Chained Indices Unchained:

On the Welfare Foundations of Income Growth Measurement II

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February 2025

Abstract

KEYWORDS: Chained quantity indexes, GDP, equivalent variation, Divisia index, Fisher-ideal index and Fisher-Shell index.

JEL CLASSIFICATION NUMBERS: E01, O47, C43, O41.

1 Introduction

2 Structural Change Model

Description of technology. Let's follow Herrendorf et al. (2021) by assuming that there are two sectors, goods and services. Goods and services value added, denoted by Y_{gt} and Y_{st}, respectively, are produced according to the Cobb–Douglas production functions

$$Y_{jt} = A_{jt} K_{jt}^{\theta} L_{jt}^{1-\theta}, \quad j \in \{g, s\}.$$

These functions share the same capital intensity, represented by $\theta \in (0, 1)$, but may have different total factor productivities, A_{jt}. Production in each sector relies on capital K_{jt}, and labor L_{jt}, where $j \in \{g, s\}$ indexes the goods and services sectors, respectively.

Investment is produced using the CES technology

$$I_t = A_{xt} \left(\omega^{\frac{1}{\varepsilon}} X_{gt}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega)^{\frac{1}{\varepsilon}} X_{st}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

where X_{gt} and X_{st} represent goods and services inputs, respectively, ε is the elasticity of substitution between them, and $\omega \in (0, 1)$ determines the weight of sectorial inputs. The state of investment-specific technology is neutral with respect to inputs and represented by the investment specific productivity, A_{xt}.

A representative household has an initial capital stock $K_0 > 0$ and is endowed with one unit of time per period, which is supplied inelastically. Capital depreciates at a rate $\delta > 0$, following the law of motion:

$$\dot{K}_t = X_t - \delta K_t. \tag{1}$$

Both capital and labor can move freely between sectors. At equilibrium, the feasibility conditions impose the following constraints:

$$K_{gt} + K_{st} = K_t, \quad L_{gt} + L_{st} = L_t,$$

$$C_{gt} + X_{gt} = Y_{gt}, \quad C_{st} + X_{st} = Y_{st},$$

where C_{gt} and C_{st} are the consumption of goods and services, respectively. The labor force L_t grows at the exogenous rate $n > 0$.

Equilibrium prices. Let us adopt the investment good as numeraire. Following Herrendorf et al (2021), it is easy to show that the price of goods, P_{gt}, relative to the price of services, P_{st},

$$\frac{P_{gt}}{P_{st}} = \frac{A_{st}}{A_{gt}},$$

is equal to the inverse of the relative sectorial TFPs. Notice that a dupla of production factors (K, L) that produces $K^\theta L^{1-\theta} = 1$ has the same value in the goods and service sectors, since $P_{gt}A_{gt} = P_{st}A_{st}$.

Moreover, since we have adopted the investment good as numeraire, the prices of goods and services, relative to the investment good, read

$$P_{jt} = \frac{\mathcal{A}_t}{A_{jt}}, \quad j \in \{g, s\}. \tag{2}$$

where

$$\mathcal{A}_t = A_{xt} \left(\omega A_{gt}^{\varepsilon-1} + (1-\omega)A_{st}^{\varepsilon-1} \right)^{\frac{1}{\varepsilon-1}}. \tag{3}$$

As shown below, $\mathcal{A}_t$ is the productivity of the investment sector.

In the investment sector, at equilibrium, the ratio of expenditure shares and input quantities on goods and services are, respectively, given by

$$\frac{P_{gt}X_{gt}}{P_{st}X_{st}} = \frac{\omega}{1-\omega} \left(\frac{A_{st}}{A_{gt}} \right)^{1-\varepsilon} \quad \text{and} \quad \frac{X_{gt}}{X_{st}} = \frac{\omega}{1-\omega} \left(\frac{A_{gt}}{A_{st}} \right)^{\varepsilon}.$$

Notice that if goods and services are complementary in the investment technology, i.e. $\varepsilon \in (0, 1)$, and technical progress is faster in the production of goods, relative to services, then the value added of the goods sector shrinks while real production increases, relative to the service sector.

