Chained Indices Unchained:

On the Welfare Foundations of Income Growth Measurement II

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Abstract

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1 Introduction

2 Structural Change Model

This section fundamentally follows Herrendorf et al. (2021).

Description of technology. Let's assume there are two sectors, goods and services. Value added in both sectors is produced according to the Cobb–Douglas production technologies

$$Y_{jt} = A_{jt} K_{jt}^{\theta} L_{jt}^{1-\theta}, \tag{1}$$

where $j \in \{g, s\}$ indexes the goods and services sectors, respectively. These production functions share the same capital intensity, represented by $\theta \in (0, 1)$, but may have different total factor productivities, A_{jt}. Production in each sector relies on homogeneous production factors, capital K_{jt} and labor L_{jt}, which can move freely between sectors. Consequently,

$$K_{gt} + K_{st} = K_t, \quad L_{gt} + L_{st} = L_t,$$

where K_t and L_t represent total capital and total labor, respectively.

Investment is produced using the CES technology

$$I_t = A_{xt} \left(\omega^{\frac{1}{\varepsilon}} X_{gt}^{\frac{\varepsilon-1}{\varepsilon}} + (1-\omega)^{\frac{1}{\varepsilon}} X_{st}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

where X_{gt} and X_{st} represent goods and services inputs, respectively, ε is the elasticity of substitution between them, and $\omega \in (0, 1)$ determines the weight of sectorial inputs. The state of investment-specific technology is neutral with respect to inputs and represented by the investment specific productivity, A_{xt}.

Capital depreciates at a rate $\delta > 0$, following the law of motion:

$$\dot{K}_t = I_t - \delta K_t. \tag{2}$$

At equilibrium, the feasibility conditions impose the following constraints:

$$C_{gt} + X_{gt} = Y_{gt}, \quad C_{st} + X_{st} = Y_{st},$$

where C_{gt} and C_{st} are the consumption of goods and services, respectively.

Equilibrium prices. Let us adopt the investment good as numeraire. It is easy to show that the price of goods, P_{gt}, relative to the price of services, P_{st},

$$\frac{P_{gt}}{P_{st}} = \frac{A_{st}}{A_{gt}},$$

is equal to the inverse of the relative sectorial TFPs. Notice that a dupla of production factors (K, L) that produces $K^{\theta}L^{1-\theta} = 1$ has the same value in the goods and service sectors, since $P_{gt}A_{gt} = P_{st}A_{st}$.

Moreover, since we have adopted the investment good as numeraire, the prices of goods and services, relative to the investment good, read

$$\boxed{P_{jt} = \frac{\mathcal{A}_t}{A_{jt}}, \quad j \in \{g, s\}.} \tag{3}$$

where

$$\boxed{\mathcal{A}_t = A_{xt}\left(\omega A_{gt}^{\varepsilon-1} + (1-\omega)A_{st}^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-1}}.} \tag{4}$$

As shown below, $\mathcal{A}_t$ is the productivity of the investment sector.

In the investment sector, at equilibrium, the ratio of expenditure shares and input quantities on goods and services are, respectively, given by

$$\frac{P_{gt}X_{gt}}{P_{st}X_{st}} = \frac{\omega}{1-\omega}\left(\frac{A_{st}}{A_{gt}}\right)^{1-\varepsilon} \quad \text{and} \quad \frac{X_{gt}}{X_{st}} = \frac{\omega}{1-\omega}\left(\frac{A_{gt}}{A_{st}}\right)^{\varepsilon}.$$

Notice that if goods and services are complementary in the investment technology, i.e. $\varepsilon \in (0, 1)$, and technical progress is faster in the production of goods, relative to services, then the value added of the goods sector shrinks while real production increases, relative to the service sector.

In the following, we assume that $A_{gt} = A_{g0}\,\mathrm{e}^{\gamma_g t}$, $A_{st} = A_{s0}\,\mathrm{e}^{\gamma_s t}$, and $\mathcal{A}_t = \mathcal{A}_0\,\mathrm{e}^{\gamma_{\mathcal{A}} t}$, $\gamma_x > \gamma_g > \gamma_s$.[1] As a consequence, the growth rate of P_{gt} and P_{st} are g_{P_g} and g_{P_s}, respectively, with $0 < g_{P_g} = \gamma_x - \gamma_g < \gamma_x - \gamma_s = g_{P_s}$. Notice that A_{xt} is residually given by equation (4).

