Chained Indices Unchained:

On the Welfare Foundations of Income Growth Measurement

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June 2025

Abstract

This paper studies the welfare foundations of chained quantity indices used in national accounts to measure GDP growth. We examine how alternative methods of intertemporal welfare evaluation perform within a structural transformation framework, where preferences are non-homothetic, evolve over time, and aggregate dynamics converge to an Aggregate Balanced Growth Path (ABGP). We show that evaluating welfare gains through a sequence of local, preference-consistent comparisons across contiguous periods yields a chained index that aligns with the ABGP and provides a welfare-based measure of income growth. In contrast, fixed-base approaches, which assess historical gains from the standpoint of current preferences, are prone to substitution bias and generate significant revisions as preferences evolve. After calibrating the structural transformation model for the U.S. economy, we show that chained indices better capture the welfare-relevant dimensions of economic growth and provide a more robust basis for measuring real income dynamics in the presence of structural transformation.

KEYWORDS: Structural transformation, Investment specific technical change, Chained quantity indexes, GDP measurement, Equivalent variation, Divisia index, Fisher-ideal index and Fisher-Shell index.

JEL CLASSIFICATION NUMBERS: C43, E01, E13, O11, O14, O41, O47.

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*The author expresses gratitude to Ariel Burstein for a highly beneficial discussion held during a visit to UCLA in April 2024.

1 Introduction

In the 1990s, the Bureau of Economic Analysis (BEA) implemented a fundamental change in its approach to measuring GDP growth, replacing fixed-base Laspeyres indices with chained Fisher-ideal indices. This shift was driven by the tendency of standard fixed base methods to produce systematic reductions in historical growth rates after base-year revisions.[1] The urgency of this issue increased in the 1980s as durable goods' prices, especially in the computer sector, declined sharply due to substantial quality improvements.

The BEA began to adjust its GDP measurement framework in the mid-1980s by introducing quality adjustments to price indices, particularly for high-tech durable goods. While this improved the accuracy of real GDP estimates, it highlighted the limitations of fixed-base indices, necessitating more frequent base-year revisions to maintain the reliability of the measures. In 1996, the need for a more flexible, continuously updated index led the BEA to adopt the chained Fisher-ideal index for measuring GDP growth. By updating weights continuously, this index not only accounts for shifts in relative prices but also reduces substitution bias, reflecting structural changes in the economy and aligning growth measurement with current economic conditions without requiring periodic base-year revisions.[2]

While the chained Fisher-ideal index offers practical advantages in accuracy and stability, its welfare foundations have yet to be fully understood. Central to this discussion is the question of whether GDP growth measured through chained indices can accurately reflect changes in welfare over time, a critical aspect for assessing the true impact of economic growth on society's well-being. This paper aims to answer this question from the perspective of the economic theory of index numbers.[3]

This paper investigates the welfare implications of chained indices for the measure-

[1] See Parker and Triplett (1996) for a discussion of how fixed base indices misrepresent economic growth, understating it before the base year and overstating it thereafter.

[2] See Whelan (2002).

[3] For some foundational papers on the literature, see Fisher (1922), Fisher and Shell (1968), Diewert (1976), Diewert (1978), and Caves et al. (1982).

ment of income growth in models of structural transformation. Explain here the main issues of the structural transformation literature, the importance of a welfare-based measure of aggregate income growth, and the role of investment as pointed out by HRV.

This paper focuses on the Divisia index, which is well approximated in practice by a Fisher-ideal quantity index, and contrast two recent theoretical frameworks that provide differing welfare interpretations for chained indices used in national accounts. Both approaches, those of Baqaee and Burstein (2023) and Durán and Licandro (2025), utilize the Fisher-Shell principle as a theoretical foundation, but with key differences in their application.[4].

Index number theory, the theoretical foundation for GDP measurement, relies on stable preferences. However, when preferences evolve over time, as it is the case of the type of non-homothetic preferences used in the structural transformation literature, according to the *Fisher-Shell Principle* suggested by Fisher and Shell (1968), welfare comparisons should adopt a consistent preference order. Following this principle, welfare comparisons between two different moments in time require a common preference order, or *reference order*, to accurately capture the welfare gains or losses associated with changes in income and expenditure. Determining how this principle should be applied in practice —particularly in the context of evolving preferences and rapid changes in price structures— still presents a theoretical and methodological challenge.

Baqaee and Burstein (2023) propose an equivalent variation measure of welfare gains, using the preferences of the current period as a reference order to evaluate the entire history of income growth. Their approach requires stability and homotheticity of preferences to ensure that the chained index accurately reflects welfare gains. Any deviation from these conditions makes the chained Divisia index inconsistent with their welfare measure.

Durán and Licandro (2025), by contrast, present an alternative approach that applies the Fisher-Shell principle in a dynamic context by chaining welfare gains between contiguous periods. They refer to it as the chained Fisher-Shell quantity index. A salient advantage of this method is that it does not need the stability and homotheticity assumptions required by Baqaee and Burstein (2023). Instead, in the framework of

[4]See also the seminal paper by Licandro et al. (2002)

continuous-time dynamic general equilibrium models, Durán and Licandro (2025) show that a Fisher-Shell true quantity index aligns with the Divisia index. Then, chained Divisia indices continuously adapt to changes in preferences, income, and prices, offering a flexible welfare-based measure of income growth. Under this approach, at any moment in time, welfare gains are measured using the preferences relevant to that specific time and then chained.

The result in Durán and Licandro (2025) does not invalidate the measure proposed by Baqaee and Burstein (2023); rather, it highlights that chained Divisia indices are also welfare-based. Thus, it illustrates the broader principle that multiple true quantity indices can capture welfare gains in the same context. In light of this result, Proposition 1 in Baqaee and Burstein (2023) can be understood to demonstrate that the chained Divisia index and the Baqaee-Burstein equivalent variation measure, being alternative measures of welfare gains aiming to answer different questions, converge to each other only under homothetic and stable preferences.

To illustrate the implications of these contrasting approaches, we quantitatively evaluate an endogenous growth model with learning-by-doing and embodied technical change. The model captures the sustained decline in the price of investment goods relative to consumption goods, a trend that has been a notable feature of the U.S. economy for several decades. In this model, we compare the Baqaee and Burstein equivalent variation measure (BBEV) and the chained Fisher-Shell index. Our analysis reveals that the BBEV amplifies the shortcomings of traditional fixed-base indices, exhibiting an even greater substitution bias as the relative prices of investment goods decrease over time. This finding suggests that the fixed-base approach, like a standard Paasche index, systematically understates past welfare gains. The chained Fisher-Shell index, in contrast, maintains stability and provides a time-invariant measure of welfare gains that aligns with the balanced growth path of the model. By continuously updating the reference order, the chained index avoids the need for frequent revisions while capturing welfare-relevant growth, reflecting the stationary nature of the dynamic economy.

The findings in this paper highlight the limitations of fixed-base indices, particularly in the context of modern economies where relative prices and consumption patterns are in constant flux. The substitution bias inherent in fixed-base indices may distort long-term growth measurements, making chained indices, such as the Fisher-ideal quantity index, a more reliable tool for capturing welfare-relevant growth. Our results suggest that the chained Fisher-Shell index provides a consistent welfare-based measure that can accommodate evolving economic conditions without imposing rigid assumptions on the stability or homotheticity of preferences. This has important implications for national accounts, as it supports the use of chained indices as a more accurate reflection of welfare gains over time, aligning growth measurement with the economic realities of today's rapidly changing economies.

In sum, this paper contributes to the literature on welfare-based growth measurement by clarifying the theoretical foundations of chained indices and evaluating their practical implications within a general equilibrium framework. By comparing two distinct interpretations of the Fisher-Shell principle, we provide a deeper understanding of the welfare properties of chained indices and offer a rationale for their adoption in national accounts. Our findings support the view that chained indices offer a flexible, welfare-consistent measure that can adapt to evolving preferences and economic conditions, thereby providing an accurate measure of welfare-relevant growth.

2 Structural Change Model

To compare the behavior of fixed-base Fisher-Shell indices with that of the chained Fisher-Shell index, this section adopts the structural transformation model developed by Herrendorf et al. (2021). Hereafter, we refer to it as the HRV structural transformation model. The model features three sectors: goods, services, and investment, with non-homothetic preferences defined over consumption of goods and services. Both goods and services are also used in the production of investment, which is governed by a CES technology. This framework provides a rich environment for evaluating how fixed-base and chained true quantity indices capture welfare-relevant income growth along a balanced growth path with non-homothetic preferences and time-varying income elasticities and elasticities of substitution.

Description of technology. We assume that goods and services are produced by distinct sectors. Value added in each sector is generated using Cobb–Douglas production technologies

$$Y_{j,t} = A_{j,t} K_{j,t}^{\theta} L_{j,t}^{1-\theta}, \tag{1}$$

where j, $j \in \{g, s\}$, indexes the goods and services sectors, respectively. These production functions share a common capital intensity parameter θ, $\theta \in (0, 1)$, but differ in total factor productivity, denoted $A_{j,t}$. Each sector employs the same homogeneous production factors, capital $K_{j,t}$ and labor $L_{j,t}$, which are freely mobile across sectors. It follows that

$$K_{g,t} + K_{s,t} = K_t \quad \text{and} \quad L_{g,t} + L_{s,t} = L_t,$$

where K_t and L_t denote the economy's total capital and labor endowments, respectively.

Investment is produced using a CES technology

$$I_t = A_{x,t} \left(\omega^{\frac{1}{\varepsilon}} X_{g,t}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega)^{\frac{1}{\varepsilon}} X_{s,t}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

where $X_{g,t}$ and $X_{s,t}$ denote inputs from the goods and services sectors, respectively. The parameter ε, $\varepsilon > 0$, governs the elasticity of substitution between these inputs, and ω, $\omega \in (0, 1)$, captures their relative weight in the production of investment. The term $A_{x,t}$ represents investment-specific productivity, which is neutral with respect to input composition.

Capital depreciates at a constant rate $\delta > 0$ and evolves according to the standard law of motion

$$\dot{K}_t = I_t - \delta K_t. \tag{2}$$

In equilibrium, efficiency requires that output in each sector be allocated between consumption and investment inputs, s.t.,

$$C_{g,t} + X_{g,t} = Y_{g,t} \quad \text{and} \quad C_{s,t} + X_{s,t} = Y_{s,t},$$

where $C_{g,t}$ and $C_{s,t}$ denote consumption of goods and services, respectively.

Equilibrium prices. It is straightforward to show that the relative price of goods, $P_{g,t}$, to services, $P_{s,t}$, satisfies

$$\frac{P_{g,t}}{P_{s,t}} = \frac{A_{s,t}}{A_{g,t}},$$

that is, it equals the inverse of the relative sectoral TFPs. Note that a bundle of production factors (K, L) such that $K^{\theta} L^{1-\theta} = 1$ has the same value in both sectors, since $P_{g,t} A_{g,t} = P_{s,t} A_{s,t}$.

We adopt the investment good as the numeraire. It is easy to see that, at equilibrium, the prices of goods and services, relative to the prices of investment, are given by

$$P_{j,t} = \frac{\mathcal{A}_t}{A_{j,t}}, \quad j \in \{g, s\}, \tag{3}$$

where

$$\mathcal{A}_t = A_{x,t} \left(\omega A_{g,t}^{\varepsilon-1} + (1-\omega) A_{s,t}^{\varepsilon-1} \right)^{\frac{1}{\varepsilon-1}} . \tag{4}$$

As shown below, $\mathcal{A}_t$ corresponds to total factor productivity in the investment sector.

At equilibrium in the investment sector, the ratios of expenditure shares and input quantities on goods and services are given by

$$\frac{P_{g,t} X_{g,t}}{P_{s,t} X_{s,t}} = \frac{\omega}{1-\omega} \left(\frac{A_{s,t}}{A_{g,t}} \right)^{1-\varepsilon} \quad \text{and} \quad \frac{X_{g,t}}{X_{s,t}} = \frac{\omega}{1-\omega} \left(\frac{A_{g,t}}{A_{s,t}} \right)^{\varepsilon} .$$

In what follows, and consistently with the data, we impose the following assumption on sectoral TFP:

Assumption 1. *Total factor productivity evolves according to $A_{g,t} = A_{g,0}\, e^{\gamma_g t}$, $A_{s,t} = A_{s,0}\, e^{\gamma_s t}$, and $\mathcal{A}_t = \mathcal{A}_0\, e^{\gamma_{\mathcal{A}} t}$, where the growth rates satisfy $\gamma_{\mathcal{A}} > \gamma_g > \gamma_s$.*[5]

As a consequence, the growth rates of $P_{g,t}$ and $P_{s,t}$, denoted respectively g_{P_g} and g_{P_s}, are given by

$$0 < g_{P_g} = \gamma_{\mathcal{A}} - \gamma_g < \gamma_{\mathcal{A}} - \gamma_s = g_{P_s}.$$

Under Assumption 1, the model delivers the prediction, consistent with the data, that consumption service prices grow faster than consumption goods prices, and that the overall consumption price index increases more rapidly than the price of investment goods.

Aggregate production technology. Like in National Accounts, we define aggregate final output, measured in units of the investment good, as

$$Y_t = P_{g,t} C_{g,t} + P_{s,t} C_{s,t} + I_t. \tag{5}$$

[5] In equation (4), $A_{x,t}$ adjusts to be consistent with this assumption.

