Program Locality Homework

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1. **Consider the abc examples in Figure 1. Give the order that has the best locality and show that the order is indeed optimal.**

   Answer: The order with the best locality and its corresponding *reuse distances* are:

   $$\begin{array}{lcccccc} rd: & \infty & 1 & \infty & 1 & \infty & 1 \\ trace: & a & a & b & b & c & c \end{array}$$

   The *miss ratio* when $c = 1$ for the above sequence is:

   $mr(1) = P(rd > 1) = \frac{3}{6} = 0.5.$

   Assume there exists an order with a miss ratio less than 0.5, in other words, we must have less than 3 misses during the access. However, apparently, this is not possible because 3 misses must happen in this example ($m = 3$) when we first visit *a*, *b* and *c*.

   That is to say, for any cache with size $c \geq 1$ in this example, the miss ratio $mr(c) \geq 0.5$, so "a a b b c c" is an optimal order with the best locality.

2. - **Prove that in the limit form Xiang formula, we always have fp(0) = 0.**

     Answer: The limit form Xiang formula is:

     $\lim_{n\to\infty} fp(x) = m - \sum_{i=x+1}^{\infty} (i-x)P(rt=i)$

     For $x = 0$, $\lim_{n\to\infty} fp(0) = m - \sum_{i=1}^{\infty} iP(rt=i)$ **— *Eq. 1***

     Now, consider the infinite long $n = \infty$ trace has a cycle C with a size of *c*, indicates that all data blocks will be visited in a single loop, such that the size of distinct data blocks m within the whole trace is the same as the m in C.

     The ***Eq. 1*** defined $P(rt = i)$ as the portion of accesses that have reuse time *i*, thus

     $\sum_{i=1}^{\infty} iP(rt=i) = \sum_{i=1}^{\infty} i\frac{rt(i)}{c}$ **— *Eq. 2***

As we can see, c equals to the number of accesses in within a cycle.

Then, we define $B^j(j \in [1, m])$ as the j_{th} distinct data block in a cycle, so $rt(B^j, i)$ represents the number of accesses on the j_{th} block whose reuse time is i, therefore

$\sum_{i=1}^{\infty} iP(rt = i) = \sum_{j=1}^{m} \sum_{i=1}^{\infty} i\frac{rt(B^j,i)}{c}$ — ***Eq. 3***

Combined ***Eq. 3*** with ***Eq. 1***, so that

$\lim_{n\to\infty} fp(0) = m - \sum_{j=1}^{m} \sum_{i=1}^{\infty} i\frac{rt(B^j,i)}{c}$ — ***Eq. 4***

For each distinct data block B, we know that $\sum_{i=1}^{\infty} i \times rt(B, i) = c$, substituting is into ***Eq. 4***, we get

$\lim_{n\to\infty} fp(0) = m - \frac{m\times c}{c} = m - m = 0$

Therefore, $fp(0) = 0$.

- **Does the conclusion hold for finite length traces? If yes, give a proof; otherwise, show a counter example.**

  Answer: The Xiang formula is:

  $\lim_{n\to\infty} fp(x) = m - \frac{1}{n-x+1}\Big(\sum_{i=x+1}^{n-1}(i-x)rt(i) + \sum_{k=1}^{m}(f_k - x)I(f_k > x) + \sum_{k=1}^{m}(l_k - x)I(l_k > x)\Big)$

  For $x = 0$, $\lim_{n\to\infty} fp(0) = m - \frac{1}{n+1}\Big(\sum_{i=1}^{n-1} i \times rt(i) + \sum_{k=1}^{m} f_k \times I(f_k > 0) + \sum_{k=1}^{m} l_k \times I(l_k > 0)\Big)$

  Since $\sum_{k=1}^{m} f_k \times I(f_k > 0) + \sum_{k=1}^{m} l_k \times I(l_k > 0) = \sum_{k=1}^{m}(f_k + l_k + rt_k - rt_k\Big)$, so

  $\lim_{n\to\infty} fp(0) = m - \frac{1}{n+1}\Big(\sum_{i=1}^{n-1} i \times rt(i) + \sum_{k=1}^{m}(f_k + l_k + rt_k - rt_k\Big)$

  For $f_k + l_k + rt_k = n + 1$, we can derive

  $\lim_{n\to\infty} fp(0) = m - \frac{1}{n+1}\Big(\sum_{i=1}^{n-1} i \times rt(i) + m(n+1) - \sum_{k=1}^{m} rt_k\Big)$

  Besides, as two representatives of total reuse distance in the trace, $\sum_{i=1}^{n-1} i \times rt(i) = \sum_{k=1}^{m} rt_k$, so that

  $\lim_{n\to\infty} fp(0) = m - \frac{1}{n+1}\Big(m(n+1)\Big) = m - m = 0$

  Therefore, for finite length traces, $fp(0) = 0$.

3. ▪ **Show the derivation from P (A, rt = i) in Figure 3 to fp(A, x) in Figure 4.**

   Answer: Reference from the reuse time distributions in *Figure 3*:

i	$P(A, rt = i)$	$P(B, rt = i)$	$P(rt = i)$
1	1/3	0	1/3
2	1/3	0	1/3
3M	0	1/3	1/3

$fp(A,0) = 1 - \Big(P(A, rt = 1) + 2 \times P(A, rt = 2)\Big) = 1 - \left(\frac{1}{3} + 2 \times \frac{1}{3}\right) = 0$

$fp(A,1) = 1 - P(A, rt = 2) = 1 - \frac{1}{3} = \frac{2}{3}$

$fp(A,2) = fp(A, 3 \leq x < 3M) = fp(A, x \geq 3M) = 1 - 0 = 1$

▪ **Show the derivation from fp(B, x) to the miss ratio of array B in its single-array cache mrB(c) and the two-array cache mrAB(B,c).**

$mr_B(0) = fp(B,1) - fp(B,0) = \frac{1}{3} - 0 = \frac{1}{3}$

$mr_B(\frac{1}{3}) = fp(B,2) - fp(B,1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$mr_B(c < M) = fp(B, x+1) - fp(B,x)|_{fp(B,x)=c} = \frac{x+1}{3} - \frac{x}{3} = \frac{1}{3}$

$mr_B(c \geq M) = fp(B, x+1) - fp(B,x)|_{fp(B,x)=c} = M - M = 0$

$mr_{AB}(B,0) = fp(B,1) - fp(B,0) = \frac{1}{3} - 0 = \frac{1}{3}$

$mr_{AB}(B,1) = fp(B,2) - fp(B,1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$mr_{AB}(B,\frac{5}{3}) = fp(B,3) - fp(B,2) = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$

$mr_{AB}(B, c \leq M) = fp(B, x+1) - fp(B,x)|_{fp(x)=c} = \frac{x+1}{3} - \frac{x}{3} = \frac{1}{3}$

$mr_{AB}(B, c > M) = fp(B, x+1) - fp(B,x)|_{fp(x)=c} = M - M = 0$