Introduction to SAT Solving

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OR Meetup Leipzig – 20.06.19

About me

- Using SAT solvers for around 10 years
- SAT solver in Rust github.com/jix/varisat
- Blog jix.one

Outline

- What are SAT solvers?
- How to use them?
- How do they work?

Propositional Logic

Formulas

- Constants $1, 0$
- Boolean variables $a, b, \ldots$
- Negation $\neg$
- Binary connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus$

- $a \leftrightarrow ((\neg b \rightarrow \neg c) \vee d \vee c)$

Propositional Logic

Models

- Assignment $\mathcal{A} = \{a \mapsto 1, b \mapsto 0, \ldots\}$
- Model $\mathcal{A} \models \varphi$ or not $\mathcal{A} \not\models \varphi$

Propositional Logic

Satisfiability

For a formula φ, does there exist an assignment $\mathcal{A}$ so that $\mathcal{A}$ is a model of φ?

$$(\exists \mathcal{A})\, \mathcal{A} \models \varphi$$

Propositional Logic

Satisfiability

For a formula φ, does there exist an assignment $\mathcal{A}$ so that $\mathcal{A}$ is a model of φ?

$$(\exists \mathcal{A})\, \mathcal{A} \models \varphi$$

NP

Propositional Logic

Satisfiability

For a formula φ, does there exist an assignment $\mathcal{A}$ so that $\mathcal{A}$ is a model of φ?

$$(\exists \mathcal{A})\, \mathcal{A} \models \varphi$$

NP-complete

Propositional Logic

Validity

For a formula φ, and for every assignment $\mathcal{A}$ is $\mathcal{A}$ a model of φ?

$$(\forall \mathcal{A})\, \mathcal{A} \models \varphi$$

Propositional Logic

Validity

For a formula φ, and for every assignment $\mathcal{A}$ is $\mathcal{A}$ a model of φ?

$$\begin{aligned} & (\forall \mathcal{A})\, \mathcal{A} \models \varphi \\ \iff & \neg(\exists \mathcal{A})\, \mathcal{A} \models \neg\varphi \end{aligned}$$

Applications

- Hardware and software verification
- Scheduling
- Versioned dependency resolution
- Other NP-complete problems
- Backend for other solvers and tools

Conjunctive Normal Form (CNF)

- Boolean variables $a, b, \ldots$
- Literals $a, \neg a, b, \neg b, \ldots$
- Clauses $C_1 = a,\; C_2 = b \vee c,$
  $C_3 = \neg d \vee e \vee f,\; \ldots,\; C_n = \neg x$
- Formula $\varphi = C_1 \wedge C_2 \wedge C_3 \wedge \ldots \wedge C_n$
- Often $\varphi = \{\{a\}, \{b, c\}, \ldots, \{\neg x\}\}$

DIMACS CNF

$$
\begin{aligned}
&(\neg x_1 \lor x_2) \land \\
&(\neg x_1 \lor \neg x_3) \land \\
&(\neg x_1 \lor x_4) \land \\
&(x_2 \lor \neg x_3) \land \\
&(x_2 \lor x_4) \land \\
&(\neg x_3 \lor x_4) \land \\
&(x_1 \lor \neg x_2 \lor x_3 \lor \neg x_4)
\end{aligned}
$$

DIMACS CNF

$$
\begin{aligned}
&(\neg x_1 \lor x_2) \land \\
&(\neg x_1 \lor \neg x_3) \land \\
&(\neg x_1 \lor x_4) \land \\
&(x_2 \lor \neg x_3) \land \\
&(x_2 \lor x_4) \land \\
&(\neg x_3 \lor x_4) \land \\
&(x_1 \lor \neg x_2 \lor x_3 \lor \neg x_4)
\end{aligned}
$$

```
p cnf 4 7
-1 2 0
-1 -3 0
-1 4 0
2 -3 0
2 4 0
-3 4 0
1 -2 3 -4 0
```

Typical Solver API

```
solver.add_clause([-1, 2])
solver.add_clause([-1, -3])
solver.add_clause([-1, 4])
...
solver.solve()
```

Tseytin Transformation

$$a \leftrightarrow ((\neg b \rightarrow \neg c) \vee (d \vee c))$$

Tseytin Transformation

$$a \leftrightarrow \underbrace{(\overbrace{(\neg b \rightarrow \neg c)}^{t_1} \vee \overbrace{(d \vee c)}^{t_2})}_{t_3}$$

Tseytin Transformation

$$a \leftrightarrow \underbrace{(\overbrace{(\neg b \rightarrow \neg c)}^{t_1} \vee \overbrace{(d \vee c)}^{t_2})}_{t_3}$$

$$(t_1 \leftrightarrow \neg b \rightarrow \neg c) \quad \wedge$$

Tseytin Transformation

$$a \leftrightarrow \underbrace{(\overbrace{(\neg b \rightarrow \neg c)}^{t_1} \vee \overbrace{(d \vee c)}^{t_2})}_{t_3}$$

$$(t_1 \leftrightarrow \neg b \rightarrow \neg c) \quad \wedge$$

$$(t_2 \leftrightarrow d \vee c) \quad \wedge$$

Tseytin Transformation

$$a \leftrightarrow (\underbrace{(\overbrace{\neg b \rightarrow \neg c}^{t_1}) \lor (\overbrace{d \lor c}^{t_2})}_{t_3})$$

$$(t_1 \leftrightarrow \neg b \rightarrow \neg c) \quad \land$$

$$(t_2 \leftrightarrow d \lor c) \quad \land$$

$$(t_3 \leftrightarrow t_1 \lor t_2) \quad \land$$

Tseytin Transformation

$$a \leftrightarrow \underbrace{(\overbrace{(\neg b \rightarrow \neg c)}^{t_1} \vee \overbrace{(d \vee c)}^{t_2})}_{t_3}$$

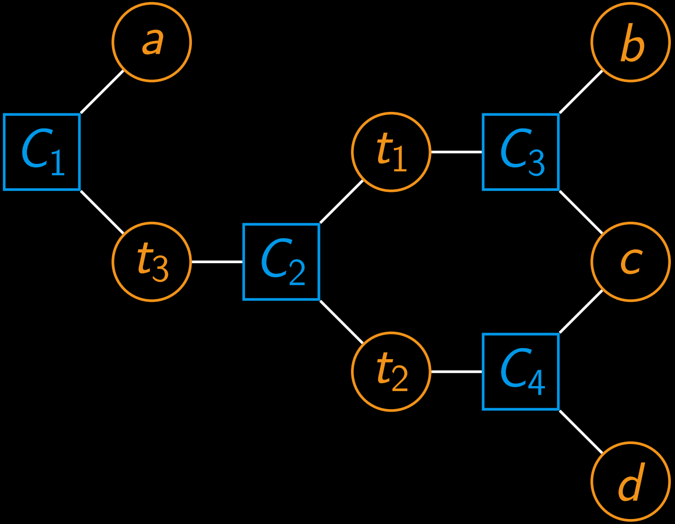
$$\begin{aligned}
&(t_1 \leftrightarrow \neg b \rightarrow \neg c) \quad \wedge \\
&(t_2 \leftrightarrow d \vee c) \quad \wedge \\
&(t_3 \leftrightarrow t_1 \vee t_2) \quad \wedge \\
&(a \leftrightarrow t_3)
\end{aligned}$$

Tseytin Transformation

$$a \leftrightarrow \underbrace{(\overbrace{(\neg b \rightarrow \neg c)}^{t_1} \vee \overbrace{(d \vee c)}^{t_2})}_{t_3}$$

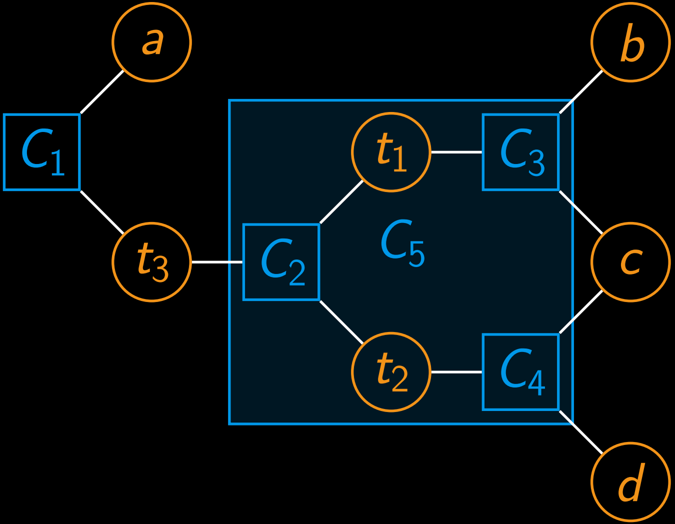
$$\begin{aligned} &(t_1 \leftrightarrow \neg b \rightarrow \neg c) && \wedge \\ &(t_2 \leftrightarrow d \vee c) && \wedge \\ &(t_3 \leftrightarrow t_1 \vee t_2) && \wedge \\ &(a \leftrightarrow t_3) \end{aligned}$$

Constraint Graphs



$$C_1(a, t_3) \wedge$$
$$C_2(t_1, t_2, t_3) \wedge$$
$$C_3(t_1, b, c) \wedge$$
$$C_4(t_2, c, d)$$

Constraint Graphs



$$C_1(a, t_3) \wedge$$
$$C_2(t_1, t_2, t_3) \wedge$$
$$C_3(t_1, b, c) \wedge$$
$$C_4(t_2, c, d)$$

Implications

$$x \to y$$

Implications

$$x \rightarrow y$$

$$\Leftrightarrow \quad \neg x \vee y$$

Implications

$$x \to y$$

$$\Leftrightarrow \quad \neg x \lor y$$

$$x_1 \land \cdots \land x_i \to y_1 \lor \cdots \lor y_j$$

Implications

$$
\begin{aligned}
& x \to y \\
\Leftrightarrow\quad & \neg x \lor y
\end{aligned}
$$

$$
\begin{aligned}
& x_1 \land \cdots \land x_i \to y_1 \lor \cdots \lor y_j \\
\Leftrightarrow\quad & \neg(x_1 \land \cdots \land x_i) \lor (y_1 \lor \cdots \lor y_j)
\end{aligned}
$$

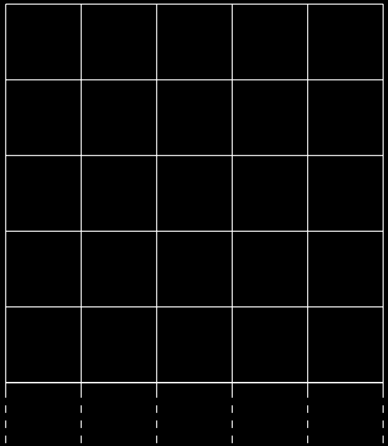
Implications

$$\begin{aligned} & x \to y \\ \Leftrightarrow\ & \neg x \lor y \end{aligned}$$

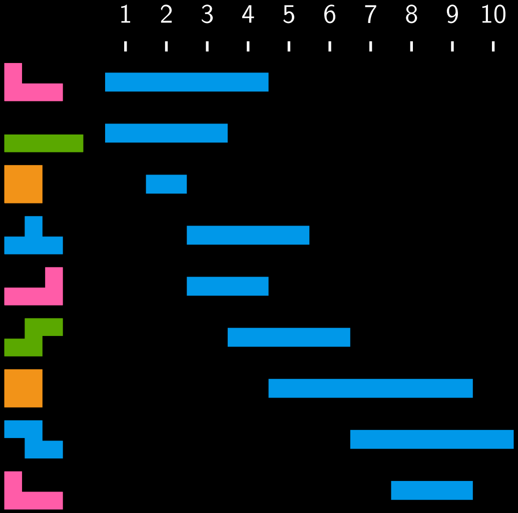
$$\begin{aligned} & x_1 \land \cdots \land x_i \to y_1 \lor \cdots \lor y_j \\ \Leftrightarrow\ & \neg(x_1 \land \cdots \land x_i) \lor (y_1 \lor \cdots \lor y_j) \\ \Leftrightarrow\ & \neg x_1 \lor \cdots \lor \neg x_i \lor y_1 \lor \cdots \lor y_j \end{aligned}$$

