DD2380

Artificial Intelligence

Module: Taming uncertainty

Patric Jensfelt

DD2380

Artificial Intelligence

Probabilistic reasoning

Patric Jensfelt

Credits

- Based partly on material from
  - Kevin Murphy, MIT, UBC, Google
  - Danica Kragic, KTH
  - W. Burgard, C. Stachniss, M. Benewitz and K. Arras, when at Albert-Ludwigs-Universität Freiburg

Motivation: Why use probabilities

- An artificial system (or human) does not have perfect knowledge of its environment and of the results of its actions, and it needs to deal with uncertainty at many levels

- Uncertainty plays an important role in: sensor interpretation, sensor fusion, map making, path planning, self-localization, control, etc, etc

Motivational examples

- Diagnos diseases
  - Doctor knows (Dr Watson)
    - How common a certain disease is
    - Connection with factors such as age, sex, habits, …
    - Connection with probing results of, e.g. temperature
  - Observe
  - Diagnos

Motivational examples

- Autonomous car: Cross intersection safely?
  - From manufacturer or learned by car
    - Sensor models
    - Statistic from different roads
    - Weather models
    - …
  - Observations from own car and others?
  - Q:
    - Can I cross if I want to be 99.99999% safe?
    - Can I cross if I want to be 99% safe?

Motivational examples

- A1: Hunt ducks!
  - Have some knowledge about how different birds move
  - Have some rather sparse observations of birds
  - How will they move next so I can shoot?
  - How to tell which birds belong to the same species?

Recap of probability theory

- Probability of event X: $p(X)$
- Joint probability of X and Y: $p(X,Y)$, i.e. X AND Y
- Conditional probability of X given Y: $p(X|Y)$

Notation

- For boolean variables we will use
  - p(X) to mean the probability that X is true and
  - p(¬X) the probability that X is false.

Rules of probability

- $p(X) \in [0,1]$ (i.e. $0 \leq p(X) \leq 1$)
- $p(X) = 1 - p(\neg X)$
- Sum rule (marginalization): $p(X) = \sum_{Y} p(X,Y)$
- Product rule: $p(X,Y) = p(Y|X)p(X)$
- Prob sum to one: $1 = \sum_{\text{all } X} p(X)$

Conditioning

- Combining

$$p(X)=\sum_Y p(X,Y) \qquad \text{(sum rule)}$$

$$p(X,Y)=p(X|Y)p(Y) \quad \text{(product rule)}$$

gives

$$\mathbf{p(X)=\sum_Y p(X|Y)p(Y)}$$

Bayes rule

- p(Y|X) = p(X|Y) p(Y) / p(X)

  posterior = likelihood * prior / evidence

- What is the probability of Y given some evidence X can be expressed in factors that are sometimes easier to determine.

- Exercise1: Prove it using the previous rules!

Bayes rule cont'd

- Rewrite $p(Y|X) = p(X|Y)\, p(Y) / p(X)$

  using the sum rule $p(X) = \sum_Y p(X|Y)p(Y)$

$$p(Y|X) = p(X|Y)\, p(Y) / \sum_Y p(X|Y)p(Y)$$

$p(X) = \sum_Y p(X|Y)p(Y)$ is a normalization factor $1/\eta$

$\rightarrow$

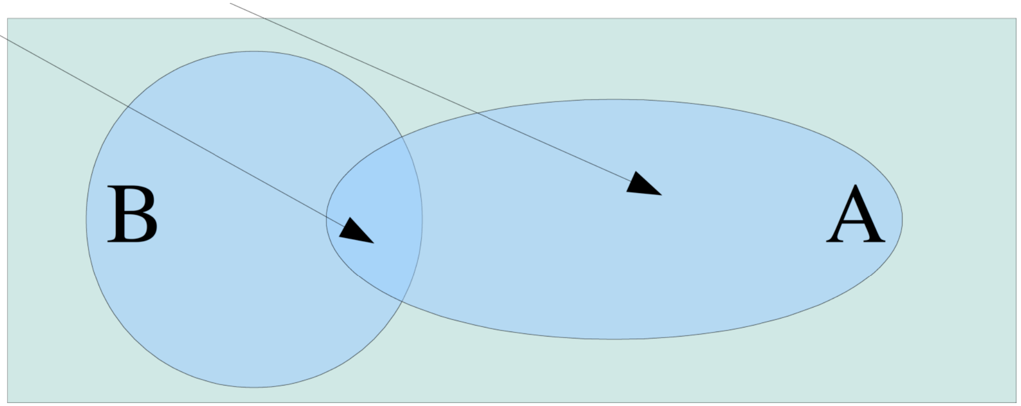
$$p(Y|X) = \eta\, p(X|Y)\, p(Y)$$

Conditional probability

- Consider two dependent events A and B

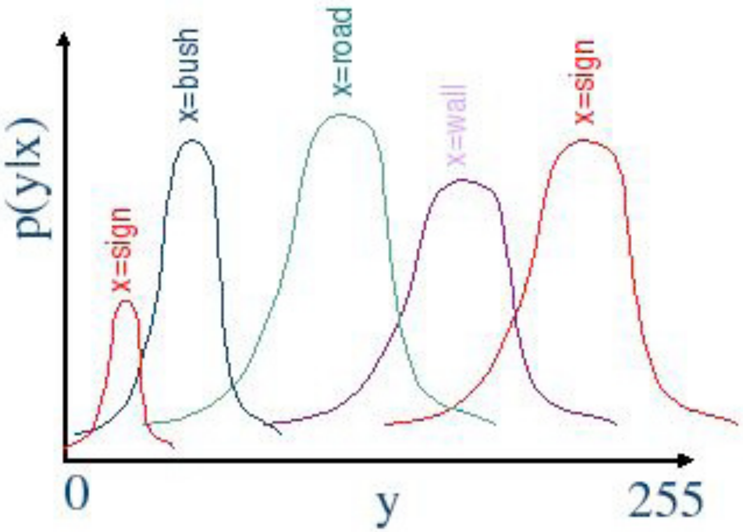
B \ A	True	False
True	0.1	0.3
False	0.4	0.2

- Probability of B given that A is true
  p(B|A) = p(A,B) / P(A)



Example conditional dependence

- Knowing what we look at gives a much better idea of what to expect to measure.
- Ex: If we know that we look for a sign we expect either bright (near 255) for the white background or dark (near 0) for pixel values for the text



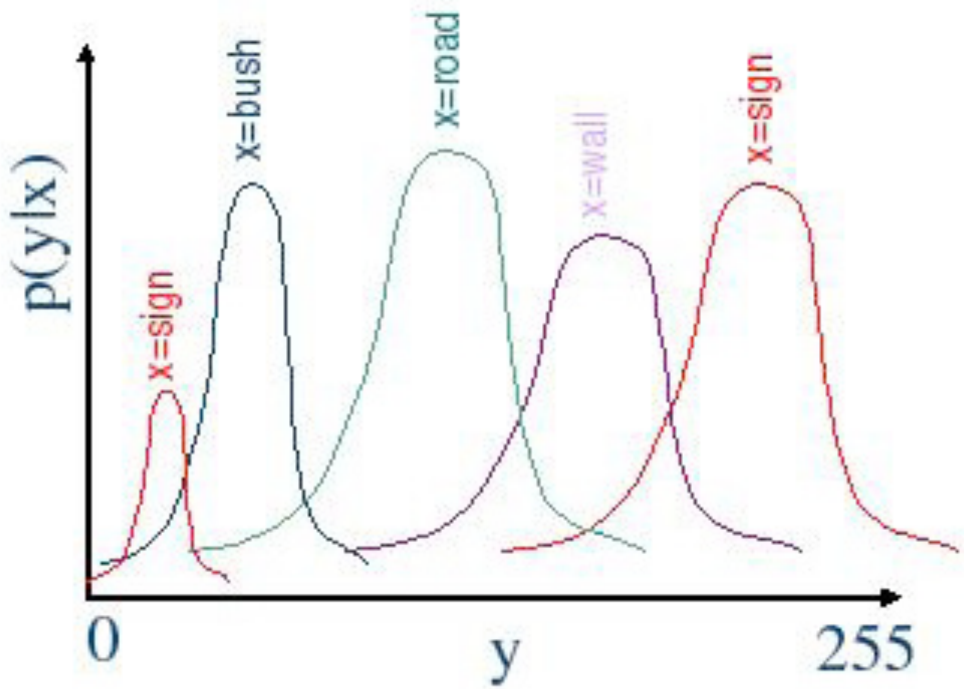
So what?

