PROGRESS ON
Endogenous Production Networks under Supply Chain Uncertainty

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July 30, 2024

Contents

1 Some Notes on Introdution Part

1.1 Definitions

Production Network A production network refers to a complex system of interconnected entities and processes involved in the production and distribution of goods and services. This network includes suppliers, manufacturers, distributors, retailers, and end customers, all working together to ensure the efficient flow of products from raw materials to finished goods. The production network encompasses various stages such as procurement, manufacturing, logistics, and sales, each playing a crucial role in maintaining the overall efficiency and effectiveness of the production process. Effective management of a production network can lead to improved productivity, cost savings, and competitive advantage.

Domar Weights Domar weights, named after the economist Evsey Domar, are used to measure the contribution of each sector to the overall economy. Specifically, in a production network, the Domar weight of a sector is the ratio of that sector's output to the total GDP. This weight reflects the relative importance of a sector in the economy, considering both direct and indirect contributions through the production network. The concept is crucial in understanding how shocks to different sectors can propagate through the economy and affect overall productivity and welfare.

Rish-averse representative household A risk-averse representative household is a theoretical construct used in economic models to represent the behavior of a typical household that prefers to avoid risk. This household supplies a fixed amount of labor and makes consumption decisions to maximize its utility, which depends on the consumption of various goods. The utility function used in the model typically exhibits constant relative risk aversion (CRRA), meaning the household's aversion to risk remains constant regardless of its wealth level. The household makes consumption decisions after uncertainty in the economy is resolved, facing a budget constraint based on the prices of goods and the household's income. The risk aversion parameter (ρ) in the utility function quantifies how much the household dislikes risk: a higher (ρ) indicates greater risk aversion. The household's decisions influence the production network because firms take into account the household's preferences and risk aversion when making their own production and pricing decisions.

TFP Process The TFP process refers to the Total Factor Productivity process, which is a crucial component in understanding economic growth and production efficiency. TFP measures the efficiency with which labor and capital are used together in the production process. The TFP process involves both the endogenous and exogenous factors that affect productivity in different sectors of the economy.

Risk exposure Risk exposure refers to the extent to which an entity (such as a firm, household, or economy) is vulnerable to various types of risks that can affect its performance or stability. In an economic context, risk exposure often involves uncertainties related to price fluctuations, supply chain disruptions, productivity shocks, and other external factors that can impact costs, revenues, and overall economic welfare.

Variance of Unit Costs: Firms prefer inputs with stable prices and avoid techniques relying on inputs with positively correlated prices. This helps in diversifying risk and minimizing cost volatility.

Correlation with Productivity Shocks: Firms prefer inputs whose prices are positively correlated with their productivity shocks. This means that during a negative shock, input prices are likely to be low, reducing expected cost increases.

Risk-Adjusted Prices: Firms' technique choices are influenced by risk-adjusted prices, which account for the expected price of inputs and their covariance with the stochastic discount factor. Goods that are cheaper when aggregate consumption is low are particularly attractive.

Impact on Supply Chain: Higher supplier volatility increases the likelihood of link destruction in supply relationships. Firms tend to move away from riskier suppliers to ensure stability.

Hulten's Theorem Hulten's theorem, named after economist Charles R. Hulten, is a fundamental result in the field of growth accounting and productivity analysis. The theorem states that the aggregate output (GDP) of an economy is a weighted sum of the outputs of its individual sectors, with the weights being the sectoral shares in total output. In simple terms, it implies that the proportional change in aggregate output is equal to the weighted sum of the proportional changes in the output of individual sectors.

Mathematically, if ΔY represents the change in aggregate output and Δy_i represents the change in the output of sector i, Hulten's theorem can be expressed as:

$$\Delta Y = \sum_i w_i \Delta y_i$$

where w_i is the Domar weight of sector i, reflecting its importance in the overall economy. It simplifies the analysis of how shocks to individual sectors affect the whole economy. It assumes a fixed production network, meaning the input-output relationships between sectors do not change in response to the shocks.

Alternative Economy An alternative economy refers to an economic system or a set of practices that differ from the traditional market-driven economy. It encompasses a wide range of economic models and activities that prioritize social, environmental, and ethical considerations over profit maximization. These alternative economic systems often emphasize community-oriented, cooperative, and sustainable practices.

In the context of the provided document, alternative economies are used as benchmarks to evaluate the impact of various factors such as uncertainty on the production network and macroeconomic aggregates. Specifically, the document compares the baseline economy to alternative economies where firms are either unconcerned about risk when making sourcing decisions or have perfect foresight of productivity shocks. These comparisons help isolate the impact of uncertainty on the production network and its subsequent effect on GDP and welfare.

Multi-sector economy A multi-sector economy refers to an economic model that includes multiple sectors or industries, each producing different goods or services. This approach allows for a more detailed and realistic analysis of the economy by capturing the interactions and dependencies between various sectors. In a multi-sector economy, each sector may have its own production function, input requirements, and productivity shocks, and the outputs of some sectors serve as inputs for others, creating a complex network of interconnections.

Productivity shifter the productivity shifter is a function that represents how effectively a sector combines its inputs to produce output. It reflects the total factor productivity (TFP) of the sector, which varies depending on the chosen production technique α_i. This shifter function is crucial in determining the productivity level of a sector and is influenced by the allocation of input shares among different suppliers.

Aggregate Risk refers to the overall level of risk that affects the entire economy or a significant portion of it. It encompasses the uncertainties and potential fluctuations in economic variables that can impact multiple sectors simultaneously. Unlike idiosyncratic risk, which affects only individual firms or sectors, aggregate risk involves macroeconomic factors that can influence the entire economic system.

Pareto Efficient Allocations A Pareto efficient allocation is a state of resource distribution where it is impossible to make any individual better off without making at least one individual worse off. In other words, an allocation is Pareto efficient if no further reallocation can improve someone's situation without harming another person's situation. This concept is named after the Italian economist Vilfredo Pareto.

1.2 Summary for innovations

Modeling Supply Chain Uncertainty The authors construct a model of endogenous network formation to investigate how firms' decisions to mitigate supply chain risks affect the production network and macroeconomic aggregates. This model builds on and extends the work of Acemoglu and Azar (2020).

Focus on Uncertainty Unlike previous models that assume firms know the realization of shocks when choosing production techniques, this model incorporates uncertainty and beliefs about future productivity shocks into the decision-making process. This change allows the model to capture the impact of uncertainty on the structure of the production network.

Technique Choice and Production Network The model allows firms to choose production techniques that specify which intermediate inputs to use and how to combine them. These techniques can vary in terms of productivity, and firms can adjust the importance of suppliers or drop them altogether. This flexibility captures adjustments in the production network along both intensive and extensive margins.

Risk-Adjusted Prices Firms in the model choose techniques by considering risk-adjusted prices, reflecting the risk attitude of the representative household. This approach shows how aggregate risk and firms' sourcing decisions interact to shape the production network.

Empirical Relevance The authors provide a basic calibration of the model using U.S. data to evaluate the importance of these mechanisms. They also highlight the model's ability to predict that increased uncertainty leads firms to prefer more stable suppliers, which reduces macroeconomic volatility but also lowers aggregate output.

Comparative Analysis with Alternative Economies The paper compares the baseline economy with alternative economies where firms either do not consider risk in their sourcing decisions or have perfect foresight of productivity shocks. This comparison helps to isolate the impact of uncertainty on the production network and macroeconomic outcomes.

2 Model

Notations and Symbols

Notations	Meanings
ρ	The utility function quantifies how much the household dislikes risk
$i \in \{1, \cdots, n\}$	n sectors
$\mathcal{A}_i$	The representative firm in sector $i$has access to a set of production techniques
$\alpha_i = (\alpha_{i1}, \cdots, \alpha_{in}) \in \mathcal{A}_i$	Inputs used in production and combined in production
$A_i(\alpha_i)$	a productivity shitfer
L_i	Labor
$X_i = (X_{i1}, \cdots, X_{in})$	A vector of intermediate inputs
ε_i	Stochastic component of sector i's total factor productivity
$\varepsilon \sim \mathcal{N}(\mu, \Sigma)$	Collect the previous shock $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$
$\zeta(\alpha_i)$	A normalization to simplify future expressions
$\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$	Cartesian product
$C = (C_1, \cdots, C_n)$	consumption vector
$u(\cdot)$	CRRA with a coefficient of relative risk aversion $\rho \geqslant 1$
P_i	the price of good i
Λ	Stochastic discount factor
$\overline{P}$	Price index
β	consumption shares
$K_i(\alpha_i, P)$	The unit cost of production
Q_i	the equilibrium demand for good i
$\mathcal{L}(\alpha) = (I - \alpha)^{-1}$	The Leontief inverse
ω_i	Domar weight of sector i
α_i^*	a technique to maximize expected discounted profits
$\lambda(\alpha^*)$	stochastic discount factor
$k_i(\alpha_i, \alpha^*)$	The log of unit cost
$\mathcal{R}(\alpha^*)$	The vector of equilibirum risk-adjusted price

2.1 Firms and production functions

The corresponding production function

$$F(\alpha_i, L_i, X_i) = e^{\varepsilon_i} A_i(\alpha_i)\zeta(\alpha_i) L_i^{1-\sum_{j=1}^{n}\alpha_{ij}} \prod_{j=1}^{n} X_{ij}^{\alpha_{ij}} \tag{1}$$

where L_i is labor and $X_i = (X_{i1}, \cdots, X_{in})$ is a vector of intermediate inputs. The term ε_i is the stochastic component of sector i's total factor productivity. Finally, $\zeta(\alpha_i)$ is a normalization to simplify future expressions.

Set of feasible production techniques

$$\mathcal{A}_i = \left\{ \alpha_i \in [0,1]^n : \sum_{j=1}^{n} \alpha_{ij} \leqslant \bar{\alpha}_i \right\}$$

where $0 < 1 - \bar{\alpha}_i < 1$ provides a lower bound on the share of labor in the production of good i.

Assumption 1. $A_i(\alpha_i)$ is smooth and strictly log-concave.

For each sector i, there is a unique vector of ideal input shares $\alpha_i^\circ \in \mathcal{A}_i$ that maximize A_i and that represents the most productive way to combine intermediate inputs to produce good i. **We normalize $A_i(\alpha_i^\circ) = 1$ for all i.**

Example One example of a function $A_i(\alpha_i)$ that satisfies Assumption 1 is the quadratic form

$$\log A_i(\alpha_i) = \frac{1}{2}(\alpha_i - \alpha_i^\circ)^T \bar{H}_i (\alpha_i - \alpha_i^\circ) \tag{2}$$

where $\bar{H}_i$ is a negative-definite matrix that is also the Hessian of $\log A_i$.