Aggregate investment technology. At equilibrium, from Lemma 1 in Herrendorf et al. (2021), the aggregate investment technology is

$$I_t = \mathcal{A}_t\left(\frac{X_{gt}}{A_{gt}} + \frac{X_{st}}{A_{st}}\right) = \mathcal{A}_t K_{xt}^{\theta} L_{xt}^{1-\theta}, \tag{5}$$

[1] In (4), A_{xt} accommodates for this assumption to be true.

where L_{xt} and K_{xt} are defined as

$$L_{xt} = \lambda_t L_t \quad \text{and} \quad K_{xt} = \lambda_t K_t, \quad \text{with} \quad \lambda_t = \frac{X_{gt}}{A_{gt} K_t^\theta} + \frac{X_{st}}{A_{st} K_t^\theta}.$$

Let's think on $\lambda \in (0,1)$ as the fraction of employment and capital allocated to produce inputs for the investment sector. (check this!)

Aggregate production technology. Let us define aggregate final output, measured in units of final investment, as:

$$Y_t = P_{gt} C_{gt} + P_{st} C_{st} + I_t. \tag{6}$$

It can be easily shown that the aggregate production technology, as measured in units of the investment good, is

$$Y_t = \mathcal{A}_t K_t^\theta L_t^{1-\theta}. \tag{7}$$

(check it!)

Aggregate dynamics. From the equations (2), (6) and (7), for $t \geq 0$, given the exogenous path of $\mathcal{A}_t$ and an initial stock of capital $K_0 > 0$, the law of motion for capital reads

$$\dot{K}_t = \underbrace{\mathcal{A}_t K_t^\theta L_t^{1-\theta} - E_t}_{I_t} - \delta K_t.$$

where

$$E_t = P_{gt} C_{gt} + P_{st} C_{st} \tag{8}$$

is total consumption expenditure in units of the investment good.

Non-homothetic preferences. Population is a mass N_t growing at rate $n > 0$. At any time t, each individual offers h_t units of human capital, exogenously growing at the rate $\gamma_h > 0$. Households inelastically supply $L_t = h_t N_t$.

The economy features an infinitely lived representative household which preferences are represented by the following intertemporal utility function

$$\int_0^\infty U(c_{gt}, c_{st}) \, \mathrm{e}^{(n-\rho)t} \, \mathrm{d}t.$$

The discount rate is given by $\rho > n$. The instantaneous utility function $U(\cdot,\cdot)$ is assumed to be a price-independent-generalized-linear indirect utility (PIGL henceforth). It depends on per capita consumption $c_{gt} = C_{gt}/N_t$ and $c_{st} = C_{st}/N_t$. Since the PIGL class generally lacks a known direct utility representation, we work with its indirect utility formulation $V(e_t, P_{gt}, P_{st})$, where $e_t = E_t/N_t$ is per capita consumption expenditure. Following Boppart (2014), let us assume

Assumption 1. *The instantaneous utility function $U(c_{gt}, c_{st})$ is PIGL with indirect utility representation*

$$V(e_t, P_{gt}, P_{st}) = \frac{1}{\chi}\left(\frac{e_t}{P_{st}}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}, \tag{9}$$

where e_t is consumption expenditure per capita, $\eta > 0$ and $1 > \gamma > \chi > 0$.

Intertemporal problem and intratemporal allocation. Under Assumption 1, the representative household choses a path $\{e_t, k_t\}$, for consumption expenditure and capital per capita, that solves the following dynamic problem

$$v(k_t) = \max \int_0^{\infty} \frac{e_t^{\chi}}{\chi}\,\Gamma_t\,dt$$

subject to

$$\dot{k}_t = \widehat{\mathcal{A}}_t k_t^{\theta} - e_t - (\delta + n)k_t, \tag{10}$$

where $k_t = K_t/N_t$ and $\widehat{\mathcal{A}}_t = \mathcal{A}_t h_t^{1-\theta}$. The discount factor, $\Gamma_t = P_{st}^{-\chi}\mathrm{e}^{(n-\rho)t}$, is smaller than one and declining over time since P_{st}, as in the data, is assumed to be increasing over time. Preferences are then CIES on the path of consumption expenditure, with elasticity of substitution larger than one.[2] All other terms in (9) are excluded as they are additive; their discounted integral remains independent of control and state variables, having no impact on the determination of the optimal path.

The Euler equation associated to the household problem above is

$$\frac{\dot{e}_t}{e_t} = \frac{1}{1-\chi}\left(\theta\widehat{\mathcal{A}}_t k_t^{\theta-1} - \rho - \delta - \chi\frac{\dot{P}_{st}}{P_{st}}\right). \tag{11}$$

The equilibrium path solves then (10) and (11), given k_0.

[2] Notice that in this framework, the intertemporal elasticity of substitution is $\frac{1}{1-\chi}$.

