Approximation of Pufferfish Privacy for Gaussian Priors

Ni Ding, *Member, IEEE*

Abstract—**This paper studies how to approximate pufferfish privacy when the adversary's prior belief of the published data is Gaussian distributed. Using Monge's optimal transport plan, we show that (ϵ, δ)-pufferfish privacy is attained if the additive Laplace noise is calibrated to the differences in mean and variance of the Gaussian distributions conditioned on every discriminative secret pair. A typical application is the private release of the summation (or average) query, for which sufficient conditions are derived for approximating ϵ-statistical indistinguishability in individual's sensitive data. The result is then extended to arbitrary prior beliefs trained by Gaussian mixture models (GMMs): calibrating Laplace noise to a convex combination of differences in mean and variance between Gaussian components attains (ϵ, δ)-pufferfish privacy.**

Index Terms—**Pufferfish privacy, noise calibration, Monge-Kantorovich optimal transport plan.**

I. INTRODUCTION

WHEN participating in data sharing activities, we want to provide useful information to others but keep secret our personal data, e.g., race, gender, etc. To do so, some privacy metric is applied to quantify the confidentiality of the sensitive attributes in the released data. A data regulation scheme is devised thereafter to mitigate the privacy leakage. Differential privacy (DP) [1], [2] is a rigorous definition of data privacy based on a typical inference setting [3]. For an analyst who is able to compute the data statistics, DP ensures a restricted probabilistic resolution on the secret that is nested in the released data.

Specifically, for a (deterministic) query function $f(\cdot)$ that returns distinct values when it is applied to a database D given two secrets s_i and s_j. For example, s_i refers to the event that some user is present in D, while s_j denotes the event that this user is absent. To protect privacy, the query answer $f(D)$ should be randomized (e.g., by injecting noise) to ensure some statistical indistinguishability between s_i and s_j. That is, denoting $\Pr(\tilde{f}(D)|s_i)$ and $\Pr(\tilde{f}(D)|s_j)$ the probability of noised $f(D)$ given s_i and s_j, respectively, the difference between them should be upper bounded by a nonnegative threshold ϵ. This is so-called ϵ-indistinguishability, where the *privacy budget* ϵ denotes the privacy level: a smaller ϵ indicates higher indistinguishability between $\Pr(\tilde{f}(D)|s_i)$ and $\Pr(\tilde{f}(D)|s_j)$ and therefore more privacy.[1]

While in DP the uncertainty is introduced by the data regulation scheme only, a more realistic scenario is that the original dataset exhibits an inherent uncertainty, too. For example, a database could be drawn from a probabilistic space, where the chances for getting each realization D conditioned on distinct secrets s_i and s_j are different; or, the query function $f(\cdot)$ could be a randomized response function. To this end, a more general framework is formulated by [4], [5] called *pufferfish privacy*.

For the original data X that is statistically correlated with the nesting secret S, the purpose of pufferfish privacy is to have the noised data $\tilde{X}$ probabilistically indistinguishable on S. DP can be treated as a special case of pufferfish privacy for deterministic X such that $X = f(D)$. This framework also incorporates Bayesian inference setting (as seen in quantitative information flow [6] and information leakage studies [7], [8]) via a parameter ρ that denotes the prior belief of the adversary. The prior belief is usually a probability distribution, e.g., the conditional probability of X given secret S, before privatization, which could be learned from the previous data release. In this sense, the posterior probability refers to the statistics of the noised data $\tilde{X}$, where the purpose is to guarantee ϵ-indistinguishability between $\Pr(\tilde{X}|s_i, \rho)$ and $\Pr(\tilde{X}|s_j, \rho)$.

The difficulty is how to calibrate the noise given the intrinsic uncertainty in the original dataset. It is shown in [9] that scaling Laplace noise by the Wasserstein metric of infinite order W_∞ is sufficient to attain ϵ-pufferfish privacy.[2] [10] pointed out the infeasibility of this approach due to the non-convexity of W_∞ [11], [12] and proposed a realistic W_1 noise calibrating method by the corresponding Kantorovich optimal transport plan. However, the W_1 method can only be applied to X taking values in a countable alphabet, as it involves the computation of the second derivative of a joint probability, which is hard to obtain for continuous random variables or probability distributions that do not have a closed-form expression. In addition, for continuous X and the corresponding probability density function, the maximum pairwise distance over the Kantorovich optimal transport plan could be infinitely large, which would result in an excessive amount of noise and therefore severely deteriorate the data utility.

On the other hand, an adversary would be very likely to adopt machine learning techniques to infer the prior knowledge ρ, i.e., train or fit a parameterized probability density function out of the past observations. In particular, for X being aggregated statistics, e.g., the counting or summation query, Gaussian prior would be a good choice as it is closest to the true statistics. This is the reason why the prior knowledge is usually modeled by a normal probability density function such as [13].

---

[1]In DP, it is assumed that the database D given s_i and s_j differs in one entry only.

[2]This method reduces to the famous ℓ_1-sensitivity noise calibration for DP [2] if X is deterministic, which also proves that less noise is required for attaining pufferfish privacy than DP [5], [9].

A. Our Contributions

This paper studies how to calibrate Laplace noise for attaining pufferfish privacy when the adversary's prior belief ρ is Gaussian distributed. The result is further extended to arbitrarily distributed prior belief ρ that is trained by Gaussian mixture model (GMM). The main results in this paper will be derived under the Monge's optimal transport plan [14]–[16], the W_2 solution for Gaussian couplings.

The main contributions of this paper are listed below.

1) To release data X which is known to be normally distributed given all instances of secret S but differ in mean and variance, we apply Monge's optimal transport plan to show that a δ-approximation of ϵ-pufferfish privacy, i.e., (ϵ, δ)-pufferfish privacy, can be achieved by adding Laplace noise to X. The scale parameter b of Laplace noise should be calibrated to the differences in both mean and variance conditioned on each pair of secrets s_i and s_j. This method is shown to be a generalization of the ℓ_1-sensitivity noise calibration method for DP [2].

2) The result above is applied to the problem of privatizing the summation query in a multi-user system containing a finite number of participants. It is assumed that each user obtains an independent random variable. To privately release the summation over all users, we derive a sufficient condition for ensuring the statistical indistinguishability about the individual's presence in the system. It is proved that ϵ-indistinguishability about each participant's data can be guaranteed.

3) Assuming the adversary learns the prior knowledge ρ on the arbitrarily distributed X by GMMs, we show that (ϵ, δ)-pufferfish privacy is guaranteed by calibrating the scale parameter b of Laplace noise to the convex combination of differences in mean and variance of Gaussian components. We apply this method to the `adult` and `Hungarian heart disease` datasets in the UCI machining learning repository [17] to show how to scale the parameter b to achieves δ-approximation of ϵ-indistinguishability on real-world data.

B. Related Works

Consider publishing a table having two columns "`age`" and "`cholesterol level`". Even if column "`age`" is excluded, it can still be inferred by an adversary who can exploit the correlation between the two attributes. Here, the DP setting [1], [2] does not fit, as the published data "`cholesterol level`" is a random variable, not a deterministic query answer, the statistics of which depend on the hidden secret "`age`". Instead, a pufferfish privacy framework is proposed in [4], [5] to study how to achieve indistinguishability in the presence of intrinsic randomness. It has been applied to temporally correlated data, e.g., the privacy measure in [18], [19], monitoring web browsing behavior [20] and the trajectory data, e.g., [21], etc. Besides these specific applications, an efficient noise calibrating method remains missing until the proposal of Wasserstein approach in [9]. It is shown that pufferfish privacy can be attained by a Laplace mechanism calibrated by the Wasserstein distance W_∞. To deal with the

difficulty in calculating W_∞, apart from the relaxation by a Rényi measure [22], [10] shows that W_∞ method can be computed by Kantorovich optimal transport plan (the solution to W_1 distance), which is easy to obtain for finite and discrete alphabet cases, e.g., the W_1 approach for a trajectory clustering task in [23, Algorithm 2]. This motivates the study of a more realistic case as to how an adversary infers the intrinsic randomness.

Pufferfish privacy is a guarantee of indistinguishability against the adversary's prior knowledge ρ [4], [5], the belief or side information on the statistical dependence of published data on hidden secret, which might be obtained from previous data releases. For 50 distinct cholesterol levels, an adversary needs to store 50 probability values to express the prior belief for only one secret instance. Instead, fitting a probabilistic model can largely reduce space complexity, e.g., two values, mean and variance, determine a Gaussian probability. This coincides with the idea of statistical machine learning [24], but in return causes a problem in the Wasserstein approach: the noise calibrated by the maximum pairwise distance of W_∞ or W_1 solution might be too large.[3] Therefore, the existing study is restricted to special cases of parameterized priors, e.g., Gaussian priors that only differ in mean [13], [25]. This paper considers pufferfish privacy where the prior ρ is Gaussian distribution with the mean and variance varying with secret instance. The underlying assumption is that the published data is a continuous random variable, not the bounded discrete ones as in [18]–[21].

Meanwhile, an important use case for data privacy is reporting aggregated statistics for a finite number of users/participants, e.g., how to ensure ϵ-DP in counting query so that an adversary cannot easily tell the existence of individual users [1], [2]. This also motivates the proposal of pufferfish privacy (see hedging privacy in [5, Section 7]) where the data obtained by each user is randomized. This paper also studies how to attain (ϵ, δ)-pufferfish privacy in this multi-user system.

C. Notation

The capital and lower case letters denote random variable and its realization, respectively. For example, x denotes an instance/realization of random variable X. Notation $P_X(x)$ refers to the probability $\Pr(X = x)$. The calligraphic $\mathcal{X}$ denotes the alphabet of X. We use $X|s$ to denote the random variable X conditioned on the event $S = s$ and $P_{X|S}(\cdot|s)$ to denote the corresponding conditional probability distribution. Normal probability density distribution with mean μ and variance σ is denoted by $\mathcal{G}(\mu, \sigma^2)$ and Laplace distribution with the scale parameter b is denoted by $\mathcal{L}(b)$. We only consider zero mean Laplace distribution in this paper. We use $\mathbb{R}$ and $\mathbb{R}_+$ to denote real number set and nonnegative real number set, respectively.

