving trees.

Chapter 4

Queueing Theory

4.1 Queueing system

Queueing happens in everyday life. At the supermarket or airport, bank and so on. A typical queueing system (say at the airport check-in):

Suppose the n^{th} customer C_n arrives at time $\tau_n \geqslant 0$, where $\tau_n = 0$ and $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant$. The interarrival time between customers c_n and c_{n+1} is $T_n = \tau_{n+1} - \tau_n$ $(T_0 = \tau_1)$. The service time for customer C_n is $X_n \geqslant 0$. Assume the T_n are independent random variables, and the some for the X_n. The distribution function (DF) of $T_n (n \geqslant 0)$ is $A_n : \mathbb{R} \to [0, 1]$, where

$$A_n(t) = \mathbb{P}(T_n \leqslant t)$$

and the probability density function (PDF) is

$$a_n(t) = A_n'(t)$$

Similarly, X_n $(n \geqslant 1)$ has DF $B_n(t) = \mathbb{P}(X_n \leqslant t)$ and PDF $b_n(t) = B_n'(t)$.

Kendall-Lee notation A queueing system description is written as:

Maximum number of customers allowed in the queueing system (in queue and in service)

Service time distribution

Interarrival time distribution

Customer population

$$A \,/\, B \,/\, s \,/\, D \,/\, q \,/\, p \tag{4.1}$$

Number of servers

Queueing discipline

Notable cases: For A and B:

- M: Markovian, with exponential distribution

$$A_n(t) = 1 - e^{-\lambda_n t}, \quad a_n(t) = \lambda_n e^{-\lambda_n t}$$

for some parameter $\lambda_n > 0$ $(n \geqslant 0)$. Similarly for $B_n(t)$, $b_n(t)$ with parameter $\mu_n > 0$ $(n \geqslant 1)$.

- D: Deterministic, with "constant distribution".
- E_k: Erlang-k
- G: General, not specified, usually the mean and variance are known.

For D:

- FCFS: First come first served. Services is performed according to the order of the queue. New arrival join the back of the queue.
- LCFS: Last come first served.
- SIRO: Service in random order.

Note also Nucleic acid test queueing discipline.

Arrival order: $1, 2, \cdots, 20, 21, 22, \cdots, 40, 41, \cdots$

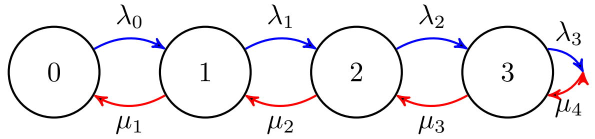
Service order: $20, 19, \cdots, 40, 39, \cdots, 21, 60, \cdots$

For example, $M/G/2/SIRO/50/1000$. If $D = FCFS$, $q = p = \infty$, then we write 4.1 as $A/B/s$. The simplest case is $M/M/s$, especially $M/M/1$.

Definition 4.1.

- State/Queue length= Number of customers in the queueing system/queue (12 and 8 in the diagram).
- $N(t)$ = State at time $t \geqslant 0$.
- $\mathbb{P}_k(t)$ = Probability of state k at time $t \geqslant 0$ (assume $\mathbb{P}_k(0) = 0$).
- λ_k = Mean arrival rate = Expected number of new customer arrivals per unit time, when state $= k \geqslant 0$.
- μ_k = Mean service rate for overall system = Expected number of customers served per unit time between all busy servers (those serving customers), when state $= k \geqslant 1$.

When A and B are both Markovian, we have the birth-death process rate diagram:



If λ_k is constant $\forall k \geqslant 0$, write $\lambda_k = \lambda$. If the mean service rate per busy server is constant $\forall k \geqslant 1$, denote this constant by μ. Then

$$\mu_k = \begin{cases} k_\mu & 1 \leqslant k \leqslant s \\ s\mu & k > s. \end{cases}$$

Under these circumstance, $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ are the expected interarrival time and the expected service time. Also,

$$\varrho = \frac{\lambda}{s\mu}$$

is the utilisation factor of the service facility, which is the expected fraction of time the service capacity $(s\mu)$ is being utilised by arriving customers (λ).