Herrendorf et al. (2021) show that at equilibrium the aggregate production function is

$$Y_t = \mathcal{A}_t K_t^{\theta} L_t^{1-\theta}, \tag{6}$$

where $\mathcal{A}_t$ is the investment-sector productivity index defined in equation (4). Notice, however, that Y_t is arbitrarily measured in units of the investment good.

Aggregate dynamics. Combining equations (2), (5), and (6), the law of motion for capital, for $t \geq 0$, given the exogenous path of $\mathcal{A}_t$ and an initial capital stock $K_0 > 0$, can be written as

$$\dot{K}_t = \underbrace{\mathcal{A}_t K_t^{\theta} L_t^{1-\theta} - E_t}_{I_t} - \delta K_t,$$

where

$$E_t = P_{g,t} C_{g,t} + P_{s,t} C_{s,t} \tag{7}$$

denotes total consumption expenditure, measured in units of the investment good.

Non-homothetic preferences. Let population be denoted by N_t, growing at a constant rate n, $n > 0$. At each point in time t, every individual supplies h_t units of human capital, which grows exogenously at rate γ_h, $\gamma_h > 0$. Total labor supplied is thus $L_t = h_t N_t$, and is offered inelastically.

The economy features an infinitely lived representative household whose preferences are represented by the intertemporal utility function

$$\int_0^{\infty} U(c_{g,t}, c_{s,t})\, \mathrm{e}^{(n-\rho)t}\, \mathrm{d}t, \tag{8}$$

where $\rho > n$ is the subjective discount rate and $U(\cdot,\cdot)$ is per capita utility. The instantaneous utility function $U(\cdot,\cdot)$ is assumed to belong to the price-independent generalised linear (PIGL) class. It depends on per capita consumption of goods and services, defined as $c_{g,t} = C_{g,t}/N_t$ and $c_{s,t} = C_{s,t}/N_t$, respectively.

Since the PIGL class generally lacks a closed-form direct utility representation, we work with its indirect utility form $V(e_t, P_{g,t}, P_{s,t})$, where $e_t = E_t/N_t$ denotes per capita consumption expenditure. Following Boppart (2014), we make the following assumption

Assumption 2. *The instantaneous utility function $U(c_{g,t}, c_{s,t})$ belongs to the PIGL class and has the indirect utility representation*

$$V(e_t, P_{g,t}, P_{s,t}) = \frac{1}{\chi}\left(\frac{e_t}{P_{s,t}}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}, \tag{9}$$

where $\eta > 0$ and $1 > \gamma \geq \chi \geq 0$.

Under this specification, the relevant expenditure measure is in terms of services and preferences are non-homothetic if $\chi \neq 0$. These preferences also feature the two traditional mechanisms shaping structural change: income and price effects. To understand how parameter values govern the direction and intensity of these affects, we derive the expenditure share on goods using Roy's identity, which is given by

$$\frac{P_{g,t}c_{g,t}}{e_t} = \eta\left(\frac{e_t}{P_{s,t}}\right)^{-\chi}\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma}. \tag{10}$$

Notice that χ regulates the income effect, with $\chi > 0$ implying that goods are necessities and services a luxury. Furthermore, under $\gamma > 0$, goods and services are complements, the price effect being stronger the higher value of γ.

Intertemporal problem and intratemporal allocation. Under Assumption 2, the representative household chooses a path $\{e_t, k_t\}$ for per capita consumption expenditure and capital that solves the dynamic program

$$v(k_t) = \max_{\{e_t,k_t\}} \int_0^{\infty} \frac{e_t^{\chi}}{\chi}\,\Gamma_t\,dt,$$

subject to the law of motion for capital per capita

$$\dot{k}_t = \widehat{\mathcal{A}}_t k_t^{\theta} - e_t - (\delta + n)k_t, \tag{11}$$

where $k_t = K_t/N_t$ and $\widehat{\mathcal{A}}_t = \mathcal{A}_t h_t^{1-\theta}$. The discount factor is $\Gamma_t = P_{s,t}^{-\chi}\mathrm{e}^{(n-\rho)t}$, which from Assumption 1 declines over time since $P_{s,t}$ is increasing like in the data. Preferences are constant intertemporal elasticity of substitution (CIES) with respect to consumption expenditure.[6] The remaining terms in equation (9) are excluded from the objective

[6] In this framework, the intertemporal elasticity of substitution is $1/(1-\chi)$.

function, as they are additive and their discounted integral is independent of the control and state variables; thus, they do not affect the optimal path.

The Euler equation characterising the solution to the household's problem is[7]

$$\frac{\dot{e}_t}{e_t} = \frac{1}{1-\chi}\left(\theta \widehat{\mathcal{A}}_t k_t^{\theta-1} - \rho - \delta - \chi\, g_{P_s})\right). \tag{12}$$

An equilibrium path for $\{e_t, k_t\}$ is given by the system formed by equations (11) and (12), given the initial condition k_0, $k_0 > 0$, and a standard transversality condition.

The intratemporal allocation of expenditure between goods and services can be characterized by equation (10). Given this expression, $c_{g,t}$ can be solved for directly, and $c_{s,t}$ follows by inverting the identity $e_t = P_{g,t}c_{g,t} + P_{s,t}c_{s,t}$.

Aggregate Balanced Growth Path (ABGP). Along the aggregate balanced growth path (ABGP), the per capita variables $\{k_t, e_t, y_t\}$ grow at the constant rate

$$g_k = \frac{\gamma_{\mathcal{A}}}{1-\theta} + \gamma_h.$$

From the Euler equation (12), capital per capita evolves according to

$$k_t^* = \kappa^{\frac{1}{\theta-1}} \widehat{\mathcal{A}}_t^{\frac{1}{1-\theta}}, \qquad \text{where} \qquad \kappa \doteq \frac{\rho + \delta + \chi g_{P_s} + (1-\chi) g_k}{\theta}. \tag{13}$$

Substituting into equation (11), the path of per capita consumption expenditure satisfies

$$e_t^* = (\kappa - \delta - n - g_k) k_t^*. \tag{14}$$

From the aggregate production function (6), gross and net nominal income per capita are given by

$$y_t^* = \widehat{\mathcal{A}}_t {k_t^*}^{\theta} = \kappa k_t^* \quad \text{and} \quad m_t^* = (\kappa - \delta) k_t^*, \tag{15}$$

where the second equality in the first equation follows directly from equation (13). The consumption shares of gross and net income, respectively, are

$$\frac{e_t^*}{y_t^*} = \frac{\kappa - \delta - n - g_k}{\kappa} \quad \text{and} \quad \frac{e_t^*}{m_t^*} = \frac{\kappa - \delta - n - g_k}{\kappa - \delta}.$$

[7] See Lemma 4 in Boppart (2014).

On the Divisia index. In the following section, we will discuss the role of the Divisia index in measuring income growth and welfare gains. Before that, we now compute the Divisia index at the ABGP of the structural transformation economy.

At the ABGP, the Divisia index for gross and net domestic product per capita is defined as follows:

$$g_t^D = s_e \underbrace{(s_{g,t} g_g + (1 - s_{g,t}) g_{s,t})}_{\widehat{g}_{e,t}} + (1 - s_e) g_x, \tag{16}$$

where $\widehat{g}_{e,t}$ is the growth rate of real consumption expenditure per capita, and $g_x = g_k$ is the growth rate of per capita investment. The shares of consumption expenditure in gross and net income are given by $s_e = \left\{ \frac{e_t^*}{y_t^*}, \frac{e_t^*}{m_t^*} \right\}$, respectively. Note that $\widehat{g}_{e,t}$ differs from the growth rate of consumption expenditure measured in units of the investment good, $g_e = g_k$.

From (10), the growth rate of goods consumption is

$$g_g = (1 - \chi) \frac{\theta}{1 - \theta} \gamma_{\mathcal{A}} + (1 - \chi) \gamma_h + (1 - \gamma) \gamma_g + (\gamma - \chi) \gamma_s > 0.$$

Along the ABGP, goods consumption directly benefit from sectoral technical progress, γ_g, human capital accumulation, γ_h, and embodied technical progress, $\frac{\theta}{1-\theta}\gamma_{\mathcal{A}}$, but suffers from the indirect effect of consumption shifting to services, as represented by all terms related to χ and γ.

The share of goods in total consumption expenditure is, from equation (10),

$$s_{g,t} = \eta \left(\frac{e_t}{P_{s,t}} \right)^{-\chi} \left(\frac{P_{g,t}}{P_{s,t}} \right)^{\gamma}.$$

It is easy to see that $s_{g,t}$ is decreasing, converging to zero as time goes to infinity.

Finally, to derive the growth rate $g_{s,t}$, observe from equation (10) that real consumption services satisfy

$$c_{s,t} = \frac{e_t}{P_{s,t}} - \eta \left(\frac{e_t}{P_{s,t}} \right)^{1-\chi} \left(\frac{P_{g,t}}{P_{s,t}} \right)^{1+\gamma} = \left(1 - s_{g,t} \frac{P_{g,t}}{P_{s,t}} \right) \frac{e_t}{P_{s,t}}.$$

The growth rate $g_{s,t} \doteq \frac{\dot{c}_{s,t}}{c_{s,t}}$ can be computed from this expression. Notice that as time goes to infinity, $g_{s,t}$ converges from above to $g_k - g_{P_s} = \frac{\theta}{1-\theta}\gamma_{\mathcal{A}} + \gamma_h + \gamma_s$. The consumption of services benefits not only from direct gains in sectoral TFP, as given by γ_s, but also from gains in human capital, γ_h, and investment-specific technical progress, as given by $\frac{\theta}{1-\theta}\gamma_{\mathcal{A}}$.

Notice that real consumption of services is systematically growing faster than real consumption of goods, since the difference $g_{s,t} - g_c$ converges from above to

$$\lim_{t\to\infty} \left(g_{s,t} - g_c\right) = \chi \left(\frac{\theta}{1-\theta}\gamma_{\mathcal{A}} + \gamma_h + \gamma_s\right) + \gamma\left(\gamma_g - \gamma_s\right) > 0.$$

3 Index Number Theory and GDP Growth

Bellman representation. Following Durán and Licandro (2025), the Bellman representation of the representative household's preferences at time t is given by

$$W(c_{g,t}, c_{s,t}, x_t; \nu_t) = U(c_{g,t}, c_{s,t}) + \nu_t x_t, \tag{17}$$

where $x_t = \dot{k}_t$ denotes net investment per capita and $\nu_t = v'(k_t)$ is the marginal value of capital per capita at time t. In this representation, preferences at time t are non-homothetic and time varying, indexed by the marginal value of capital, ν_t. Notice that the quasi-linearity of the Bellman representation is an artefact of the additively separability of intertemporal preferences, which makes the marginal value of capital independent of consumption decisions.

To assess the welfare implications of output growth in a structural transformation economy, it is necessary to represent household preferences over per capita current consumption in goods and services, and per capita current investment in a way that is compatible with intertemporal optimisation. The Bellman representation offers such a framework by mapping the recursive structure of preferences into a static formulation that depends only on current choices and the marginal value of capital. This representation captures the trade-off between present consumption and future utility derived from investment, thus enabling meaningful welfare comparisons over time. Crucially, because it summarises the value of postponed consumption through the marginal value of capital, the Bellman representation provides a consistent basis for applying index number theory—such as the Fisher–Shell true quantity index—in dynamic settings. This allows the construction of output growth indices that accurately reflect welfare changes without requiring full knowledge of future consumption paths and the entire flow of consumption utility.[8]

At any time t, the representative household maximises the Bellman representation of preferences (17) with respect to $\{c_{g,t}, c_{s,t}, x_t\}$, subject to the per capita budget constraint

$$P_{g,t}c_{g,t} + P_{s,t}c_{s,t} + x_t = m_t, \tag{18}$$

where m_t denotes current net income per capita. Note that at equilibrium $m_t = y_t - \delta k_t$. It is important to emphasise that y_t and m_t refer to nominal income per capita, gross and net, as they are expressed in units of the numeraire. They do not represent real expenditure, even though they are measured in units of the investment good. Accordingly, aggregate nominal income is given by $Y_t = y_t N_t$ and $M_t = m_t N_t$, which we take as our measures of nominal GDP and NDP, respectively. All arguments that follow are independent of this arbitrary choice of numeraire.

Prior to applying index number theory to the Bellman representation in (17), Proposition 1 below derives the corresponding indirect utility function and the associated expenditure function.

Proposition 1. *The indirect utility and expenditure functions associated with the Bellman representation of preferences in equation* (17) *are, respectively, given by*

$$u(m_t, P_{g,t}, P_{s,t}; \nu_t) = V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right) + \nu_t\left(m_t - \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}\right), \tag{19}$$

and

$$e(w_t, P_{g,t}, P_{s,t}; \nu_t) = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}} + \frac{w_t}{\nu_t} - \frac{V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right)}{\nu_t}. \tag{20}$$

Proof: See Appendix A.

An important property of the HRV structural transformation model is that the marginal value of consumption expenditure, $\frac{\partial V(\cdot)}{\partial e}$, must be equal to the marginal value of capital, ν, which implies that consumption expenditure at equilibrium is[9]

$$e_t = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}. \tag{21}$$

[8] See Durán and Licandro (2025) for a more detailed discussion.

[9] A formal derivation is in the proof of Proposition 1 in Appendix A.

Thanks to the quasi-linearity of the Bellman representation, which as explained above is a direct implication of intertemporal separable preferences, the indirect utility function is linear in net income m_t, and the expenditure function is linear in the contribution of current consumption and current net investment to welfare, w_t.