So what?

- If variables are dependent we can get information about one variable by measuring another!!
- Foundation for most of the probabilistic reasoning and statistical machine learning

Simplistic probabilistic reasoning

- If we measure 115, what is the most likely category?



Road classification

Bayes rule example

- Vision system for detecting zebras Z
- Prior: p(Z)=0.02 (zebra in 2% of images)
- Detector for "stripey areas" gives observations, O
- Detector gives yes/no
- Detector performance
  - $p(O|Z)=0.8$ (true pos)
  - $p(O|\neg Z)=0.1$ (false pos, e.g. gate)

- Task: Calculate $p(Z|O)$

  What does it represent?

  How big roughly?

Bayes rules example solution

- p(Z|O) represent prob that there is a zebra if our detector says there is one.
- How big? Use Bayes rule!

$$p(Z|O)=\frac{p(Z|O)\,p(Z)}{p(O)}$$

- And then?

Bayes rules example solution

- Expand p(O) using conditioning

$$
\begin{aligned}
p(Z|O) &= \frac{p(Z|O)\,p(Z)}{p(O)} \\
&= \frac{p(Z|O)\,p(Z)}{p(O|Z)\,p(Z) + p(O|\neg Z)\,p(\neg Z)} \\
&= \frac{0.8 \cdot 0.02}{0.8 \cdot 0.02 + 0.1 \cdot 0.98} \\
&= \frac{0.016}{0.016 + 0.098} \\
&\approx 0.1404
\end{aligned}
$$

Bayes rule example discussion

- Intuition tells most people that the detector is much better than this, i.e. we would expect to see a much higher $p(Z|O)$ since the detector is correct in 80% of the cases
- However, only 1 out of 50 images has a zebra
  → 49 out of 50 do not contain a zebra
  \+ detector not perfect
- Failing to account for negative evidence properly is a typical failing of human intuitive reasoning

Conditional independence

- If X is conditionally independent of Y given Z
  $p(X|Y,Z) = P(X|Z)$

- Which also means

  $p(X,Y|Z)$ = {product rules} =

  $p(X|Y,Z)p(Y|Z)$ = {conditional independence}

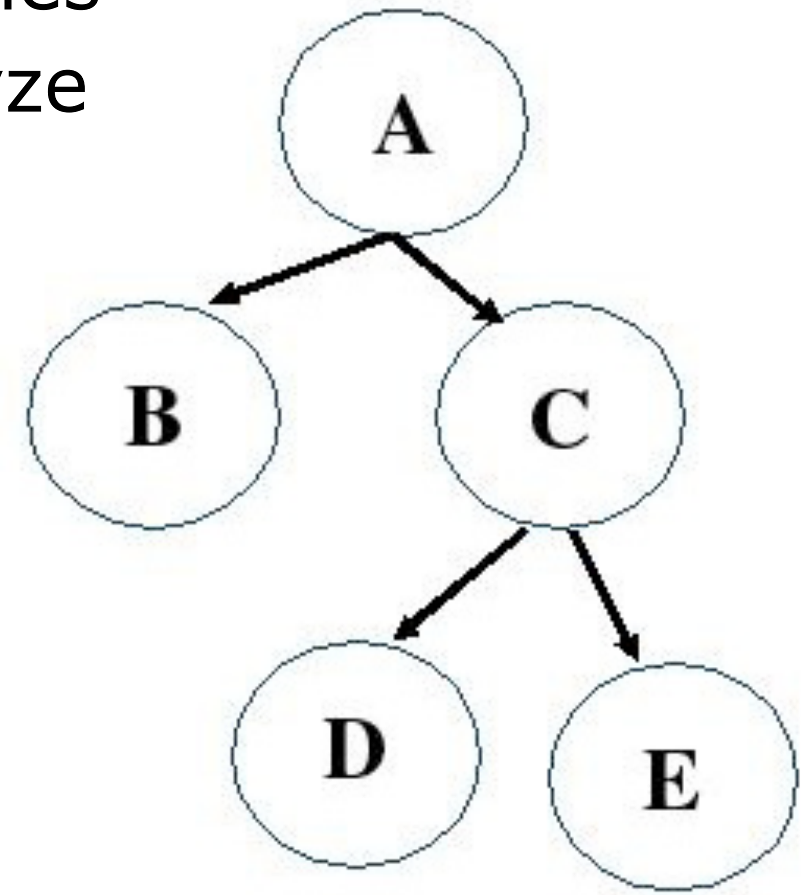
  $p(X|Z)p(Y|Z)$

- NOTE: Not the same as $p(X,Y) = P(X)p(Y)$

Shifting gear, hang on!

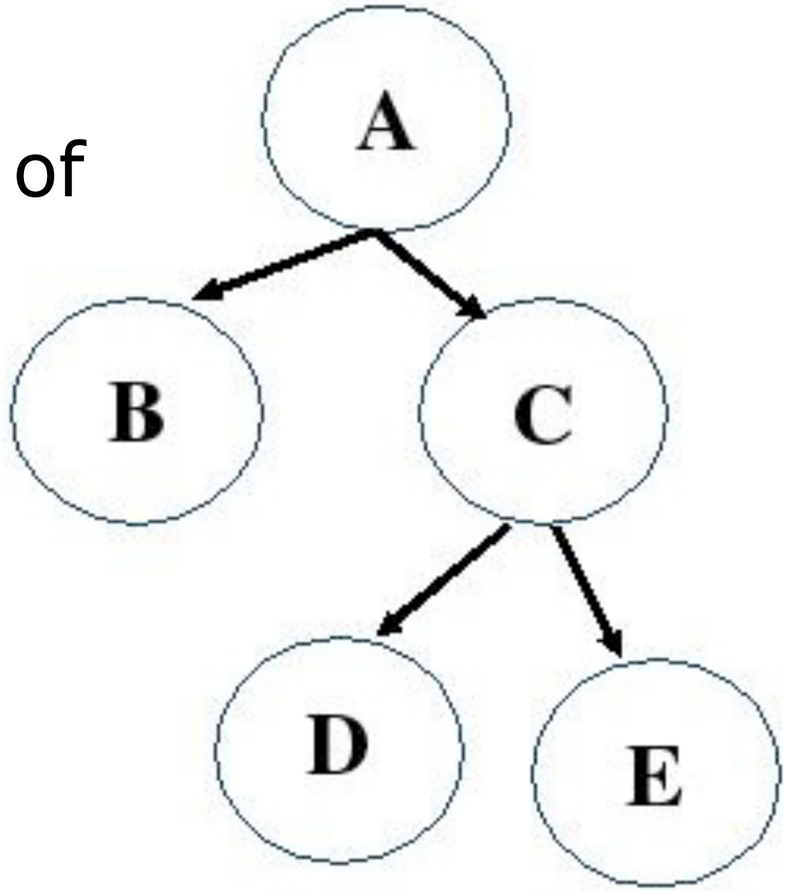
Probabilistic Graphical Models

- Compact repr. of the joint distribution over a set of variables
- Graphical repr. that helps analyze and structure prob. information
- Each variable is encoded as a node
- Variables discrete or continues
- Conditional independence assumptions coded as arcs
- Here: a Bayesian network (directed acyclic graph (DAG))