Example Problem

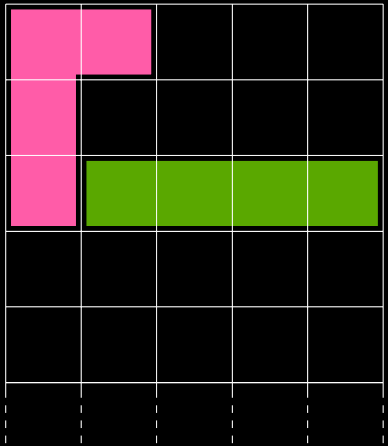
Example Problem



↑ Minimize



Example Problem



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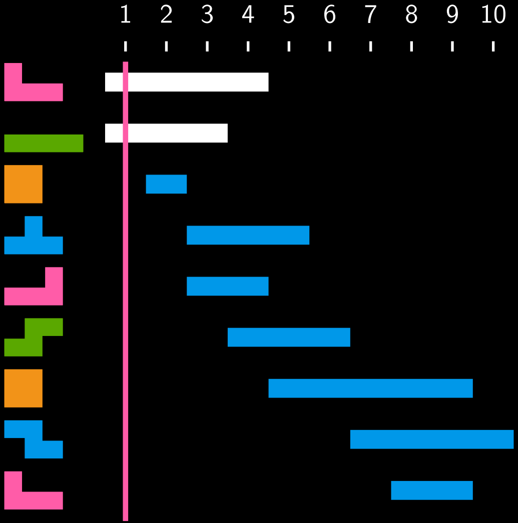
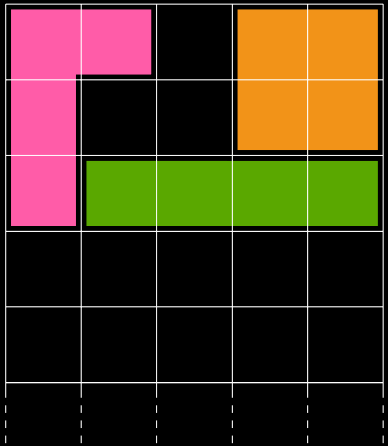


Example Problem



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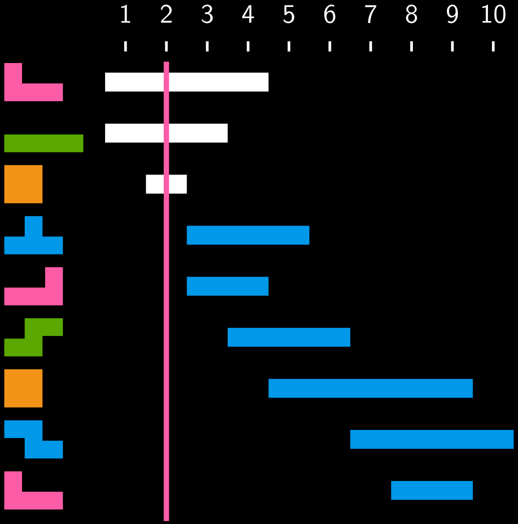

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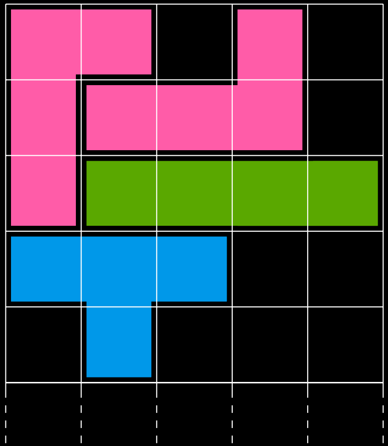

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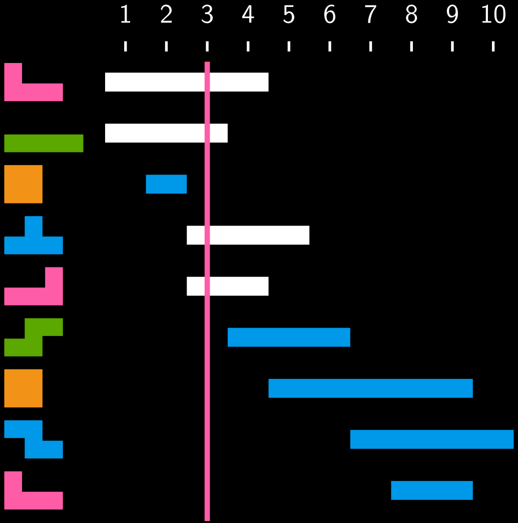
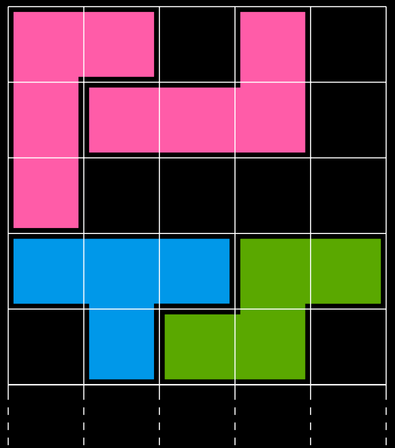


Example Problem



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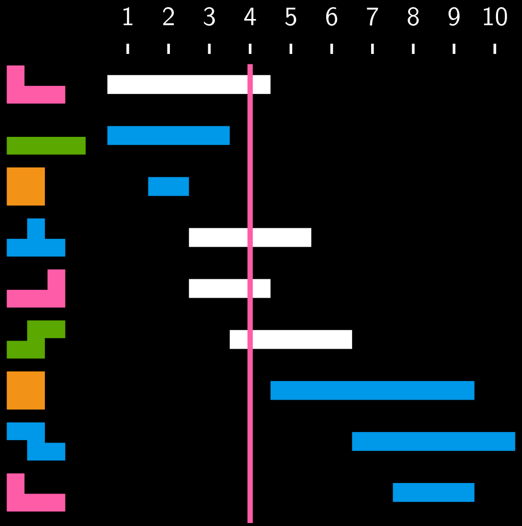
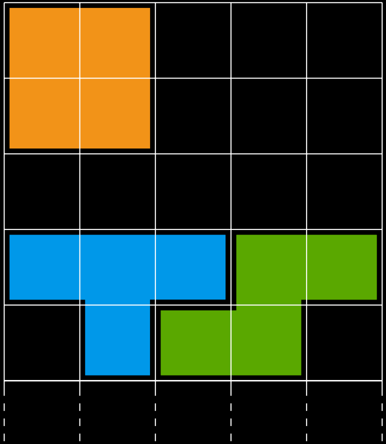


Example Problem



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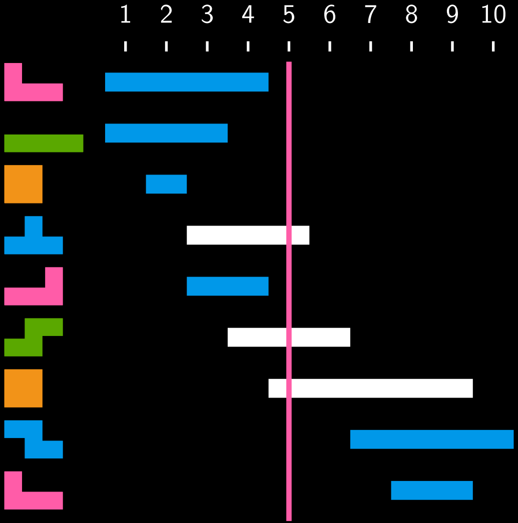

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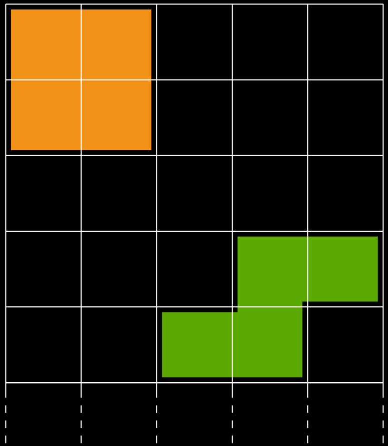

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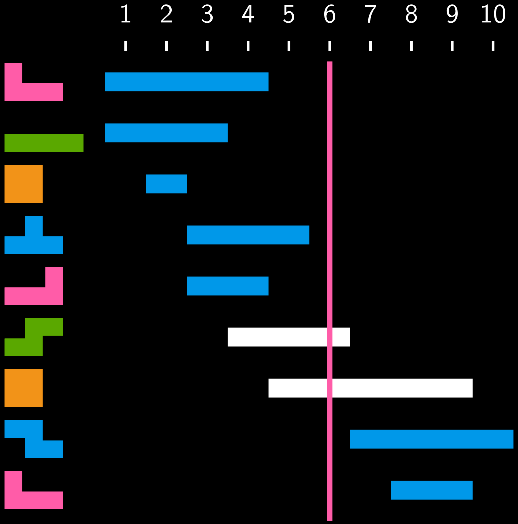
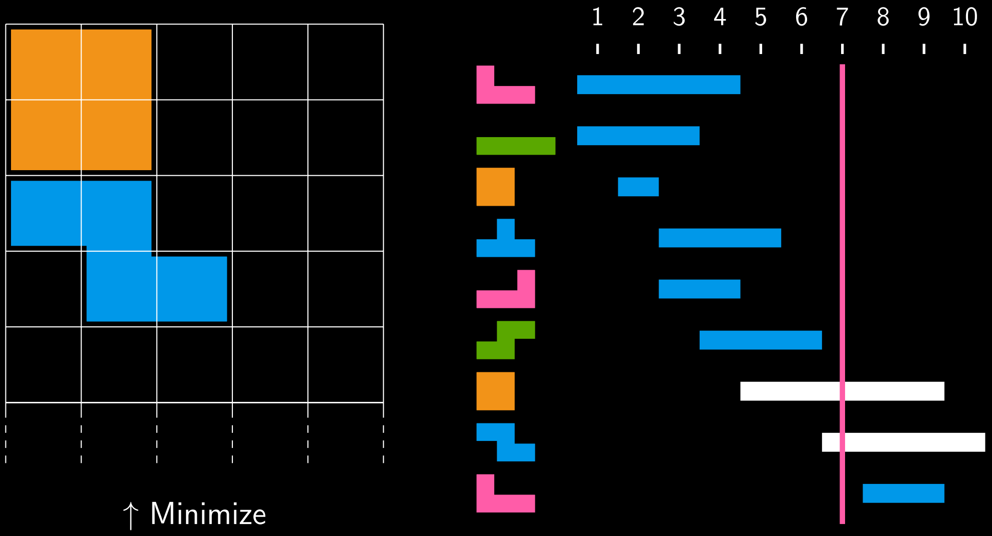


Example Problem



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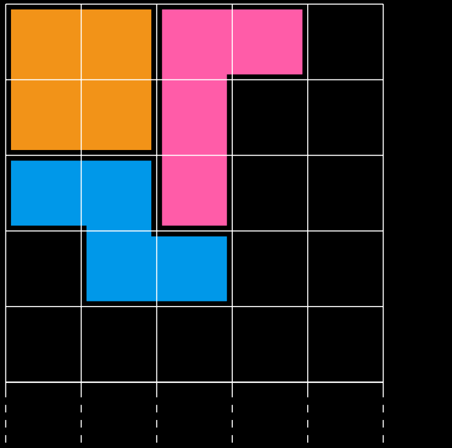

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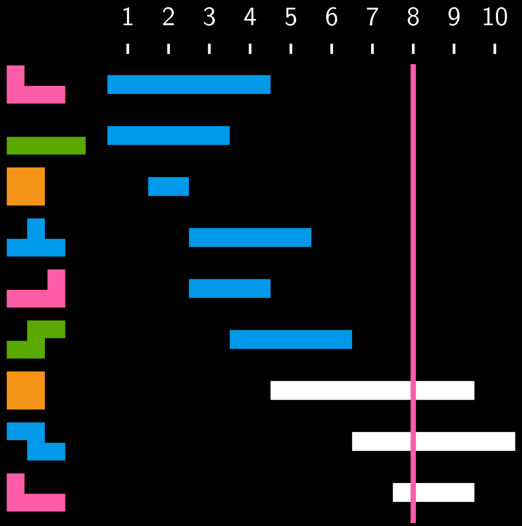

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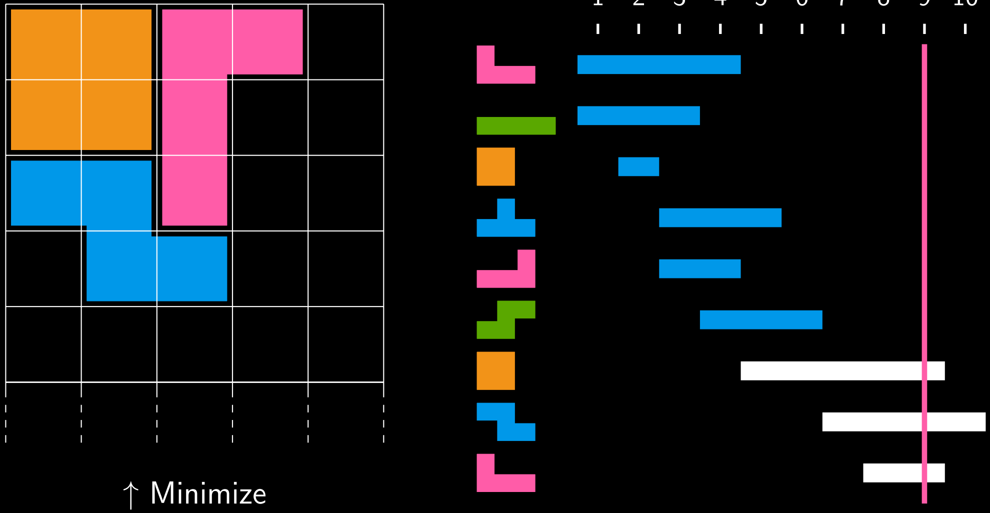