2.2 Household preferences

CRRA A risk-averse representative household supplies one unit of labor in elastically and chooses aconsumption vector $C = (C_1, \cdots, C_n)$ to maximize

$$u\left(\left(\frac{C_1}{\beta_1}\right)^{\beta_1} \cdots \left(\frac{C_n}{\beta_n}\right)^{\beta_n}\right) \tag{3}$$

where $\beta_i > 0$ for all i and $\sum_{i=1}^{n} \beta_i = 1$. We refer to $Y = \prod_{i=1}^{n} (\beta_i^{-1} C_i)^{\beta_i}$ as aggregate consumption or, equivalently in this setting, GDP. The utility function $u(\cdot)$ is CRRA[1] with a coefficient of relative risk aversion $\rho \geqslant 1$. The household makes consumption decisions after uncertainty is resolved and so in each state of the world it faces the budget constraint

$$\sum_{i=1}^{n} P_i C_i \leqslant 1 \tag{4}$$

where P_i is the price of good i, and the wage is used as the numeraire.

Stochastic discount factor Firms are owned by the representative household and maximize expected profits discounted by the household's stochastic discount factor

$$\Lambda = u'(Y)/\overline{P} \tag{5}$$

wbere $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$ is the price index.

Log GDP From the optimization problem of the household it is straightforward to show that

$$y = -\beta^T p \tag{6}$$

where $y = \log Y$, $p = (\log P_1, \cdots, \log P_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$. Log GDP is thus the negative of the sum of log prices weighted by the consumption shares β. Intuitively, as prices decrease relative to wages, the household can purchase more goods, and aggregate consumption increases.

2.3 Unit cost minimization

The second stage problem Under a given technique α_i, the cost minimization problem of a firm in sector i is

$$K_i(\alpha_i, P) = \min_{L_i, X_i} \left(L_i + \sum_{j=1}^{n} P_j X_{ij} \right), \quad \text{subject to } F(\alpha_i, L_i, X_i) \geqslant 1 \tag{7}$$

[1]CRRA stands for Constant Relative Risk Aversion. It is a type of utility function used in economics to describe the behavior of agents who have a consistent attitude towards risk, regardless of their wealth level. The CRRA utility function is commonly used in models of consumer behavior, finance, and macroeconomics because it has several desirable properties, including scalability and tractability.

Thus we construct a Lagrangian Function as:

$$\mathcal{L} = L_i + \sum_{j=1}^{n} P_j X_{ij} + \lambda \left(1 - e^{\varepsilon_i} A_i(\alpha_i)\zeta(\alpha_i) \left(\prod_{j=1}^{n} X_{ij}^{\alpha_{ij}} \right) L_i^{\left(1-\sum_{j=1}^{n}\alpha_{ij}\right)} \right)$$

First-Order Conditions: Taking the first-order conditions with respect to L_i, X_{ij}, and λ, we get:

$$0 = 1 - \left(1 - \sum_{j=1}^{n} \alpha_{ij}\right) e^{\varepsilon_i} \lambda A_i(\alpha_i)\zeta(\alpha_i) \left(\prod_{j=1}^{n} X_{ij}^{\alpha_{ij}} \right) L_i^{\left(-\sum_{j=1}^{n}\alpha_{ij}\right)}$$

$$0 = P_j - \lambda e^{\varepsilon_i} A_i(\alpha_i)\zeta(\alpha_i) L_i^{\left(1-\sum_{j=1}^{n}\alpha_{ij}\right)} \left(\prod_{j=1}^{n} X_{ij}^{\alpha_{ij}} \right) X_{ij}^{-1}\alpha_{ij}$$

Thus we could get the following things:

$$L_i = \left(1 - \sum_{j=1}^{n} \alpha_{ij}\right) \lambda$$

$$X_{ij} = \frac{\lambda \alpha_{ij}}{P_j}$$

Thus we could substitute to the equation and get the following:

$$K_i(\alpha_i, P) = \frac{1}{e^{\varepsilon_i} A_i(\alpha_i)} \prod_{j=1}^{n} P_j^{\alpha_{ij}} \tag{8}$$

2.4 Technique choice

The first stage problem Given an expression for K_i, the first stage of the representative firm's problem is to pick a technique $\alpha_i \in \mathcal{A}_i$ to maximize expected discounted profits, that is,

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} \mathbb{E}\left[\Lambda Q_i (P_i - K_i(\alpha_i, P))\right] \tag{9}$$

where Q_i is the equilibrium demand for good i, and where the profits in different states of the world are weighted by the household's stochastic discount factor Λ. The representative firm takes P, Q_i and Λ as given, and so the only term in (9) over which it has any control is the unit cost $K_i(\alpha_i, P)$.

2.5 Equilibrium conditions

Competitive Pressure In equilibrium, competitive pressure pushes prices to be equal to unit costs, so that

$$P_i = K_i(\alpha_i, P) \quad \text{for all } i \in \{1, 2, \cdots, n\} \tag{10}$$

Definition 1. An equilibirum is a choice of technique $\alpha^* = (\alpha_1^*, \cdots, \alpha_n^*)$ and a stochastic tuple $(P^*, C^*, L^*, X^*, Q^*)$ such that

1. (Optimal technique choice) For each $i \in \{1, 2, \cdots, n\}$, the technique choice $\alpha_i^* \in \mathcal{A}_i$ solves (9) given price P^*, demand Q_i^* and the stochastic discount factor Λ^* given by (5).

2. (Optimal input choice) For each $i \in \{1, 2, \cdots, n\}$, factor demands per unit of output L_i^*/Q_i^* and X_i^*/Q_i^* are a solution to (7) given price P^* and the chosen technique α_i*.

3. (Consumer maximization) The consumption vector C^* maximizes (3) subject to (4) given prices P^*.

4. (Unit cost pricing) For each $i \in \{1, 2, \cdots, n\}$, P_i^* solves (10) where $K_i(\alpha_i^*, P^*)$ is given by (8).

5. (Market clearning) For each $i \in \{1, 2, \cdots, n\}$,

$$C_i^* + \sum_{j=1}^{n} X_{ji}^* = Q_i^* = F_i(\alpha_i^*, L_i^*, X_i^*), \quad \text{and } \sum_{i=1}^{n} L_i^* = 1 \tag{11}$$

3 Equilibirum prices and GDP in a fixed-network economy

Domar weight We also define the Domar weight ω_i of sector i as the ratio of its sales to nominal GDP, such that

$$\omega_i = \frac{P_i Q_i}{P^T C}$$

Also $\omega^T = \beta^T \mathcal{L}(\alpha) > 0$ in the model.

Lemma 1. Under a given network α, the vector of log prices is given by

$$p(\alpha) = -\mathcal{L}(\alpha)(\varepsilon + a(\alpha)) \tag{12}$$

and log GDP is given by

$$y(a) = \omega(a)^T(\varepsilon + a(\alpha)) \tag{13}$$

where $a(\alpha) = (\log A_i(\alpha_i), \cdots, \log A_n(\alpha_n))$

Proof. Combining the unit cost equation (8) with the equilibrium condition (10) and taking the log we could get

$$p_i = \log P_i = \log K_i(\alpha, P) = -\varepsilon_i - \log A_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} \log P_j = -\varepsilon_i - a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} p_j$$

where $a_i(\alpha_i) = \log(A_i(\alpha_i))$. This is a system of linear equations whose solution is (12). The log price vector is also normally distributed since it is a linear transformation of normal random variable. Combining with (6) yields (13).

$$y = -\beta^T p = \underbrace{\boxed{-\beta^T - \mathcal{L}}}_{\omega^T = \beta^T \mathcal{L}(\alpha)}(\alpha)(\varepsilon + a(\alpha)) = \omega^T(\varepsilon + a(\alpha))$$

□

The first and second moments

$$\mathbb{E}[y(\alpha)] = \omega(a)^T(\mu + a(\alpha)) \quad \mathbb{V}[y(\alpha)] = \omega(a)^T \Sigma \omega(\alpha) \tag{14}$$

Corollary 1. For a fixed production network α, the following holds:

1. The impact of a change in expected TFP μ_i on the moments of log GDP is given by

$$\frac{\partial \mathbb{E}[y]}{\partial \mu_i} = \omega_i \quad \frac{\partial \mathbb{V}}{\partial \mu_i} = 0$$

2. The impact of a change in volatility Σ_{ij} on the moments of log GDP is given by

$$\frac{\partial \mathbb{E}[y]}{\partial \Sigma_{ij}} = 0 \quad \frac{\partial \mathbb{V}}{\partial \Sigma_{ij}} = \omega_i \omega_j$$

4 Firm decisions

Log of those things Log of the stochastic discount factor

$$\lambda(\alpha^*) = \log \Lambda(\alpha^*)$$

The log of the unit cost

$$k_i(\alpha_i, \alpha^*) = \log K_i(\alpha_i, P^*(\alpha^*))$$

where α^* denotes the equilibrium network.

Probelm of the firm Using this notation, we can reorganize the problem of the firm (9) as

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathbb{E}[k_i(\alpha_i, \alpha^*)] + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)] \tag{15}$$

Combining the equation with (5) we can write $\lambda = \log(\Lambda)$ as

$$\lambda(\alpha^*) = -(1-\rho)\sum_{i=1}^{n} \beta_i p_i(\alpha^*)$$

Taking the log of (8) yields

$$k_i(\alpha_i, \alpha^*) = -(\varepsilon_i + a(\alpha_i)) + \sum_{j=1}^{n} \alpha_{ij} p_j(\alpha^*)$$

Both $\lambda(\alpha^*)$ and $k_i(\alpha_i, \alpha^*)$ are normally distributed since they are linear combinations of ε and the log price vector, which is normally distributed by Lemma 1.