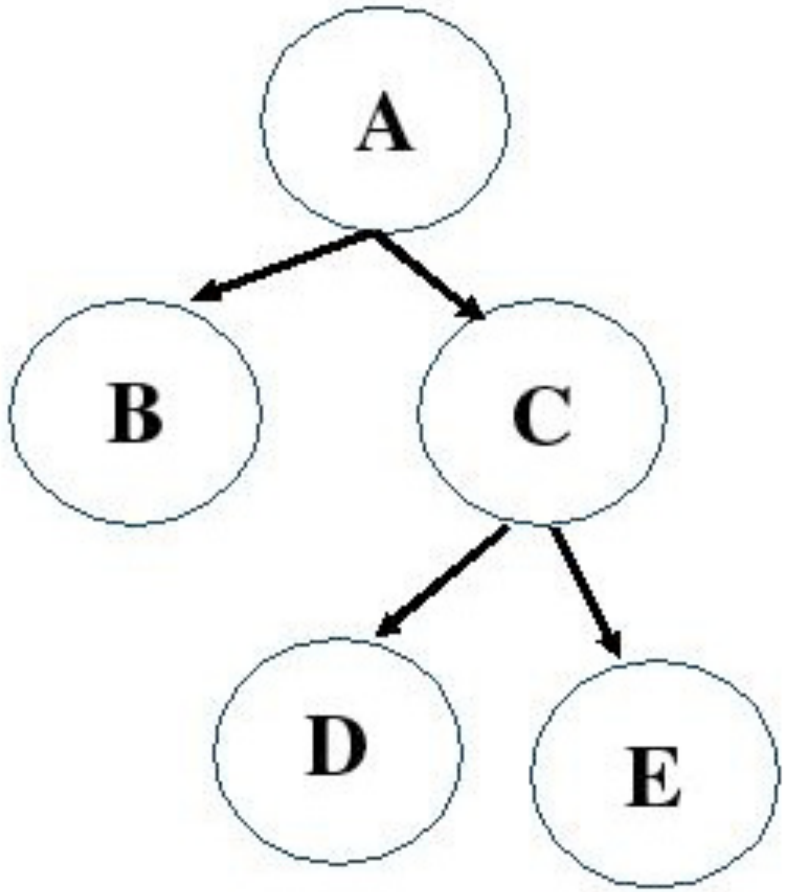
Bayesian network

- A root node
- B, D, E leaf nodes
- A “causes” B and C
  → Value of A influences value of B and C
- A parent to B and C
- B and C children of A
- A ancestor of D and E



Bayesian network

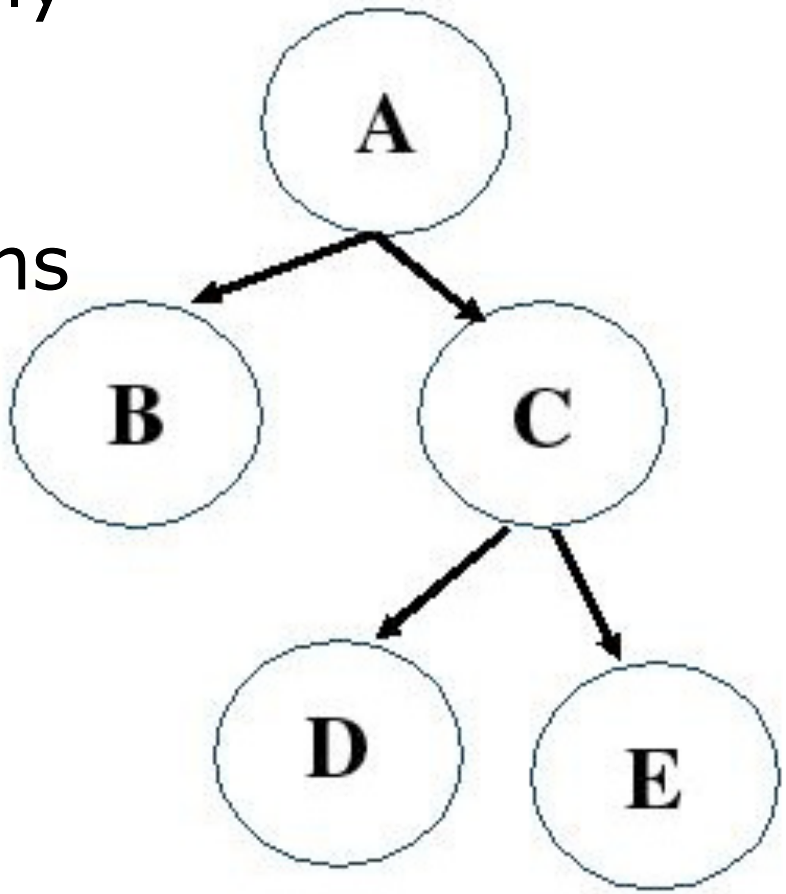
- Arrow → "direct influence over"
  A has direct influence over B



Bayesian network

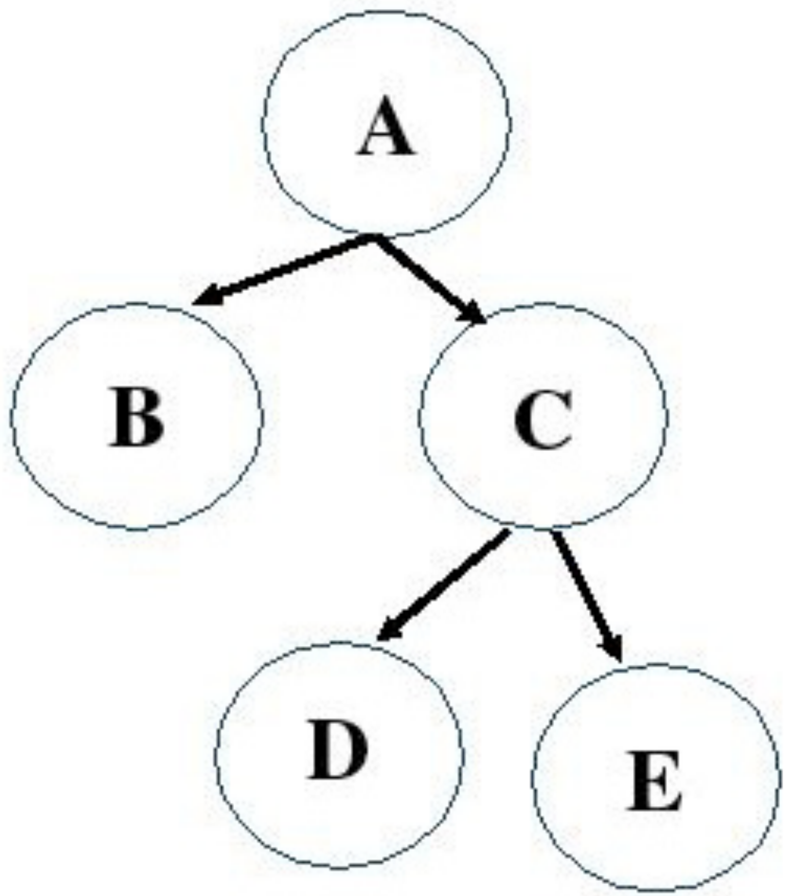
- B and C are dependent
- HOWEVER, they are conditionally independent given A

- Q: What does this mean in terms of formulas?