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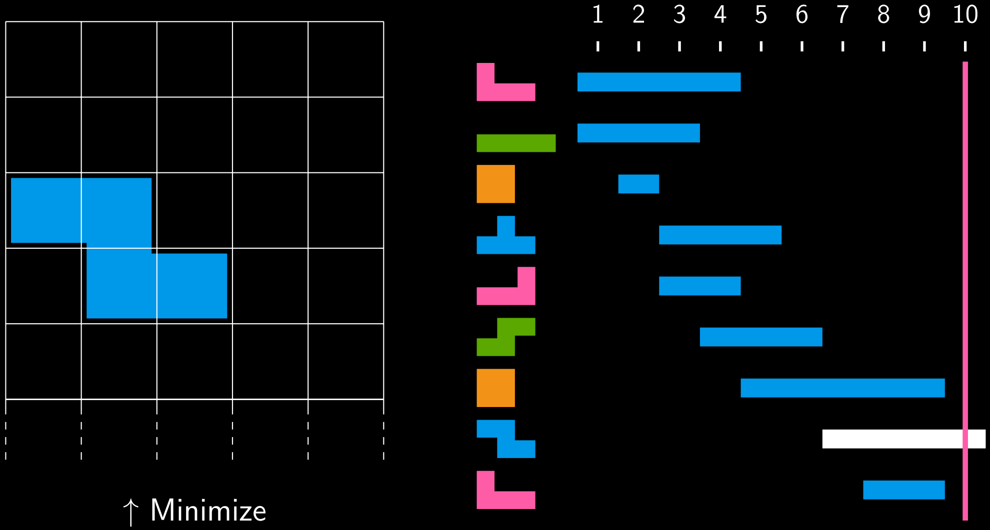
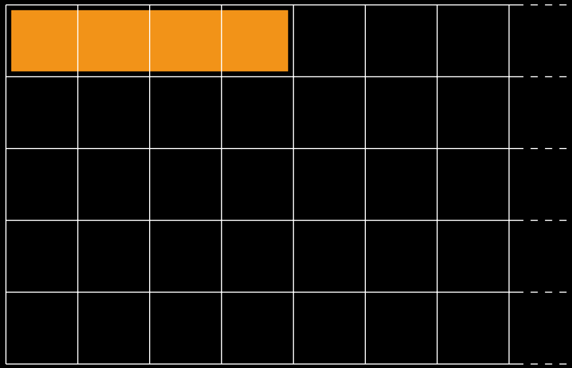