Turning to the firm problem 9, we can write

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathbb{E}\left[\Lambda \frac{\beta^T \mathcal{L}(\alpha^*) \mathbb{1}_i}{P_i} K_i(\alpha_i, P)\right],$$

where we have used (A.7) from Supplemental Appendix A in Kopytov et al.(2024). We can drop $\beta^T \mathcal{L}(\alpha^*) \mathbb{1}_i > 0$ since it is a deterministic scalar that does not depend on α_i. Rewriting thisequation in terms of log quantities yields

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathbb{E}[k_i(\alpha_i, \alpha^*)] + \mathrm{Cov}[\lambda(\alpha^*), k_i(\alpha_i, \alpha^*)]$$

The objective function in (15) captures how beliefs and uncertainty affect the production network. Its first term implies that the firm prefers to adopt techniques that provide, in expectation, a lower unit cost of production. Taking the expected value of the log of (8), we can write this term as

$$\mathbb{E}[k_i(\alpha_i, \alpha^*)] = -\mu_i - a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} \mathbb{E}[p_j]$$

Thus we could substitute $k_i(\alpha_i, \alpha^*)$ to the (15):

$$\mathbb{E}[k_i(\alpha_i, \alpha^*)] = \mathbb{E}[\log K_i(\alpha_i, \alpha^*)] = \mathbb{E}[-\varepsilon_i - \log A_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} P_j]$$

$$= -\mu_i - \boxed{a_i(\alpha_i)} + \sum_{j=1}^{n} \alpha_{ij} \mathbb{E}[P_j]$$

By the definition $a(\alpha) = (\log A_i(\alpha_i), \cdots, \log A_n(\alpha_n))$

so that, unsurprisingly, the firm prefers techniques that have high productivity ai and that rely on inputs that are expected to be cheap.

The second term in (15) captures the importance of aggregate risk for the firm's decision. It implies that the firm prefers to have a low unit cost in states of the world in which the marginal utility of consumption is high. As a result, the coefficient of risk aversion ρ of the household indirectly determines how risk-averse firms are. We can expand this term as

$$\mathrm{Cov}[\lambda, k_i] = \mathrm{Corr}[\lambda, k_i] \sqrt{\mathbb{V}[\lambda]} \sqrt{\mathbb{V}[k_i]}$$

which implies that the firm tries to minimize the correlation of its unit cost with λ. Furthermore, since prices and GDP tend to move in opposite directions (see Lemma 1), $\mathrm{Corr}[\lambda, k_i]$ is typically positive, and so firms seek to minimize the variance of their unit cost. This has several implications for their choice of suppliers. To see this, we can use (8) to write

$$\mathbb{V}[k_i(\alpha_i, \alpha)] = \mathbb{V}[\log K_i(\alpha_i, \alpha)] = \mathbb{V}[-\varepsilon_i - \log A_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} P_j]$$

$$= \Sigma_{ii} + \sum_{j=1}^{n} \alpha_{ij} \mathbb{V}[p_j] + \sum_{j \neq k} \alpha_{ij} \alpha_{ik} \mathrm{Cov}[p_j, p_k] + 2\mathrm{Cov}\left[-\varepsilon_i, \sum_{j=1}^{n} \alpha_{ij} p_j\right]$$

Thus we could conclude:

$$\mathbb{V}[k_i(\alpha_i, \alpha)] = \sum_{j=1}^{n} \alpha_{ij} \mathbb{V}[p_j] + \sum_{j \neq k} \alpha_{ij} \alpha_{ik} \mathrm{Cov}[p_j, p_k] + 2\mathrm{Cov}\left[-\varepsilon_i, \sum_{j=1}^{n} \alpha_{ij} p_j\right] + \Sigma_{ii} \tag{16}$$

Lemma 2. In equilibirum, the technique choice problem of the representative firm in sector i is

$$\alpha_i^* \in \arg\max_{\alpha_i \in \mathcal{A}_i} a_i(\alpha_i) - \sum_{j=1}^{n} \alpha_{ij} \mathcal{R}_j(\alpha^*) \tag{17}$$

where

$$\mathcal{R}(\alpha^*) = \mathbb{E}[p(\alpha^*)] + \mathrm{Cov}[p(\alpha^*), \lambda(\alpha^*)] \tag{18}$$

is the vector of equilibirum risk-adjusted price, and where

$$\mathbb{E}[p(\alpha^*)] = -\mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) \quad \mathrm{Cov}[p(\alpha^*), \lambda(\alpha^*)] = (\rho - 1)\mathcal{L}(\alpha^*)\Sigma[\mathcal{L}(\alpha^*)]^T \beta$$

First-order Condition Se can take the first-order condition for an interior solution of problem (17) and use the implicit function theorem to write

$$\frac{\partial \alpha_{ij}}{\partial \mathcal{R}_k} = [H_i^{-1}(\alpha_i)]_{jk} \tag{19}$$

where H_i^{-1} is the inverse of the Hessian matrix of a_i and where $[\cdot]_{jk}$ denotes its element j, k. This equation implies that if a good k becomes marginally more expensive or more risky (higher $\mathcal{R}_k$), firm i responds by changing its share α_{ik} of good k by $[H_i^{-1}(\alpha_i)]_{kk}$. Since a_i is strictly concave by Assumption 1, the diagonal elements of H_i^{-1} are negative, and so a higher $\mathcal{R}_k$ always leads to a decline in α_{ik}. The size of that decline depends on the curvature of a_i.

Substitutes and Complements Whether the increase in $\mathcal{R}_k$ leads to a decline or an increase in the share of other inputs $j \neq k$ depends on whether the shares of j and k are complements or substitutes in the production of good i. If $[H_i^{-1}]_{jk} > 0$ we say that they are **substitutes**, and in that case a higher risk-adjusted price $\mathcal{R}_k$ leads to an increase in α_{ij}. As the firm decreases α_{ik}, the incentives embedded in a_i to increase α_{ij} get stronger, and the firm substitutes α_{ij} for α_{ik}. In contrast, if $[H_i^{-1}]_{jk} < 0$ we say that the shares of j and k are **complements**, and an increase in $\mathcal{R}_k$ leads to a decline in α_{ij}. One sufficient condition for a Hessian matrix H_i to feature complementarities for all sectors is $[H_i]_{jk} \geqslant 0$ for all $j \neq k$.

Example: Substitutability and complementarity in partial equilibrium

To show how the substitution patterns embedded in ai affect technique choices, we can revisit the car manufacturer example from the introduction. Suppose that this manufacturer primarily uses steel (input

1) to produce cars, and that it relies on equipment (input 2) such as milling machines and lathes to transform raw steel into usable components. As before, the manufacturer also has the option to purchase carbon fiber (input 3) to replace steel components if needed. It would be natural to endow this manufacturer (sector $i = 4$) with a TFP shifter function of the form

$$a_4(\alpha_4) = -\sum_{j=1}^{4} \kappa_j(\alpha_{4j} - \alpha_{4j}^{\circ})^2 - \psi_1(\alpha_{41} - \alpha_{42})^2 - \psi_2[(\alpha_{41} + \alpha_{43}) - (\alpha_{41}^{\circ} + \alpha_{43}^{\circ})]^2, \tag{20}$$

where $\kappa_j > 0$, $\psi_1 > 0$ and $\psi_2 > 0$. From the second term, we see that any increase in the share α_{41} of steel would incentivize the firm to increase the share α_{42} of steel machinery as well. Inputs 1 and 2 are therefore complements in the production of cars. In contrast, the third term implies that any increase in the share α_{41} of steel would make it optimal to reduce the share α_{43} of carbon fiber, and so the shares of inputs 1 and 3 are substitutes. These patterns can be confirmed by computing the inverse Hessian of a_4 directly and inspecting the off-diagonal terms. The parameters $\psi_1 > 0$ and $\psi_2 > 0$ determine the strength of these substitution-complementarity patterns.

Figure 1 shows what happens to the production technique chosen by this car manufacturer if the risk-adjusted price of steel increases. In panel (a) the increase in $\mathcal{R}_1$ comes from a higher expectedprice $\mathbb{E}[p_1]$, while in panel (b) the price of steel becomes more volatile (higher $\mathbb{V}[p_1]$). Naturally, when the risk-adjusted price of steel rises, the manufacturer relies less on steel in production, and α_{41} falls. Since steel machinery is only useful when steel is used in production, the share α_{42} falls as well. If the increase in $\mathcal{R}_1$ is large enough, the manufacturer severs the link with its steel and steelmachinery suppliers completely so that both $\alpha_{41} = \alpha_{42} = 0$. At the same time, as steel becomes more expensive in ris-adjusted terms, the firm finds a carbon fiber supplier and progressively increases the share α_{i3}.

5 Equilibrium existence, uniqueness and efficiency

5.1 The efficient allocation

Lemma 3. An efficient production network α^* solves

$$\mathcal{W} := \max_{\alpha \in \mathcal{A}} W(a, \mu, \Sigma)$$

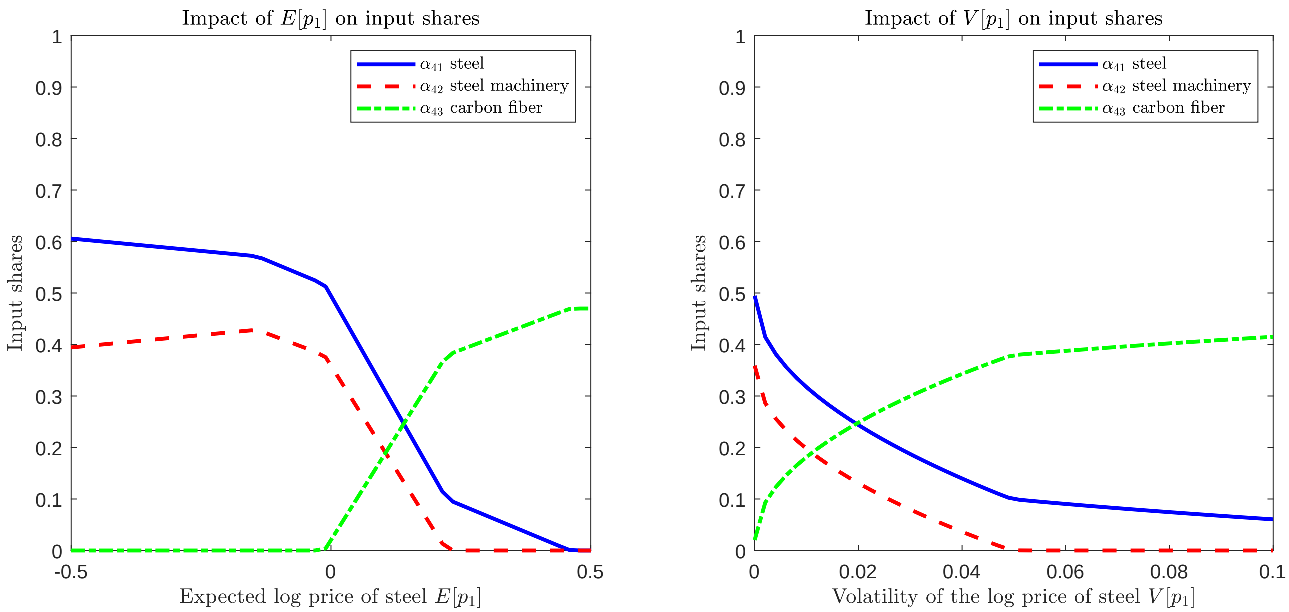
where $\mathcal{W}$ is a measure of the welfare of the household, and where

$$W(a, \mu, \Sigma) := \mathbb{E}[y(\alpha)] - \frac{1}{2}(\rho - 1)\mathbb{V}[y(\alpha)] \tag{21}$$

Risk aversion parameter

is a welfare under a given network α.