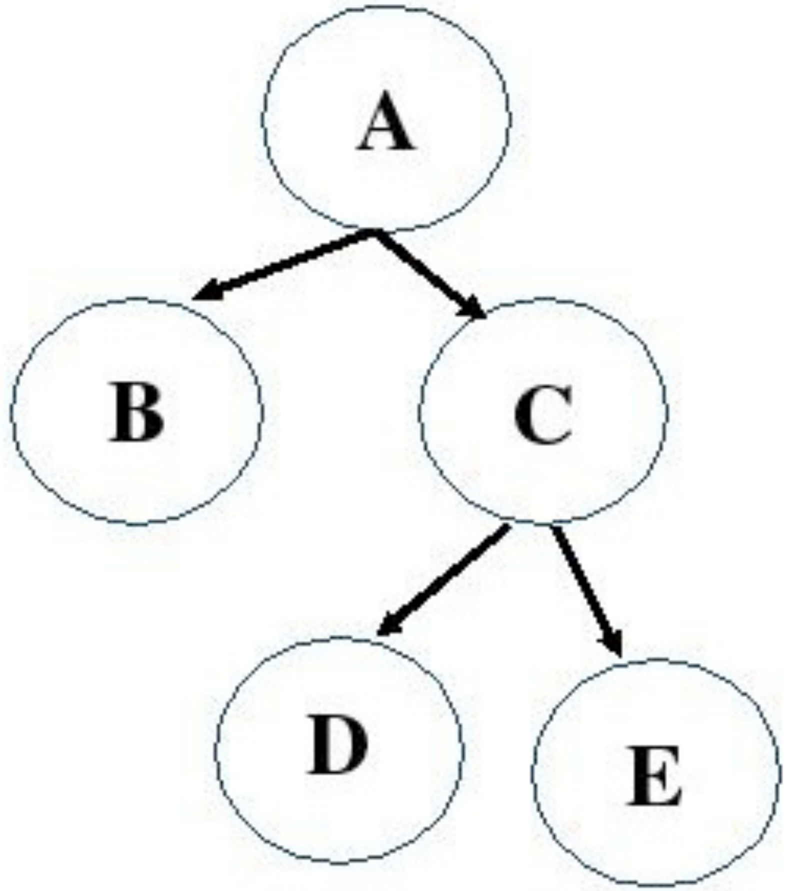
B and C independent given A

- $P(B,C|A)=P(B|A)P(C|A)$
- If we do not know A, knowing something about B will tell us something about C (tells us about A which tells us about C)
- but if we know A then knowing B does not tell us anything more about C than we already knew because of A.



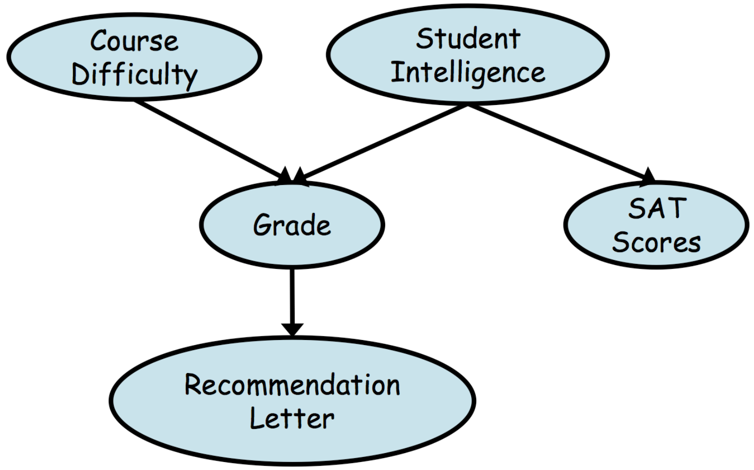
Bayesian network

- C depends on A
- E depends on A and C. **HOWEVER**, E is conditionally independent of A given C.
- That is, C captures all the information in A relevant to determine E.

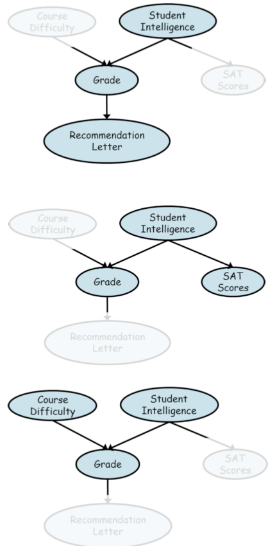


Flow of probabilistic influence

- When can X influence Y?
  - Case1: No evidence about Z
  - Case2: Evidence about Z



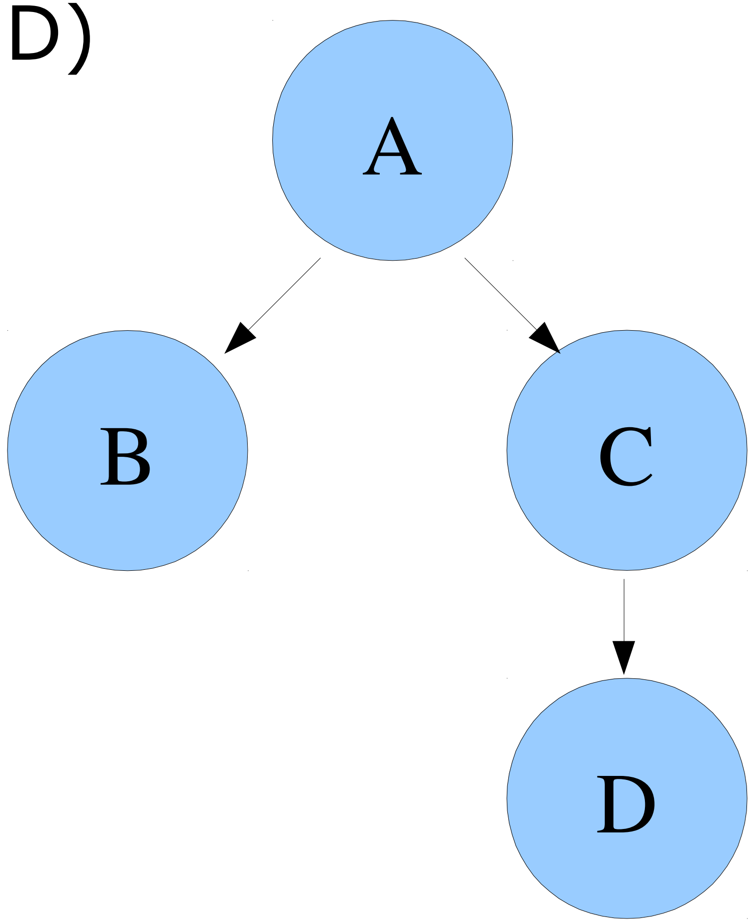
Graphical Structure	NO evidence about Z	YES evidence about Z
$X \rightarrow Z \rightarrow Y$	✓	✗
$X \leftarrow Z \rightarrow Y$	✓	✗
$X \rightarrow Z \leftarrow Y$	✗	✓