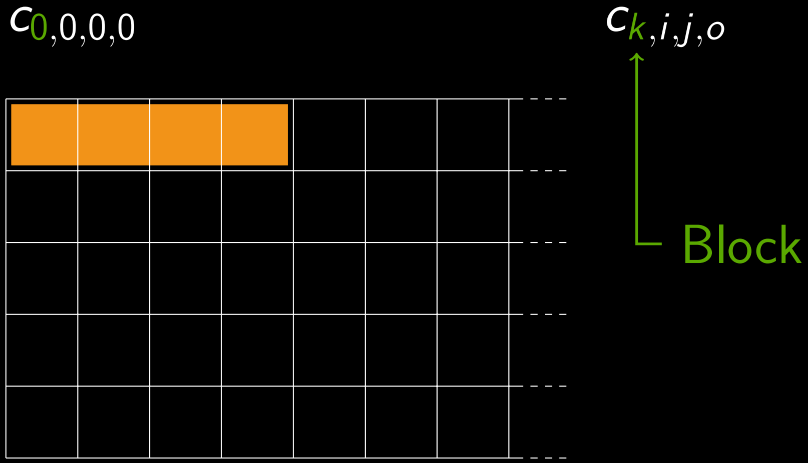
Example: Variables

$c_{0,0,0,0}$

$c_{k,i,j,o}$



Example: Variables



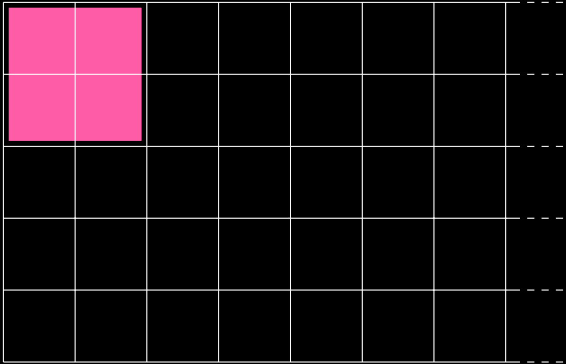
$c_{0,0,0,0}$

$c_{k,i,j,o}$

Block

Example: Variables

$c_{1,0,0,0}$



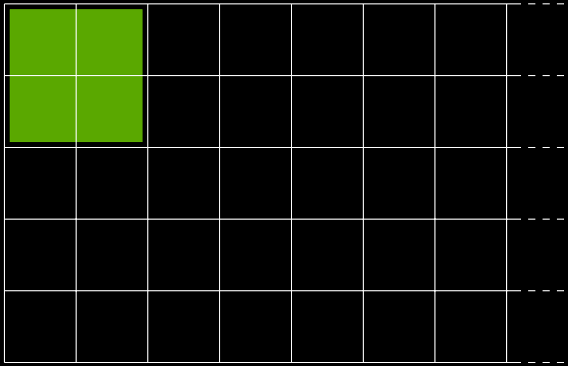
$c_{k,i,j,o}$

Block

Example: Variables

$c_{2,0,0,0}$

$c_{k,i,j,o}$



Block

Example: Variables

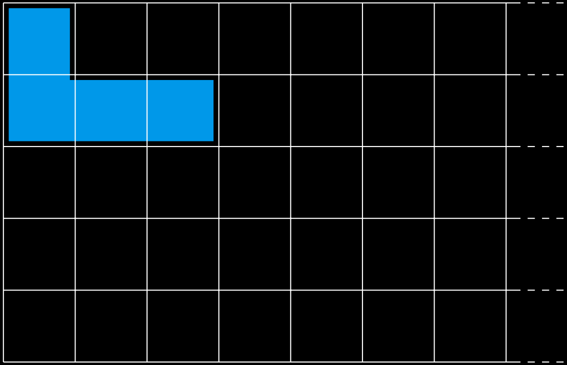
$c_{3,0,0,0}$

$c_{k,i,j,o}$

Block

Example: Variables

$c_{3,0,0,0}$



$c_{k,i,j,o}$

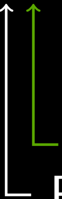
Row

Block

Example: Variables

$c_{3,1,0,0}$

$c_{k,i,j,o}$

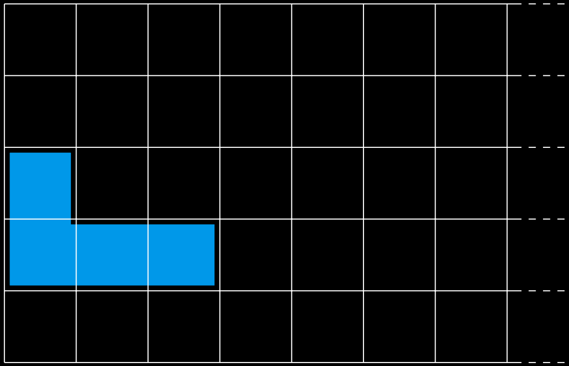


Row

Block

Example: Variables

$c_{3,2,0,0}$



$c_{k,i,j,o}$

Row

Block

Example: Variables

$c_{3,2,0,0}$

$c_{k,i,j,o}$

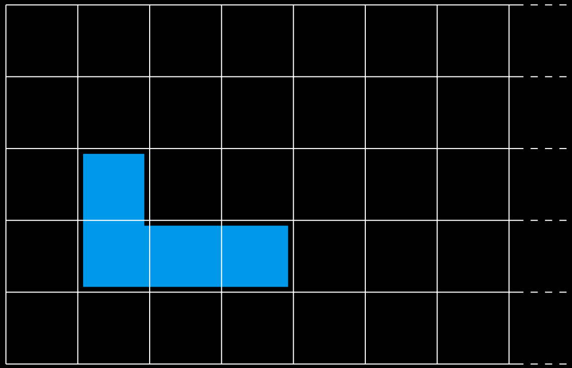
Column

Row

Block

Example: Variables

$c_{3,2,1,0}$



$c_{k,i,j,o}$

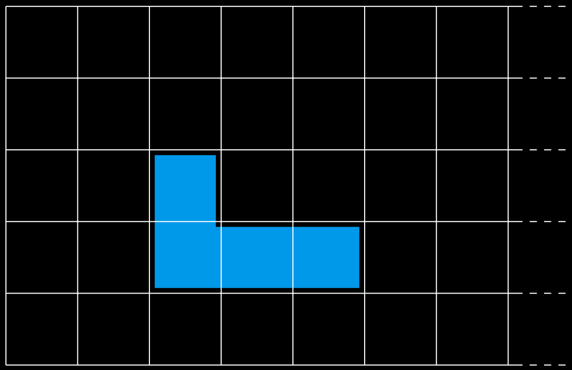
Column

Row

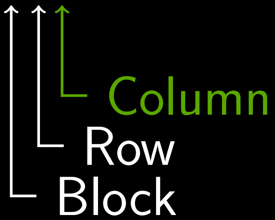
Block

Example: Variables

$c_{3,2,2,0}$



$c_{k,i,j,o}$



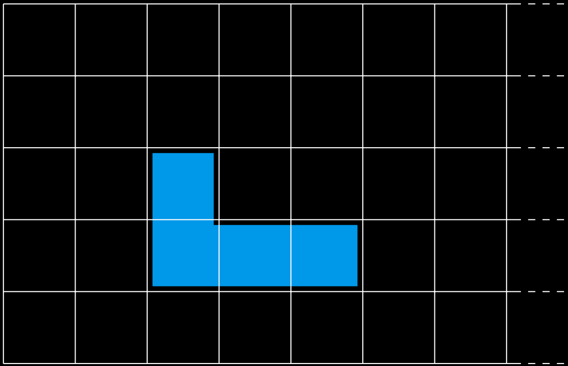
Column

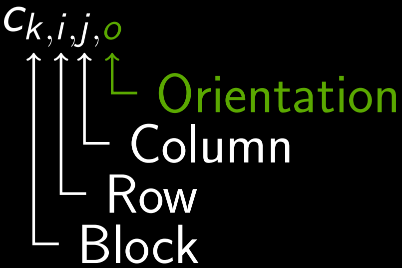
Row

Block

Example: Variables

$c_{3,2,2,0}$





$c_{k,i,j,o}$

Orientation

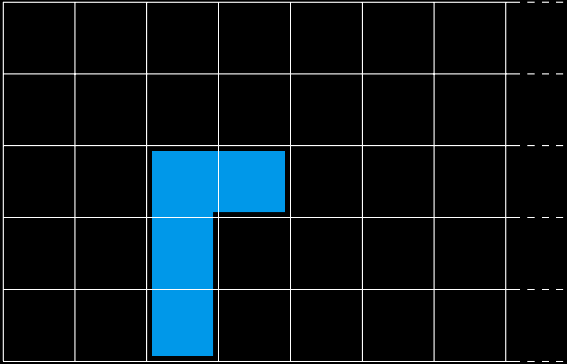
Column

Row

Block

Example: Variables

$c_{3,2,2,1}$



$c_{k,i,j,o}$

Orientation

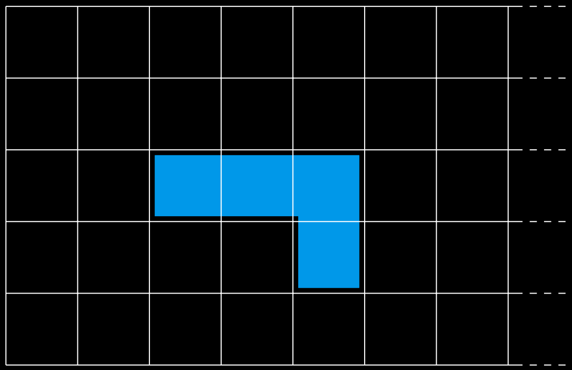
Column

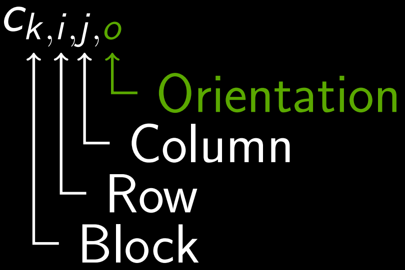
Row

Block

Example: Variables

$c_{3,2,2,2}$





$c_{k,i,j,o}$

Orientation

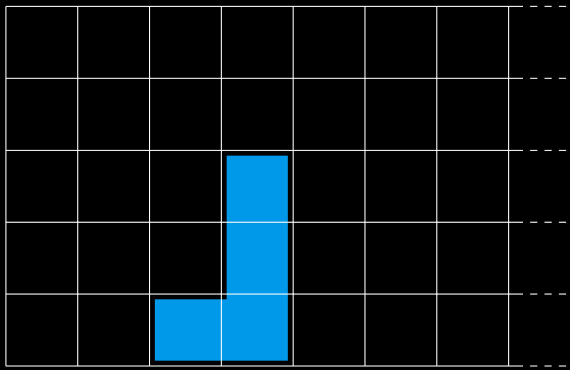
Column

Row

Block

Example: Variables

$c_{3,2,2,3}$

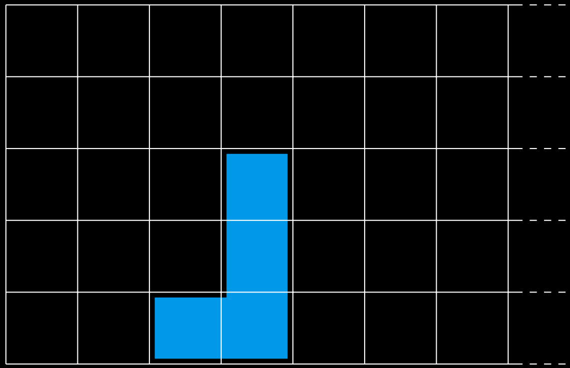


$c_{k,i,j,o}$

- Orientation
- Column
- Row
- Block

Example: Variables

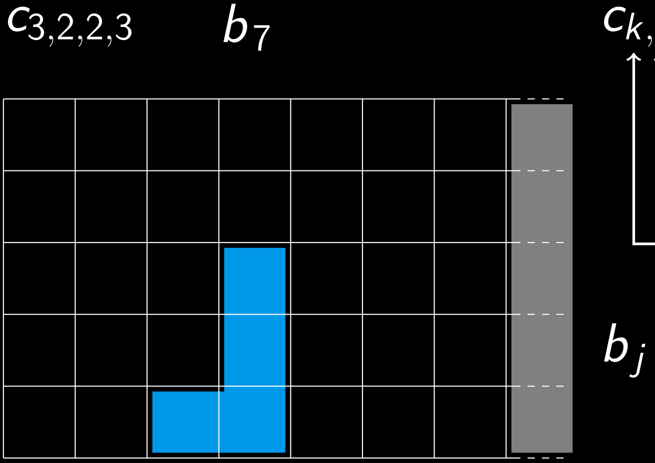
$c_{3,2,2,3}$



$c_{k,i,j,o}$

Orientation
Column
Row
Block

Example: Variables



$c_{3,2,2,3}$

b_7

b_j

$c_{k,i,j,o}$

Orientation

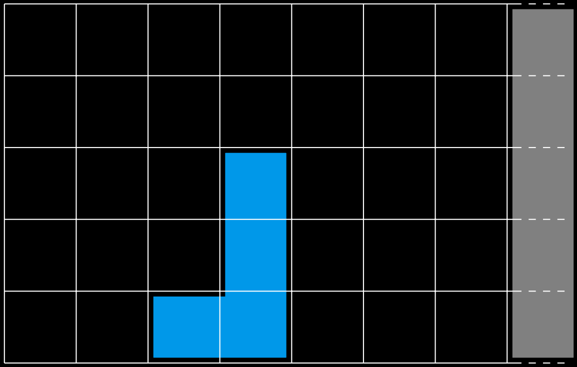
Column

Row

Block

Example: Variables

$c_{3,2,2,3}$ b_7



$c_{k,i,j,o}$

Orientation

Column

Row

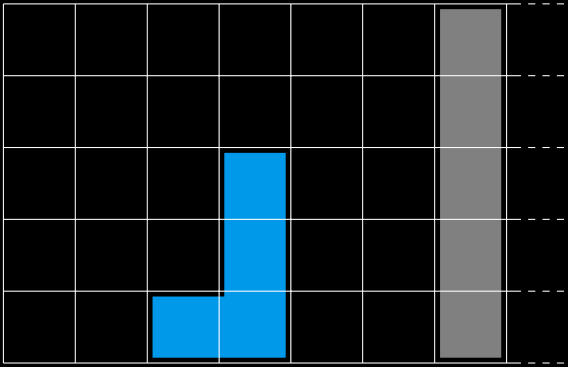
Block

b_j

Barrier Column

Example: Variables

$c_{3,2,2,3}$ b_6



$c_{k,i,j,o}$

Orientation

Column

Row

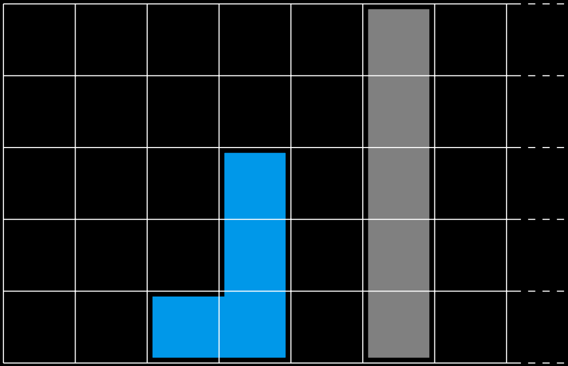
Block

b_j

Barrier Column

Example: Variables

$c_{3,2,2,3}$ b_5



$c_{k,i,j,o}$

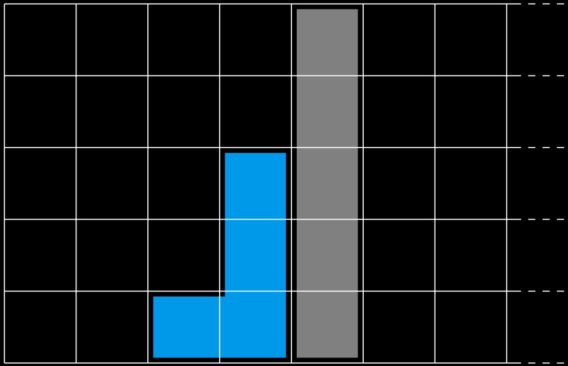
- Orientation
- Column
- Row
- Block

b_j

- Barrier Column

Example: Variables

$c_{3,2,2,3}$ b_4



$c_{k,i,j,o}$

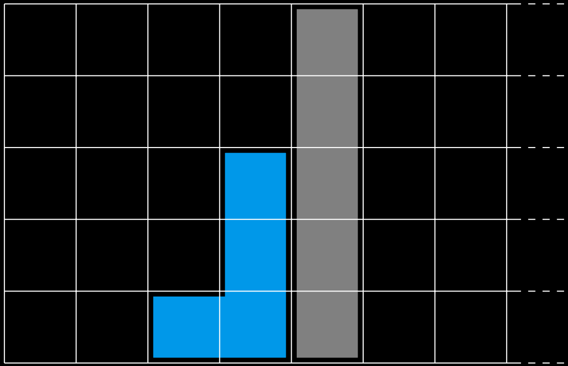
- o: Orientation
- j: Column
- i: Row
- k: Block

b_j

- j: Barrier Column

Example: Variables

$c_{3,2,2,3}$ b_4



$c_{k,i,j,o}$

- o: Orientation
- j: Column
- i: Row
- k: Block

b_j

- j: Barrier Column

Example: Constraints

$$C_k = \{c_{k,i,j,o} \mid i, j, o \text{ in bounds}\}$$

Example: Constraints

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$$F_{t,i,j} = \{c_{k,i',j',o} \mid c_{k,i',j',o} \text{ occupies } i, j \text{ at time } t\}$$

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- $(\forall t, i, j)$ at-most-one-of$(F_{t,i,j} \cup \{b_j\})$

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Example: Constraints

$C_k = \{c_{k,i,j,o} \mid i, j, o \text{ in bounds}\}$
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- $(\bigvee C_k) \wedge (\forall k)$ at-most-one-of(C_k)
- $(\forall t, i, j)$ at-most-one-of$(F_{t,i,j} \cup \{b_j\})$
- $(\forall j)\ b_j \rightarrow b_{j+1}$

Example: at-most-one-of

Binomial

x_0

x_8 x_1

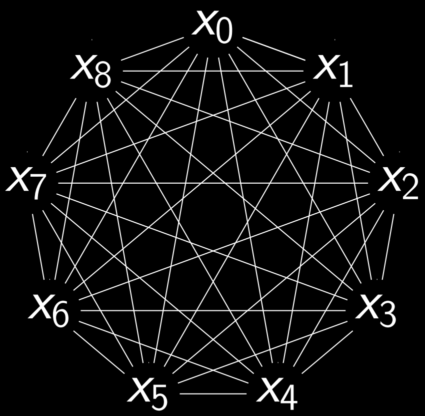
x_7 x_2

x_6 x_3

x_5 x_4

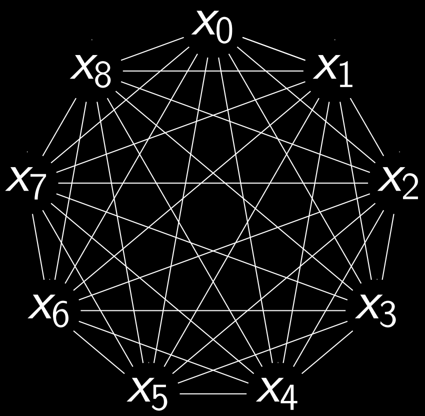
Example: at-most-one-of

Binomial



Example: at-most-one-of

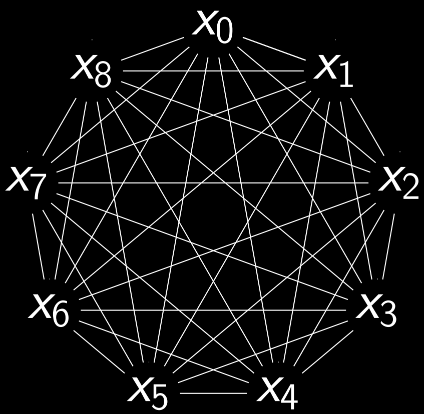
Binomial



$$(\neg x_0 \lor \neg x_1) \land$$
$$(\neg x_0 \lor \neg x_2) \land$$
$$\vdots$$
$$(\neg x_6 \lor \neg x_8) \land$$
$$(\neg x_7 \lor \neg x_8)$$

Example: at-most-one-of

Binomial

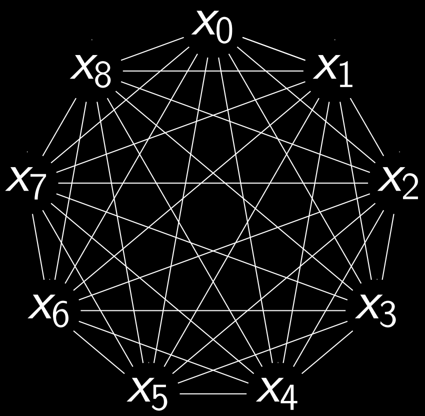


$$(\neg x_0 \vee \neg x_1) \;\wedge$$
$$(\neg x_0 \vee \neg x_2) \;\wedge$$
$$\vdots$$
$$(\neg x_6 \vee \neg x_8) \;\wedge$$
$$(\neg x_7 \vee \neg x_8)$$

$$\binom{n}{2}$$

Example: at-most-one-of

Binomial



$$(\neg x_0 \vee \neg x_1) \;\wedge$$
$$(\neg x_0 \vee \neg x_2) \;\wedge$$
$$\vdots$$
$$(\neg x_6 \vee \neg x_8) \;\wedge$$
$$(\neg x_7 \vee \neg x_8)$$

$$\binom{n}{2} = \Theta(n^2)$$

Example: at-most-one-of

Product

x_0	x_1	x_2
x_3	x_4	x_5
x_6	x_7	x_8

Example: at-most-one-of

Product

c_0	c_1	c_2
x_0	x_1	x_2
x_3	x_4	x_5
x_6	x_7	x_8

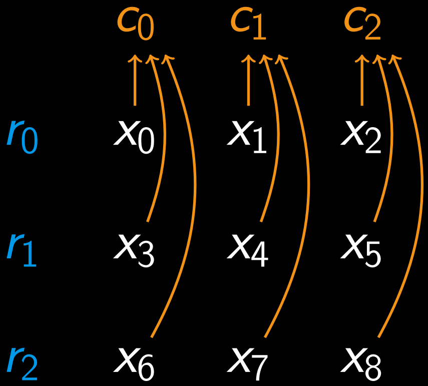
Example: at-most-one-of

Product

$$\begin{array}{cccc} & c_0 & c_1 & c_2 \\ r_0 & x_0 & x_1 & x_2 \\ r_1 & x_3 & x_4 & x_5 \\ r_2 & x_6 & x_7 & x_8 \end{array}$$