The intratemporal allocation of e_t to c_{gt} and c_{st} results from the use of Roy's Identity to derive the expenditure share of goods:

$$\frac{P_{gt}c_{gt}}{e_t} = \eta \left(\frac{e_t}{P_{st}}\right)^{-\chi} \left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}. \tag{12}$$

The last equation solves for c_{gt} and then c_{st} can be obtained inverting the definition of consumption expenditure e_t.

Aggregate Balanced Growth Path (ABGP). At the ABGP, per capita aggregates $\{k_t, e_t, y_t\}$, they all grow at the constant rate $g_k = \frac{\gamma_{\mathcal{A}}}{1-\theta} + \gamma_h$. From the Euler equation (11), the stock of capital follows

$$k_t^* = \kappa^{\frac{1}{\theta-1}} \widehat{\mathcal{A}}_t^{\frac{1}{1-\theta}} \qquad \text{where} \qquad \kappa \doteq \frac{\rho + \delta + \chi g_{P_s} + (1-\chi) g_k}{\theta} \tag{13}$$

is the user cost of capital divided by θ. From (10), consumption expenditure follows

$$e_t^* = (\kappa - \delta - n - g_k) k_t^*. \tag{14}$$

From (7), gross nominal income per capita is

$$y_t^* = \widehat{\mathcal{A}}_t k_t^{*\,\theta} = \kappa k_t^*. \tag{15}$$

The last equality directly derives from (13). Notice that the consumption share of gross income is

$$\frac{e_t^*}{y_t^*} = \frac{\kappa - \delta - n - g_k}{\kappa}.$$

Bellman representation. Following Duran and Licandro (2025), the Bellman representation at time t of the representative household problem is

$$W(c_{gt}, c_{st}, x_t; \nu_t) = U(c_{gt}, c_{st}) + \nu_t x_t, \tag{16}$$

where $x_t = \dot{k}_t$ is net investment and $\nu_t = v'(k_t)$ is the marginal value of capita per capital at time t. In the Bellman representation, preferences at t are indexed by the marginal value of capital ν_t. However, it's very important to stress that y_t is not real expenditure per capital but nominal expenditure per capita, and as such $Y_t = y_t N_t$ is our measure of nominal GDP. The representative household maximises (16) with respect to $\{c_g, c_s, x\}$, subject to the budget constraint

$$P_{gt}c_{gt} + P_{st}c_{st} + x_t = m_t, \tag{17}$$

where $m_t = y_t - \delta k_t$ is current net income per capita. Notice that consumption expenditure per capita is $e_t = m_t - x_t$.

Proposition 1. *The indirect utility and expenditure functions associated to the Bellman representation of preferences in (16) are, respectively,*

$$u(m_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t\left(m_t - (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}\right) \tag{18}$$

and

$$e(W_t, P_{gt}, P_{st}; \nu_t) = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} + \frac{W_t}{\nu_t} - \frac{V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right)}{\nu_t}. \tag{19}$$

Proof: The time-t household primal problem of maximising (16) subject to (17) may be solved in two stages. At the first stage, by definition of an indirect utility function,

$$V(e_t, P_{gt}, P_{st}) = \max_{\{c_{gt}, c_{st}\}: P_{gt}c_{gt} + P_{st}c_{st} = e_t} U(c_{gt}, c_{st}).$$

At the second stage, solve

$$\max_x V(m_t - x, P_{gt}, P_{st}) + \nu_t x.$$

The F.O.C. is

$$\left(\frac{e_t}{P_{st}}\right)^{\chi} \frac{1}{e_t} = \nu_t,$$

or equivalently,

$$e_t = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}.$$

Since

$$x_t = m_t - e_t,$$

then

$$x_t = m_t - (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}.$$

Consequently, the indirect utility function associated to the Bellman representation of preferences (16) is

$$u(m_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t\left(m_t - (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}\right).$$

The expenditure function is the solution of the corresponding dual problem

$$\max_{e,x} \; e + x \quad \text{s.t.} \quad V(e, P_{gt}, P_{st}) + \nu_t x = w_t$$

The F.O.C.'s are

$$1 = \lambda e^{\chi-1} P_{st}^{-\chi} \quad \text{and} \quad 1 = \lambda \nu_t.$$

Then,

$$e = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}},$$

and

$$x = \frac{w_t - V\Big((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\Big)}{\nu_t}.$$

Consequently

$$e(w_t, P_{gt}, P_{st}; \nu_t) = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} + \frac{w_t - V\Big((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\Big)}{\nu_t}.$$

The last equation completes the proof. $\square$

Equivalent variation measure. Based on the Fisher and Shell principle that welfare comparisons must be done using the same preference set, and in line with Baqaee and Burstein (2023) equivalent variation measure –see their Definition 4, let us define the hypothetical income at z, for $z < t$,

$$\widehat{m}_{tz} = e\Big(u\Big(m_z, P_{gz}, P_{sz}; \nu_t\Big), P_{gt}, P_{st}; \nu_t\Big). \tag{20}$$

$\widehat{m}_{tz}$ is the level of income per capita at current prices that the representative household would have needed at time z to attain the utility achievable with past income and prices but evaluated using the current Bellman representation of preferences.