Figure 1: Impact of rising the risk-adjusted price of steel



Recasting household welfare in terms of Domar weights

Since Domar weights play a crucial role in determining the expected value and the variance of GDP, it will be useful to recast the problem of the social planner in the space of ω. Using (14), we can write the objective function (21) as

$$W(a,\mu,\Sigma) := \mathbb{E}[y(\alpha)] - \frac{1}{2}(\rho-1)\mathbb{V}[y(\alpha)] = \omega(\alpha)^T(\mu + a(\alpha)) - \frac{1}{2}(\rho-1)\omega(\alpha)^T\Sigma\omega(\alpha)$$

Thus we conclude that:

$$\omega^T\mu + \omega^T a(\alpha) - \frac{1}{2}(\rho-1)\omega^T\Sigma\omega \tag{22}$$

The only term in this expression that does not depend exclusively on ω is $\omega^T a(\alpha)$, which corresponds to the contribution of the TFP shifter functions $(a_1, \cdots, a_n)$ to aggregate TFP. We want to write this object in terms of ω alone. For that purpose, notice that several networks α areconsistent with a given Domar weight vector ω, but that not all of them are equivalent in termsof welfare. Indeed, to achieve a given ω the planner will only select the network α that maximizes welfare, which amounts to maximizing $\omega^T a(\alpha)$.

Formally, consider the optimization problem

$$\bar{a}(\omega) := \max_{\alpha \in \mathcal{A}} w^T a(\alpha) \tag{23}$$

subject to the definition of the Domar weights given by $\omega^T = \beta^T \mathcal{L}(\alpha)$. We refer to the value function $\bar{a}$ as the aggregate TFP shifter function. It provides the maximum value of TFP $\omega^T a(\alpha)$ that can be achieved under the constraint that the Domar weights must be equal to some given vector ω. We denote by $\alpha(\omega)$ the solution to (23). Since both $\bar{a}(\omega)$ and $\alpha(\omega)$ depend exclusively on the TFP shifter functions $(a_1, \cdots, a_n)$ and on the preference vector β, these two functions will be invariant, for a given ω, to the changes in beliefs (μ, Σ) that we consider in the next sections.

Example.

We can solve explicitly for $\bar{a}(\omega)$ and $\alpha(\omega)$ under the quadratic TFP shifter function specified in (2). At an interior solution $\alpha \in \mathrm{int}\mathcal{A}$, the optimal production network $\alpha(\omega)$ that solves (23) for a given vector of Domar weights ω is

$$\alpha_i(\omega) - \alpha_i^\circ = H_i^{-1} \left(\sum_{j=1}^{n} \omega_j H_j^{-1} \right)^{-1} \left(\omega - \beta - \sum_{j=1}^{n} \omega_j \alpha_j^\circ \right), \tag{24}$$

for all i, and the associated value function $\bar{a}$ is

$$\bar{a}(\omega) = \frac{1}{2} \sum_{i=1}^{n} \omega_i (\alpha_i(\omega) - \alpha_i^\circ)^T H_i (\alpha_i(\omega) - \alpha_i^\circ). \tag{25}$$

Corollary 2. The efficient Domar Weight vector ω^* solves

$$\mathcal{W} = \max_{w \in \mathcal{O}} \underbrace{\omega^T \mu + \bar{a}(\omega)}_{\mathbb{E}[y]} - \frac{1}{2}(\rho - 1) \underbrace{\omega^T \Sigma \omega}_{\mathbb{V}[y]} \tag{26}$$

where $\mathcal{O} = \{\omega \in \mathbb{R}_+^n : \ \omega \geqslant \beta \text{ and } 1 \geqslant \omega^T(\mathbb{1} - \bar{\alpha})\}$ and $\bar{a}(\omega)$ is given by (23)

Lemma 4. The objective function of the planner's problem (26) is strictly concave. Furthermore, there is a unique vector of Domar weights ω^* that solves that problme, and there is a unique production network $\alpha(\omega^*)$ associated with that solution.

5.2 Fundamental properties of the equilibrium

Proposition 1. There exists a unique equilibrium, and it is efficient.

6 Beliefs and the production network

In this section, we characterize how beliefs (μ, Σ) affect the equilibrium production network. We begin with a general result that describes how a change in a sector's risk or expected TFP impacts its own Domar weight. We then provide an expression that characterizes how the full vector ofDomar weights responds to a marginal change in (μ, Σ). Finally, we investigate how beliefs affect the structure of the underlying production network α. As we only consider the equilibrium networkfrom now on, we lighten the notation by dropping the superscript $*$ when referring to equilibrium variables.

6.1 Domar weights

In contrast, when the network isendogenous, they are equilibrium objects that vary with (μ, Σ). The next proposition describes the relationship between these quantities.

Proposition 2. The Domar weight ω_i of sector i is (weakly) increasing in μ_i and (weakly) decreasing in Σ_{ii}.

Risk-adjusted productivity shocks

Proposition 2 describes how the Domar weight of a sector responds to a change in its own TFP process, and it holds generally. At an interior equilibrium, we can also characterize how any change in beliefs affects the full vector ω. For that purpose, we introduce a risk-adjusted version of the productivity vector ε defined as

$$\mathcal{E} = \underbrace{\mu}_{\mathbb{E}[\varepsilon]} - \underbrace{(\rho - 1)\Sigma\omega}_{\text{Cov}[\varepsilon,\lambda]} \tag{27}$$

The vector $\mathcal{E}$ captures how higher exposure to the productivity process ε_i affects the representative household's utility. It depends on how productive each sector i is in expectation, and on how its ε_i covaries with the stochastic discount factor λ. If we denote by $\mathbb{1}_i$ the column vector with a 1 as ith element and zeros elsewhere, we can write

$$\frac{\partial \mathcal{E}}{\partial \mu_i} = \mathbb{1}_i, \tag{28}$$

such that an increase in μ_i makes sector i more attractive. It however leaves the risk-adjusted TFP of other sectors unchanged. Similarly, for a change in Σ_{ij}, we can compute

$$\frac{\partial \mathcal{E}}{\partial \Sigma_{ij}} = -\frac{1}{2}(\rho - 1)(\omega_j \mathbb{1}_i + \omega_i \mathbb{1}_j) \tag{29}$$

Using the definition of $\mathcal{E}$, we can write the first-order condition of the planner's problem (26) at an interior solution as

$$\nabla \bar{a}(\omega) + \mathcal{E} = 0 \tag{30}$$

where ∇ is the gradient of the aggregate TFP shifter function $\bar{a}$. This first-order condition shows that the planner balances the benefit of a sector in terms of risk-adjusted TFP against its impact on the aggregate TFP shifter.

Proposition 3. Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. If $\omega \in \mathrm{int}\mathcal{O}$, then the response of the equilibrium Domar weights to a change in γ is given by

$$\frac{d\omega}{d\gamma} = \underbrace{-\mathcal{H}^{-1}}_{\text{propagation}} \times \underbrace{\frac{\partial \mathcal{E}}{\partial \gamma}}_{\text{impulse}} \tag{31}$$

where the $n \times n$ negative definite matrix $\mathcal{H}$ is given by

$$\mathcal{H} = \nabla^2 \bar{a} + \frac{\partial \mathcal{E}}{\partial \omega} \tag{32}$$

and where the matrix $\nabla^2 \bar{a}$ is the Hessian of the aggregate TFP shifter function $\bar{a}$, and $\frac{\partial \mathcal{E}}{\partial \omega} = -\frac{d\mathrm{Cov}[\varepsilon, \lambda]}{d\omega} = -(\rho - 1)\Sigma$ is the Jacobian matrix of the risk-adjusted TFP vector $\mathcal{E}$.

The response of the Domar weights to a change in beliefs, as given by (31), can be decomposed into an impulse component and a propagation component. The impulse captures the direct impact of the change on risk-adjusted TFP. It is simply given by the partial derivative of $\mathcal{E}$ with respect to the moment of interest (see (28) and (29) above). This impulse is then propagated through $\mathcal{H}^{-1}$ to capture its full equilibrium effect on the Domar weights.

Global complements and substitutes Just as $\mathcal{H}_i^{-1}$ captured local substitution patterns between inputs in the problem of firm i, $\mathcal{H}^{-1}$ captures global, economy-wide substitution patterns between sectors.

If $\mathcal{H}_{ij}^{-1} < 0$, we say that i and j are **global complements**. If instead $\mathcal{H}_{ij}^{-1} > 0$, we say that i and j are **global substitutes**.

The following corollary justifies this terminology by showing that the sign of $\mathcal{H}_{ij}^{-1}$ is sufficient to characterize how Domar weights respond to a change in the productivity process.

Corollary 3. If $w \in \operatorname{int}\mathcal{O}$, then the following holds.

1. An increase in the expected value μ_i or a decline in the variance Σ_{ii} leads to an increase in ω_j if i and j are global complements, and to a decline in ω_j if i and j are global substitutes.
2. An increase in the covariance Σ_{ij}, $i \neq j$, leads to a decline in ω_k if k is global complement with i and j, and to an increase in ω_k if k is global substitute with i and j.

Σ and global substition patterns The following lemma describes how an increase in covariance Σ_{ij} between any two sectors affects the degree of global substitution between them.

Lemma 5. An increase in the covariance Σ_{ij} induces stronger global substitution between i and j, in the sense that $\frac{\partial \mathcal{H}_{ij}^{-1}}{\partial \Sigma_{ij}} > 0$.

Intuitively, if the correlation between ε_i and ε_j becomes larger, the planner has stronger incentives to lower ω_j after an increase in ω_i in order to reduce aggregate risk. From (32), we see that the strength of that diversification mechanism depends on the household's risk aversion through ρ.