Example: at-most-one-of

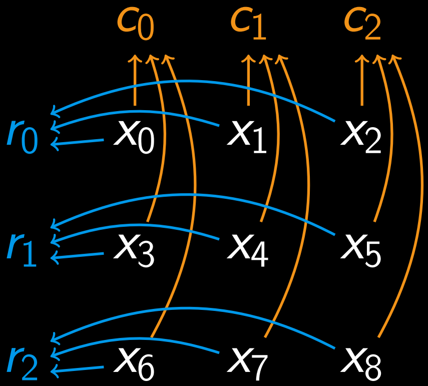
Product



	c_0	c_1	c_2
r_0	x_0	x_1	x_2
r_1	x_3	x_4	x_5
r_2	x_6	x_7	x_8

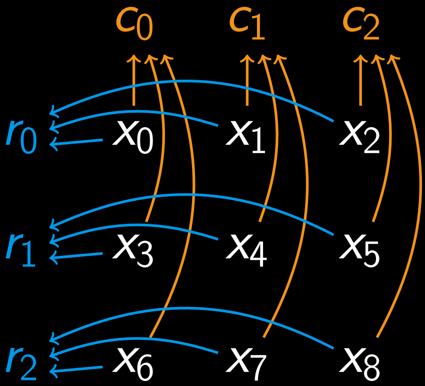
Example: at-most-one-of

Product



Example: at-most-one-of

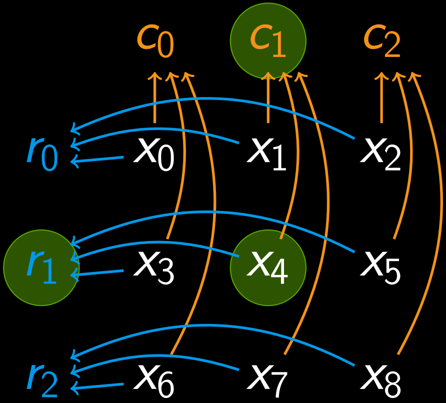
Product



$$
\begin{aligned}
&(\neg x_0 \vee c_0) && \wedge \\
&(\neg x_0 \vee r_0) && \wedge \\
&\quad\vdots && \\
&(\neg x_8 \vee c_2) && \wedge \\
&(\neg x_8 \vee r_2) &&
\end{aligned}
$$

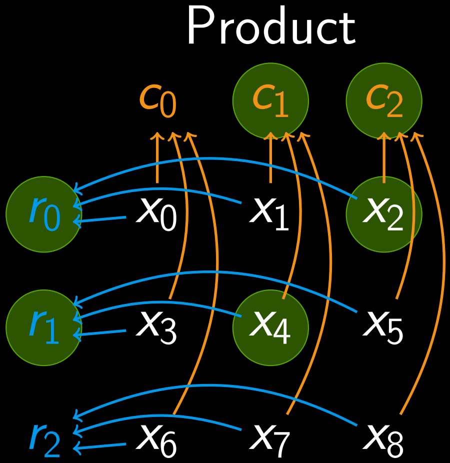
Example: at-most-one-of

Product



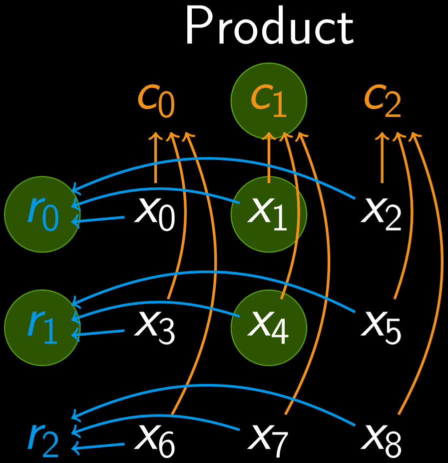
$$\begin{array}{ll}
(\neg x_0 \vee c_0) & \wedge \\
(\neg x_0 \vee r_0) & \wedge \\
\vdots & \\
(\neg x_8 \vee c_2) & \wedge \\
(\neg x_8 \vee r_2) &
\end{array}$$

Example: at-most-one-of



$$
\begin{aligned}
&(\neg x_0 \vee c_0) && \wedge \\
&(\neg x_0 \vee r_0) && \wedge \\
&\quad\vdots \\
&(\neg x_8 \vee c_2) && \wedge \\
&(\neg x_8 \vee r_2)
\end{aligned}
$$

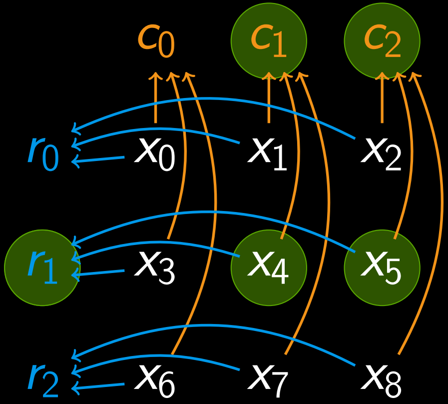
Example: at-most-one-of



$$
\begin{aligned}
&(\neg x_0 \vee c_0) \quad \wedge \\
&(\neg x_0 \vee r_0) \quad \wedge \\
&\quad \vdots \\
&(\neg x_8 \vee c_2) \quad \wedge \\
&(\neg x_8 \vee r_2)
\end{aligned}
$$

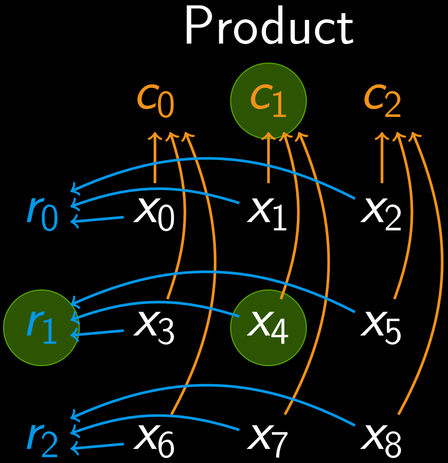
Example: at-most-one-of

Product



$$
\begin{aligned}
&(\neg x_0 \lor c_0) && \land \\
&(\neg x_0 \lor r_0) && \land \\
&\quad\vdots && \\
&(\neg x_8 \lor c_2) && \land \\
&(\neg x_8 \lor r_2) &&
\end{aligned}
$$

Example: at-most-one-of



$$
\begin{array}{ll}
(\neg x_0 \vee c_0) & \wedge \\
(\neg x_0 \vee r_0) & \wedge \\
\vdots & \\
(\neg x_8 \vee c_2) & \wedge \\
(\neg x_8 \vee r_2) & \wedge \\
\leq_1 (c_0, c_1, c_2) & \wedge \\
\leq_1 (r_0, r_1, r_2) &
\end{array}
$$

Example: Helping the Solver

$$\sum_{i,j}\left(\bigvee F_{t,i,j}\right) \text{ constant and known for every } t$$

Example: Counting

x_0

x_1

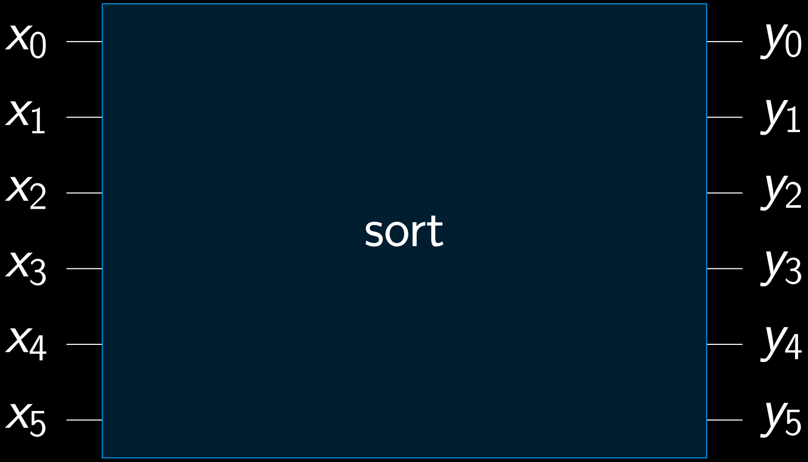
x_2

x_3

x_4

x_5

Example: Counting

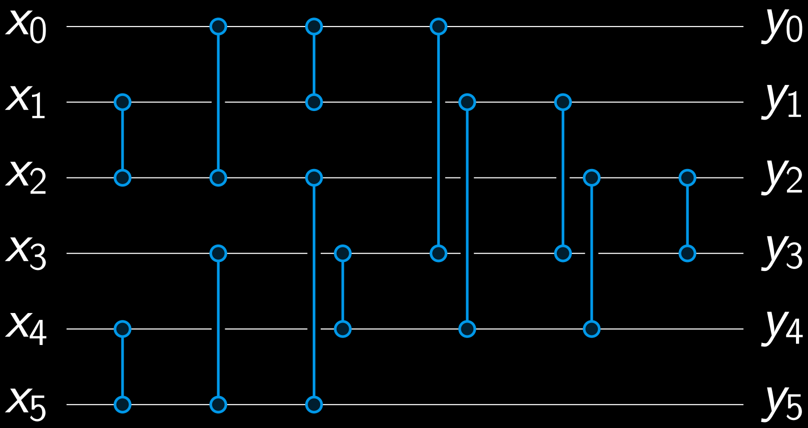


x_0
x_1
x_2
x_3
x_4
x_5

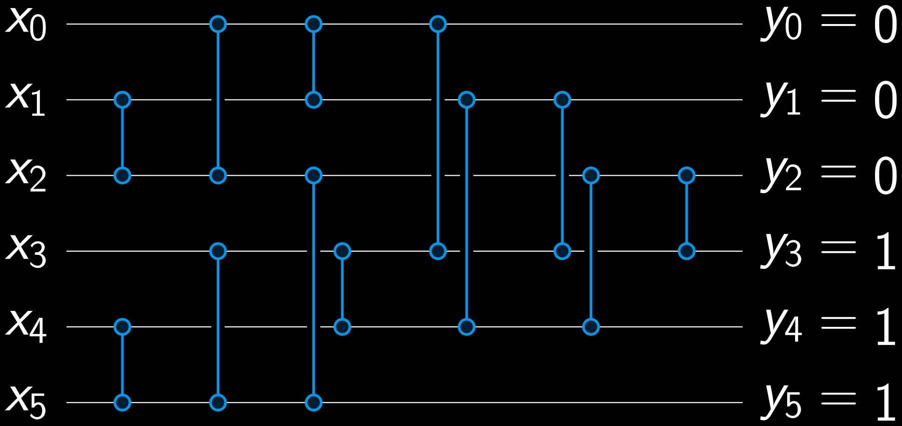
sort

y_0
y_1
y_2
y_3
y_4
y_5

Example: Counting



Example: Counting



Demo

SAT Solver Internals

Resolution

$$\frac{a \vee b \vee x \qquad \neg x \vee c \vee d}{}$$

Resolution

$$\frac{a \lor b \lor x \qquad \neg x \lor c \lor d}{a \lor b \lor c \lor d}$$

Resolution

$$\frac{\neg x \lor x \quad \neg x \rightarrow a \lor b \quad x \rightarrow c \lor d}{a \lor b \lor c \lor d}$$

DP Algorithm

$$\begin{array}{rcrcr} c & \vee & \neg b & \vee & \neg a \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \\ \neg d & \vee & \neg a & \vee & \neg c \end{array}$$

DP Algorithm

$$\begin{array}{rcrcr}
c & \vee & \neg b & \vee & \neg a \\
\neg e & \vee & \neg b & \vee & \neg a \\
d & \vee & \neg e & \vee & a \\
\neg b & \vee & \neg e & \vee & a \\
\neg e & \vee & \neg a & \vee & b \\
\neg d & \vee & \neg a & \vee & \neg c
\end{array}$$

DP Algorithm

$$\begin{array}{rcrcr} c & \vee & \neg b & \vee & \neg a \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \\ \neg d & \vee & \neg a & \vee & \neg c \end{array} \Rightarrow \begin{array}{rcrcrcr} \neg d & \vee & \neg a & \vee & \neg b & \vee & \neg a \\ \neg e & \vee & \neg b & \vee & \neg a & & \\ d & \vee & \neg e & \vee & a & & \\ \neg b & \vee & \neg e & \vee & a & & \\ \neg e & \vee & \neg a & \vee & b & & \end{array}$$