Aggregate investment technology. [We can drop this paragarph.] At equilibrium, from Lemma 1 in Herrendorf et al. (2021), the aggregate investment technology is

$$I_t = \mathcal{A}_t \left(\frac{X_{gt}}{A_{gt}} + \frac{X_{st}}{A_{st}} \right) = \mathcal{A}_t K_{xt}^\theta L_{xt}^{1-\theta} \tag{4}$$

where L_{xt} and K_{xt} are defined as

$$L_{xt} = \frac{X_{gt}}{A_{gt}K_t^\theta} + \frac{X_{st}}{A_{st}K_t^\theta}, \quad \text{and} \quad K_{xt} = L_{xt}K_t.$$

Let's think on $L_x \in (0, 1)$ as the fraction of employment and capital allocated to produce inputs for the investment sector.

Aggregate production technology. Let us define aggregate final output, measured in units of final investment, as:

$$Y_t = P_{gt}C_{gt} + P_{st}C_{st} + I_t. \tag{5}$$

It can be easily shown that the aggregate production technology is

$$Y_t = \mathcal{A}_t K_t^\theta L_t^{1-\theta}. \tag{6}$$

Aggregate dynamics. From the equations (5) and (6), for $t \geq 0$, given the exogenous path of $\mathcal{A}_{xt}$ and an initial stock of capital $K_0 > 0$, the $\{K_t\}$ law of motion reads

$$\dot{K}_t = \underbrace{\mathcal{A}_t K_t^\theta - E_t}_{I_t} - \delta K_t.$$

where

$$E_t = P_{gt}C_{gt} + P_{st}C_{st} \tag{7}$$

is consumption expenditure in units of the investment good.

Non-homothetic preferences. The economy features an infinitely lived representative household which preferences are represented by the following utility function

$$\int_0^\infty U(C_{gt}, C_{st}) \, e^{-\rho t} \, dt.$$

The discount rate is given by $\rho > 0$. The instantaneous utility function $U(\cdot, \cdot)$ is assumed to be a price-independent-generalized-linear indirect utility (PIGL henceforth). Since the PIGL class generally lacks a known direct utility representation, we work with its indirect utility formulation $v(E_t, P_{gt}, P_{st})$. Following Boppart (2014), let us assume

Assumption 1. *The instantaneous utility function $U(C_{gt}, C_{st})$ is PIGL with indirect utility representation*

$$V(E_t, P_{gt}, P_{st}) = \frac{1}{\chi}\left(\frac{E_t}{P_{st}}\right)^\chi - \frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^\gamma - \frac{1}{\chi} + \frac{\eta}{\gamma}, \tag{8}$$

where E_t is consumption expenditure, $\eta > 0$ and $1 > \gamma > \chi > 0$.

Intertemporal problem and intratemporal allocation. Under Assumption 1, the representative household choses a path $\{E_t, K_t\}$, for consumption expenditure and capital, that solves the following dynamic problem

$$v(K_t) = \max \int_0^\infty \frac{E_t^\chi}{\chi} \Gamma_t \, dt$$

subject to

$$\dot{K}_t = \mathcal{A}_t K_t^\theta - E_t - \delta K_t, \tag{9}$$

where the discount factor, $\Gamma_t = P_{st}^{-\chi} e^{-\rho t}$, is smaller than one and declining over time since P_{st}, as in the data, is assumed to be increasing over time. E_t is nothing else that our measure, in units of the investment good, of total consumption. Preferences are then CIES, with elasticity of substitution larger than one. All other terms in (8) are excluded as they are additive; their discounted integral remains a constant, having no impact on the determination of the optimal path.

The Euler equation associated to the household problem above is

$$\frac{\dot{E}_t}{E_t} = \frac{1}{1-\chi} \left(\theta \mathcal{A}_t K_t^{\theta-1} - \rho - \delta - \chi \frac{\dot{P}_{st}}{P_{st}} \right). \tag{10}$$

The equilibrium path solves then (9) and (10), given K_0.

The intratemporal allocation of E_t to C_{gt} and C_{st} results from the use of Roy's Identity to derive the expenditure share of goods:

$$\frac{P_{gt} C_{gt}}{E_t} = \eta \left(\frac{E_t}{P_{st}} \right)^{-\chi} \left(\frac{P_{gt}}{P_{st}} \right)^\gamma. \tag{11}$$

The last equation solves for C_{gt} and then C_{st} can be obtained inverting the definition of consumption expenditure E_t.