D. Organization

Section II clarifies the definition of pufferfish privacy and reviews the Monge's optimal transport plan. Sections III and

---

[3]This is largely due to the fact that the support of a parameterized probability is usually the overall real number set.

IV derive sufficient conditions for attaining (ϵ, δ)-pufferfish privacy for Gaussian and GMM priors, respectively. Section III-B shows how to compute the scaling parameter b for publishing pufferfish private summation query in a multi-user system. Section V presents the experimental result on the `adult` dataset. In this paper, the main results (Theorems 1 and 2 and Corollaries 1 and 2) are stated in the main context with corresponding proofs presented in the appendix.

II. PRELIMINARIES

Denote S the secret and $\mathcal{S}$ the alphabet of the secret, where each $s \in \mathcal{S}$ denotes an elementary event or outcome of S, e.g., s ="the patient has type 2 diabetes", s ="the user is female", etc. There is a statistical correlation between the data X to be published and secret S, which can be described by the conditional probability $P_{X|S}(\cdot|s, \rho)$. Here, ρ denotes an adversary's prior belief on the probability distribution of X given secret $S = s$, which could be different from others, i.e., for two adversaries obtaining prior beliefs ρ and ρ', $P_{X|S}(\cdot|s, \rho)$ and $P_{X|S}(\cdot|s, \rho')$ are two different probability distributions even for the same secret s. In this paper, $P_{X|S}(\cdot|s, \rho)$ is assumed to be Gaussian probability density function (See Section II-B) characterized by its mean and variance. Thus, each adversary obtains his own prior belief ρ constituted by pairs of means and variances, each of which corresponds to one secret s.

To protect secret S, a privatized version Y of the original data X is generated to ensure the indistinguishability between two distinct secrets s_i and s_j. Let $\mathbb{S} \subseteq \mathcal{S}^2$ be the *discriminative pair set* containing all secret pairs (s_i, s_j) on which a certain degree of statistical indistinguishability should be guaranteed against each adversary's prior belief ρ.

Definition 1 $((\epsilon, \delta)$-pufferfish privacy$)$**.** *For $\delta \in [0, 1)$ and $\epsilon > 0$, Y attains δ-approximation of ϵ-pufferfish privacy on the discriminative secret pair set $\mathbb{S}$ if for all $(s_i, s_j) \in \mathbb{S}$ and ρ,*

$$P_{Y|S}(B|s_i, \rho) \le e^\epsilon P_{Y|S}(B|s_j, \rho) + \delta, \ \forall B \subseteq \mathbb{R}. \quad (1)$$

If the condition (1) holds for $\delta = 0$, $(\epsilon, 0)$-pufferfish privacy is also called ϵ-pufferfish privacy [4], [5].

A. Privatization mechanism

We consider additive noise mechanism to generate Y. Let N be the zero mean noise that is independent of X. The goal is to attain pufferfish privacy in

$$Y = X + N.$$

See Fig. 1. Denoting $P_N(\cdot)$ the probability of N, we have the conditional probability

$$P_{Y|S}(y|s, \rho) = \int P_N(y - x) P_{X|S}(x|s, \rho) \, dx,$$

for which the condition (1) is equivalent to

$$
\begin{aligned}
& P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho) \\
& = \int_B \int P_N(y - x) P_{X|S}(x|s_i, \rho) \, dx \, dy \\
& \quad - e^\epsilon \int_B \int P_N(y - x') P_{X|S}(x'|s_j, \rho) \, dx' \, dy \\
& = \int_B \int \left(P_N(y - x) - e^\epsilon P_N(y - x') \right) d\pi(x, x') \, dy \le \delta.
\end{aligned}
\tag{2}
$$

The coupling π is the joint probability $\pi(x, x')$ with two marginals being $P_{X|S}(x|s_i, \rho)$ and $P_{X|S}(x'|s_j, \rho)$, i.e., $\int \pi(x, x') \, dx' = P_{X|S}(x|s_i, \rho)$ for all x and $\int \pi(x, x') \, dx = P_{X|S}(x'|s_j, \rho)$ for all x'.

B. Monge's Optimal Transport Plan $\hat{\pi}$

Assume that the priors are Gaussian distributed: $X|s_i \sim \mathcal{G}(\mu_i, \sigma_i)$ and $X|s_j \sim \mathcal{G}(\mu_j, \sigma_j)$ for each discriminative pair $(s_i, s_j) \in \mathbb{S}$. Consider the Monge's optimal transport plan $\hat{\pi}$ [14]–[16]:

$$d\hat{\pi}(x, x') = dP_{X|S}(x|s_i, \rho) \cdot \mathbb{I}\{x' = T(x)\}$$

where $\mathbb{I}\{\cdot\}$ is the indicator function and T is the linear mapping

$$T(x) = \mu_j + \frac{\sigma_j}{\sigma_i}(x - \mu_i), \quad \forall x. \tag{3}$$

Inequality (2) under $\hat{\pi}$ reduces to

$$\int \int_B \left(P_N(y - x) - e^\epsilon P_N(y - T(x)) \right) dy \, dP_{X|S}(x|s_i, \rho) \le \delta. \tag{4}$$

The main results in this paper are derived by proving the inequality (4). The transport plan $\hat{\pi}$ is in fact the minimizer of the Wasserstein metric $W_\alpha = \left(\inf_\pi \int d^\alpha(x, x') \, d\pi(x, x') \right)^{1/\alpha}$ in the order of $\alpha = 2$, where the infimum is taken over all couplings π of two Gaussian marginals [16] [26, Remark 2.31].

Noise reduction: It is clear that any coupling π can be adopted to derive a sufficient condition for (ϵ, δ)-pufferfish privacy, e.g., determining the minimum noise power that satisfies (2) under transport plan $\pi(x, x') = P_{X|S}(x|s_i, \rho) P_{X|S}(x'|s_j, \rho), \forall x, x'$. But, it might result in a high noise power that jeopardizes data utility. We adopt the W_2 or Monge's optimal transport method in an attempt to minimize the noise for attaining (ϵ, δ)-pufferfish privacy. See Appendix E, where we explain in detail how the minimization $\inf_\pi$ in Monge's optimal transport plan contributes to a noise reduction over all couplings.

III. GAUSSIAN PRIORS

We consider Laplace noise $N \sim \mathcal{L}(b)$ with the noise distribution $P_N(z) = \frac{1}{2b} e^{-\frac{|z|}{b}}, \forall z \in \mathbb{R}$.

Theorem 1. *For $X|s_i \sim \mathcal{G}(\mu_i, \sigma_i)$ and $X|s_j \sim \mathcal{G}(\mu_j, \sigma_j)$ for all $(s_i, s_j) \in \mathbb{S}$, adding Laplace noise $N \sim \mathcal{L}(b)$ with*

$$b \ge \frac{1}{\epsilon} \max_{\rho, (s_i, s_j) \in \mathbb{S}} \left\{ |\mu_i - \mu_j| + |\sigma_i - \sigma_j| \tau^*(\delta) \right\} \tag{5}$$

attains (ϵ, δ)-pufferfish private on $\mathbb{S}$ in Y.
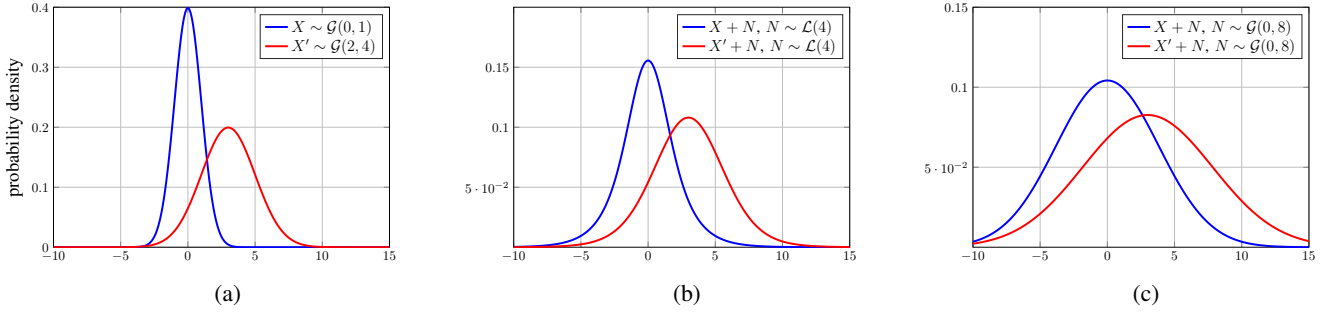
Fig. 1: For the original data X and X' in (a) that is normal distributed with different mean and variance, (b) shows the resulting probability density of $Y = X + N$ and $Y' = X' + N$ for Laplace noise $N \sim \mathcal{L}(4)$, where the maximum logarithmic difference in probability density is $\max_y \left| \log \frac{P_Y(y)}{P_{Y'}(y)} \right| = 0.2992$. (c) shows the resulting probability density of Y and Y' for Gaussian noise $N \sim \mathcal{G}(0, 8)$, where $\max_y \left| \log \frac{P_Y(y)}{P_{Y'}(y)} \right| = 0.0156$. Note, the Laplace noise in (b) and Gaussian noise in (c) have the same variance.

The proof is in Appendix A. In (5), $\tau^*(\delta) = \min\{\tau \colon \Pr(Z > \tau) \leq \frac{\delta}{2}\}$ or $\tau^*(\delta) = Q^{-1}(\frac{\delta}{2})$, where $Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} \, dx$ is the tail probability of standard normal distribution. Alternatively, one can derive the value of $\tau^*(\delta)$ by the Lambert-W function. See Appendix A. Theorem 1 states that to attain sufficient statistical indistinguishability in the released data, the additive noise should be large enough to compensate the difference in both mean and variance. The second term $|\sigma_i - \sigma_j| \tau^*(\delta)$ in (5) corresponds to the difference in variance. The value of $\tau^*(\delta)$ is decreasing in $\delta \in (0, 1)$ and can be obtained numerically. See Fig. 5 in Appendix A.