$\nabla^2 \bar{a}$ and global substitution patterns The next lemma establishes sufficient conditions under which local complementarities translate into global complementarities.

Lemma 6. Suppose that all input shares are (weak) local complements in the production of all goods, that is $[H_i^{-1}]_{kl} \leqslant 0$ for all i and all $k \neq l$. If $\alpha \in \operatorname{int}\mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ such that if $\|\Sigma\| \leqslant \bar{\Sigma}$, all sectors are global complements, that is $\mathcal{H}_{ij}^{-1} < 0$ for all $i \neq j$.

Impact of Lemma 6

1. **Generation of Global Complementarities**: Even if the local TFP shifter functions are neutral (i.e., $\left[H_i^{-1}\right]_{kl} = 0$ for all i and $k \neq l$), the equilibrium forces of the model generate global complementarities between sectors. This means that the model itself induces sectors to be globally complementary without requiring local TFP shifter functions to exhibit local complementarities.

2. **Equilibrium Forces** Suppose a sector i becomes more attractive, for instance due to an increase in μ_i. Any other sector j that relies on i (either directly or indirectly, if $L_{ji} > 0$) would benefit from that change and also become more attractive. This triggers an increase in Domar weights throughout the network and a shift away from labor, generating global complementarities between sectors.

3. **Policy and Practical Applications** Understanding the conditions under which local complementarities translate into global complementarities can help in formulating more effective economic policies, especially regarding resource allocation and inter-sector coordination. This is crucial for improving overall economic efficiency and welfare.

4. **Role of Covariance Matrix (Σ)** The lemma highlights that the degree of global substitution or complementarity between sectors can be influenced by the covariance matrix Σ. If Σ is sufficiently small, local complementarities can lead to global complementarities, while larger Σ might induce stronger global substitution forces due to diversification effects.

Parametrize H_i Let

$$H_i^{-1} = \begin{bmatrix} -1 & \frac{s}{n-1} & \cdots & \frac{s}{n-1} \\ \frac{s}{n-1} & -1 & & \vdots \\ \vdots & & \ddots & \frac{s}{n-1} \\ \frac{s}{n-1} & \cdots & \frac{s}{n-1} & -1 \end{bmatrix} \tag{33}$$

where we impose $-(n-1) < s < 1$ to guarantee that H_i^{-1} is negative definite. When $s < 0$ all input shares are complements in the production of good i, and when $s > 0$ they are substitutes. The next lemma describes sufficient conditions under which local substitution imply global substitution.

Lemma 7. Suppose that all the TFP shifter functions $(a_1, \cdots, a_n)$ take the form (2), with $\alpha_i^\circ = \alpha_j^\circ$ for all i, j, and that H_i^{-1} is of the form (33) for all i. If $\alpha \in \text{int}\mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ and a threshold $0 < \bar{s} < 1$ such that if $\|\Sigma\| \leqslant \bar{\Sigma}$ and $s > \bar{s}$, then all sectors are global substitutes, that is $\mathcal{H}_{ij}^{-1} > 0$ for all $i \neq j$.

An approximate equation for the equilibrium Domar weights This section discusses how to derive an approximate equation for the equilibrium Domar weights using a Taylor expansion of $\nabla \bar{a}$. The key steps and impacts are outlined as follows:

First, we define the ideal shares α°, which maximize the values of the TFP shifters $(a_1, \ldots, a_n)$. Based on this, we can write:

$$\nabla \bar{a}(\omega) \approx \nabla \bar{a}(\omega^\circ) + \nabla^2 \bar{a}(\omega^\circ)(\omega - \omega^\circ) \tag{34}$$

This approximation is accurate if the cost of deviating from the ideal shares embedded in the local TFP shifters is large.

Using this approximation, the first-order condition (30) becomes linear in ω, allowing us to solve for the equilibrium Domar weights.

Lemma 8. If $\omega \in \text{int}\mathcal{O}$, the equilibrium Domar weights are approximately given by:

$$\omega = \omega^\circ - [\mathcal{H}^\circ]^{-1}\mathcal{E}^\circ + O\left(\|\omega - \omega^\circ\|^2\right) \tag{35}$$

where the superscript $\circ$ indicates that $\mathcal{H}$ and $\mathcal{E}$ are evaluated at ω°.

Impacts of Lemma 8

1. **Global Substitution Patterns** This approximation shows that the equilibrium Domar weights can be explained in terms of the global substitution patterns embedded in $[\mathcal{H}^\circ]^{-1}$ and the expected attractiveness of all sectors, captured by the risk-adjusted productivity $\mathcal{E}^\circ$.

2. **Inter-Sector Interactions** If a sector i is endowed with a productivity process that is high in expectation or has a low covariance with the stochastic discount factor, $\mathcal{E}_i^\circ$ will be large. Since the diagonal elements of $[\mathcal{H}^\circ]^{-1}$ are negative, ω_i tends to be larger than ω_i°.

3. **Relative Weight Changes** A large $\mathcal{E}_i^\circ$ also contributes to increasing the Domar weights of all sectors that are global complements with i and to decreasing the Domar weights of sectors that are global substitutes with i.

6.2 The production network

Proposition 4. If $\alpha \in \text{int}\mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ such that if $\|\Sigma\| \leqslant \bar{\Sigma}$ the following holds.

1. (Complementarity) Suppose that input shares are local complements in the production of good i, that is $[H_i^{-1}]_{kl} < 0$ for all $k \neq l$. Then a beneficial change to k $(\partial \mathcal{E}_k / \partial \gamma > 0)$ increases α_{ij} for all j.

2. (Substitution) Suppose that the conditions of Lemma 7 about the TFP shifters $(a_1, \cdots, a_n)$ hold. Then there exists a threshold $0 < \bar{s} < 1$ such that if $s > \bar{s}$, a beneficial change to k $(\partial \mathcal{E}_k / \partial \gamma > 0)$ decreases α_{ij} for all i and all $j \neq k$, and increases α_{ik} for all i.

Proposition 4 illustrates the impact of complementarity and substitution of input shares on the adjustment of production networks. When input shares are locally complementary in the production of a product, a beneficial change to one input increases its share in the production of all products. Conversely, in the presence of strong substitution effects, a beneficial change to one input decreases the shares of other inputs in production while increasing its own share.

An approximate equation for the equilibrium production network

As for the Domar weights, one must in general use numerical methods to find the equilibrium network α. We can, however, derive an approximation for the equilibrium production network when the cost of deviating from the ideal shares α° is large. Specifically, let $a_i(\alpha_i) = \bar{\kappa} \times \hat{a}_i(\alpha_i)$, where $\hat{\alpha}$ does not depend on κ, and suppose that $\alpha_i^\circ \in \mathrm{int}\mathcal{A}_i$. The parameter $\hat{\kappa} > 0$ captures how costly it is for the firms to deviate from α° in terms of TFP loss. When $\hat{\kappa}$ is large, we can use perturbation theory to derive an approximate equation for α.

Lemma 9. If $\alpha \in \mathrm{int}\mathcal{A}$, the equilibrium input shares in sector i are approximately given by

$$\alpha_i = \alpha_i^\circ + \bar{\kappa}^{-1} \left(\hat{H}_i^\circ\right)^{-1} \mathcal{R}^\circ + O(\kappa^{-2}) \tag{36}$$

where $\hat{H}_i^\circ$ is the Hessian of $\hat{a}_i$ at α_i°, and where the vector of risk-adjusted prices at α° is given by

$$\mathcal{R}^\circ = -\mathcal{L}\mu + (\rho - 1)\mathcal{L}^\circ \Sigma \omega^\circ$$

Lemma 9 primarily addresses the approximate solution for the production network when the cost function is nonlinear. Specifically, when it is costly for firms to deviate from the ideal shares α°, the equilibrium production network can be approximated using perturbation theory. Equation (36) provides an approximation indicating that the equilibrium input shares α_i depend on the risk-adjusted prices $\mathcal{R}^\circ$. This result demonstrates that when the cost of deviating from the ideal shares is high, the equilibrium production network can be approximated by evaluating the equilibrium prices as if firms chose the ideal shares.

Example: cascading link destruction

The example of “cascading link destruction” discusses how an increase in uncertainty in a single sector can trigger a chain reaction throughout the production network. Specifically, when the volatility of a sector increases, multiple producers sequentially switch to more stable suppliers, causing a series of adjustments in the production network.

The specific example is as follows:

1. In a low-uncertainty state (left figure): Firms in sectors 1 to 3 directly or indirectly rely on sector 4 as a supplier.

2. In a high-uncertainty state (right figure): As the uncertainty in sector 4 increases, firms in sector 3, seeking a more stable supply, switch to using inputs from sector 7. This change implies that firms in sector 2, to avoid risk, switch to using inputs from sector 6, and so on, creating a cascade of adjustments.

Through this example, the paper demonstrates how the production network adjusts in response to changes in uncertainty. These adjustments not only affect the directly related firms but also propagate through the supply chain, impacting firms far removed from the initial shock.

Figure 2: Cascading impact of a change in Σ_{44}

(a) Low uncertainty about ε_4

(b) High uncertainty about ε_4

We can interpret this cascading network adjustment through the lens of Lemma 9. Differentiating the expression with respect to Σ_{44} yields

$$\frac{d\alpha_{ij}}{d\Sigma_{44}} = \bar{\kappa}^{-1}(\rho-1)\omega_4^\circ \left(\underbrace{\left[\left(\hat{H}_i^\circ\right)^{-1}\right]_{jj} \mathcal{L}_{j4}^\circ}_{\text{direct effect of } \Sigma_{44} \text{ on } j} + \underbrace{\sum_{l \neq j} \left[\left(\hat{H}_i^\circ\right)^{-1}\right]_{jl} \mathcal{L}_{jl}^\circ}_{\text{indirect effect of } \Sigma_{44} \text{ through other suppliers } l \neq j} \right) + O(\bar{\kappa}^{-2}) \quad (37)$$

Equation (37) states that if a firm j relies on sector 4 as an input (either immediate or distant, such that $\mathcal{L}_{j4}^\circ > 0$), an increase in Σ_{44} makes j less attractive. This direct effect pushes α_{ij} down (recall that $[H_i^\circ]_{jj} < 0$ by the concavity of a_i). There is also an indirect effect that operates through the second term in (37). If another sector $l \neq j$ also relies on 4 ($\mathcal{L}_{l4}^\circ > 0$), then an increase in Σ_{44} makes l less attractive as well. This indirect channel can lead to either a decrease or an increase in α_{ij}, depending on whether j and l are complements or substitutes in the production of i; that is, whether $[(H_i^\circ)^{-1}]_{jl}$ is negative or positive.