Example 1′. Suppose a time unit is one hour.

- If on average, a customer arrives every 10 minutes, then $\lambda = 6$ customers/hour. Expected interarrival time $\frac{1}{\lambda} = \frac{1}{6}$.
- If on average, a service time takes 30 minutes, then $\mu = 2$ customers/hour. Expected service time $= \frac{1}{\mu} = \frac{1}{2}$.

Utilisation factor $\varrho = \frac{\lambda}{s\mu} = \frac{3}{s}$.

- If $s = 1, 2$, then $\varrho > 1$. The state $\to \infty$ as time $\to \infty$, and the system is "unsteady".
- If $s = 3$, then $\rho = 1$. Queue is unstable and system may not become "steady".
- If $s \geqslant 4$, then $\rho < 1$. The system will become "steady".

Definition 4.2.

The average state over $[0, t]$

$$\bar{L}(t) = \frac{1}{t}\int_0^t N(u)du$$

The long term average state

$$\bar{L} = \lim_{t\to\infty} \bar{L}(t)$$

Number of arrivals in $[0, t]$

The average number of arrivals in $[0, t]$

$$\bar{\alpha}(t) = \frac{\alpha(t)}{t}$$

$$\bar{\lambda} = \lim_{t\to\infty} \bar{\alpha(t)} \tag{4.2}$$

The long term average arrival rate

- Sojourn time = Time a customer spends in queueing system.

- W_k = Sojourn time of customer C_k.
- $\bar{W}(t) = \frac{1}{\alpha(t)} \sum_{k=1}^{\alpha(t)} W_k$, the average sojourn time in $[0, t]$.

$$\bar{W} = \lim_{t \to \infty} \frac{1}{n} \sum_{k=1}^{n} W_k \tag{4.3}$$

the long term average sojourn time

Theorem 4.1 (Little' Low)**.** For any queueing system,

$$\overline{L} = \overline{\lambda}\,\overline{W}, \tag{4.4}$$

provided the limits 4.2, 4.3 exist.

Proof. Let $\beta(t)$ be the number of departures in $[0, t]$. We compare the graphs of $\alpha(t)$, $\beta(t)$, and the sojourn times.

Let $E(t)$ be the excess sojourn time of all active customers beyond time t. Then

$$\begin{aligned}\overline{L}(t) &= \frac{1}{t}\int_0^t N(u)du = \frac{1}{t}\int_0^t (\alpha(u) - \beta(u))du \\ &= \frac{\alpha}{t}\frac{1}{\alpha}\left(\sum_{k=1}^{\alpha(t)} W_k - E(t)\right) = \bar{\alpha}(t)\overline{W(t)} - \frac{E(t)}{t}\end{aligned}$$

Letting $t \to \infty$ and assuming $\frac{E(t)}{t} \to 0$, 4.4 follows. $\square$

Remark 4.1. If $\overline{L}(q)$ = Long term average queue length, $\overline{W}_q$ = Long term average queueing time, then we can similarly prove that

