Operations Research

Information about the Final Exam

18 June 2024

Date and time: 6 July 2024 (Saturday), 14:30 to 16:30. You are recommended to arrive at around 14:00.

Venue: Teaching Building 1 (第一教学楼), room 1208.

Rules for the Final Exam:

- The Final Exam is CLOSED BOOK. This means that text books and class notes MAY NOT be used during the exam. The questions may be any of the following forms:
  - State and/or prove a result that was presented during the course ("bookwork" question).
  - Calculate some answers, for example, the optimal solution and optimal value of a LP.
  - Prove a result that was not proved in the course. Such a proof should not require too much thinking.
- During the exam, you will only be allowed the use of basic stationery that may not conceal information, such as pen/pencil, eraser, and ruler. Ample paper will be provided, and personal paper is STRICTLY FORBIDDEN. The use of any electronic device, such as calculator and mobile phone, is STRICTLY FORBIDDEN.
- Each student must provide an individual effort in the exam. This means that you MUST NOT be involved in any form of assistance with another person during the exam. If you are caught under the suspicion that you have been involved in a coordinated effort with someone, then serious consequences may occur, including all of your efforts being DISQUALIFIED.
- The questions will be provided in both Chinese and English. You may give your answers in either language.
- The exam will consist of eight questions. Please answer any five questions. All eight questions carry equal credit. If you answer more than five questions, then only the points from the five best answered questions will count.

Should you believe that there may be some circumstance which may affect your best ability to sit this Final Exam, please let me know as soon as possible. Honesty is the best policy.

Operations Research

Exam style questions

18 June 2024

In order to help you to prepare for the final examination, you are strongly advised to try the following questions.

Chapter 1

1. Use the simplex method to solve the following LPs.

(a) $\max\ f = 2x_1 - x_2 + x_3$
$$\begin{cases} 3x_1 + x_2 + \ \ x_3 \le 6 \\ x_1 - x_2 + 2x_3 \le 1 \\ x_1 + x_2 - \ \ x_3 \le 2 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

(b) $\max\ f = x_1 + 4x_2 + 2x_3$
$$\begin{cases} 4x_1 + x_2 + 2x_3 \le 5 \\ -x_1 + x_2 + 2x_3 \le 10 \\ x_1 \in \mathbb{R},\ x_2, x_3 \ge 0 \end{cases}$$

2. For the following LPs, derive the dual LP. Use the dual simplex method to solve each LP, and state the primal and dual optimal solutions.

(a) $\min\ f = x_1 + 2x_2$
$$\begin{cases} -x_1 + 4x_2 \le -2 \\ -2x_1 + 2x_2 \le -7 \\ -x_1 - 3x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

(b) $\min\ f = 6x_1 + 7x_2 + 3x_3 + 5x_4$
$$\begin{cases} 5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12 \\ x_2 - 5x_3 - 6x_4 \ge 10 \\ 2x_1 + 5x_2 + \ \ x_3 + \ \ x_4 \ge 8 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

3. Use the two-phase method to solve the following LPs.

(a) $\min\ f = 2x_1 + 3x_2 + 4x_3$
$$\begin{cases} 3x_1 + 2x_2 + \ \ x_3 \le 10 \\ 2x_1 + 3x_2 + 3x_3 \le 15 \\ x_1 + \ \ x_2 - \ \ x_3 \ge 4 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

(b) $\max\ f = 2x_1 + 3x_2 + x_3$
$$\begin{cases} x_1 + x_2 + x_3 \le 40 \\ 2x_1 + x_2 - x_3 \ge 10 \\ - x_2 + x_3 \ge 10 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

4. Consider the LP
$$\max\ c^T x \qquad \begin{cases} Ax \preccurlyeq b \\ x \succcurlyeq 0 \end{cases} \tag{1}$$
where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ is a variable. Let p^* be the optimal value of this LP.

(a) State the *dual LP* of the LP (1).

(b) Show that the dual of the dual LP of (1) is the LP (1).