7 Implications for GDP and welfare

Proposition 5. Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. Under an endogenous network, welfare responds to a marginal change in γ as if the network were fixed at its equilibrium value α^*, that is

$$\frac{d\mathcal{W}(\mu,\Sigma)}{d\gamma} = \frac{\partial W(\alpha^*,\mu,\Sigma)}{\partial\gamma}$$

Let α^* be the equilibrium network, i.e., $\alpha^* = \alpha(\mu,\Sigma)$. When we make a small change to γ, the equilibrium network will adjust to accommodate the new γ. However, Proposition 5 states that the effect of this adjustment on the marginal change can be neglected.

While this proposition shows that the flexibility of the network plays no role for the response of welfare to a marginal change in beliefs, this is generally not true for non-infinitesimal changes. In that case, shifts in (μ,Σ) that are beneficial to welfare are amplified, compared to the fixed-network benchmark, while changes that are harmful are dampened (see Proposition 2). Indeed, if we denote by $\alpha^*(\mu,\Sigma)$ the equilibrium production network under (μ,Σ) and by $W(\alpha,\mu,\Sigma)$ welfare under a network α, we can write that the difference in welfare after a change in beliefs from (μ,Σ) to (μ',Σ') satisfies the inequality

$$\underbrace{\mathcal{W}(\mu',\Sigma') - \mathcal{W}(\mu,\Sigma)}_{\text{Change in welfare under a flexible network}} \geqslant \underbrace{W(\alpha^*(\mu,\Sigma),\mu',\Sigma') - W(\alpha^*(\mu,\Sigma),\mu,\Sigma)}_{\text{Change in welfare under a fixed network}}. \quad (38)$$

Corollary 4. The impact of an increase in μ_i on welfare is given by

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i \tag{39}$$

and the impact of an increase in Σ_{ij} on welfare is given by

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho - 1)\omega_i\omega_j \tag{40}$$

7.1 Beliefs and GDP

Proposition 6. The presence of uncertainty lowers expected log GDP, in the sense that $\mathbb{E}[y]$ is largest when $\Sigma = 0$.

This proposition follows directly from Lemma 3. Without uncertainty ($\Sigma = 0$), the variance $\mathbb{V}[y]$ of log GDP is zero for all networks $\alpha \in \mathcal{A}$. The social planner then maximizes $\mathbb{E}[y]$ only. When, instead, the productivity vector ε is uncertain ($\Sigma \neq 0$), the planner also seeks to lower $\mathbb{V}[y]$ which necessarily lowers expected log GDP in equilibrium.

Corollary 5. Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. The equilibrium response to a change in beliefs γ must satisfy

$$\underbrace{\frac{d\mathbb{E}[y]}{d\gamma} - \frac{\partial\mathbb{E}[y]}{\partial\gamma}}_{\text{Excess response of } \mathbb{E}[y]} = \frac{1}{2}(\rho - 1)\underbrace{\left(\frac{d\mathbb{V}[y]}{d\gamma} - \frac{\partial\mathbb{V}[y]}{\partial\gamma}\right)}_{\text{Excess response of } \mathbb{V}[y]} \tag{41}$$

Corollary 5 is a direct consequence of Proposition 5. Since the response of welfare to a marginal change in beliefs must be the same under a flexible and a fixed network, a larger increase in $\mathbb{E}[y]$ under a flexible network must come at the cost of alarger increase in the variance $\mathbb{V}[y]$.

Proposition 7. If $\omega \in \text{int}\mathcal{O}$, the following holds.

1. The impact of an increase in μ_i on log GDP is given by

$$\frac{d\mathbb{E}[y]}{d\mu_i} = \underbrace{\omega_i}_{\text{Fixed network}} - (\rho - 1)\omega^T\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\mu_i} = \underbrace{0}_{\text{Fixed network}} - 2\omega^T\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\mu_i}.$$

2. The impact of an increase in Σ_{ij} on log GDP is given by

$$\frac{d\mathbb{E}[y]}{d\Sigma_{ij}} = \underbrace{0}_{\text{Fixed network}} - (\rho - 1)\omega^T\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\Sigma_{ij}}, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\Sigma_{ij}} = \underbrace{\omega_i\omega_j}_{\text{Fixed network}} - 2\omega^T\Sigma\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\Sigma_{ij}}.$$

Corollary 6. Without uncertainty ($\Sigma = 0$) the moments of GDP respond to changes in beliefs as if the network were fixed, such that

$$\frac{d\mathbb{E}[y]}{d\mu_i} = \frac{\partial \mathbb{E}[y]}{\partial \mu_i} = \omega_i, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\Sigma_{ij}} = \frac{\partial \mathbb{V}[y]}{\partial \Sigma_{ij}} = \omega_i \omega_j$$

Corollary 7. Suppose that $\omega \in \mathrm{int}\mathcal{O}$. There exists a threshold $\bar{\Sigma} < 0$ such that if $\Sigma_{kl} > \bar{\Sigma}$ for all k, l, then the following holds.

1. If all sectors are global complements with sector i, that is $\mathcal{H}^{-1}_{ik} < 0$ for $k \neq i$, then

$$\frac{d\mathbb{E}[y]}{d\mu_i} = \frac{\partial \mathbb{E}[y]}{\partial \mu_i} > \omega_i, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\Sigma_{ij}} = \frac{\partial \mathbb{V}[y]}{\partial \mu_i} > 0$$

2. If all sectors are global complements with sectors i and j, that is $\mathcal{H}^{-1}_{ik} < 0$ and $\mathcal{H}^{-1}_{jk}$ for $k \neq i, j$, then

$$\frac{d\mathbb{E}[y]}{d\mu_i} = \frac{\partial \mathbb{E}[y]}{\partial \Sigma_{ij}} < 0, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\Sigma_{ij}} = \frac{\partial \mathbb{V}[y]}{\partial \mu_i} < \omega_i \omega_j$$

Corollary 8. Suppose that $\omega \in \mathrm{int}\mathcal{O}$. There exists a threshold $\underline{\Sigma} < 0$ and $\bar{\Sigma} > 0$ such that

1. If all sectors are global substitutes with sector i, that is $\mathcal{H}^{-1}_{ik} > 0$ for $k \neq i$, and sector i is not too risky while other sectors are sufficiently risky in the sense that $\Sigma_{ji} < \underline{\Sigma}$ for all j and $\Sigma_{jk} > \bar{\Sigma}$ for all $j, k \neq i$, then

$$\frac{d\mathbb{E}[y]}{d\mu_i} < \omega_i, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\mu_i} < 0.$$

2. If all sectors are global substitutes with sectors i and j, that is $\mathcal{H}^{-1}_{ik} > 0$ and $\mathcal{H}^{-1}_{jk} > 0$ for $k \neq i, j$, and sectors i and j are not too risky while other sectors are sufficiently risky in the sense that $\Sigma_{li} < \underline{\Sigma}$ and $\Sigma_{lj} < \underline{\Sigma}$ for all l, and $\Sigma_{lk} > \bar{\Sigma}$ for all $l, k \neq i$ and $l, k \neq j$, then

$$\frac{d\mathbb{E}[y]}{d\Sigma_{ij}} > 0, \quad \text{and} \quad \frac{d\mathbb{V}[y]}{d\Sigma_{ij}} > \omega_i \omega_j.$$

After an increase in the TFP mean (μ_i) of a sector, the Domar weight (ω_i) of that sector increases, which pushes up the variance of log GDP ($\mathbb{V}[y]$). However, if the sector's TFP variance (Σ_{ii}) is small, the increase in $\mathbb{V}[y]$ is also small. Since other sectors are global substitutes with this sector, the increase in i leads to a decline in the Domar weights of all other sectors. If the variances of these other sectors are large relative to Σii, this decline in their Domar weights results in a substantial decrease in $\mathbb{V}[y]$. According to the logic of Proposition 7, this means that the expected log GDP ($\mathbb{E}[y]$) must increase by less than the fixed-network term i. Similarly, an increase in Σii leads to an increase in $\mathbb{V}[y]$ that is larger than under a fixed network. In this case, $\mathbb{E}[y]$ increases in response to the higher Σ_{ii}, indicating that uncertainty can be beneficial to expected log GDP at the margin.

Counterintuitive implications of changes in beliefs

1. **Belief Changes and GDP Response**

   (a) Corollaries 7 and 8 indicate that the response of GDP to changes in beliefs can be different from the predictions of Hulten's theorem in a fixed-network economy. The endogenous adjustment of the network can lead to more extreme outcomes.

   (b) An increase in the mean productivity (μ) of a sector can lead to a decrease in the expected log GDP ($\mathbb{E}[y]$), and an increase in the variance (Σ) of a sector can lead to a decrease in the variance of log GDP ($\mathbb{V}[y]$).

2. **Example of a Low-Productivity but Stable Producer**

   (a) Consider a producer with low but stable productivity. The high price of its goods makes it less attractive as a supplier.

   (b) If its expected productivity increases, its risk-reward profile improves, attracting more buyers. This can lead producers to move away from more productive but riskier suppliers, potentially causing expected GDP to fall.

3. **Impact of Increased Volatility** An increase in the volatility of a sector's productivity can lead to a decline in $\mathbb{V}[y]$. This is because producers may shift away from more volatile sectors, leading to a network that is less susceptible to fluctuations.

In the economy depicted in Figure 3, sectors 4 and 5 use only labor to produce, while sectors 1 to 3 can also use goods 4 and 5 as inputs. For sectors 1 to 3, goods 4 and 5 are either local substitutes (panels (a) to (c)) or local complements (panels (d) to (f)). Sector 4 is more productive and volatile than sector 5.

