1. Simplex Method

(1)

$$\max\ f = 2x_1 - x_2 + x_3$$
$$\begin{cases} 3x_1 + x_2 + x_3 \le 6 \\ x_1 - x_2 + 2x_3 \le 1 \\ x_1 + x_2 - x_3 \le 2 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$$\max\ f = 2x_1 - x_2 + x_3$$
$$\begin{cases} 3x_1 + x_2 + x_3 + t_1 = 6 \\ x_1 - x_2 + 2x_3 + t_2 = 1 \\ x_1 + x_2 - x_3 + t_3 = 2 \\ x_1, x_2, x_3, t_1, t_2, t_3 \ge 0 \end{cases}$$

x_1	x_2	x_3	t_1	t_2	t_3	f	
3	1	1	1	0	0	0	6
1	-1	2	0	1	0	0	1
1	1	-1	0	0	1	0	2
-2	1	-1	0	0	0	1	0

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	f	
0	4	-5	1	-3	0	0	3
1	-1	2	0	1	0	0	1
0	-2	-3	0	-1	1	0	1
0	-1	3	0	2	0	1	2

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	f	
0	1	$-\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	0	0	$\frac{3}{4}$
1	0	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{7}{4}$
0	0	$-\frac{11}{2}$	$\frac{1}{2}$	$-\frac{5}{2}$	1	0	$\frac{5}{2}$
0	0	$\frac{7}{4}$	$\frac{1}{4}$	$\frac{5}{4}$	0	1	$\frac{11}{4}$

Thus the optimal solution is $(x_1, x_2, x_3, t_1, t_2, t_3) = (\frac{7}{4}, \frac{3}{4}, 0, 0, 0, \frac{5}{2})$. and optimal value $f = \frac{11}{4}$.

(2)

$$\max\ f = x_1 + 4x_2 + 2x_3$$
$$\begin{cases} 4x_1 + x_2 + 2x_3 \le 5 \\ -x_1 + x_2 + 2x_3 \le 10 \\ x_1 \in \mathbb{R},\ x_2, x_3 \ge 0 \end{cases}$$

let $x_1 = z_1 - z_2$, where $z_1, z_2 \ge 0$

$$\max\ f = z_1 - z_2 + 4x_2 + 2x_3$$
$$\begin{cases} 4z_1 - 4z_2 + x_2 + 2x_3 + t_1 = 5 \\ -z_1 + z_2 + x_2 + 2x_3 + t_2 = 10 \\ z_1, z_2, x_2, x_3, t_1, t_2 \ge 0 \end{cases}$$

z_1	z_2	x_2	x_3	t_1	t_2	f	
4	-4	1	2	1	0	0	5
-1	1	1	2	0	1	0	10
-1	1	-4	-2	0	0	1	0

$\Rightarrow$

z_1	z_2	x_2	x_3	t_1	t_2	f	
4	-4	1	2	1	0	0	5
-5	5	0	0	-1	1	0	5
15	-15	0	6	4	0	1	20

$\Rightarrow$

z_1	z_2	x_2	x_3	t_1	t_2	f	
0	0	1	2	$\frac{1}{5}$	$\frac{4}{5}$	0	9
-1	1	0	0	$-\frac{1}{5}$	$\frac{1}{5}$	0	1
0	0	0	6	1	3	1	35

Thus the optimal solution is $(z_1, z_2, x_2, x_3, t_1, t_2) = (z, 1+z, 9, 0, 0, 0)$, therefore, $(x_1, x_2, x_3) = (-1, 9, 0)$. And the optimal value is 35.

2. Dual LP.

(1)

$$\min\ f = x_1 + 2x_2$$
$$\begin{cases} -x_1 + 4x_2 \le -2 \\ -2x_1 + 2x_2 \le -7 \\ -x_1 - 3x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

$$\max\ f' = -x_1 - 2x_2$$
$$\begin{cases} -x_1 + 4x_2 \le -2 \\ -2x_1 + 2x_2 \le -7 \\ -x_1 - 3x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

Dual LP is:

$$\min\ g = -2y_1 - 7y_2 + 2y_3$$
$$\begin{cases} -y_1 - 2y_2 - y_3 \ge -1 \\ 4y_1 + 2y_2 - 3y_3 \ge -2 \\ y_1, y_2, y_3 \ge 0 \end{cases}$$

	x_1	x_2	t_1	t_2	t_3	
t_1	-1	4	1	0	0	-2
t_2	-2	2	0	1	0	-7
t_3	-1	-3	0	0	1	2
$-f$	-1	-2	0	0	0	0

$\Rightarrow$

	x_1	x_2	t_1	t_2	t_3	
t_1	0	3	1	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_1	1	-1	0	$-\frac{1}{2}$	0	$\frac{7}{2}$
t_3	0	-4	0	$-\frac{1}{2}$	1	$\frac{11}{2}$
$-f$	0	-3	0	$-\frac{1}{2}$	0	$\frac{7}{2}$

Thus the optimal solution is $(x_1, x_2, t_1, t_2, t_3) = (\frac{7}{2}, 0, \frac{3}{2}, 0, \frac{11}{2})$. and the optimal value is $\frac{7}{2}$.