How compute the solution. In the following, to measure quantity indices, information on prices and quantities for consumption, investment and income will be required. The use of the investment good as numeraire, in this framework, is inconsequential. These are the steps to follow in order to measure the needed variables.

1. Assume that A_{gt}, A_{st} and $\mathcal{A}_t$ all three grow at different constant rates, with $\widehat{A}_g > \widehat{A}_s$. In the data, $\widehat{A}_g$ is very close to $\widehat{\mathcal{A}}$.

2. Use equations (2) and (3) to solve for P_{gt} and P_{st}.

3. Solve the dynamic system (9) and (10) to compute E_t and K_t, and the value function $v(K_t)$.

4. Use (7) and (11) to solve for C_{gt} and C_{st}

5. Use (5) and (6) to solve for I_t and Y_t.

Bellman representation. Following Duran and Licandro (2025), the Bellman representation of the representative household preferences is

$$W(C_{gt}, C_{st}, X_t; \nu_t) = U(C_{gt}, C_{st}) + \nu_t X_t, \tag{12}$$

where $X_t = \dot{K}_t$ is net investment and ν_t is the marginal value of capital at time t. In the Bellman representation, preferences at t are indexed by the marginal value of capital ν_t. However, it's very important to stress that Y_t is not real expenditure but nominal expenditure, and as such it will be our measure of nominal GDP. The representative household maximises (12) with respect to $\{C_g, C_s, X\}$, subject to the budget constraint

$$P_{gt}C_{gt} + P_{st}C_{st} + X_t = M_t,$$

where M_t is current net income. Notice that consumption expenditure $E_t = M_t - X_t$.

Proposition 1. *The indirect utility and expenditure functions associated to the Bellman representation of preferences in (12) are*

$$u(M_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t\left(M_t - (\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}\right) \tag{13}$$

$$e(W_t, P_{gt}, P_{st}; \nu_t) = (\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}} + \frac{W_t}{\nu_t} - \frac{V\left((\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right)}{\nu_t}. \tag{14}$$

PROOF: Since by definition of an indirect utility function

$$\max_{\{C_{gt}, C_{st}\}: P_{gt}C_{gt} + P_{st}C_{st} = E_t} U(C_{gt}, C_{st}) = V(E_t, P_{gt}, P_{st}),$$

optimal net investment is

$$X_t = \arg\max_X V(M_t - X, P_{gt}, P_{st}) + \nu_t X = M_t - (\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}.$$

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The indirect utility function associated to the Bellman representation of preferences is

$$u(M_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t\left(M_t - (\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}\right)$$

and the expenditure function is

$$e(W_t, P_{gt}, P_{st}; \nu_t) = \frac{W_t - V\left((\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right)}{\nu_t} + (\nu_t P_{st}^\chi)^{\frac{1}{\chi-1}}. \quad \square$$

Equivalent variation measures. Based on the Fisher and Shell principle that welfare comparisons must be done using the same preference set, let us define the hypothetical income at z, for $z < t$,

$$\widehat{M}_{tz} = e\left(u\left(M_z, P_{gz}, P_{sz}; \nu_t\right), P_{gt}, P_{st}; \nu_t\right). \tag{15}$$

$\widehat{M}_{tz}$ is the level of income at current prices that the representative household would have needed at time z to attain the utility achievable with past income and prices but evaluated using the current Bellman representation of preferences.