A. Special Case: ℓ_1-sensitivity Method for Differential Privacy

We show below that Theorem 1 is a generalization of the ℓ_1-sensitivity noise calibration method proposed in [2] for attaining differential privacy (DP).

Remark 1 (Translation priors). *If for all $(s_i, s_j) \in \mathbb{S}$, $X|s_i$ and $X|s_j$ are translation rvs to each other, i.e., $P_{X|S}(x|s_i) = P_{X|S}(x - \mu_j + \mu_i | s_j)$ for all x, $Y = X + N$ for $N \sim \mathcal{L}(b)$ is ϵ-pufferfish private if*

$$b \geq \frac{1}{\epsilon} \max_{\rho, (s_i, s_j) \in \mathbb{S}} |\mu_i - \mu_j|. \tag{6}$$

Remark 1 is derived by the fact that the second term $|\sigma_i - \sigma_j| \tau^*(\delta)$ in (5) vanishes if $\sigma_i = \sigma_j$ for all $(s_i, s_j) \in \mathbb{S}$. The proof is in Appendix B showing that Remark 1 holds for any translation priors $X|s_i$ and $X|s_j$, not just Gaussian distribution. A similar result can also be found in [25, Theorem 4.1].

In DP [27], data X is deterministic, i.e., priors $P_{X|S}(.|s_i)$ and $P_{X|S}(.|s_j)$ are point masses located at means μ_i and μ_j, respectively, with the same variance $\sigma_i = \sigma_j = 0$, e.g., query answers of two neighboring databases. In this case, the sufficient condition in Theorem 1 reduces to $b \geq \frac{1}{\epsilon} \max_{\rho, (s_i, s_j) \in \mathbb{S}} |\mu_i - \mu_j|$, which is exactly the ℓ_1-sensitivity method proposed in [2] for attaining ϵ-DP. In other words, Theorem 1 extends the ℓ_1-sensitivity method for probabilistic priors with different variance.

B. Summation query in K-independent user system

Assume there are K users indexed by $\mathcal{K} = \{1, \ldots, K\}$. Each user k obtains a random variable Z_k, independently. We construct a multiple random variable $Z = (Z_k \colon k \in \mathcal{K})$ for the overall outcome of the system. The answer to the summation query is $X = \sum_{k \in \mathcal{K}} Z_k$. Let the adversary's prior belief ρ on X be a Gaussian distribution with observed mean and variance. Note, in this case, this prior belief is a good approximation of the true distribution of X by the central limit theorem (CLT).

For each user k, denote "$Z_k = \perp$" the event that user k is absent in the system, "$Z_k \neq \perp$" the event that user k is present in the system and "$Z_k = a$" the event that user is present in the system and reports the value a of random variable Z_k. Consider a discriminative pair set $\mathbb{S}$ that consists of mutually exclusive secret pairs s_i and s_j denoting whether or not user k is present, i.e., $s_i = $ "$Z_k = \perp$" and $s_j = $ "$Z_k \neq \perp$". Alternatively, we define

$$\mathbb{S}_\perp = \big\{ (\text{``}Z_k \neq \perp \text{''}, \text{``}Z_k = \perp \text{''}) \colon k \in \mathcal{K} \big\}.$$

It is clear that the pufferfish privacy on $\mathbb{S}_\perp$ guarantees the adversary's indistinguishability between the existence and nonexistence of individual users. Or, if the purpose is to make the actual realization of Z_k indistinguishable for each user, we can define the discriminative pair set

$$\mathbb{S}_a = \big\{ (\text{``}Z_k = a\text{''}, \text{``}Z_k = a'\text{''}) \colon k \in \mathcal{K} \big\}$$

for some $a, a' \in \mathbb{R}$ such that $a \neq a'$. For discrete rv Z_k, typically $a' = a \pm 1$.

In general, Z_k's are mutually independent, but not necessarily identically distributed. For each $k \in \mathcal{K}$, assume Z_k is a random variable with mean μ_k and variance σ_k^2. Let the adversary's prior belief ρ be that the summation query

$$X = \sum_{k \in \mathcal{K}} Z_k \sim \mathcal{G}\left(\sum_{k \in \mathcal{K}} \mu_k, \sum_{k \in \mathcal{K}} \sigma_k^2 \right) \tag{7}$$

when all K users present in the system. Based on (7) and due to the independence of Z_k's, given $Z_k = a$,

$$X = a + \sum_{k' \in \mathcal{K}_{-k}} Z_{k'} \sim \mathcal{G}\left(a + \sum_{k' \in \mathcal{K}_{-k}} \mu_{k'}, \sum_{k' \in \mathcal{K}_{-k}} \sigma_{k'}^2\right),$$
(8)

where $\mathcal{K}_{-k} = \mathcal{K} \setminus \{k\}$ containing all users except user k. Note, the prior beliefs (7) and (8) approach the true probability distributions of X for large K based on CLT, if Z_k's satisfy the Lyapunov or Lindeberg condition [28].

Using Theorem 1, we derive a sufficient condition for attaining pufferfish privacy on $\mathbb{S}_\perp$ and $\mathbb{S}_a$ below.

Corollary 1. *In the K-independent user system, let X be the summation query and $N \sim \mathcal{L}(b)$. $Y = X + N$ is*
(a) (ϵ, δ)-pufferfish private on $\mathbb{S}_\perp$ if

$$b \geq \frac{1}{\epsilon} \max_{k \in \mathcal{K}} \left\{|\mu_k| + \triangle\sigma_k \tau^*(\delta)\right\}$$
(9)

where $\triangle\sigma_k = \sqrt{\sum_{k' \in \mathcal{K}_{-k}} \sigma_{k'}^2 + \sigma_k^2} - \sqrt{\sum_{k' \in \mathcal{K}_{-k}} \sigma_{k'}^2}$.
(b) ϵ-pufferfish private on $\mathbb{S}_a$ if

$$b \geq \frac{|a - a'|}{\epsilon}.$$
(10)

Proof: For discriminative pair set $\mathbb{S}_\perp$, given $Z_k = \perp$, X has mean $\sum_{k' \in \mathcal{K}_{-k}} \mu_{k'}$ and variance $\sum_{k' \in \mathcal{K}_{-k}} \sigma_{k'}^2$; given $Z \neq \perp$, X has mean $\sum_{k' \in \mathcal{K}_{-k}} \mu_{k'} + \mu_k$ and variance $\sum_{k' \in \mathcal{K}_{-k}} \sigma_{k'}^2 + \sigma_k^2$. Applying Theorem 1, we have (a). For discriminative pair set $\mathbb{S}_a$ containing secret pairs $Z_k = a$ and $Z_k = a'$, we have X given $Z_k = a$ being a translation of X given $Z_k = a'$, i.e., $P_{X|S}(x|Z_k = a) = P_{X|S}(x - a' + a|Z_k = a')$, $\forall x \in \mathbb{R}$, where the priors have the same variance but differ in mean. By Remark 1, calibrating noise to the difference in mean $|a - a'|$ attains ϵ-pufferfish privacy. ∎

In Corollary 1(a), $\triangle\sigma_k \to 0$ as $K \to \infty$. That is, when the number of users K grows, it is getting more difficult for an adversary to distinguish the changes in variance of the summation query conditioned on the participation of individual users. Equivalently, it is easier for an individual to hide his/her presence in a larger system. This coincides with the intuition of early definition of data privacy, e.g., K-anonymity [29], t-closeness [30], to guarantee a lower bound on the number of users where indistinguishability is achievable. In addition, as $K \to \infty$, the lower bound in (9) approaches $\frac{1}{\epsilon} \max_{k \in \mathcal{K}} \mu_k$, i.e., it suffices to scale the Laplace noise to the maximum mean over all users.

Identical users: If Z_k is identically distributed, i.e., $\mu_k = \mu$ and $\sigma_k^2 = \sigma^2$ for all $k \in \mathcal{K}$, we have (9) reduced to

$$b \geq \frac{1}{\epsilon}\left(|\mu| + (\sqrt{K} - \sqrt{K-1})\sigma\tau^*(\delta)\right).$$
(11)

Here, the multiple random variable $Z = (Z_k \colon k \in \mathcal{K})$ could also refer to a K-length *i.i.d.* sample sequence, e.g., a dataset with K records/rows. In this case, (11) states that as the number of records K grows larger, the term $(\sqrt{K} - \sqrt{K-1})\sigma\tau^*(\delta)$ vanishes, and we can attain ϵ-pufferfish privacy by adding Laplace noise $N \sim \mathcal{L}(b)$ with $b = \mu/\epsilon$. See Fig. 2. If $\mu \to 0$, only small amount of noise
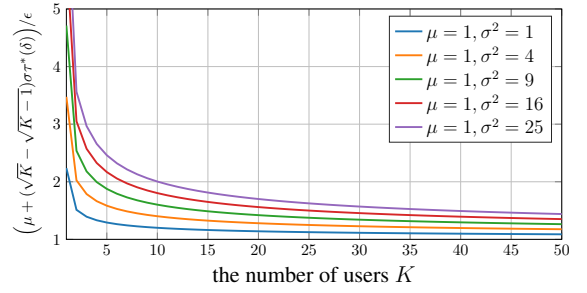

Fig. 2: For the K-independent and identical user system, the lower bound on b in (11) for attaining $(1, 0.3)$-pufferfish privacy with Laplace noise $N \sim \mathcal{L}(b)$ as the number of users K increases. We set the mean $\mu = 1$ and vary the variance σ^2 from 1 to 25.

is required. The identically distributed Z_k setting also applies to data parallelism for distributed learning, e.g., [31], where a large dataset is partitioned and assigned to users to allow parallel model training.