(c) Let d^* be the optimal value of the dual LP of the LP (1). Prove that $p^* \le d^*$.

5. Consider the non-linear programming problem

$$\begin{array}{ll} \min & f(x) \\ & g_i(x) \leq 0, \quad 1 \leq i \leq m \end{array} \tag{2}$$

where $x \in \mathbb{R}^n$ is a variable, and $f : X_0 \to \mathbb{R}$, $g_i : X_i \to \mathbb{R}$ where $X_0, X_1, \ldots, X_m \subset \mathbb{R}^n$ are the domains of $f, g_1, \ldots, g_m$. Let p^* be the optimal value of the problem (2).

(a) Define the *domain* D, the *feasible set* R, and the *optimal set* R^* of the problem (2).

(b) What does it mean to say that the problem (2) is a *convex programming problem (CP)*?

(c) Suppose that the problem (2) is a CP. Prove that the sets R and R^* are convex. Prove also that if f is a strictly convex function, and $R^* \neq \emptyset$, then the CP (2) has a unique optimal solution.

6. Use the KT conditions to solve the following non-linear programs.

(a) $\max\ f(x_1, x_2) = 3x_1 + 4x_2$
$\quad x_1^2 + x_2^2 \leq 3$

(b) $\min\ f(x_1, x_2) = 2x_1^2 + 9x_2^2$
$\quad x_1 + 3x_2 \geq 1$

7. Consider the unconstrained convex programming problem

$$\min \quad f(x_1, x_2) = 4x_1^2 - 4x_1x_2 + 2x_2^2$$

Let $x^{(0)} = (\,2\ \ 3\,)^T$.

(a) Using the steepest descent method, calculate the terms $x^{(1)}$ and $x^{(2)}$ in a minimising sequence.

(b) What happens if Newton's method is used?
Hint: The inverse of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

8. Use a recurrence relation method to solve the following knapsack problems.

(a) $\max\ f = 4x_1 + 7x_2 + 5x_3 + 6x_4$
$\quad \begin{cases} 3x_1 + 5x_2 + 6x_3 + 8x_4 \leq 16 \\ x_1, x_2, x_3, x_4 \in \mathbb{Z}_{\geq 0} \end{cases}$

(b) $\max\ 8x_1 + 11x_2 + 6x_3 + 4x_4$
$\quad \begin{cases} 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ x_1, x_2, x_3, x_4 \in \{0, 1\} \end{cases}$

Chapter 2

9. Find some optimal probability distributions and the value for the following zero-sum games.

(a) $\begin{pmatrix} -1 & -2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 3 & 5 & -2 \\ 0 & -3 & 2 & 1 \\ 3 & -1 & 0 & 2 \end{pmatrix}$

10. Find all mixed strategy Nash equilibriums and Nash equilibrium payoffs for the following non-constant-sum games.

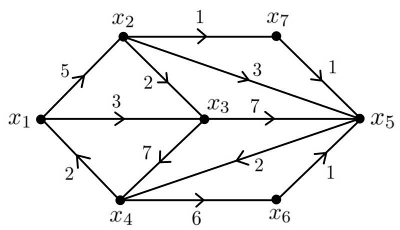
$$\text{(a)}\quad \begin{pmatrix} (0,-2) & (-1,-1) & (2,0) \\ (1,2) & (-1,0) & (0,1) \\ (-1,1) & (-2,0) & (1,-2) \end{pmatrix} \qquad\qquad \text{(b)}\quad \begin{pmatrix} (0,4) & (3,4) & (2,2) \\ (2,-3) & (-2,-1) & (1,2) \\ (1,1) & (4,0) & (4,-1) \end{pmatrix}$$

11. Let (A, B) be a non-constant-sum game between Alice and Bob. Suppose that for Alice, the strategies $\alpha_{i_1}, \ldots, \alpha_{i_t}$ strictly dominate the strategy α_k. Prove that every mixed strategy Nash equilibrium $(\bar{x}, \bar{y})$ must have $\bar{x}_k = 0$.

