PS 1: Mathemathical Foundation of ML

Juan Vila

October 14, 2019

Question 1

1. The matrix represent in row the different elements to build (Roads, Settlement, Cities and Development Cards), the columns instead are the necessary resources to build those structures(Limber, Wheath, Sheep, Iron and Bricks). The Catan input matrix is a matrix of 4x5.

$$CatanInputMatrix = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

2. In general the matrix multiplication needed for the calculating the cost of producing is:

$$Cost = Q * CatanInputMatrix * P$$

Where P is a vector of 5X1 with the following values:

$$P = \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

And Q is a vector of 1X4, which indicates the quantities of production of Roads, Settlement, Cities and Development Cards.

$$Q = \begin{pmatrix} nRoads & nSettlement & nCities & nDvCards \end{pmatrix}$$

Then if we replace the vectors into the cost function we get:

$$Cost = \begin{pmatrix} nRoads & nSettlement & nCities & nDvCards \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Let suppose that we want to build one of each one, then the total cost for each structure is:

$$Cost = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$Cost = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 34 \\ 16 \end{pmatrix}$$

If we would like to get the total cost we need to multiply by a matrix of 4x1 fill of one's the returning cost vector

$$TotalCost = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} * Cost$$

$$TotalCost = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 3 \\ 11 \\ 34 \\ 16 \end{pmatrix}$$

$$TotalCost = 3 + 11 + 34 + 16 = 64$$

3. Suppose you want to build a city, two settlements, and six road lengths connecting them. Again using matrix multiplication, find the total resources required to fill the order.
   The cost Q vector take values (6,2,1,0), then the cost calculation would look like:

$$Cost = \begin{pmatrix} 6 & 2 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$Cost = \begin{pmatrix} 6+2 & 2+2 & 2 & 3 & 6+2 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$Cost = \begin{pmatrix} 8 & 4 & 2 & 3 & 8 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

We need 10 unit of Limber, 4 of Weath, 2 of Sheeps, 3 Iron Ore and 8 Brics.

4. Continuing with the operation we can see that we have to multiplying by the price vector we get:

$$Cost = \begin{pmatrix} 8 & 4 & 2 & 3 & 8 \end{pmatrix} * \begin{pmatrix} 1 \\ 5 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

$$TotalCost = 8 + 20 + 6 + 24 + 16 = 74$$

5. import numpy as np
   def cost_calculator(prices, input_matrix, quantities):
   ”’
   Description:
   inputs:
   prices= vector of prices of the inputs quantities = vector of the desire quantities to produce
   input_matrix = matrix that specifiy the quantities of factors to produce one quantitie.
   output: total cost
   ”’
   p_by_matrix = np.dot(input_matrix, prices)
   total_cost = np.dot(q, p_by_matrix)
   rv =print("Total Cost equal to "+str(total_cost)+"\n"+"The price for one " +"Road is $"+str(p_by_matrix[0])+"\n"+"The price for one " +"Settlement is $"+str(p_by_matrix[1])+"\n"+"The price for one " +"City is $"+str(p_by_matrix[2])+"\n"+"The price for one " +"Dev. Card is $"+str(p_by_matrix[3]))
   return rv
   prod_matrix =[[1,0,0,0,1],[1,1,1,0,1],[0,2,0,3,0],[0,1,1,1,0]]
   p = [1,5,3,8,2]
   q = [1,1,1,1]
   cost_calculator(p,prod_matrix,q)
   I attach the jupyter notebook file at the end

Question 2

1. We can multiply X^tX using the outer product representation as $\sum_{i=1}^{n} x_i x_i^t$, then we can write $C = \frac{\sum_{i=1}^{n} x_i x_i^t}{n}$, where $x_i x_i^t$ is the multiplication of two matrix of dimension Px1 and 1xP, which have the less possible rank of 1.

2. As every x_i is Lineal independent, and the dimension of X^tX is PxP, then the rank(X^tX) should be the min(p,p) which is equal to p.