DP Algorithm

$$\begin{array}{rcrcr}
c & \vee & \neg b & \vee & \neg a \\
\neg e & \vee & \neg b & \vee & \neg a \\
d & \vee & \neg e & \vee & a \\
\neg b & \vee & \neg e & \vee & a \\
\neg e & \vee & \neg a & \vee & b \\
\neg d & \vee & \neg a & \vee & \neg c
\end{array}
\Rightarrow
\begin{array}{rcrcrcr}
\neg d & \vee & \neg a & \vee & \neg b & \vee & \neg a \\
\neg e & \vee & \neg b & \vee & \neg a & & \\
d & \vee & \neg e & \vee & a & & \\
\neg b & \vee & \neg e & \vee & a & & \\
\neg e & \vee & \neg a & \vee & b & &
\end{array}$$

DP Algorithm

$$\begin{array}{rcrcr} c & \vee & \neg b & \vee & \neg a \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \\ \neg d & \vee & \neg a & \vee & \neg c \end{array} \Rightarrow \begin{array}{rcrcr} \neg d & \vee & \neg a & \vee & \neg b \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \end{array}$$

DP Algorithm

$$\begin{array}{rcrcr} c & \vee & \neg b & \vee & \neg a \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \\ \neg d & \vee & \neg a & \vee & \neg c \end{array} \Rightarrow \begin{array}{rcrcr} \neg d & \vee & \neg a & \vee & \neg b \\ \neg e & \vee & \neg b & \vee & \neg a \\ d & \vee & \neg e & \vee & a \\ \neg b & \vee & \neg e & \vee & a \\ \neg e & \vee & \neg a & \vee & b \end{array}$$

DP Algorithm

$$\begin{array}{rcrcr}
c & \vee & \neg b & \vee & \neg a \\
\neg e & \vee & \neg b & \vee & \neg a \\
d & \vee & \neg e & \vee & a \\
\neg b & \vee & \neg e & \vee & a \\
\neg e & \vee & \neg a & \vee & b \\
\neg d & \vee & \neg a & \vee & \neg c
\end{array}
\Rightarrow
\begin{array}{rcrcr}
\neg d & \vee & \neg a & \vee & \neg b \\
\neg e & \vee & \neg b & \vee & \neg a \\
d & \vee & \neg e & \vee & a \\
\neg b & \vee & \neg e & \vee & a \\
\neg e & \vee & \neg a & \vee & b
\end{array}$$

Unit Resolution

$$\frac{a \lor b \lor x \qquad \neg x}{}$$

Unit Resolution

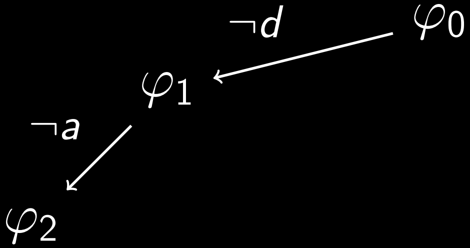
$$\frac{a \lor b \lor x \qquad \neg x}{a \lor b}$$