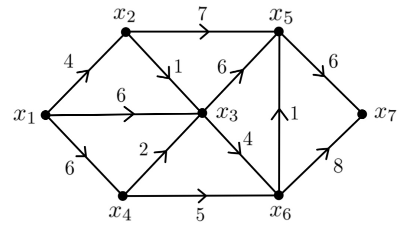
Chapter 3

12. For the following network digraphs, use Dijkstra's Algorithm to find the shortest path from x_1 to all other vertices. Show also the shortest paths tree.

(a)

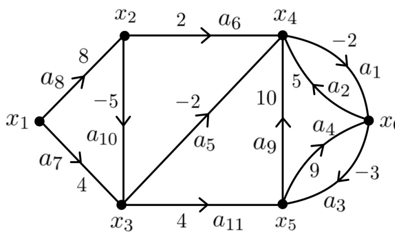


(b)

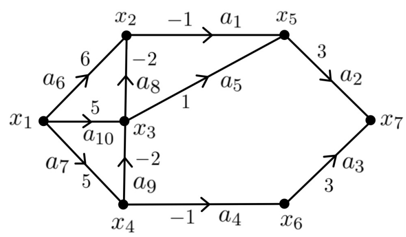


13. Consider the following network digraphs, with the arcs a_i as indicated. Use the Bellman-Ford Algorithm to find the shortest path from x_1 to all other vertices. Show also the shortest paths tree.

(a)



(b)



14. Let (D, c, s, t) be flow network.

(a) What is meant by a *flow* f of (D, c, s, t)?

(b) Let f be a flow of (D, c, s, t). Prove that

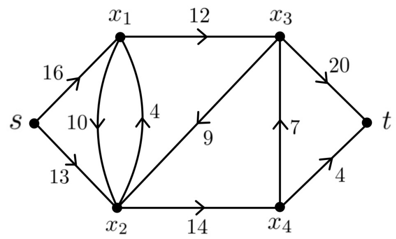
$$\sum_{y \in \Gamma^+(s)} f(sy) - \sum_{z \in \Gamma^-(s)} f(zs) = \sum_{z \in \Gamma^-(t)} f(zt) - \sum_{y \in \Gamma^+(t)} f(ty). \tag{3}$$

(c) Let $v(f)$ be the common value in (3), and $A(X, \bar{X})$ be a cut of (D, c, s, t). Prove that

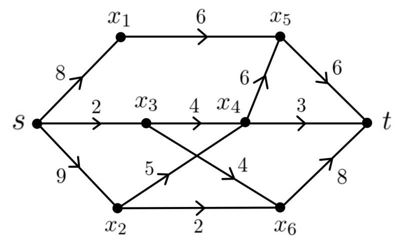
$$v(f) = \sum_{xy \in A(X,\bar{X})} f(xy) - \sum_{yx \in A(\bar{X},X)} f(yx).$$

15. Use the Ford-Fulkerson Algorithm to find a maximum flow and a minimum cut of the following network flows.

(a)



(b)



Chapter 4

16. (a) Let T be an exponential random variable, with parameter $\gamma > 0$. Prove that

    (i) $\mathbb{P}(T > t + h \mid T > h) = \mathbb{P}(T > t)$, for all $t, h > 0$.

    (ii) $\mathbb{P}(T \leq t + h \mid T > t) \approx \gamma h$, for all $t > 0$ and small $h > 0$.

    (b) Let $T_1, \ldots, T_n$ be exponential random variables, with parameters $\gamma_1, \ldots, \gamma_n > 0$. Prove that the random variable

$$U = \min(T_1, \ldots, T_n)$$

has an exponential distribution with parameter $\gamma_1 + \cdots + \gamma_n$.

17. Suppose that in a M/M/s queueing system, a time unit is one hour. On average, a customer arrives every 20 minutes, and a customer is served every 15 minutes.

    (a) Show that this queueing system is in steady state, for every $s \geq 1$.

    (b) For $k \geq 0$, let p_k be the probability that the state of the queueing system is equal to k. Calculate each probability p_k when $s = 1$. State any result that you use.