Dual optimal solutions is $(y_1, y_2, y_3) = (\frac{3}{2}, 0, \frac{11}{2})$

(2)

$$\min\ f = 6x_1 + 7x_2 + 3x_3 + 5x_4$$
$$\begin{cases} 5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12 \\ x_2 - 5x_3 - 6x_4 \ge 10 \\ 2x_1 + 5x_2 + x_3 + x_4 \ge 8 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

$$\max\ f' = -6x_1 - 7x_2 - 3x_3 - 5x_4$$
$$\begin{cases} -5x_1 - 6x_2 + 3x_3 - 4x_4 \le -12 \\ -x_2 + 5x_3 + 6x_4 \le -10 \\ -2x_1 - 5x_2 - x_3 - x_4 \le -8 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

Dual LP: $\min\ g = -12y_1 - 10y_2 - 8y_3$

$$\begin{cases} -5y_1 - 2y_3 \ge -6 \\ -6y_1 - y_2 - 5y_3 \ge -7 \\ 3y_1 + 5y_2 - y_3 \ge -3 \\ -4y_1 + 6y_2 - y_3 \ge -5 \\ y_1, y_2, y_3 \ge 0 \end{cases}$$

	x_1	x_2	x_3	x_4	t_1	t_2	t_3	
t_1	-5	-6	3	-4	1	0	0	-12
t_2	0	-1	5	6	0	1	0	-10
t_3	-2	-5	-1	-1	0	0	1	-8
$-f$	-6	-7	-3	-5	0	0	0	0

$\Rightarrow$

	x_1	x_2	x_3	x_4	t_1	t_2	t_3	
x_2	$\frac{5}{6}$	1	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{6}$	0	0	2
t_2	$\frac{5}{6}$	0	$\frac{9}{2}$	$\frac{20}{3}$	$-\frac{1}{6}$	1	0	-8
t_3	$\frac{13}{6}$	0	$-\frac{7}{2}$	$\frac{7}{3}$	$-\frac{5}{6}$	0	1	2
$-f$	$-\frac{1}{6}$	0	$-\frac{13}{2}$	$-\frac{1}{3}$	$-\frac{7}{6}$	0	0	14

$\Rightarrow$

	x_1	x_2	x_3	x_4	t_1	t_2	t_3	
x_2	0	1	-5	-6	0	-1	0	10
t_1	-5	0	-27	-40	1	-6	0	48
t_3	-2	0	-26	-31	0	-5	1	42
$-f$	-6	0	38	47	0	7	0	70

Thus the optimal solution is $(x_1, x_2, x_3, x_4) = (0, 10, 0, 0)$.

$(y_1, y_2, y_3) = (48, 0, 42)$. the optimal value is $f = 70$.

3. Two - Phase Method.

(1) $\min\ f = 2x_1 + 3x_2 + 4x_3$

$$\begin{cases} 3x_1 + 2x_2 + x_3 \le 10 \\ 2x_1 + 3x_2 + 3x_3 \le 15 \\ x_1 + x_2 - x_3 \ge 4 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$\max f' = -2x_1 - 3x_2 - 4x_3 \to \max h = -s$

$$\begin{cases} 3x_1 + 2x_2 + x_3 + t_1 = 10 \\ 2x_1 + 3x_2 + 3x_3 + t_2 = 15 \\ x_1 + x_2 - x_3 - t_3 + s = 4 \\ x_1, x_2, x_3, t_1, t_2, t_3, s \ge 0 \end{cases}$$

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x_1	x_2	x_3	t_1	t_2	t_3	s	f'	h	
3	2	1	1	0	0	0	0	0	10
2	3	3	0	1	0	0	0	0	15
1	1	-1	0	0	-1	1	0	0	4
2	3	4	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	1	0

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	s	f'	h	
3	2	1	1	0	0	0	0	0	10
2	3	3	0	1	0	0	0	0	15
1	1	-1	0	0	-1	1	0	0	4
2	3	4	0	0	0	0	1	0	0
-1	-1	1	0	0	-1	0	0	1	-4

x_1	x_2	x_3	t_1	t_2	t_3	s	f'	h	
1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	0	$\frac{10}{3}$
0	$\frac{5}{3}$	$\frac{7}{3}$	$-\frac{2}{3}$	1	0	0	0	0	$\frac{25}{3}$
0	$\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	0	-1	1	0	0	$\frac{2}{3}$
0	$\frac{5}{3}$	$\frac{10}{3}$	$-\frac{2}{3}$	0	0	0	1	0	$-\frac{20}{3}$
0	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	0	-1	0	0	1	$-\frac{2}{3}$

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	s	f'	h	
1	0	3	1	0	2	-2	0	0	2
0	0	9	1	1	5	-5	0	0	5
0	1	-4	-1	0	-3	3	0	0	2
0	0	-10	1	0	5	-5	1	0	-10
0	0	0	0	0	0	1	0	1	0

Thus $s = h = 0$. and $(x_1, x_2, x_3, t_1, t_2, t_3) = (2, 2, 0, 0, 5, 0)$.

II

x_1	x_2	x_3	t_1	t_2	t_3	f'	
1	0	3	1	0	2	0	2
0	0	9	1	1	5	0	5
0	1	-4	-1	0	-3	0	2
0	0	-10	1	0	0	5	-10

From dual method, we could find $(x_1, x_2, x_3, t_1, t_2, t_3) = (2, 2, 0, 0, 5, 0)$ and the optimal value is $\max f' = 10$. thus $\min f = 10$

