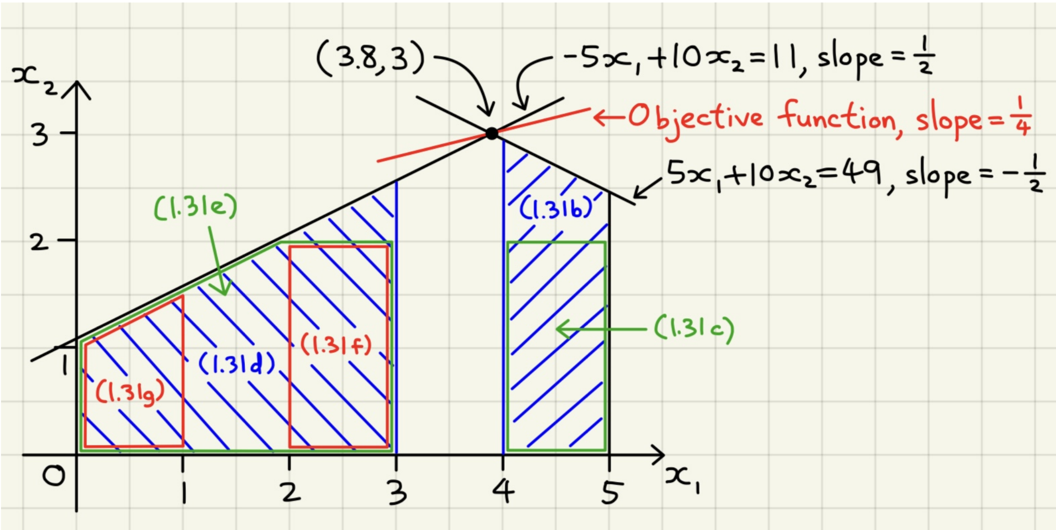
Branch and Bound Method

Example 11. Use the branch and bound method to solve the IP

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ \ \ 5x_1 + 10x_2 \leq 49 \\ \ \ x_1 \qquad\quad\ \leq 5 \\ x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{cases} \tag{1.31a}$$

A diagram looks like



Solving the LP relaxation of (1.31a), we find the optimal solution of $(x_1, x_2) = (3.8, 3)$. We consider $x_1 \geq 4$ and $x_1 \leq 3$ in turn. If $x_1 \geq 4$, we have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ \ \ 5x_1 + 10x_2 \leq 49 \\ 4 \leq x_1 \leq 5 \\ x_2 \geq 0 \end{cases} \tag{1.31b}$$

The optimal solution of (1.31b) is $(x_1, x_2) = (4, 2.9)$. Now consider $x_2 \leq 2$ and $x_2 \geq 3$ in (1.31b). If $x_2 \geq 3$, then (1.31b) becomes infeasible. If $x_2 \leq 2$, we have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ \ \ 5x_1 + 10x_2 \leq 49 \\ 4 \leq x_1 \leq 5 \\ 0 \leq x_2 \leq 2 \end{cases} \tag{1.31c}$$

The optimal solution of (1.31c) is $(x_1, x_2) = (4, 2)$. We have an incumbent optimal value of $-4 + 4 \cdot 2 = 4$.

Now consider $x_1 \le 3$ in the LP relaxation of (1.31a). We have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \le 11 \\ 5x_1 + 10x_2 \le 49 \\ 0 \le x_1 \le 3 \\ x_2 \ge 0 \end{cases} \tag{1.31d}$$

The optimal solution of (1.31d) is $(x_1, x_2) = (3, 2.6)$. Now consider $x_2 \le 2$ and $x_2 \ge 3$ in (1.32d). If $x_2 \ge 3$, then (1.31d) becomes infeasible. If $x_2 \le 2$, we have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \le 11 \\ 5x_1 + 10x_2 \le 49 \\ 0 \le x_1 \le 3 \\ 0 \le x_2 \le 2 \end{cases} \tag{1.31e}$$

The optimal solution of (1.31e) is $(x_1, x_2) = (1.8, 2)$. Now consider $x_1 \le 1$ and $x_1 \ge 2$ in (1.31e). If $x_1 \ge 2$, we have the subproblem