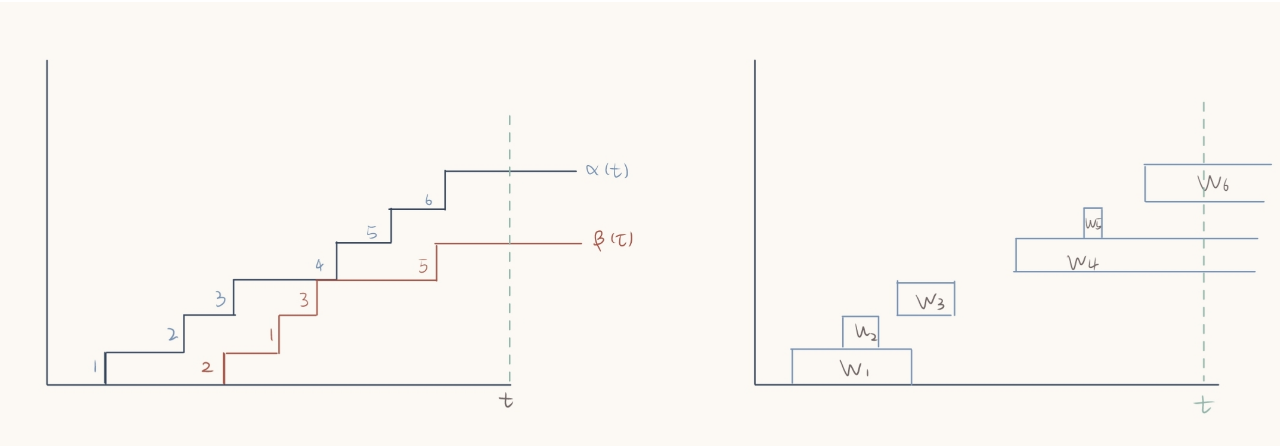
$$\overline{L}_q = \overline{\lambda}\,\overline{W}_q$$

Also, if the expected service time $\frac{1}{\mu}$ is constant, then

$$\overline{W} = \overline{W}_q + \frac{1}{\mu}$$

Example 2′. Bob owns a small coffee shop. He estimates that on average, in one hour, 40 customers arrive at his shop, and each customer spends 6 minutes in the queueing system. Then the average number of customers in the queueing system is

$$\overline{L} = \underbrace{40}_{\overline{\lambda}} \times \underbrace{0.1}_{\overline{W}} = 4$$



4.2 The exponential distribution

Recall that in a Markovian queueing system. $A/B/s$ both $A = M$(interarrival time) and $B = M$(service time) follow the exponential distributions, with DF and PDF of the form

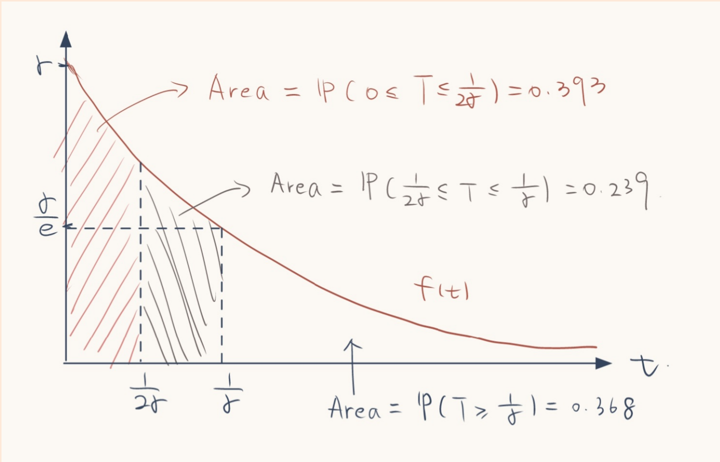
$$F(t) = \mathbb{P}(T \leqslant t) = \begin{cases} 1 - e^{-\gamma t} & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases}$$
$$f(t) = \begin{cases} \gamma e^{-\gamma t} & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases}$$

for some $\gamma > 0$, where T is the A or B RV. We have

$$\mathbb{E}T = \int_0^\infty \gamma u e^{-\gamma u} du = \frac{1}{\gamma}$$
$$\text{var}T = \mathbb{E}(T^2) - (\mathbb{E}T)^2 = \int_0^\infty \gamma u^2 e^{-\gamma u} du - \frac{1}{\gamma^2} = \frac{1}{\gamma^2}$$

Proposition 4.2. $f(t)$ is strictly decreasing for $t \geqslant 0$. Thus,

$$\mathbb{P}(0 \leqslant T \leqslant h) > \mathbb{P}(t \leqslant T \leqslant t + h), \quad \forall t, h > 0$$



Recall For events E_1, E_2 in a probability space, the conditional probability of E_1, gives E_2, is

$$\mathbb{P}(E_1 \mid E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)}$$

Proposition 4.3 (Lack of memory)**.**

$$\mathbb{P}(T > t + h \mid T > h) = \mathbb{P}(T > t) \quad \forall t, h > 0$$

That is, the probability distribution of the remaining time until the event (arrival or service completion) occurs is always the same, regardless of how much time h has already passed. There is no memory of what has already occurred.