Equivalent variation measure. Based on the Fisher–Shell principle —which states that intertemporal welfare comparisons must be made using a consistent preference ordering— and following the logic of the equivalent variation measure introduced by Durán and Licandro (2025), we adopt t as the base time and define the hypothetical income at time z as[10]

$$\widehat{m}_{t,z} = e\Big(u\big(m_z, P_{g,z}, P_{s,z}; \nu_t\big), P_{g,t}, P_{s,t}; \nu_t\Big). \tag{22}$$

The quantity $\widehat{m}_{t,z}$ represents the level of income per capita, valued at time t prices, that the representative household would have needed at time z to attain the utility achievable under the historical income and prices at z, but evaluated using the Bellman representation of preferences at time t.

Fixed-base Fisher-Shell indices. Let us adopt the following convention for an economy in which a national statistical agency has recorded National Accounts data from an initial time t_0 to the current time t. Following the strategy suggested by Baqaee and Burstein (2023), in this context, we define a current-base Fisher–Shell index as

$$\mathcal{P}_{t,z} = \log(\widehat{m}_{t,z}) - \log(\widehat{m}_{t,t_0}), \qquad z \in (t_0, t). \tag{23}$$

By construction, the index is normalised so that $\mathcal{P}_{t,t_0} = 0$. The value at time t then satisfies

$$\mathcal{P}_{t,t} = \log(m_t) - \log(\widehat{m}_{t,t_0}),$$

which is an equivalent variation measure of welfare gains from the initial time t_0 to the current time t, as evaluated using current preferences and prices. For any intermediate time $z \in (t_0, t)$, the difference $\mathcal{P}_{t,t} - \mathcal{P}_{t,z} = \log(m_t) - \log(\widehat{m}_{t,z})$ captures the welfare gains from time z to t.

[10]This hypothetical income is consistent with the equivalent variation measure suggested by Baqaee and Burstein (2023, Definition 4). It's important to point out that Durán and Licandro (2025) follow closely the methodology suggested in the seminal paper by Licandro et al (2002).

The term $\widehat{m}_{t,z}$ can be interpreted as the level of income per capita at time z, after correcting for changes in the price level, corrections being made from the perspective of the current Bellman representation of preferences. Then, the correction is made using current prices and current preferences. As such, the ratio $\widehat{m}_{t,z}/m_t$ provides a money-metric measure of the relative welfare level of the past when judged from today's standpoint. This approach mirrors the logic of a Paasche index, in that it values past allocations using current preferences and prices.

The current-base Fisher-Shell index implicitly defines a notion of instantaneous, welfare-based growth. For any $z < t$, the growth rate is given by

$$\frac{\partial \mathcal{P}_{t,z}}{\partial z} = \frac{\partial \log \widehat{m}_{t,z}}{\partial z} = \frac{1}{\widehat{m}_{t,z}} \frac{\partial \widehat{m}_{t,z}}{\partial z}. \tag{24}$$

As we demonstrate in Proposition 3 below, the instantaneous growth rate at the base time t, i.e. $\left.\frac{\partial \mathcal{P}_{t,z}}{\partial z}\right|_{z=t}$, coincides with the Divisia index. For all $z < t$, the current base Fisher-Shell growth rate is strictly lower and declines monotonically the further one moves backward in time. In economies undergoing structural transformation —characterised by relative price changes and shifting expenditure patterns— this decline can be significant. In some cases, as we illustrate below, the current base Fisher-Shell index may even register negative welfare growth when evaluating sufficiently distant past periods. This arises because the evaluation freezes preferences and prices at the base time t.

Proposition 2. *The current-base Fisher-Shell index $\mathcal{P}_{t,z}$ in (23), for all time $z < t$, grows at a rate smaller than the Divisia index. Then:*

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} < g^D_z, \quad \textit{for all } z < t.$$

PROOF: See Appendix **??**. THE PROOF NEEDS TO BE FINISHED.

We can also adopt the opposite view and measure welfare gains from the perspective of any past time $\tau < t$, by using (22) to measure welfare gains of moving from τ to $z \in (\tau, t)$. This alternative index will be like a Laspeyres index. To fix ideas, let us adopt t_0 as base time. In the following, we will represent the past-base Fisher-Shell index, for $z \in (t_0, t)$, as

$$\mathcal{L}_{t_0,z} = \log \widehat{m}_{t_0,z} - \log m_{t_0}. \tag{25}$$

The index is normalised to $\mathcal{L}_{t_0,t_0} = 0$ and $\mathcal{L}_{t_0,t} = \log \widehat{m}_{t_0,t} - \log m_{t_0}$. Implicit on this index, the equilibrium instantaneous growth rate of the economy at z is measured by

$$\frac{\partial \mathcal{L}_{t_0,z}}{\partial z} = \frac{\partial \log \widehat{m}_{t_0,z}}{\partial z} = \frac{1}{\widehat{m}_{t_0,z}} \frac{\partial \widehat{m}_{t_0,\tau}}{\partial z}. \tag{26}$$

Interestingly, the instantaneous growth rate at t_0, as measured by the past-base Fisher-Shell index, i.e. $\frac{\partial \mathcal{L}_{t_0,\tau}}{\partial \tau}|_{\tau=t_0}$, is also equal to the Divisia index at t_0. For any $z > t_0$, the instantaneous growth rate of the past-base Fisher-Shell index is higher than the Divisia index and increases as the welfare evaluation refers to a more distant point in the future. PROVE IT!!

Finally, Appendix G shows that for all $z \in (t_0, t)$, $\mathcal{L}_{t_0,z} > \mathcal{P}_{t,z}$, meaning that growth rates measured by means of a current-base Fisher–Shell index, like a Paasche index, are systematically lower than growth rates measured by a past-base Fisher–Shell index, like a Laspeyres index. REVISE THE PROOF.

Fisher–Shell Index and Divisia Index. Following Durán and Licandro (2025), we show that the growth rate of the base-time Fisher–Shell index coincides with the Divisia index when evaluated at the base time.

Proposition 3. *Let $\widehat{m}_{tz}$ be defined by equation* (22)*, and let preferences be represented as in equations* (19), (20), *and* (9). *Then, the instantaneous growth rate of the current-base Fisher–Shell index at current time t is equal to the Divisia index*

$$g_t^{FS} \doteq \left. \frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} \right|_{z=t} = g_t^D \quad \textit{where} \quad g_t^D \doteq s_{et} \underbrace{\left(s_{g,t}\, g_{g,t} + (1 - s_{g,t}) g_{s,t}\right)}_{g_{e_t}} + (1 - s_{et}) g_{x,t},$$

$s_{et} = e_t/m_t$, $s_{g,t} = P_{g,t}c_{g,t}/e_t$ *and* $s_{s,t} = P_{s,t}c_{s,t}/e_t$.

Proof: See Appendix D.

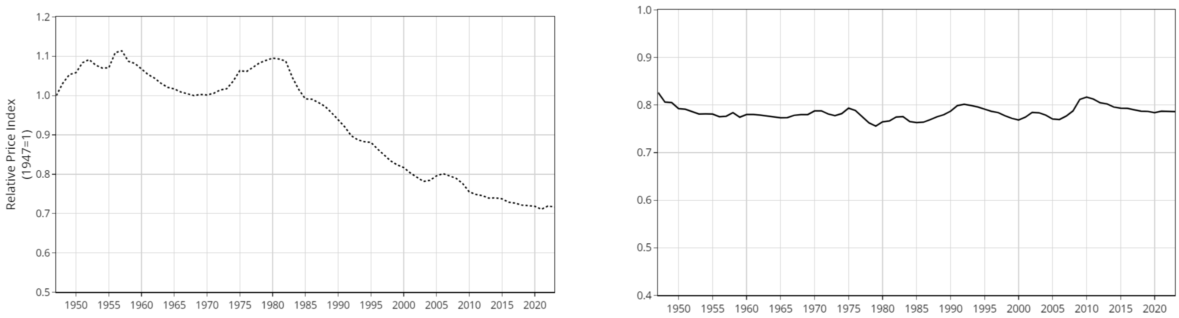
It is important to note that both the fixed-base Fisher–Shell indices and the chained Fisher–Shell index are welfare-based and grounded in the Fisher–Shell principle, though they rely on different reference orders. While they yield different quantitative measures of real income, both are derived from the same underlying preference representation. The fixed-base indices apply the Fisher–Shell principle globally by evaluating welfare gains across time using a fixed preference structure —either current or historical— while the chained index applies the principle locally, assessing instantaneous welfare gains at each point in time and chaining them to construct a consistent intertemporal measure. As a result, the fixed-base and chained indices may exhibit different properties in dynamic settings. In the following section, we quantitatively examine the behaviour of these indices in the context of the structural transformation model discussed above.

4 Mapping U.S. Data

In this section, we examine the aggregation of consumption and investment within a balanced growth path (BGP) framework, focusing on the measurement of welfare gains through alternative indices. To make the data in the US National Accounts consistent with the model, we follow the approach proposed by Herrendorf et al. (2021) and define goods consumption as the sum of personal consumption expenditure in goods and net exports of goods. Services consumption groups personal consumption expenditures on services, government consumption expenditure, and net exports of services. In turn, investment includes domestic private investment and government investment. We then apply Fisher-ideal quantity and price indices to generate real measures of consumption of goods, consumption of services, and investment. The consumption of goods and services is further aggregated into real total consumption using a Fisher-ideal index. The final measure of real GDP is yet another Fisher-ideal index, in this case aggregating real consumption and total investment.

4.1 The Relative Price of Investment and Chained Indices

~~Based on U.S. National Account data, we define consumption as non-durable consumption and services, and investment as durable consumption, equipment, and intellectual property. We use Fisher-ideal quantity and price indices to aggregate the components of consumption and investment into real measures, along with the corresponding price deflators. Nominal GDP is defined as the total private expenditure on consumption and investment. Structures, both residential and non-residential, public expenditure and net exports are excluded due to their distinct price behavior. Including a third type of expenditure in our GDP definition would not fundamentally alter the analysis but would make the argument less straightforward.~~



(a) Relative Investment Price (b) Consumption Share

Figure 1: Relative Prices and Income Shares.

Note: BEA data. Consumption aggregates consumption of goods and services. Goods include personal consumption expenditures in goods and net exports of goods. Services consumption groups personal consumption expenditures on services, government consumption expenditure, and net exports of services. Investment includes domestic private investment and government investment. GDP is consumption plus investment.

Figure 1 illustrates the evolution of the price of our investment measure relative to our consumption aggregate since 1947 ~~, the first year for which official US National Accounts data are available~~. ~~Since the 1960s,~~ The relative price of investment has shown a clear downward trend since 1980, declining at an average annual rate of 0.98% since the beginning of the sample. By 2023, the relative price of investment goods had fallen to approximately 72% of its original level. As shown in Figure 1, the share of nominal consumption in our measure of nominal GDP averages around 78% during the sample period, fluctuating around 80% towards the end of the sample period. ~~dipping slightly below 76% between 1960 and 2004 and~~ rising slightly from 1980 onward.

In what follows, we measure real GDP using a Fisher-ideal index based on prices and quantities of the consumption and investment measures defined above. The annual growth rate between 1947 and 2023, as measured by a chained Fisher-ideal index, is 3.2%. To assess the consistency of the Fisher-ideal index with the Fisher-Shell index, we also compute a Divisia index using past expenditure shares.[11] The differences between these

[11] I THINK THIS IS OLD AND NEEDS TO BE DELETED. DO WE WANT TO BUILD THIS INDEX STILL? Compare the Divisia with current and past shares, and the Tornquist index that uses the simple average.

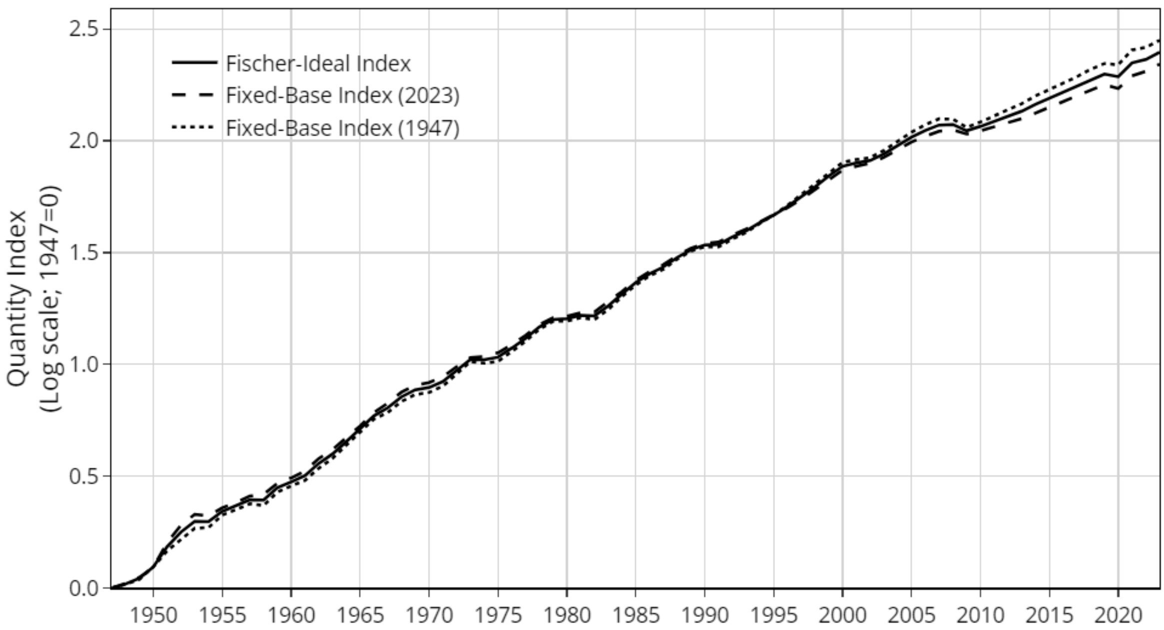


Figure 2: Real GDP

Note: BEA data. Real GDP is measured as a chained Fisher-ideal index (solid), a 1947 base-year index (dotted) and a 2023 base-year index (dashed). All indices are normalized to one in 1947.

two indices are negligible, visually indistinguishable in the graphs, as the sum of squared differences is of the order of 8×10^{-10}, with the cumulative difference between 1947 and 2023 amounting to approximately 0.5% of the initial GDP. Consequently, the chained Fisher-ideal approximates well the chained Fisher-Shell index suggested by Durán and Licandro (2025).