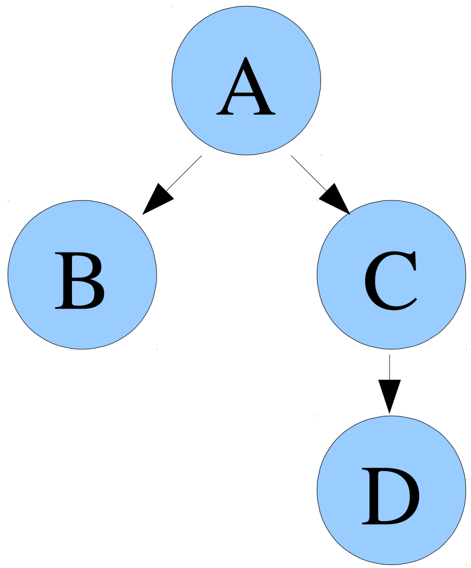
Adapted from Daphne Koller

Joint distribution

- Exercise2: Factorize p(A,B,C,D)
- Remember product rule
  p(X,Y) = P(Y|X)P(X)
- Tip: Work from the top and factor out A, then B, C and D



Derivation



$$
\begin{aligned}
p(A,B,C,D) &= \{\text{product rule with X=A, Y=B,C,D}\} \\
&= p(B,C,D|A)p(A) \\
&= \{\text{product rule with X=B, Y=C,D}\} \\
&= p(C,D|A,B)p(B|A,C,D)p(A) \\
&= \{\text{B conditionally independent of C,D given A}\} \\
&= p(C,D|A,B)p(B|A)p(A) \\
&= \{\text{product rule with X=C, Y=D}\} \\
&= p(D|A,B,C)p(C|A,B)p(B|A)p(A) \\
&= \{\text{C conditionally independent of B given A}\} \\
&= p(D|A,B,C)p(C|A)p(B|A)p(A) \\
&= \{\text{D conditionally independent of A and B given C}\} \\
&= p(D|C)p(C|A)p(B|A)p(A)
\end{aligned}
$$

- In general

$$p(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{i=n} p(X_i|Parents(X_i))$$

Note

- We could have used the product rule in any order
- Looking at the graph we can make use of the conditional independencies
- Other factorization possible, but not as compact

Alarm example

- You have an alarm. It reacts reliably to burglaries but is sometimes triggered also by small earthquakes.
- Two neighbors John and Mary promises to call you at work when they hear the alarm
- John calls almost every time there is an alarm but sometimes confuses it with a phone ringing and calls then too
- Mary plays loud music and sometimes misses it completely but rarely mix other things with it
- Draw Bayesian network!

Alarm example

Structure

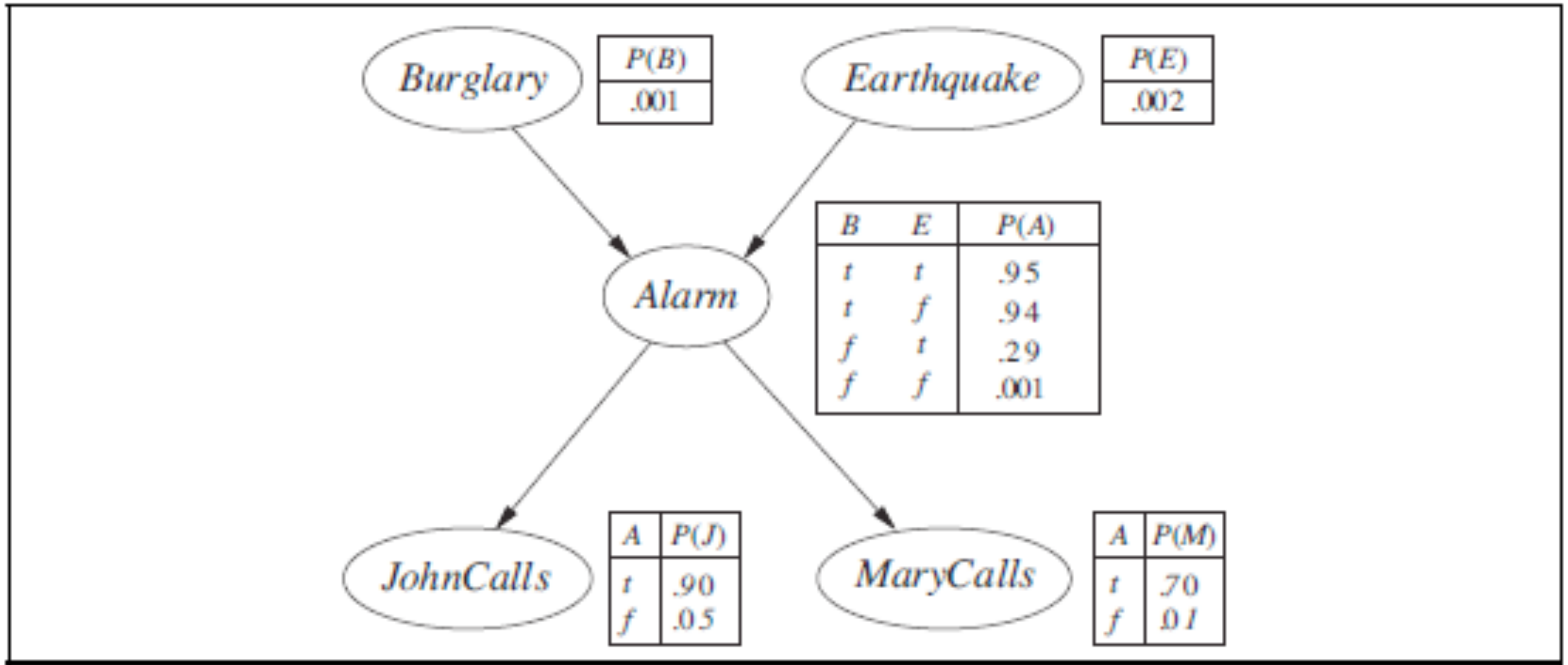
- You have an alarm. It reacts reliably to burglaries but is sometimes triggered also by small earthquakes.
- Two neighbors John and Mary promises to call you at work when they hear the alarm (not earthquake or burglary!)

Probabilities

- John calls almost every time there is an alarm but sometimes confuses it with a phone ringing and calls then too
- Mary plays loud music and sometimes misses it completely but rarely mix other things with it
- Draw Bayesian network! What variables?

Alarm example cont'd

- John and Mary calling does not depend on what triggered the alarm only the alarm itself (simplification)
- CPT – conditional probability table. (Values not specified in the text before!! Made up here)



Exercise4: Alarm example calculation

- Calculate p(J,M,A,¬B,¬E) Means what??

  J : JohnCalls
  M : MaryCalls
  A : Alarm
  B : Burglary
  E : Earthquake

- Remember that p(¬X) = 1-p(X) and

$$p(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{i=n} p(X_i | Parents(X_i))$$

Alarm example calculation

- $p(J,M,A,\neg B,\neg E) = 0.00062$
- So what does this tell us?