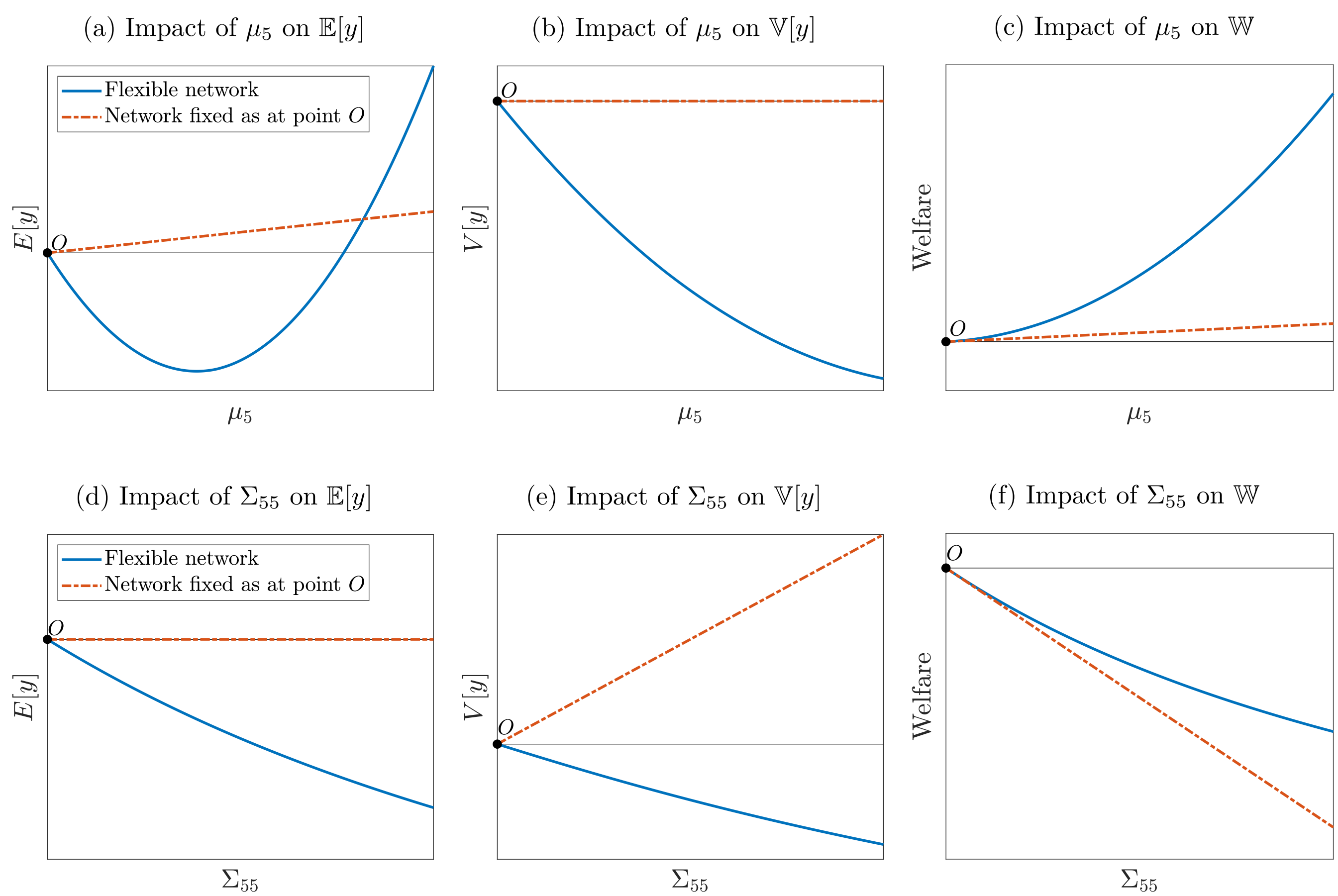
Consider the impact of a positive shock to μ_5 when inputs 4 and 5 are substitutes. Initially, the increase in μ_5 negatively impacts expected log GDP ($\mathbb{E}[y]$) because sector 5, although less productive, now offers a better risk-reward trade-off. This causes sectors 1 to 3 to shift towards good 5 and away from good 4, reducing $\mathbb{E}[y]$ since $\mu_4 > \mu_5$. V[y] also declines because sector 5 is less volatile, aligning with Proposition 7. Ultimately, the overall effect on welfare is positive, as shown in panel (c), because the welfare gain from reduced volatility outweighs the initial drop in $\mathbb{E}[y]$.

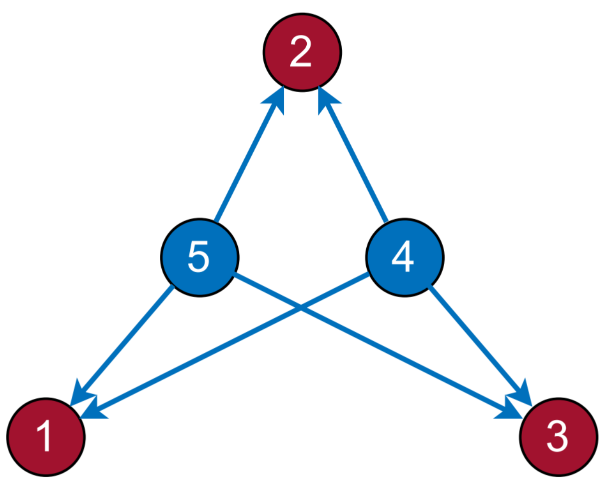
To emphasize the role of the endogenous network for this mechanism, Figure 3 also shows the effect of the same increase in μ_5 when the network is kept fixed (dashed red lines). From Corollary 1, the marginal impact of μ_5 on expected log GDP is equal to its Domar weight, and increasing μ_5 has a positive impact on $\mathbb{E}[y]$. At the same time, $\mathbb{V}[y]$ is unaffected by changes in μ. Whilean increase in μ_5 is welfare-improving in this case, the effect is less pronounced than in the flexible network economy. Indeed, in the latter case the equilibrium network adjusts precisely to maximize the beneficial impact of the change in beliefs on welfare, as implied by (38).

We can use a small variation of this economy to illustrate how an increase in an element of Σ can lower the variance of log GDP, and simultaneously lower welfare. Start again from the economy in the left column of Figure 3 (point O) but suppose that inputs 4 and 5 are complements in the production of goods 1 to 3. Consider an increase in the volatility of sector 5. In response, sectors 1 to 3 start to rely less on sector 5. But since inputs 4 and 5 are complements, sectors 1 to 3 also reduce their shares of input 4, thus increasing the overall share of labor which is a safe input. As a result, the variance of log GDP declines (panel e). Expected log GDP also goes down by Proposition 7 (panel d). The combined effect on welfare is negative, as predicted by Corollary 4 (panel f). In this case, the reorganization of the network mitigates the adverse effect of the increase in volatility on welfare. Instead, if the network is fixed, an increase in Σ_{55} does not affect expected log GDP but leads to an increase in the variance of log GDP. As a result, welfare drops substantially more than under an endogenous network, as implied by (38).

Notes: There is an arrow from j to i if $\alpha_{ij} > 0$. Household: $\rho = 2.5$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{3}\varepsilon$, $\beta_4 = \beta_5 = \frac{3}{2}\varepsilon$, where $\varepsilon > 0$ is very small. $\mu = (0.1, 0.1, 0.1, 0.1, 0.08)$, Σ is diagonal, with $\operatorname{diag}(\Sigma) = (0.2, 0.2, 0.2, 0.2, 0.02)$. a is as in (2) with $\alpha^\circ_{14} = \alpha^\circ_{15} = \alpha^\circ_{24} = \alpha^\circ_{25} = \alpha^\circ_{34} = \alpha^\circ_{35} = 0.25$; all other α°_{ij} are zero. $H_4 = H_5$ are matrices with 50 on the diagonal. $H_1 = H_2 = H_3$ with $[H_1]_{11} = [H_1]_{22} = [H_1]33 = 50$, $[H_1]_{44} = [H_1]_{55} = 2$. In panels (a)-(c), μ_5 goes from -0.08 to 0.1; 4 and 5 are substitutes, $[H_1]_{45} = 1.9$. In panels (d)-(f), Σ_{55} goes from 0.02 to 0.2; 4 and 5 are complements, $[H_1]_{45} = 1.9$.

Figure 3: The non-monotone impact of beliefs on GDP





8 A basic calibration of the model

The analysis above highlights the economic forces that determine how the production network, GDP and welfare respond to changes in the productivity process. Clearly, the model is too stylized to capture all the fluctuations in the production network observed in reality, and other mechanisms, not present in our model, may also be important in practice. With that caveat in mind, we present in this section results from a basic calibration of the model to the United States economy to get a sense of the quantitative potential of our main mechanisms.

Below, we first describe how the model is parameterized and briefly go over which features of the US economy the model matches well, and in what dimensions it falls short. Finally, we explore how beliefs shape the production network and investigate how the changing structure of the network influences aggregate output and welfare in our stylized model. We keep the analysis succinct but provide more details in Appendix B.

8.1 Parametrization

The Bureau of Economic Analysis (BEA) provides U.S. sectoral input-output tables for $n = 37$ sectors at an annual frequency from 1948 to 2020. From these data, we compute the input shares α_{ijt} of each sector in each year t, the average consumption expenditure share of each sector β_i, and sectoral TFP measured as the Solow residual.

To calibrate the model, we need to make explicit assumptions about the process for TFP. For the endogenous productivity shifter $A_i(\alpha_{it})$ we adopt a particular version of form (2) which includes a diagonal component for $\bar{H}_i$ are a penalty for deviating from an ideal labor share (see (69) inthe appendix). We set the ideal shares $(\alpha_1^\circ, \cdots, \alpha_n^\circ)$ equal to the time average of the input shares observed in the data. The exogenous sectoral productivity process ε_t is assumed to follow a random walk with drift,

$$\varepsilon_t = \gamma + \varepsilon_{t-1} + u_t, \tag{42}$$

where γ is an $n \times 1$ vector of deterministic drifts and $u_t \sim \text{iid}\mathcal{N}(0, \Sigma_t)$ is a vector of shocks. We further assume that firms know γ and ε_t at time t, so that the conditional mean and the covariance of beliefs are given by $\mu_t = \gamma + \varepsilon_{t-1}$ and Σ_t. Importantly, we allow uncertainty Σ_t to vary over time and estimate it from TFP data using a rolling window that puts more weight on more recent observations.

We use a simple moment-matching strategy to pin down the 1) relative risk aversion parameter ρ of the household, 2) the TFP shifter functions $\bar{H}_i$ and 3) the time-varying beliefs (μ_t, Σ_t). We describe this procedure in Appendix B.