(2) $\max\ f = 2x_1 + 3x_2 + x_3$

$$\begin{cases} x_1 + x_2 + x_3 \le 40 \\ 2x_1 + x_2 - x_3 \ge 10 \\ -x_2 + x_3 \ge 10 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$\max h = -s_1 - s_2$

$$\begin{cases} x_1 + x_2 + x_3 + t_1 = 40 \\ 2x_1 + x_2 - x_3 - t_2 + s_1 = 10 \\ -x_2 + x_3 - t_3 + s_2 = 10 \\ x_1, x_2, x_3, t_1, t_2, t_3, s_1, s_2 \ge 0 \end{cases}$$

x_1	x_2	x_3	t_1	t_2	t_3	s_1	s_2	f	h	
1	1	1	1	0	0	0	0	0	0	40
2	1	-1	0	-1	0	1	0	0	0	10
0	-1	1	0	0	-1	0	1	0	0	10
0	0	0	0	0	0	1	1	0	1	0
-2	-3	-1	0	0	0	0	0	1	0	0

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	s_1	s_2	f	h	
1	1	1	1	0	0	0	0	0	0	40
2	1	-1	0	-1	0	1	0	0	0	10
0	-1	1	0	0	-1	0	1	0	0	10
-2	0	0	0	1	1	0	0	0	1	-20
-2	-3	-1	0	0	0	0	0	1	0	0

x_1	x_2	x_3	t_1	t_2	t_3	s_1	s_2	f	h	
0	$-\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	0	35
1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	5
0	-1	1	0	0	-1	0	1	0	0	10
0	1	-1	0	-1	1	1	0	0	1	-10
0	-2	-2	0	-1	0	1	0	1	0	10

$\Rightarrow$

x_1	x_2	x_3	t_1	t_2	t_3	s_1	s_2	f	h	
0	1	0	1	0	$\frac{3}{2}$	0	$-\frac{3}{2}$	0	0	20
1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	10
0	-1	1	0	0	-1	0	1	0	0	10
0	0	0	0	0	0	1	1	0	1	0
0	-4	0	0	-1	-2	1	2	1	0	30

Therefore $(x_1, x_2, x_3, t_1, t_2, t_3, S_1, S_2) = (10, 0, 10, 20, 0, 0, 0, 0)$ and $h = S_1 = S_2 = 0$.

x_1	x_2	x_3	t_1	t_2	t_3	f	
0	1	0	1	0	$\frac{3}{2}$	0	20
1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	10
0	-1	1	0	0	-1	0	10
0	-4	0	0	-1	-2	1	30

Thus the optimal value is 30. where $(x_1, x_2, x_3) = (10, 0, 10)$

4. Consider the LP

$\max\ c^T x$

$$\begin{cases} Ax \le b \\ x \ge 0 \end{cases}$$

(a) The dual LP is

$\min\ b^T y$

$$\begin{cases} A^T y \ge c \\ y \ge 0 \end{cases}$$

(b) Let's transform the dual LP:

$\max\ -b^T y$

$$\begin{cases} -A^T y \le -c \\ y \ge 0 \end{cases}$$

thus the dual of dual LP is

$\min\ -c^T z$

$$\begin{cases} -Az \ge -b \\ z \ge 0 \end{cases}$$

, which equals to

$\max\ c^T z$

$$\begin{cases} Az \le b \\ z \ge 0 \end{cases}$$

, which is the LP(1)

(c) Prove that $p^* \le d^*$

$$d^* = b^T \bar{y},\quad p^* = c^T \bar{x} \le c^T A^{-1} b \le (A^T y)^T A^{-1} b = y^T A A^{-1} b = y^T b = b^T y = d^*$$

5. Non-linear programming

min $f(x)$

$g_i(x) \le 0 \quad 1 \le i \le m$

(a) $D = \bigcap_{i=0}^{m} X_i$, $R = \{x \in D: g_i(x) \le 0, 1 \le i \le m\}$

$R^* = \{x \in R \mid f(x) = p^*\}$ (2) f & g_i are convex functions.

(3) ① The sets R and R^* are convex.

If F is strictly convex, and $R^* \ne \emptyset$, then

$\exists$ unique optimal solution.

$f((1-\lambda)x + \lambda y) < (1-\lambda) f(x) + \lambda f(y) = p^*$

contradiction

6. (a)

max $f(x_1, x_2) = 3x_1 + 4x_2$

min $f'(x_1, x_2) = -3x_1 - 4x_2$

$x_1^2 + x_2^2 - 3 \le 0$

$$\Rightarrow \begin{cases} \bar{x}_1^2 + \bar{x}_2^2 - 3 \le 0 \\ \lambda \ge 0 \\ \lambda(\bar{x}_1^2 + \bar{x}_2^2 - 3) = 0 \\ \nabla f'(\bar{x}) + \lambda \nabla g(\bar{x}) = \begin{pmatrix} -3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2\bar{x}_1 \\ 2\bar{x}_2 \end{pmatrix} = 0 \end{cases}$$

① $\lambda = 0$. $-3 = 0, -4 = 0$ contradiction

② $\lambda \ne 0$ $\bar{x}_1^2 + \bar{x}_2^2 - 3 = 0$

$\bar{x}_1 = \frac{3}{2\lambda}$, $\bar{x}_2 = \frac{2}{\lambda}$

$\left(\frac{9}{4} + 4\right) \frac{1}{\lambda^2} = 3$, $\lambda \ge 0$ $\Rightarrow \lambda = \frac{5}{2\sqrt{3}}$

Thus $\bar{x}_1 = \frac{3\sqrt{3}}{5}$

$\bar{x}_2 = \frac{4\sqrt{3}}{5}$

The optimal value is:

$f = 3\bar{x}_1 + 4\bar{x}_2 = 5\sqrt{3}$

(b) min $f(x_1, x_2) = 2x_1^2 + 9x_2^2$

$g(x_1, x_2) = -x_1 - 3x_2 + 1 \le 0$

$$\begin{cases} -x_1 - 3x_2 + 1 \le 0 \\ \lambda \ge 0 \\ \lambda(-x_1 - 3x_2 + 1) = 0 \\ \begin{pmatrix} 4x_1 \\ 18x_2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \end{pmatrix} = 0 \end{cases}$$