IV. GMM PRIORS

For arbitrary distributed $X|s_i$, assume the adversary trains a Gaussian mixture model (GMM) $X|s_i \sim \mathcal{GM}(D_i)$ that is constituted by a finite number D_i of Gaussian components. That is, under the adversary's prior belief ρ, the probability distribution of X given secret s_i is

$$P_{X|S}(x|s_i, \rho) = \sum_{m=1}^{D_i} \alpha_{im}\mathfrak{g}(x; \mu_{im}\sigma_{im}^2),$$

where $\sum_{m=1}^{D_i} \alpha_{im} = 1$ and $\mathfrak{g}(\cdot; \mu_{im}, \sigma_{im}^2)$ denotes the mth Gaussian component with mean μ_{im} and variance σ_{im}^2.

We apply the optimal transport plan between two GMMs recently derived in [32], [33]. For $X|s_i \sim \mathcal{GM}(D_i)$ and $X|s_j \sim \mathcal{GM}(D_j)$, the transport plan $\hat{\pi}$ for W_2 distance is [4]

$$\hat{\pi}(x, x') = \sum_{m,l} w_{kl}^*\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)\mathbb{I}\{x' = T_{ml}(x)\}$$
(12)

where T_{ml} is the linear mapping (3) between the mth component in $\mathcal{GM}(D_i)$ and lth component in $\mathcal{GM}(D_j)$ and $w_{ml}^*, \forall m, l$ is the minimizer of linear programming $\min \sum_{ml} w_{ml} W_2^2(\mathfrak{g}(\cdot; \mu_{im}, \sigma_{im}^2), \mathfrak{g}(\cdot; \mu_{jl}, \sigma_{jl}^2))$. [5]

We show in the following theorem that it suffices to calibrate the noise to the weighted sum of means and variance for GMM priors.

---

[4]The transport plan in (12) gives rise to W_2 distance if the infimum is taken over all couplings that are GMMs, i.e., it provides an upper bound on the actual W_2. See [33, Sec. 4].

[5]$W_2^2(\mathfrak{g}(\cdot; \mu_{im}, \sigma_{im}^2), \mathfrak{g}(\cdot; \mu_{jl}, \sigma_{jl}^2)) = (\mu_{im} - \mu_{jl})^2 + (\sigma_{im} - \sigma_{jl})^2$ denotes the square of W_2 distance between mth component in $\mathcal{GM}(D_i)$ and lth component in $\mathcal{GM}(D_j)$ [33, Sec. 2.4.1]. That is, the optimal transport plan in (12) is fully determined by the parameters of GMMs.
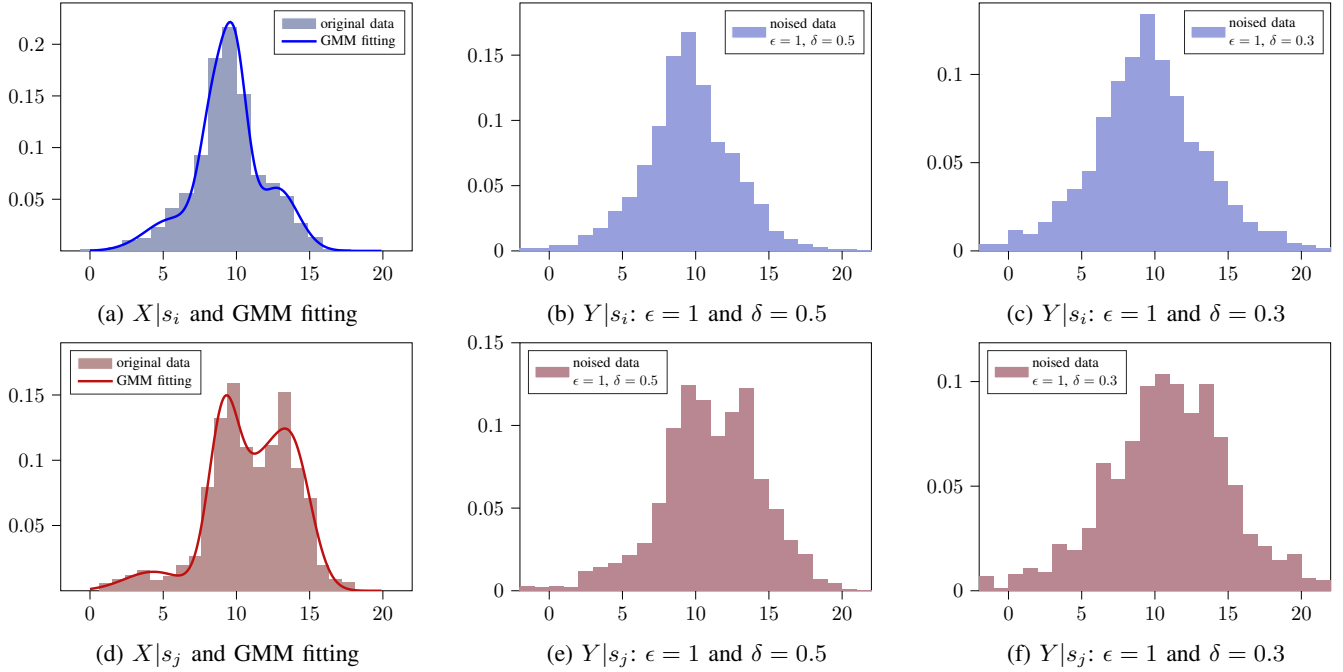
Fig. 3: The `Adult` dataset in UCI machine learning repository [17]: X and S denote the attributes `education-num` and `race`, respectively. To attain the statistical indistinguishability between secrets s_i ="race is `Black`" and s_j ="race is `Asian-Pac-Islander`", the privatized data $Y = X + N$ is generated, where in Laplace noise $N \sim \mathcal{L}(b)$ is calibrated by Theorem 2 based on the GMM fitting for attaining $(1, 0.5)$-pufferfish privacy and $(1, 0.3)$-pufferfish privacy.

Theorem 2. *For $X|s_i \sim \mathcal{GM}(D_i)$ and $X|s_j \sim \mathcal{GM}(D_j)$ for all $(s_i, s_j) \in \mathbb{S}$, adding Laplace noise $N \sim \mathcal{L}(b)$ with*[6]

$$b \geq \frac{1}{\epsilon} \max_{\rho, (s_i, s_j) \in \mathbb{S}} \sum_{m,l} w_{ml}^* \Big(|\mu_{im} - \mu_{jl}| + \tau^*(\delta)|\sigma_{im} - \sigma_{jl}| \Big). \quad (13)$$

attains (ϵ, δ)-pufferfish privacy on $\mathbb{S}$ in Y.

Consider a special case when all GMMs have the same number of components and differ in mean only. That is,

$$P_{X|S}(x|s_i) = \sum_{m=1}^{D} \alpha_m \mathfrak{g}(x; \mu_{im}, \sigma_m^2) \quad (14)$$

for all secrets s_i, where $\mu_{im} \neq \mu_{jm}$ for each secret pair $(s_i, s_j) \in \mathbb{S}$. This is a common situation in audio pattern recognition, the so-called GMM-UBM method [34]: train a universal background model (UBM); then, adapt to the GMM for each individual by changing only the means of Gaussian components in UBM. It is a maximum a posteriori (MAP) approach, where variance adaptation does not improve the performance of estimation and therefore remains unchanged.

GMM-UBM would very likely be the inference method adopted by an adversary, who is maliciously estimating the arbitrarily distributed statistics for each secret s_i. In this case, for each secret pair (s_i, s_j), all Gaussian components are translations to each other: for each component m, $\mathfrak{g}(x; \mu_{im}, \sigma_m^2) = \mathfrak{g}(x'; \mu_{jm}, \sigma_m^2), \forall x' = x - \mu_{im} + \mu_{jm}$. In this case, the

---

[6]Note that there is an optimal weight w_{kl}^* for each pair of secret $(s_i, s_j) \in \mathbb{S}$.

following corollary shows that it suffices to only scale b to a convex combination of differences in mean.

Corollary 2. *If $X|s_i \sim \mathcal{GM}(D)$ with probability distribution (14) for all s_i, $Y = X + N$ for $N \sim \mathcal{L}(b)$ is ϵ-pufferfish private if*

$$b \geq \frac{1}{\epsilon} \max_{\rho, (s_i, s_j) \in \mathbb{S}} \sum_m \alpha_m |\mu_{im} - \mu_{jm}|. \quad (15)$$

V. EXPERIMENT

In the UCI machine learning repository [17], the `adult` dataset was extracted from the census bureau database containing 32652 instances/individuals/records and 15 attributes. In this experiment, S refers to the sensitive attribute `race`, a categorical variable, and X refers to attribute `education-num`, an integer number indicating an individual's education level. We simulate a scenario that `education-num` column is to be published. It is assumed that the adversary have access to all published records of `education-num` and can statistically infer the information on `race`. Therefore, the data curator requires a certain level of statistical indistinguishability (by specifying the values of ϵ and δ) between the secrets: s_i ="race is `Black`" and s_j ="race is `Asian-Pac-Islander`".