The calibrated coefficient of relative risk aversion $\hat{\rho}$ is 4.3, which is similar to values used orestimated in the macroeconomics literature. Our procedure also provides time-series for the vector μ_t and the matrix Σ_t, and we aggregate these variables across sectors to obtain economy-wide measures of the expected value $\bar{\mu}_t$ and the variance $\bar{\Sigma}_t$ of aggregate TFP. As we might expect, these measures are cyclical, with $\bar{\mu}_t$ falling and $\bar{\Sigma}_t$ rising during recessions. Overall, our measure of aggregate uncertainty $\bar{\Sigma}_t$ has been relatively stable since 1980, with occasional sharp spikes, most notably during the Great Recession of 2007–2009 (see Figure 5 in Appendix B.3).

We next assess how well the calibrated model fits key moments in the data. As we have seen above, the Domar weights, and how they react to changes in μ_t and Σ_t, are central for the mechanisms of the model. The model is able to roughly replicate features of the empirical Domar weights, with a cross-sectional correlation between the time-averaged Domar weights in the model and in the data of 0.96. However, the average Domar weight in the model (0.03) is lower than its data counterpart (0.05). Overall, the model can account for about 40% of the over-time standard deviation of Domar weights, which indicates that other mechanisms, such as technological progress that might expand the set of available techniques, might be at work in reality.

The mechanisms of the model predict that a decline in the expected productivity of a sector μ_i, or an increase in its variance Σ_{ii}, should push firms to reduce the importance of that sector as an input provider, leading to a decline in its Domar weight. Reassuringly, these correlations are visible in the data, where $\text{Corr}(\omega_{jt}, \mu_{jt}) = 0.1$, and $\text{Corr}(\omega_{jt}, \Sigma_{jjt}) = -0.4$. The calibrated model is also able to roughly match these correlations, and the corresponding numbers are 0.1 and -0.3.

8.2 The production network, welfare and output

To evaluate the quantitative potential of an endogenous production network for welfare and GDP, we compare the calibrated model to two sets of alternative economies. First, we compare our baseline model to an economy in which the network is kept completely fixed at its sample average. This exercise therefore informs us about the overall impact of changes in the structure of the production network. We then investigate the role of uncertainty alone in shaping the production network. We do so by considering 1) an economy in which production techniques are chosen as if $\Sigma_t = 0$, and 2) a perfect-foresight economy in which firms observe the realization of ε_t before making technique choices (the "known ε_t" economy). In both cases, uncertainty is irrelevant for decisions, and so these exercises allow us to isolate the impact of uncertainty on the production network and, through that channel, on macroeconomic aggregates.

We find that expected log GDP in the "fixed network" economy is 2.1% lower than in our baseline calibration with a flexible network. Intuitively, as some sectors become more productive over time, the goods that they produce become cheaper, and firms would like to rely more on them. With a flexible network this is possible, and the aggregate economy becomes more productive as aresult. The difference in welfare between the two models is about 2.1% as well.

When we isolate the role of uncertainty, however, these numbers become smaller. In line withthe theory, the baseline economy is on average less productive and less volatile than under the "as if $\Sigma_t = 0$" alternative but the numbers are small, on the order of 0.01% for $\mathbb{E}[y]$ and 0.10% for $\mathbb{V}[y]$. This suggests that, for most of the sample period, uncertainty is sufficiently low that firms simply buy their inputs from the most productive suppliers without much concern for any risk involved.

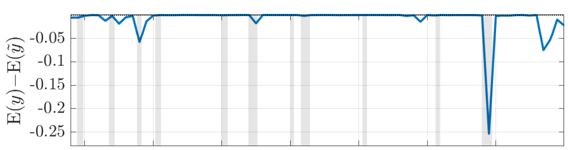
The differences between our calibrated economy and the "no uncertainty" alternatives are however larger during high-uncertainty episodes like the Great Recession. The top row of Figure 4 shows that expected log GDP in the baseline economy is about 0.25% lower in 2009 than in the alternative "as if $\Sigma_t = 0$" economy. Because of the large increase in uncertainty, firms adjust their production techniques toward safer but less productive suppliers to avoid potentially large increases in costs. The result in terms of aggregate volatility is visible in the top-right panel, where we see that log GDP is about 2.4% less volatile in 2009 in the baseline economy. Interestingly, realized log GDP, shown in the left-bottom panel, is substantially higher in the baseline economy than in the "as if $\Sigma_t = 0$" alternative. Essentially, firms took out an insurance against particularly bad TFP draws and opted for safer suppliers. When these fears were realized, this insurance policy paid off so that the baseline economy fared about 2.7% better in terms of realized log GDP compared to the alternative.

The right-bottom panel provides the same information for the "known ε_t" alternative. In this case, beliefs (μ_t, Σ_t), and in particular uncertainty, play no role in shaping the network and, from the planner's problem, the optimal network is simply the one that maximizes (realized) consumption. It follows that realized consumption (or GDP) is always larger than in the baseline model. Unsurprisingly, the difference is particularly pronounced during episodes of high uncertainty, when knowing ε_t provides a larger advantage, and reaches a high of 3% during the Great Recession.

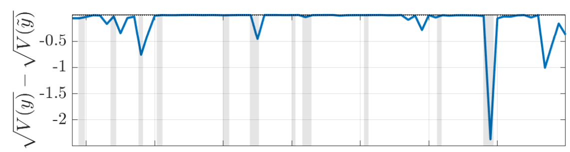
Overall, our findings suggest that, while uncertainty might have a limited impact on the economy on average, it may play a larger role in shaping the production network during high-uncertainty periods, with consequences for expected and realized GDP, as well as for welfare. Given the stylized nature of the model, these findings should be interpreted with caution. The model abstracts from other forces that might affect the production network, such as changes in demand and technological progress that would expand the set of production techniques. Similarly, the production function might not be Cobb-Douglas in reality, in which case changes in prices would affect Domar weights. We also made the implicit assumption that it takes one year (the frequency of our data) for firms to change production techniques. While this assumption might be reasonable for some sectors, it is likely that the time it takes to retool a factory varies significantly by industry, or even depending on what the new and the old techniques are. While we believe that the mechanisms that we explore in this paper would still be present in a richer model, more work would be needed to fully assess their importance.

Figure 4: The role of uncertainty in the postwar period.

First row: "as if $\Sigma_t = 0$" as the alternative

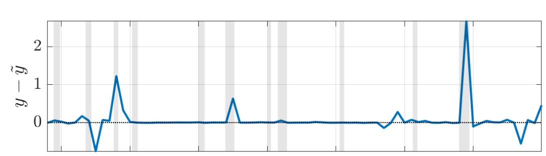


(a) Difference in expected log GDP [%]

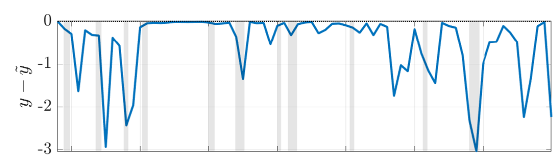


(b) Difference in expected st. dev. of log GDP [%]

(c) Left column: "as if $\Sigma_t = 0$" as alternative
Difference in realized log GDP [%]



(d) Right column: "known ε_t" as alternative
Difference in realized log GDP [%]



9 Model-free evidence for the mechanisms

The model proposed in this paper relies on simplifying assumptions for tractability. In this section, we present additional evidence in support of the main mechanisms of the model that does not rely on this structure. Through firm-level regressions that closely follow Alfaro, Bloom, and Lin (2019) we document that 1) higher uncertainty about a firm leads to a decline in its Domar weight, and 2) network connections involving riskier suppliers are more likely to break down. We test these predictions at the firm level to take advantage of the abundance of data and of instrumental variables that are available at this level of aggregation. Supplemental Appendix E in Kopytov et al. (2024) describes the data and the instruments in detail.

9.1 Uncertainty and Domar weights

We first test the model's prediction that Domar weights decrease with uncertainty. We use annual U.S. data from 1963 to 2016 provided by Compustat. Our main variables of interest area firm's Domar weight, constructed by dividing its sales by nominal GDP, and a measure of itsstock price volatility, which we use as a proxy for uncertainty. We then regress the change in Domar weight on the change in stock price volatility. The results are presented in the first column of Table I. In column (2), we follow Alfaro et al. (2019) and address potential endogeneity concerns by instrumenting stock price volatility with industry-level exposure to ten aggregate sources of uncertainty shocks. In column (3), we use option prices to back out an implied measure of future volatility. In all cases, we find a negative and significant relationship between uncertainty and Domar weights. The effect is also economically large with a decline in Domar weight of about 18% following a doubling in firm-level volatility (roughly a 3.3 standard deviation volatility shock), according to the IV estimates. Overall, these results provide evidence that higher uncertainty leads to lower Domar weights, in line with the predictions of our theoretical model.

9.2 Uncertainty and link destruction

We conduct a similar exercise, this time at the firm-to-firm relationship level, to investigate whether higher supplier uncertainty is associated with a higher likelihood of link destruction. We proceed by combining the uncertainty data described above with data from 2003 to 2016 about firm-level supply relationships provided by Factset. We then regress a dummy variable that equals one in the last year of a relationship on the change in the supplier's stock price volatility. The results are presented in column

Table 1: Domar weights and uncertainty

	Change in Domar weight		
	(1): OLS	(2): IV	(3): IV
Δ Volatility$_{i,t-1}$	-0.058***	-0.137***	-0.218***
	(0.004)	(0.034)	(0.073)
1st moment 10IV$_{i,t-1}$	No	Yes	Yes
Type of volatility	Realized	Realized	Implied
Fixed effects	Yes	Yes	Yes
Observations	112,563	27,380	17,151
F-statistic	—	14.2	9.8

(1) of Table II. As in the last exercise, column (2) uses industry level sensitivity to aggregate shocks as instruments, and column (3) uses implied volatility from option prices as a measure of uncertainty. In all cases, we find a positive and statistically significant relationship between supplier volatility and the end of supply relationships, which is consistent with buyers moving away from riskier suppliers. The effect is also economically large with a doubling in volatility associated with a 12 percentage point increase in the likelihood that a relationship is destroyed, according to the IV estimates.

Table 2: Link destruction and supplier volatility

	Dummy for last year of supply relationship		
	(1): OLS	(2): IV	(3): IV
ΔVolatility$_{t-1}$ of supplier	0.026**	0.097***	0.144**
	(0.012)	(0.035)	(0.063)
1st moment 10IV$_{t-1}$ of supplier	No	Yes	Yes
Type of volatility	Realized	Realized	Implied
Fixed effects	Yes	Yes	Yes
Observations	35,629	35,620	26,195
F-statistic	—	22.9	10.4

10 Appendix