① $\lambda = 0$ $x_1 = x_2 = 0 \Rightarrow 1 \le 0$ contradiction

② $\lambda \ne 0$ $x_1 = \frac{1}{4}\lambda$, $x_2 = \frac{1}{6}\lambda$

$\lambda\left(-\frac{1}{4}\lambda - \frac{1}{2}\lambda + 1\right) = \lambda\left(1 - \frac{3}{4}\lambda\right) = 0 \Rightarrow \lambda = \frac{4}{3}$

$x_1 = \frac{1}{3}$, $x_2 = \frac{2}{9}$ min $f = 2 \times \frac{1}{9} + 9 \times \frac{4}{81} = \frac{6}{9} = \frac{2}{3}$

7. Unconstrained CP.

min $f(x_1, x_2) = 4x_1^2 - 4x_1x_2 + 2x_2^2$

$x^{(0)} = (2 \quad 3)^T$

(a) STEEP. DESCENT METHOD

$$-\nabla f(x_1, x_2) = -\begin{pmatrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 \end{pmatrix} = 4\begin{pmatrix} x_2 - 2x_1 \\ x_1 - x_2 \end{pmatrix}$$

$$z^{(0)} = -\nabla f(x^{(0)}) = 4\begin{pmatrix} -1 \\ -1 \end{pmatrix} = -4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$f(x^{(0)} + \lambda z^{(0)}) = f(2-4\lambda, 3-4\lambda)$

$= 4(2-4\lambda)^2 - 4(2-4\lambda)(3-4\lambda) + 2(3-4\lambda)^2$

$\frac{\partial f}{\partial \lambda} = 8(2-4\lambda)\cdot(-4) + \underline{16(3-4\lambda)} + 16(2-4\lambda) + \underline{4(3-4\lambda)(-4)}$

$= -16(2-4\lambda) = 0 \Rightarrow \lambda = \frac{1}{2}$

$x^{(1)} = \left(2 - 4 \times \frac{1}{2}, 3 - 4 \times \frac{1}{2}\right) = (0, 1)$

$$z^{(1)} = -\nabla f(x^{(1)}) = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$f(x^{(1)} + \lambda z^{(1)}) = f(4\lambda, 1-4\lambda) = 4(4\lambda)^2 - 4(4\lambda)(1-4\lambda) + 2(1-4\lambda)^2$

$\frac{\partial f}{\partial \lambda} = 128\lambda + 128\lambda - 16 + 4(1-4\lambda)(-4) = 0 \Rightarrow \lambda = \frac{1}{10}$

$x^{(1)} = \left(\frac{2}{5}, \frac{3}{5}\right)$

(b) $$\nabla^2 f(x) = \begin{pmatrix} 16 & -4 \\ -4 & 4 \end{pmatrix}$$

$$z = -\nabla^2 f(x)^{-1} \nabla f(x) = \frac{1}{32-16}\begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix} 4 \begin{pmatrix} x_2 - 2x_1 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$z^{(0)} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$f(x^{(0)} + z^{(0)}) = f(2 - 2\lambda, 3 - 3\lambda) = 4 \cdot 4(1-\lambda)^2 - 4 \cdot 2 \cdot 3(1-\lambda)^2 + 2 \cdot 9(1-\lambda)^2$

$\frac{\partial f}{\partial z} = 0 \Rightarrow \lambda = 1 \Rightarrow x^{(1)} = (0, 0)$ and $z^{(1)} = (0, 0)$. Thus the circulation ends.

8. (a) $\max\ f = 4x_1 + 7x_2 + 5x_3 + 6x_4$

$$\begin{cases} 3x_1 + 5x_2 + 6x_3 + 8x_4 \le 16 \\ x_1, x_2, x_3, x_4 \in \mathbb{Z}_{\ge 0} \end{cases}$$

$$f_4(d) = \begin{cases} 0, & 0 \le d \le 7 \\ 6, & 8 \le d \le 15 \\ 12, & d = 16 \end{cases}$$

$f_3(d) = \max(f_4(d),\ 5x_3 + f_4(d - 6x_3))$

$$f_3(d) = \begin{cases} 0, & 0 \le d \le 5 \\ 5, & 6 \le d \le 7 \\ 6, & 8 \le d \le 11 \\ 10, & 12 \le d \le 15 \\ 12, & d = 16 \end{cases}$$

$f_2(d) = \max(f_3(d),\ 7x_2 + f_3(d - 5x_2))$

$$f_2(d) = \begin{cases} 0, & 0 \le d \le 4 \\ 7, & 5 \le d \le 9 \\ 14, & 10 \le d \le 14 \\ 21, & 15 \le d \le 16 \end{cases}$$

$f_1(d) = \max(f_2(d),\ 4x_1 + f_2(d - 3x_1))$

$$f_1(d) = \begin{cases} 0, & 0 \le d \le 2 \\ 4, & 3 \le d \le 4 \\ 7, & 5 \le d \le 7 \\ 11, & 8 \le d \le 9 \\ 14, & d = 10 \\ 15, & 11 \le d \le 13 \\ 19, & d = 14 \\ \underline{\underline{21}}, & 15 \le d \le 16. \end{cases}$$

Thus the maximum f is 21,

when $(x_1, x_2, x_3, x_4) = (0, 3, 0, 0)$.