  Very unlikely that both John and Mary calls and the alarm has gone off when there is no burglary or earthquake

Two views on Bayesian networks

- Representation of the joint probability distribution
  - Helps to understand how to construct it
- Encoding of a collection of independence statements
  - Helpful in designing inference procedure

Tip

- When constructing the network try to use ordering based on
  cause → symptom (causal) rather than
  symptom → cause (diagnostic)

- Need to specify fewer numbers and numbers are easier to get

- Ex: Alarm → MaryCalls means have to specify p(MaryCalls|Alarm) which is a lot easier than p(Alarm|MaryCalls) for MaryCalls → Alarm

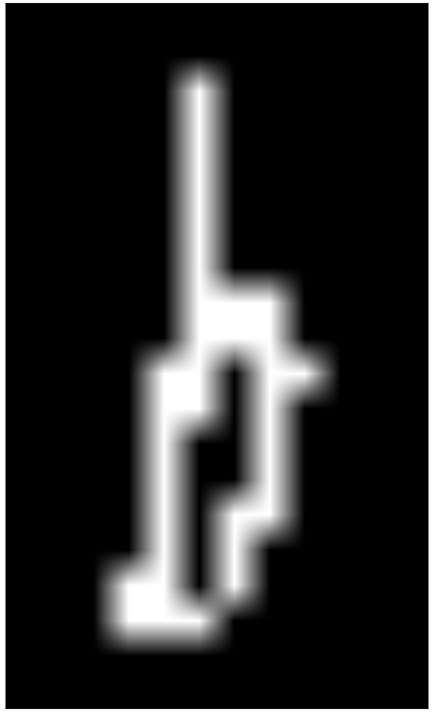
Sequential data

- Often we have data that is sampled in time or space

Sequential data

- Measurement of time series
- Example: Sign recognition

- Measure: drawn path
- Want: characters



Sequential data

- Measurement of time series
- Example: Activity recognition

- Measure: images
- Want: what happens?

Sequential data

- Measurement of time series
- Example: Speech recognition

- Measure: audio signal
- Want: Words/sentences

Material from
Giampiero Salvi
DD2118 Speech and speaker recognition

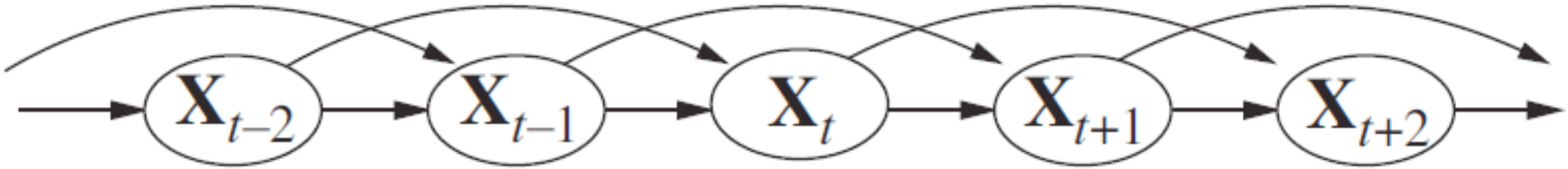
Let's get cracking with the theory again

Sequential data
Markov model

- The (first order) Markov assumption
  - The present (current state) can be predicted using local knowledge of the past (state at the previous step)
  - X_t is conditionally independent of all X_k ,k=t-2, …, given X_{t-1} ,i.e.
    $p(X_t|X_{t-1},X_{t-2},X_{t-3},\ldots) = p(X_t|X_{t-1})$

Second-order Markov Model

- State at time k depends on the states at times k-1 and k-2



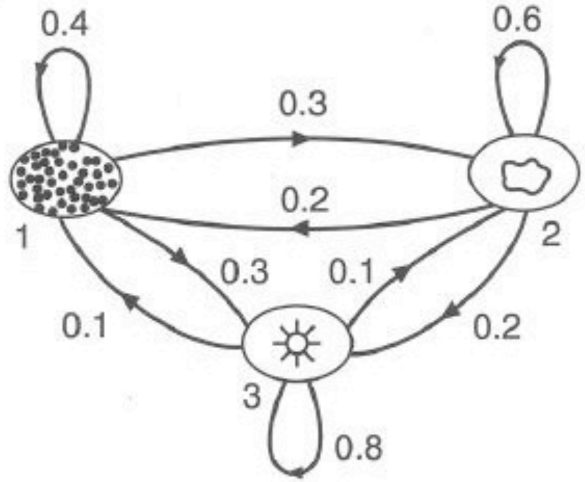
Example: Weather prediction

Once each day (e.g., at noon), the weather is observed and classified as being one of the following:

- State 1—Rain (or Snow; e.g. precipitation)
- State 2—Cloudy
- State 3—Sunny

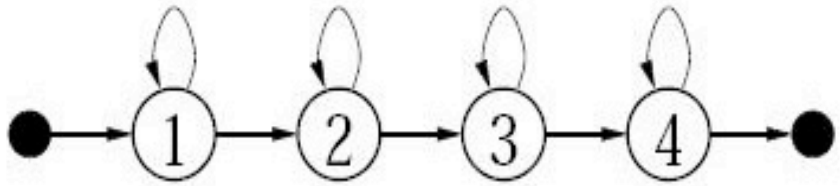
with state transition probabilities:

$$A = \{a_{ij}\} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$



Ex: 30% chance to transition from state 1 to state 2 (and 3).

Calculation example



Transition probabilities:

$$A = \{a_{ij}\} = \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

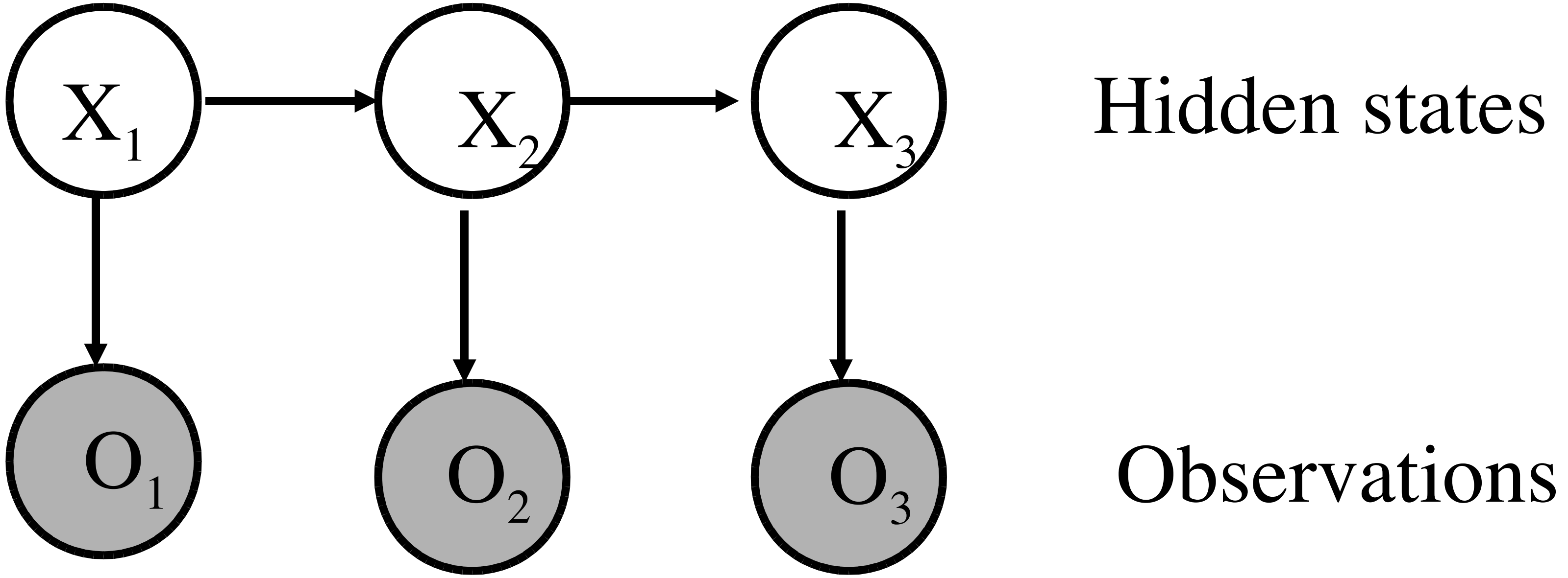
→ We start in state 1 at t=1

- p(X(t=2)=3) = ?
- p(X(t=2)=2) = ?
- p(X(t=3)=2) = ?

