CMSC 25300 / 35300
Homework 1: Vectors and Matrices

1. **Matrix multiplication.** *Settlers of Catan* is an awesome game. In this game, participants build roads, settlements, and cities by using resources such as wood, bricks, wheat, sheep, and ore. The number of resources required for each building project are reflected in Figure 1. (Ignore the text at the bottom about cities replacing already-built settlements.)

Figure 1: Building costs in Settlers of Catan

   a) Write the information about how many of what resources are required to build roads, settlements, cities, or development cards in a matrix. What do the rows represent? What do the columns represent?

   b) Departing from the game somewhat, suppose resources cost \$1 for each unit of wood, \$2 for brick, \$3 for sheep, \$5 for wheat, and \$8 for ore. Write this information in a vector. Write out a matrix-vector multiplication that calculates the total cost of buying roads, settlements, and cities. (Ignore the fact that a city had to replace a settlement.)

   c) Suppose you want to build a city, two settlements, and six road lengths connecting them. Again using matrix multiplication, find the total resources required to fill the order.

   d) Calculate the total cost for the order (using, you guessed it, matrix multiplication)

   e) Get up and running with either Matlab or Python. In your language of choice, write a script that computes the matrix multiplications in the previous parts of this problem.

2. Let

$$\boldsymbol{X} = \begin{bmatrix} - & \boldsymbol{x}_1^\top & - \\ - & \boldsymbol{x}_2^\top & - \\ & \vdots & \\ - & \boldsymbol{x}_n^\top & - \end{bmatrix} \in \mathbb{R}^{n \times p}$$

or $\boldsymbol{X}^\top = [\boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \cdots \ \boldsymbol{x}_n] \in \mathbb{R}^{p \times n}$, where $\boldsymbol{x}_i \in \mathbb{R}^p$ is the ith column of $\boldsymbol{X}^\top$. Consider the matrix

$$\boldsymbol{C} = \frac{\boldsymbol{X}^\top \boldsymbol{X}}{n} .$$

a) Express $\boldsymbol{C}$ as a sum of rank-1 matrices (i.e., columns of $\boldsymbol{X}^\top$ times rows of $\boldsymbol{X}$).

b) Assuming $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly independent, what is the rank of $\boldsymbol{C}$?

3. Let

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} .$$

a) What is the rank of $\boldsymbol{X}$?

b) What is the rank of $\boldsymbol{X}\boldsymbol{X}^\top$?

c) Find a largest set of linearly independent columns in $\boldsymbol{X}$.

4. Answer the following questions. Make sure to explain your reasoning.

a) Are the columns of the following matrix linearly independent?

$$\boldsymbol{X} = \begin{bmatrix} +0.92 & +0.92 \\ -0.92 & +0.92 \\ +0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$$

b) Are the columns of the following matrix linearly independent?

$$\boldsymbol{X} = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$$

c) Are the columns of the following matrix linearly independent?

$$\boldsymbol{X} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

d) What is the rank of the following matrix?

$$\boldsymbol{X} = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$$