(b) ① $x_2 = 0$. $x_1 = x_3 = x_4 = 1$ will make f locally maximum

$f = 8 + 6 + 4 = 18$

② $x_2 = 1$. $5x_1 + 4x_3 + 3x_4 \le 7$

⟨1⟩ $x_1 = 1$. Then $x_3 = x_4 = 0$. $f = 8 + 11 = 19$

⟨2⟩ $x_1 = 0$ $4x_3 + 3x_4 \le 7$ $x_3 = x_4 = 1$ $f = 11 + 6 + 4 = 21$

Thus $(x_1, x_2, x_3, x_4) = (0, 1, 1, 1)$ and $f = 21$.

9. (a) $\begin{pmatrix} -1 & -2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & -1 \end{pmatrix}$ The row 1 is dominated by row 2. Then we delete row 1. And the column 2 dominates column 1 and 3. Then delete column 2.

$$\begin{array}{c|cc} & y_1 & y_3 \\ \hline x_2 & 0 & 2 \\ x_3 & 1 & -1 \end{array}$$

max u

$$\begin{cases} x_3 \ge u \\ 2x_2 - x_3 \ge u \\ x_2 + x_3 = 1 \\ x_2, x_3, u \ge 0 \end{cases} \Rightarrow \begin{cases} x_2 + u \le 1 \\ -3x_2 + u \le -1 \\ x_2, u \ge 0 \end{cases}$$

max $u = \frac{1}{2}$, when $(x_1, x_2, x_3) = (0, \frac{1}{2}, \frac{1}{2})$

min v.

$$\begin{cases} 2y_3 \le v \\ y_1 - y_3 \le v \\ y_1 + y_3 = 1 \\ y_1, y_3, v \ge 0 \end{cases} \Rightarrow \begin{cases} -2y_3 + v \ge 0 \\ 2y_3 + v \ge 1 \\ y_3, v \ge 0 \end{cases}$$

Thus min $v = \frac{1}{2}$, where $(y_1, y_2, y_3) = (\frac{3}{4}, 0, \frac{1}{4})$

(b) $\begin{pmatrix} -1 & 3 & 5 & -2 \\ 0 & -3 & 2 & 1 \\ 3 & -1 & 0 & 2 \end{pmatrix}$ Column 3 dominates Column 2. Delete Col. 3.
Row 3 dominates Row 2. Delete Row 2.
Column 1 dominates Column 4. Detele Col. 1

$$\begin{array}{c|cc} & y_2 & y_4 \\ \hline x_1 & 3 & -2 \\ x_3 & -1 & 2 \end{array}$$

max $f = u$.

$$\begin{cases} 3x_1 - x_3 \ge u \\ -2x_1 + 2x_3 \ge u \\ x_1 + x_3 = 1 \\ x_1, x_3, u \ge 0 \end{cases} \Rightarrow \begin{cases} -4x_1 + u \le -1 \\ 4x_1 + u \le 2 \\ x_1, u \ge 0 \end{cases} \Rightarrow$$

max $u = \frac{1}{2}$, when $(x_1, x_2, x_3) = (\frac{3}{8}, 0, \frac{5}{8})$

min $g = v$.

$$\begin{cases} 3y_2 - 2y_4 \le v \\ -y_2 + 2y_4 \le v \\ y_2 + y_4 = 1 \\ y_2, y_4, v \ge 0 \end{cases} \Rightarrow \begin{cases} -5y_2 + v \le -2. \\ 3y_2 + v \le 2 \\ y_2, v \ge 0 \end{cases} \Rightarrow \min v = \tfrac{1}{2}, \text{ when } (y_1, y_2, y_3, y_4) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2})$$

10. MSNE

(a)
$$\begin{pmatrix} (0,-2) & (-1,-1) & (2,0) \\ (1,2) & (-1,0) & (0,1) \\ (-1,1) & (-2,0) & (1,-2) \end{pmatrix}$$

$(\frac{1}{2} \times \text{Row } 1 + \frac{1}{2} \times \text{Row } 2)$ dominates Row 3. delete Row 3.

$(\frac{1}{2} \times \text{Col } 1 + \frac{1}{2} \times \text{Col } 3)$ dominates col 2. delete col 2.

$$\begin{array}{cc|cc} & & y_1 & y_3 \\ & & q & (1-q) \\ x_1 & p & (0,-2) & (2,0) \\ x_2 & 1-p & (1,2) & (0,1) \end{array}$$

$$\mathbb{E}_A(p,q) = 2p(1-q) + (1-p)q = p(2-3q) + q$$

$$\mathbb{E}_B(p,q) = -2pq + 2(1-p)q + (1-p)(1-q) = q(1-3p) + (1-p)$$

Best response to $p = \begin{cases} 1, & q < \frac{2}{3} \\ 0, & q > \frac{2}{3} \\ [0,1], & q = \frac{2}{3} \end{cases}$ Best response to $q = \begin{cases} 1, & p < \frac{1}{3} \\ 0, & p > \frac{1}{3} \\ [0,1], & p = \frac{1}{3} \end{cases}$

MSNE		Payoff	
Alice	Bob	Alice	Bob
$\bar{x} = (1,0,0)^T$	$\bar{y} = (0,1,0)^T$	2	0
$\bar{x} = (0,1,0)^T$	$\bar{y} = (1,0,0)^T$	1	2
$\bar{x} = (\frac{1}{3}, \frac{2}{3}, 0)^T$	$\bar{y} = (\frac{2}{3}, 0, \frac{1}{3})$	$\frac{2}{3}$	$\frac{2}{3}$

(b)
$$\begin{pmatrix} (0,4) & (3,4) & (2,2) \\ (2,-3) & (-2,-1) & (1,2) \\ (1,1) & (4,0) & (4,-1) \end{pmatrix}$$

Row 3 could dominate Row 1. Thus delete Row 1.