Fixed-base indices. Let us adopt the following convention for an economy where an Office for National Statistics have recorded National Accounts data from some initial time t_0 to the current time t. In this economy, a current-base equivalent variation index, for $s \in \{t_0, t\}$, is

$$\mathcal{P}_{t,s} = \widehat{m}_{t,s} - \widehat{m}_{t,t_0}, \tag{16}$$

where $\widehat{m}_{t,z} = \log \widehat{M}_{tz}$. Notice that the index is normalized to $\mathcal{P}_{t,t_0} = 0$, such that $\mathcal{P}_{t,t} = m_t - \widehat{m}_{t,t_0}$ measures welfare gains from the initial time t_0 to the current time t, and $\mathcal{P}_{t,t} - \mathcal{P}_{t,s} = m_t - \widehat{m}_{t,s}$ welfare gains from any time $s \in (t_0, t)$ to t. Implicit on this index, the equilibrium instantaneous growth rate of the economy at s is measured by

$$\frac{\partial \mathcal{P}_{t,s}}{\partial s} = \frac{\partial \widehat{m}_{t,s}}{\partial s}. \tag{17}$$

As we show below, the instantaneous growth rate at t of the current-base equivalent variation index, $\frac{\partial \mathcal{P}_{t,s}}{\partial s}|_{s=t}$, is equal to the Divisia index at t. For any $s < t$, the instantaneous growth rate is lower than the Divisia index and declines as the welfare evaluation refers to a more distant point in the past. (prove it)

Chained Divisia. Following Duran and Licandro (2025),

Parameter values. Table 1 shows the parameters values used by Herrendorf et al. (2021) in their quantitative exercise. In the measurement of sectoral total factor productivity (TFP) they have followed a structured approach based on observable economic data. First, sector-specific TFP (A_{jt}), $j = \{g, s\}$, was estimated using data from WORLD KLEMS on real value added, capital and labor inputs. Given the aggregate capital share θ, they computed the sectoral TFP growth rates using

$$\widehat{A}_{jt} = \widehat{Y}_{jt} - \theta \widehat{K}_{jt} - (1 - \theta)\widehat{L}_{jt}, \quad j \in \{g, s\},$$

where $\widehat{x}$ measures the growth rate of variable x. All variables except $\widehat{A}_{jt}$ are directly observable. Normalizing initial TFP levels to $A_{j0} = 1$, they use the estimated growth rates to construct the time series for A_{jt}.

Next, they estimated aggregate investment-specific TFP (A_{xt}). Again, setting the initial condition $A_{x0} = 1$, they computed its growth rates using

$$\widehat{A}_{xt} = \widehat{X}_t - \frac{P_{gt}X_{gt}}{X_t}\widehat{X}_{gt} - \frac{P_{st}X_{st}}{X_t}\widehat{X}_{st},$$

where all components at the right-hand-side of this equation are observable.

Parameter	Value
Preferences	
ρ (discount rate)	0.04
χ	0.55
η	0.44
γ	0.69
Technology	
θ (capital share)	1/3
δ (depreciation rate)	0.08
ω	0.65
ε	0.00

Table 1: Herrendorf et al. (2021) calibration

A ABGP

Let us assume that at the ABGP, $A_{gt} = A_{g0} \, \mathrm{e}^{\gamma_g t}$, $A_{st} = A_{s0} \, \mathrm{e}^{\gamma_s t}$, and $\mathcal{A}_t = \mathcal{A}_0 \, \mathrm{e}^{\gamma_\mathcal{A} t}$, $\gamma_x > \gamma_g > \gamma_s$. Consequently, the growth rate of P_{gt} and P_{st} are g_{P_g} and g_{P_s}, respectively, with $0 < g_{P_g} = \gamma_x - \gamma_g < \gamma_x - \gamma_s = g_{P_s}$.

At the ABGP, from the Euler equation (10), the stock of capital follows

$$K_t^* = k^{* \frac{1}{\theta-1}} \mathcal{A}_t^{\frac{1}{1-\theta}} \quad \text{where} \quad k^* = \frac{\rho + \delta + \chi g_{P_s} + (1-\chi)g_E}{\theta} \tag{18}$$

is the user cost of capital divided by θ. From (9), consumption expenditure follows

$$E_t^* = (k^* - \delta - g_K)K_t^*. \tag{19}$$

From (6), gross income is

$$Y_t^* = \mathcal{A}_t K_t^{*\theta} = k^* K_t^*. \tag{20}$$

Notice that the consumption share of gross income is

$$\frac{E_t^*}{Y_t^*} = \frac{k^* - \delta - g_K}{k^*}.$$

All three variables, K_t, E_t and Y_t are measure in units of the investment good and grow at the rate $g_K = \frac{\gamma_\mathcal{A}}{1-\theta}$.