Fixed-base indices. Let us adopt the following convention for an economy where an Office for National Statistics have recorded National Accounts data from some initial time t_0 to the current time t. In this economy, a current-base equivalent variation index, for $s \in \{t_0, t\}$, is

$$\mathcal{P}_{t,s} = \log(\widehat{m}_{t,s}) - \log(\widehat{m}_{t,t_0}). \tag{21}$$

Notice that the index is normalized to $\mathcal{P}_{t,t_0} = 0$, such that $\mathcal{P}_{t,t} = \log(m_t) - \log(\widehat{m}_{t,t_0})$ measures welfare gains from the initial time t_0 to the current time t, and $\mathcal{P}_{t,t} - \mathcal{P}_{t,s} = \log(m_t) - \log(\widehat{m}_{t,s})$ welfare gains from any time $s \in (t_0, t)$ to t. Implicit on this index, the equilibrium instantaneous growth rate of the economy at s is measured by

$$\frac{\partial \mathcal{P}_{t,s}}{\partial s} = \frac{\partial \widehat{m}_{t,s}}{\partial s}. \tag{22}$$

As we show below, the instantaneous growth rate at t of the current-base equivalent variation index, $\frac{\partial \mathcal{P}_{t,s}}{\partial s}|_{s=t}$, is equal to the Divisia index at t. For any $s < t$, the instantaneous growth rate is lower than the Divisia index and declines as the welfare evaluation refers to a more distant point in the past. (prove it)

Fisher-Shell index. Following Duran and Licandro (2025), in order to show the Fisher-Shell index is equal to the Divisia index, we must compute the total derivative of $\widehat{M}_{tz}$ with respect to z and evaluate it at $z = t$. From the main text,

$$\widehat{m}_{tz} = e\Big(u\big(m_z, P_{gz}, P_{sz}; \nu_t\big), P_{gt}, P_{st}; \nu_t\Big), \tag{20}$$

with

$$u(m_t, P_{gt}, P_{st}; \nu_t) = V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right) + \nu_t \left(m_t - (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}\right), \tag{18}$$

$$e(w_t, P_{gt}, P_{st}; \nu_t) = (\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} + \frac{w_t}{\nu_t} - \frac{V\left((\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}}, P_{gt}, P_{st}\right)}{\nu_t}, \tag{19}$$

and

$$V(e_t, P_{gt}, P_{st}) = \frac{1}{\chi}\left(\frac{e_t}{P_{st}}\right)^{\chi} - \frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma} - \frac{1}{\chi} + \frac{\eta}{\gamma}. \tag{9}$$

Combining them, we get

$$\widehat{m}_{tz} = \frac{1-\chi}{\chi}(\nu_t P_{st}^{\chi})^{\frac{1}{\chi-1}} - \frac{1}{\nu_t}\frac{\eta}{\gamma}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma} + m_z + \text{other terms that don't depend on } z$$

In order to define the per capita Fisher-Shell index, we take time derivatives with respect to z and evaluate them at $z = t$, such that,

$$g_t^{\mathrm{FS}} \doteq \left.\frac{\mathrm{d}\log \widehat{m}_{tz}}{\mathrm{d}z}\right|_{z=t} = \frac{\dot{m}_t}{m_t} - \frac{e_t}{m_t} g_{P_{st}} - \frac{\eta}{\nu_t m_t}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}\big(g_{P_g t} - g_{P_s t}\big).$$

Since

$$\frac{\dot{m}_t}{m_t} = s_{et}\, g_{e_t} + (1 - s_{et}) g_{xt},$$

where $s_e = e/m$=E/M, and

$$g_{et} = s_{gt}\big(g_{gt} + g_{P_g t}\big) + s_{st}\big(g_{st} + g_{P_s t}\big),$$

the Fisher-Shell index becomes

$$g_t^{\mathrm{FS}} = g_t^{D} + \frac{P_{gt} c_{gt}}{m_t}\big(g_{P_g t} - g_{P_g t}\big) - \frac{\eta}{\nu_t m_t}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}\big(g_{P_g t} - g_{P_s t}\big).$$