What if we cannot observe the state?

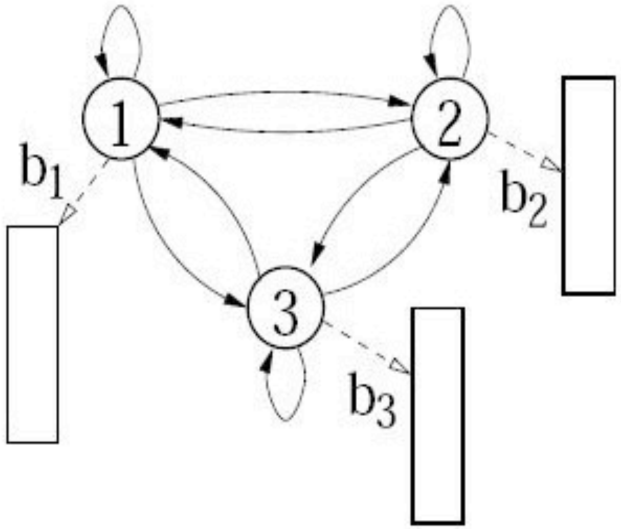
- Examples:
  - Cannot observe the weather, only the temperature
    - State=weather, observation=temperature
  - Cannot observe the words spoken, only the sound uttered
    - State=word, observation=sound
  - Cannot observe the letter written, only the connected points drawn
    - State=letter, observation=what is drawn

Hidden Markov Models (HMMs)



- State transition model:
  $p(X_t=j|X_{t-1}=i)=A(i,j)=a_{ij}$
- Observation model:
  $p(O_t=j|X_t=i)=b_{ij}$

Hidden Markov Models



Probability of state i producing a *discrete* observation o_t, which can have one of a finite set of values, is:

Elements of HMM

1. Number of states N, $x \in \{1, \ldots, N\}$;
2. Number of events K, $k \in \{1, \ldots, K\}$;
3. Initial-state probabilities,
   $\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \leq i \leq N$;
4. State-transition probabilities,
   $A = \{a_{ij}\} = \{P(x_t = j | x_{t-1} = i)\}$ for $1 \leq i, j \leq N$;
5. Discrete output probabilities,
   $B = \{b_i(k)\} = \{P(o_t = k | x_t = i)\}$ for $1 \leq i \leq N$ and $1 \leq k \leq K$.

Elements of HMM

1. Number of states N, $x \in \{1, \ldots, N\}$;

2. Number of events K, $k \in \{1, \ldots, K\}$;

The number of possible observations types

3. Initial-state probabilities,
   $\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \leq i \leq N$;

4. State-transition probabilities,
   $A = \{a_{ij}\} = \{P(x_t = j | x_{t-1} = i)\}$ for $1 \leq i, j \leq N$;

5. Discrete output probabilities,
   $B = \{b_i(k)\} = \{P(o_t = k | x_t = i)\}$ for $1 \leq i \leq N$ and $1 \leq k \leq K$.

Elements of HMM

1. Number of states N, $x \in \{1, \ldots, N\}$;

2. Number of events K, $k \in \{1, \ldots, K\}$;

Note that we can start with the probability spread over several states

3. Initial-state probabilities,
$\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \leq i \leq N$;

4. State-transition probabilities,
$A = \{a_{ij}\} = \{P(x_t = j | x_{t-1} = i)\}$ for $1 \leq i, j \leq N$;

5. Discrete output probabilities,
$B = \{b_i(k)\} = \{P(o_t = k | x_t = i)\}$ for $1 \leq i \leq N$ and $1 \leq k \leq K$.

The model if often called λ

- λ is the model, i.e.,
  $\lambda=(A, B, \pi)$

state transition matrix

Distribution for initial state

Output Matrix
Output sometimes called “emissions”

- λ is sometimes called $\mathcal{M}$
- A, B and π are row-stochastic matrices (their rows sum to 1)

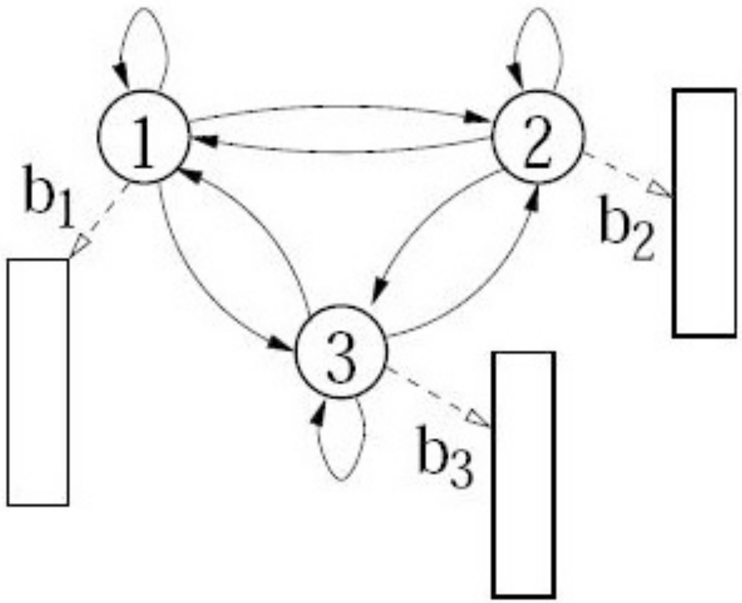
Elements of HMM, λ

- Initial Distribution : contains the probability of the (hidden) model being in a particular hidden state at time t = 1 (sometimes t=0).
- Often referred to as π
- Ex: $\pi=[0.5\ 0.2\ 0.3]$, i.e.,

  $p(X_1=1)=0.5$

  $p(X_1=2)=0.2$

  $p(X_1=3)=0.3$



Elements of HMM, λ

- State transition matrix : holding the probability of transitioning from one hidden state to another hidden state.
- Ex: a_{21} gives $p(X_{t+1}=1|X_t=2)$, i.e. probability to transition from state 2 to state 1

	$X_{t+1}=1$	$X_{t+1}=2$	...	$X_{t+1}=N$
$X_t=1$	a_{11}	a_{12}	...	a_{1N}
$X_t=2$	a_{21}	a_{22}	...	a_{2N}
...	...	...	...	...
$X_t=N$	a_{N1}	a_{N2}	...	a_{NN}