Proof.

$$\begin{aligned}\mathbb{P}(T > t + h \mid T > h) &= \frac{\mathbb{P}(T > t + h \text{ and } T > h)}{\mathbb{P}(T > h)} = \frac{\mathbb{P}(T > t + h)}{\mathbb{P}(T > h)} \\ &= \frac{e^{-\gamma(t+h)}}{e^{-\gamma h}} = e^{-\gamma t} = \mathbb{P}(T > t)\end{aligned}$$

□

Proposition 4.4. Let $T_1, \cdots, T_n$ be independent exponential RVs with parameters $\gamma_1, \cdots, \gamma_n$. Then the RV

$$U = \min(T_1, \cdots, T_n)$$

has an exponential distribution with parameter $\gamma = \gamma_1 + \cdots + \gamma_n$. Thus, if T_i represents the time until the i^{th} event occurs, then U represents the time until the first of the n events occurring.

Proof.

By independence

$$\begin{aligned}\mathbb{P}(U > t) &= \mathbb{P}(T_1 > t, \cdots, T_n > t \text{ all hold}) \\ &= \mathbb{P}(T_1 > t) \cdots \mathbb{P}(T_n > t) \\ &= e^{-\gamma_1 t} \cdots e^{-\gamma_n t} = e^{-\gamma t}\end{aligned}$$

□

In particular, if $X_1, \cdots, X)s$ are the independence service times, each exponential with parameter μ, then U is the time until the next service completion, which is the exponential with parameter $s\mu$.

Relationship with Poisson distribution and process

Suppose the time between consecutive occurrences of some kind of event (For example, interarrivals) has an exponential distribution with parameter γ. Let $Y(t)$ be the number of event occurrences at time $t \geqslant 0$. Then $Y(t)$ is a RV with the Poisson Distribution with parameters γt, given by

$$\mathbb{P}(Y(t) = k) = \frac{(\gamma t)^k e^{-\gamma t}}{k!}, \quad \text{for } k = 0, 1, 2, \cdots$$

When the events are counted on a continuing basis, the counting process $\{Y(t) : t \geqslant 0\}$ is a Poisson process with mean rate γ.

Proposition 4.5.

$$\mathbb{P}(T \leqslant t + h \mid T > t) \approx \gamma h, \quad \forall t > 0 \text{ and small } h$$

Proof.

$$\begin{aligned}\mathbb{P}(T \leqslant t + h \mid T > t) &= \frac{\mathbb{P}(T \leqslant t + h \text{ and } T > t)}{\mathbb{P}(T > t)} \\ &= \frac{(1 - e^{-\gamma(t+h)}) - (1 - e^{-\gamma t})}{e^{-\gamma t}} \\ &= 1 - e^{-\gamma h} \approx \gamma h \text{ for small } h\end{aligned}$$

□

Note that Prop 4.3 says that the probability of an event occurring within an interval of fixed length h is constant, regardless of how much time has passed. Prop 4.5 says that if h is small, then this constant probability is approximately γh.

4.3 Birth-Death Process and M/M/s Systems

Recall that $\mathbb{P}_k(t) =$ Probability that state $= k \geqslant 0$ at time $t \geqslant 0$ in a queueing system.

Definition 4.3. A queueing system is in transient state if $\mathbb{P}_k(t)$ depends on t. The system becomes steady state if $\mathbb{P}_k(t)$ becomes independent of t if $t \geqslant t_1$ for some (large) time t_1. If the system is in steady state, let

$$p_k = \text{Probability that state} = k \geqslant 0$$

The birth-death process describe the state of a queueing system over time. If the system is Markovian, we assume the following:

(1) Given $N(t) = k$, the probability distributions of the remaining times until the next "birth" (arrival) and next "death" (departure) are exponential with parameters λ_k, μ_k.