From (12),

$$\frac{P_{gt}c_{gt}}{e_t} = \eta \left(\frac{e_t}{P_{st}}\right)^{-\chi} \left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}.$$

Since at equilibrium $e_t^{\chi-1} = \nu_t P_{st}^{\chi}$, we can easily show that

$$g_t^{\mathrm{FS}} = g_t^D.$$

Divisia index. At the ABGP of the HRV's economy, the Divisia index for NDP per capita is

$$g_t^D = s_e\big(s_{gt}g_g + (1-s_{gt})g_s\big) + (1-s_e)g_x - n = g_t^D - n,$$

where the shares are $s_e \doteq \frac{e_t}{m_t} = \frac{\kappa-\delta-n-g_K}{\kappa}$ and, from (12) $s_{gt} = \eta\left(\frac{e_t}{P_{st}}\right)^{-\chi}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}$. The growth rate of investment is $g_x = g_K$ and the growth rates of aggregate goods and service consumptions are g_g and g_s, respectively. We are only missing the growth rate of consumption services. Notice that, from (12), real consumption services are

$$c_{st} = \frac{e_t}{P_{st}} - \eta\left(\frac{e_t}{P_{st}}\right)^{1-\chi}\left(\frac{P_{gt}}{P_{st}}\right)^{\gamma}.$$

Parameter	**Value**
Preferences	
ρ (discount rate)	0.04
χ	0.55
η	0.44
γ	0.69
Technology	
θ (capital share)	1/3
δ (depreciation rate)	0.08
ω	0.65
ε	0.00

Table 1: Herrendorf et al. (2021) calibration

Parameter values. Table 1 shows the parameters values used by Herrendorf et al. (2021) in their quantitative exercise. In the measurement of sectoral total factor productivity (TFP) they have followed a structured approach based on observable economic data. First, sector-specific TFP (A_{jt}), $j = \{g, s\}$, was estimated using data from WORLD KLEMS on real value added, capital and labor inputs. Given the aggregate capital share θ, they computed the sectoral TFP growth rates using

$$\widehat{A}_{jt} = \widehat{y}_{jt} - \theta \widehat{k}_{jt}, \quad j \in \{g, s\},$$

where $\widehat{x}$ measures the growth rate of variable x. All variables except $\widehat{A}_{jt}$ are directly observable. Normalizing initial TFP levels to $A_{j0} = 1$, they use the estimated growth rates to construct the time series for A_{jt}.

Next, they estimated aggregate investment-specific TFP (A_{xt}). Again, setting the initial condition $A_{x0} = 1$, they computed its growth rates using

$$\widehat{A}_{xt} = \widehat{X}_t - \frac{P_{gt}X_{gt}}{X_t}\widehat{X}_{gt} - \frac{P_{st}X_{st}}{X_t}\widehat{X}_{st},$$

where all components at the right-hand-side of this equation are observable.

Aggregate Balanced Growth Path (ABGP). Since from the primal problem of the household

$$\max V(e, P_{gt}, P_{st}) + \nu_t x \quad \text{subject to} \quad e + x = m_t.$$

From the FOC for e,

$$\nu_t = e_t^{1-\chi} P_{st}^{\chi}.$$

Consequently, at the ABGP, $g_\nu = (1 - \chi)g_e$. We have then all information required to compute the current-base equivalent variation measure in (21).

From (12), real consumption on goods is

$$c_{gt} = \eta \left(\frac{e_t}{P_{st}}\right)^{1-\chi} \left(\frac{P_{gt}}{P_{st}}\right)^{\gamma-1}, \tag{23}$$

which grows at the constant rate $g_g = (1 - \chi)g_k + (\gamma - 1)g_{P_g} + (\chi - \gamma)g_{P_s}$.

How compute the solution. In the following, to measure quantity indices, information on prices and quantities for consumption, investment and income will be required. The use of the investment good as numeraire, in this framework, is inconsequential. These are the steps to follow in order to measure the needed variables.

1. Assume that A_{gt}, A_{st} and $\mathcal{A}_t$ all three grow at different constant rates, with $\widehat{A}_g > \widehat{A}_s$.
2. Use equations (3) and (4) to solve for P_{gt} and P_{st}.
3. Solve the dynamic system (10) and (11) to compute e_t and k_t, and the value function $v(k_t)$.
4. Use (8) and (12) to solve for c_{gt} and c_{st}
5. Use (6) and (7) to solve for $i_t = I_t/L_t$ and $y_t = Y_t/L_t$.