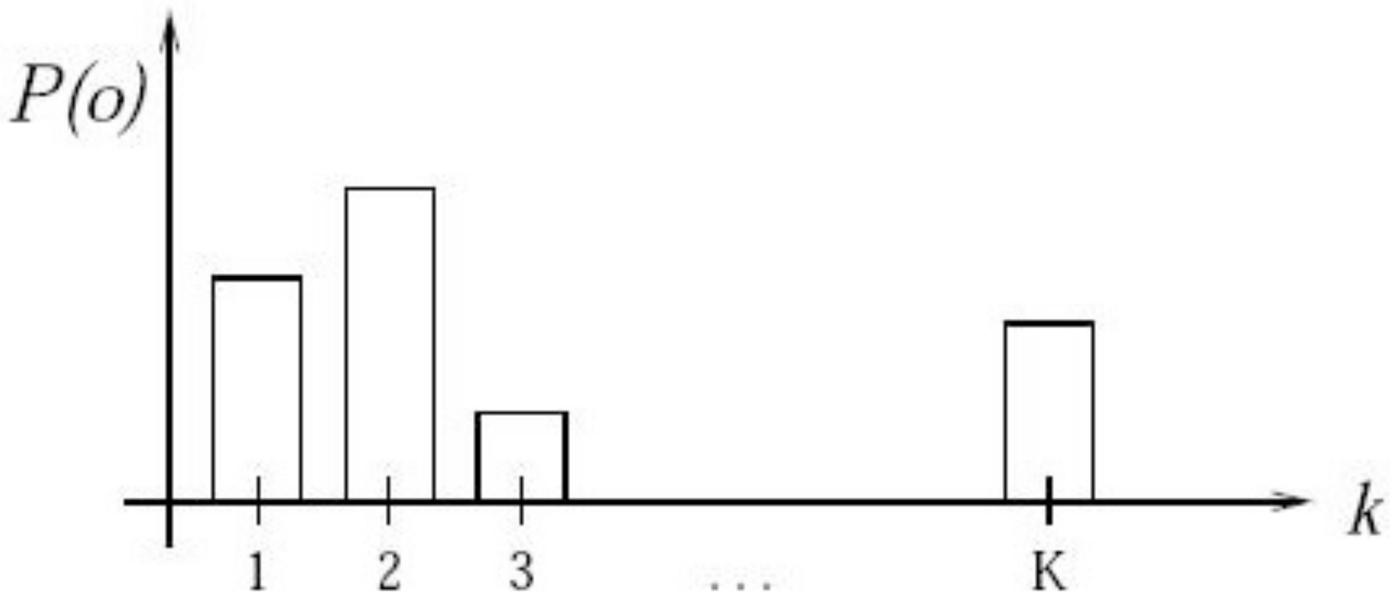
Elements of HMM, λ

- Output matrix : Contains the probability of observing a particular measurement given that the hidden model is in a particular hidden state.

	$O_t=1$	$O_t=2$	...	$O_t=K$
$X_t=1$	$b_1(1)$	$b_1(2)$	...	$b_1(K)$
$X_t=2$	$b_2(1)$	$b_2(2)$	...	$b_2(K)$
...	...	...	...	...
$X_t=N$	$b_N(1)$	$b_N()$	...	$b_N(K)$

- $b_j(O_t)$ is the probability to observe O_t in state j

Discrete output probabilities



- NOTE: The probability for a certain observation/event typically depends on the state we are in

What is an HMM, reeealllly?!?!?

- It is a **model** (not necessarily a perfect one) for system
- It can be used to (e.g.)
  - Generate predictions about how the system will behave
  - Analyze if a sequence of measurements match a certain model, e.g., did the person say "Bayesian" (i.e. match our model for that word) or "Bearnaise" (i.e. match that model)?
  - Learn something about a system by learning the model parameters.

A1

- At A-B level in A1 you will **learn models** for the motion of birds to be able to **predict** where they will be in the next time step. You will also identify what type of bird you see by comparing the observations to the models you built.

End of lecture

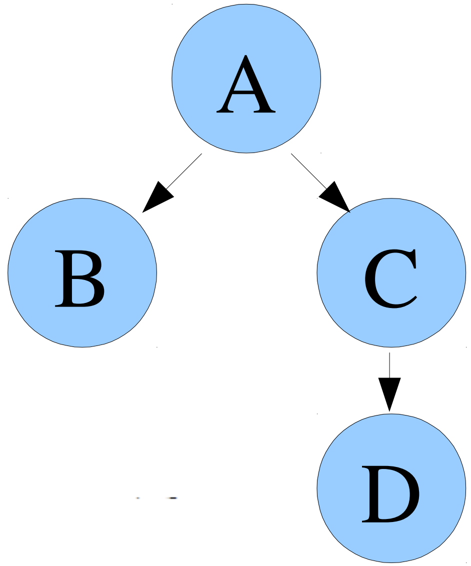
See exercises after this...

Exercise 1: Deriving Bayes rules

- Product rule
  - $p(X,Y)=p(Y|X)p(X)$
  - $p(Y,X)=p(X|Y)p(Y)$
- $p(X,Y)=p(Y,X)$ (symmetry)
- $p(Y|X)p(X) = p(X|Y)p(Y)$

$\rightarrow \boxed{p(Y|X) = p(X|Y)\,p(Y) / p(X)}$

Exercise2: Factorization

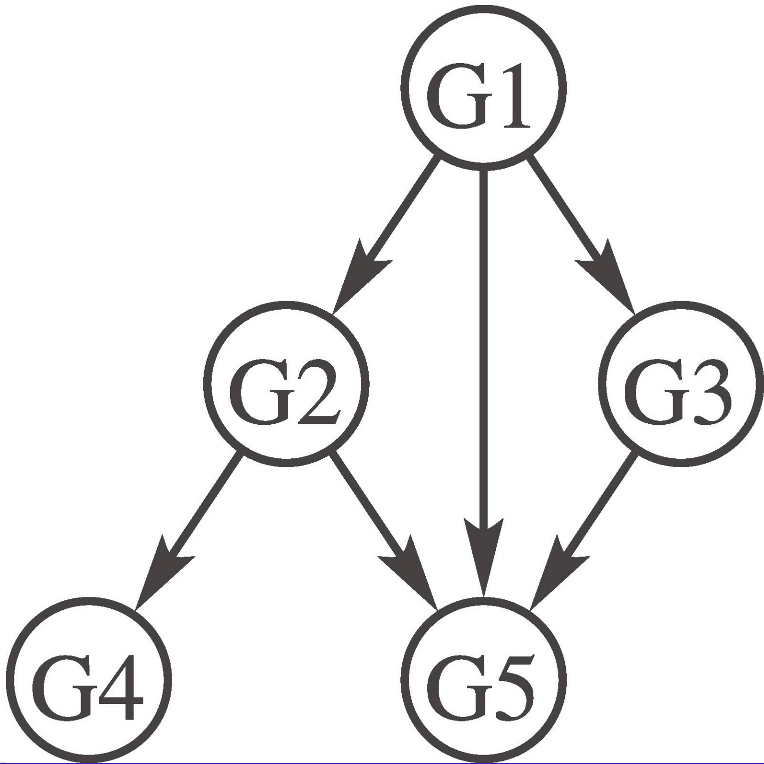


$$
\begin{aligned}
p(A,B,C,D) &= \{\text{product rule with X=A, Y=B,C,D}\} \\
&= p(B,C,D|A)p(A) \\
&= \{\text{product rule with X=B, Y=C,D}\} \\
&= p(C,D|A,B)p(B|A,C,D)p(A) \\
&= \{\text{B conditionally independent of C,D given A}\} \\
&= p(C,D|A,B)p(B|A)p(A) \\
&= \{\text{product rule with X=C, Y=D}\} \\
&= p(D|A,B,C)p(C|A,B)p(B|A)p(A) \\
&= \{\text{C conditionally independent of B given A}\} \\
&= p(D|A,B,C)p(C|A)p(B|A)p(A) \\
&= \{\text{D conditionally independent of A and B given C}\} \\
&= p(D|C)p(C|A)p(B|A)p(A)
\end{aligned}
$$

Exercise3: Factorize the graph

- $p(G1,G2,G3,G4,G5)$

  $= p(G1)\, p(G2|G1)\, p(G3|G1)\, p(G4|G2)\, p(G5|G1,G2,G3)$



Exercie4: Alarm example calculation

- $p(J,M,A,\neg B,\neg E)$ = {use graph!!}

  $= p(\neg B)p(\neg E)p(A|\neg B,\neg E)p(J|A)p(M|A)$

  = {read from CPT and use $p(\neg X) = 1\text{-}p(X)$}

  $= (1-0.001) * (1-0.002) * 0.001 * 0.9 * 0.7$

  $= 0.00062$