(2) All RVs in (1) are mutually independent. The next change in state is

$$\begin{aligned} &\text{either:} && k \to k+1 && \text{(a birth)} \\ &\text{or:} && k \to k-1 && \text{(a death)} \end{aligned}$$

Remark 4.2. By Prop 4.3, for the Markovian birth-death process, the future depends only on the current state, and is independent of any past events. The process is thus a continuous time Markov chain.

The relation of the exponential distribution to the Poisson process implies that the λ_k and μ_k are mean rates.

Assume that a queueing system can reach steady state. Let $u_k(t)$ and $v_k(t)$ be the number of times the birth-death process enters and leave state k in $[0, t]$. Then $|u_k(t) - v_k(t)| \leqslant 1$, so

$$\lim_{t\to\infty} \left| \frac{u_k(t)}{t} - \frac{v_k(t)}{t} \right| = 0$$

Since $\frac{u_k(t)}{t}$ and $\frac{v_k(t)}{t}$ are the mean rates at which the process enters and leaves state k, we have:

Proposition 4.6 (Rate in = Rate out Principle)**.** For any state $k \geqslant 0$ in a Markovian birth-death Process,

$$\text{Mean entering rate} = \text{Mean leaving rate} \tag{4.5}$$

An equation of the form 4.5 is a balance equation.

For state 0, the process may only enter from state 1. In steady state, since p_1 is the proportion of time that the process can enter state 0, the mean entering rate to state 0 is $\mu_1 p_1$. Similarly, the mean leaving rate from state 0 is $\lambda_0 p_0$. Thus 4.5$\Rightarrow$

$$\mu_1 p_1 = \lambda_0 p_0 \tag{4.6}$$

For any other state $k \geqslant 1$, we similarly have

$$\lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} = (\lambda_k + \mu_k) p_k \tag{4.7}$$

Now, 4.6 $\Rightarrow$ $\mu_1 p_1 - \lambda_0 p_0 = 0$. And 4.7 $= \mu_{k+1} p_{k+1} - \lambda_k p_k = \mu_k p_k - \lambda_{k-1} p_{k-1}$, $\forall k \geqslant 1$. Thus $\mu_k p_k - \lambda_{k-1} p_{k-1} = 0$, $\forall k \geqslant 0$.

We have

$$\begin{aligned} p_k &= \frac{\lambda_{k-1}}{\mu_k} p_{k-1} + \frac{1}{\mu_k} \left(\mu_{k-1} p_{k-1} - \lambda_{k-2} p_{k-2} \right) = \frac{\lambda_{k-1}}{\mu_k} p_{k-1} \\ &= \cdots = \frac{\lambda_{k-1} \lambda_{k-2} \cdots \lambda_0}{\mu_k \mu_{k-1} \cdots \mu_0} p_0 \end{aligned}$$

Let

$$r_k = \frac{\lambda_{k-1} \lambda_{k-2} \cdots \lambda_0}{\mu_k \mu_{k-1} \cdots \mu_0}, \ \forall k \geqslant 1, \quad \text{and } r_0 = 1 \tag{4.8}$$

Then $p_k = r_k p_0$, $\forall k \geqslant 0$. Since $\sum\limits_{k=0}^{\infty} p_k = 1$, we have

$$p_0 = \left(\sum_{k=0}^{\infty} r_k \right)^{-1} \tag{4.9}$$

We remark that steady state cannot be reached if λ_k, μ_k are s.t. $\sum\limits_{k=0}^{\infty} r_k = \infty$. If λ, μ and $\rho = \frac{\lambda}{s\mu}$ are as in section 4.1, then steady state can be reached if $\rho < 1$.