In this case, X refers to the values of `education-num` in all 32652 records; $X|s_i$ refers to the values of `education-num` in all records having attribute `race` being "race is `Black`"; $X|s_j$ refers to the values of `education-num` in all records having attribute `race` being

(a) $X|s_i$ and GMM fitting

(b) $Y|s_i$: $\epsilon = 1$ and $\delta = 0.5$

(c) $Y|s_i$: $\epsilon = 1$ and $\delta = 0.3$

(d) $X|s_j$ and GMM fitting

(e) $Y|s_j$: $\epsilon = 1$ and $\delta = 0.5$
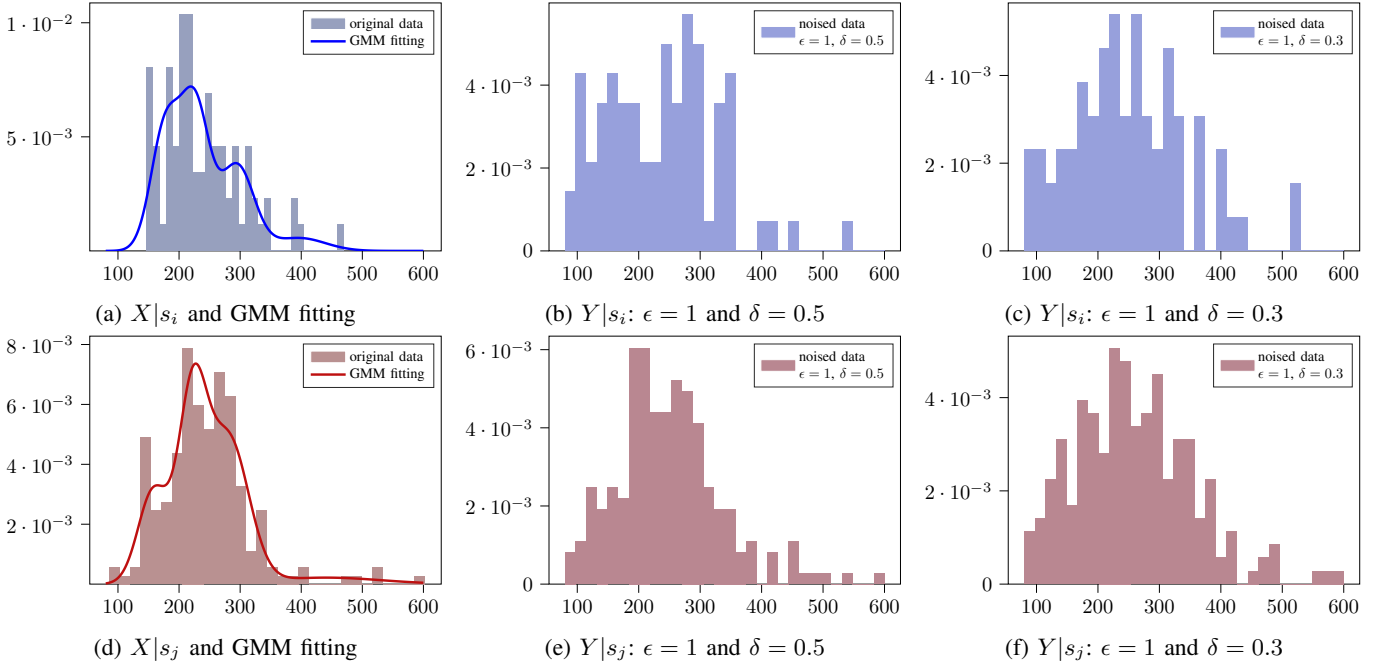
(f) $Y|s_j$: $\epsilon = 1$ and $\delta = 0.3$

Fig. 4: The `Hungarian heart disease` dataset in UCI machine learning repository [17]: X and S denote the attributes `chol`, the cholesterol level, and `sex`, respectively. To attain the statistical indistinguishability between secrets $s_i =$ "`sex is female`" and $s_j =$ "`sex is male`", the privatized data $Y = X + N$ is generated, where in Laplace noise $N \sim \mathcal{L}(b)$ is calibrated by Theorem 2 based on the GMM fitting for attaining $(1, 0.5)$-pufferfish privacy and $(1, 0.3)$-pufferfish privacy.

"`race is Asian-Pac-Islander`".[7] Noise N is added to X to generate privatized data Y referring to 32652 randomized values of `education-num`.[8] By Definition 1, the problem is to ensure $P_{Y|S}(B|s_i, \rho) \le e^\epsilon P_{Y|S}(B|s_j, \rho) + \delta$ for any real number subset B. That is, the adversary should have difficulty telling whether "`race is Black`" or "`race is Asian-Pac-Islander`" by observing the randomized `education-num`s in Y, regardless of the observing order.[9]

We first fit two three-component GMMs to the empirical distributions of $X|s_i$ and $X|s_j$, respectively (see Fig. 3(a) and (d)). Here, the GMMs denote the adversary's belief or side information ρ on X given two secrets s_i and s_j. For $\epsilon = 1$ and $\delta = 0.5$, calculating the Laplace parameter b by applying Theorem 2, we generate privatized data $Y = X + N$ where $N \sim \mathcal{L}(b)$ is calibrated by Theorem 2. The plots in Fig. 3(b) and (e) show the statistics of Y given s_i and s_j, where the differences in empirical probability is reduced. Repeating the same procedure for a more strict privacy constraint: $\epsilon = 1$ and $\delta = 0.3$, the statistical indistinguishability in the randomized data is further improved. See the two plots in Fig. 3(c) and (f). The experiment results show the noise calibration methods proposed in this paper work for any given privacy constraints ϵ and δ. The exact values of ϵ and δ can be determined by

data privacy requirements in actual applications.

In addition, we repeat the same experiment using the another dataset in the UCI machine learning repository: the `Hungarian heart disease` dataset was created by the Hungarian Institute of Cardiology, Budapest, which records 293 patients' data of 76 attributes for the purpose of identifying the presence of heart disease. We extract two attributes: `sex` as the sensitive data S and `chol`, denoting the serum cholesterol in mg/dl, as the public data X. We consider X given two secrets: $s_i =$ "`sex is female`" and $s_j =$ "`sex is male`". The results are in Fig. 4. The same as Fig. 3, they show that one can apply the sufficient condition (2) to attain (ϵ, δ)-pufferfish privacy.

VI. DISCUSSION

It is understandable that the privacy protection should not severely undermine the useful information in the released data Y. For example, the noised counting query in Section III-B should prevent the malicious inference on individual's data or existence, but still report the summation with the highest accuracy. It is noted that the conditions derived in Theorems 1 and 2 are sufficient only. Yet, there is a problem of how to improve these results to further reduce the data distortion but still guarantee a specific degree of statistical indistinguishability, i.e., to minimize the noise for (ϵ, δ)-pufferfish private Y. This section discusses possible solutions: tighten the upper bound on $P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho)$ that points out several directions for future works.

---

[7]Here, both $X|s_i$ and $X|s_j$ refer to a set of data records, instead of a query answer, with the probability for each instance x to appear in the dataset governed by $P_{X|S}(x|s_i)$ and $P_{X|S}(x|s_j)$, respectively. Therefore, the DP framework proposed [1], [2] does not fit in this case.

[8]Noise N is assumed to be a continuous random variable and therefore the value of Y is continuously changing.

[9]Even if in a typical scenario where the data curator releases the randomized `education-num` Y for all black people first and then for all Asian-Pac-Islander people, the change in statistics in Y is under control.

A. Tighter Bound on (2)

The idea of proving Theorems 1 and 2 is to derive an upper bound on (2): $P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho) \leq U$ for all $B \subseteq \mathbb{R}$ and then request $U \leq \delta$ to work out the value of scale parameter b in Laplace noise for given ϵ and δ. See Appendices A and C. It is clear the tighter U is, the smaller value of b can be derived and the smaller the amount of noise added to the (ϵ, δ)-pufferfish privacy attaining released data. It is worth studying whether a tighter U can be derived.

1) Reducing $\tau^*(\delta)$: In Theorem 1, the maximand in the sufficient condition (5) consists of the differences in mean $|\mu_i - \mu_j|$ and standard deviation $|\sigma_i - \sigma_j|$, where the latter is scaled by $\tau^*(\delta)$. Clearly, minimizing $\tau^*(\delta)$ that attains (ϵ, δ)-pufferfish privacy will result in a smaller b and therefore a reduction in noise power. In this paper, the value of $\tau^*(\delta)$ is derived by a sequence of upper bounds on $\int \left(1 - e^{\epsilon - \frac{|x - T(x)|}{b}}\right) \mathrm{d}P_{X|S}(x|s_i, \rho)$. See (18) to (19) in Appendix A. It is possible to tighten these upper bounds or the Gaussian tail bound in (21), e.g., by referring to the analytical tightening method such as [35], to further reduce $\tau^*(\delta)$.

2) Transport plan other than $\hat{\pi}$: Section II-B points out the approach of tightening the bound on (2) by a minimization over all couplings $\inf_\pi \{P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho)\}$. This paper exploits the existing Monge's optimal coupling $\hat{\pi}$ for Gaussian priors, which only searches an upper bound on this infimum (see Appendix E). It is of interest whether there exist other couplings that can improve Theorems 1 and 2.

Finally, the results in this paper can be refined or simplified for some specific settings or applications, e.g., the summation query where individual's random variable Z_k is binary, where one might be able to find better solution than Monge's optimal transport plan or a smaller $\delta^*(\delta)$.

B. Exponential Mechanism

The Laplace distribution belongs to the exponential family, for which we use the inequality of the exponential function $e^{\epsilon - \frac{|y - T(x)| - |y - x|}{b}} \geq e^{\epsilon - \frac{|x - T(x)|}{b}}$ and upper bound $\int \left(1 - e^{\epsilon - \frac{|x - T(x)|}{b}}\right) \mathrm{d}P_{X|S}(x|s_i, \rho) \leq \delta$ thereafter in the proofs of Theorems 1 and 2, in Appendices A and C, respectively. This implies the exponential mechanism with noise probability $P_N(z) \propto e^{-\eta(\theta)d(z)}$ for some metric $d(\cdot)$ can be applied to approximate the ϵ-pufferfish privacy for Gaussian priors, too. To do so, one can refer to [2] that extends the noise calibration result on Laplace mechanism to exponential mechanism for attaining ϵ-DP. Among all the probability distributions in the exponential family, we specifically discuss the Gaussian mechanism below.

Gaussian Noise: Gaussian mechanism is another commonly used privatization scheme, which might be favored to enhance data utility. One reason is that the noise variance proportional to the ℓ_2-norm is no larger than ℓ_1-norm. This can be seen from Fig. 1, showing that with the same noise power, Gaussian mechanism provides higher statistical indistinguishability than Laplace mechanism. In addition, the tail probability of Gaussian distribution decays faster than Laplace distribution, as pointed out by [36], and therefore results in an accurate query answer.