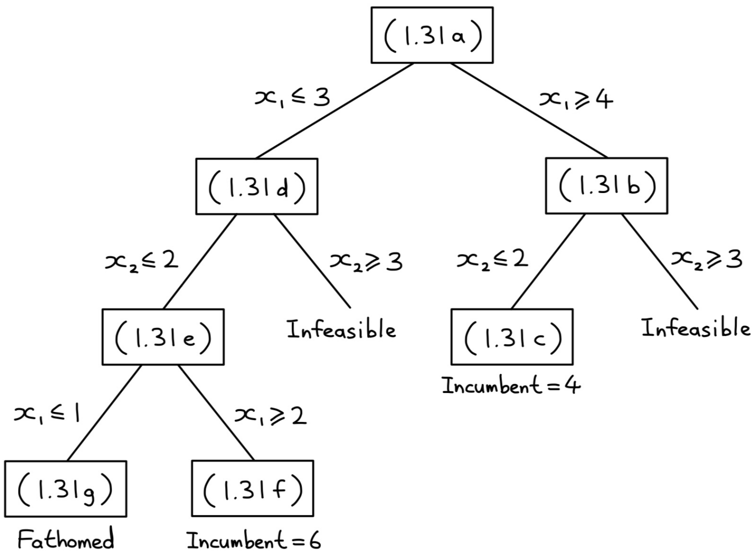
$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \le 11 \\ 5x_1 + 10x_2 \le 49 \\ 2 \le x_1 \le 3 \\ 0 \le x_2 \le 2 \end{cases} \tag{1.31f}$$

The optimal solution of (1.31f) is $(x_1, x_2) = (2, 2)$. We have a new incumbent optimal value of $-2 + 4 \cdot 2 = 6$. If $x_1 \le 1$ in (1.31e), we have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \le 11 \\ 5x_1 + 10x_2 \le 49 \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 2 \end{cases} \tag{1.31g}$$

The optimal solution of (1.31g) is $(x_1, x_2) = (1, 1.6)$. The optimal value is $-1 + 4(1.6) = 5.4$, which is smaller than the incumbent optimal value of 6. We say that the subproblem (1.31g) is fathomed.

We conclude that the optimal solution of the IP (1.31a) is $(x_1, x_2) = (2, 2)$, and the optimal value is $f = 6$. We have the following branching procedure.



Note that if we have more variables x_i, then we will have to solve each LP subproblem with some method that we know, such as the simplex method.

Example 11′. Use the branch and bound method to solve the MIP

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ 5x_1 + 10x_2 \leq 49 \\ x_1 \leq 5 \\ x_1 \in \mathbb{Z}_{\geq 0},\ x_2 \geq 0 \end{cases} \tag{1.32a}$$

We proceed similarly, but we accept any optimal solution to a subproblem where $x_1 \in \mathbb{Z}_{\geq 0}$. Solving the LP relaxation of (1.32a), we find the optimal solution of $(x_1, x_2) = (3.8, 3)$. We consider $x_1 \geq 4$ and $x_1 \leq 3$ in turn. If $x_1 \geq 4$, we have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ 5x_1 + 10x_2 \leq 49 \\ 4 \leq x_1 \leq 5 \\ x_2 \geq 0 \end{cases} \tag{1.32b}$$

The optimal solution of (1.32b) is $(x_1, x_2) = (4, 2.9)$. We have an incumbent optimal value of $-4 + 4(2.9) = 7.6$.

Now consider $x_1 \leq 3$ in the LP relaxation of (1.32a). We have the subproblem

$$\max\ f = -x_1 + 4x_2$$
$$\begin{cases} -5x_1 + 10x_2 \leq 11 \\ 5x_1 + 10x_2 \leq 49 \\ 0 \leq x_1 \leq 3 \\ x_2 \geq 0 \end{cases} \tag{1.32c}$$

The optimal solution of (1.32c) is $(x_1, x_2) = (3, 2.6)$, and the optimal value is $-3 + 4(2.6) = 7.4$, which is smaller than the incumbent optimal value of 7.6.

We conclude that the optimal solution of the MIP (1.32a) is $(x_1, x_2) = (4, 2.9)$, and the optimal value is $f = 7.6$.