We also compute fixed-base indices using 1947 and 2023 as alternative base years. As shown in Figure 2, the chained Fisher-ideal index falls between the 1947 and 2023 fixed-base indices, all normalized to one in 1947.[12] FOR OMAR: NOW OUR INDICES ARE VERY CLOSE, SO THE DIFFERENCE IS NEGLIGIBLE. ALSO, THE 1947-BASE INDEX IS NOT ALWAYS HIGHER. THE 1947 INDEX GOES BELOW THE 2023 INDEX AROUND 2015. ~~By 2023, the difference between these two fixed-base indices is 44.5%. In this context, as pointed out by~~

[12] A Fisher-ideal based on the 1947 and 2003 base-year index is very similar to the chained Fisher-ideal.

Table 1: Calibrated Parameter Values

θ	ρ	δ	n	γ_h	$\gamma_{\mathcal{A}}$	γ_s	γ_g
0.333	0.04	0.08	0.0098	0.0041	0.0122	0.0004	0.0109

Note: θ, γ, δ are taken from HRV. TFP growth rates are computed using the indices from HRV. Population growth is taken from the US Census Bureau. γ_h is computed using the labor services index calculated by HRV.

Calibration. To calibrate the model, we assume that the economy is in an ABGP since 1980. We chose this period because the effective measured TFP in the investment sector exhibits a clear linear trend from 1980 onward, satisfying the requirement that $\mathcal{A}$ must grow at a constant rate along the ABGP. [13]

Our strategy involves a combination of calibration and structural estimation. We take the capital share parameter θ, the depreciation rate δ, and the time discount rate ρ directly from HRV. We follow the same procedure for the parameters that discipline the sectoral weights ω and the elasticity of substitution ε in the investment aggregator.

For sectoral and aggregate TFP, we assume that productivity grows at a constant exponential rate γ_j, such that $A_{j,t} = A_{j,0}e^{\gamma_j t}$ for $j \in \{g, s, \mathcal{A}\}$. Using the measured TFP series computed by HRV via growth accounting, we set $A_{j,0} = 1\ \forall j$ and set γ_j to match the value of $A_{j,t}$ in 2023. Investment-specific TFP is estimated residually to guarantee that $\mathcal{A}$ grows at a constant rate.

Table 2: Preference Parameters Estimated via SMM

χ	η	γ
0.550	0.239	0.900

Note: χ is taken directly from HRV. η and γ are estimated via SMM targeting the share of consumption expenditure on goods in the data.

With these parameters in hand, we simulate the economy in ABGP and estimate the preference parameters η, χ and γ via SMM, targeting the share of consumption

[13] See Herrendorf et al. (2021). The bottom panel of Figure 6 presents the measured investment TFP, while Proposition 1 discusses the conditions for the existence of the ABGP.

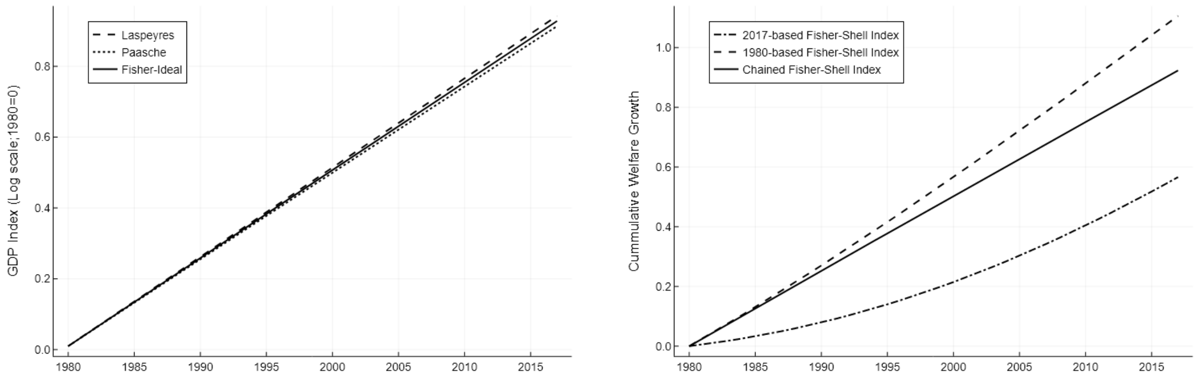
expenditure in goods the data. [14]

In the calibrated structural transformation economy at the ABGP, a statistical agency uses the model's predicted quantities and prices for goods, services, and investment to measure standard fixed-base and chained indices. Panel (a) in Figure 3 compares the chained Fisher-ideal index (solid line) with the 1980- and 2017-base indices for real GDP –dashed and dotted lines, respectively. All three measures track the long-run trend of the U.S. economy well. The x-axis covers the period from 1980 to 2017, and the y-axis plots the logarithm of real income –normalized to zero at 1980 and comparable to GDP in the data. As expected, the Laspeyres-type index slightly overstates growth, the Paasche-type index understates it, and the Fisher-ideal index lies in between. Differences across measures remain modest.[15]

Fixed-base vs chained Fisher-Shell indices. As part of our main discussion on the welfare-based measurement of output growth, panel (b) in Figure 3 displays the evolution of real GDP at the ABGP of the calibrated structural transformation economy. GDP is shown using three alternative measures: the chained Fisher-Shell index, given by the Divisia index $\mathcal{FS}_t$; the 1980-base Fisher-Shell index, $\mathcal{L}_{1980,t}$; and the 2017-base index, $\mathcal{P}_{2017,t}$. As in panel (a), the x-axis covers the period from 1980 to 2017, and the y-axis plots the logarithm of real income. The solid line corresponds to the chained Divisia index, which closely tracks the Fisher-ideal index shown in panel (a). This measure is normalized to zero in 1980 and exhibits a declining growth rate, from 1.535% in 1980 to 1.481% in 2017. The slowdown reflects a central implication of structural transformation models: as the share of goods in total consumption falls and services rise, overall growth becomes increasingly driven by productivity in the service sector—the slowest-growing sector—resulting in a gradual decline in welfare gains over time.

[14]The SMM minimizes the sum of squared residuals of the distance between the share of consumption on goods in the model and the data. We choose χ to take the same value as in HRV ($\chi = 0.55$). We estimate η and γ under the restrictions that $\eta > 0$ and $1 > \gamma > \chi > 0$. We also require that the elasticity of substitution between goods and services must be nonnegative in all periods.

[15]It is important to notice that for the three indices in Figure 3, we use the same Fisher-ideal indices for goods consumption, service consumption and investment taken from the data. Very likely, computing a Laspayers index using the disaggregated will generate a larger overvaluation of GDP growth.



(a) Real GDP

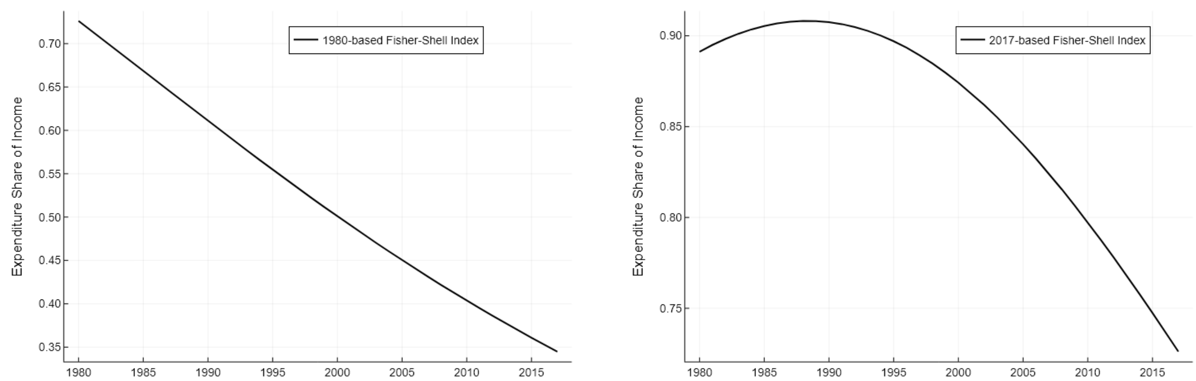
(b) Fisher-Shell (GDP)

Figure 3: Real GDP and Welfare Based Metrics.

Panel (a): Model generated data. Real GDP measured as chained Fisher-ideal (solid), 1947-base Laspeyres index (dashed), and 2023-base Paasche index (dotted). Panel (b): Chained Fisher-Shell (solid), 1980-base Fisher-Shell (dashed), and 217-base Fisher-Shell (dotted).

The fixed-base measures behave differently. Let us understand first the behavior of past-base Fisher-Shell indices. When the representative agent adopts 1980 as the base-time, as shown by the dashed line in panel (b) of Figure 3, the 1980-base Fisher-Shell index ends up 15.56% higher than the chained index in 2017, when measured relative to 1980, substantially overestimating income growth. The fundamental reason is the following. Under past preferences and prices, optimal consumption expenditure per capita remains constant at the 1980 level, i.e., $\widehat{e}_{1980,t} = e_{1980} = \left(\nu_{1980} P^{\chi}_{s,1980}\right)^{\frac{1}{\chi-1}}$ for all $t \in (1980, 2017)$. As realised income m_t, as well as hypothetical income $\widehat{m}_{1980,t}$, grow along the balanced growth path, the hypothetical consumption share $\widehat{e}_{1980,t}/\widehat{y}_{1980,t}$, where $\widehat{y}_{1980,t} = \widehat{m}_{1980,t} + \delta k_t$, declines steadily, as shown in panel (a) of Figure 4. When computing the past-base Fisher-Shell index, the agent maintains a fixed level of consumption expenditure even as income rises, implying a decreasing hypothetical consumption expenditure share and an increasing hypothetical investment share. Consequently, when adopting past preferences and prices, the agent places increasing weight on investment growth –whose growth rate exceeds that of consumption– leading to an overestimation of real GDP growth.

The 2017-base Fisher-Shell index evaluates past outcomes using 2017 preferences and prices. At each earlier date z, $z < 2017$, the ideal consumption expenditure per



(a) 1980-base Fisher-Shell (b) 2017-base Fisher-Shell

Figure 4: Hypothetical Consumption Expenditure Shares.

capita of the representative agent is fixed at $e_{2017} = \left(\nu_{2017} P^{\chi}_{s,2017}\right)^{\frac{1}{\chi-1}}$. Since income generally declines when going back in time, the hypothetical consumption share becomes increasingly large relative to income. For example, in 1980, the representative agent would aspire to a consumption level exceeding 1.6 times their income, as shown in panel (b) of Figure 4. This implies a negative investment share, revealing a strong substitution effect. Consequently, the 2017-base Fisher-Shell index is 56.4% smaller than the chained index in 2017, relative to 1980.

Base-time upgrade and GDP history revision. Figure 5 illustrates how the fixed-base Fisher-Shell index evolves with each revision of the base. When measured by the chained Divisia index, represented by the solid line, gross income grew around 25% over the ten-year period between 1980 and 1990. The 1980-base Fisher-Shell index overstates the Divisia index by 1.4% in 1990, implying a total income growth of 26.4% over the same ten-year period. As the base is revised forward over time, the current-base Fisher-Shell index increasingly diverges from the chained Divisia index for the same ten-year period. By 1990, for instance, the 1990-base Fisher-Shell index significantly underestimates growth, estimating income gains about 2.7% below the chained Divisia index for the full ten-year period. This gap widens with each subsequent revision, as shown in Figure 5, reaching a 17.6% reduction in measured growth for the 2017-base Fisher-Shell index.

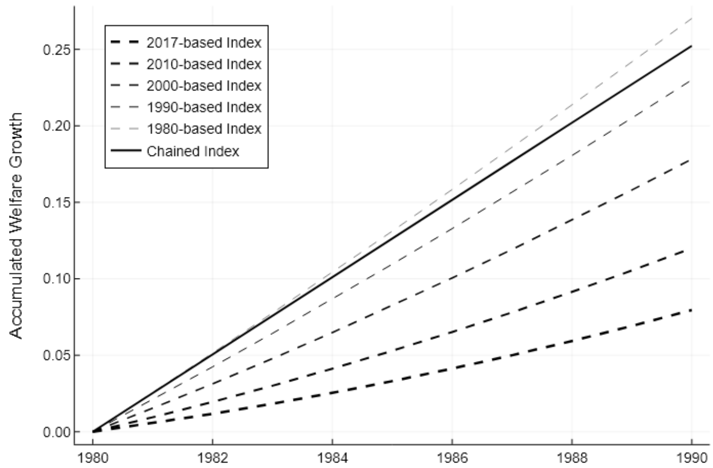


Figure 5: Changing the base

Summary. In line with the main contribution of the paper, the chained Fisher-Shell index provides a consistent measure of welfare-based income growth. As formally established in Proposition 3, this index coincides with the Divisia index, which measures instantaneous welfare gains along the economy's equilibrium path. This property aligns the chained Fisher-Shell index with the growth rates reported in National Accounts, which are also constructed using a chained Fisher-ideal index.