While this paper focuses on Laplace mechanism, we suggest the design of Gaussian mechanism as follows. For Gaussian noise $N \sim \mathcal{G}(0, \theta^2)$ and Gaussian prior ρ, the privatized data $Y = X + N$ conditioned on each secret s_i is necessarily Gaussian distributed with mean μ_i and variance $\theta^2 + \sigma_i^2$. Then, the problem of attaining (ϵ, δ)-pufferfish privacy is to derive a sufficient condition in the form of $\theta^2 \geq \xi(\epsilon, \delta, \mu_i, \mu_j, \sigma_i^2, \sigma_j^2)$ such that $P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho) \leq \delta, \forall B \subseteq \mathbb{R}$ where $Y|s_i \sim \mathcal{G}(\mu_i, \theta^2 + \sigma_i^2)$ and $Y|s_j \sim \mathcal{G}(\mu_j, \theta^2 + \sigma_j^2)$. This reduces to determining the sufficient condition on $\sigma_1^2 = \theta^2 + \sigma_i^2$ or $\sigma_2^2 = \theta^2 + \sigma_j^2$ such that [37]

$$\int [\mathfrak{g}(y; \mu_1, \sigma_1^2) - e^\epsilon \mathfrak{g}(y; \mu_2, \sigma_2^2)]_+ \, \mathrm{d}y \leq \delta, \qquad (16)$$

where $[z]_+ = \max\{z, 0\}$. The main difficulty is to obtain an estimate of the left hand side of (16), for which one can still refer to the proofs of Theorem 1 or the FFT method proposed in [38]. Once the Gaussian mechanism is designed, one can compare it experimentally to the Laplace mechanism in Figs. 3 and 4 for the same values of ϵ and δ.

VII. CONCLUSION

Considering adding Laplace noise $N \sim \mathcal{L}(b)$ to the normally distributed data, we derived a lower bound on the scale parameter b for attaining (ϵ, δ)-pufferfish privacy in the noised data. It is shown that the ϵ-indistinguishability is guaranteed if the noise is sufficient enough to compromise the difference in mean and variance conditioned on the discriminative secret pair (s_i, s_j). An application of this result to the summation query revealed that as the number of participants increases, it is more difficulty for an adversary to differentiate individual participants and therefore requires less noise for attaining pufferfish privacy. When the noise is pairwisely calibrated to the convex combination of difference in mean and variance of Gaussian components, (ϵ, δ)-pufferfish privacy is attained for any arbitrarily distributed data that is modeled by GMM. Finally, we pointed out two ways for improving the results in this paper for a higher data utility.

APPENDIX A
PROOF OF THEOREM 1

For each $(s_i, s_j) \in \mathbb{S}$, we have (4) equal to

$$\int \int_B \frac{1}{2b} \left(e^{-\frac{|y - x|}{b}} - e^{\epsilon - \frac{|y - T(x)|}{b}}\right) \mathrm{d}y \, \mathrm{d}P_{X|S}(x|s_i, \rho)$$

$$= \int \int_B \frac{1}{2b} e^{-\frac{|y - x|}{b}} \left(1 - e^{\epsilon - \frac{|y - T(x)| - |y - x|}{b}}\right) \mathrm{d}y \, \mathrm{d}P_{X|S}(x|s_i, \rho)$$

$$\leq \int_A \int_B \frac{1}{2b} e^{-\frac{|y - x|}{b}} \, \mathrm{d}y \left(1 - e^{\epsilon - \frac{|x - T(x)|}{b}}\right) \mathrm{d}P_{X|S}(x|s_i, \rho)$$

$$(17)$$

$$\leq \int_A \left(1 - e^{\epsilon - \frac{|x - T(x)|}{b}}\right) \mathrm{d}P_{X|S}(x|s_i, \rho)$$

$$\leq P_{X|S}(A|s_i, \rho), \qquad \forall B \subseteq \mathbb{R}, \qquad (18)$$

where inequality (17) is because of the triangular inequality $|y - T(x)| - |y - x| \leq |x - T(x)|$ and $A = \{x : |x - T(x)| > \epsilon b\}$
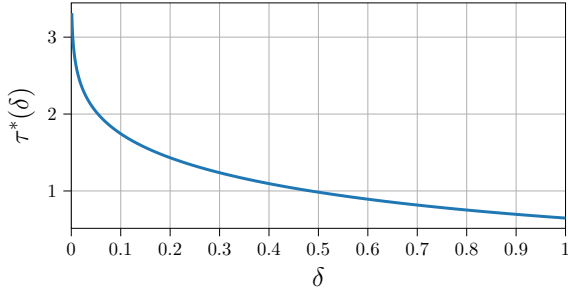
Fig. 5: The value of $\tau^*(\delta)$ in (22) when δ varies from 0.001 to 0.999.

in (18). Assume $b \geq \frac{|\mu_i - \mu_j| + c}{\epsilon}$, where $c \in \mathbb{R}_+$. For $X \sim \mathcal{G}(\mu_i, \sigma_i)$, let $Z = \frac{X - \mu_i}{\sigma_i} \sim \mathcal{G}(0, 1)$. We have

$$
\begin{aligned}
P_{X|S}(A|s_i, \rho) &= \Pr(|X - T(X)| > \epsilon b) \\
&= \Pr\left(\left|X - \mu_j - \frac{X - \mu_i}{\sigma_i}\sigma_j\right| > \epsilon b\right) \\
&= \Pr(|Z(\sigma_i - \sigma_j) + (\mu_i - \mu_j)| > \epsilon b) \\
&= \Pr\left(\left|Z + \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j}\right| > \frac{\epsilon b}{|\sigma_i - \sigma_j|}\right) \\
&= \begin{cases} \Pr\left(\left|Z + \frac{|\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right| > \frac{\epsilon b}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} \geq 0 \\ \Pr\left(\left|Z - \frac{|\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right| > \frac{\epsilon b}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} < 0 \end{cases} \\
&\leq \begin{cases} 2\Pr\left(Z > \frac{\epsilon b - |\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} \geq 0 \\ 2\Pr\left(Z < \frac{-\epsilon b + |\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} < 0 \end{cases} \\
&= \begin{cases} 2\Pr\left(Z > \frac{\epsilon b - |\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} \geq 0 \\ 2\Pr\left(Z > \frac{\epsilon b - |\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right) & \frac{\mu_i - \mu_j}{\sigma_i - \sigma_j} < 0 \end{cases}
\end{aligned}
$$
(19)

where (19) utilizes the symmetric property of standard normal distribution. We then just need to have $\Pr\left(Z > \frac{\epsilon b - |\mu_i - \mu_j|}{|\sigma_i - \sigma_j|}\right) = \Pr\left(Z > \frac{c}{|\sigma_i - \sigma_j|}\right) \leq \frac{\delta}{2}$.

Using the Q-function for tail probability of standard normal distribution: $Q(t) = \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-x^2/2}\,dx$, define

$$
\begin{aligned}
\tau^*(\delta) &= \min\left\{\tau: \Pr(Z > \tau) \leq \frac{\delta}{2}\right\} \\
&= Q^{-1}(\delta/2)
\end{aligned}
$$
(20)

We then have sufficient condition $c \geq |\sigma_i - \sigma_j|\tau^*(\delta)$ which gives $b \geq \frac{1}{\epsilon}\left(|\mu_i - \mu_j| + |\sigma_i - \sigma_j|\tau^*(\delta)\right)$. Maximizing over ρ and $\mathbb{S}$, we have (5).

One can derive an upper bound on $\tau^*(\delta)$ without using Q-function. For example, following an (ϵ, δ)-DP proof in [27, Appendix A] by using Gaussian tail bound

$$
\Pr(Z > \tau) \leq \frac{1}{\sqrt{2\pi}\tau}e^{-\frac{\tau^2}{2}}, \quad \forall \tau > 0.
$$
(21)

To ensure $\Pr(Z > \tau) \leq \frac{\delta}{2}$, it suffices to have $\frac{1}{\sqrt{2\pi}\tau}e^{-\frac{\tau^2}{2}} \leq \frac{\delta}{2}$, which is equivalent to $\tau e^{\frac{\tau^2}{2}} \geq \frac{2}{\sqrt{2\pi}\delta} \implies \frac{\tau^2}{2}e^{\frac{\tau^2}{2}} \geq \frac{\tau}{\sqrt{2\pi}\delta} \implies \tau^2 \geq 2W_0\left(\frac{\tau}{\sqrt{2\pi}\delta}\right)$ and

$$
\tau^*(\delta) \leq \min\left\{\tau \in \mathbb{R}_+: \tau^2 \geq 2W_0\left(\frac{\tau}{\sqrt{2\pi}\delta}\right)\right\},
$$
(22)

where $W_0(\cdot)$ is the Lambert-W function such that $W_0(x)e^{W_0(x)} = x$ for nonnegative x. $W_0(x)$ is increasing in

$x \in \mathbb{R}_+$. See Fig. 5. Or, $\tau e^{\frac{\tau^2}{2}} \geq \frac{2}{\sqrt{2\pi}\delta} \implies \tau^2 e^{\tau^2} \geq \frac{2}{\pi\delta^2} \implies \tau^2 \geq W_0\left(\frac{2}{\pi\delta^2}\right)$ and

$$
\tau^*(\delta) \leq \sqrt{W_0\left(\frac{2}{\pi\delta^2}\right)},
$$
(23)

It should be noted that the method of determining $\tau^*(\delta)$ that satisfies $\Pr\left(Z > \frac{c}{|\sigma_i - \sigma_j|}\right) \leq \frac{\delta}{2}$ is not unique. ∎

APPENDIX B
PROOF OF REMARK 1

For translation priors $X|s_i$ and $X|s_j$, we have linear push-forward $T(x) = \mu_j + (x - \mu_i), \forall x \in \mathbb{R}$. Then, for Laplace noise $N \sim \mathcal{L}(b)$, we have (4) equal to

$$
\begin{aligned}
&\int\int_B \left(P_N(y - x) - e^\epsilon P_N(y - x - \mu_j + \mu_i)\right)\,dy\,dP_{X|S}(x|s_i, \rho) \\
&= \int\int_B \frac{1}{2b}\left(e^{-\frac{|y - x|}{b}} - e^{\epsilon - \frac{|y - x - \mu_j + \mu_i|}{b}}\right)\,dy\,dP_{X|S}(x|s_i, \rho) \\
&\leq \left(1 - e^{\epsilon - \frac{|\mu_i - \mu_j|}{b}}\right)\int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy\,dP_{X|S}(x|s_i, \rho), \forall B \subseteq \mathbb{R}.
\end{aligned}
$$