Question 3

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

1. The rank is at least 4, but as the 4th row is full of zero we know that the rank is 3. Another way to do it is to swicth the last row with third row and we have a row-echelon matrix with three pivot vectors. Also, we see

that $v_5 = v_2 + v_3$ and $v_4 = v_2 + v_3 - v_1$, then we can argue that the only LI vectors in the matrix X are $v_{1,}, v_2$ and v_3. Then, are three LI vectors then the rank is equal to three.

2. The result of $X^t X$ is :

$$X^t X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$$

With a simple examination, we can see that v_1 is equal to v_3 and $v_1 + v_4 = v_5$, then there is three LI vectors, then the $rank\,(X^t X) = 3$

3. With a simple examination we see that $v_5 = v_2 + v_3$ and $v_4 = v_2 + v_3 - v_1$, then we can argue that the only LI vectors in the matrix X are $v_{1,}, v_2$ and v_3.

Question 4

1. For calculating which vectors are LI, we need to show that there is a solution different to the trivial to the following system:

$$c_1 \begin{pmatrix} .92 \\ -.92 \\ .92 \\ -.92 \end{pmatrix} + c_2 \begin{pmatrix} .92 \\ .92 \\ -.92 \\ -.92 \end{pmatrix} = 0$$

$$\begin{pmatrix} .92 & .92 \\ -.92 & .92 \\ .92 & -.92 \\ -.92 & -.92 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{gathered} .92c_1 + .92c_2 = 0 \\ -.92c_1 + .92c_2 = 0 \\ .92c_1 - .92c_2 = 0 \\ -.92c_1 - .92c_2 = 0 \end{gathered}$$

Then we divide all equation by 0.92. We get the following result:

$$\begin{gathered} (1)c_1 + c_2 = 0 \\ (2) - c_1 + c_2 = 0 \\ (3)c_1 - c_2 = 0 \\ (4) - c_1 - c_2 = 0 \end{gathered}$$

If we add up eq.1 and eq.2 we get that$2c_2 = 0$ -> $c_2 = 0$, this imply that

$c_1 = 0$, then we have not found anoy other solution that not is the trivial one then both vectors are LI.

2. For calculating which vectors are LI, we need to show that there is a solution different to the trivial to the following system:

$$(1)a + b + c = 0$$
$$(2) - a + b - c = 0$$
$$(3)a - b - c = 0$$

if we add up 1 and 2, we get $2b = 0$, the replacing this result into 3, we get that $a = c$, then replacing this two results into 1, we get that $2a = 0$ -$>a = 0$,then $a = b = c = 0$.Then the only solution is the trivial one then the three vectors are LI.

3. We have to see which of the vectors of the following matrix are independent:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$$

Then we have to show that exist a different solution that the trivial one for the following system $Ax = 0$, where $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$:

$$(1)a + 2b + 2c = 0$$
$$(2)3a + 4b + 5c = 0$$
$$(3)5a + 6b + 8c = 0$$

From eq.2 we can solve for b and get the following result, $(4)b = -\frac{3a+5c}{4}$, then if we replace in eq.1 we get:

$$a - \frac{3a + 5c}{2} + 2c = 0$$
$$-\frac{a}{2} - \frac{c}{2} = 0$$
$$(5)a = -c$$

Now, if we replace eq5 into eq 4 we get that $(6)b = -\frac{1}{2}c$. Then if we replace 5 and 6 into the whole system for find a value for c, it clear the three equation meaning that we have find a non-trivial solution for the system, this imply that the columns are LD, and LI columns are $v_1, v_2 = \left\{ \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \right\}, \left\{ \begin{array}{c} 2 \\ 4 \\ 6 \end{array} \right\}$.

4. Now for find the rank of the following matrix:

$$\begin{bmatrix} 5 & 2 \\ -5 & 2 \\ 5 & -2 \end{bmatrix}$$

we can see that the should have a rank of at least 2 is matrix of 3 by 2. Now we have to see if there is some column that is LD with another one. For doing this we requiere to see if $a * v_1 = v_2$, this is no feasible because in row one, we get that $5a = 2$->$a = \frac{2}{5}$, then from row two we get that $a = -\frac{2}{5}$.This imply, that v_1 and v_2 are LI, then the rank is 2.