By contrast, current-base Fisher-Shell indices systematically underestimate past growth. This underestimation becomes more severe the further back in time one goes, as the evaluation is conducted using today's prices and preferences. Moreover, current-base indices require permanent revisions: as time passes the base is updated, past growth is re-evaluated and typically revised downward. This feature underscores a key limitation of fixed-base indices and highlights the advantages of chaining for capturing welfare-relevant growth dynamics.

5 Conclusions

In the framework of a continuous-time, learning-by-doing economy, this paper examines the welfare foundations of chained indices, particularly the Fisher-ideal quantity index, as a preferred tool for measuring GDP growth in national accounts. Baqaee and Burstein

(2023) and Durán and Licandro (2025) offer alternative measures of GDP growth.

By comparing the frameworks of Durán and Licandro (2025) and Baqaee and Burstein (2023), we have highlighted the implications of fixed-base versus chained approaches for accurately reflecting welfare gains over time. Our findings underscore the theoretical and practical limitations of fixed-base indices, which suffer from substitution bias as economic conditions evolve, and require permanent revisions leading to distorted representations of welfare gains. This is particularly evident in a model with balanced growth and embodied technical change, where the chained Fisher-Shell index coincides with a chained Divisia index, but fixed-base measures like the BBEV tend to mimic the behavior of traditional Paasche indices, underestimating welfare gains due to the structural bias inherent in fixed weights.

In contrast, Durán and Licandro (2025) chained Divisia approach offers a dynamic alternative that does not require stability and homotheticity of preferences, thus accommodating a wider range of economic environments. By chaining welfare gains between contiguous periods, this approach reflects welfare changes as they unfold, making it particularly suitable for modern economies with rapidly evolving technology. The chained Fisher-Shell index derived from this approach maintains a consistent measure of welfare gains, aligned with the steady-state growth of the balanced growth path model. This time-invariant property of the chained index avoids the need for frequent revisions and provides a flexible welfare-based measure that better captures the economic realities of today's dynamic economies.

Our analysis also speaks to the broader policy implications of adopting chained indices in national accounts. The findings suggest that chained indices like the Fisher-ideal quantity index not only reduce substitution bias but also offer a more reliable metric for tracking welfare-relevant economic growth. This advantage is particularly pertinent for policymakers and economists who rely on GDP growth figures as indicators of national progress and societal well-being. A chained approach better reflects the changing structure of the economy, ensuring that measures of growth remain consistent with people's actual welfare over time.

Overall, this paper contributes to the ongoing dialogue on the role of chained indices in national accounts and their relevance for welfare measurement. By clarifying the welfare properties of chained indices and highlighting the potential for bias in fixed-base measures, we provide a theoretical foundation and empirical rationale for their continued adoption. The findings support the view that chained indices, by adapting to structural changes in the economy, offer a more accurate reflection of welfare gains over time, ultimately contributing to a more nuanced understanding of economic growth and its impact on well-being.

The fixed-base BBEV index indeed encounters a similar issue to that highlighted by Parker and Triplett (1996) regarding fixed-base Laspeyres indices. Both suffer from the substitution bias, where the fixed-base method does not account for shifts in spending or investment patterns over time. This leads to distortions in growth measurement, as fixed-base indices can increasingly misstate growth the further away they move from the base period. As a result, the fixed-base BBEV index may overestimate or underestimate growth, particularly in environments with dynamic changes in relative prices, much like the issues seen with the traditional fixed-base Laspeyres index.

Appendices

A Proof of Proposition 1

The household's time-t primal problem of maximizing (17), subject to the budget constraint (18), can be solved in two stages. In the first stage, by the definition of the indirect utility function,

$$V(e_t, P_{g,t}, P_{s,t}) = \max_{\{c_{g,t}, c_{s,t}\}: P_{g,t} c_{g,t} + P_{s,t} c_{s,t} = e_t} U(c_{g,t}, c_{s,t}).$$

In the second stage, the household chooses x_t to solve

$$\max_{x} \; V(m_t - x, P_{g,t}, P_{s,t}) + \nu_t x.$$

The first-order condition is

$$\frac{\partial V}{\partial e_t}(e_t, P_{g,t}, P_{s,t}) = \nu_t.$$

Using the PIGL functional form in (9), $\frac{\partial V}{\partial e_t} = e_t^{\chi-1} P_{s,t}^{-\chi}$, we obtain

$$e_t^{\chi-1} P_{s,t}^{-\chi} = \nu_t \quad \Rightarrow \quad e_t = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}.$$

It follows that

$$x_t = m_t - e_t = m_t - \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}.$$

Substituting back, the indirect utility function is

$$u(m_t, P_{g,t}, P_{s,t}; \nu_t) = V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right) + \nu_t \left(m_t - \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}\right).$$

The expenditure function is derived from the dual problem

$$\min_{e,x} \; e + x \quad \text{s.t.} \quad V(e, P_{g,t}, P_{s,t}) + \nu_t x = w_t,$$

where w_t is an arbitrary level of utility. The F.O.C.s with respect to e and x are

$$1 = \mu \frac{\partial V}{\partial e} = \lambda e^{\chi-1} P_{s,t}^{-\chi} \quad \text{and} \quad 1 = \mu \nu_t,$$

where μ is the Lagrangian multiplier associated to the constraint. Equating multipliers yields

$$e^{\chi-1} P_{s,t}^{-\chi} = \nu_t \quad \Rightarrow \quad e = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}.$$

The same condition as in the primal problem. Solving for x

$$x = \frac{w_t - V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right)}{\nu_t}.$$

Thus, the expenditure function is

$$e(w_t, P_{g,t}, P_{s,t}; \nu_t) = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}} + \frac{w_t - V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right)}{\nu_t},$$

which completes the proof. □

B Proof of Proposition

By definition, the hypothetical income at time $z < t$, evaluated using the Bellman representation of preferences at time t, is given by:

$$\widehat{m}_{t,z} = e\left(u(m_z, P_{g,z}, P_{s,z}; \nu_t), P_{g,t}, P_{s,t}; \nu_t\right),$$

where $e(\cdot)$ is the expenditure function associated with the indirect utility function $u(\cdot)$.

From the definition of the indirect utility function under the Bellman representation:

$$u(m_z, P_{g,z}, P_{s,z}; \nu_t) = V(\bar{e}_z, P_{g,z}, P_{s,z}) + \nu_t(m_z - \bar{e}_z),$$

where $\bar{e}_z = \left(\nu_t P_{s,z}^{\chi}\right)^{\frac{1}{\chi-1}}$ and

$$V(e, P_g, P_s) = \frac{1}{\chi}\left(\frac{e}{P_s}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_g}{P_s}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}.$$

Now substitute $u(\cdot)$ into the expenditure function:

$$\begin{aligned}
\widehat{m}_{t,z} &= e\left(V(\bar{e}_z, P_{g,z}, P_{s,z}) + \nu_t(m_z - \bar{e}_z), P_{g,t}, P_{s,t}; \nu_t\right) \\
&= \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}} + \frac{1}{\nu_t}\left(V(\bar{e}_z, P_{g,z}, P_{s,z}) + \nu_t(m_z - \bar{e}_z)\right) - \frac{1}{\nu_t}V(e_t, P_{g,t}, P_{s,t}),
\end{aligned}$$

where $e_t = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}$, so $V(e_t, P_{g,t}, P_{s,t}) = \frac{1}{\chi}\left(\frac{e_t}{P_{s,t}}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}$.

Likewise, substitute $V(\bar{e}_z, \cdot)$, and simplify:

$$\begin{aligned}
\widehat{m}_{t,z} &= e_t + m_z - \bar{e}_z + \frac{1}{\nu_t}\left[V(\bar{e}_z, P_{g,z}, P_{s,z}) - V(e_t, P_{g,t}, P_{s,t})\right] \\
&= m_z + \frac{1}{\nu_t}\left[\left(\frac{\bar{e}_z}{P_{s,z}}\right)^{\chi} - \left(\frac{e_t}{P_{s,t}}\right)^{\chi} - \eta\left(\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma}\right)\right] - \bar{e}_z + e_t.
\end{aligned}$$

We start from the expression for the hypothetical income at time $z < t$, evaluated using the Bellman representation of preferences at time t:

$$\widehat{m}_{t,z} = m_z + \frac{1}{\nu_t}\left[\left(\frac{\bar{e}_z}{P_{s,z}}\right)^{\chi} - \left(\frac{e_t}{P_{s,t}}\right)^{\chi} - \eta\left(\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma}\right)\right] - \bar{e}_z + e_t.$$

We now use the model's equilibrium condition that characterises the optimal level of consumption expenditure under the Bellman representation:

$$\bar{e}_z = \frac{1}{\nu_t}\left(\frac{\bar{e}_z}{P_{s,z}}\right)^{\chi}, \qquad e_t = \frac{1}{\nu_t}\left(\frac{e_t}{P_{s,t}}\right)^{\chi}.$$

Substituting both expressions into the original equation yields:

$$\begin{aligned}\widehat{m}_{t,z} = m_z &+ \frac{1}{\nu_t}\left[\left(\frac{\bar{e}_z}{P_{s,z}}\right)^{\chi} - \left(\frac{e_t}{P_{s,t}}\right)^{\chi} - \eta\left(\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma}\right)\right] \\ &- \frac{1}{\nu_t}\left(\frac{\bar{e}_z}{P_{s,z}}\right)^{\chi} + \frac{1}{\nu_t}\left(\frac{e_t}{P_{s,t}}\right)^{\chi}.\end{aligned}$$

The first and fourth terms cancel, as do the second and fifth. We are left with:

$$\widehat{m}_{t,z} = m_z - \frac{\eta}{\nu_t}\left[\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma}\right].$$

This completes the proof. □

C Proof of Proposition

We proceed step-by-step.

Step 1. Differentiating $\log \widehat{m}_{t,z}$**.** By the chain rule:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} &= \frac{\mathrm{d}}{\mathrm{d}z}\log \widehat{m}_{t,z} \\ &= \frac{1}{\widehat{m}_{t,z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\widehat{m}_{t,z} \\ &= \frac{1}{\nu_t \widehat{m}_{t,z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_t),\end{aligned}$$

using the fact that the partial derivative $\partial e(u, P_g, P_s; \nu)/\partial u = 1/\nu$.

Step 2. Differentiating $u(m_z, P_{g,z}, P_{s,z}; \nu_t)$. From Proposition 1:

$$u(m_z, P_{g,z}, P_{s,z}; \nu_t) = V(\widehat{e}_z, P_{g,z}, P_{s,z}) + \nu_t(m_z - \widehat{e}_z),$$
$$\widehat{e}_z = \left(\nu_t P_{s,z}^{\chi}\right)^{\frac{1}{\chi-1}}.$$

It represents hypothetical consumption expenditure at z prices but evaluated at t preferences. We then get:

$$\frac{\mathrm{d}}{\mathrm{d}z}u = \nu_t \dot{m}_z + \frac{\partial V}{\partial P_{g,z}}\dot{P}_{g,z} + \frac{\partial V}{\partial P_{s,z}}\dot{P}_{s,z},$$

as the terms in $\dot{\widehat{e}}_z$ cancel, and

$$\frac{\partial V}{\partial P_g} = -\eta\left(\frac{P_g}{P_s}\right)^{\gamma}\cdot\frac{1}{P_g},$$
$$\frac{\partial V}{\partial P_s} = -\left(\frac{\widehat{e}}{P_s}\right)^{\chi}\cdot\frac{1}{P_s} + \eta\left(\frac{P_g}{P_s}\right)^{\gamma}\cdot\frac{1}{P_s}.$$

Step 3. Using Roy's identity.

$$\widehat{s}_g = \frac{P_g\widehat{c}_g}{\widehat{e}} = \eta\left(\frac{\widehat{e}}{P_s}\right)^{-\chi}\left(\frac{P_g}{P_s}\right)^{\gamma} \quad \Rightarrow \quad \eta\left(\frac{P_g}{P_s}\right)^{\gamma} = \widehat{s}_g\left(\frac{\widehat{e}}{P_s}\right)^{\chi},$$

where $\widehat{c}_g$ and $\widehat{s}_g$ represent hypothetical per capital goods consumption and hypothetical goods consumption share of total hypothetical consumption expenditure. Then,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} &= \frac{\dot{m}_z}{\widehat{m}_{t,z}} + \frac{1}{\nu_t\widehat{m}_{t,z}}\cdot\left(\widehat{s}_{g,z}\left(\frac{\widehat{e}_z}{P_{s,z}}\right)^{\chi}\left(g_{P_s,z} - g_{P_g,z}\right) - \left(\frac{\widehat{e}_z}{P_{s,z}}\right)^{\chi} g_{P_s,z}\right)\\ &= \frac{\dot{m}_z}{\widehat{m}_{t,z}} + \frac{\widehat{e}_t}{\widehat{m}_{t,z}}\cdot\left(\widehat{s}_{g,z}\ \left(g_{P_s,z} - g_{P_g,z}\right) - g_{P_s,z}\right)\end{aligned}$$

Step 4. Evaluate it at $t = z$. In this case, $\widehat{m}_{z,z} = m_z$, $\widehat{s}_{g,z} = s_{g,z}$ and $\widehat{e}_z = e_z$. Then, using $e_z = \left(\nu_z P_{s,z}^{\chi}\right)^{\frac{1}{\chi-1}}$,

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}\right|_{t=z} &= \frac{1}{\nu_z m_z}\cdot\frac{\mathrm{d}}{\mathrm{d}z}u(m_z, P_{g,z}, P_{s,z}; \nu_z)\\ &= \frac{\dot{m}_z}{m_z} - s_{e,z}\Big(s_{g,z}\, g_{P_g,z} + (1 - s_{g,z})\, g_{P_s,z}\Big).\end{aligned}$$

Step 4. The Divisia index at z. The Divisia index is:

$$g_z^D = s_{e,z}\Big(s_{g,z}\, g_{g,z} + (1 - s_{g,z})\, g_{s,z}\Big) + (1 - s_{e,z})\, g_{x,z}.$$

From the definition of current net income m_z,

$$\frac{\dot{m}_z}{m_z} = g_z^D + s_{e,z}\Big(s_{g,z}\, g_{P_g,z} + (1 - s_{g,z})\, g_{P_s,z}\Big).$$

Then,

$$g_z^D = \frac{\dot{m}_z}{m_z} - s_{e,z}\Big(s_{g,z}\, g_{P_g,z} + (1 - s_{g,z})\, g_{P_s,z}\Big) = \left.\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}\right|_{t=z}.$$

At any time z, the growth rate of the z-base Fisher-Shell index is equal to the corresponding Divisia index.