Thus, any $b \geq \frac{\mu_i - \mu_j}{\epsilon}$ ensures $P_{Y|S}(y \in B|s_i, \rho) - e^\epsilon P_{Y|S}(y \in B|s_j, \rho) \leq 0$ for all $B \subseteq \mathbb{R}$. Taking the maximum of the right hand side over all ρ and $(s_i, s_j) \in \mathbb{S}$, we have (6). ∎

APPENDIX C
PROOF OF THEOREM 2

For $P_{X|S}(x|s_i, \rho) = \sum_{m=1}^{D_i}\alpha_{im}\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)$ and $P_{X|S}(x|s_j, \rho) = \sum_{l=1}^{D_j}\alpha_{jl}\mathfrak{g}(x; \mu_{jl}, \sigma_{jl}^2)$ and $N \sim \mathcal{L}(b)$, we have (4) equal to

$$
\begin{aligned}
&\sum_m \alpha_{im}\int\int_B \frac{1}{2b}\left(e^{-\frac{|y - x|}{b}} - \right. \\
&\qquad\left. e^\epsilon \sum_l \frac{w_{ml}^*}{\alpha_{im}}e^{-\frac{|y - T_{ml}(x)|}{b}}\right)\,dy\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2) \\
&\leq \sum_m \alpha_{im}\int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy\Big(1 - \\
&\qquad \sum_l \frac{w_{ml}^*}{\alpha_{im}}e^{\epsilon - \frac{|x - T_{ml}(x)|}{b}}\Big)\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2) \\
&= \int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy \sum_m \alpha_{im}\Big(1 - \\
&\qquad \sum_l \frac{w_{ml}^*}{\alpha_{im}}e^{\epsilon - \frac{|x - T_{ml}(x)|}{b}}\Big)\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2) \\
&= \int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy\Big(1 - \sum_{m,l} w_{ml}^* e^{\epsilon - \frac{|x - T_{ml}(x)|}{b}}\Big)\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)
\end{aligned}
$$
(24)

$$
\leq \int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy\Big(1 - e^{\epsilon - \frac{\sum_{m,l} w_{ml}^*|x - T_{ml}(x)|}{b}}\Big)\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)
$$
(25)

$$
\leq \int\int_B \frac{1}{2b}e^{-\frac{|y - x|}{b}}\,dy\Big[1 - e^{\epsilon - \frac{\sum_{m,l} w_{ml}^*|x - T_{ml}(x)|}{b}}\Big]_+\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)
$$
(26)

$$
\leq \int\Big[1 - e^{\epsilon - \frac{\sum_{m,l} w_{ml}^*|x - T_{ml}(x)|}{b}}\Big]_+\,d\mathfrak{g}(x; \mu_{im}, \sigma_{im}^2)
$$
(27)

$$
= \int\Big[1 - e^{\epsilon - \sum_{m,l} w_{ml}^*\frac{|(\sigma_{im} - \sigma_{jl})z + (\mu_{im} - \mu_{jl})|}{b}}\Big]_+\,d\mathfrak{g}(z; 0, 1)
$$
(28)

$$
\leq \int\Big[1 - e^{\epsilon - \sum_{m,l} w_{ml}^*\frac{|\sigma_{im} - \sigma_{jl}||z| + |\mu_{im} - \mu_{jl}|}{b}}\Big]_+\,d\mathfrak{g}(z; 0, 1)
$$
(29)

$$
\leq \int_A d\mathfrak{g}(z; 0, 1), \quad \forall B \subseteq \mathbb{R},
$$
(30)

where

$$A = \Big\{ z : \sum_{m,l} w_{ml}^* \big(|\sigma_{im} - \sigma_{jl}||z| + |\mu_{im} - \mu_{jl}| \big) > \epsilon b \Big\}.$$

Equality (24) is using the fact that $\sum_l w_{ml}^* = \alpha_{im}, \forall m$ and $\sum_m \alpha_{im} = 1$ [33], (25) is using the Jensen inequality of the exponential function $\mathbb{E}[e^X] \geq e^{E[X]}$, in (26), $[z]_+ = \max\{z, 0\}$, inequality (27) is because $\int_B \frac{1}{2b} e^{-\frac{|y-x|}{b}} \, dy \leq 1$ for all B, equality (28) is by substituting $T_{ml}(x) = \mu_{jl} + \frac{\sigma_{jl}}{\sigma_{im}}(x - \mu_{im})$ and the change of variable $Z = \frac{X - \mu_{im}}{\sigma_{im}}$, (29) is using triangular inequality $|(\sigma_{im} - \sigma_{jl})z + (\mu_{im} - \mu_{jl})| \leq |\sigma_{im} - \sigma_{jl}||z| + |\mu_{im} - \mu_{jl}|$ and the monotonicity of $[\cdot]_+$ and (30) is because $\Big[1 - e^{\epsilon - \sum_{m,l} w_{ml}^* \frac{|\sigma_{im} - \sigma_{jl}||z| + |\mu_{im} - \mu_{jl}|}{b}} \Big]_+ \in [0, 1]$.

For $b \geq \frac{1}{\epsilon} \big(\sum_{m,l} w_{ml}^* |\mu_{im} - \mu_{jl}| + c \big)$,

$$\int_A d\mathfrak{g}(z; 0, 1) = \Pr\Big(|Z| > \frac{c}{\sum_{m,l} w_{ml}^* |\sigma_{im} - \sigma_{jl}|} \Big)$$
$$= 2\Pr\Big(Z > \frac{c}{\sum_{m,l} w_{ml}^* |\sigma_{im} - \sigma_{jl}|} \Big).$$

by (21), it suffices to have $c \geq \tau^*(\delta) \sum_{m,l} w_{ml}^* |\sigma_{im} - \sigma_{jl}|$ such that $b \geq \frac{1}{\epsilon} \sum_{m,l} w_{ml}^* \big(|\mu_{im} - \mu_{jl}| + \tau^*(\delta)|\sigma_{im} - \sigma_{jl}| \big)$. Maximizing over ρ and $\mathbb{S}$, we get (13). ∎

APPENDIX D
PROOF OF COROLLARY 2

For $P_{X|S}(x|s_i, \rho) = \sum_{m=1}^D \alpha_m \mathfrak{g}(x; \mu_{im}, \sigma_m^2)$, $P_{X|S}(x|s_j, \rho) = \sum_{m=1}^D \alpha_m \mathfrak{g}(x; \mu_{jm}, \sigma_m^2)$ and $N \sim \mathcal{L}(b)$, (4) equals to

$$\sum_m \alpha_m \int \int_B \Big(P_N(y - x) - $$
$$e^\epsilon P_N(y - x + \mu_{im} - \mu_{jm}) \Big) \, dy \, d\mathfrak{g}(x; \mu_{im}, \sigma_m^2)$$
$$\leq \sum_m \alpha_m \Big(1 - e^{\epsilon - \frac{|\mu_{im} - \mu_{jm}|}{b}} \Big) \int \int_B P_N(y - x) \, dy \, d\mathfrak{g}(x; \mu_{im}, \sigma_m^2).$$

So, $b \geq \frac{1}{\epsilon} \sum_m \alpha_m |\mu_{im} - \mu_{jl}|$ ensures (4) ≤ 0. Maximizing over ρ and $\mathbb{S}$, we get (15). ∎

APPENDIX E
INTERPRETATION OF $\hat{\pi}$

In this paper, the purpose of Laplace mechanism is to attain (ϵ, δ)-pufferfish privacy: for given ϵ and δ, $P_{Y|S}(B|s_i, \rho) \leq e^\epsilon P_{Y|S}(B|s_j, \rho) + \delta, \forall B \subseteq \mathbb{R}, (s_i, s_j) \in \mathbb{S}$. For $X|s_i \sim \mathcal{G}(\mu_i, \sigma_i^2)$, $X|s_j \sim \mathcal{G}(\mu_j, \sigma_j^2)$ and $N \sim \mathcal{L}(b)$, consider

the problem of searching the minimum value of δ over all couplings:

$$\inf_\pi \{ P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho) \}$$
$$= \inf_\pi \int_B \int \big(P_N(y - x) - e^\epsilon P_N(y - x') \big) \, d\pi(x, x') \, dy$$
$$= \inf_\pi \int_B \int \frac{1}{2b} \big(e^{-\frac{|y-x|}{b}} - e^{-\frac{|y-x'|}{b}} \big) \, d\pi(x, x') \, dy$$
$$\leq \inf_\pi \int \big(1 - e^{\epsilon - \frac{|x-x'|}{b}} \big) \int_B \frac{1}{2b} e^{-\frac{|y-x|}{b}} \, dy \, d\pi(x, x')$$
$$\leq \inf_\pi \int \Big[1 - e^{\epsilon - \frac{|x-x'|}{b}} \Big]_+ \, d\pi(x, x')$$
$$\leq \int \Big[1 - e^{\epsilon - \frac{|x-x'|}{b}} \Big]_+ \, d\hat{\pi}(x, x') \tag{31}$$
$$\leq \int \Big[\frac{|x-x'|}{b} - \epsilon \Big]_+ \, d\hat{\pi}(x, x') \tag{32}$$
$$\leq \Big(\int \big(\frac{|x-x'|}{b} - \epsilon \big)^2 \, d\hat{\pi}(x, x') \Big)^{\frac{1}{2}} \tag{33}$$
$$= \Big(\inf_\pi \int \big(\frac{|x-x'|}{b} - \epsilon \big)^2 \, d\pi(x, x') \Big)^{\frac{1}{2}}, \forall B \subseteq \mathbb{R}, \tag{34}$$

where inequality $e^x \geq 1 + x$ is applied in (32) and inequality (33) is due to the monotonicity of ℓ_α-norm. In (34), $\inf_\pi$ gives rise to the Monge's transport plan $\hat{\pi}$ in (33), which tightens the upper bound on $\inf_\pi \{ P_{Y|S}(B|s_i, \rho) - e^\epsilon P_{Y|S}(B|s_j, \rho) \}$. The reason for having this tighter upper bound is to obtain a smaller value of b, indicating less noise power, as a sufficient condition for (ϵ, δ)-pufferfish privacy. Therefore, adopting Monge's optimal transport plan contributes to a reduction in noise for attaining pufferfish privacy.