If s, $\overline{L}$, $\overline{L}_q$ as defined in section 4.1, and $\overline{\lambda}$ is as in 4.2, then

$$\overline{L} = \sum_{k=0}^{\infty} k p_k, \quad \overline{L}_q = \sum_{k=s}^{\infty} (k-s) p_k, \quad \text{and} \quad \overline{\lambda} = \sum_{k=0}^{\infty} \lambda_k p_k \tag{4.10}$$

Now, consider the M/M/1 system. For λ, μ, ρ as above, with $\rho = \frac{\lambda}{\mu} < 1$, we have

4.8 $\Rightarrow$ $r_k = \left(\frac{\lambda}{\mu}\right)^k = \rho^k$, $\forall k \geqslant 0$.

4.9 $\Rightarrow$ $p_o = \left(\sum\limits_{k=0}^{\infty} r_k\right)^{-1} = \left(\sum\limits_{k=0}^{\infty} \rho^k\right)^{-1} = 1 - \rho$.

4.10 $\Rightarrow$ $\overline{L} = \sum\limits_{k=0}^{\infty} k p_k = \sum\limits_{k=0}^{\infty} k r_k p_0 = (1-\rho) \sum\limits_{k=0}^{\infty} k \rho^k = (1-\rho)\rho \frac{d}{d\rho} \sum\limits_{k=0}^{\infty} \rho^k = (1-\rho)\rho \frac{d}{d\rho} \frac{1}{1-\rho} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

Similarly, 4.10 $\Rightarrow$

$$\overline{L}_q = \sum_{k=1}^{\infty}(k-1)p_k = \overline{L} - (1-p_0) = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Note that if $\rho = \frac{\lambda}{\mu} > 1$, then $\sum\limits_{k=0}^{\infty} r_k = \sum\limits_{k=0}^{\infty} \rho^k = \infty$, and we cannot have steady state.

Back to $\rho < 1$. Let W be a sojourn time RV. If the customer C finds k customers in the queue, then C has to wait through $k+1$ service times, including C's own, but not the customer currently in service (see Prop 4.3). If $X_1, \cdots, X_{k+1}$ are the RVs of the service times, each exponential with parameter μ, then letting $S_{k+1} = X_1 + \cdots + X_{k+1}$, $k \geqslant 0$, we have

$$\mathbb{P}(W > t) = \sum_{k=0}^{\infty} p_k \mathbb{P}(S_{k+1} > t)$$

It can be shown from this that

$$\mathbb{P}(W > t) = e^{-\mu(1-\rho)t}, \quad t \geqslant 0$$

Thus, W has the exponential distribution with parameter $\mu(1\rho)$, and

$$\mathbb{E}W = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}$$

Similarly, let W_q be a queueing time RV. If C finds no customers already in the system, then $\mathbb{P}(W_q = 0) = 1 - \rho$. If c finds $k \geqslant 1$ customers in the system, then C has to wait through k service times, so

$$\begin{aligned}
\mathbb{P}(W_q > t) &= \sum_{k=1}^{\infty} p_k \mathbb{P}(S_k > t) = \sum_{k=1}^{\infty}(1-\rho)\rho^k \mathbb{P}(S_k > t) \\
&= \rho \sum_{k=0}^{\infty} p_k \mathbb{P}(S_{k+1} > t) = \rho \mathbb{P}(W > t) \\
&= \rho e^{-\mu(1-\rho)t}, \quad t \geqslant 0
\end{aligned}$$

And $\mathbb{E}W_q = \mathbb{E}W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$.

We see that W_q does not have an exponential distribution. But the conditional distribution

$$\mathbb{P}(W_q > t \mid W_q > 0) = \frac{\mathbb{P}(W_q > t)}{\mathbb{P}(W_q > 0)} = e^{-\mu(1-\rho)t}, \quad t \geqslant 0$$

is exponential with parameter $\mu(1-\rho)$.