$(\frac{1}{2} \times \text{col } 1 + \frac{1}{2} \times \text{col } 3)$ dominate col. 2 Delete Col. 2.

B.R. $p = \begin{cases} 1, & q > \frac{3}{4} \\ 0, & q < \frac{3}{4} \\ [0,1], & q = \frac{3}{4} \end{cases}$

B.R. $q = \begin{cases} 1, & p < \frac{2}{7} \\ 0, & p > \frac{2}{7} \\ [0,1], & p = \frac{2}{7} \end{cases}$

Only have

$\bar{x} = (0, \frac{2}{7}, \frac{5}{7})^T$

$\bar{y} = (\frac{3}{4}, 0, \frac{1}{4})^T$

NE payoffs are.

$\mathbb{E}_A(\frac{2}{7}, \frac{3}{4}) = \frac{7}{4}$

$\mathbb{E}_B(\frac{2}{7}, \frac{3}{4}) = -\frac{1}{7}$

$$\begin{array}{cc|cc} & & y_1 & y_3 \\ & & q & 1-q \\ x_2 & p & (2,-3) & (1,2) \\ x_3 & 1-p & (1,1) & (4,-1) \end{array}$$

$$\mathbb{E}_A(p,q) = 2pq + p(1-q) + q(1-p) + 4(1-p)(1-q) = p(4q-3) + 4 - 3q$$

$$\mathbb{E}_B(p,q) = -3pq + 2p(1-q) + (1-p)q - (1-p)(1-q) = q(2-7p) + 3p - 1$$

12. By permuting the a_i, we may assume $\alpha_1, \dots, \alpha_t$ $(1 \le t < m)$ strictly dominate α_m. Then $\exists \delta \in \Delta_t$, s.t. $\delta_1 a_{1j} + \dots + \delta_t a_{tj} > a_{mj}$, $\forall 1 \le j \le n$. Suppose $\exists$ a MSNE $(\bar{x}, \bar{y})$ s.t. $\exists \bar{x}_m > 0$, Consider

$x' \in \Delta_m$ where

$$x_i' = \begin{cases} x + \varepsilon \delta_i \bar{x}_m & \text{if } 1 \le i \le t \\ \bar{x}_i, & \text{if } t < i < m \\ (1-\varepsilon)\bar{x}_m & \text{if } i = m \end{cases}$$

for small $\varepsilon > 0$. Note that $x' \in \Delta_m$ since $\bar{x}_m > 0 \Rightarrow \bar{x}_i < 1$ $\forall 1 \le i \le m$.

Then $E_A(x', \bar{y}) = \sum_{\substack{1\le i\le t\\ 1\le j\le n}} a_{ij}(x_i + \varepsilon\delta_i \bar{x}_m)\bar{y}_j + \sum_{\substack{t<i\le m\\ 1\le j\le m}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\le j\le n} a_{mj}(1-\varepsilon)\bar{x}_m\bar{y}_j$

$$= \sum_{\substack{t<j\le m\\ 1\le j\le t}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\le j\le t} a_{mj}(1-\varepsilon)\bar{x}_m\bar{y}_j + \sum_{\substack{1\le j\le t\\ 1\le j\le n}} a_{ij}\,\varepsilon\,\underline{\delta_i\bar{x}_m\bar{y}_j}$$

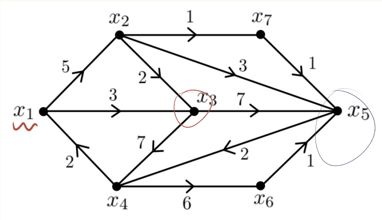
$>$

$$> \sum_{\substack{1\le i\le t\\ 1\le j\le n}} a_{ij}\bar{x}_i\bar{y}_j + \sum_{1\le j\le t} a_{mj}(1-\underline{\varepsilon})\bar{x}_m\bar{y}_j + \sum_{1\le j\le n} a_{mj}\,\underline{\varepsilon}\,\bar{x}_m\bar{y}_j = E_A(\bar{x}, \bar{y})$$

which contradicts that $\bar{x}$ is Alice best response to $\bar{y}$. Vice versa.

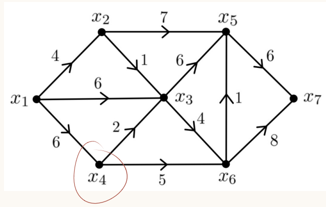
Chapter 3.

12. Dijkstra's Algorithm



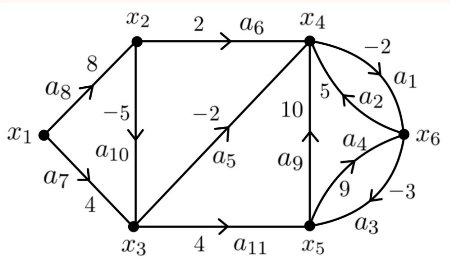
x_1	x_2	x_3	x_4	x_5	x_6	x_7
0^*	∞	∞	∞	∞	∞	∞
$\underline{0^*}$	5	3^{*}	∞	∞	∞	∞
0^*	5^{*}	$\underline{3^*}$	10	10	∞	∞
0^*	$\underline{5^*}$	3^*	10	8	∞	6^{*}
0^*	5^*	3^*	10	7^{*}	∞	$\underline{6^*}$
0^*	5^*	3^*	9^{*}	$\underline{7^*}$	∞	6^*
0	5	3	9	7	10	6