Step 4. Comparing $\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}$ with the Divisia index at z. The Divisia index is:

$$g_z^D = \frac{\dot{m}_z}{m_z} - s_{e,z} g_{P_{s,z}} - s_{e,z} s_{g,z}(g_{P_{g,z}} - g_{P_{s,z}}),$$

and equals:

$$\frac{1}{\nu_z m_z} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_z).$$

But $\widehat{m}_{t,z}$ is evaluated with ν_t, not ν_z, and at $P_{g,t}, P_{s,t}$, not $P_{g,z}, P_{s,z}$. So:

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \widehat{m}_{t,z} = \frac{1}{\nu_t \widehat{m}_{t,z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_t) < \frac{1}{\nu_z m_z} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_z) = g_z^D.$$

The inequality follows because:

- $\nu_t \neq \nu_z$ (preferences differ over time),
- $\widehat{m}_{t,z} < m_z$ (evaluating past income at current preferences/prices undervalues it).

Step 5. Strict inequality for $z < t$. The inequality is strict unless $z = t$, where all evaluations align. Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}z} \log \widehat{m}_{t,z}\right|_{z=t} = g_t^D, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z} \log \widehat{m}_{t,z} < g_z^D \text{ for all } z < t.$$

□

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Step 1. Differentiating $\log \widehat{m}_{t,z}$. By the chain rule:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} &= \frac{\mathrm{d}}{\mathrm{d}z} \log \widehat{m}_{t,z} \\
&= \frac{1}{\widehat{m}_{t,z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \widehat{m}_{t,z} \\
&= \frac{1}{\nu_t \widehat{m}_{t,z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_t),
\end{aligned}$$

using that $\partial e/\partial u = 1/\nu_t$.

Step 2. Differentiating $u(m_z, P_{g,z}, P_{s,z}; \nu_t)$. From

$$u(m_z, P_{g,z}, P_{s,z}; \nu_t) = V(\widehat{e}_z, P_{g,z}, P_{s,z}) + \nu_t(m_z - \widehat{e}_z),$$
$$\widehat{e}_z = \left(\nu_t P_{s,z}^{\chi}\right)^{\frac{1}{\chi-1}},$$

we get:

$$\frac{\mathrm{d}}{\mathrm{d}z}u = \nu_t \dot{m}_z + \frac{\partial V}{\partial P_{g,z}}\dot{P}_{g,z} + \frac{\partial V}{\partial P_{s,z}}\dot{P}_{s,z},$$

where $\dot{\widehat{e}}_z$ terms cancel due to the envelope condition.

The derivatives of V are:

$$\frac{\partial V}{\partial P_{g,z}} = -\eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma}\cdot\frac{1}{P_{g,z}},$$
$$\frac{\partial V}{\partial P_{s,z}} = -\left(\frac{\widehat{e}_z}{P_{s,z}}\right)^{\chi}\cdot\frac{1}{P_{s,z}} + \eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma}\cdot\frac{1}{P_{s,z}}.$$

Step 3. Using Roy's identity.

$$s_{g,z} = \frac{P_{g,z}c_{g,z}}{e_z} = \eta\left(\frac{e_z}{P_{s,z}}\right)^{-\chi}\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} \quad\Rightarrow\quad \eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} = s_{g,z}\left(\frac{e_z}{P_{s,z}}\right)^{\chi}.$$

Then:

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} = \frac{\dot{m}_z}{\widehat{m}_{t,z}} + \frac{1}{\nu_t\widehat{m}_{t,z}}\cdot\left(s_{g,z}\left(\frac{e_z}{P_{s,z}}\right)^{\chi}(g_{P_{s,z}} - g_{P_{g,z}}) - \left(\frac{\widehat{e}_z}{P_{s,z}}\right)^{\chi}g_{P_{s,z}}\right)$$

Step 4. Evaluation at $t = z$. In this case, $\widehat{m}_{z,z} = m_z$ and $\widehat{e}_z = e_z$. Using that $e_z = \left(\nu_z P_{s,z}^{\chi}\right)^{\frac{1}{\chi-1}}$, we obtain:

$$\left.\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}\right|_{t=z} = \frac{1}{\nu_z m_z}\cdot\left.\frac{\mathrm{d}}{\mathrm{d}z}u(m_z, P_{g,z}, P_{s,z}; \nu_z)\right|_{t=z}$$
$$= \frac{\dot{m}_z}{m_z} - s_{e,z}\Big(s_{g,z}\, g_{P_{g,z}} + (1 - s_{g,z})\, g_{P_{s,z}}\Big),$$

where $s_{e,z} = \frac{e_z}{m_z}$.

Step 5. Divisia index at z. By definition,

$$g_z^D = s_{e,z}\Big(s_{g,z}\, g_{g,z} + (1 - s_{g,z})\, g_{s,z}\Big) + (1 - s_{e,z})\, g_{x,z}.$$

From the definition of net income,

$$\frac{\dot{m}_z}{m_z} = g_z^D + s_{e,z}\Big(s_{g,z}\, g_{P_{g,z}} + (1 - s_{g,z})\, g_{P_{s,z}}\Big),$$

so:

$$g_z^D = \frac{\dot{m}_z}{m_z} - s_{e,z}\Big(s_{g,z}\, g_{P_{g,z}} + (1 - s_{g,z})\, g_{P_{s,z}}\Big).$$

Thus:

$$\left.\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}\right|_{t=z} = g_z^D.$$

Step 6. Comparing growth at $z < t$.

From Step 1, we have:

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} = \frac{1}{\widehat{m}_{t,z}} \cdot \frac{1}{\nu_t} \cdot \frac{\mathrm{d}}{\mathrm{d}z} u(m_z, P_{g,z}, P_{s,z}; \nu_t).$$

From Step 2, recall:

$$\frac{\mathrm{d}}{\mathrm{d}z} u = \nu_t \dot{m}_z + \frac{\partial V}{\partial P_{g,z}} \dot{P}_{g,z} + \frac{\partial V}{\partial P_{s,z}} \dot{P}_{s,z}.$$

Therefore:

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} = \frac{\dot{m}_z}{\widehat{m}_{t,z}} + \frac{1}{\nu_t \widehat{m}_{t,z}} \left(\frac{\partial V}{\partial P_{g,z}} \dot{P}_{g,z} + \frac{\partial V}{\partial P_{s,z}} \dot{P}_{s,z} \right).$$

Now consider the Divisia index at time z, from Step 5:

$$g_z^D = \frac{\dot{m}_z}{m_z} - s_{e,z} \left(s_{g,z} g_{P_{g,z}} + (1 - s_{g,z}) g_{P_{s,z}} \right).$$

Let us rewrite the derivative of $\mathcal{P}_{t,z}$ similarly:

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} = \frac{\dot{m}_z}{\widehat{m}_{t,z}} - \underbrace{\left(\left(\frac{\widehat{e}_z}{P_{s,z}} \right)^{\chi} - s_{g,z} \left(\frac{e_z}{P_{s,z}} \right)^{\chi} \right) \cdot \frac{g_{P_{s,z}}}{\nu_t \widehat{m}_{t,z}}}_{(*)} - \underbrace{s_{g,z} \left(\frac{e_z}{P_{s,z}} \right)^{\chi} \cdot \frac{g_{P_{g,z}}}{\nu_t \widehat{m}_{t,z}}}_{(**)}.$$

Meanwhile, g_z^D contains:

$$-s_{e,z} \left(s_{g,z} g_{P_{g,z}} + (1 - s_{g,z}) g_{P_{s,z}} \right) = -\left(\frac{e_z}{m_z} \right) \left[s_{g,z} g_{P_{g,z}} + (1 - s_{g,z}) g_{P_{s,z}} \right].$$

To compare both expressions:

- Note that $\widehat{m}_{t,z} < m_z$, because it reflects past income evaluated at current preferences, which undervalues it. - Also $\nu_t \neq \nu_z$, so the marginal utility weights differ. - And $\widehat{e}_z = \left(\nu_t P_{s,z}^{\chi}\right)^{\frac{1}{\chi - 1}} \neq e_z = \left(\nu_z P_{s,z}^{\chi}\right)^{\frac{1}{\chi - 1}}$.

Hence, the differences in numerators and denominators in the adjustment terms (*) and (**) lead to a smaller sum for $\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z}$ than for g_z^D, since the adjustment is over-scaled relative to the true marginal valuation of the past economy.

Therefore:

$$\frac{\mathrm{d}\mathcal{P}_{t,z}}{\mathrm{d}z} < g_z^D \quad \text{for all } z < t.$$

□

D Proof of Proposition 3

From the definition of the current-base Fisher-Shell index in (23) and the corresponding hypothetical income in (22), we have

$$\mathcal{P}_{t,z} = \log \widehat{m}_{t,z} - \log \widehat{m}_{t,t_0} \quad \text{with} \quad \widehat{m}_{t,z} = e\Big(u\big(m_z, P_{g,z}, P_{s,z}; \nu_t\big), P_{g,t}, P_{s,t}; \nu_t\Big).$$

Then, the growth rate of the current-base Fisher-Shell index at current time t is given by

$$g_t^{\mathrm{FS}} = \frac{\mathrm{d}}{\mathrm{d}z} \log \widehat{m}_{t,z}\bigg|_{z=t} = \frac{1}{\widehat{m}_{t,z}} \cdot \frac{\mathrm{d}\widehat{m}_{t,z}}{\mathrm{d}z}\bigg|_{z=t}.$$

By the chain rule:

$$\frac{\mathrm{d}\widehat{m}_{t,z}}{\mathrm{d}z} = \frac{\partial e}{\partial u} \cdot \frac{\mathrm{d}u}{\mathrm{d}z}.$$

From the expenditure function in (20),

$$e(u, P_{g,t}, P_{s,t}; \nu_t) = \big(\nu_t P_{s,t}^{\chi}\big)^{\frac{1}{\chi-1}} + \frac{u}{\nu_t} - \frac{V\left(\big(\nu_t P_{s,t}^{\chi}\big)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right)}{\nu_t},$$

we have $\partial e/\partial u = 1/\nu_t$. Hence,

$$g_t^{\mathrm{FS}} = \frac{1}{\nu_t\, m_t} \cdot \frac{\mathrm{d}u}{\mathrm{d}z}\bigg|_{z=t},$$

since $\widehat{m}_{t,t} = m_t$. Next, from the indirect utility function in (19),

$$u(m_z, P_{g,z}, P_{s,z}; \nu_t) = V\left(\widehat{e}_z, P_{g,z}, P_{s,z}\right) + \nu_t(m_z - \widehat{e}_z), \quad \text{with} \quad \widehat{e}_z = \big(\nu_t P_{s,z}^{\chi}\big)^{\frac{1}{\chi-1}},$$

we differentiate:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\partial V}{\partial e} \cdot \frac{\mathrm{d}\widehat{e}_z}{\mathrm{d}z} + \frac{\partial V}{\partial P_{g,z}} \cdot \frac{\mathrm{d}P_{g,z}}{\mathrm{d}z} + \frac{\partial V}{\partial P_{s,z}} \cdot \frac{\mathrm{d}P_{s,z}}{\mathrm{d}z} - \nu_t \cdot \frac{\mathrm{d}\widehat{e}_z}{\mathrm{d}z} + \nu_t \cdot \frac{\mathrm{d}m_z}{\mathrm{d}z}.$$

From

$$V(e, P_g, P_s) = \frac{1}{\chi}\left(\frac{e}{P_s}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_g}{P_s}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma},$$

we compute the partial derivatives:

$$\frac{\partial V}{\partial e} = \left(\frac{e}{P_s}\right)^{\chi-1} \cdot \frac{1}{P_s}, \quad \frac{\partial V}{\partial P_g} = -\eta\left(\frac{P_g}{P_s}\right)^{\gamma} \cdot \frac{1}{P_g},$$
$$\frac{\partial V}{\partial P_s} = -\left(\frac{e}{P_s}\right)^{\chi} \cdot \frac{1}{P_s} + \eta\left(\frac{P_g}{P_s}\right)^{\gamma} \cdot \frac{1}{P_s}.$$

From the definition of $\widehat{e}_t$, we find that the terms in $\mathrm{d}\widehat{e}_z/\mathrm{d}z$ cancel. Substituting into the total derivative:

$$\begin{aligned}\left.\frac{\mathrm{d}u}{\mathrm{d}z}\right|_{z=t} &= -\eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} \cdot g_{P_g,t} \\ &\quad + \left(\eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} - \left(\frac{e_t}{P_{s,t}}\right)^{\chi}\right) \cdot g_{P_s,t} \\ &\quad + \nu_t \cdot \dot{m}_t.\end{aligned}$$

Note that $\widehat{e}_t = e_t$. Now we express the price effects in terms of observable shares using Roy's identity:

$$s_{g,t} = \frac{P_{g,t}c_{g,t}}{e_t} = \eta\left(\frac{e_t}{P_{s,t}}\right)^{-\chi}\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} \quad \Rightarrow \quad \eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} = s_{g,t}\left(\frac{e_t}{P_{s,t}}\right)^{\chi}.$$

Rewriting the price derivative terms in the total derivative:

$$\begin{aligned}&-\eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} \cdot g_{P_g,t} + \left(\eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} - \left(\frac{e_t}{P_{s,t}}\right)^{\chi}\right) \cdot g_{P_s,t} \\ &= -\left(\frac{e_t}{P_{s,t}}\right)^{\chi} \cdot \left(s_{g,t}g_{P_g,t} + (1-s_{g,t})g_{P_s,t}\right) \\ &= e_t\nu_t\left(s_{g,t}g_{P_g,t} + (1-s_{g,t})g_{P_s,t}\right).\end{aligned}$$

The last equality derives from $e_t = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}$.