In this paper, we use the Monge's solution of $d\hat{\pi}(x, x') = dP_{X|S}(x|s_i, \rho) \cdot \mathbb{I}\{x' = T(x)\}$ and the main results, Theorems 1 and 2, are essentially derived by the idea of imposing an upper bound δ on (31):

$$\int \Big[1 - e^{\epsilon - \frac{|x-x'|}{b}} \Big]_+ \, d\hat{\pi}(x, x')$$
$$= \int \Big[1 - e^{\epsilon - \frac{|x-T(x)|}{b}} \Big]_+ \, dP_{X|S}(x|s_i, \rho) \leq \delta. \tag{35}$$

REFERENCES

[1] C. Dwork, "Differential privacy," in *Automata, Languages and Programming*, M. Bugliesi, B. Preneel, V. Sassone, and I. Wegener, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 1–12.
[2] C. Dwork, F. McSherry, K. Nissim, and A. Smith, "Calibrating noise to sensitivity in private data analysis," in *Theory of Cryptography*, S. Halevi and T. Rabin, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 265–284.
[3] C. Dwork, "Differential privacy: A survey of results," in *Theory and Applications of Models of Computation*, M. Agrawal, D. Du, Z. Duan, and A. Li, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 1–19.
[4] D. Kifer and A. Machanavajjhala, "A rigorous and customizable framework for privacy," in *Proceedings of the 31st ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, ser. PODS '12. New York, NY, USA: Association for Computing Machinery, 2012, p. 77–88.
[5] ——, "Pufferfish: A framework for mathematical privacy definitions," *ACM Transactions on Database Systems*, vol. 39, no. 1, Jan. 2014.
[6] G. Smith, "On the foundations of quantitative information flow," in *Foundations of Software Science and Computational Structures*, L. de Alfaro, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 288–302.

[7] F. du Pin Calmon and N. Fawaz, "Privacy against statistical inference," in *2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, 2012, pp. 1401–1408.

[8] L. Sankar, S. R. Rajagopalan, and H. V. Poor, "Utility-privacy tradeoffs in databases: An information-theoretic approach," *IEEE Transactions on Information Forensics and Security*, vol. 8, no. 6, pp. 838–852, 2013.

[9] S. Song, Y. Wang, and K. Chaudhuri, "Pufferfish privacy mechanisms for correlated data," in *Proceedings of the 2017 ACM International Conference on Management of Data*, New York, NY, USA, 2017, p. 1291–1306.

[10] N. Ding, "Kantorovich mechanism for pufferfish privacy," in *International Conference on Artificial Intelligence and Statistics*. Valencia, Spain: PMLR, 2022, pp. 5084–5103.

[11] T. Champion, L. De Pascale, and P. Juutinen, "The ∞-Wasserstein distance: Local solutions and existence of optimal transport maps," *SIAM Journal on Mathematical Analysis*, vol. 40, no. 1, pp. 1–20, 2008.

[12] L. De Pascale and J. Louet, "A study of the dual problem of the one-dimensional l_∞-optimal transport problem with applications," *Journal of Functional Analysis*, vol. 276, no. 11, pp. 3304–3324, 2019.

[13] W. Zhang, O. Ohrimenko, and R. Cummings, "Attribute privacy: Framework and mechanisms." New York, NY, USA: Association for Computing Machinery, 2022, p. 757–766.

[14] D. Dowson and B. Landau, "The fréchet distance between multivariate normal distributions," *Journal of multivariate analysis*, vol. 12, no. 3, pp. 450–455, 1982.

[15] C. R. Givens and R. M. Shortt, "A class of Wasserstein metrics for probability distributions." *Michigan Mathematical Journal*, vol. 31, no. 2, pp. 231–240, 1984.

[16] A. Takatsu, "Wasserstein geometry of Gaussian measures," *Osaka Journal of Mathematics*, vol. 48, no. 4, pp. 1005–1026, 2011.

[17] A. Asuncion and D. Newman, "UCI machine learning repository https://archive.ics.uci.edu/ml/index.php," 2007. [Online]. Available: https://archive.ics.uci.edu/ml/index.php

[18] C. Niu, Z. Zheng, S. Tang, X. Gao, and F. Wu, "Making big money from small sensors: Trading time-series data under pufferfish privacy," in *IEEE INFOCOM 2019 - IEEE Conference on Computer Communications*. IEEE, Apr. 2019.

[19] J. Ding, A. Ghosh, R. Sarkar, and J. Gao, "Publishing asynchronous event times with pufferfish privacy," in *2022 18th International Conference on Distributed Computing in Sensor Systems (DCOSS)*. IEEE, May 2022.

[20] W. Liang, H. Chen, R. Liu, Y. Wu, and C. Li, "A pufferfish privacy mechanism for monitoring web browsing behavior under temporal correlations," *Computers & Security*, vol. 92, p. 101754, 2020.

[21] L. Ou, Z. Qin, S. Liao, H. Yin, and X. Jia, "An optimal pufferfish privacy mechanism for temporally correlated trajectories," *IEEE Access*, vol. 6, pp. 37 150–37 165, 2018.

[22] C. Pierquin, A. Bellet, M. Tommasi, and M. Boussard, "Rényi pufferfish privacy: General additive noise mechanisms and privacy amplification by iteration via shift reduction lemmas," *hal-04363020*, 2023.

[23] Y. Zeng, Y. Sang, S. Luo, and M. Song, *Communications in Computer and Information Science*. Springer Singapore, 2021, ch. A Pufferfish Privacy Mechanism for the Trajectory Clustering Task, pp. 307–317.

[24] M. Sugiyama, *Introduction to statistical machine learning*. Morgan Kaufmann, 2015.

[25] M. Chen and O. Ohrimenko, "Protecting global properties of datasets with distribution privacy mechanisms," in *Proceedings of The 26th International Conference on Artificial Intelligence and Statistics*, ser. Proceedings of Machine Learning Research, F. Ruiz, J. Dy, and J.-W. van de Meent, Eds., vol. 206. PMLR, 25–27 Apr 2023, pp. 7472–7491.

[26] G. Peyré, M. Cuturi *et al.*, "Computational optimal transport," *Center for Research in Economics and Statistics Working Papers*, no. 2017-86, 2017.

[27] C. Dwork, A. Roth *et al.*, "The algorithmic foundations of differential privacy." *Found. Trends Theor. Comput. Sci.*, vol. 9, no. 3-4, pp. 211–407, 2014.

[28] H. Fischer, *A history of the central limit theorem: from classical to modern probability theory*. Springer, 2011.

[29] P. Samarati and L. Sweeney, "Protecting privacy when disclosing information: k-anonymity and its enforcement through generalization and suppression," Harvard Data Privacy Lab., Tech. Rep., 1998.

[30] N. Li, T. Li, and S. Venkatasubramanian, "t-closeness: Privacy beyond k-anonymity and l-diversity," in *Proceeding of 2007 IEEE 23rd International Conference on Data Engineering*, 2007, pp. 106–115.

[31] Y. Li, Z. Zeng, J. Li, B. Yan, Y. Zhao, and J. Zhang, "Distributed model training based on data parallelism in edge computing-enabled elastic optical networks," *IEEE Communications Letters*, vol. 25, no. 4, pp. 1241–1244, Apr. 2021.

[32] Y. Chen, T. T. Georgiou, and A. Tannenbaum, "Optimal transport for Gaussian mixture models," *IEEE Access*, vol. 7, pp. 6269–6278, 2019.

[33] J. Delon and A. Desolneux, "A Wasserstein-type distance in the space of Gaussian mixture models," *SIAM Journal on Imaging Sciences*, vol. 13, no. 2, pp. 936–970, 2020.

[34] F. Bimbot, J.-F. Bonastre, C. Fredouille, G. Gravier, I. Magrin-Chagnolleau, S. Meignier, T. Merlin, J. Ortega-García, D. Petrovska-Delacrétaz, and D. A. Reynolds, "A tutorial on text-independent speaker verification," *EURASIP Journal on Advances in Signal Processing*, vol. 2004, pp. 1–22, 2004.

[35] B. Balle and Y.-X. Wang, "Improving the Gaussian mechanism for differential privacy: Analytical calibration and optimal denoising," in *Proceedings of the 35th International Conference on Machine Learning*, ser. Proceedings of Machine Learning Research, J. Dy and A. Krause, Eds., vol. 80. PMLR, 10–15 Jul 2018, pp. 394–403.

[36] I. Mironov, "Rényi differential privacy," in *2017 IEEE 30th Computer Security Foundations Symposium (CSF)*, 2017, pp. 263–275.

[37] S. Meiser and E. Mohammadi, "Tight on budget? tight bounds for r-fold approximate differential privacy," in *Proceedings of the 2018 ACM SIGSAC Conference on Computer and Communications Security*, ser. CCS '18. New York, NY, USA: Association for Computing Machinery, 2018, p. 247–264.

[38] A. Koskela, J. Jälkö, and A. Honkela, "Computing tight differential privacy guarantees using FFT," in *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics*, ser. Proceedings of Machine Learning Research, S. Chiappa and R. Calandra, Eds., vol. 108. PMLR, 26–28 Aug 2020, pp. 2560–2569.

Ni Ding received the PhD degree from the Australian National University, Australia, in 2017. She was a postdoctoral fellow at Data 61, CSIRO, Australia, from 2017 to 2020 and a Doreen Thomas Postdoctoral Fellow at the University of Melbourne from 2020-2023. She is now a lecturer at the School of Computer Science, University of Auckland, New Zealand. Her research interests generally include optimizations in information theory, signal processing and machine learning. She is currently interested in data privacy, discrete and combinatorial optimization problems raised in discrete event control in cross-layer adaptive modulation, source coding and game theory (in particular, the games with strong structures, e.g., supermodular and convex games).