DPLL Algorithm

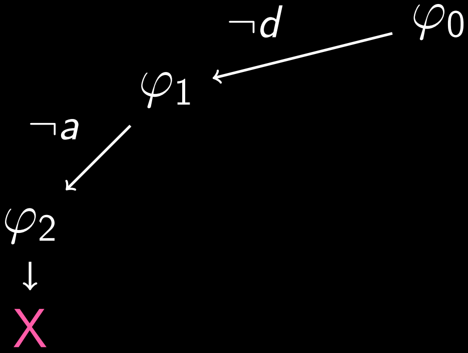
$$\varphi_0$$

DPLL Algorithm

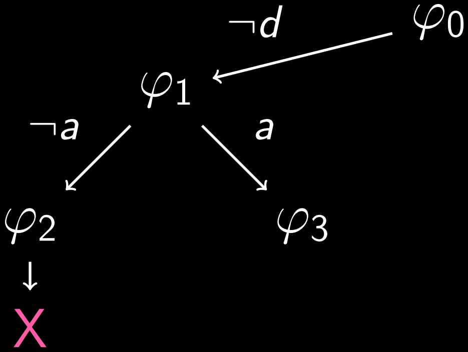
DPLL Algorithm



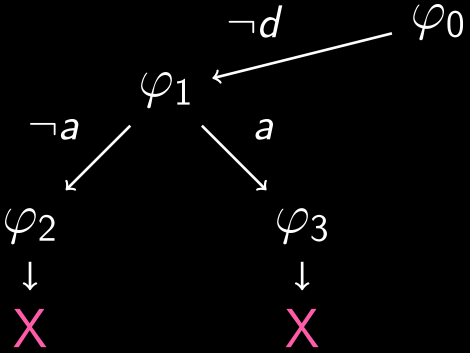
DPLL Algorithm



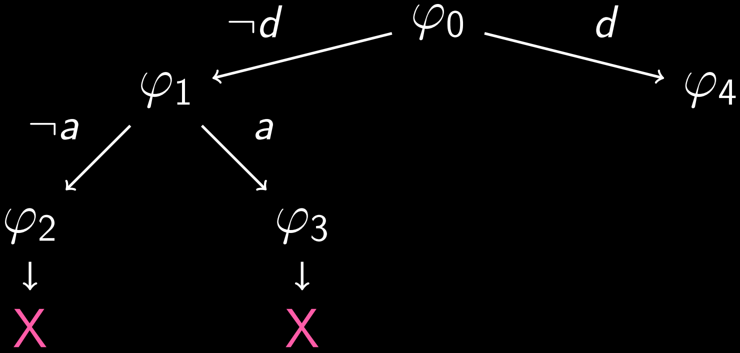
DPLL Algorithm



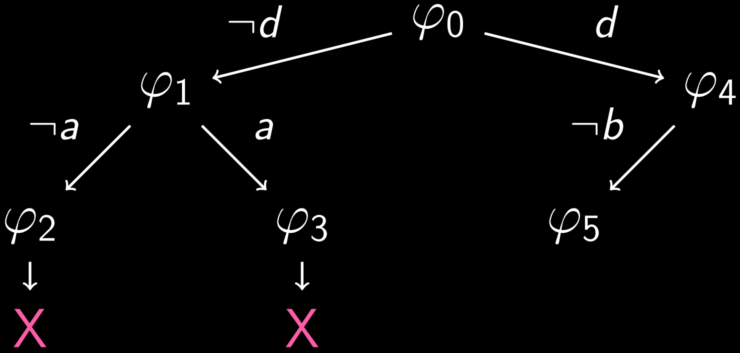
DPLL Algorithm



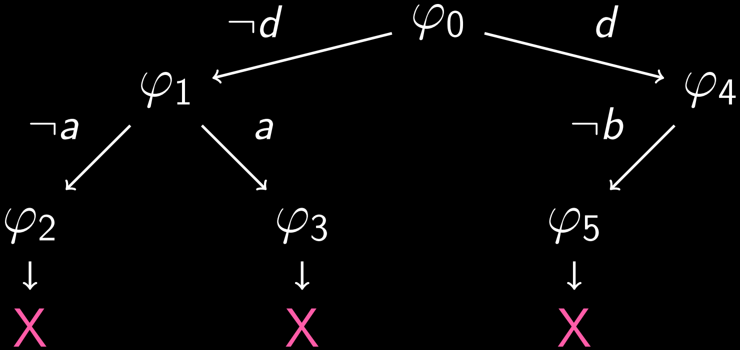
DPLL Algorithm



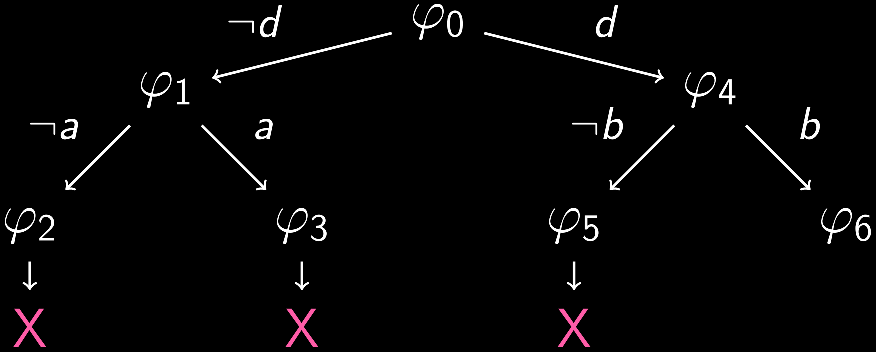
DPLL Algorithm



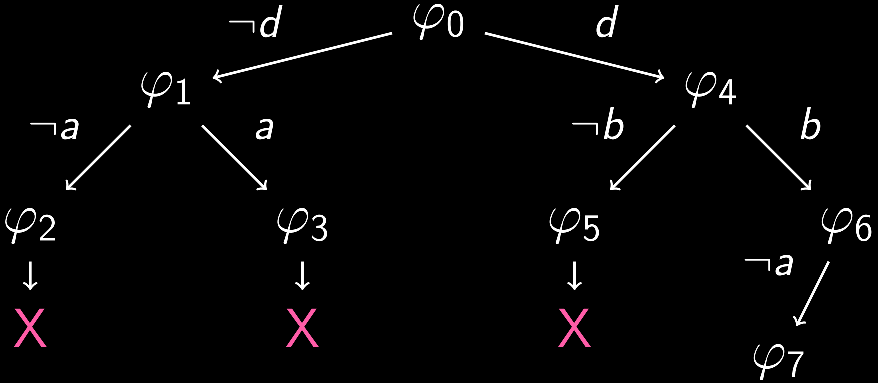
DPLL Algorithm



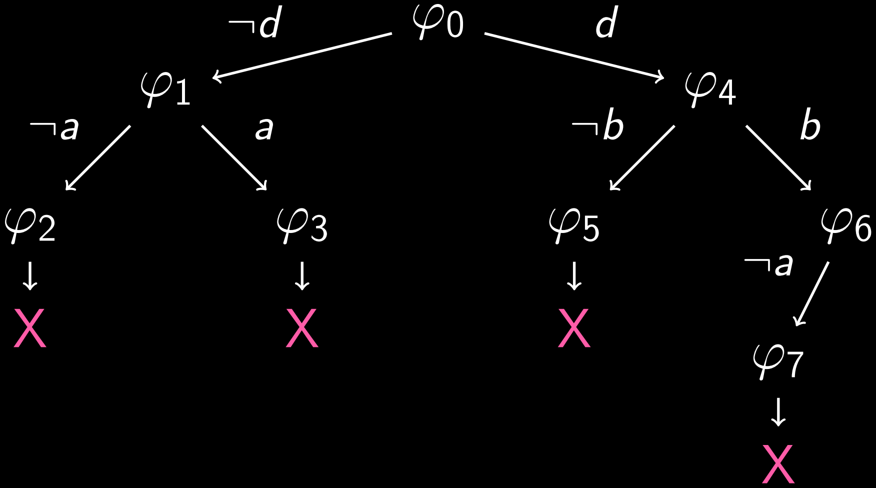
DPLL Algorithm



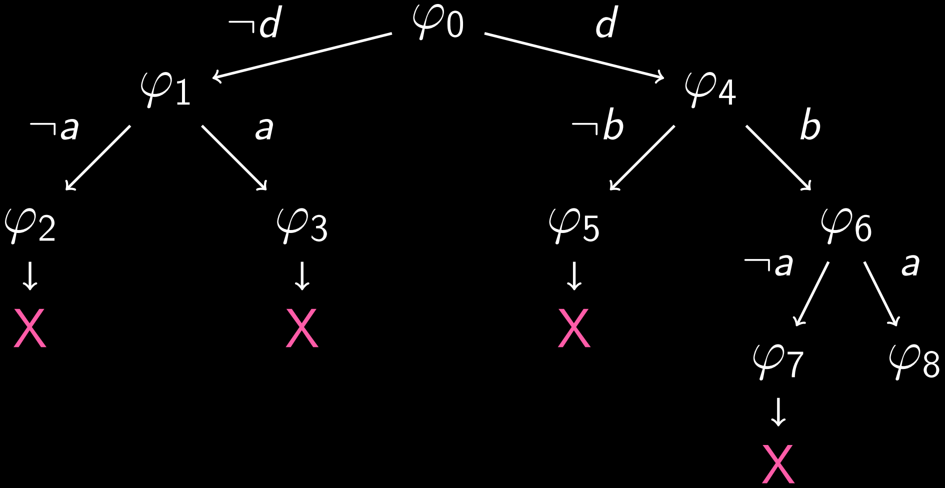
DPLL Algorithm



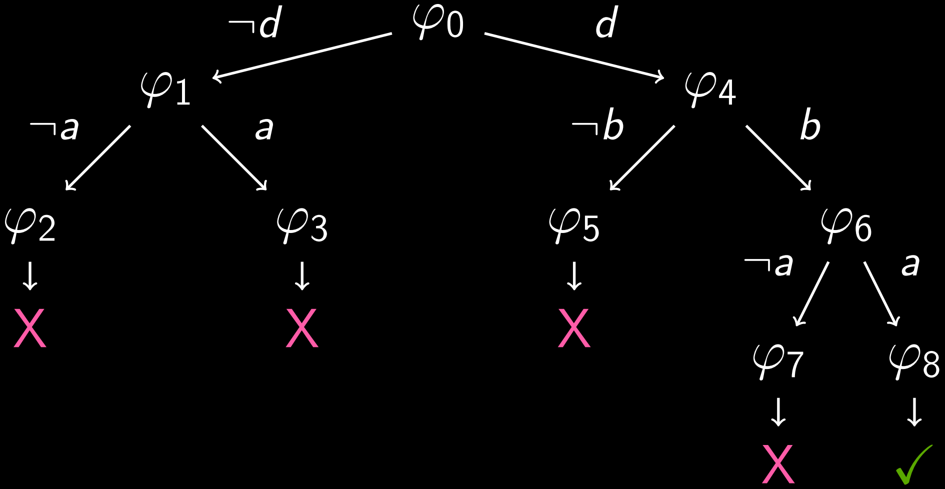
DPLL Algorithm



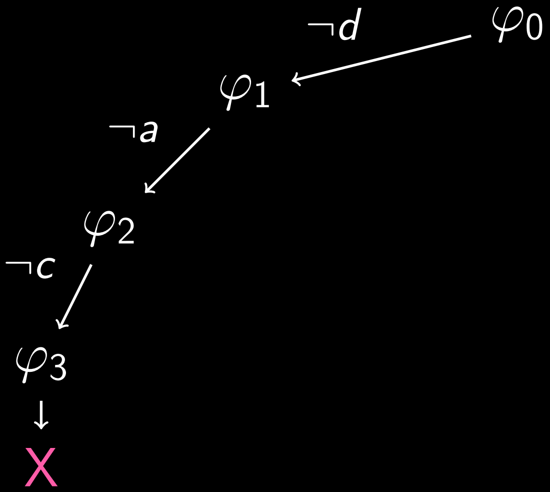
DPLL Algorithm



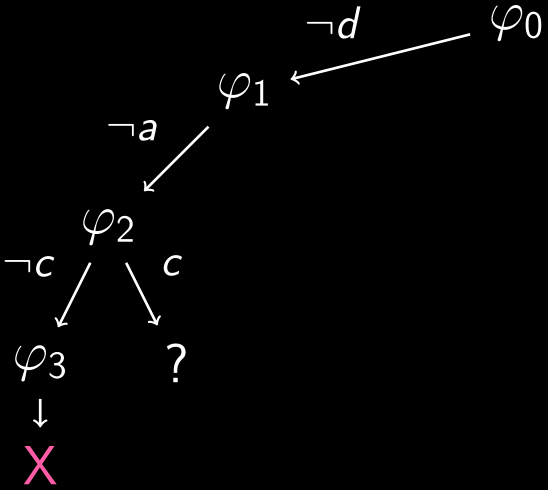
DPLL Algorithm



Non-Chronological Backtracking



Non-Chronological Backtracking



Non-Chronological Backtracking

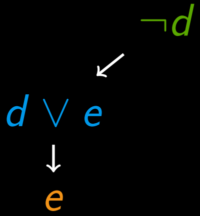
$\neg d$

Non-Chronological Backtracking

$\neg d$

$d \vee e$

Non-Chronological Backtracking

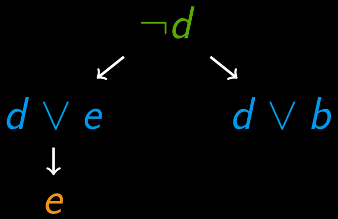


$\neg d$

$d \vee e$

e

Non-Chronological Backtracking



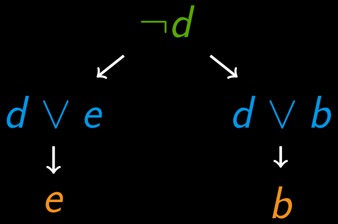
$\neg d$

$d \vee e$

$d \vee b$

e

Non-Chronological Backtracking

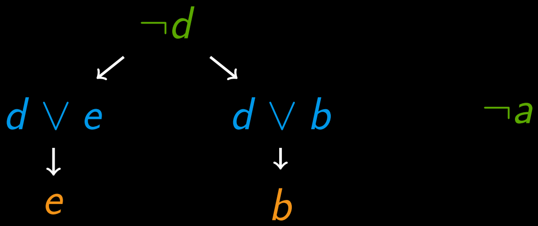


$\neg d$

$d \vee e$ $d \vee b$

e b

Non-Chronological Backtracking



$\neg d$

$d \lor e$

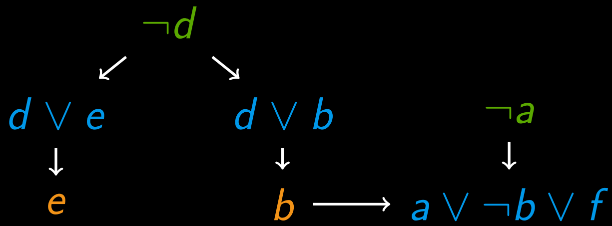
$d \lor b$

$\neg a$

e

b

Non-Chronological Backtracking



$\neg d$

$d \vee e$

$d \vee b$

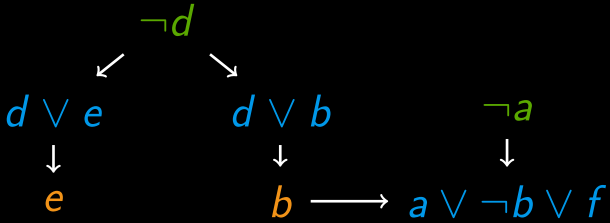
$\neg a$

e

b

$a \vee \neg b \vee f$

Non-Chronological Backtracking



$\neg d$

$d \lor e$

$d \lor b$

$\neg a$

e

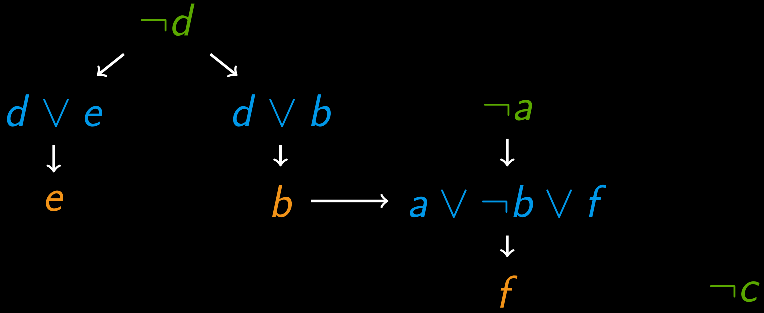
b

$a \lor \neg b \lor f$

Non-Chronological Backtracking

$$\begin{tikzcd}
 & \neg d \arrow[dl] \arrow[dr] & & \\
d \lor e \arrow[d] & & d \lor b \arrow[d] & \neg a \arrow[d] \\
e & & b \arrow[r, long] & a \lor \neg b \lor f \arrow[d] \\
 & & & f
\end{tikzcd}$$