Now return to the full expression:

$$g_t^{\mathrm{FS}} = \frac{\dot{m}_t}{m_t} - \frac{e_t}{m_t}\Big(s_{g,t}g_{P_g,t} - (1-s_{g,t})g_{P_s,t}\Big).$$

From $m_t = e_t + x_t$, and $e_t = P_{g,t}c_{g,t} + P_{s,t}c_{s,t}$, we get

$$\frac{\dot{m}_t}{m_t} = s_{e,t}g_{e,t} + (1-s_{e,t})g_{x,t},$$

$$g_{e,t} = s_{g,t}(g_{g,t} + g_{P_{g,t}}) + (1 - s_{g,t})(g_{s,t} + g_{P_{s,t}}),$$

so that all price terms cancel:

$$\begin{aligned} g_t^{\mathrm{FS}} &= s_{e,t}\left[s_{g,t}g_{g,t} + (1 - s_{g,t})g_{s,t} + s_{g,t}g_{P_{g,t}} + (1 - s_{g,t})g_{P_{s,t}}\right] \\ &\quad + (1 - s_{e,t})g_{x,t} - s_{e,t}g_{P_{s,t}} - s_{e,t}s_{g,t}(g_{P_{g,t}} - g_{P_{s,t}}) \\ &= s_{e,t}\left(s_{g,t}g_{g,t} + (1 - s_{g,t})g_{s,t}\right) + (1 - s_{e,t})g_{x,t} = g_t^D. \quad \square \end{aligned}$$

E Robustness

An alternative to the previous chained Fisher-Shell index, that evaluates welfare gains at any time $z < t$, using time z preferences and prices, would evaluate welfare gains using z prices but t preferences. Let us define the following hypothetical income at $h < z < t$,

$$\widehat{m}_{t,z,h} = e\Big(u\big(m_h, P_{g,h}, P_{s,h}; \nu_t\big), P_{g,z}, P_{s,z}; \nu_t\Big). \tag{27}$$

The quantity $\widehat{m}_{t,z,h}$ represents the level of income per capita valued at time z prices, that the representative household would have needed at time h to attain the utility achievable under the historical income and prices at h, but evaluated using the Bellman representation of preferences at time t.

From the definition of the current-base Fisher-Shell index in (23) and the corresponding hypothetical income in (22), we have

$$\mathcal{P}_{t,z,h} = \log \widehat{m}_{t,z,h} - \log \widehat{m}_{t,t_0,h} \quad \text{with} \quad \widehat{m}_{t,z,h} = e\Big(u\big(m_h, P_{g,h}, P_{s,h}; \nu_t\big), P_{g,z}, P_{s,z}; \nu_t\Big).$$

Then, the growth rate of $\mathcal{P}_{t,z,h}$ at current time z is given by

$$\widehat{g}_z^{\mathrm{FS}} = \frac{\mathrm{d}}{\mathrm{d}h} \log \widehat{m}_{t,z,h}\bigg|_{h=z} = \frac{1}{\widehat{m}_{t,z,h}} \cdot \frac{\mathrm{d}\widehat{m}_{t,z,h}}{\mathrm{d}h}\bigg|_{h=z}.$$

By the chain rule:

$$\frac{\mathrm{d}\widehat{m}_{t,z,h}}{\mathrm{d}h} = \frac{\partial e}{\partial u} \cdot \frac{\mathrm{d}u}{\mathrm{d}h}.$$

From the expenditure function in (20),

$$e(u, P_{g,t}, P_{s,t}; \nu_t) = \left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}} + \frac{u}{\nu_t} - \frac{V\left(\left(\nu_t P_{s,t}^{\chi}\right)^{\frac{1}{\chi-1}}, P_{g,t}, P_{s,t}\right)}{\nu_t},$$

we have $\partial e/\partial u = 1/\nu_t$. Hence,

$$\widehat{g}_z^{\text{FS}} = \frac{1}{\nu_t\, \widehat{m}_{t,z,h}} \cdot \left.\frac{\mathrm{d}u}{\mathrm{d}h}\right|_{h=z}.$$

Next, from the indirect utility function in (19),

$$u(m_h, P_{g,h}, P_{s,h}; \nu_t) = V\left(\widehat{e}_{t,h}, P_{g,h}, P_{s,h}\right) + \nu_t(m_h - \widehat{e}_{t,h}), \quad \text{with} \quad \widehat{e}_{t,h} = \left(\nu_t P_{s,h}^{\chi}\right)^{\frac{1}{\chi-1}},$$

we differentiate:

$$\frac{\mathrm{d}u}{\mathrm{d}h} = \frac{\partial V}{\partial e} \cdot \frac{\mathrm{d}\widehat{e}_h}{\mathrm{d}h} + \frac{\partial V}{\partial P_{g,h}} \cdot \frac{\mathrm{d}P_{g,h}}{\mathrm{d}h} + \frac{\partial V}{\partial P_{s,h}} \cdot \frac{\mathrm{d}P_{s,h}}{\mathrm{d}h} - \nu_t \cdot \frac{\mathrm{d}\widehat{e}_h}{\mathrm{d}h} + \nu_t \cdot \frac{\mathrm{d}m_h}{\mathrm{d}h}.$$

From

$$V(e, P_g, P_s) = \frac{1}{\chi}\left(\frac{e}{P_s}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_g}{P_s}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma},$$

we compute the partial derivatives:

$$\frac{\partial V}{\partial e} = \left(\frac{e}{P_s}\right)^{\chi-1} \cdot \frac{1}{P_s}, \quad \frac{\partial V}{\partial P_g} = -\eta\left(\frac{P_g}{P_s}\right)^{\gamma} \cdot \frac{1}{P_g},$$

$$\frac{\partial V}{\partial P_s} = -\left(\frac{e}{P_s}\right)^{\chi} \cdot \frac{1}{P_s} + \eta\left(\frac{P_g}{P_s}\right)^{\gamma} \cdot \frac{1}{P_s}.$$

From the definition of $\widehat{e}_h$, we find that the terms in $\mathrm{d}\widehat{e}_h/\mathrm{d}h$ cancel. Substituting into the total derivative:

$$\begin{aligned} \left.\frac{\mathrm{d}u}{\mathrm{d}h}\right|_{h=z} &= -\eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} \cdot g_{P_g,z} \\ &\quad + \left(\eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{\widehat{e}_{t,z}}{P_{s,z}}\right)^{\chi}\right) \cdot g_{P_s,z} \\ &\quad + \nu_t \cdot \dot{m}_z. \end{aligned}$$

Now we express the price effects in terms of observable shares using Roy's identity:

$$s_{g,t} = \frac{P_{g,t}c_{g,t}}{e_t} = \eta\left(\frac{e_t}{P_{s,t}}\right)^{-\chi}\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} \quad \Rightarrow \quad \eta\left(\frac{P_{g,t}}{P_{s,t}}\right)^{\gamma} = s_{g,t}\left(\frac{e_t}{P_{s,t}}\right)^{\chi}.$$

Rewriting the price derivative terms in the total derivative:

$$\begin{aligned} &-\eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} \cdot g_{P_g,z} + \left(\eta\left(\frac{P_{g,z}}{P_{s,z}}\right)^{\gamma} - \left(\frac{\widehat{e}_{t,z}}{P_{s,z}}\right)^{\chi}\right) \cdot g_{P_s,z} \\ &= -\left(\frac{\widehat{e}_{t,z}}{P_{s,z}}\right)^{\chi} \cdot \left(s_{g,z}\, g_{P_g,z} + (1 - s_{g,z}) g_{P_s,z}\right) \\ &= \widehat{e}_{t,z}\nu_t\left(s_{g,z}\, g_{P_g,z} + (1 - s_{g,z}) g_{P_s,z}\right). \end{aligned}$$

The last equality derives from $\widehat{e}_{t,z} = \left(\nu_t P^{\chi}_{s,z}\right)^{\frac{1}{\chi-1}}$.

Now return to the full expression:

$$\widehat{g}^{\text{FS}}_z = \frac{\dot{m}_z}{\widehat{m}_{t,z,h}} - \frac{\widehat{e}_{t,z}}{\widehat{m}_{t,z,h}}\Big(s_{g,z}\, g_{P_g,z} - (1-s_{g,z})g_{P_s,z}\Big).$$

From $m_t = e_t + x_t$, and $e_t = P_{g,t}c_{g,t} + P_{s,t}c_{s,t}$, we get

$$\frac{\dot{m}_t}{m_t} = s_{e,t}g_{e,t} + (1-s_{e,t})g_{x,t},$$

$$g_{e,t} = s_{g,t}(g_{g,t} + g_{P_{g,t}}) + (1-s_{g,t})(g_{s,t} + g_{P_{s,t}}),$$

so that all price terms cancel: (CHECK FROM HERE)

$$\begin{aligned}\widehat{g}^{\text{FS}}_z = s_{e,z}\Big(s_{g,z}g_{g,z} + (1-s_{g,z})g_{s,z} + s_{g,z}g_{P_{g,z}} + (1-s_{g,z})g_{P_{s,z}}\Big) + (1-s_{e,z})g_{x,z} - \\ - \bar{s}_{e,t,z}\Big(s_{g,z}\, g_{P_{g,z}} + (1-s_{g,z})g_{P_{s,z}}\Big),\end{aligned}$$

where $\bar{s}_{e,t,z} = \frac{\widehat{e}_{t,z}}{m_z}$. Then

$$\widehat{g}^{\text{FS}}_z = s_{e,t}\left(s_{g,t}g_{g,t} + (1-s_{g,t})g_{s,t}\right) + (1-s_{e,t})g_{x,t} = g^D_t. \quad \square$$

F Laspeyres and Paasche indices

This appendix shows that, in an economy with constant income shares, the substitution bias does not explain why the Laspeyres index exceeds the Paasche index. Instead, this discrepancy results from a more fundamental property: the Laspeyres index is a weighted aritmethic mean while the Paasche index is a weighted harmonic mean. This fundamental property was already known in the XIX century and is clearly mentioned by Irving Fisher (1911), who wrote "the Paasche price index can be written as a current-period revenue share–weighted harmonic average." Let us develop the argument in our framework.

In the context of the LBD economy of Section 2 with two goods, consumption and investment, the Laspeyres and Paasche indices between s and t, $s < t$, are defined as

$$\mathcal{L}_{t,s} = \frac{c_t + p_s x_t}{c_s + p_s x_s} \quad \text{and} \quad \mathcal{P}_{t,s} = \frac{c_t + p_t x_t}{c_s + p_t x_s}.$$

In an economy with constant income shares, at it is the case in the LBD economy, the indices read

$$\mathcal{L}_{t,s} = s_c g_c + (1 - s_c) g_x \quad \text{and} \quad \mathcal{P}_{t,s} = \left(s_c g_c^{-1} + (1 - s_c) g_x^{-1}\right)^{-1},$$

where the time-invariant consumption share $s_c = \frac{c_t}{c_t + p_t c_t}$; g_c and g_x are the time-invariant growth rates between s and t of consumption and investment, respectively. The Laspeyres index measures the weighted average gains from moving forward, being equal to a Divisia index, while the Paasche index measures the inverse of the weighted average loses from moving backward. They are arithmetic and harmonic means, respectively

After some simple algebra, we can easily show that the ratio

$$\frac{\mathcal{L}}{\mathcal{P}} = 1 + s_c(1 - s_c)\left(\frac{g_c}{g_x} + \frac{g_x}{g_c} - 2\right) \geq 1.$$

Since $g_c^2 + g_x^2 \geq 2 g_c g_x$, by the Cauchy-Schwarz inequality, we have the well-know result in index number theory that $\mathcal{L} \geq \mathcal{P}$, with equality only if $g_c = g_x$.

Let us call z to the ratio of both growth rates, then

$$\frac{\mathcal{L}}{\mathcal{P}} = 1 + s_c(1 - s_c)\left(z + \frac{1}{z} - 2\right) \quad \text{and} \quad \frac{\partial \frac{\mathcal{L}}{\mathcal{P}}}{\partial z} = s_c(1 - s_c)\left(1 - \frac{1}{z^2}\right) \geq 0,$$

irrespective of z being g_c/g_x or the inverse. The distance between $\mathcal{L}$ and $\mathcal{P}$ increases with the distance between g_c and g_x. The result does not depend on whether investment is growing faster than consumption, but on the distance between both growth rates as measured by z.