Since from the primal problem of the household

$$\max V(E, P_{gt}, P_{st}) + \nu_t X \quad \text{subject to} \quad E + X = M_t.$$

From the FOC for E

$$\nu_t^* = E_t^{*\,1-\chi} P_{st}^\chi.$$

We have then all information required to compute the current-base equivalent variation measure in (16).

From (11), real consumption on goods is

$$C_{gt} = \eta \left(\frac{E_t}{P_{st}}\right)^{1-\chi} \left(\frac{P_{gt}}{P_{st}}\right)^{\gamma-1}, \tag{21}$$

which grows at the constant rate $g_g = (1-\chi)g_k + (\gamma-1)g_{P_g} + (\chi-\gamma)g_{P_s}$.

Divisia index. The Divisia index

$$g_t^D = s_e\big(s_{gt}g_g + (1 - s_{gt})g_s\big) + (1 - s_e)g_x,$$

where the shares are $s_e \doteq \frac{E_t}{M_t} = \frac{k^* - \delta - g_K}{k^*}$ and, from (11) $s_{gt} = \eta\left(\frac{E_t}{P_{st}}\right)^{-\chi}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}$. The growth rate of investment is $g_x = g_K$. We are only missing the growth rate of consumption services. Notice that, from (11), real consumption services are

$$C_{st} = \frac{E_t}{P_{st}} - \eta\left(\frac{E_t}{P_{st}}\right)^{1-\chi}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma},$$

Fisher-Shell index. In order to show the Fisher-Shell index is equal to the Divisia index, we must compute the total derivative of $\widehat{M}_{tz}$ with respect to z and evaluate it at $z = t$. From the main text,

$$\widehat{M}_{tz} = e\Big(u\big(M_z, P_{gz}, P_{sz}; \nu_t\big), P_{gt}, P_{st}; \nu_t\Big), \tag{15}$$

with

$$u(M_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t\left(M_t - (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}\right), \tag{13}$$

$$e(W_t, P_{gt}, P_{st}; \nu_t) = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} + \frac{W_t}{\nu_t} - \frac{V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right)}{\nu_t}, \tag{14}$$

and

$$V(E_t, P_{gt}, P_{st}) = \frac{1}{\chi}\left(\frac{E_t}{P_{st}}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}. \tag{8}$$

Combining them, we get

$$\widehat{M}_{tz} = \frac{1-\chi}{\chi}(\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} - \frac{1}{\nu_t}\frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma} + M_z + \text{other terms that don't depend on } z$$

In order to define the Fisher-Shell index, we take time derivatives with respect to z and evaluate them at $z = t$, such that,

$$g_t^{\text{FS}} \doteq \left.\frac{\mathrm{d}\widehat{M}_{tz}}{\mathrm{d}z}\right|_{z=t}\frac{1}{M_t} = \frac{\dot{M}_t}{M_t} - \frac{E_t}{M_t}g_{P_{st}} - \frac{\eta}{\nu_t M_t}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}\big(g_{P_g t} - g_{P_s t}\big).$$

Since

$$\frac{\dot{M}_t}{M_t} = s_{et}\, g_{Et} + (1 - s_{et})g_{Xt},$$

where $s_e = E/M$, and

$$g_{Et} = s_{gt}\big(g_{gt} + g_{P_g t}\big) + s_{st}\big(g_{st} + g_{P_s t}\big),$$

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the Fisher-Shell index becomes

$$g_t^{\text{FS}} = g_t^D + \frac{P_{gt}C_{gt}}{M_t}\left(g_{P_gt} - g_{P_gt}\right) - \frac{\eta}{\nu_t M_t}\left(\frac{P_{gt}}{P_{st}}\right)^\gamma \left(g_{P_gt} - g_{P_st}\right).$$

From (11),

$$\frac{P_{gt}C_{gt}}{E_t} = \eta \left(\frac{E_t}{P_{st}}\right)^{-\chi}\left(\frac{P_{gt}}{P_{st}}\right)^\gamma.$$

Since at equilibrium $E_t^{\chi-1} = \nu_t P_{st}^\chi$, we can easily show that

$$g_t^{\text{FS}} = g_t^D.$$