Next, consider an M/M/s system ($s \geqslant 2$). We can obtain (still λ, μ constants, and $\rho = \frac{\lambda}{s\mu}$)

$$r_k = \begin{cases} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k, & 1 \leqslant k \leqslant s \\ \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \rho^{k-s} = \frac{1}{s! s^{k-s}}\left(\frac{\lambda}{\mu}\right)^k & k \geqslant s \end{cases} \tag{4.11}$$

If $\rho = \frac{\lambda}{s\mu} < 1$, then 4.9, 4.11⇒

$$p_0 = \left(\sum_{k=0}^{\infty} r_k\right)^{-1} = \left(\sum_{k=0}^{n-1} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k + \sum_{k=s}^{\infty} \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{1}{1-\rho}\right)^{-1}$$

Also, $p_k = r_k p_0$ for $k \geqslant 1$, where r_k is as in 4.11, 4.10⇒

$$\overline{L}_q = \frac{p_0 \rho}{s!(1-\rho)}\left(\frac{\lambda}{\mu}\right)^s$$

and Th 4.1 (Little's Law), 4.4⇒

$$\overline{W}_q = \frac{\overline{L}_q}{\lambda}, \quad \overline{W} = \overline{W}_q + \frac{1}{\mu}, \quad \overline{L} = \lambda\overline{W} = \lambda\overline{W} + \frac{\lambda}{\mu} = \overline{L}_q + \frac{\lambda}{\mu}$$

We can also obtain:

$$\mathbb{P}(W > t) = e^{-\mu t}\left(1 + \frac{p_0}{s!(1-\rho)}\left(\frac{\lambda}{\mu}\right)^s \frac{1 - e^{-\mu t(s-1-\frac{\lambda}{\mu})}}{s - 1 - \frac{\lambda}{\mu}}\right)$$

$$\mathbb{P}(W_q > t) = (1 - \mathbb{P}(W_q = 0))e^{-s\mu(1-\rho)t},$$

where $\mathbb{P}(W_q = 0) = \sum_{k=0}^{s-1} p_k$.

Example 3′. Suppose in a M/M/s system, a time unit is one hour. On average, a customer arrives every 30 minutes, and is served every 20 minutes. Then $\lambda = 2$ customers/hour, and $\mu = 3$ customers/hour. Since $\rho = \frac{2}{3s} < 1$, the system is in steady state $\forall s \geqslant 1$. We have the following various values for $s = 1, 2$.

	$s=1$	$s=2$
ρ	$\frac{2}{3}$	$\frac{1}{3}$
p_0	$\frac{1}{3}$	$\frac{1}{2}$
p_1	$\frac{2}{9}$	$\frac{1}{3}$
p_k, $k \geqslant 2$	$\frac{1}{3}\left(\frac{2}{3}\right)^k$	$\left(\frac{1}{3}\right)^k$
$\overline{L}_q$	$\frac{4}{3}$	$\frac{1}{12}$
$\overline{L}$	2	$\frac{3}{4}$
$\overline{W}_q$	$\frac{2}{3}$ hour	$\frac{1}{24}$ hour
$\overline{W}$	1 hour	$\frac{3}{8}$ hour

Chapter 5

Exam

5.1 Not in the Final Exam

5.1.1 Chapter 1

- Th 1.2, 1.3, Cor 1.4
- Proof of Th 1.7 (remember statement)
- Th 1.8, 1.9, 1.10 (proofs in handout, remember statements)
- Proofs of Th 1.13, 1.14 (remember statements)
- Penalty function method, Lemma 1.17, Th 1.18, Examples 8,9
- Algorithm method with linear constraints, up to end of 1.2.
- Branch and bound method, Example 11.
- Cutting Plane Method, Example 13.

5.1.2 Chapter 2

- Proof of Th 2.2.
- Cooperative games, Example 2',7.
- All of 2.4.

5.1.3 Chapter 3

- Proofs of Th 3.2, 3.3, 3.6, 3.10, but know every Algorithm!

5.1.4 Chapter 4

- Proof of Th 4.1 (remember statement)
- Expressions for $\mathbb{P}(W > t)$, $\mathbb{E}W$, $\mathbb{P}(W_q > t)$, $\mathbb{E}W_q$.

5.2 Example Questions