Finally, for given growth rates, provided $g_c \neq g_x$, the ratio $\frac{\mathcal{L}}{\mathcal{P}}$ is hump-shaped with respect to s_c, reaching its maximum when $s_c = \frac{1}{2}$. This can be explained as follows: when $s_c = \frac{1}{2}$, the Laspeyres index $\mathcal{L}$ and the Paasche index $\mathcal{P}$ are balanced in terms of the weight given to g_c and g_x. This equal weighting maximizes the effect of the divergence between g_c and g_x on $\frac{\mathcal{L}}{\mathcal{P}}$, making the ratio the largest at $s_c = \frac{1}{2}$. For values of s_c closer to 0 or 1, the ratio decreases as one of the growth rates dominates the calculation, reducing the impact of the difference between g_c and g_x.

In this context, a larger decline rate of the relative investment price does not generate any substitution effect, since at equilibrium consumption and investment shares remain constant, but raises the distance between the growth rate of consumption and the

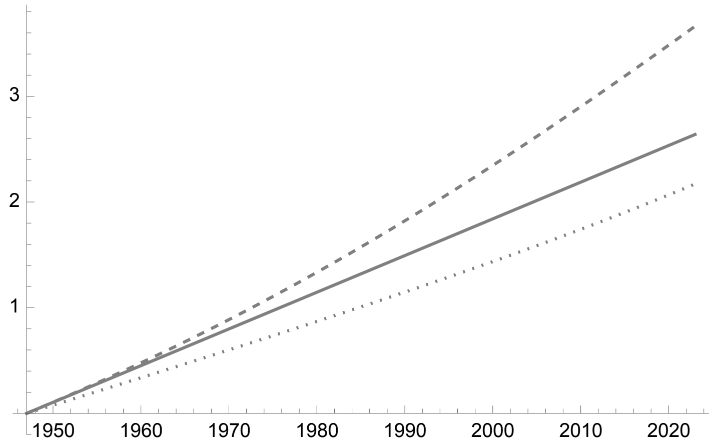


Figure 6: Real GDP

BEA data. Real GDP is measured as a chained Fisher-ideal index (solid), a 1947 base-year index (dashed) and a 2023 base-year index (dotted)

growth rate of investment. As a consequence, it also increases the distance between the corresponding Paasche and Laspeyres indices. Figure 6 shows the 1947 and 2023 fixed-base indices for an economy with a 3.3% annual decline of the relative investment price, instead of the observed 2.06% as reported in Table **??**. When compared to Figure 2, the distance in 2023 between the two indices jumped up to 106%. The fundamental reason is that the difference between the annual investment growth rate and the annual consumption growth rate raised from 2.59% in the benchmark economy to 3.25% in the economy with larger decline in the relative price of investment.

Notice that, in our framework, a Paasche index can be written as

$$\mathcal{P}_{t,s} = \widehat{s}_c g_c + (1 - \widehat{s}_c) g_x = \left(s_c g_c^{-1} + (1 - s_c) g_x^{-1} \right)^{-1},$$

where

$$\widehat{s}_c = \frac{s_c g_x}{(1 - s_c) g_c + s_c g_x} > s_c.$$

The inequality holds because $g_x > g_c$. Since the Paasche index gives to consumption a weight larger than its income share, and consumption growth at a lower rate than investment, the Paasche index is smaller than the Laspeyres index, irespective of any substitution bias. The larger g_x is, the higher the consumption weight $\widehat{s}_c$ and the lower the Paasche index.

More generally, let us assume there is a set of n items with quantities and prices $\mathbf{x}_t = \{x_{1,t}x_{2,t}, ..., x_{n,t}\}$ and $\mathbf{p}_t = \{p_{1,t}, p_{2,t}, ..., p_{n,t}\}$, respectively. For this set of items, let us compare times t and s, $t > s$, by the mean of the ratio of the corresponding Laspeyres and Paasche quantity indices, s.t.,

$$\frac{\mathcal{L}_{t,s}}{\mathcal{P}_{t,s}} = \frac{\mathbf{p}_s\mathbf{x}_t}{\mathbf{p}_s\mathbf{x}_s} \times \frac{\mathbf{p}_t\mathbf{x}_s}{\mathbf{p}_t\mathbf{x}_t}$$

Let us assume that the shares $s_i = \frac{p_{i,t}x_{i,t}}{\mathbf{p}_t\mathbf{x}_t}$ are time invariant, then

$$\frac{\mathcal{L}_{t,s}}{\mathcal{P}_{t,s}} = \left(\sum s_i g_{i,t}\right)\left(\sum s_i g_{i,t}^{-1}\right) \geq 1$$

where $g_{i,t} = \frac{x_{i,t}}{x_{i,s}}$ is the growth factor of item i between s and t. The object at the right hand side is the product of the mean and the harmonic mean, represented by the Laspeyres and the inverse of the Paasche, respectively. The property that $\frac{\mathcal{L}_{t,s}}{\mathcal{P}_{t,s}}$ is larger than one is known as the *arithmetic-harmonic mean inequality.* The left-hand side of the inequality is increasing on the variance of vector $\mathbf{g} = \{g_1, g_2, ..., g_n\}$.

In an economy with log preferences defined on a vector $\mathbf{x}_t$, income shares are equal over time irrespective of prices, since income effect and substitution effect compensate each other. In this framework, the disperion of quantities will depend on relative price trends, making the Laspeyres index to overestimate growth relative to the Paasche index. It makes clear that this bad property of fixed-base indices may be unrelated to the substitution bias.

Interestingly, the introduction of quality corrections in prices, by definition, changes the growth rate of quantities without affecting the income shares. They cannot then produce any substitution bias. However, by changing the growth rate of quantities differently, affect the variance of their growth rates. There is no economic reason, but it is a property of the indices themselves.

G Past- and Current-base BBEV indices

It is easy to show that the past-based BBEV is always larger than the current-base BBEV. Let us define the current-base BBEV index as

$$\mathcal{P}_{t,s}^{\mathrm{BB}} = \frac{m_t}{m_{t,s}} = m_t\left(\frac{p_t}{\nu_t}(\log p_s - \log p_t) + \frac{p_t m_s}{p_s}\right)^{-1}$$

and the past-base BBEV index as

$$\mathcal{L}^{\mathrm{BB}}_{t,s} = \frac{m_{s,t}}{m_s} = \frac{1}{m_s}\left(\frac{p_s}{\nu_s}\left(\log p_t - \log p_s\right) + \frac{p_s m_t}{p_t}\right),$$

where from equation (??)

$$m_{t,s} = \frac{p_t}{\nu_t}\left(\log p_s - \log p_t\right) + \frac{p_t m_s}{p_s}.$$

Consequently, at the equilibrium of the LBD economy,

$$\begin{aligned}
\frac{\mathcal{L}^{\mathrm{BB}}_{t,s}}{\mathcal{P}^{\mathrm{BB}}_{t,s}} &= \left(\frac{p_s}{\nu_s}\frac{1}{m_t}\left(\log p_t - \log p_s\right) + \frac{p_s}{p_t}\right)\left(\frac{p_t}{\nu_t}\frac{1}{m_s}\left(\log p_s - \log p_t\right) + \frac{p_t}{p_s}\right) \\
&= \left(\frac{p_t}{\nu_s}\frac{1}{m_t}\left(\log p_t - \log p_s\right) + 1\right)\left(\frac{p_s}{\nu_t}\frac{1}{m_s}\left(\log p_s - \log p_t\right) + 1\right) \\
&= \left(1 - s_c\lambda g_k\, \mathrm{e}^{-g_k(t-s)}(t-s)\right)\left(1 + s_c\lambda g_k\, \mathrm{e}^{g_k(t-s)}(t-s)\right)
\end{aligned}$$

The right hand side of the last line can be written as

$$h(x) \doteq \left(1 - a g_k\, \mathrm{e}^{-g_k x} x\right)\left(1 + a g_k\, \mathrm{e}^{g_k x} x\right),$$

with $h(0) = 1$, where $x = t - s \geq 0$ and $a = s_c\lambda \in (0,1)$. The first and second derivatives are

$$h'(x) = a\mathrm{e}^{-g_k x} g\left(\mathrm{e}^{2g_k x} - 1 + g_k x\left(1 + \mathrm{e}^{g_k x}(\mathrm{e}^{g_k x} - 2a)\right)\right)$$

and

$$h''(x) = a\mathrm{e}^{-g_k x} g_k^2\left(2 + 2\mathrm{e}^{g_k x}(\mathrm{e}^{g_k x} - a) + g_k x(\mathrm{e}^{2g_k x} - 1)\right) > 0,$$

with $h'(0) = 0$ and $h''(0) = 4 - 2a > 0$, since $a \in (0,1)$ and $g_k > 0$. Consequently, $\mathcal{L}^{\mathrm{BB}}_{t,s} > \mathcal{P}^{\mathrm{BB}}_{t,s}$ for all $s < t$. The ratio of the past-base BBEV to the current-base BBEV behaves like the ratio of the Laspeyres to the Paasche indices.

H Learning-by-Doing and the Intertemporal Elasticity of Substitution

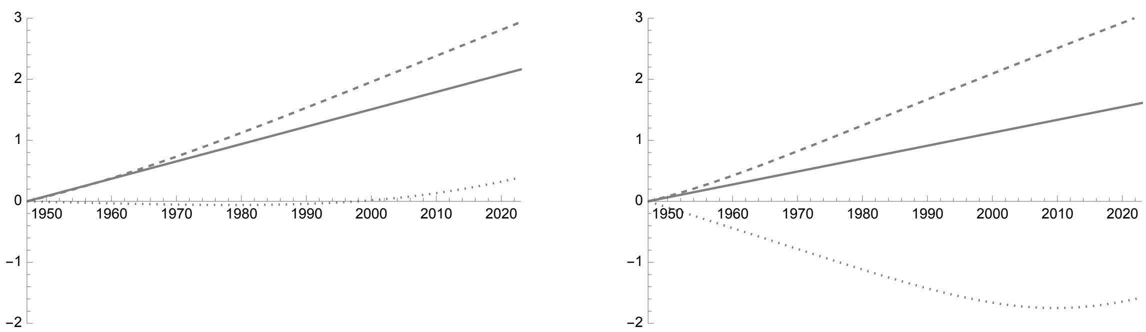
In the general case, when the constant intertemporal elasticity of substitution (CIES) σ is different from one, the indirect utility function (19) and the expenditure functions (??) read

$$u(u_t, p_t; \nu_t) = \frac{\nu_t}{p_t} m_t + \frac{1}{\sigma - 1}\left(\frac{p_t}{\nu_t}\right)^{\sigma-1} \quad \text{and} \quad e(u_t, p_t; \nu_t) = \frac{p_t}{\nu_t} u_t - \frac{1}{\sigma - 1}\left(\frac{p_t}{\nu_t}\right)^{\sigma}.$$

Consequently, for $s < t$, t being the base-time, the hypothetical income at time s is

$$\widehat{m}_{t,s} = \frac{\nu_t^{-\sigma}}{\sigma - 1} p_t \left(p_s^{\sigma-1} - p_t^{\sigma-1} \right) + \frac{p_t m_s}{p_s},$$

which converges to (**??**) when $\sigma \to 1$. We have simulated the economy for a large range of values of $\sigma > 0$. The result is qualitatively similar to Figure 7. The larger the intertemporal elasticity of substitution, the larger the substitution bias, making the fixed-base BBEV measure to diverge more relative to the chained Divisia. Moreover, discrepancies are still substantial even for low levels of the IES.



(a) $\sigma = 0.075$

(b) $\sigma = 2.5$

Figure 7: Comparison of Fisher-Shell and Fixed-Base BBEV metrics.

Chained Fisher-Shell (solid), BBEV 1947-fixed-base (dashed), and BBEV 2023-fixed-based (dotted)

Elasticity of substitution between c and x in the Bellman representation. In the general case of CIES, from the household problem, the ratio of consumption to net investment is

$$\frac{c}{x} = \frac{p \left(\frac{p}{\nu}\right)^{\sigma}}{m - \left(\frac{p}{\nu}\right)^{\sigma}} = p \left(m \left(\frac{\nu}{p} \right)^{\sigma} - 1 \right)^{-1}.$$

The first derivative with respect to the relative price p is

$$\frac{\partial \left(\frac{c}{x}\right)}{\partial p} = \frac{c}{px} \left(1 + \sigma \frac{m}{px} \right).$$

The elasticity of substitution between c and x in the Bellman representation is then $\xi = 1 + \frac{\sigma}{1-\widehat{s}_c}$, with $1 - \widehat{s}_c = \frac{\delta + g_k}{zq}$ and $g_k = \frac{\sigma(\alpha z q - \delta - \rho)}{1 - \lambda + \sigma\lambda}$. It is clear that the elasticity of substitution between c and x in the Bellman representation, ξ, is positively related to the intertemporal elasticity of substitution, σ. A higher σ implies a greater responsiveness of consumption to an increase in the relative price of investment, as households become more willing to substitute present for future consumption.

Consequently, as the relative price of investment declines permanently, a higher intertemporal elasticity of substitution leads to a greater fictitious substitution bias in the fixed-base BBEV index, amplifying the distortion in the BBEV measure when comparing the present with the distant past.

Replication Files

Supplementary replication materials ca be found at XXXX.

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