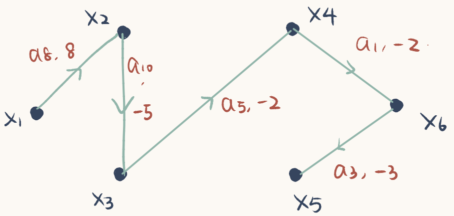
(b)



x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\underline{0^*}$	4^{*}	6	6	∞	∞	∞
0^*	$\underline{4^*}$	5^{*}	6	11	∞	∞
0^*	4^*	$\underline{5^*}$	6^{*}	11	9	∞
0^*	4^*	5^*	$\underline{6^*}$	11	9^{*}	∞
0	4^*	5^*	6^*	11^{*}	$\underline{9^*}$	13
0	4^*	5^*	6^*	$\underline{11^*}$	9^*	13^{*}
0	4^*	5^*	6^*	11^*	9^*	$\underline{13^*}$

13. Bellman-Fold's Algorithm.





Initial, Round 1,

	x_1	x_2	x_3	x_4	x_5	x_6
$a_1 - a_7$	0	∞	4	∞	∞	∞
a_8	0	8	4	∞	∞	∞
$a_9 - a_{10}$	0	8	3	∞	∞	∞
a_{11}	0	8	3	∞	4	∞

Round 2

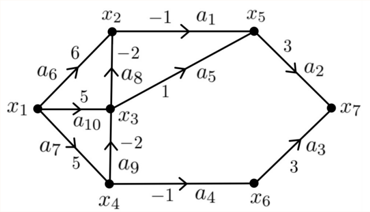
	x_1	x_2	x_3	x_4	x_5	x_6
$a_1 - a_4$	0	8	3	∞	4	13
a_5	0	8	3	1	4	13
$a_6 - a_{10}$	0	8	3	1	4	13

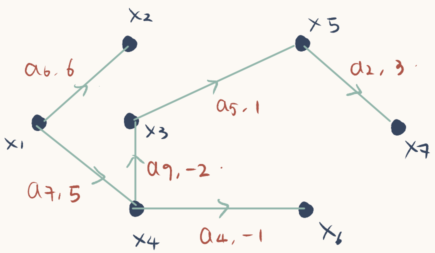
Round 3.

	x_1	x_2	x_3	x_4	x_5	x_6
a_1	0	8	3	1	4	−1
$a_2 - a_3$	0	8	3	1	−4	−1
$a_4 - a_{10}$	0	8	3	1	−4	−1

Round 4 doesn't change any weights.

(2)





Round 1:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$a_1 - a_6$	0	6	∞	∞	∞	∞	∞
a_7	0	6	∞	5	∞	∞	∞
$a_8 - a_9$	0	6	3	5	∞	∞	∞
a_{10}	0	6	3	5	∞	∞	∞

Round 2.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
a_1	0	6	3	5	5	∞	∞
a_2	0	6	3	5	5	∞	8
$a_3 - a_4$	0	6	3	5	5	4	8
a_5	0	6	3	5	4	4	8
$a_6 - a_{10}$	0	6	3	5	4	4	8

Round 3.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$a_1 - a_2$	0	6	3	5	4	4	7
$a_3 - a_{10}$	0	6	3	5	4	4	7

Round 4 doesn't change any weights.

14. (D, c, s, t) be flow network.

(a) A flow f in a flow network is a function $f: A(D) \to \mathbb{R}_{\geq 0}$ which satisfies the following:

$0 \leq f(a) \leq f(c)$, $\forall a \in A(D)$, and $\sum_{z \in \Gamma_-(x)} f(zx) = \sum_{y \in \Gamma_+(x)} f(xy)$, $\forall x \in V(D) \setminus \{s, t\}$

That is, under f, the amount flowing into x and out of x is same.

(b) For all $v \in V(D)$

$$\sum_{v \in V(D)} \Bigg(\underbrace{\sum_{y \in \Gamma_+(v)} f(xy) - \sum_{z \in \Gamma_-(v)} f(zx)}_{(*)} \Bigg) = 0$$

Since at x, the terms $f(xy)$ and $-f(zx)$ can cancelled by the terms. $-f(xy)$ and $f(zy)$ from y and z.

Thus the terms $(*) = 0$ except when $x = s$ & t.

Thus. $\sum_{y \in \Gamma_+(s)} f(sy) - \sum_{z \in \Gamma_-(s)} f(zs) = \sum_{z \in \Gamma_-(t)} f(zt) - \sum_{y \in \Gamma_+(t)} f(ty)$

(c). $v(f)$ $A(X, \bar{X})$ Pf: $v(f) = \sum_{xy \in A(X, \bar{X})} f(xy) - \sum_{yx \in A(\bar{X}, x)} f(yx)$

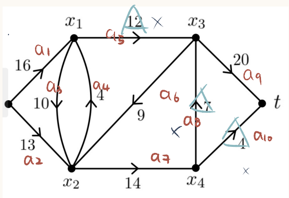
$$v(f)=\sum_{y\in\Gamma_+(s)} f(sy)-\sum_{z\in\Gamma_-(s)} f(zs)=\sum_{x\in X}\left(\sum_{y\in\Gamma^+(s)} f(xy)-\sum_{z\in\Gamma^-(s)} f(zx)\right)$$

由于在内部X内流量会相互抵消，只剩下跨越X与$\bar{X}$的边，即为

$$v(f)=\sum_{xy\in A(X,\bar{X})} f(xy)-\sum_{yx\in A(\bar{X},X)} f(yx).$$

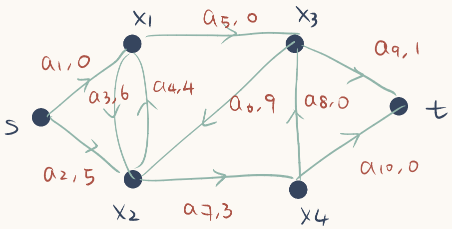
15.