Non-Chronological Backtracking



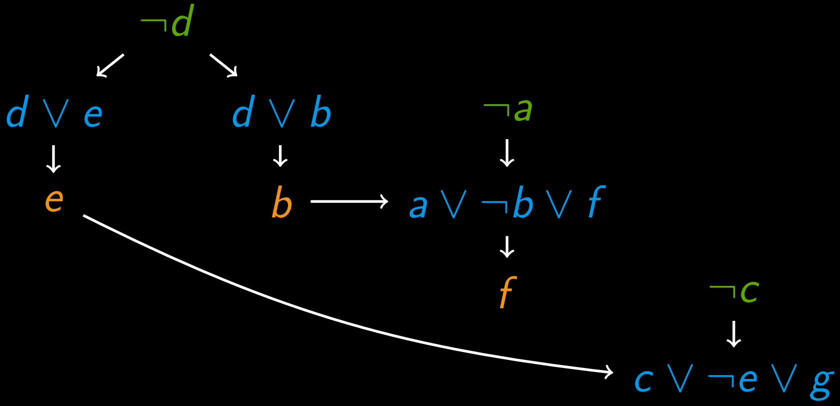
$\neg d$

$d \lor e$ $\quad$ $d \lor b$ $\quad$ $\neg a$

e $\quad$ $b \longrightarrow a \lor \neg b \lor f$

f $\quad$ $\neg c$

Non-Chronological Backtracking



$\neg d$

$d \vee e$

$d \vee b$

$\neg a$

e

b

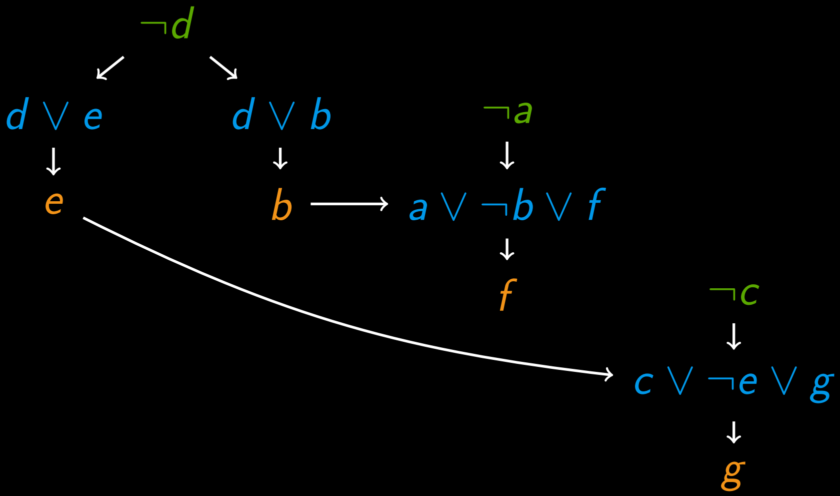
$a \vee \neg b \vee f$

f

$\neg c$

$c \vee \neg e \vee g$

Non-Chronological Backtracking



$\neg d$

$d \lor e$

$d \lor b$

$\neg a$

e

b

$a \lor \neg b \lor f$

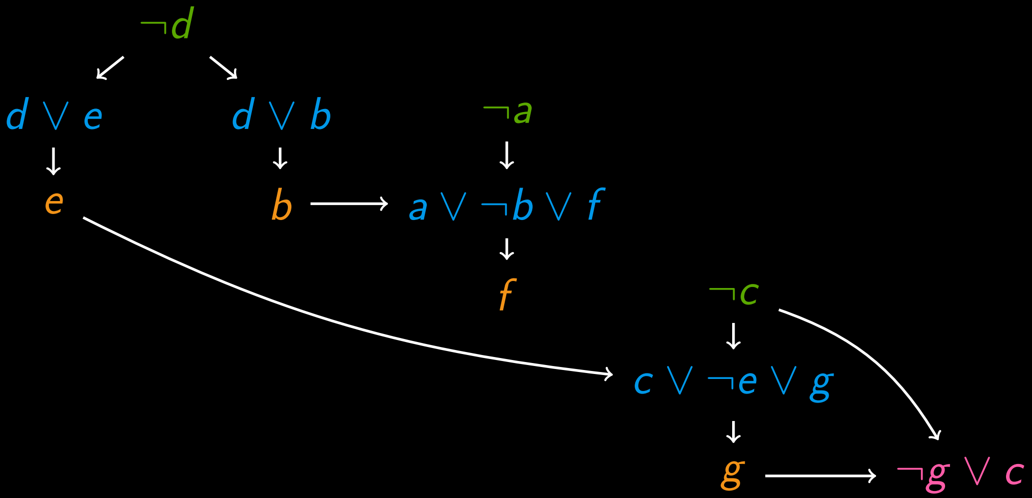
f

$\neg c$

$c \lor \neg e \lor g$

g

Non-Chronological Backtracking



$\neg d$

$d \lor e$

$d \lor b$

$\neg a$

e

b

$a \lor \neg b \lor f$

f

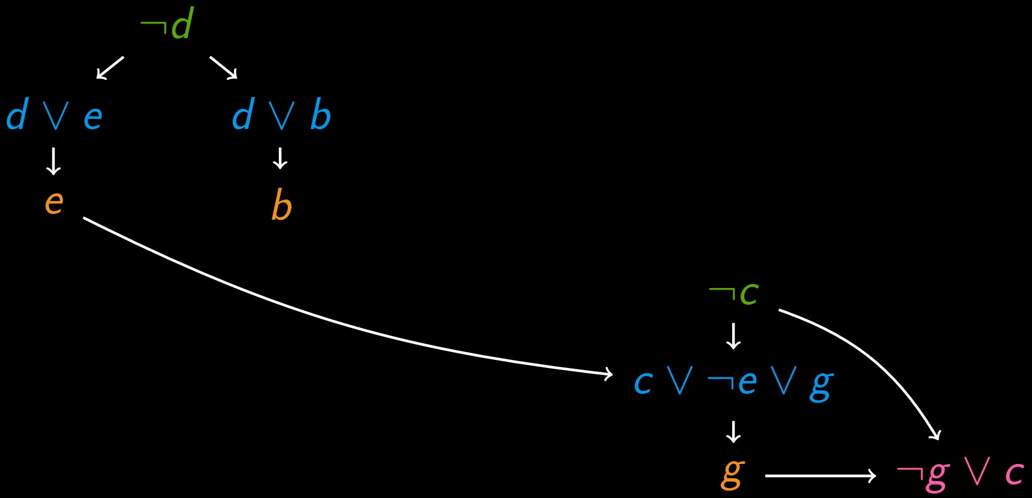
$\neg c$

$c \lor \neg e \lor g$

g

$\neg g \lor c$

Non-Chronological Backtracking



$\neg d$

$d \vee e$

$d \vee b$

e

b

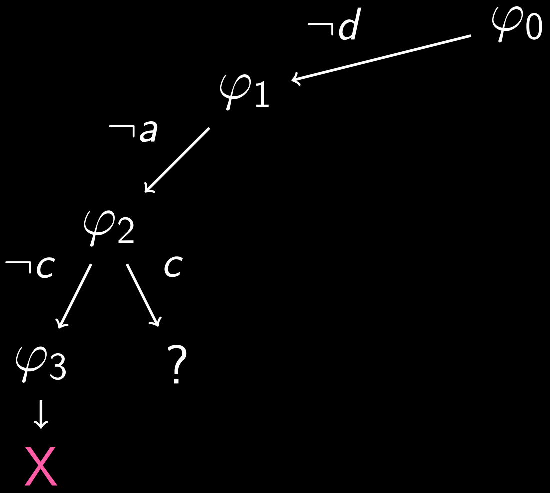
$\neg c$

$c \vee \neg e \vee g$

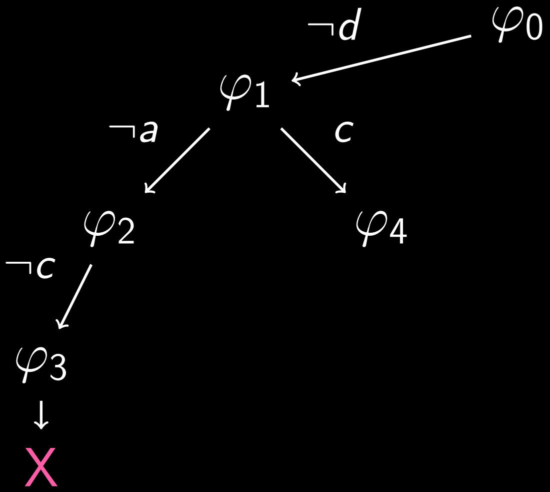
g

$\neg g \vee c$

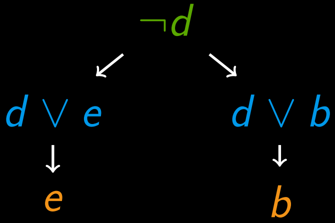
Non-Chronological Backtracking



Non-Chronological Backtracking



Conflict Driven Clause Learning



$\neg d$

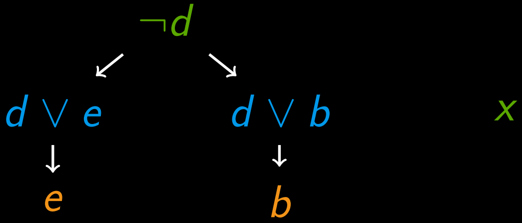
$d \vee e$

$d \vee b$

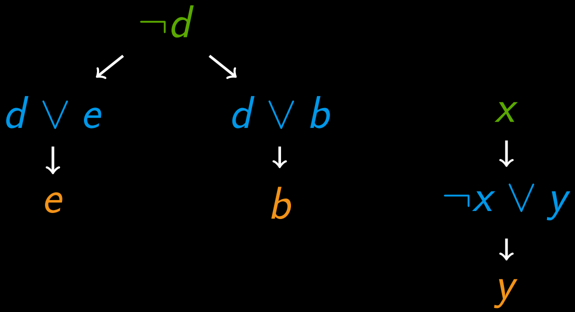
e

b

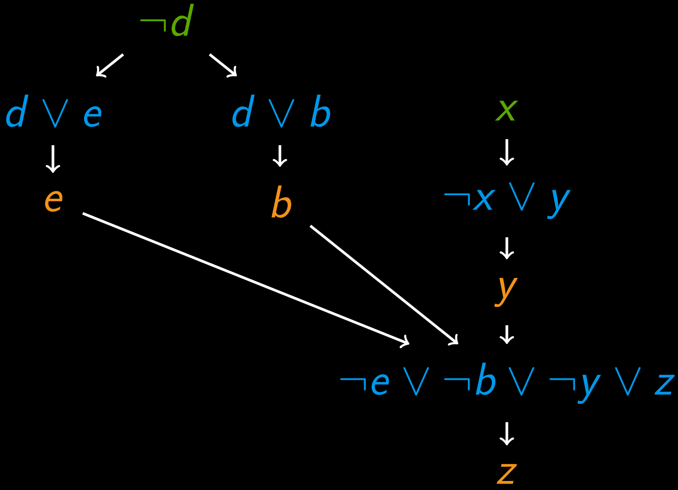
Conflict Driven Clause Learning



Conflict Driven Clause Learning



Conflict Driven Clause Learning



$\neg d$

$d \vee e$

$d \vee b$

x

e

b

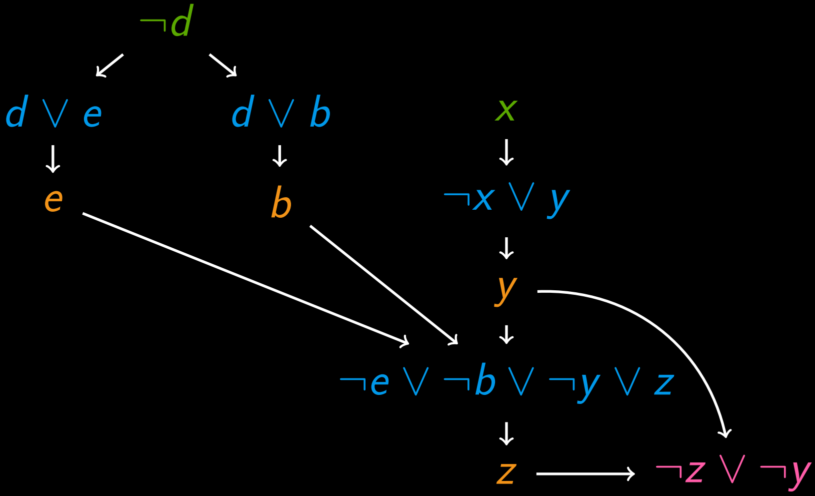
$\neg x \vee y$

y

$\neg e \vee \neg b \vee \neg y \vee z$

z

Conflict Driven Clause Learning



$\neg d$

$d \lor e$ $d \lor b$ x

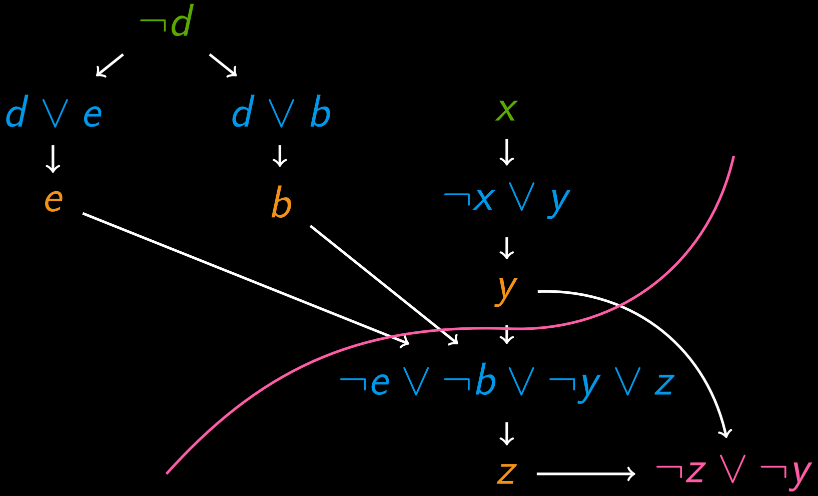
e b $\neg x \lor y$

y

$\neg e \lor \neg b \lor \neg y \lor z$

z $\neg z \lor \neg y$

Conflict Driven Clause Learning



$\neg d$

$d \vee e$

$d \vee b$

x

e

b

$\neg x \vee y$

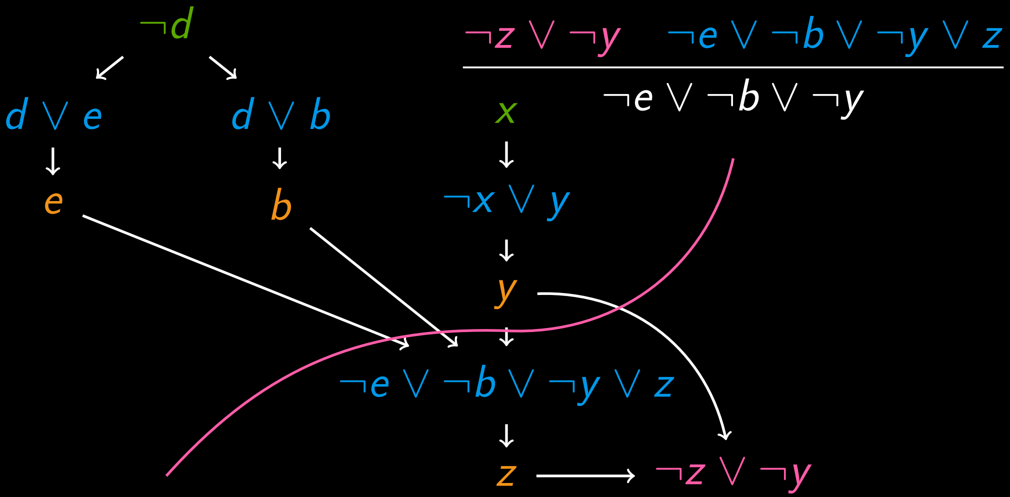
y

$\neg e \vee \neg b \vee \neg y \vee z$

z

$\neg z \vee \neg y$

Conflict Driven Clause Learning



$$\frac{\neg z \lor \neg y \qquad \neg e \lor \neg b \lor \neg y \lor z}{\neg e \lor \neg b \lor \neg y}$$

$\neg d$

$d \lor e$ $\quad$ $d \lor b$

e $\quad$ b

x

$\neg x \lor y$

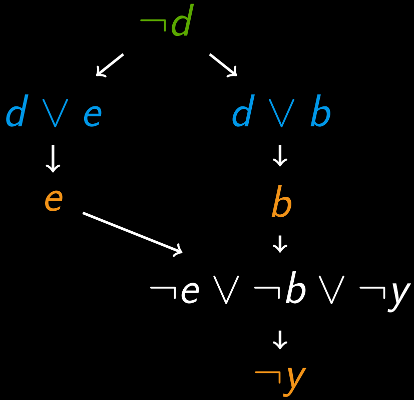
y

$\neg e \lor \neg b \lor \neg y \lor z$

z

$\neg z \lor \neg y$

Conflict Driven Clause Learning



$\neg d$

$d \lor e$

$d \lor b$

e

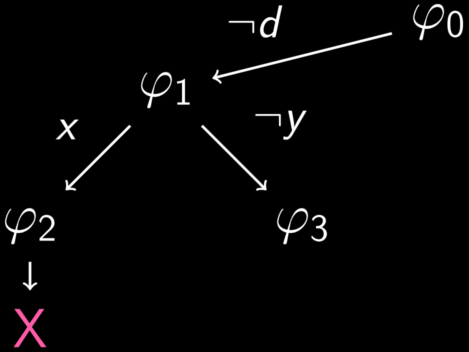
b

$\neg e \lor \neg b \lor \neg y$

$\neg y$

$$\frac{\neg z \lor \neg y \qquad \neg e \lor \neg b \lor \neg y \lor z}{\neg e \lor \neg b \lor \neg y}$$

Conflict Driven Clause Learning



What else?

- Optimized Data Structures
- Decision Heuristics
- Clause Removal
- Rapid Restarts
- Pre- & Inprocessing
- Local Search

Beyond SAT

- Model Counting
- MaxSAT
- Non-Disjunctive Constraints
- SAT modulo theories (SMT) using CDCL(T)
- Non-Boolean CDCL
- Quantified SAT (QBF)

- “Handbook of Satisfiability”
  Biere et al. (Eds.), IOS Press, 2009
- “The Art of Computer Programming”
  Vol. 4, F. 6, Knuth, Addison-Wesley, 2015
- “An Extensible SAT-solver” (MiniSat)
  Een, Sörensson, SAT 2003.