(a)



a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	Path	flow
16/0	13/0	10/0	4/0	12/0	9/0	14/0	7/0	20/0	4/0	$a_1a_5a_9$	12
4/12	13/0	10/0	4/0	0/12	9/0	14/0	7/0	8/12	4/0	$a_1a_3a_7a_8a_9$	4
0/12	13/0	6/4	4/0	0/12	9/0	10/4	3/4	4/16	4/0	$a_2a_7a_8a_9$	3
0/12	9/4	6/4	4/0	0/12	9/0	7/7	0/4	1/15	4/0	$a_2a_7a_{10}$	4
0/12	5/8	6/4	4/0	0/12	9/0	3/11	0/4	1/15	0/4		

The maximum flow is $12+4+3+4=23$.

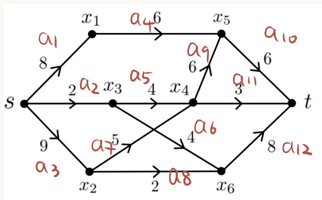


$X=\{s,x_1,x_2,x_4\}$, $\bar{X}=\{x_3,t\}$

minimum cut is.

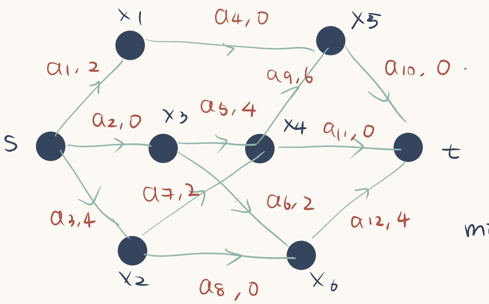
$C(X,\bar{X})=12+7+4=23$.

(2).



a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	Path	flow
8/0	2/0	9/0	6/0	4/0	4/0	5/0	2/0	6/0	6/0	3/0	8/0	$a_1a_4a_{10}$	6
2/6	2/0	9/0	0/6	4/0	4/0	5/0	2/0	6/0	0/6	3/0	8/0	$a_2a_5a_{11}$	2
2/6	0/2	9/0	0/6	2/2	4/0	5/0	2/0	6/0	0/6	1/2	8/0	$a_3a_7a_{11}$	1
2/6	0/2	8/1	0/6	2/2	4/0	4/1	2/0	6/0	0/6	0/3	8/0	$a_3a_8a_{12}$	2
2/6	0/2	6/3	0/6	2/2	4/0	4/1	0/2	6/0	0/6	0/3	6/2	$a_3a_7\overleftarrow{a_5}a_6a_{12}$	2
2/6	0/2	4/5	0/6	4/0	2/2	2/3	0/2	6/0	0/6	0/3	4/4		

The maximum flow is $6+2+1+2+2=13$.



$X = \{s, x_1, x_2, x_4, x_5\}$

$\bar{X} = \{x_3, x_6, t\}$

minimum cut $C(X, \bar{X}) = 2 + 2 + 6 + 3 = 13$

Chapter 4.

16. (a) T be an exponential r.v. $\gamma > 0$.

(i) $\mathbb{P}(T > t+h \mid T > h) = \dfrac{P(T>t+h,\ T>h)}{P(T>h)} = \dfrac{\mathbb{P}(T>t+h)}{\mathbb{P}(T>h)} = \dfrac{e^{-\gamma(t+h)}}{e^{-\gamma h}} = e^{-\gamma t} = \mathbb{P}(T>t)$

(ii) $P(T \le t+h \mid T > t) = \dfrac{P(t < T \le t+h)}{P(T>t)} = \dfrac{P(T>t) - P(T>t+h)}{P(T>t)} = \dfrac{e^{-\gamma t} - e^{-\gamma(t+h)}}{e^{-\gamma t}} = 1 - e^{-\gamma t} \approx 1 - (1 + \gamma h) = \gamma h$

(b) $U = \min(T_1, \dots, T_n)$

$P(U \ge t) = \mathbb{P}(T_1 \ge t, T_2 \ge t, \dots, T_n \ge t) = \mathbb{P}(T_1 \ge t)\mathbb{P}(T_2 \ge t) \dots \mathbb{P}(T_n \ge t) = e^{-\gamma_1 t} e^{-\gamma_2 t} \dots e^{-\gamma_n t} = e^{-(\gamma_1 + \gamma_2 + \dots + \gamma_n)t}$

Thus it has an exponential dist. $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$

17. (a) $\lambda = 3$, $\mu = 4$. $\rho = \dfrac{\lambda}{s\mu} \le \dfrac{\lambda}{\mu} < 1$ Thus it's in a steady state.

(b). P_K. when $s = 1$, $\rho = \dfrac{\lambda}{\mu} = \dfrac{3}{4}$. $r_K = \left(\dfrac{\lambda}{\mu}\right)^K = \left(\dfrac{3}{4}\right)^K$ and $P_0 = \left(\sum_{K=0}^{\infty} r_K\right)^{-1} = 1 - \dfrac{\lambda}{\mu} = \dfrac{1}{4}$.

and $P_K = \left(\dfrac{\lambda}{\mu}\right)^K P_0 = \left(\dfrac{3}{4}\right)^K \cdot \dfrac{1}{4} = \dfrac{3^{K-1}}{4}$.